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Optimal dynamics in networks

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Abstract

This thesis describes the solution strategies for a model from the mathematical field of systems and control are described. The question is how to control a given system in such a way that a prespecified cost criterion is minimized. The system describes the dynamics on a network and two different types of dynamics are considered. The first form of dynamics corresponds to barrels with water which are connected by pipes. The second form of dynamics corresponds to a mass-spring system. Both problems are solved using a direct method and a method using the Algebraic Riccati Equation.

Contents

1 Introduction

In this thesis we will discuss the solution strategies for a model from the mathematical field of systems and control. The question is how to control a given system in such a way that a prespecified cost criterion is minimized. The problem is defined on a network and we will consider two different types of dynamics. The cost function is in both cases an infinite integral over time where the integrand is the sum of the squared norm of the input vector and the squared norm of a specified output vector. The output vector will differ in the different cases. Chapter 2 summarizes the definitions and theorems which we will use throughout this thesis. Chapter 3 considers the first model and chapter 4 the second model. In those chapters we will first describe each model and after that give detailed information about the mathematical model, solution method and the optimal solution. In the appendices we added some Matlab code in which the found solution can be demonstrated.

2 Our framework

In this chapter we review some definitions and theorems which will be used throughout the paper. It serves as background information and we will sometimes refer to this definitions and theorems. We also make some small assumptions about the model.

2.1 Graph theory

Because we consider a model on a network, we use a graph to define our network.

Definition 2.1. A *directed graph* G = (V,E) consists of a set V with n vertices and a set E of m ordered pairs (a, b), called edges. Here a indicates the tail vertex and b the head vertex of the edge.

Every directed graph is completely specified by its incidence matrix B.

Definition 2.2. For every directed graph we can construct an *incidence matrix* B. The matrix has dimensions $n \times m$, corresponding to the n vertices and m edges of the graph. The value of the (i, j)-th entry of B is given by:

$$b_{ij} = \begin{cases} 1 & \text{if the j-th edge is directed to the i-th vertex} \\ -1 & \text{if the j-th edge is directed from the i-th vertex} \\ 0 & \text{otherwise} \end{cases}$$

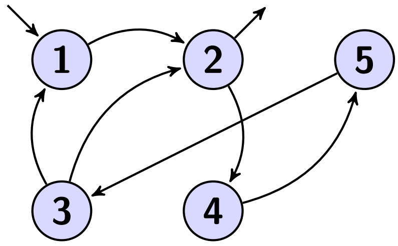
In some of our models we will connect some of the vertices to the outer world. In these cases we add edges which are only connected to one vertex. They are not included in the incidence matrix, but we make use of another additional matrix.

Definition 2.3. The matrix E is used to specify which vertices are connected to the outer world. The matrix has dimensions $n \times k$, where k is the number of vertices which are connected to the outer world. The (i, j)-th entry of E is given by:

$$e_{ij} = \begin{cases} 1 & \text{if the j-th external connection is directed to the i-th vertex} \\ -1 & \text{if the j-th external connection is directed from the i-th vertex} \\ 0 & \text{otherwise} \end{cases}$$

The orientation of the edges is only a matter of choice, a so-called sign convention.

Example 1. To clarify the definitions stated above, follows here an example of a directed graph and the corresponding incidence matrix B and matrix E.



$$B = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \qquad E = \begin{pmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Definition 2.4. A graph is called *connected* if you can find a path of edges from any vertex to any other vertex.

Theorem 2.1. *The incidence matrix B of a connected graph has the following property:*

$$\ker(B^T) = \text{span}(\mathbb{1}) \qquad \mathbb{1} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

2.2 Algebraic Riccati Equation

One of the solution methods uses solutions of the Algebraic Riccati Equation (ARE). We will now state the definitions and theorems which we will use. They all appear in [van der Schaft, 2009, p. 70-85].

Definition 2.5. For a linear system $\dot{x} = Ax + Bu$ with cost criterion

$$J_\infty(x_0, u) = \frac{1}{2}\int_0^\infty (x^T(t)Qx(t) + u^T(t)u(t))\ \mathrm{d}t \tag{1}$$

the Algebraic Riccati Equation (ARE) is defined as:

$$PA + A^TP - PBB^TP + Q = 0 \tag{2}$$

A solution P is a square matrix with the same dimensions as A. In general there are multiple solutions.

Theorem 2.2. *For the system $\dot{x} = Ax + Bu$ with cost criterion (1) the optimal control u has the following special form*

$$u_{opt} = -B^T Px(t) \tag{3}$$

with total minimal cost

$$J_\infty(x_0, u_{opt}) = \frac{1}{2} x_0^T P x_0 \tag{4}$$

Here P is one of the positive solutions of the ARE.

Not every solution P of the ARE gives rise to the correct optimal solution. In many cases we can apply this theorem:

Theorem 2.3. *If the linear system $\dot{x} = Ax+Bu$ with artificial output $y = Qx$ is minimal, then there exists exactly one positive solution to the Algebraic Riccati Equation.*

Theorem 2.4. *If the system $\dot{x} = Ax + Bu$ with artificial output $y = Qx$ is minimal, let P_∞ be the unique positive solution of the Algebraic Riccati equation. The infinite horizon LQ-problem with cost criterion (1) has a unique solution:*

$$u_{opt} = -B^T P_\infty x(t) \tag{5}$$

with total minimal cost

$$J_\infty(x_0, u_{opt}) = \frac{1}{2} x_0^T P_\infty x_0 \tag{6}$$

Furthermore $A - BB^T P_\infty$ is asymptotically stable.

2.3 Assumptions

We would like to keep our model and solution as general as possible. We make the following assumptions which will not restrict the generality and usability of the found solutions.

Assumption 1. We assume that in the graphs we consider every edge has a different head and tail vertex. This means we have no self loops.

Assumption 2. We assume that all the graphs we consider are connected. If not, we can consider every connected component seperately.

3 First model

3.1 Model description

We consider the following problem. Imagine some barrels with water which are connected by pipes. Every barrel has a given water height. Water can flow through the given pipes and we are able to control these flows. Setting up a flow from one barrel to another comes at a certain price. But differences in water height between two barrels are also valuable. The object is to find a flow protocol which minimizes the total cost over time.

3.2 Mathematical description

The model described above can be translated into a mathematical model. The dynamics is given by:

$$\begin{cases} \dot{x} = Bu \\ y = B^T x \end{cases} \qquad \begin{matrix} x \in \mathbb{R}^n \\ u \in \mathbb{R}^m \\ y \in \mathbb{R}^m \end{matrix} \tag{7}$$

Here B is the incidence matrix of the graph. We call x the state vector of the system, where x_i corresponds to the state of the i-th vertex, let's say water height. It depends on time and $\dot{x}$ is the time derivative of x. We call u the input vector, since we can set its value. Here u_i is the amount of flow through edge i. We call y the output vector; the value of y_i is the difference between the state on the tail vertex and the state on the head vertex of edge i.

The cost criterion is defined by:

$$J_\infty(x_0, u) = \frac{1}{2} \int_0^\infty (\|y\|^2 + \|u\|^2) \, \mathrm{d}t \tag{8}$$

The objective is to find the input u which minimizes the cost criterion, which we will call the optimal control u_{opt}.

3.3 Calculations

We will use two different approaches in finding the optimal control for the system (7) with cost criterion (8).

3.3.1 First approach

The first approach uses the solutions of the Algebraic Riccati Equation (ARE) to find the optimal control, as described in Section 2.2. The ARE simplifies in our case to:

$$BB^T = PBB^T P \tag{9}$$

Of course we have the trivial solution $P_1 = I$. But we also have solutions of the form $P_2 = I + \alpha \mathbb{1}\mathbb{1}^T$ with $\alpha \in \mathbb{R}$. We can show this by using Theorem 2.1:

$$\begin{aligned} PBB^T P &= (I + \alpha \mathbb{1}\mathbb{1}^T) BB^T (I + \alpha \mathbb{1}\mathbb{1}^T) \\ &= BB^T + \alpha BB^T \mathbb{1}\mathbb{1}^T + \alpha \mathbb{1}\mathbb{1}^T BB^T + \alpha^2 \mathbb{1}\mathbb{1}^T BB^T \mathbb{1}\mathbb{1}^T \\ &= BB^T \end{aligned}$$

In fact the solution set P_2 contains P_1 (take $\alpha = 0$). At this point we did not prove that there are no solutions outside the solution set P_2. In the rest of this first approach we assume that there are no other (positive) solutions than P_2.

By Theorem 2.2 the optimal control u_{opt} is given by $u_{\text{opt}} = -B^T P x$. We will show that for every P with $\alpha \in \mathbb{R}$ we end up with the same control u and therefore the same closed-loop dynamics:

$$u_{\text{opt}} = -B^T P_2 x = -B^T x - \alpha B^T \mathbb{1}\mathbb{1}^T x = -B^T x \qquad \begin{cases} \dot{x} = -BB^T x \\ y = B^T x \end{cases} \tag{10}$$

From Theorem 2.2 the value function is given by $\frac{1}{2}x_0^T P x_0$. Even though the closed-loop dynamics are equal for every α, the value function is not:

$$P_2 = I + \alpha \mathbb{1}\mathbb{1}^T : \qquad \frac{1}{2}x_0^T P_2 x_0 = \frac{1}{2}\sum_{i=1}^{N} x_{0i}^2 + \alpha \frac{1}{2}\left(\sum_{i=1}^{N} x_{0i}\right)^2 \tag{11}$$

The value of $\frac{1}{2}x_0^T P x_0$ should correctly match the real value function given by the integral (8). If the initial vector $x_0 = \beta\mathbb{1}$, the total costs are zero, namely; the vector $y = B^T x_0 = 0$ and as a consequence $d = 0$ is optimal, which leads to $J_\infty = 0$. In other words: $J_\infty(\beta\mathbb{1}, u_{\text{opt}}) = 0$ and this can be used to determine the value of α in P_2 (assume $\beta \neq 0 \neq n$) :

$$J_\infty(\beta\mathbb{1}, u_{\text{opt}}) = 0 \tag{12}$$
$$\Longrightarrow \qquad \frac{1}{2}\beta^2 \mathbb{1}^T \left(I + \alpha\mathbb{1}\mathbb{1}^T\right)\mathbb{1} = 0 \tag{13}$$
$$\Longleftrightarrow \qquad \mathbb{1}^T \left(I + \alpha\mathbb{1}\mathbb{1}^T\right)\mathbb{1} = 0 \tag{14}$$
$$\Longleftrightarrow \qquad \mathbb{1}^T \left(\mathbb{1} + \alpha n\mathbb{1}\right) = 0 \tag{15}$$
$$\Longleftrightarrow \qquad n + \alpha n^2 = 0 \tag{16}$$
$$\Longleftrightarrow \qquad \alpha = -\frac{1}{n} \tag{17}$$

We see that there is only one possible value for α and therefore we found the solution $P_\infty = I - \frac{1}{n}\mathbb{1}\mathbb{1}^T$. We will now discuss another solution method in which we show that the solution above is correct. In Section 3.4 we will discuss several properties of the found solution.

3.3.2 Second approach

The second approach uses some straightforward calculations on the cost criterion. We will use two properties given by these equations:

$$\frac{1}{2}\|u + y\|^2 = \frac{1}{2}\|u\|^2 + \frac{1}{2}\|y\|^2 + u^T y \tag{18}$$

$$\frac{d}{dt}\frac{1}{2}\|x\|^2 = \dot{x}^T x = u^T B^T x = u^T y \Longrightarrow \int_0^\infty u^T y \text{ dt} = \frac{1}{2}\|x_\infty\|^2 - \frac{1}{2}\|x_0\|^2 \tag{19}$$

We start of with the cost criterion and rewrite it in such a way that we can argue what the optimal control should be:

$$J_\infty(x_0, u) = \frac{1}{2}\int_0^\infty (\|y(t)\|^2 + \|u(t)\|^2) \text{ dt} \tag{20}$$
$$= \frac{1}{2}\int_0^\infty \|u(t) + y(t)\|^2 \text{ dt} - \int_0^\infty (u(t)^T y(t)) \text{ dt} \tag{21}$$
$$= \frac{1}{2}\int_0^\infty \|u(t) + y(t)\|^2 \text{ dt} - \frac{1}{2}\|x_\infty\|^2 + \frac{1}{2}\|x_0\|^2 \tag{22}$$

The last term does clearly not depend on u and so it is not necessary to mention it while searching for the optimal control u.

We can also say something about the second term: $\frac{1}{2}\|x_\infty\|^2$. Since we are

minimizing the cost criterion, we implicitly assume that the integral in the cost criterion is finite. Therefore, we should have the integrand to go to zero. The integrand consists of two quadratic terms, namely $\|y\|^2$ and $\|u\|^2$ and therefore they should both go to zero. If $\|y\|^2$ goes to zero, $y = B^T x$ also goes to zero. This means that x goes to a vector from the kernel of B^T, which is given in equation (2.1). From this we know that x goes to $\alpha\mathbb{1}$ for some α.
In words: all x_i's go to the same value and will be equal as $t \to \infty$; we say x reaches consensus.

We do not yet know to which value of α all x_i's will converge. Therefore we need another argument. The sum of the state x does not change and this has a useful consequence:

$$\frac{d}{dt}\mathbb{1}^T x = \mathbb{1}^T \dot{x} = \mathbb{1}^T Bu = 0 \Rightarrow \mathbb{1}^T x_0 = \mathbb{1}^T x_\infty \tag{23}$$

The sum of x_0 and x_∞ are equal and since x_∞ reaches consensus, every x_i goes to the average of the total starting state x_0, and thus we found α:

$$x_\infty = \frac{\mathbb{1}^T x_0}{n}\mathbb{1} \tag{24}$$

The nice thing about this outcome is that the second term in (22) does not depend on the input d as well. Only the first term is left: $\frac{1}{2}\int_0^\infty \|u + y\|^2 \text{ d}t$. Because this integral will always be bigger than or equal to zero, the control $u = -y$ is the optimal control. The first term will then vanish.

What's left for the value function is only this:

$$\begin{aligned}
J_\infty(x_0, u) &= \frac{1}{2}\|x_0\|^2 - \frac{1}{2}\|x_\infty\|^2 \\
&= \frac{1}{2}x_0^T x_0 - \frac{1}{2n}x_0^T \mathbb{1}\mathbb{1}^T x_0 \\
&= \frac{1}{2}x_0^T \left(I - \frac{1}{n}\mathbb{1}\mathbb{1}^T\right) x_0
\end{aligned}$$

So with this approach we found $u_{\text{opt}} = -y = -B^T x$ with the resulting cost function above.

3.4 Result

In the preceding sections we came to the following result.

Theorem 3.1. *For the system* (7) *with cost criterion* (8)*, the optimal input is given by:*

$$u_{opt} = -B^T x \tag{25}$$

which leads to the minimal total cost:

$$J_\infty(x_0, u_{opt}) = \frac{1}{2}x_0^T(I - \frac{1}{n}\mathbb{1}\mathbb{1}^T)x_0 \tag{26}$$

One of the properties of the value function is that it must be greater than, or equal to zero. Therefore $P_2 = I - \frac{1}{n}\mathbb{1}\mathbb{1}^T$ should be positive semidefinite. This is characterized by non-negative eigenvalues. We can use Gershgorin's Circle Theorem to show this. Gershgorin's Circle Theorem allocates the region in the complex plane where the eigenvalues of a given square matrix lie. Since P_2 is real and symmetric, all it's eigenvalues are real and therefore we only need to mention the real numbers. All the Gershgorin Circles are equal for P_2 and are given by:

$$\left|\lambda - (1 - \frac{1}{n})\right| \leq (n-1)\left|-\frac{1}{n}\right| \quad \Leftrightarrow \quad \begin{cases} \lambda & \geq & (1-\frac{1}{n}) - (n-1)\frac{1}{n} \\ \lambda & \leq & (1-\frac{1}{n}) + (n-1)\frac{1}{n} \end{cases} \tag{27}$$

These bounds hold for every eigenvalue λ of P_2. The lower bound is equal to 0:

$$\lambda \geq (1 - \frac{1}{n}) - (n-1)\frac{1}{n} = 1 - \frac{1}{n} - 1 + \frac{1}{n} = 0$$

And thus all eigenvalues of P_2 are in particular greater than, or equal to 0. This shows that P_2 is positive semidefinite and therefore the minimal cost (26) is never negative.

Another property is that if we start of with a state x_0 in which every x_i is equal, thus $x_0 = \beta\mathbb{1}$, then $u_{\text{opt}} = 0$ and $\dot{x} = 0$. This means that the costs are zero and $x(t) = x_0$ for every $t \geq 0$.

Example 2. In Figure 1 you see a plot of the state vector x over time, controlled by u_{opt} from (25). We used a cycle graph with 20 vertices, where the initial state at each vertex is a random number between 0 and 20. The average of the initial state lies close to 10 and you can clearly see that each x_i converges to that value.

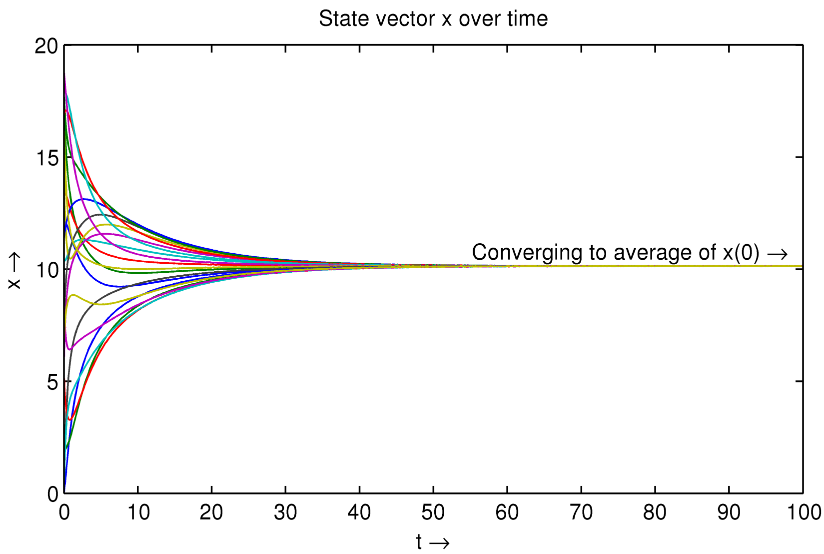


Figure 1: Example of optimal flow in case of a cycle graph

3.5 Interpretation

The results we found in Section 3.4 can be interpreted very well in our model. The control u_{opt} sets up a flow in every pipe equal to the difference in water height between the two connected barrels. So if two barrels are connected, there will be a flow from the barrel with the highest water height to the other. This will cause the difference to go to zero and that is indeed the result we expected. If we start of with equal water height in every barrel, there will be no water flow and the costs are zero.

4 Second model

4.1 Model description

We consider the following problem. Imagine a network of springs connecting some unit masses. Every mass has a certain moving speed and the springs keep these masses together. At every spring we have a value corresponding to the extension of the spring. There is some interaction between the speed of the masses and the springs. The masses and springs are vibrating. We can adjust the acceleration of some of the masses, but this comes at a certain price. If the masses where we have control over have some speed, that's valuable. The aim is to find a way of adjusting the speed of the masses we can control to keep the total costs over time as low as possible. It seems logical to bring the masses we can control to stop and then keep their speed fixed. We will describe a mathematical model and investigate its behavior.

4.2 Mathematical description

The model described above can be translated into the mathematical model:

$$\begin{cases} \dot{x} &= -Bx_c + Ed \\ \dot{x}_c &= B^T x \\ z &= E^T x \end{cases} \qquad \begin{array}{l} x \in \mathbb{R}^n \\ x_c \in \mathbb{R}^m \\ d \in \mathbb{R}^k \\ y \in \mathbb{R}^m \\ z \in \mathbb{R}^k \end{array} \tag{28}$$

The vector x corresponds to the vertices, where x_i is the speed of the i-th mass. The vector x_c corresponds to the edges, where x_{ci} is the extension of the i-th spring. Just as in our first model, we have a graph with n vertices and m edges. The matrix B is the incidence matrix as defined in Definition (2.2) and E indicates of which masses we can adjust the acceleration in the way of Definition (2.3). We have control over k masses and the k-dimensional vector d indicates how we adjust its accelerations. The output vector z is used in the cost criterion and contains the speed of the k masses we can control.

We can rewrite the first two equations in (28) in block matrix notation to:

$$\begin{pmatrix} \dot{x} \\ \dot{x}_c \end{pmatrix} = \begin{pmatrix} 0 & -B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} x \\ x_c \end{pmatrix} + \begin{pmatrix} E \\ 0 \end{pmatrix} d \tag{29}$$

With this in mind, it is sometimes useful to define the vector s:

$$s = \begin{pmatrix} x \\ x_c \end{pmatrix}$$

The defined cost criterion looks like:

$$J_\infty(x_0, d) = \frac{1}{2} \int_0^\infty (\|z\|^2 + \|d\|^2)\ \mathrm{d}t \tag{30}$$

The object is to find the input d which minimizes the cost criterion, which we will call the optimal control d.

4.3 Calculations

The system in (28) exhibits oscillatory motions when the input d is zero; Figure 2 illustrates the behaviour of the state vector x. Since we want to minimize the cost criterion, we need to have sufficient control over the system. We will do a mathematical assumptions about this:

Assumption 3. We assume that the system in (28) is controllable.

As a consequence of the special form of the output vector z, controllability implies observability. And therefore our model is minimal.

We will again use two different approaches to find the optimal control for the system (28) with cost criterion (30).

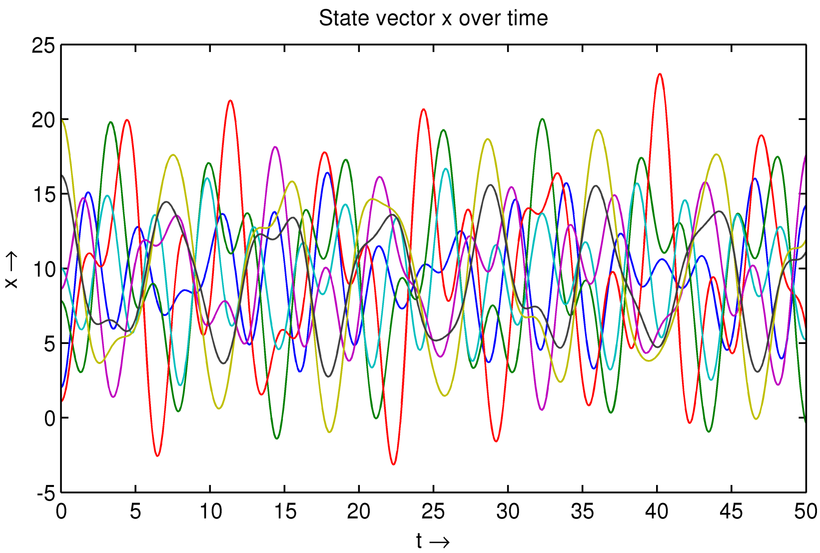


Figure 2: Example of the spring system model with input $d = 0$.

4.3.1 First approach

We will consider the Algebraic Riccati Equation of the system in (28):

$$P \begin{bmatrix} 0 & -B \\ B^T & 0 \end{bmatrix} + \begin{bmatrix} 0 & -B \\ B^T & 0 \end{bmatrix}^T P - P \begin{bmatrix} EE^T & 0 \\ 0 & 0 \end{bmatrix} P + \begin{bmatrix} EE^T & 0 \\ 0 & 0 \end{bmatrix} = 0 \tag{31}$$

The matrix P is symmetric and we can rewrite it using

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} \qquad \begin{matrix} P_{11} \in \mathbb{R}^{nxn} \\ P_{12} \in \mathbb{R}^{nxm} \\ P_{22} \in \mathbb{R}^{mxm} \end{matrix} \tag{32}$$

to the following four equations:

$$\begin{cases} P_{12}B^T + BP_{12}^T + EE^T & = P_{11}EE^TP_{11} \\ -P_{11}B + BP_{22} & = P_{11}EE^TP_{12} \\ P_{22}B^T - B^TP_{11} & = P_{12}^TEE^TP_{11} \\ -P_{12}^TB - B^TP_{12} & = P_{12}^TEE^TP_{12} \end{cases} \tag{33}$$

The second and third equations are equal. In this way get:

$$\begin{cases} P_{12}B^T + BP_{12}^T + EE^T & = P_{11}EE^TP_{11} \\ BP_{22} - P_{11}B & = P_{11}EE^TP_{12} \\ -P_{12}^TB - B^TP_{12} & = P_{12}^TEE^TP_{12} \end{cases} \tag{34}$$

It is clear that $P_{11} = I, P_{12} = 0, P_{22} = I$ is a solution for these equations. This corresponds to the trivial solution $P = I$. Since we assume that our system is minimal, the theory from [van der Schaft, 2009, p. 70-85] in Theorem 2.3 guarantees that there is only one possible solution to the Algebraic Riccati Equation. We conclude that $P = I$ is unique.
Using Theorem 2.4 we find the optimal control for our second model:

$$u_{\text{opt}} = -\begin{bmatrix} E^T & 0 \end{bmatrix} \begin{bmatrix} x \\ x_c \end{bmatrix} \tag{35}$$

With the corresponding minimal cost:

$$J_\infty(s_0, d_{\text{opt}}) = \frac{1}{2}s_0^Ts_0 = \frac{1}{2}\|s_0\|^2 \tag{36}$$

We will now discuss another solution method. In Section (4.4) we discuss and state all our findings.

4.3.2 Second approach

Just like in the simple case, we will rewrite the cost criterion. We will use two properties given by the following equations:

$$\frac{1}{2}\|z + d\|^2 = \frac{1}{2}\|z\|^2 + \|d\|^2 + z^Td \tag{37}$$

$$\frac{d}{dt}\left(\frac{1}{2}\|s\|^2\right) = \dot{x}^Tx + x_c^T\dot{x}_c = -x_c^TB^Tx + d^TE^Tx + x_c^TB^Tx = d^Tz \tag{38}$$

$$\implies \qquad \int_0^\infty d^Tz \text{ dt} = \frac{1}{2}\|s_\infty\|^2 - \frac{1}{2}\|s_0\|^2 \tag{39}$$

We start of with the cost criterion and rewrite it in such a way that we can argue what the optimal control should be:

$$J_\infty(x_0, d) = \frac{1}{2}\int_0^\infty \|z\|^2 + \|d\|^2 \ \mathrm{d}t \tag{40}$$

$$= \frac{1}{2}\int_0^\infty \|z + d\|^2 \ \mathrm{d}t - \int_0^\infty d^T z \ \mathrm{d}t \tag{41}$$

$$= \frac{1}{2}\int_0^\infty \|z + d\|^2 \ \mathrm{d}t + \frac{1}{2}\|s_0\|^2 - \frac{1}{2}\|s_\infty\|^2 \tag{42}$$

The second term does clearly not depend on d and so we don't have to mention it while minimizing the cost criterion. From Theorem 2.4 we know that our system is asymptotically stable and therefore $x_\infty = 0$. The third term will therefore be zero and only the first term is left. Because this integral will always be bigger than, or equal to zero, the control $d = -z$ is the optimal control. The first term will then vanish.

What's left for the value function is only this:

$$J_\infty(s_0, d_{\mathrm{opt}}) = \frac{1}{2}\|s_0\|^2 \tag{43}$$

So with this approach we found $d_{\mathrm{opt}} = -z = -E^T x$ with the resulting minimal cost above.

4.4 Result

In the preceding sections we found the following result.

Theorem 4.1. *For the system* (28) *with cost criterion* (30)*, the optimal input is given by:*

$$d_{opt} = -E^T x \tag{44}$$

which leads to the minimal total cost:

$$J_\infty(s_0, d_{opt}) = \frac{1}{2}\|s_0\|^2 \tag{45}$$

A property of the cost criterion is that it's value is always greater than, or equal to zero. The found expression $\frac{1}{2}\|s_0\|^2$ clearly satisfies this property.

Another property of the system (28) is that the equilibrium $s = 0$ is assymptotically stable. So if we start off with $s_0 = 0$, we have $s(t) = 0$ and thus $z = 0$ for all $t \geq 0$. A best response to this state would be to do nothing, say $d = 0$ and so the total minimal costs should be zero. If we look at the solution in Theorem 4.1 for $s_0 = 0$ we see that the minimal costs and the optimal input are zero indeed.

Example 3. In Figure 3 you see a plot of the state vector x over time, controlled by u_{opt} from (44). We used the graph from Example 1 exactly as given there. On every vertex we started of with a random number between 0 and 20. The initial values on the edges were zero. From the picture you clearly see that the state x converges to zero.

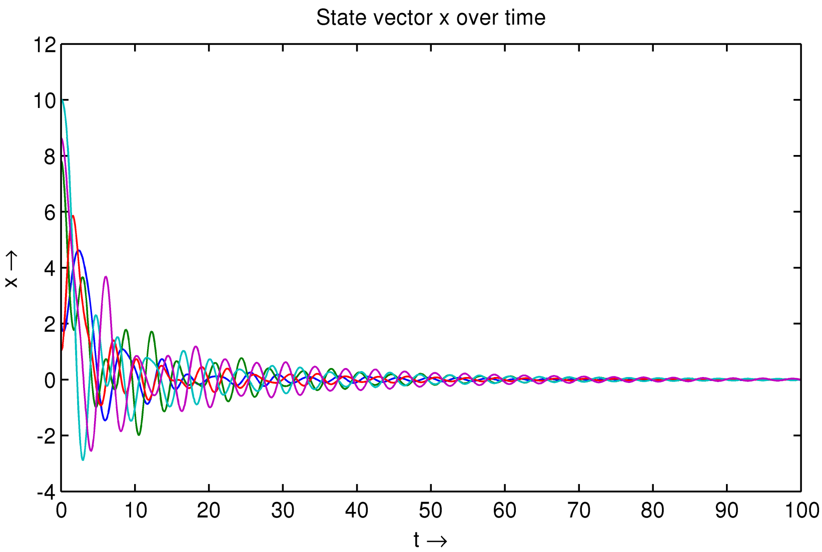


Figure 3: Example of optimal flow in case of a cycle graph

4.5 Interpretation

We can translate the results of Section 4.4 to our model. The optimal control $d = -z$ is like a linear damper. The damping on each mass we can control is equal to the speed of that mass in opposite direction. Investigation of different graphs, starting vectors and corresponding solutions using the Matlab code from Appendix B , shows that the speed of the masses we can control go to zero much faster than the other masses. The speed of all the other masses also go to zero, but this takes more time. If the speed of the controllable masses is approximately zero, it remains zero.
The time it takes to stabilize the whole system, depends heavily on the structure of the graph. If you take graphs with a simple structure, like line graphs and circle graphs, the whole system converges to zero very fast. A complete graph, even with only five vertices and with two dampers, is already too complex to bring it to zero within short time. Additional dampers on other vertices are needed to damp the system. One can not easily predict how long it takes to stabilize the system for the different setups. However, you can forecast the total minimal costs.

5 Conclusions

For both problems we used two different solutions methods and those give both rise to the correct solution. The first method using the Algebraic Riccati Equation, uses the special linear form of the system and the infinite cost criterion. The second, direct method is also applicable to other systems; for example non-linear systems.
The results we found cannot say anything about the convergence of the systems. We know this strongly depends on the network. Investigation of this behaviour would be an interesting subject for another research.

References

A.J. van der Schaft. *Calculus of Variations and Optimal Control*. University of Groningen, 2009.

A Code for first model

This code can be used to simulate the behaviour of the first model.

```
function flow1
% -n: number of vertices
% -starttime: starttime of the flow
% -endtime:  endtime of the flow
% -bbt: The matrix B*B', where B is the incidence matrix of the graph
% we consider
% -x: random initial state vector
% -alpha: used in the definition of P
% -P: the solution of the Algebraic Riccati Equation (ARE).
% -d: function to calculate the input vector.
% -f: function to calculate the RHS of the system of ODE's.
% -[T,Y]: solution of our system of ODE's with Y(i) the solution at
% time T(i).
% -integrand: vector where integrand(i) is the integrand of the cost
% criterion at time T(i).
% -input: vector where input(i) is the inputvector at time T(i).
% -waardefunctie: the theoretical outcome of the cost criterion using
% the matrix P.
% -integraal: The approximated value of the cost function

n = 40;
starttime = 0;
endtime = 180;
figure(1);
[bbt,B,m] = circlegraph(n);
title('The graph we consider');

x = (rand(n,1)*n);

alpha = -1/n;
P = eye(n)+alpha*ones(n,n);
d = @(x) -B'*P*x;
f = @(t,x) B*d(x);

[T,Y] = ode45(f,[starttime endtime],x);

for i = 1:size(Y,1)
    integrand(i) = 0.5*(Y(i,:)*B*B'*Y(i,:)'+d(Y(i,:)')'*d(Y(i,:)'));
    input(:,i) = d(Y(i,:)');
end

figure(2);
plot(T,Y);
title('State vector x over time');
xlabel('t \rightarrow')
ylabel('x \rightarrow')
text(endtime - (endtime-starttime)/50,mean(Y(end,:)),'Converging to
average of x(0) \rightarrow','VerticalAlignment','bottom',
'HorizontalAlignment','right')
```

```

netto = sum(Y,2);
figure(3);
plot(T,netto);
title('The sum of the state x over time');

figure(4)
plot(T,input);
title('The input vector d');

figure(5)
plot(T,integrand);
title('Integrand van kostenfunctie over tijd');

% The theoretical and approximated values of the cost function:
waardefunctie = 0.5*(x'*P*x)
integraal = trapz(T,integrand)

end
```

B Code for second model

This code can be used to simulate the behaviour of the second model.

```
function flow2

% -n: number of vertices
% -k: number of externally connected vertices
% -starttime: starttime of the flow
% -endtime:  endtime of the flow
% -bbt: The matrix B*B', where B is the incidence matrix of the graph
%  we consider
% -W: the matrix from the system of ODE's
% -x: random initial state vector
% -xc: zero initial co-state vector
% -S: the total state vector
% -E: The matrix which describes which vertices are externally
%  connected
% -F: Giving E the right dimensions.
% -P: the solution of the Algebraic Riccati Equation (ARE).
% -d: function to calculate the input vector.
% -f: function to calculate the RHS of the system of ODE's.
% -[T,Y]: solution of our system of ODE's with Y(i) the solution at
% time T(i).
% -integrand: vector where integrand(i) is the integrand of the cost
% criterion at time T(i).
% -input: vector where input(i) is the inputvector at time T(i).
% -waardefunctie: the theoretical outcome of the cost criterion using
% the matrix P.
% -integraal: The approximated value of the cost function

n = 5;
k = 2;
starttime = 0;
endtime = 100;
figure(1);
[bbt,B,m] = circlegraph(n);
title('The graph we consider');

```

```
W = [ zeros(n,n) -B ; B' zeros(m,m)];

x = (rand(n,1)*n);
xc = zeros(m,1);
S = [x;xc];

E = zeros(n,k);
E(1,1) = 1;
E(3,2) = 1;

F = [ E ; zeros(m,k)];

P11 = eye(n);
P12 = zeros(n,m);
P22 = eye(m,m);

P = [P11 P12 ; P12' P22];

d = @(S) -F'*S;
f = @(t,S) W*S+F*d(S);

[T,Y] = ode45(f,[starttime endtime],S);

for i = 1:size(Y,1)
    integrand(i) = 0.5*(Y(i,:)*F*F'*Y(i,:)'+d(Y(i,:)')'*d(Y(i,:)'));
    input(:,i) = d(Y(i,:)');
end

figure(2);
plot(T,Y(:,1:n));
title('State vector x over time');
xlabel('t \rightarrow');
ylabel('x \rightarrow');

figure(3);
plot(T,Y(:,n+1:n+m));
title('Co-state vector xc over time');

netto = sum(Y(:,1:n),2);
figure(5);
plot(T,netto);
title('The sum of the state x over time');
%
figure(6)
plot(T,input);
title('The input vector d');
%
figure(7)
plot(T,integrand);
title('Integrand van kostenfunctie over tijd');

figure(8)
plot(T,0.5*sum(Y(:,1:n).*Y(:,1:n),2));
title('1/2 ||x||^2 over time');

waardefunctie = 0.5*(S'*P*S)
integraal = trapz(T,integrand)

end
```