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An overview of microscopic and macroscopic traffic models

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Abstract

Traffic models can be used for several applications. For example when adjusting the infrastructure or trying to solve the current known traffic problems. There are different types of traffic models: microscopic, mesoscopic and macroscopic models. In this thesis three microscopic models: the follow-the-leader model, the optimal velocity model and the generalized force model, and two macroscopic traffic models: the LW-model and Payne's model are described. The difference between these models is appointed. In the end the influence of adding a PI-controller to the optimal velocity model is discussed.

1 Introduction

Traffic models can be used for several applications. For example they can be used when adjusting the infrastructure or trying to solve the current known traffic problems. Because traffic models are used for several applications much attention is given to research on traffic models. The first traffic flow models were developed in the fifties. [8] Additions are made after the development of these traffic models and other models have been developed.

There are different types of traffic flow models and they can be classified in various ways. For example they can be classified to the level of detail. Then there are three types of models: microscopic, mesoscopic, and macroscopic traffic models. Microscopic traffic models give attention to the details of traffic flow. These models simulate single vehicle-driver units. [9] Macroscopic traffic models assume a sufficiently large number of vehicles on a road such that each stream of vehicles can be treated as flowing in a tube or a stream. Mesoscopic traffic models look at vehicle groups. In this thesis we focus on microscopic and macroscopic traffic models. [12]

In the first section three microscopic traffic models: the follow-the-leader model, the optimal velocity model and the generalized force model are discussed. These three models are compared with each other. Two macroscopic models: the LW-model and Payne's model are described in section 2. These models consist of a continuity equation based on the conservation of vehicels and an equation describing the velocity. The difference between these models is discussed. In the last section we look specifically at the optimal velocity model described in section 1 and the impact of adding a PI-controller to this model.

2 Microscopic traffic models

Microscopic traffic models describe the details of traffic flow and the interaction taking place within it. Microscopic traffic models simulate single vehicle-driver units. The dynamic variables of the models represent microscopic properties like the position and velocity of single vehicles. These models can be divided into two categories: cell automata models, which are discrete in time and space, and continuous models, which are continuous in time. The latter are required for detailed studies of car-following behaviour and traffic instabilities. A few of them will be discussed below.

The first microscopic traffic models were developed in the sixties. [7] The following strategy is used to build such a dynamical model. The equation of motion of each vehicle is based on the assumption that each driver responds to a stimulus from other vehicles in some special way. These response is expressed in terms of acceleration. The stimulus may be a function of the positions of vehicles and their time derivatives, and so on. This function is decided by supposing that drivers of vehicles obey postulated traffic regulations at all times in order to avoid traffic accidents.

For the regulations there are two major types of theories. The first one is based on the idea that each vehicle must maintain the legal safe distance of the preceding vehicle. This legal safe distance depends on the velocity difference of these two successive vehicles. These theories are called the follow-the-leader theories. The second type is based on the idea that each vehicle has the legal velocity. The legal velocity is equal to the desired velocity of the driver. If there is little traffic this velocity is assumed to be the maximum allowed velocity. This velocity decreases when the traffic density increases. So the legal velocity depends on the relative position between vehicles. [3]

Most of the earlier microscopic traffic models were based on the first idea. The follow-the-leader model proposed by Gazis, Herman and Rothery for example.[5] Before this model is discussed it is important that the following notation is mentioned.

In microscopic traffic models each vehicle is numbered. In this paper the i-th vehicle follows the $(i+1)$-th vehicle. The position of the i-th vehicle is equal to p_i and the velocity of this vehicle is equal to v_i. Besides the following variables are important:

- $\Delta p_{ij} = p_j - p_i$: the relative position between vehicle i and j. Vehicle j is a neighboring vehicle influencing vehicle i. The relative position Δp is denoted as $p_{i+1} - p_i$.
- $\dot{v}_i$: the acceleration of vehicle i.
- $\Delta v_{ij} = v_j - v_i$: the relative velocity between vehicle i and j. Vehicle j is a neighboring vehicle influencing vehicle i. The relative velocity Δv is in contrast to Δp defined as $v_i - v_{i+1}$.

2.1 The Follow-the-leader Model

The follow-the-leader model was proposed by Gazis, Herman and Rothery. [5] In the follow-the-leader model it is assumed that the dynamics of vehicle i are given by the equation of motion:

$$\dot{p}_i(t) = v_i(t) \tag{1}$$

and the acceleration equation:

$$\dot{v}_i(t+T) = \kappa_i[v_{i+1}(t) - v_i(t)]. \tag{2}$$

In equation (2) the acceleration of vehicle i is slowed down by the adaptation time T. Therefore the following vehicle is assumed to accelerate at time $t+T$. The parameter κ_i reflects the sensitivity of the driver of vehicle i. The following functions can be assumed for this sensitivity:

1. constant: $\kappa_i = a_i$.
2. step function:
$$\kappa_i = \begin{cases} a_i & for \quad \Delta p \leq \Delta p_{crit} \\ b_i & for \quad \Delta p > \Delta p_{crit} \end{cases}$$
3. reciprocal spacing: $\kappa_i = c_i/\Delta p$.

Here a_i, b_i and c_i are constants. In this thesis the parameter κ_i is assumed to be a constant. [5] [7]

2.2 The Optimal Velocity Model

The optimal velocity model (OVM) is proposed by Bando *et al.* [7] This model is based on the second idea previously mentioned: each vehicle has the legal velocity, which depends on the relative position between vehicles. The acceleration equation in this model becomes:

$$\dot{v}_i(t) = \kappa_i[V_i(\Delta p_{ij}) - v_i(t)]. \tag{3}$$

In equation (3) the term $V_i(\Delta p_{ij})$ is the legal velocity or optimal velocity of vehicle number i and the parameter κ_i is again equal to the sensitivity of the driver of vehicle i. As said the legal velocity depends on the relative position between vehicles. The velocity must be reduced when the relative position between vehicles decreases. The velocity has to be small enough to prevent crashing into the preceding car. When the relative position between the vehicles increases, the vehicle can move with a higher velocity although the vehicle does not exceed the maximum velocity. So V must be a monotonically increasing function with an upper bound. In the optimal velocity model $V_i(\Delta p_{ij})$ is given by the following equation:

$$V_i(\Delta p_{ij}) = V_i^0 + V_i^1 \sum_{j \in N(i)} \tanh(p_j - p_i). \tag{4}$$

The constants V_i^0 are "the preferred velocities" and the constants V_i^1 represent the sensitivities of the drivers. The neighboring vehicles influencing vehicle i

are denoted by the set $N(i)$.

Combining formula (3) and formula (4) gives the equation:

$$\dot{v}_i(t) = \kappa_i[-v_i(t) + V_i^0 + V_i^1 \sum_{j \in N(i)} \tanh(\Delta p_{ij})]. \quad (5)$$

[3] [7]

2.3 Generalized Force Model

The generalized force model (GFM) is proposed by Helbing and Tilch. [7] They developed this model motivated by the success of the so-called social force models in the description of behavorial changes. According to the social force concept the amount and direction of a behavioral change, the acceleration, is given by a sum of generalized forces. These forces reflect the different motivations which a driver feels at the same time in response to his respective environment. The success of this approach in describing traffic dynamics is based on the fact that driver reactions to typical traffic situations are mostly the same. These reactions are determined by the optimal behavorial strategy.

The driver behavior is mainly given by the motivation to reach a certain desired velocity of the driver. The desired velocity of the driver of vehicle i is denoted by v_i^0 and will be reflected by an acceleration force f_i^0. The driver will also keep a safe distance from other cars j. This will be described by the repulsive interaction forces $f_{i,j}$. The equation which describes this behavior is given by:

$$\dot{v}_i(t) = f_i^0(v_i) + \sum_{j(\neq i)} f_{i,j}(\Delta p_{ij}, v_i) + \xi_i(t). \quad (6)$$

In this equation ξ_i is the fluctating force of vehicle i. The fluctating force may be include individual variations of driver behavior. Here this force is set to zero. The acceleration force is proportional to the difference between the desired and actual velocity:

$$f_i^0(v_i) = \frac{v_i^0 - v_i}{\tau_i}. \quad (7)$$

The time τ_i is equal to the acceleration time. Suppose that the most important interaction concerns the vehicle in front. Therefore equation (6) becomes:

$$\dot{v}_i(t) = \frac{v_i^0 - v_i}{\tau_i} + f_{i,i+1}(\Delta p, v_i). \quad (8)$$

The only thing which have to be specified is the interaction force $f_{i,i+1}$. If this force is equal to:

$$f_{i,i+1} = \frac{V_i(\Delta p) - v_i^0}{\tau_i} \quad (9)$$

and $\tau_i = 1/\kappa_i$ then the optimal velocity model (see 2.2) is obtained. The only difference is that this optimal velocity model assumes that the acceleration of a vehicle is only influenced by the vehicle in front. In the generalized force model this relation is extended by a term which should guarantee early enough and sufficient braking in cases of large relative velocities Δv. This deceleration term is equal to:

$$\frac{\Delta v \cdot \Theta(\Delta v)}{\tau_i'} e^{-[\Delta p - \Delta p(v_i)]/R_i'}. \tag{10}$$

Here the Heaviside function $\Theta(\Delta v)$ is defined as:

$$\Theta(\Delta v) = \begin{cases} 1 & if \quad \Delta v \geq 0 \\ 0 & if \quad \Delta v < 0 \end{cases}. \tag{11}$$

So the deceleration term (10) is only added if the velocity of the vehicle in front is smaller. The parameter τ_i' in this term is the braking time. This time is smaller than τ_i, because deceleration capabilities of vehicles are greater than acceleration capabilities. The term:

$$e^{-[\Delta p - \Delta p(v_i)]/R_i'}$$

is added because the deceleration term should be zero for large relative positions Δp. If the relative position decreases the deceleration term should increase. The variable $\Delta p(v_i)$ is a certain safe distance depending on the velocity of vehicle i. The constant R_i' can be interpreted as a range of the braking interaction.

The deceleration term (10) has to be substracted from equation (9) which express the acceleration. This yields the following interaction force:

$$f_{i,i+1} = \frac{V_i(\Delta p) - v_i^0}{\tau_i} - \frac{\Delta v \cdot \Theta(\Delta v)}{\tau_i'} e^{-[\Delta p - \Delta p(v_i)]/R_i'}. \tag{12}$$

The number of parameters in this model can be reduced if the function $V_i(\Delta p)$ is replaced by:

$$V_i(\Delta p, v_i) = V_i^0\{1 - e^{-[\Delta p - \Delta p(v_i)]/R_i}\}. \tag{13}$$

The model consisting of equations (8), (12) and (13) is called the generalized force model. Combining these equations gives:

$$\dot{v}_i(t) = \kappa_i[-v_i(t) + V_i^0 - V_i^1 e^{-[\Delta p - \Delta p(v_i)]/R_i}] - \kappa_i'[\Delta v \cdot \Theta(\Delta v) e^{-[\Delta p - \Delta p(v_i)]/R_i'}]. \tag{14}$$

[7]

2.4 Comparison microscopic traffic models

The three microscopic traffic models described in the sections before are all based on the assumption that each driver responds to a stimulus from other vehicles in terms of acceleration. How the drivers are stimulated is different in these microscopic models. The follow-the-leader model and the generalized force model assume that the acceleration of one vehicle is only influenced by the vehicle in front. The optimal velocity model assumes that the driver also responds to other neigboring vehicles. The follow-the-leader model assumes that this acceleration depends on the relative velocity between the so-called follower and leader vehicle. The acceleration of vehicle i in the optimal velocity model and the generalized force model is determined by the difference between the own velocity and a legal velocity which depends on the relative position between vehicles.

The optimal velocity model and the generalized force model are almost the same. One difference between these models is the assumption that in the generalized force model a vehicle is only influenced by the vehicle in front. An other difference between these models is a deceleration term added in the generalized force model. This term guarantees early enough and sufficient braking in case of large velocities to avoid accidents. This braking term takes into account that vehicles prefer to keep a certain safe distance. This safe distance depends on the velocity of the vehicle.

The follow-the-leader model and the optimal velocity model can be expressed as port-Hamiltonian system. The follow-the-leader model written as port-Hamiltonian system is:

$$\dot{p} = RB^T \frac{\partial H}{\partial p} \tag{15}$$

where B is the incidence matrix (see Appendix A), R is a diagonal matrix with elements κ_i on the diagonal and the Hamiltonian H is equal to:

$$H(p) = \sum_i \frac{p^2}{2m_i}. \tag{16}$$

Here it's neglected that the acceleration is slowed down by an adaptation time T. The optimal velocity model can be expressed as port-Hamiltonian system in the following way:

$$\frac{d}{dt}\begin{bmatrix} p \\ \eta \end{bmatrix} = \begin{bmatrix} -I & B \\ -B^T & O \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial p} \\ \frac{\partial H}{\partial \eta} \end{bmatrix}. \tag{17}$$

Here the Hamiltonian is equal to:

$$H(p, \eta) = \sum_i \frac{p^2}{2m_i} + \sum_k P_k(\eta_k) \tag{18}$$

where i is the number of vehicles and k is the number of relative positions between vehicles. The relative position between vehicles, Δp_{ij}, is now denoted by η_k to avoid confusion with the impuls p. The function $P_k(\eta_k)$ is equal to:

$$P_k(\eta_k) = m_i V_i^1 \ln\cosh(\eta_k). \tag{19}$$

[4] [11]

3 Macroscopic traffic models

Macroscopic traffic modeling assumes a sufficiently large number of vehicles on a road such that each stream of vehicles can be treated as flowing in a tube or a stream. There are three variables important in macroscopic traffic modeling. These variables are:

- the rate of flow, $q(x,t)$: the number of vehicles passing a fixed point x per unit time.
- the speed of traffic flow, $v(x,t)$: the distance covered per unit time.
- the traffic density, $\rho(x,t)$: the number of vehicles in a traffic line of given length.

These traffic variables are connected in the following way:

$$q(x,t) = \rho(x,t)v(x,t). \tag{20}$$

There is also an other relation between these variables. This relation is based on the conservation of vehicles [12]:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial x} = 0. \tag{21}$$

Equation (20) and equation (21) constitute a model with two independent equations and three unknown variables. To get a complete description of traffic flow a third independent equation is needed. Macroscopic traffic models use different equations for the velocity to complete the description of traffic flow. Below the LW-Model and Payne's model are discussed.

3.1 LW-Model

The most straightforward approach is to assume that the expected velocity v can be described as a function of the traffic density:

$$v(x,t) = V_e(\rho(x,t)). \tag{22}$$

The first macroscopic model uses this relation given in equation (22). This model was proposed by Lighthill and Whitham in 1955. [8] The model is called the LW-model named after its authors. The model consists of equation (20) and equation (21) and of the non-linear first-order partial differential equation resulting from (22):

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x}\left(V_e + \rho \frac{dV_e}{d\rho}\right) = 0. \tag{23}$$

This model does not describe the nonequilibrium situations such as emergent traffic jams and stop-and-go traffic adequate because it is assumed that each time instant the mean flow speed $v(x,t)$ is equal to the equilibrium value $V_e(\rho)$ for the given vehicle density. Therefore instead of equation (22), a differential equation modeling the mean speed dynamics was suggested for describing the non-equilibrium situations. This is done by Payne. [8]

3.2 Payne's Model

Payne proposed the first continuum traffic model by a coupled system of two partial differential equations in 1971. [8] That means: equation (20) and (21) are in this model extended by a partial differential equation describing the dynamics of the velocity v. Payne's model is derived from a simple car-following rule:

$$v(x(t+T), t+T) = V_e(\rho(x+\Delta p), t). \tag{24}$$

where $x(t)$ is the position of the vehicle at time t, $v(t)$ is its velocity and V_e is the equilibrium velocity as a function of the density ρ at $x + \Delta p$. Furthermore T is the reaction time and Δp is the relative position between the follower and leader vehicle. Equation (24) shows that drivers of vehicles adapt their velocity to the equilibrium velocity which depends on the density. The equilibrium velocity represents a trade-off between the desired velocity of a driver and a reduction in the velocity due to worsening traffic conditions.

To derive the partial differential equation of Payne's model, the Taylor's expansion rule have to be applied to the left- and rightside of equation (24). The reaction time T is relatively small so the higher order terms can be neglected. This yields:

$$v(x(t+T), t+T) \approx v(x,t) + T \cdot v(x,t)\frac{\partial v}{\partial x} + T\frac{\partial v}{\partial t} \tag{25}$$

and

$$V_e(\rho(x+\Delta p), t) \approx V_e(\rho(x,t)) + \Delta p \cdot \frac{\partial \rho}{\partial x}\frac{d}{d\rho}V_e(\rho(x,t)). \tag{26}$$

The traffic density ρ is equal to $1/\Delta p$. Now substitute equation (25) and (26) in the car-following rule (24) gives:

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = \underbrace{\frac{1}{T}(V_e(\rho) - v)}_{\text{relaxation term}} - \underbrace{\frac{c^2(\rho)}{\rho}\frac{\partial \rho}{\partial x}}_{\text{anticipation term}} \tag{27}$$

where $c^2(\rho)$ is the anticipation function associated with the density and is equal to:

$$c^2(\rho) = -\frac{1}{T}\frac{dV_e}{d\rho}. \tag{28}$$

So this model relies on the microscopic description of movements of individual cars according to the follow-the-leader model.

Equation (27) is the general form of the dynamic velocity equation of most macroscopic models. The term $v\frac{\partial v}{\partial x}$ is called the convection term and describes the variation of the mean velocity due to inflowing and outflowing vehicles. The anticipation term describes the drivers' response to the situation ahead of them. The relaxation term describes the tendency of traffic flow to relax to an equilibrium velocity. [8]

4 The Optimal Velocity Model and PI-control

The optimal velocity model (see 2.2) is given by the equation:

$$\dot{v}_i(t) = \kappa_i[V_i(\Delta p_{ij}) - v_i(t)]. \tag{29}$$

The term $V_i(\Delta p_{ij})$ is the optimal velocity of vehicle i. This optimal velocity depends on the relative position of neighboring vehicles in the following way:

$$V_i(\Delta p_{ij}) = V_i^0 + V_i^1 \sum_{j \in N(i)} \tanh(p_j - p_i). \tag{30}$$

The optimal velocity model can also be written as:

$$\begin{bmatrix} \dot{v}_1 \\ \vdots \\ \vdots \\ \dot{v}_i \end{bmatrix} = \begin{bmatrix} -\kappa_1 & 0 & \cdots & 0 \\ 0 & -\kappa_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -\kappa_i \end{bmatrix} \begin{bmatrix} v_1 - V_1(\Delta p_{ij}) \\ \vdots \\ \vdots \\ v_i - V_i(\Delta p_{ij}) \end{bmatrix}. \tag{31}$$

The driver of vehicle i want to reach the optimal velocity $V_i(\Delta p_{ij})$. But if κ_i is very large this will never happen. The velocity of vehicle i is in that case never equal to the optimal velocity $V_i(\Delta p_{ij})$. PI-control is used to be sure that the velocity will always go to the optimal velocity. An integral term is added to equation (29). This gives the following equation:

$$\dot{v}_i(t) = -\kappa_i[v_i(t) - V_i(\Delta p)] - k_i \int_0^t (v_i - V_i(\Delta p_{ij}))dt \tag{32}$$

where κ_i is the proportional gain and k_i is the integral gain. The velocity v_i creeps slowly toward the optimal velocity $V_i(\Delta p_{ij})$ for small values of k_i and goes faster for larger integral gains, but the system also becomes more oscillatory. Equation (32) can be written as the system:

$$\begin{aligned} \dot{z}_i &= v_i - V_i(\Delta p_{ij}) \\ \dot{v}_i &= -\kappa_i[v_i - V_i(\Delta p_{ij})] - k_i z_i. \end{aligned} \tag{33}$$

This system can also be written as:

$$\begin{bmatrix} \dot{v}_i \\ \dot{z}_i \end{bmatrix} = \begin{bmatrix} -\widetilde{K} & -K \\ I & O \end{bmatrix} \begin{bmatrix} v_i - V_i(\Delta p_{ij}) \\ z_i \end{bmatrix} \tag{34}$$

where $\widetilde{K}$ is a diagonal matrix with elements κ_i on the diagonal, K is a diagonal matrix with elements k_i on the diagonal, I is the identity matrix and O is the null matrix.

What happens if a disturbance such as rainfall or wind is add to the system? To investigate this, apply a disturbance d_i to the system given in equation (33):

$$\begin{aligned} \dot{z}_i &= v_i - V_i(\Delta p_{ij}) \\ \dot{v}_i &= -\kappa_i[v_i - V_i(\Delta p_{ij})] - k_i z_i + d_i. \end{aligned} \tag{35}$$

After a while this system goes to an equilibrium value: $v_i = V_i(\Delta p_{ij})$ and $-k_i z_i + d_i = 0$. If the variable $\widetilde{z}_i = z_i - \bar{z}_i$ is introduced, system (35) can be written as:

$$\begin{bmatrix} \dot{x}_i \\ \dot{\widetilde{z}} \end{bmatrix} = \begin{bmatrix} -\widetilde{K} & -K \\ I & O \end{bmatrix} \begin{bmatrix} v_i - V_i(\Delta p_{ij}) \\ \widetilde{z}_i \end{bmatrix}. \tag{36}$$

Now the velocity v_i goes to the optimal velocity $V_i(\Delta p_{ij})$ and $\widetilde{z}_i(t)$ goes to zero, i.e. $z_i(t)$ goes to $\bar{z}_i(t)$. So adding a PI-controller attenuates disturbances effictively. This means if a disturbance such as rainfall or wind is add to the optimal velocity model, the velocity of vehicle i becomes still equal to the optimal velocity $V_i(\Delta p_{ij})$. [2]

5 Conclusion

Traffic flow models can be used for several applications. For example when adjusting the infrastucture and trying to solve the current known traffic problems. There are different types of traffic models: microscopic, mesoscopic and macroscopic models. In this thesis three microscopic models were discussed: the follow-the-leader model, the optimal velocity model and the generalized force model. These models are based on the assumption that each driver responds to a stimulus from other vehicles in terms of acceleration. The follow-the-leader model and the generalized force model assume that this acceleration is only influenced by the vehicle in front. The optimal velocity model assumes that the driver also responds to other neighboring vehicles. The acceleration in the follow-the-leader model only depends on the relative velocity between the follower and leader vehicle. The optimal velocity model and the generalized force model assume that the acceleration is determined by the difference between the own velocity and a desired velocity of the driver depending on the relative position between vehicles. The largest difference between the optimal velocity model and the generalized force model is a deceleration term added in the generalized force model. The follow-the leader model and the optimal velocity model can be expressed as port-Hamiltonian system.

There were also two macroscopic traffic models described: the LW-model and Payne's model. Macroscopic models assume a sufficiently large number of vehicles on a road such that each stream of vehicles can be treated as flowing in a tube or stream. The macroscopic models discussed in this thesis consist of a continuity equation based on the conservation of vehicles and an equation describing the velocity. The difference between this models is the equation for the velocity. In the LW-model it is assumed that the velocity is equal to a certain equilibrium value depending on the vehicle's density. Payne's model uses a partial differential equation describing the dynamics of the velocity.

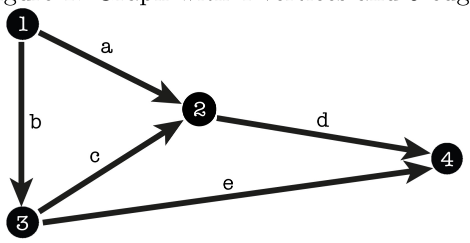
In the last section the impact of adding a PI-controller to the optimal velocity model described before was discussed. Adding a PI-controller to this system ensures that the velocity of vehicle i always becomes equal to the optimal velocity. Even if a disturbance such as wind or rainfall influence the system this is the case.

6 Appendix A: The incidence matrix

An incidence matrix is a matrix that shows the relationship between two classes of objects. In graph theory it shows the relationship between the edges (E) and the vertices (V) of a graph (G). The incidence matrix is an $m \times n$ matrix with elements a_{ij}. Here m is the number of vertices and n is the number of edges. Element a_{ij} is 0 if vertex i and edge j are not connected. Element a_{ij} is -1 if vertex i is a starting point of edge j and element a_{ij} is equal to 1 if vertex i is the endpoint of edge j.

To explain this a little bit more the following example is given. We look to the graph given in figure (1).

Figure 1: Graph with 4 vertices and 5 edges.



The graph given in figure (1) consist of 4 vertices and 5 edges. So the incidence matrix is a 4×5 matrix. Now we have to specify the elements of this matrix. First look to edge a. This edge starts in vertex 1 and ends in vertex 2. So element a_{11} is equal to -1 and a_{12} is equal to 1. Then we look to edge b. This edge starts in vertex 1 and ends in vertex 3. This means that element a_{21} is equal to -1 and element a_{23} is 1. This can be continued until we end up with the following matrix:

$$B = \begin{pmatrix} -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}. \tag{37}$$

References

[1] Rainer Ansorge. What does the entropy condition mean in traffic flow theory? *Transportation Research part B: Methodological*, 24(2):133–143, 1990.

[2] Karl Johan Astrom and Richard M. Murray. *Feedback systems: an introduction for Scientists and Engineers*, volume 396. Princeton University Press, 41 Willem Street, February 2009.

[3] Masako Bando, Katsuya Hasebe, Akihiro Nakayama, Akihiro Shibata, and Yuki Sugiyama. Dynamical model of traffic congestion and numerical simulation. *Physical Review E*, 51(1):1035–1042, 1995.

[4] Mathias Bürger, Daniel Zelazo, and Frank Allgöwer. Duality and network theory in passivity-based cooperative control. *arXiv preprint arXiv:1301.3676*, (3), 2013.

[5] Denos C Gazis, Robert Herman, and Richard W Rothery. Nonlinear follow-the-leader models of traffic flow. *Operations Research*, 9(4):545–567, 1961.

[6] Dirk Helbing, Ansgar Hennecke, Vladimir Shvetsov, and Martin Treiber. Micro-and macro-simulation of freeway traffic. *Mathematical and computer modelling*, 35(5):517–547, 2002.

[7] Dirk Helbing and Benno Tilch. Generalized force model of traffic dynamics. *Physical Review E*, 58(2):133, 1998.

[8] Serge P Hoogendoorn and P HL Bovy. State-of-the-art of vehicular traffic flow modelling. *Proceedings of the Institution of Mechanical Engineers, Part I: Journal of Systems and Control Engineering*, 215(4):283–303, 2001.

[9] S. Miller. Traffic modeling, December 2011.

[10] VI Shvetsov. Mathematical modeling of traffic flows. *Automation and remote control*, 64(6):1651–1689, 2003.

[11] A.J. Van der Schaft and J. Wei. A hamiltonian perspective on the control of dynamical distribution networks. 2012.

[12] R.W.C.P. Verstappen. Traffic flow models (lecture notes wiskundig modelleren). 2013.