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# An Introduction to Differintegrals and Fractional Calculus

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**Abstract**

We attempt to introduce differintegrals and fractional calculus to undergraduates. After a brief history of fractional calculus and some preliminary definitions, the concept of differintegrals is introduced: an operator connecting differentiation and integration. Next we look at the Grünwald-Letnikov and Riemann-Liouville differintegrals, what their construction is based on and what connects them. We will be checking whether some of the regular rules for differentiation still apply for the Riemann-Liouville differintegral and conclude by calculating fractional integrals and fractional derivatives of some basic functions.

# An Introduction to Differintegrals and Fractional Calculus

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## 1 Introduction

In this paper we attempt to introduce differintegrals and fractional calculus to undergraduates. After a brief history of fractional calculus and some basic definitions we will introduce the concept of differintegrals. Next we look at the Grünwald-Letnikov and Riemann-Liouville differintegrals and see what their construction is based on and what connects them. We will check whether some regular rules for differentiation still apply for the Riemann-Liouville differintegral. We conclude by calculating fractional integrals and fractional derivatives of some basic functions.

### 1.1 History of Fractional Calculus

Leibniz invented infinitesimal calculus around 1674, independent from Newton. In 1695 Leibniz wondered whether his calculus was compatible with non-integer order derivatives, giving birth to the concept of fractional calculus. When suggesting his question to l'Hôpital in a letter, L'Hôpital replied "What if the order will be $\frac{1}{2}$?", referring to order of the derivative. Leibniz replied: "It leads to a paradox, from which one day useful consequences will be drawn".

Though the nature of this paradox is unclear, a relatively small group of mathematicians started working on extending the regular methods of integration and differentiation for non-integer orders. Fractional calculus started to mature over the last 300 years. Over the course of time many different definitions for fractional derivatives and integrals arose from different perspectives. Some even seemed inconsistent with each other.

As Nishimoto wrote, fractional calculus is the calculus of the 21st century. It slowly began to find applications in- and outside of mathematics, like in electrical circuits and fluid mechanics. One particular case was solving the problem for tautochrone motion using fractional calculus, found by Swedish mathematician Abel in the 1820s (`[3] p.31-35`).

## 1.2 Preliminary Definitions

This paper will concern two common formulas for integration and differentiation. We will be extending these formulas to non-integer order. We will address these two formulas in a moment. The first one will be the *backward difference derivative* and gives rise to the *Grünwald-Letnikov differintegral*. The second one is the *Cauchy formula for repeated integration* and gives rise to the *Riemann-Liouville differintegral*.

It is crucial to extend the factorial to non-integer values, in order to extend the two formulas properly. That's why we will also introduce the Gamma function, along with the Beta function.

### 1.2.1 Forward and Backward Difference

The forward difference of a function in a point $t$ is given by

$$\triangle_h f(t) = f(t+h) - f(t)$$

Likewise, the backward difference is given by

$$\bigtriangledown_h f(t) = f(t) - f(t-h)$$

From this we can construct the *forward difference derivative* (FDD) and the *backward difference derivative* (BDD):

$$\text{(FDD)} \qquad \tfrac{\mathrm{d}}{\mathrm{d}t} f(t) = \lim_{h \to 0} \frac{\triangle_h f(t)}{h} = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h} \tag{1}$$

$$\text{(BDD)} \qquad \tfrac{\mathrm{d}}{\mathrm{d}t} f(t) = \lim_{h \to 0} \frac{\bigtriangledown_h f(t)}{h} = \lim_{h \to 0} \frac{f(t) - f(t-h)}{h} \tag{2}$$

Applying the FDD $n$ times to a function $f(t)$ we find

$$\triangle_h^n f(t) = \underbrace{\triangle_h \triangle_h \cdots \triangle_h}_{n} f(t) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(t + (n-k)h)$$

For the BDD we find

$$\bigtriangledown_h^n f(t) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(t - kh)$$

We can express the $n$-th derivative by using the above formula's.

$$\text{(FDD)} \quad \left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^n f(t) = \lim_{h \to 0} \frac{\triangle_h^n f(t)}{h^n} = \lim_{h \to 0} \frac{1}{h^n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(t + (n-k)h) \tag{3}$$

$$\text{(BDD)} \quad \left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^n f(t) = \lim_{h \to 0} \frac{\bigtriangledown_h^n f(t)}{h^n} = \lim_{h \to 0} \frac{1}{h^n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(t - kh) \tag{4}$$

Note that $\triangle_h f(t) = \bigtriangledown_h f(t+h)$ and from the above formulas it follows that

$$\triangle_h^n f(t) = \bigtriangledown_h^n f(t + nh) \tag{5}$$

### 1.2.2 Cauchy formula for repeated integration

Integral of a function $f(t)$ is denoted by $\int_a^t f(\nu)\mathrm{d}\nu$. Integrating $f(t)$ twice is given by

$$\int_a^t \int_a^{\nu_1} f(\nu_2)\mathrm{d}\nu_2\mathrm{d}\nu_1$$

If we use integration by parts we find

$$\begin{aligned}\int_a^t \int_a^{\nu_1} f(\nu_2)\mathrm{d}\nu_2\mathrm{d}\nu_1 &= \int_a^t 1\cdot\left[\int_a^{\nu_1} f(\nu_2)\mathrm{d}\nu_2\right]\mathrm{d}\nu_1 \\ &= \Big[-(t-\nu_1)\Big]\left[\int_a^{\nu_1} f(\nu_2)\mathrm{d}\nu_2\right]\Bigg|_{\nu_1=a}^{\nu_1=t} - \int_a^t \Big[-(t-\nu_1)\Big]f(\nu_1)\mathrm{d}\nu_1 \\ &= \int_a^t (t-\nu_1)f(\nu_1)\mathrm{d}\nu_1\end{aligned}$$

In a similar manner we see for $k \in \mathbb{Z}, k \geq 0$

$$\begin{aligned}&\int_a^t (t-\nu_1)^k\cdot\left[\int_a^{\nu_1} f(\nu_2)\mathrm{d}\nu_2\right]\mathrm{d}\nu_1 \\ &= \frac{-(t-\nu_1)^{k+1}}{k+1}\left[\int_a^{\nu_1} f(\nu_2)\mathrm{d}\nu_2\right]\Bigg|_{\nu_1=a}^{\nu_1=t} - \int_a^t \frac{-(t-\nu_1)^{k+1}}{k+1}f(\nu_1)\mathrm{d}\nu_1 \\ &= \frac{1}{k+1}\int_a^t (t-\nu_1)^{k+1}f(\nu_1)\mathrm{d}\nu_1 \qquad (6)\end{aligned}$$

Would we integration $f(t)$ $n$ times and use the results above we would find

$$\underbrace{\int_a^t \int_a^{\nu_1} \cdots \int_a^{\nu_{n-1}}}_{n} f(\nu_n)\mathrm{d}\nu_n\cdots\mathrm{d}\nu_1 = \frac{1}{(n-1)!}\int_a^t (t-\nu)^{n-1}f(\nu)\mathrm{d}\nu \qquad (7)$$

This formula is known as the *Cauchy formula for repeated integration.*

### 1.2.3 Gamma and Beta Function

The *Gamma function* $\Gamma(z)$ is defined by

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\mathrm{d}t$$

and extends the factorial by $n! = \Gamma(n+1)$, for $n \in \mathbb{Z}$, and $\Gamma(z+1) = z\Gamma(z), z \in \mathbb{C}$.

The *Beta function* $B(x,y)$ is defined by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t$$

where $\mathrm{Re}(x), \mathrm{Re}(y) > 0$.

The Gamma and Beta function share the following relation:

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

A different representation of the Gamma function can be found in `Krantz 1999, p.156`

$$\Gamma(z) = \lim_{n\to\infty} \frac{(n+1)^z n!}{z(z+1)\cdots(z+n)}$$

There is no $z \in \mathbb{C}$ such that $\Gamma(z) = 0$. As a consequence, the reciprocal Gamma function $\frac{1}{\Gamma(z)}$ never tends to infinity on a compact subset of $\mathbb{C}$. In addition, $\frac{1}{\Gamma(k)} = 0$ for all $k \in \mathbb{Z}$, $k \leq 0$.

# 2 Differintegrals

In fractional calculus, a *differintegal* is an operator that connects differentiation and integration. The operation of differentiation or integration using a differintegral is called *differintegration*. It is important to note that there are many ways to define a differintegral, as we shall see later on. In this section we will discuss the Grünwald-Letnikov differintegral (represented by ${}_aD_t^\gamma$) and the Riemann-Liouville Differintegral (represented by ${}_a\mathbf{D}_t^\gamma$).

In the notation, $\gamma$ is the *order* of differintegration. For positive integers, $\gamma$ corresponds with regular differentiation. For negative integers, $\gamma$ corresponds with regular integration. The important part of defining differintegrals is making sure that it also handles non-integer $\gamma$ well.

$a$ and $t$ are called the *terminals* of the differintegral. We will get to them later.

From time to time it might be unclear whether we are integrating of differentiating, due to the fact that differintegrals can do both. To prevent this confusion we define $p, q \geq 0$ and use $p$ and $q$ to represent the order of differintegration throughout the rest of the paper. $p$ will be used when we are differentiating, and $q$ will be used when we are integrating.

## 2.1 Grünwald-Letnikov Differintegral `([2] p.43-48)`

In this section we will discuss the *Grünwald-Letnikov* (GL) differintegral by constructing it from the bottom up. The GL differintegral is based on generalizing the *backward difference derivative* (BDD). We shall see how this generalization coincides with integration and extend it to arbitrary (non-integer) order.

### 2.1.1 Differentiation

Consider the real valued function $f(t)$. Let $f(t)$ be $p$ times differentiable. Using the BDD, the $p$-th derivative of $f$ is given by

$$\left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^p f(t) = \lim_{h\to 0} \frac{1}{h^p} \sum_{k=0}^{p} (-1)^k \binom{p}{k} f(t-kh)$$

Next we define the operator $Q$:

$$Q^p_{h,n} f(t) := \frac{1}{h^p} \sum_{k=0}^{n} (-1)^k \binom{p}{k} f(t-kh) \tag{8}$$

where $p, n, k \in \mathbb{N}$ and $h \in \mathbb{R}$.

From this definition it follows that, for $n \geq p$,

$$\lim_{h \to 0} Q^p_{h,n} f(t) = f^{(p)}(t) = \tfrac{\mathrm{d}^p}{\mathrm{d}t^p} f$$

since $\binom{p}{k} = 0$ for $k > p$.

### 2.1.2 Integration

We will now show that for negative $p$ this generalisation corresponds to regular integration. Note that binomial coefficient can be expressed as

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}$$

We will define

$$\begin{bmatrix} n \\ k \end{bmatrix} := \frac{n(n+1)\cdots(n+k-1)}{k!} \tag{9}$$

Plugging in negative values for $n$ in the binomial coefficient we get

$$\begin{aligned} \binom{-n}{k} &= \frac{-n(-n-1)\cdots(-n-k+1)}{k!} \\ &= (-1)^k \frac{n(n+1)\cdots(n+k-1)}{k!} \\ &= (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} \end{aligned}$$

Substituting this in (8) we find

$$Q^p_{h,n} f(t) = \frac{1}{h^p} \sum_{k=0}^{n} \begin{bmatrix} -p \\ k \end{bmatrix} f(t-kh)$$

In particular, for $p = -q$,

$$Q^{-q}_{h,n} f(t) = h^q \sum_{k=0}^{n} \begin{bmatrix} q \\ k \end{bmatrix} f(t-kh)$$

Would we fix $n$ and $q$ and take the limit $h \to 0$ we would find $\lim_{h\to 0} f^{(-q)}_{h,n} = 0$, which is not very interesting. Instead we let $n$ be defined by the relation $nh = t - a$, such that $n \to \infty$ as $h \to 0$. As stated before, $a$ and $t$ are called the *terminals*, and we will soon see that these form the region of integration. We will denote all this as by

$$_aD_t^{-q} f(t) := \lim_{\substack{h \to 0 \\ nh = t-a}} Q^{-q}_{h,n} f(t)$$

If we pick $q = 1$ we find

$$\begin{aligned}
{}_aD_t^{-1} f(t) &= \lim_{\substack{h\to 0\\ nh=t-a}} Q_{h,n}^{-1} f(t) = \lim_{\substack{h\to 0\\ nh=t-a}} h \sum_{k=0}^{n} \begin{bmatrix} 1 \\ k \end{bmatrix} f(t-kh) \\
&= \lim_{\substack{h\to 0\\ nh=t-a}} h \sum_{k=0}^{n} f(t-kh) = \int_0^{t-a} f(t-\tau)\mathrm{d}\tau \\
&= \int_a^t f(\nu)\mathrm{d}\nu
\end{aligned}$$

provided that $f(t)$ is integrable. We shall prove by induction to $q$ that in general

$$ {}_aD_t^{-q} f(t) = \lim_{\substack{h\to 0\\ nh=t-a}} h^q \sum_{k=0}^{n} \begin{bmatrix} q \\ k \end{bmatrix} f(t-kh) = \frac{1}{(q-1)!} \int_a^t (t-\nu)^{q-1} f(\nu)\mathrm{d}\nu \tag{10}$$

Recall the Cauchy formula for repeated integrals (7)

$$\frac{1}{(q-1)!} \int_a^t (t-\nu)^{q-1} f(\nu)\mathrm{d}\nu = \underbrace{\int_a^t \int_a^{\nu_1} \cdots \int_a^{\nu_{q-1}}}_{q} f(\nu_q)\mathrm{d}\nu_q \cdots \mathrm{d}\nu_1$$

***Proof*** Suppose we have for $q = r$ that (10) holds. We will show that it follows that (10) also holds for $q = r + 1$. We have

$$ {}_aD_t^{-(r+1)} f(t) = \lim_{\substack{h\to 0\\ nh=t-a}} h^{(r+1)} \sum_{k=0}^{n} \begin{bmatrix} r+1 \\ k \end{bmatrix} f(t-kh) \tag{11}$$

Say $f^{(-1)}(t) = \int_a^t f(\tau)\mathrm{d}\tau$, then

$$f(t) = \lim_{h\to 0} \frac{1}{h} \left[ f^{(-1)}(t) - f^{(-1)}(t-h) \right]$$

Equivalently,

$$f(t-kh') = \lim_{h\to 0} \frac{1}{h} \left[ f^{(-1)}(t-kh') - f^{(-1)}(t-kh'-h) \right]$$

If we take $h = h'$ and plug this into (11) we obtain

$$ {}_aD_t^{-(r+1)} f(t) = \lim_{\substack{h\to 0\\ nh=t-a}} h^{r} \sum_{k=0}^{n} \begin{bmatrix} r+1 \\ k \end{bmatrix} \left( f^{(-1)}(t-kh) - f^{(-1)}(t-(k+1)h) \right) \tag{12}$$

From definition (9) it follows that

$$\begin{aligned}
\begin{bmatrix} r+1 \\ k \end{bmatrix} &= \frac{(r+1)(r+2)\cdots(r+k)}{k!} \\
&= (r+k)\frac{(r+1)(r+2)\cdots(r+k-1)}{k!} \\
&= \frac{r(r+1)\cdots(r+k-1)}{k!} + \frac{(r+1)(r+2)\cdots(r+k-1)}{(k-1)!} \\
&= \begin{bmatrix} r \\ k \end{bmatrix} + \begin{bmatrix} r+1 \\ k-1 \end{bmatrix}
\end{aligned}$$

Where we put $\begin{bmatrix} r+1 \\ -1 \end{bmatrix} = 0$.

Note that $f^{(-1)}(a) = 0$ since $f^{(-1)}(t) = \int_a^t f(\tau)\mathrm{d}\tau$. Using the above in (12), along with $nh = t - a$, we get

$$
\begin{aligned}
{}_aD_t^{-(r+1)} f(t) &= \lim_{\substack{h\to 0\\ nh=t-a}} h^r \sum_{k=0}^{n} \begin{bmatrix} r+1 \\ k \end{bmatrix} \Big( f^{(-1)}(t-kh) - f^{(-1)}(t-(k+1)h) \Big) \\
&= \lim_{\substack{h\to 0\\ nh=t-a}} h^r \Big( \sum_{k=0}^{n} \begin{bmatrix} r+1 \\ k \end{bmatrix} f^{(-1)}(t-kh) \\
&\qquad - \sum_{k=0}^{n} \begin{bmatrix} r+1 \\ k \end{bmatrix} f^{(-1)}(t-(k+1)h) \Big) \\
&= \lim_{\substack{h\to 0\\ nh=t-a}} h^r \Big( \sum_{k=0}^{n} \begin{bmatrix} r \\ k \end{bmatrix} f^{(-1)}(t-kh) + \sum_{k=0}^{n} \begin{bmatrix} r+1 \\ k-1 \end{bmatrix} f^{(-1)}(t-kh) \\
&\qquad - \sum_{k=0}^{n} \begin{bmatrix} r+1 \\ k \end{bmatrix} f^{(-1)}(t-(k+1)h) \Big) \\
&= \lim_{\substack{h\to 0\\ nh=t-a}} h^r \Big( \sum_{k=0}^{n} \begin{bmatrix} r \\ k \end{bmatrix} f^{(-1)}(t-kh) + \sum_{k=1}^{n} \begin{bmatrix} r+1 \\ k-1 \end{bmatrix} f^{(-1)}(t-kh) \\
&\qquad - \sum_{k=0}^{n} \begin{bmatrix} r+1 \\ k \end{bmatrix} f^{(-1)}(t-(k+1)h) \Big) \\
&= \lim_{\substack{h\to 0\\ nh=t-a}} h^r \Big( \sum_{k=0}^{n} \begin{bmatrix} r \\ k \end{bmatrix} f^{(-1)}(t-kh) + \sum_{k=0}^{n-1} \begin{bmatrix} r+1 \\ k \end{bmatrix} f^{(-1)}(t-(k+1)h) \\
&\qquad - \sum_{k=0}^{n} \begin{bmatrix} r+1 \\ k \end{bmatrix} f^{(-1)}(t-(k+1)h) \Big) \\
&= \lim_{\substack{h\to 0\\ nh=t-a}} h^r \Big( \sum_{k=0}^{n} \begin{bmatrix} r \\ k \end{bmatrix} f^{(-1)}(t-kh) - \begin{bmatrix} r+1 \\ n \end{bmatrix} f^{(-1)}(t-(n+1)h) \Big) \\
&= {}_aD_t^{-r} f^{(-1)}(t) - \lim_{\substack{h\to 0\\ nh=t-a}} h^r \begin{bmatrix} r+1 \\ n \end{bmatrix} f^{(-1)}(t-(n+1)h) \\
&= {}_aD_t^{-r} f^{(-1)}(t) - \lim_{\substack{h\to 0\\ nh=t-a}} h^r \begin{bmatrix} r+1 \\ n \end{bmatrix} f^{(-1)}(t-nh-h) \\
&= {}_aD_t^{-r} f^{(-1)}(t) - \lim_{\substack{h\to 0\\ nh=t-a}} h^r \begin{bmatrix} r+1 \\ n \end{bmatrix} f^{(-1)}(t-(t-a)-h) \\
&= {}_aD_t^{-r} f^{(-1)}(t) - \lim_{\substack{h\to 0\\ nh=t-a}} h^r \begin{bmatrix} r+1 \\ n \end{bmatrix} f^{(-1)}(a-h) \\
&= {}_aD_t^{-r} f^{(-1)}(t) - f^{(-1)}(a) \lim_{\substack{h\to 0\\ nh=t-a}} h^r \begin{bmatrix} r+1 \\ n \end{bmatrix} \\
&= {}_aD_t^{-r} f^{(-1)}(t) - 0
\end{aligned}
$$

From this we deduce that

$$
\begin{aligned}
{}_aD_t^{-(r+1)}f(t) = {}_aD_t^{-r}f^{(-1)}(t) &= \frac{1}{(r-1)!}\int_a^t (t-\nu)^{r-1}f^{(-1)}(\nu)\mathrm{d}\nu \\
&= \frac{1}{(r-1)!}\int_a^t (t-\nu)^{r-1}\int_a^\nu f(\tau)\mathrm{d}\tau\mathrm{d}\nu \\
&= \frac{1}{r!}\int_a^t (t-\nu)^r f(\nu)\mathrm{d}\nu
\end{aligned}
$$

where we use (6). This proves the claim. □

### 2.1.3 Arbitrary (Non-Integer) Order

So far we have defined

$$
{}_aD_t^\gamma f(t) := \lim_{\substack{h\to 0\\ nh=t-a}} \frac{1}{h^\gamma}\sum_{k=0}^{n}(-1)^k\binom{\gamma}{k}f(t-kh) \tag{13}
$$

We have shown that for integer values of $\gamma$, ${}_aD_t^\gamma f(t)$ corresponds to derivatives of $f$ when $\gamma$ is positive and antiderivative of $f$ when $\gamma$ is negative. It also follows from this definition that for $\gamma = 0, {}_aD_t^0 f(t) = f(t)$. We now simply extend this definition for non-integer order differintegrals by allowing non-integer $\gamma$ in the formula, substituting the Gamma function for the binomial product. The existence of the limit is proved in `[2] p.52-55`.
Note that when $\gamma$ is positive $f(t)$ needs to be at least $m$ times differentiable, where $m \geq \gamma$. When $\gamma$ is negative, $f(t)$ needs to be integrable.

### 2.1.4 Alternative Representation

There is an other way to express of the Grünwald-Letnikov derivative. From the binomial coefficient we know that

$$
\begin{aligned}
\binom{p}{k} &= \frac{p!}{k!(p-k)!} = \frac{(k+p-k)!}{k!(p-k)!} = \frac{(p-1)!k}{k!(p-k)!} + \frac{(p-1)!(p-k)}{k!(p-k)!} \\
&= \frac{(p-1)!}{(k-1)!(p-k)!} + \frac{(p-1)!}{k!(p-k-1)!} = \binom{p-1}{k-1} + \binom{p-1}{k}
\end{aligned} \tag{14}
$$

where we put $\binom{p-1}{-1} = 0$.

Using this repeatedly in (8) we get

$$h^p \cdot f_{h,n}^{(p)}(t) = \sum_{k=0}^{n} (-1)^k \binom{p}{k} f(t-kh)$$

*(Equation (14))*

$$= \sum_{k=0}^{n} (-1)^k \binom{p-1}{k} f(t-kh) + \sum_{k=0}^{n} (-1)^k \binom{p-1}{k-1} f(t-kh)$$

$$= \sum_{k=0}^{n} (-1)^k \binom{p-1}{k} f(t-kh) + \sum_{k=1}^{n} (-1)^k \binom{p-1}{k-1} f(t-kh)$$

$$= \sum_{k=0}^{n} (-1)^k \binom{p-1}{k} f(t-kh) + \sum_{k=0}^{n-1} (-1)^{k+1} \binom{p-1}{k} f(t-(k+1)h)$$

$$= \sum_{k=0}^{n-1} (-1)^k \binom{p-1}{k} \Big[ f(t-kh) - f(t-(k+1)h) \Big] + (-1)^n \binom{p-1}{n} f(t-nh)$$

$$= \sum_{k=0}^{n-1} (-1)^k \binom{p-1}{k} \bigtriangledown_h f(t-kh) + (-1)^n \binom{p-1}{n} f(a)$$

*(Repeat once more)*

$$= \sum_{k=0}^{n-2} (-1)^k \binom{p-2}{k} \Big[ \bigtriangledown_h f(t-kh) - \bigtriangledown_h f(t-(k+1)h) \Big]$$

$$+ (-1)^{n-1} \binom{p-2}{n-1} \bigtriangledown_h f(t-(n-1)h) + (-1)^n \binom{p-1}{n} f(a)$$

$$= \sum_{k=0}^{n-2} (-1)^k \binom{p-2}{k} \bigtriangledown_h^2 f(t-kh)$$

$$+ (-1)^{n-1} \binom{p-2}{n-1} \bigtriangledown_h f(a+h) + (-1)^n \binom{p-1}{n} f(a)$$

*(After $m$ times, $m \le p \le n$)*

$$= \sum_{k=0}^{n-m} (-1)^k \binom{p-m}{k} \bigtriangledown_h^m f(t-kh) + \sum_{l=0}^{m-1} (-1)^{n-l} \binom{p-1-l}{n-l} \bigtriangledown_h^l f(a+lh)$$

*(Using identity (5))*

$$= \sum_{k=0}^{n-m} (-1)^k \binom{p-m}{k} \bigtriangledown_h^m f(t-kh) + \sum_{l=0}^{m-1} (-1)^{n-l} \binom{p-1-l}{n-l} \bigtriangleup_h^l f(a)$$

Thus we see that

$$_aD_t^p f(t) = \lim_{\substack{h\to 0 \\ nh=t-a}} f_{h,n}^{(p)}(t) =$$

$$= \lim_{\substack{h\to 0 \\ nh=t-a}} \frac{1}{h^p} \sum_{k=0}^{n-m} (-1)^k \binom{p-m}{k} \bigtriangledown_h^m f(t-kh)$$

$$+ \lim_{\substack{h\to 0 \\ nh=t-a}} \frac{1}{h^p} \sum_{l=0}^{m-1} (-1)^{n-l} \binom{p-1-l}{n-l} \bigtriangleup_h^l f(a)$$

Using equations (10), (4), (3) and the identity $\Gamma(z) = \lim_{n\to\infty} \frac{(n+1)^z n!}{z(z+1)\cdots(z+n)}$, we find that

$$ {}_aD_t^p f(t) = \frac{1}{\Gamma(-p+m)} \int_a^t (t-\nu)^{m-p-1} f^{(m)}(\nu)\mathrm{d}\nu + \sum_{l=0}^{m-1} \frac{f^{(l)}(a)(t-a)^{-p+l}}{\Gamma(-p+l+1)} \tag{15} $$

We omit the full proof of this statement. All details of this construction can be found in `[2] p.52-55`.
For this representation we see that $f(t)$ needs to be at least $m$ times differentiable.

The Grünwald-Letnikov differintegral is a nice definition for fractional integrals and derivatives, but it is not a quick tool to determine the representation of a derivative or integral of arbitrary order. However, it is very suitable for numerical approximations `([2] p.199 and on)`.

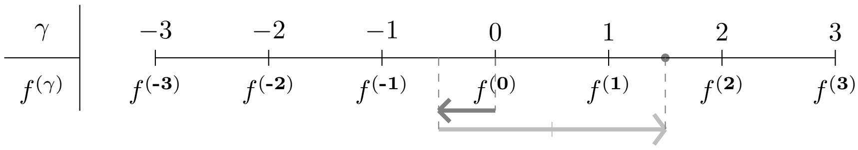
## 2.2 Riemann-Liouville Differintegral

The *Riemann-Liouville* (RL) differintegral is very similar to the Grünwald-Letnikov differintegral. It is based on extending the *Cauchy formula for repeated integration* (see (7)) for non-integer values, in order to define fractional integration. Fractional differentiation is defined by first applying fractional integration and afterwards performing regular differentiation, in a unique way.

To illustrate this concept, we picture a "time line" of $\gamma$, where $\gamma$ is the order of differintegration. Positive values of $\gamma$ correspond with differentiation. Negative values of $\gamma$ correspond with integration.

| $\gamma$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ |
|---|---|---|---|---|---|---|---|
| $f^{(\gamma)}$ | $f^{(\mathbf{-3})}$ | $f^{(\mathbf{-2})}$ | $f^{(\mathbf{-1})}$ | $f^{(\mathbf{0})}$ | $f^{(\mathbf{1})}$ | $f^{(\mathbf{2})}$ | $f^{(\mathbf{3})}$ |

Lets interpret the rules above. First we take a move to the left, then we move to the right. We can take arbitrary size steps to the left on our time line, but only integer steps to the right. Thus, to calculate a fractional derivative of $f$, we first have to move a certain amount to the left and subsequently move with integer steps to the right.
For example, would we like to know the result for $\gamma = 3/2$, we first have to take a half step to the left and subsequently two steps to the right.




Obviously, this can be done in multiple ways, like taking one and a half steps to the left and three steps to the right. In order to make the fractional derivative unique we require the step to the left to be between 0 and -1, thus of size less than 1.

### 2.2.1 Fractional Integral

The Riemann-Liouville definition of the fractional integral is given by

$$ {}_a\mathbf{D}_t^{-q} f(t) = \frac{1}{\Gamma(q)} \int_a^t (t-\nu)^{q-1} f(\nu) \mathrm{d}\nu $$

where $q \in \mathbb{R}^+$ and where we put ${}_a\mathbf{D}_t^0 f(t) = f(t)$. For details on why we can put ${}_a\mathbf{D}_t^0 f(t) = f(t)$, see `[2] p.65`.

It is merely the non-integer extension of Cauchy formula for repeated integration (see (7)), substituting the Gamma function for the factorial. Note that we need $f(t)$ to be integrable.

As a warning, in Miller-Ross `([1])` this definition is referred to as the Riemann version of the fractional integral, whereas they say the Riemann-Liouville version is where $a = 0$.

### 2.2.2 Fractional Derivative

For the fractional derivative we decompose the fractional order $p$ by $p =: m - q$, where $m \in \mathbb{Z}^+$ and $q \in (0, 1)$. If $p$ is an integer we will use the regular derivative. To calculate the fractional derivative we first preform a fractional integration of order $q$, following regular differentiation of order $m$. Thus, we define that the Riemann-Liouville definition of the fractional derivative is given by

$$ {}_a\mathbf{D}_t^p f(t) = \left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^m {}_a\mathbf{D}_t^{-q} f(t) = \frac{\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^m}{\Gamma(q)} \int_a^t (t-\nu)^{q-1} f(\nu) \mathrm{d}\nu $$

In contrast with the GL differintegral, we only require $f(t)$ to be integrable and we don't need that $f(t)$ is $m$ times differentiable.

### 2.2.3 The Fractional Integral and the Leibniz Integral Rule

**Theorem 2.1 (Leibniz Integral Rule)** *Consider the function $f(t, \nu)$ and the integral $\int_{a(t)}^{b(t)} f(t, \nu) \mathrm{d}\nu$. Let $f(t, \nu)$ be a continuous function that has a continuous derivative with respect to t within the region of integration. Suppose that $a(t)$ and $b(t)$ are also continuous and have continuous derivatives inside this region. Then, if $t$ is inside this region*

$$ \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{a(t)}^{b(t)} f(t,\nu)\mathrm{d}\nu\right) = f(x, b(x))b'(x) - f(x, a(x))a'(x) + \int_{a(t)}^{b(t)} \frac{\mathrm{d}}{\mathrm{d}t} f(t,\nu)\mathrm{d}\nu $$

Applying the theorem above to our definition of the fractional integral, we find that for $q > 1$

$$ \begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\Gamma(q)} \int_a^t (t-\nu)^{q-1} f(\nu)\mathrm{d}\nu\right) &= \frac{(t-t)^{q-1}}{\Gamma(q)} \cdot 1 - 0 \\ &\quad + \frac{1}{\Gamma(q)} \int_a^t \left[\frac{\mathrm{d}}{\mathrm{d}t}(t-\nu)^{q-1}\right] f(\nu)\mathrm{d}\nu \\ &= \frac{1}{\Gamma(q-1)} \int_a^t (t-\nu)^{q-2} f(\nu)\mathrm{d}\nu \end{aligned} $$

In general, for $q > 1$ and $m \geq 1$,

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^m \left(\frac{1}{\Gamma(q)} \int_a^t (t-\nu)^{q-1} f(\nu)\mathrm{d}\nu\right) = \frac{1}{\Gamma(q-m)} \int_a^t (t-\nu)^{q-1-m} f(\nu)\mathrm{d}\nu$$

Or in notation

$$\left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^m {}_a\mathbf{D}_t^{-q} f(t) = {}_a\mathbf{D}_t^{m-q} f(t), \qquad q > 1 \text{ and } m \geq 1 \tag{16}$$

In addition, this holds for non-integer values of $q$ provided that $t - \nu \geq 0$, implying $t \geq a$.

#### 2.2.4 Interchanging Derivation and Fractional Integration

Note that $\frac{\mathrm{d}}{\mathrm{d}t}(t-\nu)^\alpha = -\frac{\mathrm{d}}{\mathrm{d}\nu}(t-\nu)^\alpha$. We will use this a lot in the rest of the paper when we are using integration by parts.

Taking the $m$-th derivative of a fractional integral ${}_a\mathbf{D}_t^{-q} f(t)$ we get

$$\left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^m {}_a\mathbf{D}_t^{-q} f(t) = \frac{\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^m}{\Gamma(q)} \int_a^t (t-\nu)^{q-1} f(\nu)\mathrm{d}\nu$$

$$= \frac{\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{m-1}}{\Gamma(q)} \left[\int_a^t \left[\tfrac{\mathrm{d}}{\mathrm{d}t}(t-\nu)^{q-1}\right] f(\nu)\mathrm{d}\nu + \lim_{\nu\to t}(t-\nu)^{q-1} f(\nu)\right]$$

$$= \frac{\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{m-1}}{\Gamma(q)} \left[\int_a^t \left[-\tfrac{\mathrm{d}}{\mathrm{d}\nu}(t-\nu)^{q-1}\right] f(\nu)\mathrm{d}\nu + \lim_{\nu\to t}(t-\nu)^{q-1} f(\nu)\right]$$

*(integration by parts)*

$$= \frac{\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{m-1}}{\Gamma(q)} \left[\int_a^t \left[(t-\nu)^{q-1}\right] f^{(1)}(\nu)\mathrm{d}\nu - (t-\nu)^{q-1} f(\nu)\Big|_{\nu=a}^{\nu=t} + \lim_{\nu\to t}(t-\nu)^{q-1} f(\nu)\right]$$

$$= \frac{\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{m-1}}{\Gamma(q)} \left[\int_a^t (t-\nu)^{q-1} f^{(1)}(\nu)\mathrm{d}\nu + (t-a)^{q-1} f(a)\right]$$

$$= \frac{\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{m-1}}{\Gamma(q)} \int_a^t (t-\nu)^{q-1} f^{(1)}(\nu)\mathrm{d}\nu + \frac{(t-a)^{q-m} f(a)}{\Gamma(q-m+1)}$$

*(repeat once more)*

$$= \frac{\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{m-2}}{\Gamma(q)} \int_a^t (t-\nu)^{q-1} f^{(2)}(\nu)\mathrm{d}\nu + \frac{(t-a)^{q-m+1} f^{(1)}(a)}{\Gamma(q-m+2)} + \frac{(t-a)^{q-m} f(a)}{\Gamma(q-m+1)}$$

*(after m times)*

$$= \frac{1}{\Gamma(q)} \int_a^t (t-\nu)^{q-1} f^{(m)}(\nu)\mathrm{d}\nu + \sum_{l=0}^{m-1} \frac{(t-a)^{q-m+l} f^{(l)}(a)}{\Gamma(q-m+l+1)} \tag{17}$$

$$= {}_a\mathbf{D}_t^{-q} f^{(m)}(t) + \sum_{l=0}^{m-1} \frac{(t-a)^{q-m+l} f^{(l)}(a)}{\Gamma(q-m+l+1)} \tag{18}$$

From this we deduce the following theorem:

**Theorem 2.2** *Let $f(t)$ be an $m$ times differentiable function. Then*
$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^m {}_a\boldsymbol{D}_t^{-q} f(t) = {}_a\boldsymbol{D}_t^{-q} f^{(m)}(t)$ *if and only if* $f^{(l)}(a) = 0$ *for* $0 \le l \le m-1$.

If we recall our time line example, the above statement implies that it doesn't matter whether we step to the left or to right first *if and only if* $f^{(l)}(a) = 0$ for $0 \le l \le m-1$, provided that $f(t)$ is $m$ times differentiable.

### 2.2.5 Relation to the Grünwald-Letnikov Differintegral

We will show that the Riemann-Liouville differintegral equals the Grünwald -Letnikov differintegral. We already know that ${}_a\mathbf{D}_t^0 f(t) = f(t) = {}_aD_t^0 f(t)$. From (10) it follows that ${}_a\mathbf{D}_t^{-q} f(t) = {}_aD_t^{-q} f(t)$ for $q \in \mathbb{R}^+$, provided that $f(t)$ is integrable.
Now for $p \in \mathbb{R}^+$, we decompose $p$ by $p =: m - q$, $m \in \mathbb{Z}^+$ and $q \in (0,1)$. Let $f(t)$ be $m$ times differentiable. Using equation (17) we see that

$$
\begin{aligned}
{}_a\mathbf{D}_t^p f(t) &= \left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^m {}_a\mathbf{D}_t^{-q} f(t) = \\
&= \sum_{l=0}^{m-1} \frac{(t-a)^{q-m+l} f^{(l)}(a)}{\Gamma(q-m+l+1)} + \frac{1}{\Gamma(q)} \int_a^t (t-\nu)^{q-1} f^{(m)}(\nu)\mathrm{d}\nu \\
&= \sum_{l=0}^{m-1} \frac{(t-a)^{-p+l} f^{(k)}(a)}{\Gamma(-p+l+1)} + \frac{1}{\Gamma(-p+m)} \int_a^t (t-\nu)^{m-p-1} f^{(m)}(\nu)\mathrm{d}\nu
\end{aligned}
$$

Comparing this result with equation (15) we find

$$
\begin{aligned}
{}_a\mathbf{D}_t^p f(t) &= \sum_{l=0}^{m-1} \frac{(t-a)^{-p+l} f^{(k)}(a)}{\Gamma(-p+l+1)} + \frac{1}{\Gamma(-p+m)} \int_a^t (t-\nu)^{m-p-1} f^{(m)}(\nu)\mathrm{d}\nu \\
&= {}_aD_t^p f(t)
\end{aligned}
$$

We can conclude that ${}_a\mathbf{D}_t^\gamma f(t) = {}_aD_t^\gamma f(t)$ for all $\gamma \in \mathbb{R}$, provided that $f(t)$ is integrable and $m$ times differentiable.

# 3 Backwards Compatibility

It should be stressed once more that there are many different approaches to extending integrals and derivatives to non-integer order. In the literature the Riemann-Liouville definition is the most used version of the differintegral operator and the most commonly accepted fractional integral and fractional derivative. Therefore we will be using the RL differintegral throughout the end of the paper.

In this section we will check whether the RL differintegral follows the regular rules of the differentiation and integration. We will start by checking under which conditions we can interchange fractional integration and fractional differentiation. We will also discuss linearity and the product rule.

## 3.1 Integral and Derivative Compositions

From calculus we know that $\frac{\mathrm{d}^k}{\mathrm{d}t^k}\frac{\mathrm{d}^l}{\mathrm{d}t^l}f(t) = \frac{\mathrm{d}^l}{\mathrm{d}t^l}\frac{\mathrm{d}^k}{\mathrm{d}t^k}f(t) = \frac{\mathrm{d}^{(k+l)}}{\mathrm{d}t^{(k+l)}}f(t)$, provided that $f(t)$ is $k+l$ times differentiable. In this section we will check under which conditions compositions of fractional derivatives and integrals still commute and are still interchangeable.

### 3.1.1 Integral of an Integral

Let $f(t)$ be integrable. We will look at the composition of two fractional integrals.

$$\begin{aligned} {}_a\mathbf{D}_t^{-q_1}[{}_a\mathbf{D}_t^{-q_2}f(t)] =& \frac{1}{\Gamma(q_1)\Gamma(q_2)}\int_a^t (t-\nu)^{q_1-1}\int_a^\nu (\nu-\mu)^{q_2-1}f(\mu)\mathrm{d}\mu\mathrm{d}\nu \\ = (Fubini) \quad & \frac{1}{\Gamma(q_1)\Gamma(q_2)}\int_a^\nu (\nu-\mu)^{q_2-1}\int_a^t (t-\nu)^{q_1-1}f(\mu)\mathrm{d}\nu\mathrm{d}\mu \end{aligned}$$

Change of coordinates: $\nu = \mu + \gamma(t-\mu)$. The boundaries of integration become

$$\begin{cases}\mu &= \nu \\ \nu &= t\end{cases} \implies \begin{cases}\mu &= t \\ \gamma &= 1\end{cases} \qquad \begin{cases}\mu &= a \\ \nu &= t\end{cases} \implies \begin{cases}\mu &= a \\ \gamma &= 1\end{cases}$$

$$\begin{cases}\mu &= \nu \\ \nu &= a\end{cases} \implies \begin{cases}\mu &= a \\ \gamma &= 0\end{cases} \qquad \begin{cases}\mu &= a \\ \nu &= a\end{cases} \implies \begin{cases}\mu &= a \\ \gamma &= 0\end{cases}$$

Thus $[a,\nu]\times[a,t] \implies [a,t]\times[0,1]$

$$\begin{aligned} {}_a\mathbf{D}_t^{-q_1}[{}_a\mathbf{D}_t^{-q_2}f(t)] =& \frac{1}{\Gamma(q_1)\Gamma(q_2)}\int_a^t\int_0^1 \gamma^{q_2-1}(t-\mu)^{q_2-1} \\ & \cdot(1-\gamma)^{q_1-1}(t-\mu)^{q_1-1}f(\mu)\ \mathrm{d}\gamma\ (t-\mu)\ \mathrm{d}\mu \\ =& \frac{1}{\Gamma(q_1)\Gamma(q_2)}\int_a^t\left[\int_0^1 \gamma^{q_2-1}(1-\gamma)^{q_1-1}\mathrm{d}\gamma\right](t-\mu)^{q_1+q_2-1}f(\mu)\ \mathrm{d}\mu \\ =& \frac{1}{\Gamma(q_1)\Gamma(q_2)}\int_a^t\left[\int_0^1 \gamma^{q_2-1}(1-\gamma)^{q_1-1}\mathrm{d}\gamma\right](t-\mu)^{q_1+q_2-1}f(\mu)\ \mathrm{d}\mu \\ =& \frac{1}{\Gamma(q_1)\Gamma(q_2)}\int_a^t B(q_1,q_2)(t-\mu)^{q_1+q_2-1}f(\mu)\ \mathrm{d}\mu \\ =& \frac{B(q_1,q_2)}{\Gamma(q_1)\Gamma(q_2)}\int_a^t (t-\mu)^{q_1+q_2-1}f(\mu)\ \mathrm{d}\mu \\ =& \frac{1}{\Gamma(q_1+q_2)}\int_a^t (t-\mu)^{q_1+q_2-1}f(\mu)\ \mathrm{d}\mu \\ =& {}_a\mathbf{D}_t^{-(q_1+q_2)}f(t) \end{aligned}$$

We see that

$${}_a\mathbf{D}_t^{-q_1}[{}_a\mathbf{D}_t^{-q_2}f(t)] = {}_a\mathbf{D}_t^{-q_2}[{}_a\mathbf{D}_t^{-q_1}f(t)] = {}_a\mathbf{D}_t^{-(q_1+q_2)}f(t) \tag{19}$$

So as long as $f(t)$ is integrable, negative orders can be added.

### 3.1.2 Derivative of an Integral

We will be looking at the fractional derivative of a fractional integral.
Let $p_1 =: m_1 - q_1$, where $m_1 \in \mathbb{Z}^+$ and $q_1 \in (0,1)$, and let $q_2 \in \mathbb{Z}^+$.

$$ {}_a\mathbf{D}_t^{p_1}[{}_a\mathbf{D}_t^{-q_2} f(t)] = \left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^{m_1} {}_a\mathbf{D}_t^{-q_1}\, {}_a\mathbf{D}_t^{-q_2} f(t) $$

Using (19) to combine the integrals and (16) to reduce the order of the integral such that it is inside $(0,1)$, we see that

$$ \begin{aligned} &\left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^{m_1} {}_a\mathbf{D}_t^{-q_1}\, {}_a\mathbf{D}_t^{-q_2} f(t) = \left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^{m_1} {}_a\mathbf{D}_t^{-(q_1+q_2)} f(t) \\ &= \left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^{m_1} \left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^{q_3-(q_1+q_2)} {}_a\mathbf{D}_t^{-q_3} f(t) = \left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^{m_1+q_3-(q_1+q_2)} {}_a\mathbf{D}_t^{-q_3} f(t) \\ &= \left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^{m_3} {}_a\mathbf{D}_t^{-q_3} f(t) = {}_a\mathbf{D}_t^{m_3-q_3} f(t) = {}_a\mathbf{D}_t^{p_1-q_2} f(t) \end{aligned} \tag{20} $$

where $m_3 := m_1 + q_3 - (q_1 + q_2)$ and $q_3 \in (0,1)$ such that $m_3 \in \mathbb{Z}^+$. So

$$ {}_a\mathbf{D}_t^{p_1}[{}_a\mathbf{D}_t^{-q_2} f(t)] = {}_a\mathbf{D}_t^{p_1-q_2} f(t) $$

### 3.1.3 Integral of a Derivative

We will look at the fractional integral of a fractional derivative.
Let $p_2 =: m_2 - q_2$, where $m_2 \in \mathbb{Z}^+$ and $q_2 \in (0,1)$, and let $q_1 \in \mathbb{Z}^+$. First we observe that

$$ {}_a\mathbf{D}_t^{-q_1}[{}_a\mathbf{D}_t^{p_2} f(t)] = {}_a\mathbf{D}_t^{-q_1} \left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^{m_2} {}_a\mathbf{D}_t^{-q_2} f(t) $$

If we apply theorem 2.2 to this formula, and subsequently (18), we see that

$$ \begin{gathered} {}_a\mathbf{D}_t^{-q_1} \left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^{m_2} {}_a\mathbf{D}_t^{-q_2} f(t) = \left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^{m_2} {}_a\mathbf{D}_t^{-q_1}\, {}_a\mathbf{D}_t^{-q_2} f(t) \\ \Longleftrightarrow \\ \left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^{l} {}_a\mathbf{D}_t^{-q_2} f(t) = 0 \quad \text{at} \quad t = a, \quad 0 \le l \le m_2 - 1 \\ \Longleftrightarrow \\ {}_a\mathbf{D}_t^{-q} f^{(l)}(t) + \sum_{k=0}^{l-1} \frac{(t-a)^{q-l+k} f^{(k)}(a)}{\Gamma(q-l+k+1)} = 0 \quad \text{at} \quad t = a, \quad 0 \le l \le m_2 - 2 \end{gathered} $$

Now ${}_a\mathbf{D}_t^{-q} f^{(l)}(t) = 0$ at $t = a$, since it is an integral over zero length. This implies $\sum_{k=0}^{l-1} \frac{(t-a)^{q-l+k} f^{(k)}(a)}{\Gamma(q-l+k+1)} = 0$, which implies that $f^{(l)}(a) = 0$ for $0 \le l \le m_2 - 2$.

We end up with the following:

$$ \begin{gathered} {}_a\mathbf{D}_t^{-q_1} \left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^{m_2} {}_a\mathbf{D}_t^{-q_2} f(t) = \left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^{m_2} {}_a\mathbf{D}_t^{-q_1}\, {}_a\mathbf{D}_t^{-q_2} f(t) \\ \textit{if and only if} \\ f^{(l)}(a) = 0 \text{ for } 0 \le l \le m_2 - 2 \end{gathered} $$

Now we can use (20) on the right hand side of the equation such that we find

$$ {}_a\mathbf{D}_t^{-q_1}[{}_a\mathbf{D}_t^{p_2} f(t)] = {}_a\mathbf{D}_t^{p_2-q_1} f(t) \tag{21} $$

provided that $f^{(l)}(a) = 0$ for $0 \le l \le m_2 - 2$.

### 3.1.4 Derivative of a Derivative

Let $p_1 =: m_1 - q_1$ and $p_2 =: m_2 - q_2$, where $m_1, m_2 \in \mathbb{Z}^+$ and $q_1, q_2 \in (0,1)$. Using (21) and (16) we see that

$$\begin{aligned}{}_a\mathbf{D}_t^{p_1}[{}_a\mathbf{D}_t^{p_2} f(t)] &= \left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^{m_1} {}_a\mathbf{D}_t^{-q_1}[{}_a\mathbf{D}_t^{p_2} f(t)] \\ &= \left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^{m_1} {}_a\mathbf{D}_t^{p_2-q_1} f(t) \\ &= {}_a\mathbf{D}_t^{p_1+p_2} f(t)\end{aligned}$$

under the assumption that $f^{(l)}(a) = 0$ for $0 \leq l \leq m_2 - 2$.

Similarly, would we consider interchanging the two derivatives we end up with

$${}_a\mathbf{D}_t^{p_2}[{}_a\mathbf{D}_t^{p_1} f(t)] = {}_a\mathbf{D}_t^{p_1+p_2} f(t)$$

provided that $f^{(l)}(a) = 0$ for $0 \leq l \leq m_1 - 2$. Let $r$ be defined by $r := \max(m_1 - 2, m_2 - 2)$. We find

$$\begin{gathered}{}_a\mathbf{D}_t^{p_1}[{}_a\mathbf{D}_t^{p_2} f(t)] = {}_a\mathbf{D}_t^{p_2}[{}_a\mathbf{D}_t^{p_1} f(t)] = {}_a\mathbf{D}_t^{p_1+p_2} f(t) \\ \Longleftrightarrow \\ f^{(l)}(a) = 0 \text{ for } 0 \leq l \leq r\end{gathered}$$

We need $f(t)$ to be at least $r$ times differentiable in this case.

## 3.2 Linearity

From the linearity of the integral it follows that ${}_a\mathbf{D}_t^{-q}$ is linear.

$$\begin{aligned}{}_a\mathbf{D}_t^{-q}(cf(t) + dg(t)) &= \frac{1}{\Gamma(q)} \int_a^t (t-\nu)^{q-1}(cf(\nu) + dg(\nu))\mathrm{d}\nu \\ &= c\frac{1}{\Gamma(q)} \int_a^t (t-\nu)^{q-1} f(\nu)\mathrm{d}\nu + d\frac{1}{\Gamma(q)} \int_a^t (t-\nu)^{q-1} g(\nu)\mathrm{d}\nu \\ &= c\ {}_a\mathbf{D}_t^{-q} f(t) + d\ {}_a\mathbf{D}_t^{-q} g(t)\end{aligned}$$

In a similar fashion

$$\begin{aligned}{}_a\mathbf{D}_t^{p}(cf(t) + dg(t)) &= \left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^{m} {}_a\mathbf{D}_t^{-q}(cf(t) + dg(t)) \\ &= \left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^{m}\Big[c\ {}_a\mathbf{D}_t^{-q} f(t) + d\ {}_a\mathbf{D}_t^{-q} g(t)\Big] \\ &= c\ \left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^{m} {}_a\mathbf{D}_t^{-q} f(t) + d\ \left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^{m} {}_a\mathbf{D}_t^{-q} g(t) \\ &= c\ {}_a\mathbf{D}_t^{p} f(t) + d\ {}_a\mathbf{D}_t^{p} g(t)\end{aligned}$$

## 3.3 Product Rule and Leibniz Rule

We know the product rule as $\frac{d}{dt}(f(t) \cdot g(t)) = f(t) \cdot \frac{d}{dt} g(t) + g(t) \cdot \frac{d}{dt} f(t)$. This gives rise to the Leibniz rule:

$$\left(\tfrac{d}{dt}\right)^n (f(t) \cdot g(t)) = \sum_{k=0}^{n} \binom{n}{k} \Big[\left(\tfrac{d}{dt}\right)^k f(t)\Big]\Big[\left(\tfrac{d}{dt}\right)^{n-k} g(t)\Big]$$

Since $\binom{n}{k} = 0$ for $k > n$ we can extend the summation for any $r \geq n$:

$$\left(\tfrac{d}{dt}\right)^n (f(t) \cdot g(t)) = \sum_{k=0}^{r} \binom{n}{k} \left[\left(\tfrac{d}{dt}\right)^k f(t)\right] \left[\left(\tfrac{d}{dt}\right)^{n-k} g(t)\right]$$

The Leibniz rule for fractional derivatives will be defined by

$$_a\mathbf{D}_t^p (f(t) \cdot g(t)) = \sum_{k=0}^{r} \binom{p}{k} \left[\left(\tfrac{d}{dt}\right)^k f(t)\right] \left[{}_a\mathbf{D}_t^{p-k} g(t)\right]$$

Since $_a\mathbf{D}_t^p f(t) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^m {}_a\mathbf{D}_t^{-q} f(t)$ we may also write

$$_a\mathbf{D}_t^p (f(t) \cdot g(t)) = \sum_{k=0}^{r} \binom{p}{k} \left[\left(\tfrac{d}{dt}\right)^k f(t)\right] \left[\left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^{m-k} {}_a\mathbf{D}_t^{-q} g(t)\right]$$

The reasoning behind why we may define the Leibniz rule for fractional derivatives this way are given in `[2] p.91-97`.
The chain rule can also be expanded for non-integer orders, but we omit it here due to its complexity. Those who are interested can find it in `[2] p.97-98`.

# 4 Examples of Differintegration of Functions

This section will feature fractional derivatives of some basic function to give an indication how they are conceived. We will be using the RL differintegral throughout the end of the paper.

## 4.1 The Power function

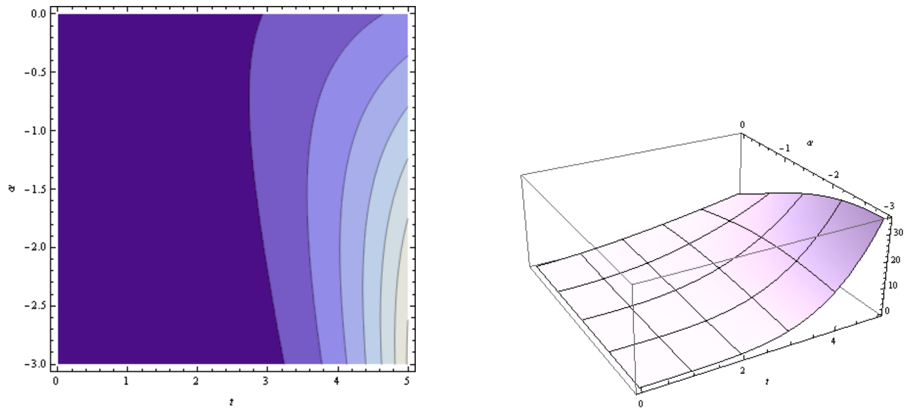
Let $f(t) = t^r$, and $a = 0$. Let $\alpha \leq r$. Then the differintegral of $f$ is given by

$$_a\mathbf{D}_t^\alpha f(t) = {}_0\mathbf{D}_t^\alpha t^r = \frac{\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^m}{\Gamma(q)} \int_0^t (t-\nu)^{q-1} \nu^r \mathrm{d}\nu$$

Using the coordinate transformation $\nu = \gamma t$ we find

$$\begin{aligned}
\frac{\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^m}{\Gamma(q)} \int_0^t (t-\nu)^{q-1} \nu^r \mathrm{d}\nu &= \frac{\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^m}{\Gamma(q)} \int_0^1 (t-\gamma t)^{q-1} (\gamma t)^r t \ \mathrm{d}\gamma \\
&= \frac{\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^m}{\Gamma(q)} \int_0^1 t^{q-1} (1-\gamma)^{q-1} \gamma^r t^{r+1} \mathrm{d}\gamma \\
&= \frac{\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^m}{\Gamma(q)} t^{q+r} \int_0^1 (1-\gamma)^{q-1} \gamma^r \mathrm{d}\gamma \\
&= \frac{\Gamma(q+r+1)}{\Gamma(q)\Gamma(q-m+r+1)} t^{q-m+r} \int_0^1 (1-\gamma)^{q-1} \gamma^r \mathrm{d}\gamma \\
&= \frac{\Gamma(q+r+1)}{\Gamma(q)\Gamma(q-m+r+1)} B(q, r+1) t^{q-m+r} \\
&= \frac{\Gamma(q+r+1)}{\Gamma(q)\Gamma(q-m+r+1)} \frac{\Gamma(q)\Gamma(r+1)}{\Gamma(q+r+1)} t^{q-m+r} \\
&= \frac{\Gamma(r+1)}{\Gamma(q-m+r+1)} t^{q-m+r} = \frac{\Gamma(r+1)}{\Gamma(r-\alpha+1)} t^{r-\alpha}
\end{aligned}$$

provided that $r > -1$



Plots for ${}_0\mathbf{D}_t^{\alpha} t^{3/2}$, with $t \in [0,5]$ and $\alpha \in [-3,0]$

## 4.2 The Constant Function

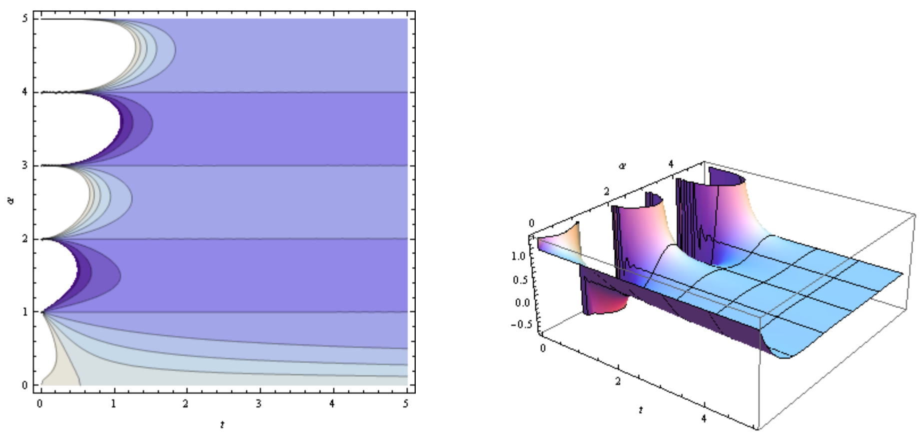
To obtain the differintegral of a constant we use the expression we found above and take the limit of $r$ to 0. We find

$$ {}_0\mathbf{D}_t^{\alpha}\, 1 = \lim_{r\to 0} {}_0\mathbf{D}_t^{\alpha} t^r = \lim_{r\to 0} \frac{\Gamma(r+1)}{\Gamma(1+r-\alpha)} t^{r-\alpha} = \frac{t^{-\alpha}}{\Gamma(1-\alpha)} $$

And due to the linearity of the RL differintegral (see section 3.2)

$$ {}_0\mathbf{D}_t^{\alpha}\, C = \frac{C t^{-\alpha}}{\Gamma(1-\alpha)} $$

An important property of the reciprocal gamma function $\frac{1}{\Gamma(z)}$ is that it equals zero if $z$ is an integer less or equal to zero. From this it follows that the derivative of a constant is zero whenever the order $\alpha$ is an integer greater than zero, and non-zero in between these integer values. Though the first outcome is exactly what we expected due to the construction of our differintegral, the second outcome is rather surprising.



Plots for ${}_{-\infty}\mathbf{D}_t^{\alpha} 1$, with $t \in [0,5]$ and $\alpha \in [0,5]$

## 4.3 The Exponential Function

Let $f(t) = e^{rt}$ with $r \in \mathbb{C}$, and $a = -\infty$. An RL differintegral where $a = -\infty$ is known as a Weyl differintegral.

$$ {}_a\mathbf{D}_t^\alpha f(t) = {}_{-\infty}\mathbf{D}_t^\alpha\ e^{rt} = \frac{\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^m}{\Gamma(q)} \int_0^t (t-\nu)^{q-1} e^{r\nu} \mathrm{d}\nu $$

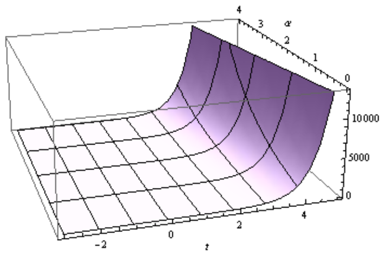
Using the coordinate transformation $\mu = r(t-\nu)$

$$ \begin{aligned} &\frac{\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^m}{\Gamma(q)} \int_0^t (t-\nu)^{q-1} e^{r\nu} \mathrm{d}\nu = r^{1-q} \frac{\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^m}{\Gamma(q)} \int_0^t \left[r(t-\nu)\right]^{q-1} e^{r\nu} \mathrm{d}\nu \\ &= r^{1-q} \frac{\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^m}{\Gamma(q)} \int_0^\infty \mu^{q-1} e^{rt-\mu} r^{-1}\ \mathrm{d}\mu = r^{-q} \frac{\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^m e^{rt}}{\Gamma(q)} \int_0^\infty \mu^{q-1} e^{-\mu} \mathrm{d}\mu \\ &= r^{-q} \frac{\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^m e^{rt}}{\Gamma(q)} \Gamma(q) = r^{-q} \left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^m e^{rt} = r^{-q} r^m e^{rt} = r^\alpha e^{rt} \end{aligned} $$

In short,

$$ {}_{-\infty}\mathbf{D}_t^\alpha\ e^{rt} = r^\alpha e^{rt} \tag{22} $$

In particular, ${}_{-\infty}\mathbf{D}_t^\alpha\ e^t = e^t$ for all $\alpha$.




Plot for ${}_{-\infty}\mathbf{D}_t^\alpha e^{2t}$, with $t \in [-3, 5]$ and $\alpha \in [0, 4]$

## 4.4 Trigonometric Functions

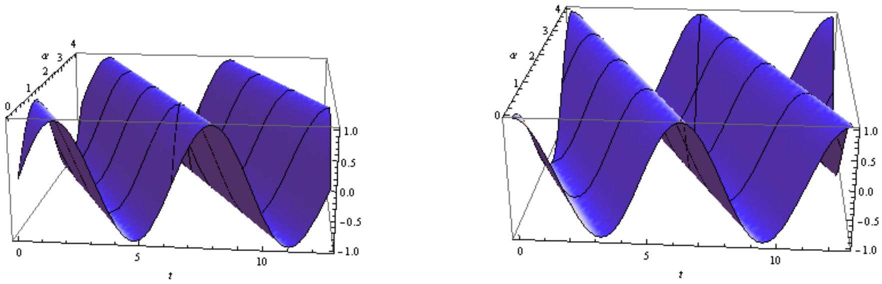
To calculate differintegals for trigonometric functions we express them using the exponential function with the following formulas

$$ \sin t = \frac{e^{it} - e^{-it}}{2i} \qquad \cos t = \frac{e^{it} + e^{-it}}{2} $$

In order to use equation (22) we put $a = -\infty$. Recalling the linearity of the RL differintegral (section 3.2), we find

$$ \begin{aligned} {}_{-\infty}\mathbf{D}_t^\alpha \sin t &= {}_{-\infty}\mathbf{D}_t^\alpha \left[\frac{e^{it} - e^{-it}}{2i}\right] = \frac{1}{2i}\left[{}_{-\infty}\mathbf{D}_t^\alpha e^{it} - {}_{-\infty}\mathbf{D}_t^\alpha e^{-it}\right] \\ &= \frac{1}{2i}\left[i^\alpha e^{it} - (-i)^\alpha e^{-it}\right] = \frac{1}{2i}\left[e^{i\frac{\pi}{2}\alpha} e^{it} - e^{-i\frac{\pi}{2}\alpha} e^{-it}\right] \\ &= \frac{1}{2i}\left[e^{i(t+\frac{\pi}{2}\alpha)} - e^{-i(t+\frac{\pi}{2}\alpha)}\right] = \sin(t + \tfrac{\pi}{2}\alpha) \end{aligned} $$

Likewise, for the cosine we find ${}_{-\infty}\mathbf{D}_t^\alpha \cos t = \cos(t + \frac{\pi}{2}\alpha)$.




Plots for ${}_{-\infty}\mathbf{D}_t^\alpha \sin(t)$ and ${}_{-\infty}\mathbf{D}_t^\alpha \cos(t)$, with $t \in [0, 4\pi]$ and $\alpha \in [0, 4]$

## 4.5 Hyperbolic Functions

Analogue to the trigonometric functions we can find an expression for differintegrals of hyperbolic functions.

$$\sinh t = \frac{e^t - e^{-t}}{2} \qquad\qquad \cosh t = \frac{e^t + e^{-t}}{2}$$

For the hyperbolic sine function

$$\begin{aligned}
{}_{-\infty}\mathbf{D}_t^\alpha \sinh t = {}_{-\infty}\mathbf{D}_t^\alpha \Big(\frac{e^t - e^{-t}}{2}\Big) &= \frac{1}{2}\Big({}_{-\infty}\mathbf{D}_t^\alpha e^t - {}_{-\infty}\mathbf{D}_t^\alpha e^{-t}\Big) \\
&= \frac{1}{2}\Big(e^t - (-1)^\alpha e^{-t}\Big) = \frac{1}{2}\Big(e^t - e^{i\pi\alpha} e^{-t}\Big) \\
= \frac{1}{2} e^{i\frac{\pi}{2}\alpha}\Big(e^{-i\frac{\pi}{2}\alpha} e^t - e^{i\frac{\pi}{2}\alpha} e^{-t}\Big) &= \frac{1}{2} e^{i\frac{\pi}{2}\alpha}\Big(e^{t - i\frac{\pi}{2}\alpha} - e^{-(t - i\frac{\pi}{2}\alpha)}\Big) \\
&= e^{i\frac{\pi}{2}\alpha} \sinh(t - i\tfrac{\pi}{2}\alpha) = i^\alpha \sinh(t + \tfrac{\pi}{2i}\alpha)
\end{aligned}$$

Likewise, for the hyperbolic cosine we find

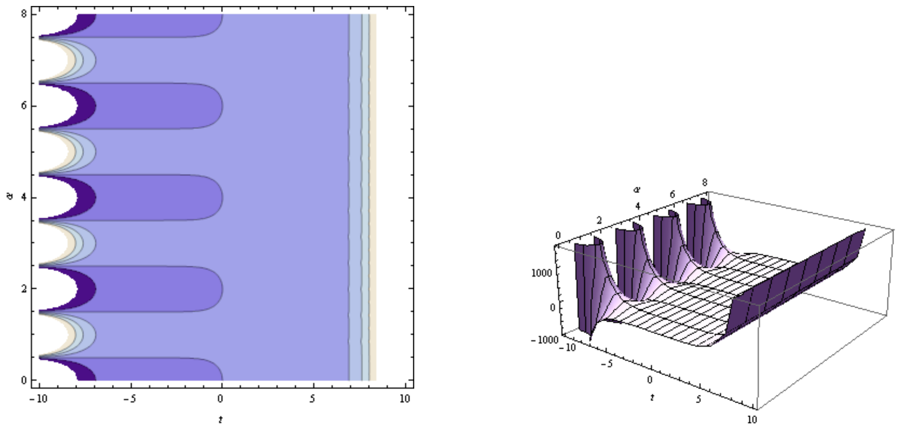
$${}_{-\infty}\mathbf{D}_t^\alpha \cosh t = e^{i\frac{\pi}{2}\alpha} \cosh(t - i\tfrac{\pi}{2}\alpha) = i^\alpha \cosh(t + \tfrac{\pi}{2i}\alpha)$$

In addition, since $\frac{\mathrm{d}}{\mathrm{d}t} \sinh t = \cosh t$ and $\frac{\mathrm{d}}{\mathrm{d}t} \cosh t = \sinh t$, the differintegrals above give rise to the following identities:

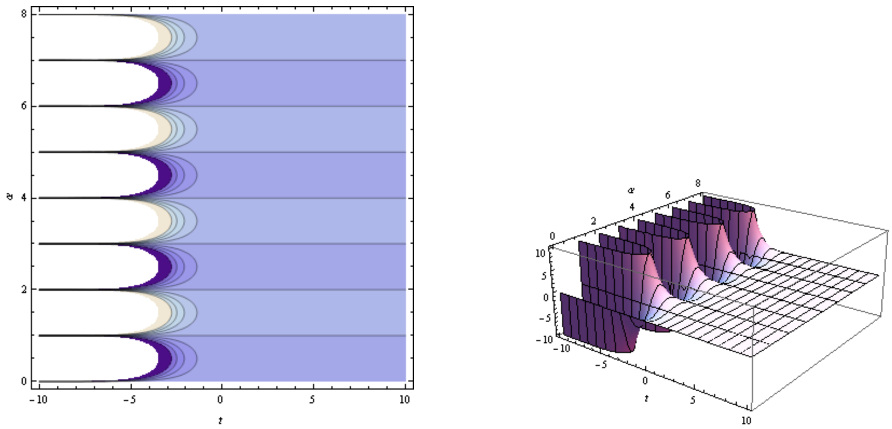
$$\begin{aligned}
\sinh t &= (-1)^k \sinh(t \pm i\pi k) \\
\cosh t &= (-1)^k \cosh(t \pm i\pi k)
\end{aligned}$$

$$\begin{aligned}
\sinh t &= (-1)^k i \cosh(t \pm i\pi k + \tfrac{\pi}{2i}) \\
\cosh t &= (-1)^k i \sinh(t \pm i\pi k + \tfrac{\pi}{2i})
\end{aligned}$$

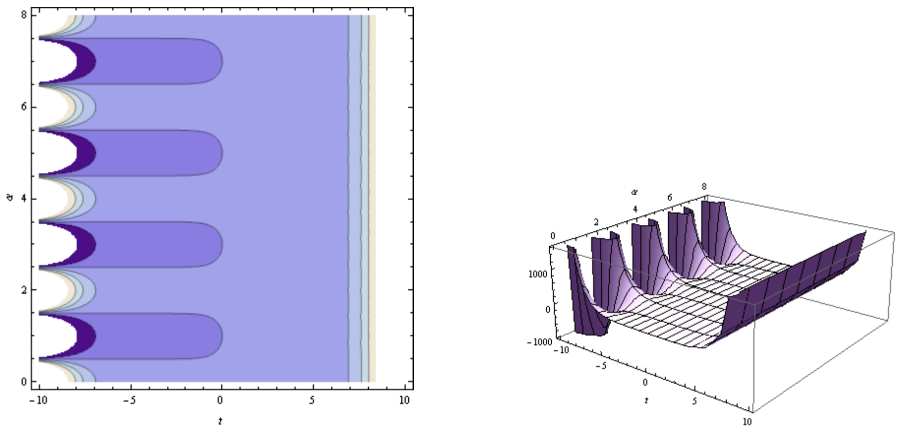
for all $k \in \mathbb{Z}$.



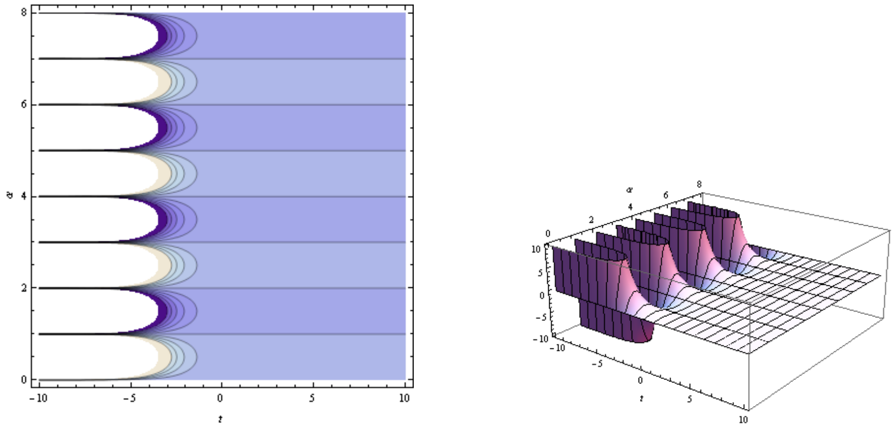

Plots for the real part of ${}_{-\infty}\mathbf{D}_t^{\alpha}\sinh(t)$ with $t\in[-10,10]$ and $\alpha\in[0,8]$




Plots for the imaginary part of ${}_{-\infty}\mathbf{D}_t^{\alpha}\sinh(t)$ with $t\in[-10,10]$ and $\alpha\in[0,8]$




Plots for the real part of ${}_{-\infty}\mathbf{D}_t^{\alpha}\cosh(t)$ with $t\in[-10,10]$ and $\alpha\in[0,8]$




Plots for the imaginary part of ${}_{-\infty}\mathbf{D}_t^{\alpha} \cosh(t)$ with $t \in [-10, 10]$ and $\alpha \in [0, 8]$

# 5 Discussion

We have seen different methods to extend regular integration and differentiation to non-integer orders. Using the forward difference derivative and the Cauchy formula for repeated integration we have defined the Grünwald-Letnikov and Riemann-Liouville differintegrals. Although the differintegrals are rather different, they give the same result, provided some conditions are met.

The RL differintegral is linear in the argument and the orders add up nicely under the right conditions. Furthermore, we have calculated differintegrals of some basic functions using the RL differintegral.

## 5.1 Other types of differintegrals

There are many other ways to represent integrals and differentials of functions. This results in various definitions of differintegrals.

### 5.1.1 Caputo Differintegral

The *Caputo* differintegral is similar to the RL differintegral. The fractional integral is also based on the *Cauchy formula for repeated integration.* In contrast with the RL differintegral, the fractional derivative is defined by first taking a regular derivative and then taking a fractional integral. As a result, all the fractional derivatives of a constant are zero.

### 5.1.2 Differintegrals based on Laplace Transform

A different approach involves the Laplace transform. If $F(t) = \int_0^t f(\nu)\mathrm{d}\nu$, then the Laplace transform of $F(t)$ is given by $\mathcal{L}\{F(t)\} = \frac{1}{s}\big(\mathcal{L}\{f(t)\}\big)(s)$. From this one may derive that

$$\mathcal{L}\Bigg\{ \underbrace{\int_a^t \int_a^{\nu_1} \cdots \int_a^{\nu_{n-1}}}_{n} f(\nu_n)\mathrm{d}\nu_n \cdots \mathrm{d}\nu_1 \Bigg\} = \frac{1}{s^n}\big(\mathcal{L}\{f(t)\}\big)(s)$$

Or, taking the inverse Laplace transform of both sides

$$\underbrace{\int_a^t \int_a^{\nu_1} \cdots \int_a^{\nu_{n-1}}}_{n} f(\nu_n)\mathrm{d}\nu_n \cdots \mathrm{d}\nu_1 = \mathcal{L}^{-1}\left\{\frac{1}{s^n}\big(\mathcal{L}\{f(t)\}\big)(s)\right\}$$

The fractional integral may be obtained by putting in non-integer values for $n$ on the right hand side. If $f(0) = 0$ and all derivatives of $f(t)$ upto order $n-1$ are zero at $t = 0$, the Laplace transform of the derivative of $f(t)$ is

$$\mathcal{L}\{\tfrac{\mathrm{d}^n}{\mathrm{d}t^n} f(t)\} = s^n\big(\mathcal{L}\{f(t)\}\big)(s)$$

which is also easy to extend to non-integer order.

Under certain conditions this approach coincides with the RL differintegral, as seen in `[2] p.104-71`. On the other hand, its easy to describe fractional integrals using the Laplace transformation, as seen in `[1] p.69-71`.

### 5.1.3 Hadamard Fractional Integral

The *Hadamard* differintegral is given by

$${}_a\mathbb{D}_t^{-q} f(t) = \frac{1}{\Gamma(q)} \int_a^t \left(\log \frac{t}{\nu}\right)^{q-1} f(\nu)\frac{\mathrm{d}\nu}{\nu}$$

It is studied very little compared to the other differintegrals.

## 5.2 Subjects of Personal Interest

There are still quite a lot of things I would like to learn about fractional calculus. For example, let ${}_a\mathcal{D}_f(t,\alpha) = {}_a\mathbf{D}_t^\alpha f(t)$. What would it mean if we let $\alpha$ be a function of $t$? Can we say that ${}_{-\infty}\mathcal{D}_{\cos}(t, -\frac{2}{\pi}t) = 1$, since we have ${}_{-\infty}\mathbf{D}_t^\alpha \cos t = \cos(t + \frac{\pi}{2}\alpha)$? And, is there a function $h(t)$ that satisfies ${}_a\mathcal{D}_h(t,\alpha) = {}_a\mathcal{D}_h(\alpha, t)$?

With the concept of fractional order derivatives came also the notion of fractional differential equations: differential equations using differintegal operators. I was wondering whether there are cases in which it would be viable to split an ordinary differential equation into derivatives of fractional order.

## References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