

# Maximum-Likelihood estimation of the Lee-Carter model

Richard de Groot

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## **Abstract**

Consider the populations of four European countries, namely France, Italy, Netherlands and Spain in the period 1950-2006. We will estimate for each country the parameters of the Lee-Carter model for human mortality using the maximum-likelihood estimation method. Furthermore, we test whether mortality developments differ between the four countries. Therefore, we determine the log-likelihoods of the unrestricted and restricted Lee-Carter models and test statistical hypotheses using the likelihood-ratio test.

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# 1 Introduction

Human mortality is globally declined during the twentieth century. As a consequence of this development, population has grown significantly. The life expectancy of human beings is increased. Consider, for example, the life expectancy at birth of females and males in the Netherlands. Between 1900 and 1999, the female life expectancy at birth rose from 49.8 to 80.4 years and male life expectancy at birth from 47.0 to 75.3. Individuals view mortality improvements as a positive change. Nevertheless, mortality trends affect pricing and reserve allocation for life annuities.

Nowadays many insurance companies and banks operate in international markets. These markets are divided by geographical regions, like a continent or a country. The human population within a region is influenced by human mortality. Insurers have to make decisions about their insurance products and regions in which they would like to offer these products. So a good insight into human mortality is important for making the right decisions about region and product choice. Therefore, insurers need information about human mortality. Detailed mortality and population data are available in mortality databases.

The mortality rate is often expressed as the number of deaths in a population, scaled to the size of the population, per time unit. For an insurance company or bank, it could be interesting to compare the mean mortality rates between different countries. Consider, for example, the female and male mean mortality rates of France, Italy, Netherlands and Spain in the period 1950-2006. In all four countries the female mean mortality rates are lower than the mean mortality rates of the males. Females on average live longer than males. The female mean mortality rate is the highest in Italy. France shows the highest mean mortality rate for males. For both females and males the lowest mean mortality rate is in Spain. So the mean mortality rate for females and males differ between the four countries. The mean mortality rates are given in table 3 of Appendix A.

We will consider the populations of France, Italy, Netherlands and Spain between 1950 and 2006. For this period we estimate for each country the parameters of the Lee-Carter model for human mortality using the maximum-likelihood estimation method. Furthermore, we test whether mortality developments differ between the four countries. The most important question, which we will investigate is:

- Do significant differences in mortality developments exist between France, Italy, Netherlands and Spain.

Therefore, we determine for these countries the log-likelihoods of the unrestricted and restricted Lee-Carter models and test some statistical hypotheses using the likelihood-ratio test.

In section 2, we model the number of deaths using the Poisson distribution, we describe the Lee-Carter model and estimate the parameters of this model for each country. We also describe the likelihood-ratio test. In section 3 we consider the unrestricted Lee-Carter model. This model is determined by the individual

Lee-Carter models of each country. In section 4 we restrict the unrestricted Lee-Carter model with respect to the model parameters and test some statistical hypotheses based on these restrictions.

## 2 Lee-Carter model

In 1992 Ronald D. Lee and Lawrence R. Carter proposed a simple statistical model of human mortality, specifying a log-bilinear form for the force of mortality  $\mu_{x,t}$ ,

$$\ln \mu_{x,t} = \alpha_x + \beta_x \kappa_t + \epsilon_{x,t}, \quad (1)$$

where

- $\mu_{x,t}$ : observed force of mortality at age  $x$  during year  $t$ ;
- $\alpha_x$ : age-specific constant;
- $\beta_x$ : age-specific patterns of mortality change;
- $\kappa_t$ : time trend;
- $\epsilon_{x,t}$ : error term.

The parameters of the model are constrained to

$$\sum_t \kappa_t = 0 \quad \text{and} \quad \sum_x \beta_x = 1. \quad (2)$$

Sum  $\kappa_t$  to 0 implies that  $\alpha_x$  equals the average of  $\ln \mu_{x,t}$  over time  $t$ . The actual forces of mortality change according to an overall mortality index  $\kappa_t$  modulated by an age response  $\beta_x$ . The shape of the  $\beta_x$  profile tells which rates decline rapidly and which slowly over time in response of change in  $\kappa_t$ . The error term  $\epsilon_{x,t}$  reflects particular age-specific historical influence not captured by the model.

Furthermore, we assume, given any integer age  $x$  and calendar year  $t$ , that

$$\mu_{x+\delta, t+\tau} = \mu_{x,t} \quad \text{for } 0 \leq \delta, \tau < 1. \quad (3)$$

The force of mortality is constant within age and within calendar year.

### 2.1 Poisson modeling for the number of deaths

Each time period an individual dies on a certain age. Let  $t$  be the time and  $x$  be the age of a death. Then  $D_{x,t}$  is the number of deaths at age  $x$  and time  $t$ . We assume that  $D_{x,t}$  is a counting random variable. Because we consider large populations with small probabilities of death, we have

$$D_{x,t} \sim \text{Poisson}(\lambda_{x,t}) \quad \text{with} \quad \lambda_{x,t} = E_{x,t} \mu_{x,t} \quad \text{and} \quad \mu_{x,t} = e^{\alpha_x + \beta_x \kappa_t}, \quad (4)$$

where  $E_{x,t}$  is the exposure-to-risk at age  $x$  and time  $t$ , the number of person years from which  $D_{x,t}$  occurred.

### 2.2 Maximum-likelihood estimation of the parameters

As mentioned in the previous section we estimate the parameters  $\alpha_x$ ,  $\beta_x$  and  $\kappa_t$  of the Lee-Carter model for France, Italy, Netherlands and Spain. We will do this using the maximum-likelihood estimation method. Therefore, we determine these parameters by maximizing the log-likelihood based on model (4). First, we consider model (4) and derive the log-likelihood. Then we define the log-likelihoods for each country. After that, we describe an updating scheme to

estimate the model parameters.

To estimate the Lee-Carter model parameters and to calculate the values of the unrestricted and restricted log-likelihoods, we need data. Therefore, we consider the number of deaths and the exposure-to-risk between 1950 and 2006, divided by gender and subgrouped in five-year age groups till age 105. The data we use is supplied by the Human Mortality Database and consists of  $5 \times 1$  matrices, where the first number refers to the age interval and the second number refers to the time interval. The domain, the five-year age groups and calendar years, is identical for all countries.

Given model (4), let  $d_{x,t}$  be the corresponding number of deaths actually observed, then the likelihood function is  $L_{x,t}(\theta; d_{x,t}) = \frac{\lambda_{x,t}^{d_{x,t}} e^{-\lambda_{x,t}}}{d_{x,t}!}$ . So the log-likelihood function is

$$\begin{aligned}\ln L_{x,t}(\theta; d_{x,t}) &= \ln\left(\frac{\lambda_{x,t}^{d_{x,t}} e^{-\lambda_{x,t}}}{d_{x,t}!}\right) \\ &= \ln(\lambda_{x,t}^{d_{x,t}} e^{-\lambda_{x,t}}) - \ln(d_{x,t}!) \\ &= d_{x,t} \ln(\lambda_{x,t}) + \ln(e^{-\lambda_{x,t}}) - \ln(d_{x,t}!) \\ &= d_{x,t} \ln(\lambda_{x,t}) - \lambda_{x,t} - \ln(d_{x,t}!),\end{aligned}$$

because all random variables  $D_{x,t}$  are assumed to be independent, we can write the joint log-likelihood as

$$\begin{aligned}l(\theta) &= \sum_{x,t} d_{x,t} \ln(\lambda_{x,t}) - \lambda_{x,t} - \ln(d_{x,t}!) \\ &= \sum_{x,t} d_{x,t} \ln(E_{x,t} \mu_{x,t}) - E_{x,t} \mu_{x,t} - \ln(d_{x,t}!) \\ &= \sum_{x,t} d_{x,t} (\ln(E_{x,t}) + \ln(\mu_{x,t})) - E_{x,t} \mu_{x,t} - \ln(d_{x,t}!) \\ &= \sum_{x,t} d_{x,t} \ln(E_{x,t}) + d_{x,t} \ln(\mu_{x,t}) - E_{x,t} \mu_{x,t} - \ln(d_{x,t}!) \\ &= \sum_{x,t} d_{x,t} \ln(\mu_{x,t}) - E_{x,t} \mu_{x,t} - \ln(d_{x,t}!) + d_{x,t} \ln(E_{x,t}) \\ &= \sum_{x,t} \{d_{x,t} \ln(\mu_{x,t}) - E_{x,t} \mu_{x,t}\} + C \\ &= \sum_{x,t} \{d_{x,t} \ln(e^{\alpha_x + \beta_x \kappa_t}) - E_{x,t} e^{\alpha_x + \beta_x \kappa_t}\} + C \\ &= \sum_{x,t} \{d_{x,t} (\alpha_x + \beta_x \kappa_t) - E_{x,t} e^{\alpha_x + \beta_x \kappa_t}\} + C,\end{aligned}$$

so the log-likelihood, is given by

$$l(\theta) = \sum_{x,t} \{d_{x,t} (\alpha_x + \beta_x \kappa_t) - E_{x,t} e^{\alpha_x + \beta_x \kappa_t}\} + C, \quad (5)$$

with  $\theta = (\alpha_x \ \beta_x \ \kappa_t)$  and  $C$  a constant.

Given log-likelihood (5), let  $l_{Fr}(\theta_{Fr})$ ,  $l_{It}(\theta_{It})$ ,  $l_{Nl}(\theta_{Nl})$  and  $l_{Sp}(\theta_{Sp})$  be the log-likelihoods of France, Italy, Netherlands and Spain respectively, defined by:

$$l_{Fr}(\theta_{Fr}) = \sum_{x,t} \{d_{x,t}^{Fr}(\alpha_x^{Fr} + \beta_x^{Fr} \kappa_t^{Fr}) - E_{x,t}^{Fr} e^{(\alpha_x^{Fr} + \beta_x^{Fr} \kappa_t^{Fr})}\} + C \quad (6)$$

$$l_{It}(\theta_{It}) = \sum_{x,t} \{d_{x,t}^{It}(\alpha_x^{It} + \beta_x^{It} \kappa_t^{It}) - E_{x,t}^{It} e^{(\alpha_x^{It} + \beta_x^{It} \kappa_t^{It})}\} + C \quad (7)$$

$$l_{Nl}(\theta_{Nl}) = \sum_{x,t} \{d_{x,t}^{Nl}(\alpha_x^{Nl} + \beta_x^{Nl} \kappa_t^{Nl}) - E_{x,t}^{Nl} e^{(\alpha_x^{Nl} + \beta_x^{Nl} \kappa_t^{Nl})}\} + C \quad (8)$$

$$l_{Sp}(\theta_{Sp}) = \sum_{x,t} \{d_{x,t}^{Sp}(\alpha_x^{Sp} + \beta_x^{Sp} \kappa_t^{Sp}) - E_{x,t}^{Sp} e^{(\alpha_x^{Sp} + \beta_x^{Sp} \kappa_t^{Sp})}\} + C. \quad (9)$$

Now we have defined the log-likelihoods for each country, we estimate the parameters by maximizing equations (6), (7), (8) and (9) with respect to  $\alpha_x$ ,  $\beta_x$  and  $\kappa_t$ . The maximum likelihood estimates  $\hat{\alpha}_x$ ,  $\hat{\beta}_x$  and  $\hat{\kappa}_t$  are obtained by setting the partial derivatives of  $l(\theta)$  to zero. We use an iterative method to estimate  $\alpha_x$ ,  $\beta_x$  and  $\kappa_t$ , because it is not possible to solve this optimization problem by a closed formula. Therefore, we have to determine an updating scheme and start the update procedure with values  $\hat{\alpha}_x^0$ ,  $\hat{\beta}_x^0$  and  $\hat{\kappa}_t^0$ .

In iteration step  $v+1$ , a single set of parameters is updated fixing the other parameters at their current estimates using the following updating scheme

$$\hat{\theta}^{(v+1)} = \hat{\theta}^{(v)} - H^{-1}(\hat{\theta}^{(v)}) \frac{\partial l}{\partial \theta}(\hat{\theta}^{(v)}),$$

where

$$H = \frac{\partial^2 l}{\partial \theta \partial \theta'}, \text{ the Hessian matrix.}$$

We use the statistical program R to do all the computations in this thesis. The parameter estimates for females and males of the Lee-Carter model for France, Italy, Netherlands and Spain are given in Appendix B.

### 2.3 Likelihood-ratio test

As mentioned before we will test some statistical hypotheses using the likelihood-ratio test. Let  $l(\hat{\theta})$  denote the log-likelihood of the unrestricted model and  $l(\tilde{\theta})$  the log-likelihood of the restricted model. Because we have many observations, we assume that the likelihood-ratio test statistic is asymptotically  $\chi^2$  distributed with degrees of freedom equal to the number of independent restrictions. We can write this as

$$-2(l(\hat{\theta}) - l(\tilde{\theta})) \sim \chi^2(p), \quad (10)$$

where  $p$  is the number of independent restrictions.

To determine whether to reject the null hypothesis  $H_0$ , we compare the value of the likelihood-ratio test statistic with a  $\chi^2(p)$ -distribution, the critical value at a specified level of significance. Let  $\Lambda$  denote the likelihood-ratio test statistic

and  $c$  the critical value of a  $\chi^2(p)$ -distribution. Then

if  $\Lambda < c$ , do not reject  $H_0$ ;  
 if  $\Lambda > c$ , reject  $H_0$ .

We test the following null hypotheses:

$$H_0^\alpha : \alpha_x^{Fr} = \alpha_x^{It} = \alpha_x^{Nl} = \alpha_x^{Sp} = \alpha_x$$

$$H_0^\beta : \beta_x^{Fr} = \beta_x^{It} = \beta_x^{Nl} = \beta_x^{Sp} = \beta_x$$

$$H_0^\kappa : \kappa_t^{Fr} = \kappa_t^{It} = \kappa_t^{Nl} = \kappa_t^{Sp} = \kappa_t$$

$$\begin{aligned} H_0^{\alpha\beta\kappa} : \quad & \alpha_x^{Fr} = \alpha_x^{It} = \alpha_x^{Nl} = \alpha_x^{Sp} = \alpha_x \quad \text{and} \\ & \beta_x^{Fr} = \beta_x^{It} = \beta_x^{Nl} = \beta_x^{Sp} = \beta_x \quad \text{and} \\ & \kappa_t^{Fr} = \kappa_t^{It} = \kappa_t^{Nl} = \kappa_t^{Sp} = \kappa_t \end{aligned}$$

against the alternatives that  $H_0^\alpha$ ,  $H_0^\beta$ ,  $H_0^\kappa$  or  $H_0^{\alpha\beta\kappa}$  are not true. We do this at 95% and 99% significance levels. Therefore, we determine in the next sections the log-likelihoods of the unrestricted model and the restricted models.

### 3 Unrestricted model

To calculate the values of the unrestricted and restricted log-likelihoods, we first determine the unrestricted and restricted Lee-Carter models.

Consider  $l_{Fr}(\theta_{Fr})$ ,  $l_{It}(\theta_{It})$ ,  $l_{Nl}(\theta_{Nl})$  and  $l_{Sp}(\theta_{Sp})$  as defined in the previous section. The value of the log-likelihood of the unrestricted model is the sum of the log-likelihoods of France, Italy, Netherlands and Spain. So the log-likelihood of the unrestricted model is given by

$$\begin{aligned} l(\hat{\theta}) &= l_{Fr}(\hat{\theta}_{Fr}) + l_{It}(\hat{\theta}_{It}) + l_{Nl}(\hat{\theta}_{Nl}) + l_{Sp}(\hat{\theta}_{Sp}) \\ &= \sum_{x,t} \{ d_{x,t}^{Fr} (\hat{\alpha}_x^{Fr} + \hat{\beta}_x^{Fr} \hat{\kappa}_t^{Fr}) - E_{x,t}^{Fr} e^{(\hat{\alpha}_x^{Fr} + \hat{\beta}_x^{Fr} \hat{\kappa}_t^{Fr})} + \\ &\quad d_{x,t}^{It} (\hat{\alpha}_x^{It} + \hat{\beta}_x^{It} \hat{\kappa}_t^{It}) - E_{x,t}^{It} e^{(\hat{\alpha}_x^{It} + \hat{\beta}_x^{It} \hat{\kappa}_t^{It})} + \\ &\quad d_{x,t}^{Nl} (\hat{\alpha}_x^{Nl} + \hat{\beta}_x^{Nl} \hat{\kappa}_t^{Nl}) - E_{x,t}^{Nl} e^{(\hat{\alpha}_x^{Nl} + \hat{\beta}_x^{Nl} \hat{\kappa}_t^{Nl})} + \\ &\quad d_{x,t}^{Sp} (\hat{\alpha}_x^{Sp} + \hat{\beta}_x^{Sp} \hat{\kappa}_t^{Sp}) - E_{x,t}^{Sp} e^{(\hat{\alpha}_x^{Sp} + \hat{\beta}_x^{Sp} \hat{\kappa}_t^{Sp})} \}. \end{aligned} \quad (11)$$

The calculated values of the unrestricted log-likelihoods for females and males are given in table 1.

### 4 Restricted models

Given unrestricted model (11). We determine the restricted models by restricting model (11) with respect to model parameters  $\alpha_x$ ,  $\beta_x$  and  $\kappa_t$ .

### 4.1 Model restricted to $\alpha_x$

We assume that parameters  $\alpha_x$  of France, Italy, Netherlands and Spain are equal. We can write this as

$$\alpha_x^{Fr} = \alpha_x^{It} = \alpha_x^{Nl} = \alpha_x^{Sp} = \alpha_x. \quad (12)$$

So when we restrict model (11) with respect to restrictions (12), the restricted model is given by

$$\begin{aligned} l_\alpha(\tilde{\theta}_\alpha) &= l_{Fr+It+Nl+Sp}(\tilde{\alpha}_x \tilde{\beta}_x^{Fr} \tilde{\beta}_x^{It} \tilde{\beta}_x^{Nl} \tilde{\beta}_x^{Sp} \tilde{\kappa}_t^{Fr} \tilde{\kappa}_t^{It} \tilde{\kappa}_t^{Nl} \tilde{\kappa}_t^{Sp}) \\ &= \sum_{x,t} \{ d_{x,t}^{Fr} (\tilde{\alpha}_x + \tilde{\beta}_x^{Fr} \tilde{\kappa}_t^{Fr}) - E_{x,t}^{Fr} e^{(\tilde{\alpha}_x + \tilde{\beta}_x^{Fr} \tilde{\kappa}_t^{Fr})} + \\ &\quad d_{x,t}^{It} (\tilde{\alpha}_x + \tilde{\beta}_x^{It} \tilde{\kappa}_t^{It}) - E_{x,t}^{It} e^{(\tilde{\alpha}_x + \tilde{\beta}_x^{It} \tilde{\kappa}_t^{It})} + \\ &\quad d_{x,t}^{Nl} (\tilde{\alpha}_x + \tilde{\beta}_x^{Nl} \tilde{\kappa}_t^{Nl}) - E_{x,t}^{Nl} e^{(\tilde{\alpha}_x + \tilde{\beta}_x^{Nl} \tilde{\kappa}_t^{Nl})} + \\ &\quad d_{x,t}^{Sp} (\tilde{\alpha}_x + \tilde{\beta}_x^{Sp} \tilde{\kappa}_t^{Sp}) - E_{x,t}^{Sp} e^{(\tilde{\alpha}_x + \tilde{\beta}_x^{Sp} \tilde{\kappa}_t^{Sp})} \}. \end{aligned}$$

### 4.2 Model restricted to $\beta_x$

To restrict model (11) with respect to  $\beta_x$ , we first write

$$\beta_x^{Fr} = \beta_x^{It} = \beta_x^{Nl} = \beta_x^{Sp} = \beta_x. \quad (13)$$

Then the restricted model is

$$\begin{aligned} l_\beta(\tilde{\theta}_\beta) &= l_{Fr+It+Nl+Sp}(\tilde{\alpha}_x^{Fr} \tilde{\alpha}_x^{It} \tilde{\alpha}_x^{Nl} \tilde{\alpha}_x^{Sp} \tilde{\beta}_x \tilde{\kappa}_t^{Fr} \tilde{\kappa}_t^{It} \tilde{\kappa}_t^{Nl} \tilde{\kappa}_t^{Sp}) \\ &= \sum_{x,t} \{ d_{x,t}^{Fr} (\tilde{\alpha}_x^{Fr} + \tilde{\beta}_x \tilde{\kappa}_t^{Fr}) - E_{x,t}^{Fr} e^{(\tilde{\alpha}_x^{Fr} + \tilde{\beta}_x \tilde{\kappa}_t^{Fr})} + \\ &\quad d_{x,t}^{It} (\tilde{\alpha}_x^{It} + \tilde{\beta}_x \tilde{\kappa}_t^{It}) - E_{x,t}^{It} e^{(\tilde{\alpha}_x^{It} + \tilde{\beta}_x \tilde{\kappa}_t^{It})} + \\ &\quad d_{x,t}^{Nl} (\tilde{\alpha}_x^{Nl} + \tilde{\beta}_x \tilde{\kappa}_t^{Nl}) - E_{x,t}^{Nl} e^{(\tilde{\alpha}_x^{Nl} + \tilde{\beta}_x \tilde{\kappa}_t^{Nl})} + \\ &\quad d_{x,t}^{Sp} (\tilde{\alpha}_x^{Sp} + \tilde{\beta}_x \tilde{\kappa}_t^{Sp}) - E_{x,t}^{Sp} e^{(\tilde{\alpha}_x^{Sp} + \tilde{\beta}_x \tilde{\kappa}_t^{Sp})} \}. \end{aligned}$$

### 4.3 Model restricted to $\kappa_t$

Now we restrict model (11) with respect to model parameter  $\kappa_t$ . The restrictions are

$$\kappa_t^{Fr} = \kappa_t^{It} = \kappa_t^{Nl} = \kappa_t^{Sp} = \kappa_t. \quad (14)$$

By restrictions (14) we can write the restricted model as

$$\begin{aligned} l_\kappa(\tilde{\theta}_\kappa) &= l_{Fr+It+Nl+Sp}(\tilde{\alpha}_x^{Fr} \tilde{\alpha}_x^{It} \tilde{\alpha}_x^{Nl} \tilde{\alpha}_x^{Sp} \tilde{\beta}_x^{Fr} \tilde{\beta}_x^{It} \tilde{\beta}_x^{Nl} \tilde{\beta}_x^{Sp} \tilde{\kappa}_t) \\ &= \sum_{x,t} \{ d_{x,t}^{Fr} (\tilde{\alpha}_x^{Fr} + \tilde{\beta}_x^{Fr} \tilde{\kappa}_t) - E_{x,t}^{Fr} e^{(\tilde{\alpha}_x^{Fr} + \tilde{\beta}_x^{Fr} \tilde{\kappa}_t)} + \\ &\quad d_{x,t}^{It} (\tilde{\alpha}_x^{It} + \tilde{\beta}_x^{It} \tilde{\kappa}_t) - E_{x,t}^{It} e^{(\tilde{\alpha}_x^{It} + \tilde{\beta}_x^{It} \tilde{\kappa}_t)} + \\ &\quad d_{x,t}^{Nl} (\tilde{\alpha}_x^{Nl} + \tilde{\beta}_x^{Nl} \tilde{\kappa}_t) - E_{x,t}^{Nl} e^{(\tilde{\alpha}_x^{Nl} + \tilde{\beta}_x^{Nl} \tilde{\kappa}_t)} + \\ &\quad d_{x,t}^{Sp} (\tilde{\alpha}_x^{Sp} + \tilde{\beta}_x^{Sp} \tilde{\kappa}_t) - E_{x,t}^{Sp} e^{(\tilde{\alpha}_x^{Sp} + \tilde{\beta}_x^{Sp} \tilde{\kappa}_t)} \}. \end{aligned}$$

#### 4.4 Model restricted to $\alpha_x$ , $\beta_x$ and $\kappa_t$

To determine the last restricted model from model (11) we use restrictions

$$\begin{aligned}\alpha_x^{Fr} &= \alpha_x^{It} = \alpha_x^{Nl} = \alpha_x^{Sp} = \alpha_x \quad \text{and} \\ \beta_x^{Fr} &= \beta_x^{It} = \beta_x^{Nl} = \beta_x^{Sp} = \beta_x \quad \text{and} \\ \kappa_t^{Fr} &= \kappa_t^{It} = \kappa_t^{Nl} = \kappa_t^{Sp} = \kappa_t.\end{aligned}\tag{15}$$

Then the restricted model is given by

$$\begin{aligned}l_{\alpha\beta\kappa}(\tilde{\theta}_{\alpha\beta\kappa}) &= l_{Fr+It+Nl+Sp}(\tilde{\alpha}_x \ \tilde{\beta}_x \ \tilde{\kappa}_t) \\ &= \sum_{x,t} \{d_{x,t}(\tilde{\alpha}_x + \tilde{\beta}_x \tilde{\kappa}_t) - E_{x,t} e^{(\tilde{\alpha}_x + \tilde{\beta}_x \tilde{\kappa}_t)}\}.\end{aligned}$$

The calculated values of the restricted log-likelihoods for females and males are given in table 1.

Model	Log-likelihood		LR test statistic	
	Female	Male	Female	Male
Unrestricted	10956508774	11583661244		
Restricted $\alpha$	12692205497	12725667693	3471393447	2284012898
Restricted $\beta$	11029472642	12018451695	145927737	869580902
Restricted $\kappa$	11425174472	12374918216	937331395	1582513945
Restricted $\alpha\beta\kappa$	11020042388	11642044801	127067229	116767115

Table 1: Values log-likelihoods and likelihood-ratio test statistics

Now we have determined the unrestricted model and restricted models and calculated the values of the corresponding log-likelihoods, we test hypotheses  $H_0^\alpha$ ,  $H_0^\beta$ ,  $H_0^\kappa$  and  $H_0^{\alpha\beta\kappa}$ , defined in section 2. Therefore, we determine the likelihood-ratio test statistics, described in section 2. We also determine the number of independent restrictions for each restricted model. We consider five-year age groups till age 105 in the period 1950-2006. So the parameter vectors  $\alpha_x$  and  $\beta_x$  of France, Italy, Netherlands and Spain contain each 22 elements. Parameter vector  $\kappa_t$  contains 57 elements. The number of independent restrictions in equations (13) and (14) are  $3 \times 22 = 66$ . Equation (15) contains  $3 \times 57 = 171$  independent restrictions. Equations (16)  $2 \times 66 + 171 = 303$ . The values of the likelihood-ratio test statistics are given in table 1. The number of independent restrictions and critical values are given in table 2.

Table 1 shows large values of the likelihood-ratio test statistics for females and males. The estimated model parameters determine the values of the unrestricted and restricted log-likelihoods. Because the parameter estimates do not differ very much between the four countries, we expect smaller values of the likelihood-ratio test statistics, which is not the case. This new problem requires further investigation. We leave it to the interesting investigator.

Model	$p$	$\chi^2(p)$	
		95%	99%
Restricted $\alpha$	66	85.96	95.63
Restricted $\beta$	66	85.96	95.63
Restricted $\kappa$	171	202.51	216.94
Restricted $\alpha\beta\kappa$	303	344.60	363.19

Table 2: Critical values

Given the results from table 1 and table 2 we can make some conclusions about the existence of significant differences in mortality developments between France, Italy, Netherlands and Spain.

## 5 Conclusion

For estimating the parameters and calculating the values of the log-likelihoods of the unrestricted and restricted Lee-Carter models, we considered the populations of the European countries France, Italy, Netherlands and Spain between 1950 and 2006. We considered females and males subgrouped in five-year age groups till age 105. Because of the assumption that  $D_{x,t}$  is Poisson distributed, which is quite realistic, we could use the maximum-likelihood estimation method to do the calculations. From the values of the unrestricted and restricted log-likelihoods we calculated the likelihood-ratio test statistics. We used these test statistics to test hypotheses  $H_0^\alpha$ ,  $H_0^\beta$ ,  $H_0^\kappa$  and  $H_0^{\alpha\beta\kappa}$  against the alternatives on 95% and 99% significance levels. From the results hypotheses  $H_0^\alpha$ ,  $H_0^\beta$ ,  $H_0^\kappa$  and  $H_0^{\alpha\beta\kappa}$  for females and males will be rejected. So we conclude that significant differences in mortality developments between France, Italy, Netherlands and Spain in the period 1950-2006 do exist.

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## 6 Appendix A: Mean mortality rates

Country	Female	Male
France	0.06789609	0.08701386
Italy	0.06960209	0.08407999
Netherlands	0.06815713	0.08283437
Spain	0.06335999	0.07497131

Table 3: Mean mortality rates  $\bar{\mu}_{x,t}$

## 7 Appendix B: Parameter estimates

### 7.1 Estimated parameters females

Age $x$	$\alpha_x$			
	France	Italy	Netherlands	Spain
0	-4.5948	-4.2057	-4.7994	-4.2650
1-4	-7.5634	-7.3760	-7.4551	-7.2160
5-9	-8.4239	-8.2887	-8.3769	-8.1232
10-14	-8.5022	-8.3670	-8.4369	-8.2334
15-19	-7.8012	-7.9954	-8.0520	-7.8379
20-24	-7.5408	-7.7273	-7.9026	-7.5904
25-29	-7.3943	-7.5320	-7.6904	-7.3547
30-34	-7.1040	-7.2505	-7.3369	-7.0970
35-39	-6.7133	-6.8879	-6.9326	-6.7847
40-44	-6.3104	-6.4757	-6.4587	-6.4213
45-49	-5.8924	-6.0220	-5.9740	-6.0282
50-54	-5.5843	-5.5765	-5.5515	-5.6213
55-59	-5.1513	-5.1398	-5.1866	-5.2169
60-64	-4.7489	-4.6618	-4.7288	-4.7314
65-69	-4.2798	-4.1377	-4.2169	-4.2089
70-74	-3.7316	-3.5608	-3.6641	-3.6051
75-79	-3.1302	-2.9580	-3.0816	-2.9962
80-84	-2.5178	-2.3742	-2.5030	-2.4170
85-89	-1.9443	-1.8336	-1.9629	-1.8822
90-94	-1.5197	-1.4098	-1.4870	-1.4962
95-99	-1.1154	-1.0162	-1.0766	-1.1048
100-104	-0.7052	-0.7991	-0.7240	-0.9140

Table 4: Estimated  $\alpha_x$

Age $x$	$\beta_x$			
	France	Italy	Netherlands	Spain
0	0.1019	0.1011	0.0914	0.1018
1-4	0.0930	0.1087	0.1189	0.1021
5-9	0.0641	0.0754	0.1025	0.0790
10-14	0.0509	0.0602	0.0650	0.0633
15-19	0.0385	0.0457	0.0442	0.0473
20-24	0.0427	0.0504	0.0450	0.0554
25-29	0.0475	0.0518	0.0455	0.0578
30-34	0.0446	0.0488	0.0427	0.0512
35-39	0.0405	0.0450	0.0392	0.0469
40-44	0.0366	0.0398	0.0302	0.0420
45-49	0.0361	0.0363	0.0255	0.0383
50-54	0.0379	0.0346	0.0259	0.0395
55-59	0.0404	0.0344	0.0286	0.0398
60-64	0.0444	0.0363	0.0344	0.0434
65-69	0.0482	0.0386	0.0428	0.0433
70-74	0.0502	0.0399	0.0474	0.0437
75-79	0.0484	0.0377	0.0477	0.0374
80-84	0.0416	0.0327	0.0410	0.0278
85-89	0.0326	0.0256	0.0316	0.0192
90-94	0.0235	0.0183	0.0205	0.0073
95-99	0.0164	0.0127	0.0133	0.0065
100-104	0.0199	0.0258	0.0165	0.0071

Table 5: Estimated  $\beta_x$

Year <i>t</i>	$\kappa_t$			
	France	Italy	Netherlands	Spain
1950	14.6166	16.3484	10.1385	19.9520
1951	15.0726	16.4545	10.5954	20.5078
1952	12.5264	14.9939	9.2662	16.3524
1953	13.0773	14.1746	10.9166	14.9904
1954	10.7904	12.2002	7.6192	13.3570
1955	9.7755	11.4604	7.0980	13.4781
1956	10.1661	12.4594	7.2977	13.1288
1957	9.4897	11.5418	7.0175	13.7236
1958	8.7214	9.9162	6.0731	10.0423
1959	7.6889	9.2806	5.6702	9.9873
1960	6.8690	9.1109	4.7037	8.9390
1961	5.6050	8.0835	3.6918	8.0307
1962	6.5068	8.9713	4.9722	8.5199
1963	6.6227	9.0298	4.3096	7.8939
1964	4.9582	6.6603	3.4978	6.1590
1965	5.1465	7.1718	3.5113	5.4674
1966	4.3503	5.4067	4.9638	4.9119
1967	4.5194	5.7115	3.3713	4.6027
1968	4.4547	5.7338	3.4933	4.1789
1969	4.8571	5.1762	3.8939	4.9332
1970	3.5831	4.4307	3.6665	3.5522
1971	3.7658	3.7970	2.7044	3.9609
1972	3.3390	3.3857	2.6977	2.4871
1973	3.1093	3.3495	2.6516	2.6701
1974	2.1576	1.7034	1.0487	1.7520
1975	1.9099	1.7088	0.8487	1.0800
1976	1.3548	1.1545	0.5625	0.0956
1977	0.4843	0.2930	-0.3884	-0.5755
1978	0.0127	-0.7866	0.0784	-0.3928
1979	-0.5076	-1.1407	-1.2101	-1.6169
1980	-0.3956	-0.3294	-1.9077	-2.5786

Table 6 *continued*

Year <i>t</i>	$\kappa_t$			
	France	Italy	Netherlands	Spain
1981	-0.6686	-2.2255	-2.3138	-3.3672
1982	-1.5362	-2.9590	-2.4034	-4.5832
1983	-1.2660	-2.5609	-2.6659	-4.1499
1984	-2.5321	-4.2053	-3.6368	-5.2358
1985	-2.8176	-4.2735	-3.9299	-5.2587
1986	-3.3515	-5.0174	-3.2172	-6.3379
1987	-4.5065	-5.7222	-4.0123	-5.6424
1988	-4.9129	-5.8625	-4.9999	-5.6107
1989	-4.9534	-6.8470	-3.6548	-5.7509
1990	-5.8559	-8.2933	-4.0558	-5.6669
1991	-5.9480	-6.4771	-4.0847	-6.1445
1992	-6.5856	-6.8741	-4.6280	-7.0814
1993	-6.5549	-6.7805	-4.0256	-7.5405
1994	-7.5020	-7.4458	-5.0155	-8.1347
1995	-7.4561	-7.9623	-4.9014	-8.2546
1996	-8.2610	-8.2445	-5.2110	-8.4807
1997	-8.9595	-9.1963	-5.8071	-9.8292
1998	-9.2743	-9.8874	-6.7884	-10.3770
1999	-9.4964	-10.9704	-5.5109	-10.5748
2000	-10.2249	-11.3127	-5.9013	-11.4635
2001	-10.3871	-12.2286	-6.6318	-12.2840
2002	-10.7736	-12.6222	-6.4401	-12.3920
2003	-11.1250	-13.4002	-6.7820	-11.9923
2004	-13.0058	-15.2610	-8.7204	-13.7813
2005	-13.1579	-15.3746	-8.6564	-14.1063
2006	-13.5150	-15.4474	-8.8591	-15.5498

Table 6: Estimated  $\kappa_t$

## 7.2 Estimated parameters males

Age $x$	$\alpha_x$			
	France	Italy	Netherlands	Spain
0	-4.2887	-4.0406	-4.5014	-4.0414
1-4	-7.2502	-7.2111	-7.1919	-6.9746
5-9	-8.0266	-7.9408	-8.0057	-7.7350
10-14	-7.9903	-7.8701	-8.0728	-7.7964
15-19	-6.8939	-6.9625	-7.3012	-6.9999
20-24	-6.4995	-6.7192	-7.0520	-6.6367
25-29	-6.4540	-6.6778	-7.0040	-6.4973
30-34	-6.2831	-6.5350	-6.8824	-6.3234
35-39	-5.9576	-6.2778	-6.6812	-6.0746
40-44	-5.5278	-5.8716	-6.0889	-5.7229
45-49	-5.0903	-5.4079	-5.5717	-5.3507
50-54	-4.6916	-4.9509	-5.0493	-4.9133
55-59	-4.3733	-4.4669	-4.5324	-4.4724
60-64	-3.9126	-4.0109	-4.0067	-4.0209
65-69	-3.5166	-3.5639	-3.4880	-3.5687
70-74	-3.0923	-3.1071	-3.0447	-3.0939
75-79	-2.6397	-2.6334	-2.6066	-2.6120
80-84	-2.1633	-2.1539	-2.2577	-2.1534
85-89	-1.7039	-1.7041	-1.8804	-1.7152
90-94	-1.2852	-1.2948	-1.5149	-1.3574
95-99	-0.9659	-0.9446	-1.2531	-1.0299
100-104	-0.5145	-0.6290	-1.5171	-0.9389

Table 7: Estimated  $\alpha_x$

Age $x$	$\beta_x$			
	France	Italy	Netherlands	Spain
0	0.1295	0.1248	0.0906	0.1460
1-4	0.1130	0.1391	0.1342	0.1449
5-9	0.0887	0.1013	0.1027	0.1136
10-14	0.0680	0.0799	0.0316	0.0919
15-19	0.0384	0.0432	0.0730	0.0477
20-24	0.0264	0.0331	0.0565	0.0422
25-29	0.0315	0.0335	0.0549	0.0455
30-34	0.0326	0.0369	0.0354	0.0392
35-39	0.0332	0.0425	0.0304	0.0368
40-44	0.0326	0.0463	0.0153	0.0365
45-49	0.0329	0.0478	0.0111	0.0339
50-54	0.0369	0.0460	0.0120	0.0344
55-59	0.0408	0.0421	0.0149	0.0346
60-64	0.0428	0.0380	0.0113	0.0366
65-69	0.0439	0.0345	0.0231	0.0353
70-74	0.0442	0.0217	0.0250	0.0365
75-79	0.0435	0.0201	0.0226	0.0339
80-84	0.0385	0.0188	0.0427	0.0266
85-89	0.0320	0.0153	0.0400	0.0217
90-94	0.0232	0.0104	0.0399	0.0072
95-99	0.0131	0.0061	0.0425	0.0034
100-104	0.0140	0.0186	0.0902	-0.0485

Table 8: Estimated  $\beta_x$

Year <i>t</i>	$\kappa_t$			
	France	Italy	Netherlands	Spain
1950	9.7631	10.0849	6.3385	15.1395
1951	9.7025	11.3450	5.9420	15.7984
1952	8.1638	9.8284	5.1195	10.7672
1953	8.7995	9.3226	6.1225	9.9772
1954	7.2519	7.5579	5.1112	7.8852
1955	6.7131	8.1264	4.3150	8.3932
1956	7.5614	9.0574	4.4136	8.3973
1957	7.1080	8.3451	4.4499	8.6236
1958	4.6941	7.1802	3.6156	5.6265
1959	5.2198	6.1005	4.7597	6.0591
1960	4.8351	6.9088	3.9873	5.3563
1961	4.5132	6.0850	3.8085	4.5546
1962	5.0128	7.3393	4.4895	4.8868
1963	4.9606	7.3905	4.4406	4.8464
1964	5.1435	5.5957	4.1549	3.7848
1965	4.3442	5.8450	4.2995	3.4198
1966	3.3821	4.5965	4.4780	2.8304
1967	3.7653	4.4297	3.1310	2.8237
1968	3.8630	5.0809	3.9311	2.4883
1969	4.3317	4.6647	4.1940	3.2286
1970	3.3874	4.0189	4.2214	2.2627
1971	3.5830	3.5486	3.6759	3.1039
1972	2.9788	3.1162	3.8986	1.2263
1973	3.0747	3.4658	3.2891	1.4956
1974	2.5998	1.9326	2.5508	1.1809
1975	2.3664	2.4054	2.1066	0.9235
1976	1.9605	1.7171	2.1465	0.2074
1977	1.4006	2.0207	1.1062	0.0466
1978	1.0851	0.8057	1.4536	-0.0180
1979	0.7401	0.4019	0.2040	-0.7303
1980	0.6906	0.7155	-0.4951	-1.6905

Table 9 *continued*

Year <i>t</i>	$\kappa_t$			
	France	Italy	Netherlands	Spain
1981	0.2737	-0.4193	-0.3305	-2.0680
1982	-0.3091	-1.3393	-0.6637	-3.1370
1983	-0.0922	-1.1489	-1.4333	-2.6014
1984	-1.0477	-2.5572	-1.3490	-3.0318
1985	-1.2773	-2.7259	-2.0216	-2.7023
1986	-1.6329	-3.5143	-1.6813	-3.5554
1987	-2.6571	-4.0122	-2.8179	-3.4097
1988	-3.1034	-4.0915	-2.8604	-3.1808
1989	-3.2935	-4.6292	-3.0444	-2.8832
1990	-3.6237	-4.2900	-2.6344	-2.4216
1991	-3.7711	-3.8293	-3.4730	-2.6757
1992	-4.2584	-4.5304	-3.4586	-3.6075
1993	-4.3541	-5.0047	-3.1114	-4.0766
1994	-5.3054	-5.7172	-4.6661	-4.6246
1995	-5.8463	-5.4966	-4.1837	-5.0318
1996	-6.1996	-6.5250	-4.5723	-5.0988
1997	-7.1692	-7.3906	-5.5294	-6.5943
1998	-7.9371	-8.1398	-5.4588	-6.8018
1999	-8.1940	-9.3156	-5.5264	-7.1858
2000	-8.6689	-10.0102	-6.2516	-7.8558
2001	-8.8711	-10.2195	-6.8355	-8.8411
2002	-9.6794	-10.5068	-7.0241	-9.1946
2003	-9.9578	-12.0117	-7.2105	-9.0303
2004	-11.8506	-13.6290	-9.1885	-10.5220
2005	-12.1335	-13.7896	-9.8251	-10.9374
2006	-12.0361	-14.1890	-10.1085	-11.8261

Table 9: Estimated  $\kappa_t$

## 8 Appendix C: R script files

### 8.1 Parameters and unrestricted log-likelihood females

```

# France
Fr<-read.csv(file="C:\\france.csv",head=TRUE,sep=";")
attach(Fr)

loglik<-function(theta,x){
  ax<-theta[1:22]
  bx<-theta[23:43]
  bx<-c(bx,1-sum(bx))
  kt<-theta[44:99]
  kt<-c(kt,-sum(kt))

  one<-rep(1,length(kt))
  a<-ax%*%t(one)

  Dxt<-matrix(DxtFFr/100,nrow=22)
  Ext<-matrix(ExtFFr/100,nrow=22)

  ll<-sum(colSums(Dxt%*%t(a+bx%*%t(kt))-Ext%*%t(exp(a+bx%*%t(kt)))))

  return(-ll)
}

ax0<-matrix(c(-4.5336584,-7.5060995,-8.3702416,-8.4522067,-7.7548003,
-7.4979048,-7.3547828,-7.0678867,-6.6804750,-6.2807173,-5.8659169,
-5.4864869,-5.1324817,-4.7329462,-4.2667481,-3.7214242,-3.1228198,
-2.5131285,-1.9422717,-1.4467931,-1.0464949,-0.6440435))

bx0<-matrix(c(-0.02814830,-0.03702005,-0.06588364,-0.07911846,-0.09154635,
-0.08734098,-0.08257363,-0.08546004,-0.08951054,-0.09347024,-0.09394742,
-0.09211971,-0.08961577,-0.08562356,-0.08180338,-0.07984893,-0.08158660,
-0.08841981,-0.09741398,-0.10652574,-0.11358619,-0.10943668))

kt0<-matrix(c(14.66918994,15.13178125,12.57509555,13.12666484,10.83968426,
9.82296771,10.21443750,9.53618215,7.31319973,7.73562022,7.02995135,
5.74891028,6.65626632,6.77779884,5.09137138,5.28505660,4.46326194,
4.64234017,4.57256903,4.98508355,3.68653826,3.87395657,3.43777355,
3.20363887,2.24745640,1.99537659,1.43600326,0.56139274,0.01206396,
-0.51077326,-0.40126041,-0.67668958,-1.54673208,-1.27891394,-2.54728323,
-2.83510050,-3.37122811,-4.52841062,-4.93695848,-4.97951656,-5.88408729,
-5.97821116,-6.61782887,-6.58904224,-7.53804578,-7.49400272,-8.30077471,
-9.00102698,-9.31752943,-9.54133508,-10.27153832,-10.43538550,-10.82356830,
-11.17654772,-13.05885766,-13.21251975,-13.81846454))

theta0<-c(ax0,bx0[1:21],kt0[1:56])
ll.model<-optim(theta0,loglik,hessian=T,x=Fr,control=list(maxit=100000))

opt.ax<-ll.model$par[1:22]
t(t(opt.ax))
opt.bx<-ll.model$par[23:43]
opt.bx<-c(opt.bx,1-sum(opt.bx))

```

```

t(t(opt.bx))
opt.kt<-ll.model$par[44:99]
opt.kt<-c(opt.kt,-sum(opt.kt))
t(t(opt.kt))

LogLikelihoodFr<-ll.model$value
LogLikelihoodFr

Convergence<-ll.model$convergence
Convergence

# Italy
It<-read.csv(file="C:\\italy.csv",head=TRUE,sep=";")
attach(It)

loglik<-function(theta,x){
ax<-theta[1:22]
bx<-theta[23:43]
bx<-c(bx,1-sum(bx))
kt<-theta[44:99]
kt<-c(kt,-sum(kt))

one<-rep(1,length(kt))
a<-ax%*%t(one)

Dxt<-matrix(DxtFIT/100,nrow=22)
Ext<-matrix(ExtFIT/100,nrow=22)

ll<-sum(colSums(Dxt%*%t(a+bx%*%t(kt))-Ext%*%t(exp(a+bx%*%t(kt)))))

return(-ll)
}

ax0<-matrix(c(-4.1979663,-7.3707033,-8.2858675,-8.3665837,-7.9404959,
-7.7327114,-7.5396789,-7.2604046,-6.8999433,-6.4898840,-6.0382387,
-5.5947742,-5.1600993,-4.6840958,-4.1618295,-3.5868913,-2.9859148,
-2.4038647,-1.8651090,-1.3942158,-1.0032159,-0.6915007))

bx0<-matrix(c(-0.01810624,-0.01044953,-0.04377010,-0.05895628,-0.07352683,
-0.06874747,-0.06736275,-0.07043614,-0.07418392,-0.07943640,-0.08291066,
-0.08455326,-0.08479366,-0.08288097,-0.08054738,-0.07929476,-0.08147614,
-0.08649366,-0.09358723,-0.10088720,-0.10648197,-0.11111743))

kt0<-matrix(c(16.4605610,16.5713770,15.0970362,14.2733082,12.2903279,
11.5423416,12.5537809,11.6277851,9.9941626,9.3546713,9.1811051,
8.1426803,9.0341261,9.0962878,6.7124602,7.2274829,5.4488774,
5.7570149,5.7825699,5.2152644,4.4666128,3.8298335,3.4155281,
3.3763164,1.7274064,1.7299583,1.1728013,0.2597824,-0.8215443,
-1.1773271,-0.3676756,-2.2653270,-3.0004216,-2.6038524,-4.2497238,
-4.3194636,-5.0647968,-5.7709559,-5.9126853,-6.8985893,-6.7219178,
-6.5962863,-6.9933316,-6.8996971,-7.5649949,-8.0814777,-8.3637043,
-9.3154926,-10.0066128,-11.0896427,-11.4319383,-12.3478044,-12.7413757,
-13.5193935,-15.3801871,-15.4937821,-16.3414583))

```

```

theta0<-c(ax0,bx0[1:21],kt0[1:56])
ll.model<-optim(theta0,loglik,hessian=T,x=It,control=list(maxit=100000))

opt.ax<-ll.model$par[1:22]
t(t(opt.ax))
opt.bx<-ll.model$par[23:43]
opt.bx<-c(opt.bx,1-sum(opt.bx))
t(t(opt.bx))
opt.kt<-ll.model$par[44:99]
opt.kt<-c(opt.kt,-sum(opt.kt))
t(t(opt.kt))

LogLikelihoodIt<-ll.model$value
LogLikelihoodIt

Convergence<-ll.model$convergence
Convergence

# Netherlands
Nl<-read.csv(file="C:\\netherlands.csv",head=TRUE,sep=";")
attach(Nl)

loglik<-function(theta,x){
  ax<-theta[1:22]
  bx<-theta[23:43]
  bx<-c(bx,1-sum(bx))
  kt<-theta[44:99]
  kt<-c(kt,-sum(kt))

  one<-rep(1,length(kt))
  a<-ax%*%t(one)

  Dxt<-matrix(DxtFNl/100,nrow=22)
  Ext<-matrix(ExtFNl/100,nrow=22)

  ll<-sum(colSums(Dxt%*%t(a+bx%*%t(kt))-Ext%*%t(exp(a+bx%*%t(kt)))))

  return(-ll)
}

ax0<-matrix(c(-4.7646219,-7.4878694,-8.4148084,-8.4760559,-8.0886569,
-7.9380057,-7.7244958,-7.3684024,-6.9627120,-6.4873747,-6.0012486,
-5.5758078,-5.1544783,-4.6885421,-4.1737722,-3.6151614,-3.0296650,
-2.4449137,-1.8983456,-1.4191059,-1.0019347,-0.6762116))

bx0<-matrix(c(-0.008121489,0.019342615,0.002954137,-0.034507399,-0.055384552,
-0.054506635,-0.054083665,-0.056799293,-0.060320522,-0.069297552,-0.074077411,
-0.073620897,-0.070950791,-0.065095943,-0.056745320,-0.052130295,-0.051876732,
-0.058508461,-0.067973643,-0.079005360,-0.086292832,-0.092997960))

kt0<-matrix(c(10.1740115,10.6332111,9.3012457,10.9595228,7.6549054,
7.1326063,7.3328675,7.0515258,6.1065179,5.7029719,4.7359579,
3.8008536,5.0042803,4.3410352,3.5946656,3.6121938,3.9406429,
3.4642979,3.5982075,3.9246011,3.7797863,2.7898251,2.7869232,

```

```

2.7334271,1.1268589,0.9198889,0.6237886,-0.3424263,0.1334250,
-1.1725819,-1.8782212,-2.2868385,-2.3187639,-2.6451349,-3.6206714,
-3.9160485,-3.1988067,-4.0029252,-4.9989181,-3.6432175,-4.0486390,
-4.0795741,-4.6289876,-4.0226038,-5.0203468,-4.9043156,-5.2177169,
-5.8174361,-6.8022773,-5.5195167,-5.9134364,-6.6090798,-6.4556552,
-6.8016615,-8.7431679,-8.6776286,-9.6734474)

theta0<-c(ax0,bx0[1:21],kt0[1:56])
ll.model<-optim(theta0,loglik,hessian=T,x=Nl,control=list(maxit=100000))

opt.ax<-ll.model$par[1:22]
t(t(opt.ax))
opt.bx<-ll.model$par[23:43]
opt.bx<-c(opt.bx,1-sum(opt.bx))
t(t(opt.bx))
opt.kt<-ll.model$par[44:99]
opt.kt<-c(opt.kt,-sum(opt.kt))
t(t(opt.kt))

LogLikelihoodNl<-ll.model$value
LogLikelihoodNl

Convergence<-ll.model$convergence
Convergence

# Spain
Sp<-read.csv(file="C:\\spain.csv",head=TRUE,sep=";")
attach(Sp)

loglik<-function(theta,x){
  ax<-theta[1:22]
  bx<-theta[23:43]
  bx<-c(bx,1-sum(bx))
  kt<-theta[44:99]
  kt<-c(kt,-sum(kt))

  one<-rep(1,length(kt))
  a<-ax%*%t(one)

  Dxt<-matrix(DxtFS/100,nrow=22)
  Ext<-matrix(ExtFS/100,nrow=22)

  ll<-sum(colSums(Dxt%*%t(a+bx%*%t(kt))-Ext%*%t(exp(a+bx%*%t(kt)))))

  return(-ll)
}

ax0<-matrix(c(-4.2585118,-7.2116195,-8.1208549,-8.2330059,-7.8394862,
-7.5940096,-7.3601497,-7.1043306,-6.7938808,-6.4322849,-6.0409136,
-5.6357697,-5.2329870,-4.7491375,-4.2282566,-3.6260439,-3.0186969,
-2.4410770,-1.9077245,-1.4808792,-1.0917044,-0.8259499))

bx0<-matrix(c(0.002202538,0.002502365,-0.020597795,-0.036285296,-0.052289923,
-0.044188922,-0.041733771,-0.048326449,-0.052659040,-0.057606947,-0.061299713,

```

```

-0.060078738,-0.059765208,-0.056207110,-0.056297886,-0.055814136,-0.062149638,
-0.071787177,-0.080314073,-0.092214396,-0.093033479,-0.102055203))

kt0<-matrix(c(20.04378043,20.70106634,16.43670506,15.07104779,13.42719824,
13.55163333,13.19561279,13.80070251,10.10575034,10.04754599,8.99606949,
8.08166930,8.57385582,7.94191734,6.20402307,5.50956955,4.94852702,
4.63665676,4.21018578,4.97259871,3.57712755,4.05186431,2.50716249,
2.69255589,1.76964210,1.09081520,0.06861619,-0.62144166,-0.43960459,
-1.66449761,-2.62692550,-3.41629833,-4.63305419,-4.20044816,-5.28714134,
-5.31078009,-5.37100447,-5.74196280,-5.71022254,-5.85042761,-5.76647843,
-6.24402410,-7.18097719,-7.64003478,-8.23429279,-8.35413000,-8.58024821,
-9.92878175,-10.47656598,-10.67435432,-11.56305919,-12.38357053,-12.49156339,
-12.09186829,-13.88090997,-14.20589628,-15.64333529))

theta0<-c(ax0,bx0[1:21],kt0[1:56])

ll.model<-optim(theta0,loglik,hessian=T,x=Sp,control=list(maxit=100000))

opt.ax<-ll.model$par[1:22]
t(t(opt.ax))
opt.bx<-ll.model$par[23:43]
opt.bx<-c(opt.bx,1-sum(opt.bx))
t(t(opt.bx))
opt.kt<-ll.model$par[44:99]
opt.kt<-c(opt.kt,-sum(opt.kt))
t(t(opt.kt))

LogLikelihoodSp<-ll.model$value
LogLikelihoodSp

Convergence<-ll.model$convergence
Convergence

LLunrestricted<-LogLikelihoodFr+LogLikelihoodIt+LogLikelihoodNl+LogLikelihoodSp
LLunrestricted

```

## 8.2 Parameters and unrestricted log-likelihood males

```

# France
Fr<-read.csv(file="C:\\france.csv",head=TRUE,sep=";")
attach(Fr)

loglik<-function(theta,x){
  ax<-theta[1:22]
  bx<-theta[23:43]
  bx<-c(bx,1-sum(bx))
  kt<-theta[44:99]
  kt<-c(kt,-sum(kt))

  one<-rep(1,length(kt))
  a<-ax%*%t(one)

  Dxt<-matrix(DxtMFr/100,nrow=22)
  Ext<-matrix(ExtMFr/100,nrow=22)

  ll<-sum(colSums(Dxt%*%t(a+bx%*%t(kt))-Ext%*%t(exp(a+bx%*%t(kt)))))

  return(-ll)
}

ax0<-matrix(c(-4.2642961,-7.2881359,-8.0673953,-8.0296754,-6.9304038,
-6.5344866,-6.4874966,-6.3149804,-5.9879343,-5.5564849,-5.1009216,
-4.6724655,-4.2731873,-3.8784187,-3.4795521,-3.0492509,-2.5903475,
-2.1106935,-1.6446770,-1.2224779,-0.8961878,-0.4623281))

bx0<-matrix(c(0.018999468,0.002487043,-0.021784967,-0.042511233,-0.072073503,
-0.084095236,-0.078960703,-0.077866709,-0.077269523,-0.077889586,-0.077569432,
-0.073616444,-0.069680416,-0.067733288,-0.066590662,-0.066298696,-0.067012126,
-0.072002227,-0.078522026,-0.087280083,-0.097364769,-0.085364884))

kt0<-matrix(c(9.80573218,9.74606068,8.20054638,8.83514458,7.28916484,
6.74930208,7.59910558,7.14469398,4.72813248,5.25540654,4.86877997,
4.54647124,5.04780997,4.99498823,4.00595493,4.37576852,3.48737658,
3.88391935,3.98622066,4.36412378,3.49702627,3.69709651,3.07562156,
3.17573893,2.69255847,2.45510832,2.04526390,1.48145204,1.16223844,
0.81348472,0.75678420,0.32956496,-0.26906764,-0.04596558,-1.02053693,
-1.25562400,-1.61636331,-2.64312922,-3.09189575,-3.28433946,-3.61691628,
-3.76661202,-4.25620494,-4.35412378,-5.30762141,-5.85068755,-6.20604881,
-7.17772453,-7.94963189,-8.20852530,-8.68530891,-8.88934434,-9.69949923,
-9.97970288,-11.87426019,-12.15886295,-12.88864401))

theta0<-c(ax0,bx0[1:21],kt0[1:56])
ll.model<-optim(theta0,loglik,hessian=T,x=Fr,control=list(maxit=100000))

opt.ax<-ll.model$par[1:22]
t(t(opt.ax))
opt.bx<-ll.model$par[23:43]
opt.bx<-c(opt.bx,1-sum(opt.bx))
t(t(opt.bx))
opt.kt<-ll.model$par[44:99]

```

```

opt.kt<-c(opt.kt,-sum(opt.kt))
t(t(opt.kt))

LogLikelihoodFr<-ll.model$value
LogLikelihoodFr

Convergence<-ll.model$convergence
Convergence

# Italy
It<-read.csv(file="C:\\italy.csv",head=TRUE,sep=";")
attach(It)

loglik<-function(theta,x){
  ax<-theta[1:22]
  bx<-theta[23:43]
  bx<-c(bx,1-sum(bx))
  kt<-theta[44:99]
  kt<-c(kt,-sum(kt))

  one<-rep(1,length(kt))
  a<-ax%*%t(one)

  Dxt<-matrix(DxtMIT/100,nrow=22)
  Ext<-matrix(ExtMIT/100,nrow=22)

  ll<-sum(colSums(Dxt%*%t(a+bx%*%t(kt))-Ext%*%t(exp(a+bx%*%t(kt)))))
}

return(-ll)
}

ax0<-matrix(c(-4.0033412,-7.2607653,-7.9919661,-7.9205202,-7.0112705,
-6.7671562,-6.7249913,-6.5813010,-6.3231858,-5.9161352,-5.4316771,
-4.9256820,-4.4364557,-3.9708024,-3.5209298,-3.0581373,-2.5813588,
-2.0987794,-1.6424462,-1.2264719,-0.8692156,-0.5715599))

bx0<-matrix(c(0.032289081,0.035862239,-0.001940854,-0.023415684,-0.060117511,
-0.070153246,-0.069775473,-0.066352166,-0.060752788,-0.057022893,-0.055459738,
-0.057250568,-0.061176510,-0.065298853,-0.068763875,-0.071014077,-0.072195413,
-0.073072379,-0.076162026,-0.080623174,-0.084541973,-0.093062117))

kt0<-matrix(c(10.1149081,11.4007474,9.8583540,9.3525720,7.5880950,
8.1568708,9.0873311,8.3748191,7.2227367,6.1401690,6.9499013,
6.1232601,7.3826897,7.4204295,5.6310288,5.8817641,4.6278154,
4.4598215,5.1149075,4.6974358,4.0476715,3.5760880,3.1410980,
3.4919661,1.9549278,2.4289392,1.7382310,1.3738454,0.8775005,
0.4602142,0.7804404,-0.3733858,-1.3115691,-1.0475899,-2.5345564,
-2.7057449,-3.4965826,-3.9992975,-4.0808408,-4.6251560,-4.2816462,
-3.8140158,-4.5242124,-5.0027702,-5.7194527,-5.4968023,-6.5292435,
-7.3967756,-8.1479653,-9.3256036,-10.0220641,-10.2331795,-10.5222632,
-12.0288477,-13.6478496,-13.8101684,-14.7789961))

theta0<-c(ax0,bx0[1:21],kt0[1:56])
ll.model<-optim(theta0,loglik,hessian=T,x=It,control=list(maxit=100000))

```

```

opt.ax<-ll.model$par[1:22]
t(t(opt.ax))
opt.bx<-ll.model$par[23:43]
opt.bx<-c(opt.bx,1-sum(opt.bx))
t(t(opt.bx))
opt.kt<-ll.model$par[44:99]
opt.kt<-c(opt.kt,-sum(opt.kt))
t(t(opt.kt))

LogLikelihoodIt<-ll.model$value
LogLikelihoodIt

Convergence<-ll.model$convergence
Convergence

# Netherlands
Nl<-read.csv(file="C:\\netherlands.csv",head=TRUE,sep=";")
attach(Nl)

loglik<-function(theta,x){
  ax<-theta[1:22]
  bx<-theta[23:43]
  bx<-c(bx,1-sum(bx))
  kt<-theta[44:99]
  kt<-c(kt,-sum(kt))

  one<-rep(1,length(kt))
  a<-ax%*%t(one)

  Dxt<-matrix(DxtMNl/100,nrow=22)
  Ext<-matrix(ExtMNl/100,nrow=22)

  ll<-sum(colSums(Dxt%*%t(a+bx%*%t(kt))-Ext%*%t(exp(a+bx%*%t(kt)))))

  return(-ll)
}

ax0<-matrix(c(-4.5067964,-7.2477738,-7.9881758,-8.1179550,-7.3238179,
-7.0759024,-7.0955331,-6.9434322,-6.6166981,-6.1417328,-5.6110750,
-5.0715717,-4.5476911,-4.0376774,-3.5449828,-3.0651769,-2.5845735,
-2.1202109,-1.6700493,-1.2587190,-0.8722722,-0.5432713))

bx0<-matrix(c(-0.04302736,-0.01118951,-0.01817998,-0.05665866,-0.09322012,
-0.10169975,-0.10560301,-0.11307681,-0.11360109,-0.11269554,-0.11127331,
-0.10938798,-0.11028248,-0.11612502,-0.12343997,-0.13049967,-0.13397399,
-0.13655291,-0.13730473,-0.13994884,-0.13995933,-0.14229995))

kt0<-matrix(c(6.37027284,5.96577283,5.11072853,6.16438512,5.08470214,
4.39444228,4.37920644,4.45761065,3.66451846,4.75077047,4.04815930,
3.84869659,4.50447980,4.47336405,4.07545599,4.17946755,4.46418908,
3.18347708,3.98197684,4.04395240,4.11013001,3.71404694,3.98938973,
3.34359807,2.50467605,2.10685621,2.11983645,1.12059735,1.49266675,
0.15795888,-0.01358775,-0.50803414,-0.67431387,-1.24131048,-1.36328423,

```

```

-2.03455701,-1.69920264,-2.85486918,-2.89184548,-3.01386927,-2.67574106,
-3.47992589,-3.46725173,-3.11993858,-4.65638339,-4.12634321,-4.55207005,
-5.52864399,-5.45268400,-5.53027226,-6.25851759,-6.52095081,-7.00720527,
-7.26793684,-9.21046033,-9.90743605,-10.74874979))

theta0<-c(ax0,bx0[1:21],kt0[1:56])
ll.model<-optim(theta0,loglik,hessian=T,x=Nl,control=list(maxit=100000))

opt.ax<-ll.model$par[1:22]
t(t(opt.ax))
opt.bx<-ll.model$par[23:43]
opt.bx<-c(opt.bx,1-sum(opt.bx))
t(t(opt.bx))
opt.kt<-ll.model$par[44:99]
opt.kt<-c(opt.kt,-sum(opt.kt))
t(t(opt.kt))

LogLikelihoodNl<-ll.model$value
LogLikelihoodNl

Convergence<-ll.model$convergence
Convergence

# Spain
Sp<-read.csv(file="C:\\spain.csv",head=TRUE,sep=";")
attach(Sp)

loglik<-function(theta,x){
  ax<-theta[1:22]
  bx<-theta[23:43]
  bx<-c(bx,1-sum(bx))
  kt<-theta[44:99]
  kt<-c(kt,-sum(kt))

  one<-rep(1,length(kt))
  a<-ax%*%t(one)

  Dxt<-matrix(DxtMSp/100,nrow=22)
  Ext<-matrix(ExtMSp/100,nrow=22)

  ll<-sum(colSums(Dxt%*%t(a+bx%*%t(kt))-Ext%*%t(exp(a+bx%*%t(kt)))))

  return(-ll)
}

ax0<-matrix(c(-4.0377526,-7.0472863,-7.8084569,-7.8705387,-7.0718650,
-6.7078716,-6.5677050,-6.3929851,-6.1434235,-5.7908761,-5.3698493,
-4.9169433,-4.4736126,-4.0146644,-3.5598531,-3.0741099,-2.5892952,
-2.1248484,-1.6803884,-1.3162565,-0.9820019,-0.7994526))

bx0<-matrix(c(0.068279430,0.062791663,0.027400835,0.004667998,-0.039003640,
-0.043425871,-0.039574027,-0.045351841,-0.047184050,-0.046836019,-0.048817537,
-0.047109770,-0.046351554,-0.043669972,-0.044386793,-0.042521023,-0.044459490,
-0.050355624,-0.054622835,-0.068391077,-0.071474630,-0.099604175))

```

```

kt0<-matrix(c(15.37748425,15.13496541,10.90139961,10.10633573,7.99496542,
8.50765078,8.51660986,8.74775014,5.72720841,6.16431002,5.45252036,
4.63813971,4.97867074,4.93411264,3.86426106,3.49519295,2.89432250,
2.88387526,2.54488039,3.30014056,2.40095736,3.17155271,1.26422878,
1.54013499,1.21254339,0.94911910,0.22435912,0.06080328,-0.00646095,
-0.72906967,-1.69646937,-2.07633078,-3.16217920,-2.61421852,-3.05500418,
-2.71943185,-3.60851334,-3.46174428,-3.20796821,-2.90446384,-2.43216402,
-2.69072250,-3.66167456,-4.13174896,-4.68075254,-5.08895763,-5.15699638,
-6.65337847,-6.86189342,-7.24679188,-7.91769766,-8.90396021,-9.25828085,
-9.09491080,-10.58742581,-11.00374170,-12.37554294))

theta0<-c(ax0,bx0[1:21],kt0[1:56])
ll.model<-optim(theta0,loglik,hessian=T,x=Sp,control=list(maxit=100000))

opt.ax<-ll.model$par[1:22]
t(t(opt.ax))
opt.bx<-ll.model$par[23:43]
opt.bx<-c(opt.bx,1-sum(opt.bx))
t(t(opt.bx))
opt.kt<-ll.model$par[44:99]
opt.kt<-c(opt.kt,-sum(opt.kt))
t(t(opt.kt))

LogLikelihoodSp<-ll.model$value
LogLikelihoodSp

Convergence<-ll.model$convergence
Convergence

LLunrestricted<-LogLikelihoodFr+LogLikelihoodIt+LogLikelihoodNl+LogLikelihoodSp
LLunrestricted

```

### 8.3 Model restricted to $\alpha_x$ females and males

#### Females

```

Fr<-read.csv(file="C:\\france.csv",head=TRUE,sep=";")
attach(Fr)
It<-read.csv(file="C:\\italy.csv",head=TRUE,sep=";")
attach(It)
Nl<-read.csv(file="C:\\netherlands.csv",head=TRUE,sep=";")
attach(Nl)
Sp<-read.csv(file="C:\\spain.csv",head=TRUE,sep=";")
attach(Sp)

loglik<-function(theta,x){
  ax<-theta[1:22]
  bxFr<-theta[23:43]
  bxFr<-c(bxFr,1-sum(bxFr))
  bxIt<-theta[44:64]
  bxIt<-c(bxIt,1-sum(bxIt))
  bxNl<-theta[65:85]
  bxNl<-c(bxNl,1-sum(bxNl))
  bxSp<-theta[86:106]
  bxSp<-c(bxSp,1-sum(bxSp))
  ktFr<-theta[107:162]
  ktFr<-c(ktFr,-sum(ktFr))
  ktIt<-theta[163:218]
  ktIt<-c(ktIt,-sum(ktIt))
  ktNl<-theta[219:274]
  ktNl<-c(ktNl,-sum(ktNl))
  ktSp<-theta[275:330]
  ktSp<-c(ktSp,-sum(ktSp))

  one<-rep(1,length(ktFr))
  a<-ax%*%t(one)

  DxtFr<-matrix(DxtFFr/100,nrow=22)
  DxtIt<-matrix(DxtFIt/100,nrow=22)
  DxtNl<-matrix(DxtFNl/100,nrow=22)
  DxtSp<-matrix(DxtFSp/100,nrow=22)

  ExtFr<-matrix(ExtFFr/100,nrow=22)
  ExtIt<-matrix(ExtFIt/100,nrow=22)
  ExtNl<-matrix(ExtFNl/100,nrow=22)
  ExtSp<-matrix(ExtFSp/100,nrow=22)

  ll<-sum(colSums(DxtFr%*%t(a+bxFr%*%t(ktFr))-ExtFr%*%t(exp(a+bxFr%*%t(ktFr)))))+
    sum(colSums(DxtIt%*%t(a+bxIt%*%t(ktIt))-ExtIt%*%t(exp(a+bxIt%*%t(ktIt)))))+
    sum(colSums(DxtNl%*%t(a+bxNl%*%t(ktNl))-ExtNl%*%t(exp(a+bxNl%*%t(ktNl)))))+
    sum(colSums(DxtSp%*%t(a+bxSp%*%t(ktSp))-ExtSp%*%t(exp(a+bxSp%*%t(ktSp)))))

  return(-ll)
}

ax0<-matrix(c(-4.3517278,-7.3669846,-8.2695969,-8.3589322,-7.8488373,
-7.6199908,-7.4371085,-7.1578999,-6.7985991,-6.3987215,-5.9710319,

```

```

-5.5606902,-5.1629180,-4.7130945,-4.2115066,-3.6433957,-3.0445857,
-2.4545040,-1.9057266,-1.4365057,-1.0455664,-0.7226119))

bxFr0<-matrix(c(-0.02814830,-0.03702005,-0.06588364,-0.07911846,-0.09154635,
-0.08734098,-0.08257363,-0.08546004,-0.08951054,-0.09347024,-0.09394742,
-0.09211971,-0.08961577,-0.08562356,-0.08180338,-0.07984893,-0.08158660,
-0.08841981,-0.09741398,-0.10652574,-0.11358619,-0.10943668))

bxIt0<-matrix(c(-0.01810624,-0.01044953,-0.04377010,-0.05895628,-0.07352683,
-0.06874747,-0.06736275,-0.07043614,-0.07418392,-0.07943640,-0.08291066,
-0.08455326,-0.08479366,-0.08288097,-0.08054738,-0.07929476,-0.08147614,
-0.08649366,-0.09358723,-0.10088720,-0.10648197,-0.11111743))

bxN10<-matrix(c(-0.008121489,0.019342615,0.002954137,-0.034507399,-0.055384552,
-0.054506635,-0.054083665,-0.056799293,-0.060320522,-0.069297552,-0.074077411,
-0.073620897,-0.070950791,-0.065095943,-0.056745320,-0.052130295,-0.051876732,
-0.058508461,-0.067973643,-0.079005360,-0.086292832,-0.092997960))

bxSp0<-matrix(c(0.002202538,0.002502365,-0.020597795,-0.036285296,-0.052289923,
-0.044188922,-0.041733771,-0.048326449,-0.052659040,-0.057606947,-0.061299713,
-0.060078738,-0.059765208,-0.056207110,-0.056297886,-0.055814136,-0.062149638,
-0.071787177,-0.080314073,-0.092214396,-0.093033479,-0.102055203))

ktFr0<-matrix(c(9.534973461,9.835657812,8.173812107,8.532332146,7.045794769,
6.384929012,6.639384375,6.198518397,4.753579825,5.028153143,4.569468378,
3.736791682,4.326573108,4.405569246,3.309391397,3.435286790,2.901120261,
3.017521111,2.972169870,3.240304307,2.396249869,2.518071770,2.234552808,
2.082365266,1.460846660,1.296994783,0.933402119,0.364905281,0.007841574,
-0.332002619,-0.260819267,-0.439848227,-1.005375852,-0.831294061,-1.655734100,
-1.842815325,-2.191298272,-2.943466903,-3.209023012,-3.236685764,-3.824656739,
-3.885837254,-4.301588766,-4.282877456,-4.899729757,-4.871101768,-5.395503562,
-5.850667537,-6.056394130,-6.201867802,-6.676499908,-6.783000575,-7.035319395,
-7.264756018,-8.488257479,-8.588137837,-8.982001951))

ktIt0<-matrix(c(10.6993646,10.7713950,9.8130735,9.2776503,7.9887131,
7.5025220,8.1599576,7.5580603,6.4962057,6.0805363,5.9677183,
5.2927422,5.8721820,5.9125871,4.3630991,4.6978639,3.5417703,
3.7420597,3.7586704,3.3899219,2.9032983,2.4893918,2.2200933,
2.1946057,1.1228142,1.1244729,0.7623208,0.1688586,-0.5340038,
-0.7652626,-0.2389891,-1.4724626,-1.9502740,-1.6925041,-2.7623205,
-2.8076513,-3.2921179,-3.7511213,-3.8432454,-4.4840830,-4.3692466,
-4.2875861,-4.5456655,-4.4848031,-4.9172467,-5.2529605,-5.4364078,
-6.0550702,-6.5042983,-7.2082678,-7.4307599,-8.0260729,-8.2818942,
-8.7876058,-9.9971216,-10.0709584,-10.6219479))

ktN10<-matrix(c(6.61310748,6.91158722,6.04580971,7.12368982,4.97568851,
4.63619410,4.76636387,4.58349177,3.96923664,3.70693173,3.07837264,
2.47055484,3.25278219,2.82167288,2.33653264,2.34792597,2.56141789,
2.25179364,2.33883487,2.55099072,2.45686110,1.81338632,1.81150008,
1.77672762,0.73245828,0.59792778,0.40546259,-0.22257710,0.08672625,
-0.76217824,-1.22084378,-1.48644503,-1.50719654,-1.71933768,-2.35343641,
-2.54543153,-2.07922436,-2.60190138,-3.24929676,-2.36809138,-2.63161535,
-2.65172317,-3.00884194,-2.61469247,-3.26322542,-3.18780514,-3.39151599,
-3.78133347,-4.42148025,-3.58768585,-3.84373366,-4.29590187,-4.19617588,

```

```

-4.42107997,-5.68305913,-5.64045859,-6.28774081))

ktSp0<-matrix(c(13.02845728,13.45569312,10.68385829,9.79618106,8.72767886,
8.80856166,8.57714831,8.97045663,6.56873772,6.53090489,5.84744517,
5.25308504,5.57300628,5.16224627,4.03261500,3.58122021,3.21654256,
3.01382689,2.73662076,3.23218916,2.32513291,2.63371180,1.62965562,
1.75016133,1.15026736,0.70902988,0.04460052,-0.40393708,-0.28574298,
-1.08192345,-1.70750157,-2.22059391,-3.01148522,-2.73029130,-3.43664187,
-3.45200706,-3.49115291,-3.73227582,-3.71164465,-3.80277795,-3.74821098,
-4.05861567,-4.66763517,-4.96602261,-5.35229031,-5.43018450,-5.57716134,
-6.45370814,-6.80976789,-6.93833031,-7.51598847,-8.04932084,-8.11951620,
-7.85971439,-9.02259148,-9.23383258,-10.16816794))

theta0<-c(ax0,bxFr0[1:21],bxIt0[1:21],bxNl0[1:21],bxSp0[1:21],ktFr0[1:56],
ktIt0[1:56],ktNl0[1:56],ktSp0[1:56])
ll.model<-optim(theta0,loglik,hessian=T,x=FrltNlSp,control=list(maxit=100000))

LogLikelihood<-ll.model$value
LogLikelihood

Convergence<-ll.model$convergence
Convergence

Males

Fr<-read.csv(file="C:\\france.csv",head=TRUE,sep=";")
attach(Fr)
It<-read.csv(file="C:\\italy.csv",head=TRUE,sep=";")
attach(It)
Nl<-read.csv(file="C:\\netherlands.csv",head=TRUE,sep=";")
attach(Nl)
Sp<-read.csv(file="C:\\spain.csv",head=TRUE,sep=";")
attach(Sp)

loglik<-function(theta,x){
ax<-theta[1:22]
bxFr<-theta[23:43]
bxFr<-c(bxFr,1-sum(bxFr))
bxIt<-theta[44:64]
bxIt<-c(bxIt,1-sum(bxIt))
bxNl<-theta[65:85]
bxNl<-c(bxNl,1-sum(bxNl))
bxSp<-theta[86:106]
bxSp<-c(bxSp,1-sum(bxSp))
ktFr<-theta[107:162]
ktFr<-c(ktFr,-sum(ktFr))
ktIt<-theta[163:218]
ktIt<-c(ktIt,-sum(ktIt))
ktNl<-theta[219:274]
ktNl<-c(ktNl,-sum(ktNl))
ktSp<-theta[275:330]
ktSp<-c(ktSp,-sum(ktSp))

one<-rep(1,length(ktFr))
a<-ax%*%t(one)

```

```

DxtFr<-matrix(DxtMFr/100,nrow=22)
DxtIt<-matrix(DxtMIT/100,nrow=22)
DxtNl<-matrix(DxtMNl/100,nrow=22)
DxtSp<-matrix(DxtMSp/100,nrow=22)

ExtFr<-matrix(ExtMFr/100,nrow=22)
ExtIt<-matrix(ExtMIT/100,nrow=22)
ExtNl<-matrix(ExtMNl/100,nrow=22)
ExtSp<-matrix(ExtMSp/100,nrow=22)

ll<-sum(colSums(DxtFr%*%t(a+bxFr%*%t(ktFr))-ExtFr%*%t(exp(a+bxFr%*%t(ktFr)))))+
sum(colSums(DxtIt%*%t(a+bxIt%*%t(ktIt))-ExtIt%*%t(exp(a+bxIt%*%t(ktIt)))))+
sum(colSums(DxtNl%*%t(a+bxNl%*%t(ktNl))-ExtNl%*%t(exp(a+bxNl%*%t(ktNl)))))+
sum(colSums(DxtSp%*%t(a+bxSp%*%t(ktSp))-ExtSp%*%t(exp(a+bxSp%*%t(ktSp)))))

return(-ll)
}

ax0<-matrix(c(-4.1221507,-7.1963862,-7.9593439,-7.9523302,-7.0124908,
-6.6835114,-6.6238091,-6.4607874,-6.1758586,-5.7667060,-5.3036172,
-4.8379047,-4.3917915,-3.9512193,-3.5144188,-3.0569006,-2.5849625,
-2.1094486,-1.6534537,-1.2462158,-0.9055158,-0.6015493))

bxFr0<-matrix(c(0.018999468,0.002487043,-0.021784967,-0.042511233,-0.072073503,
-0.084095236,-0.078960703,-0.077866709,-0.077269523,-0.077889586,-0.077569432,
-0.073616444,-0.069680416,-0.067733288,-0.066590662,-0.066298696,-0.067012126,
-0.072002227,-0.078522026,-0.087280083,-0.097364769,-0.085364884))

bxIt0<-matrix(c(0.032289081,0.035862239,-0.001940854,-0.023415684,-0.060117511,
-0.070153246,-0.069775473,-0.066352166,-0.060752788,-0.057022893,-0.055459738,
-0.057250568,-0.061176510,-0.065298853,-0.068763875,-0.071014077,-0.072195413,
-0.073072379,-0.076162026,-0.080623174,-0.084541973,-0.093062117))

bxNl0<-matrix(c(-0.04302736,-0.01118951,-0.01817998,-0.05665866,-0.09322012,
-0.10169975,-0.10560301,-0.11307681,-0.11360109,-0.11269554,-0.11127331,
-0.10938798,-0.11028248,-0.11612502,-0.12343997,-0.13049967,-0.13397399,
-0.13655291,-0.13730473,-0.13994884,-0.13995933,-0.14229995))

bxSp0<-matrix(c(0.068279430,0.062791663,0.027400835,0.004667998,-0.039003640,
-0.043425871,-0.039574027,-0.045351841,-0.047184050,-0.046836019,-0.048817537,
-0.047109770,-0.046351554,-0.043669972,-0.044386793,-0.042521023,-0.044459490,
-0.050355624,-0.054622835,-0.068391077,-0.071474630,-0.099604175))

ktFr0<-matrix(c(9.80573218,9.74606068,8.20054638,8.83514458,7.28916484,
6.74930208,7.59910558,7.14469398,4.72813248,5.25540654,4.86877997,
4.54647124,5.04780997,4.99498823,4.00595493,4.37576852,3.48737658,
3.88391935,3.98622066,4.36412378,3.49702627,3.69709651,3.07562156,
3.17573893,2.69255847,2.45510832,2.04526390,1.48145204,1.16223844,
0.81348472,0.75678420,0.32956496,-0.26906764,-0.04596558,-1.02053693,
-1.25562400,-1.61636331,-2.64312922,-3.09189575,-3.28433946,-3.61691628,
-3.76661202,-4.25620494,-4.35412378,-5.30762141,-5.85068755,-6.20604881,
-7.17772453,-7.94963189,-8.20852530,-8.68530891,-8.88934434,-9.69949923,
-9.97970288,-11.87426019,-12.15886295,-12.88864401))

```

```

ktIt0<-matrix(c(10.1149081,11.4007474,9.8583540,9.3525720,7.5880950,
8.1568708,9.0873311,8.3748191,7.2227367,6.1401690,6.9499013,
6.1232601,7.3826897,7.4204295,5.6310288,5.8817641,4.6278154,
4.4598215,5.1149075,4.6974358,4.0476715,3.5760880,3.1410980,
3.4919661,1.9549278,2.4289392,1.7382310,1.3738454,0.8775005,
0.4602142,0.7804404,-0.3733858,-1.3115691,-1.0475899,-2.5345564,
-2.7057449,-3.4965826,-3.9992975,-4.0808408,-4.6251560,-4.2816462,
-3.8140158,-4.5242124,-5.0027702,-5.7194527,-5.4968023,-6.5292435,
-7.3967756,-8.1479653,-9.3256036,-10.0220641,-10.2331795,-10.5222632,
-12.0288477,-13.6478496,-13.8101684,-14.7789961))

ktN10<-matrix(c(6.37027284,5.96577283,5.11072853,6.16438512,5.08470214,
4.39444228,4.37920644,4.45761065,3.66451846,4.75077047,4.04815930,
3.84869659,4.50447980,4.47336405,4.07545599,4.17946755,4.46418908,
3.18347708,3.98197684,4.04395240,4.11013001,3.71404694,3.98938973,
3.34359807,2.50467605,2.10685621,2.11983645,1.12059735,1.49266675,
0.15795888,-0.01358775,-0.50803414,-0.67431387,-1.24131048,-1.36328423,
-2.03455701,-1.69920264,-2.85486918,-2.89184548,-3.01386927,-2.67574106,
-3.47992589,-3.46725173,-3.11993858,-4.65638339,-4.12634321,-4.55207005,
-5.52864399,-5.45268400,-5.53027226,-6.25851759,-6.52095081,-7.00720527,
-7.26793684,-9.21046033,-9.90743605,-10.74874979))

ktSp0<-matrix(c(15.37748425,15.13496541,10.90139961,10.10633573,7.99496542,
8.50765078,8.51660986,8.74775014,5.72720841,6.16431002,5.45252036,
4.63813971,4.97867074,4.93411264,3.86426106,3.49519295,2.89432250,
2.88387526,2.54488039,3.30014056,2.40095736,3.17155271,1.26422878,
1.54013499,1.21254339,0.94911910,0.22435912,0.06080328,-0.00646095,
-0.72906967,-1.69646937,-2.07633078,-3.16217920,-2.61421852,-3.05500418,
-2.71943185,-3.60851334,-3.46174428,-3.20796821,-2.90446384,-2.43216402,
-2.69072250,-3.66167456,-4.13174896,-4.68075254,-5.08395763,-5.15699638,
-6.65337847,-6.86189342,-7.24679188,-7.91769766,-8.90396021,-9.25828085,
-9.09491080,-10.58742581,-11.00374170,-12.37554294))

theta0<-c(ax0,bxFr0[1:21],bxIt0[1:21],bxN10[1:21],bxSp0[1:21],ktFr0[1:56],
ktIt0[1:56],ktN10[1:56],ktSp0[1:56])
ll.model<-optim(theta0,loglik,hessian=T,x=FrItNlSp,control=list(maxit=100000))

LogLikelihood<-ll.model$value
LogLikelihood

Convergence<-ll.model$convergence
Convergence

```

## 8.4 Model restricted to $\beta_x$ females and males

### Females

```

Fr<-read.csv(file="C:\\france.csv",head=TRUE,sep=";")
attach(Fr)
It<-read.csv(file="C:\\italy.csv",head=TRUE,sep=";")
attach(It)
Nl<-read.csv(file="C:\\netherlands.csv",head=TRUE,sep=";")
attach(Nl)
Sp<-read.csv(file="C:\\spain.csv",head=TRUE,sep=";")
attach(Sp)

loglik<-function(theta,x){
  axFr<-theta[1:22]
  axIt<-theta[23:44]
  axNl<-theta[45:66]
  axSp<-theta[67:88]
  bx<-theta[89:109]
  bx<-c(bx,1-sum(bx))
  ktFr<-theta[110:165]
  ktFr<-c(ktFr,-sum(ktFr))
  ktIt<-theta[166:221]
  ktIt<-c(ktIt,-sum(ktIt))
  ktNl<-theta[222:277]
  ktNl<-c(ktNl,-sum(ktNl))
  ktSp<-theta[278:333]
  ktSp<-c(ktSp,-sum(ktSp))

  one<-rep(1,length(ktFr))
  aFr<-axFr%*%t(one)
  aIt<-axIt%*%t(one)
  aNl<-axNl%*%t(one)
  aSp<-axSp%*%t(one)

  DxtFr<-matrix(DxtFFr/100,nrow=22)
  DxtIt<-matrix(DxtFIt/100,nrow=22)
  DxtNl<-matrix(DxtFNl/100,nrow=22)
  DxtSp<-matrix(DxtFSp/100,nrow=22)

  ExtFr<-matrix(ExtFFr/100,nrow=22)
  ExtIt<-matrix(ExtFIt/100,nrow=22)
  ExtNl<-matrix(ExtFNl/100,nrow=22)
  ExtSp<-matrix(ExtFSp/100,nrow=22)

  ll<-sum(colSums(DxtFr%*%t(aFr+bx%*%t(ktFr))-ExtFr%*%t(exp(aFr+bx%*%t(ktFr)))))+
    sum(colSums(DxtIt%*%t(aIt+bx%*%t(ktIt))-ExtIt%*%t(exp(aIt+bx%*%t(ktIt)))))+
    sum(colSums(DxtNl%*%t(aNl+bx%*%t(ktNl))-ExtNl%*%t(exp(aNl+bx%*%t(ktNl)))))+
    sum(colSums(DxtSp%*%t(aSp+bx%*%t(ktSp))-ExtSp%*%t(exp(aSp+bx%*%t(ktSp)))))

  return(-ll)
}

axFr0<-matrix(c(-4.5336584,-7.5060995,-8.3702416,-8.4522067,-7.7548003,
-7.4979048,-7.3547828,-7.0678867,-6.6804750,-6.2807173,-5.8659169,

```

```

-5.4864869,-5.1324817,-4.7329462,-4.2667481,-3.7214242,-3.1228198,
-2.5131285,-1.9422717,-1.4467931,-1.0464949,-0.6440435))

axIt0<-matrix(c(-4.1979663,-7.3707033,-8.2858675,-8.3665837,-7.9404959,
-7.7327114,-7.5396789,-7.2604046,-6.8999433,-6.4898840,-6.0382387,
-5.5947742,-5.1600993,-4.6840958,-4.1618295,-3.5868913,-2.9859148,
-2.4038647,-1.8651090,-1.3942158,-1.0032159,-0.6915007))

axN10<-matrix(c(-4.7646219,-7.4878694,-8.4148084,-8.4760559,-8.0886569,
-7.9380057,-7.7244958,-7.3684024,-6.9627120,-6.4873747,-6.0012486,
-5.5758078,-5.1544783,-4.6885421,-4.1737722,-3.6151614,-3.0296650,
-2.4449137,-1.8983456,-1.4191059,-1.0019347,-0.6762116))

axSp0<-matrix(c(-4.2585118,-7.2116195,-8.1208549,-8.2330059,-7.8394862,
-7.5940096,-7.3601497,-7.1043306,-6.7938808,-6.4322849,-6.0409136,
-5.6357697,-5.2329870,-4.7491375,-4.2282566,-3.6260439,-3.0186969,
-2.4410770,-1.9077245,-1.4808792,-1.0917044,-0.8259499))

bx0<-matrix(c(-0.01622118,-0.01543498,-0.04352775,-0.05994233,-0.07548065,
-0.07006364,-0.06728030,-0.07167931,-0.07624606,-0.08150322,-0.08415775,
-0.08354450,-0.08241646,-0.07925217,-0.07686681,-0.07568832,-0.07865122,
-0.08527881,-0.09322626,-0.10220721,-0.10798104,-0.11335002))

ktFr0<-matrix(c(14.66918994,15.13178125,12.57509555,13.12666484,10.83968426,
9.82296771,10.21443750,9.53618215,7.31319973,7.73562022,7.02995135,
5.74891028,6.65626632,6.77779884,5.09137138,5.28505660,4.46326194,
4.64234017,4.57256903,4.98508355,3.68653826,3.87395657,3.43777355,
3.20363887,2.24745640,1.99537659,1.43600326,0.56139274,0.01206396,
-0.51077326,-0.40126041,-0.67668958,-1.54673208,-1.27891394,-2.54728323,
-2.83510050,-3.37122811,-4.52841062,-4.93695848,-4.97951656,-5.88408729,
-5.97821116,-6.61782887,-6.58904224,-7.53804578,-7.49400272,-8.30077471,
-9.00102698,-9.31752943,-9.54133508,-10.27153832,-10.43538550,-10.82356830,
-11.17654772,-13.05885766,-13.21251975,-13.81846454))

ktIt0<-matrix(c(16.4605610,16.5713770,15.0970362,14.2733082,12.2903279,
11.5423416,12.5537809,11.6277851,9.9941626,9.3546713,9.1811051,
8.1426803,9.0341261,9.0962878,6.7124602,7.2274829,5.4488774,
5.7570149,5.7825699,5.2152644,4.4666128,3.8298335,3.4155281,
3.3763164,1.7274064,1.7299583,1.1728013,0.2597824,-0.8215443,
-1.1773271,-0.3676756,-2.2653270,-3.0004216,-2.6038524,-4.2497238,
-4.3194636,-5.0647968,-5.7709559,-5.9126853,-6.8985893,-6.7219178,
-6.5962863,-6.9933316,-6.8996971,-7.5649949,-8.0814777,-8.3637043,
-9.3154926,-10.006128,-11.0896427,-11.4319383,-12.3478044,-12.7413757,
-13.5193935,-15.3801871,-15.4937821,-16.3414583))

ktN10<-matrix(c(10.1740115,10.6332111,9.3012457,10.9595228,7.6549054,
7.1326063,7.3328675,7.0515258,6.1065179,5.7029719,4.7359579,
3.8008536,5.0042803,4.3410352,3.5946656,3.6121938,3.9406429,
3.4642979,3.5982075,3.9246011,3.7797863,2.7898251,2.7869232,
2.7334271,1.1268589,0.9198889,0.6237886,-0.3424263,0.1334250,
-1.1725819,-1.8782212,-2.2868385,-2.3187639,-2.6451349,-3.6206714,
-3.9160485,-3.1988067,-4.0029252,-4.9989181,-3.6432175,-4.0486390,
-4.0795741,-4.6289876,-4.0226038,-5.0203468,-4.9043156,-5.2177169,
-5.8174361,-6.8022773,-5.5195167,-5.9134364,-6.6090798,-6.4556552,

```

```

-6.8016615,-8.7431679,-8.6776286,-9.6734474))

ktSp0<-matrix(c(20.04378043,20.70106634,16.43670506,15.07104779,13.42719824,
13.55163333,13.19561279,13.80070251,10.10575034,10.04754599,8.99606949,
8.08166930,8.57385582,7.94191734,6.20402307,5.50956955,4.94852702,
4.63665676,4.21018578,4.97259871,3.57712755,4.05186431,2.50716249,
2.69255589,1.76964210,1.09081520,0.06861619,-0.62144166,-0.43960459,
-1.66449761,-2.62692550,-3.41629833,-4.63305419,-4.20044816,-5.28714134,
-5.31078009,-5.37100447,-5.74196280,-5.71022254,-5.85042761,-5.76647843,
-6.24402410,-7.18097719,-7.64003478,-8.23429279,-8.35413000,-8.58024821,
-9.92878175,-10.47656598,-10.67435432,-11.56305919,-12.38357053,-12.49156339,
-12.09186829,-13.88090997,-14.20589628,-15.64333529))

theta0<-c(axFr0,axIt0,axNl0,axSp0,bx0[1:21],ktFr0[1:56],ktIt0[1:56],
ktNl0[1:56],ktSp0[1:56])
ll.model<-optim(theta0,loglik,hessian=T,x=FrltNlSp,control=list(maxit=100000))

LogLikelihood<-ll.model$value
LogLikelihood

Convergence<-ll.model$convergence
Convergence
```

### Males

```

Fr<-read.csv(file="C:\\france.csv",head=TRUE,sep=";")
attach(Fr)
It<-read.csv(file="C:\\italy.csv",head=TRUE,sep=";")
attach(It)
Nl<-read.csv(file="C:\\netherlands.csv",head=TRUE,sep=";")
attach(Nl)
Sp<-read.csv(file="C:\\spain.csv",head=TRUE,sep=";")
attach(Sp)

loglik<-function(theta,x){
  axFr<-theta[1:22]
  axIt<-theta[23:44]
  axNl<-theta[45:66]
  axSp<-theta[67:88]
  bx<-theta[89:109]
  bx<-c(bx,1-sum(bx))
  ktFr<-theta[110:165]
  ktFr<-c(ktFr,-sum(ktFr))
  ktIt<-theta[166:221]
  ktIt<-c(ktIt,-sum(ktIt))
  ktNl<-theta[222:277]
  ktNl<-c(ktNl,-sum(ktNl))
  ktSp<-theta[278:333]
  ktSp<-c(ktSp,-sum(ktSp))

  one<-rep(1,length(ktFr))
  aFr<-axFr%*%t(one)
  aIt<-axIt%*%t(one)
  aNl<-axNl%*%t(one)
  aSp<-axSp%*%t(one)
```

```

DxtFr<-matrix(DxtMFr/100,nrow=22)
DxtIt<-matrix(DxtMIT/100,nrow=22)
DxtNl<-matrix(DxtMNl/100,nrow=22)
DxtSp<-matrix(DxtMSp/100,nrow=22)

ExtFr<-matrix(ExtMFr/100,nrow=22)
ExtIt<-matrix(ExtMIT/100,nrow=22)
ExtNl<-matrix(ExtMNl/100,nrow=22)
ExtSp<-matrix(ExtMSp/100,nrow=22)

ll<-sum(colSums(DxtFr%*%t(aFr+bx%*%t(ktFr))-ExtFr%*%t(exp(aFr+bx%*%t(ktFr)))))+
sum(colSums(DxtIt%*%t(aIt+bx%*%t(ktIt))-ExtIt%*%t(exp(aIt+bx%*%t(ktIt)))))+
sum(colSums(DxtNl%*%t(aNl+bx%*%t(ktNl))-ExtNl%*%t(exp(aNl+bx%*%t(ktNl)))))+
sum(colSums(DxtSp%*%t(aSp+bx%*%t(ktSp))-ExtSp%*%t(exp(aSp+bx%*%t(ktSp)))))

return(-ll)
}

axFr0<-matrix(c(-4.2642961,-7.2881359,-8.0673953,-8.0296754,-6.9304038,
-6.5344866,-6.4874966,-6.3149804,-5.9879343,-5.5564849,-5.1009216,
-4.6724655,-4.2731873,-3.8784187,-3.4795521,-3.0492509,-2.5903475,
-2.1106935,-1.6446770,-1.2224779,-0.8961878,-0.4623281))

axIt0<-matrix(c(-4.0033412,-7.2607653,-7.9919661,-7.9205202,-7.0112705,
-6.7671562,-6.7249913,-6.5813010,-6.3231858,-5.9161352,-5.4316771,
-4.9256820,-4.4364557,-3.9708024,-3.5209298,-3.0581373,-2.5813588,
-2.0987794,-1.6424462,-1.2264719,-0.8692156,-0.5715599))

axNl0<-matrix(c(-4.5067964,-7.2477738,-7.9881758,-8.1179550,-7.3238179,
-7.0759024,-7.0955331,-6.9434322,-6.6166981,-6.1417328,-5.6110750,
-5.0715717,-4.5476911,-4.0376774,-3.5449828,-3.0651769,-2.5845735,
-2.1202109,-1.6700493,-1.2587190,-0.8722722,-0.5432713))

axSp0<-matrix(c(-4.0377526,-7.0472863,-7.8084569,-7.8705387,-7.0718650,
-6.7078716,-6.5677050,-6.3929851,-6.1434235,-5.7908761,-5.3698493,
-4.9169433,-4.4736126,-4.0146644,-3.5598531,-3.0741099,-2.5892952,
-2.1248484,-1.6803884,-1.3162565,-0.9820019,-0.7994526))

bx0<-matrix(c(0.0371364406,0.0309653061,0.0005653156,-0.0223959902,-0.0601398549,
-0.0694324426,-0.0666929963,-0.0671010981,-0.0657233285,-0.0645070863,-0.0639985882,
-0.0620777844,-0.0614830316,-0.0621078732,-0.0638668926,-0.0648770850,-0.0662205702,
-0.0693403281,-0.0735109054,-0.0809975395,-0.0876408156,-0.0965528519))

ktFr0<-matrix(c(9.80573218,9.74606068,8.20054638,8.83514458,7.28916484,
6.74930208,7.59910558,7.14469398,4.72813248,5.25540654,4.86877997,
4.54647124,5.04780997,4.99498823,4.00595493,4.37576852,3.48737658,
3.88391935,3.98622066,4.36412378,3.49702627,3.69709651,3.07562156,
3.17573893,2.69255847,2.45510832,2.04526390,1.48145204,1.16223844,
0.81348472,0.75678420,0.32956496,-0.26906764,-0.04596558,-1.02053693,
-1.25562400,-1.61636331,-2.64312922,-3.09189575,-3.28433946,-3.61691628,
-3.76661202,-4.25620494,-4.35412378,-5.30762141,-5.85068755,-6.20604881,
-7.17772453,-7.94963189,-8.20852530,-8.68530891,-8.88934434,-9.69949923,
-9.97970288,-11.87426019,-12.15886295,-12.88864401))

```

```

ktIt0<-matrix(c(10.1149081,11.4007474,9.8583540,9.3525720,7.5880950,
8.1568708,9.0873311,8.3748191,7.2227367,6.1401690,6.9499013,
6.1232601,7.3826897,7.4204295,5.6310288,5.8817641,4.6278154,
4.4598215,5.1149075,4.6974358,4.0476715,3.5760880,3.1410980,
3.4919661,1.9549278,2.4289392,1.7382310,1.3738454,0.8775005,
0.4602142,0.7804404,-0.3733858,-1.3115691,-1.0475899,-2.5345564,
-2.7057449,-3.4965826,-3.9992975,-4.0808408,-4.6251560,-4.2816462,
-3.8140158,-4.5242124,-5.0027702,-5.7194527,-5.4968023,-6.5292435,
-7.3967756,-8.1479653,-9.3256036,-10.0220641,-10.2331795,-10.5222632,
-12.0288477,-13.6478496,-13.8101684,-14.7789961))

ktN10<-matrix(c(6.37027284,5.96577283,5.11072853,6.16438512,5.08470214,
4.39444228,4.37920644,4.45761065,3.66451846,4.75077047,4.04815930,
3.84869659,4.50447980,4.47336405,4.07545599,4.17946755,4.46418908,
3.18347708,3.98197684,4.04395240,4.11013001,3.71404694,3.98938973,
3.34359807,2.50467605,2.10685621,2.11983645,1.12059735,1.49266675,
0.15795888,-0.01358775,-0.50803414,-0.67431387,-1.24131048,-1.36328423,
-2.03455701,-1.69920264,-2.85486918,-2.89184548,-3.01386927,-2.67574106,
-3.47992589,-3.46725173,-3.11993858,-4.65638339,-4.12634321,-4.55207005,
-5.52864399,-5.45268400,-5.53027226,-6.25851759,-6.52095081,-7.00720527,
-7.26793684,-9.21046033,-9.90743605,-10.74874979))

ktSp0<-matrix(c(15.37748425,15.13496541,10.90139961,10.10633573,7.99496542,
8.50765078,8.51660986,8.74775014,5.72720841,6.16431002,5.45252036,
4.63813971,4.97867074,4.93411264,3.86426106,3.49519295,2.89432250,
2.88387526,2.54488039,3.30014056,2.40095736,3.17155271,1.26422878,
1.54013499,1.21254339,0.94911910,0.22435912,0.06080328,-0.00646095,
-0.72906967,-1.69646937,-2.07633078,-3.16217920,-2.61421852,-3.05500418,
-2.71943185,-3.60851334,-3.46174428,-3.20796821,-2.90446384,-2.43216402,
-2.69072250,-3.66167456,-4.13174896,-4.68075254,-5.08395763,-5.15699638,
-6.65337847,-6.86189342,-7.24679188,-7.91769766,-8.90396021,-9.25828085,
-9.09491080,-10.58742581,-11.00374170,-12.37554294))

theta0<-c(axFr0,axIt0,axN10,axSp0,bx0[1:21],ktFr0[1:56],ktIt0[1:56],
ktN10[1:56],ktSp0[1:56])
ll.model<-optim(theta0,loglik,hessian=T,x=FrItNlSp,control=list(maxit=100000))

LogLikelihood<-ll.model$value
LogLikelihood

Convergence<-ll.model$convergence
Convergence

```

## 8.5 Model restricted to $\kappa_t$ females and males

### Females

```

Fr<-read.csv(file="C:\\france.csv",head=TRUE,sep=";")
attach(Fr)
It<-read.csv(file="C:\\italy.csv",head=TRUE,sep=";")
attach(It)
Nl<-read.csv(file="C:\\netherlands.csv",head=TRUE,sep=";")
attach(Nl)
Sp<-read.csv(file="C:\\spain.csv",head=TRUE,sep=";")
attach(Sp)

loglik<-function(theta,x){
  axFr<-theta[1:22]
  axIt<-theta[23:44]
  axNl<-theta[45:66]
  axSp<-theta[67:88]
  bxFr<-theta[89:109]
  bxFr<-c(bxFr,1-sum(bxFr))
  bxIt<-theta[110:130]
  bxIt<-c(bxIt,1-sum(bxIt))
  bxNl<-theta[131:151]
  bxNl<-c(bxNl,1-sum(bxNl))
  bxSp<-theta[152:172]
  bxSp<-c(bxSp,1-sum(bxSp))
  kt<-theta[173:228]
  kt<-c(kt,-sum(kt))

  one<-rep(1,length(kt))
  aFr<-axFr%*%t(one)
  aIt<-axIt%*%t(one)
  aNl<-axNl%*%t(one)
  aSp<-axSp%*%t(one)

  DxtFr<-matrix(DxtFFr/100,nrow=22)
  DxtIt<-matrix(DxtFIt/100,nrow=22)
  DxtNl<-matrix(DxtFNl/100,nrow=22)
  DxtSp<-matrix(DxtFSp/100,nrow=22)

  ExtFr<-matrix(ExtFFr/100,nrow=22)
  ExtIt<-matrix(ExtFIt/100,nrow=22)
  ExtNl<-matrix(ExtFNl/100,nrow=22)
  ExtSp<-matrix(ExtFSp/100,nrow=22)

  ll<-sum(colSums(DxtFr%*%t(aFr+bxFr%*%t(kt))-ExtFr%*%t(exp(aFr+bxFr%*%t(kt)))))+
    sum(colSums(DxtIt%*%t(aIt+bxIt%*%t(kt))-ExtIt%*%t(exp(aIt+bxIt%*%t(kt)))))+
    sum(colSums(DxtNl%*%t(aNl+bxNl%*%t(kt))-ExtNl%*%t(exp(aNl+bxNl%*%t(kt)))))+
    sum(colSums(DxtSp%*%t(aSp+bxSp%*%t(kt))-ExtSp%*%t(exp(aSp+bxSp%*%t(kt)))))

  return(-ll)
}

axFr0<-matrix(c(-4.5336584,-7.5060995,-8.3702416,-8.4522067,-7.7548003,
-7.4979048,-7.3547828,-7.0678867,-6.6804750,-6.2807173,-5.8659169,

```

```

-5.4864869,-5.1324817,-4.7329462,-4.2667481,-3.7214242,-3.1228198,
-2.5131285,-1.9422717,-1.4467931,-1.0464949,-0.6440435))

axIt0<-matrix(c(-4.1979663,-7.3707033,-8.2858675,-8.3665837,-7.9404959,
-7.7327114,-7.5396789,-7.2604046,-6.8999433,-6.4898840,-6.0382387,
-5.5947742,-5.1600993,-4.6840958,-4.1618295,-3.5868913,-2.9859148,
-2.4038647,-1.8651090,-1.3942158,-1.0032159,-0.6915007))

axNl0<-matrix(c(-4.7646219,-7.4878694,-8.4148084,-8.4760559,-8.0886569,
-7.9380057,-7.7244958,-7.3684024,-6.9627120,-6.4873747,-6.0012486,
-5.5758078,-5.1544783,-4.6885421,-4.1737722,-3.6151614,-3.0296650,
-2.4449137,-1.8983456,-1.4191059,-1.0019347,-0.6762116))

axSp0<-matrix(c(-4.2585118,-7.2116195,-8.1208549,-8.2330059,-7.8394862,
-7.5940096,-7.3601497,-7.1043306,-6.7938808,-6.4322849,-6.0409136,
-5.6357697,-5.2329870,-4.7491375,-4.2282566,-3.6260439,-3.0186969,
-2.4410770,-1.9077245,-1.4808792,-1.0917044,-0.8259499))

bxFr0<-matrix(c(-0.02814830,-0.03702005,-0.06588364,-0.07911846,-0.09154635,
-0.08734098,-0.08257363,-0.08546004,-0.08951054,-0.09347024,-0.09394742,
-0.09211971,-0.08961577,-0.08562356,-0.08180338,-0.07984893,-0.08158660,
-0.08841981,-0.09741398,-0.10652574,-0.11358619,-0.10943668))

bxIt0<-matrix(c(-0.01810624,-0.01044953,-0.04377010,-0.05895628,-0.07352683,
-0.06874747,-0.06736275,-0.07043614,-0.07418392,-0.07943640,-0.08291066,
-0.08455326,-0.08479366,-0.08288097,-0.08054738,-0.07929476,-0.08147614,
-0.08649366,-0.09358723,-0.10088720,-0.10648197,-0.11111743))

bxNl0<-matrix(c(-0.008121489,0.019342615,0.002954137,-0.034507399,-0.055384552,
-0.054506635,-0.054083665,-0.056799293,-0.060320522,-0.069297552,-0.074077411,
-0.073620897,-0.070950791,-0.065095943,-0.056745320,-0.052130295,-0.051876732,
-0.058508461,-0.067973643,-0.079005360,-0.086292832,-0.092997960))

bxSp0<-matrix(c(0.002202538,0.002502365,-0.020597795,-0.036285296,-0.052289923,
-0.044188922,-0.041733771,-0.048326449,-0.052659040,-0.057606947,-0.061299713,
-0.060078738,-0.059765208,-0.056207110,-0.056297886,-0.055814136,-0.062149638,
-0.071787177,-0.080314073,-0.092214396,-0.093033479,-0.102055203))

kt0<-matrix(c(8.24368998,8.42435296,7.13159389,6.92139824,5.84068356,
5.62600988,5.73160578,5.61136598,4.42120351,4.35316478,4.03956960,
3.48840093,3.88448147,3.83891266,2.87633412,2.90704177,2.39809911,
2.41910793,2.37165415,2.46083542,1.91886237,1.85550868,1.54064027,
1.51033464,0.88843225,0.78168845,0.45229521,0.03200915,-0.19688872,
-0.55608315,-0.54005462,-0.99715657,-1.42839440,-1.26844489,-1.92552167,
-2.00608151,-2.19674377,-2.61383323,-2.75359894,-2.85299975,-3.00554196,
-3.07344790,-3.38289553,-3.40103118,-3.78920957,-3.87246420,-4.10344523,
-4.56197120,-4.83181743,-5.00493078,-5.30723606,-5.59786644,-5.74546142,
-5.88165511,-6.82459217,-6.89412628,-7.35578306))

theta0<-c(axFr0,axIt0,axNl0,axSp0,bxFr0[1:21],bxIt0[1:21],bxNl0[1:21],
bxSp0[1:21],kt0[1:56])
ll.model<-optim(theta0,loglik,hessian=T,x=FrItNlSp,control=list(maxit=100000))

LogLikelihood<-ll.model$value

```

```

LogLikelihood

Convergence<-ll.model$convergence
Convergence

Males

Fr<-read.csv(file="C:\\france.csv",head=TRUE,sep=";")
attach(Fr)
It<-read.csv(file="C:\\italy.csv",head=TRUE,sep=";")
attach(It)
Nl<-read.csv(file="C:\\netherlands.csv",head=TRUE,sep=";")
attach(Nl)
Sp<-read.csv(file="C:\\spain.csv",head=TRUE,sep=";")
attach(Sp)

loglik<-function(theta,x){
  axFr<-theta[1:22]
  axIt<-theta[23:44]
  axNl<-theta[45:66]
  axSp<-theta[67:88]
  bxFr<-theta[89:109]
  bxFr<-c(bxFr,1-sum(bxFr))
  bxIt<-theta[110:130]
  bxIt<-c(bxIt,1-sum(bxIt))
  bxNl<-theta[131:151]
  bxNl<-c(bxNl,1-sum(bxNl))
  bxSp<-theta[152:172]
  bxSp<-c(bxSp,1-sum(bxSp))
  kt<-theta[173:228]
  kt<-c(kt,-sum(kt))

  one<-rep(1,length(kt))
  aFr<-axFr%*%t(one)
  aIt<-axIt%*%t(one)
  aNl<-axNl%*%t(one)
  aSp<-axSp%*%t(one)

  DxtFr<-matrix(DxtMFr/100,nrow=22)
  DxtIt<-matrix(DxtMIT/100,nrow=22)
  DxtNl<-matrix(DxtMNl/100,nrow=22)
  DxtSp<-matrix(DxtMSp/100,nrow=22)

  ExtFr<-matrix(ExtMFr/100,nrow=22)
  ExtIt<-matrix(ExtMIT/100,nrow=22)
  ExtNl<-matrix(ExtMNl/100,nrow=22)
  ExtSp<-matrix(ExtMSp/100,nrow=22)

  ll<-sum(colSums(DxtFr%*%t(aFr+bxFr%*%t(kt))-ExtFr%*%t(exp(aFr+bxFr%*%t(kt)))))+
  sum(colSums(DxtIt%*%t(aIt+bxIt%*%t(kt))-ExtIt%*%t(exp(aIt+bxIt%*%t(kt)))))+
  sum(colSums(DxtNl%*%t(aNl+bxNl%*%t(kt))-ExtNl%*%t(exp(aNl+bxNl%*%t(kt)))))+
  sum(colSums(DxtSp%*%t(aSp+bxSp%*%t(kt))-ExtSp%*%t(exp(aSp+bxSp%*%t(kt)))))

  return(-ll)
}

```

```

axFr0<-matrix(c(-4.2642961,-7.2881359,-8.0673953,-8.0296754,-6.9304038,
-6.5344866,-6.4874966,-6.3149804,-5.9879343,-5.5564849,-5.1009216,
-4.6724655,-4.2731873,-3.8784187,-3.4795521,-3.0492509,-2.5903475,
-2.1106935,-1.6446770,-1.2224779,-0.8961878,-0.4623281))

axIt0<-matrix(c(-4.0033412,-7.2607653,-7.9919661,-7.9205202,-7.0112705,
-6.7671562,-6.7249913,-6.5813010,-6.3231858,-5.9161352,-5.4316771,
-4.9256820,-4.4364557,-3.9708024,-3.5209298,-3.0581373,-2.5813588,
-2.0987794,-1.6424462,-1.2264719,-0.8692156,-0.5715599))

axNl0<-matrix(c(-4.5067964,-7.2477738,-7.9881758,-8.1179550,-7.3238179,
-7.0759024,-7.0955331,-6.9434322,-6.6166981,-6.1417328,-5.6110750,
-5.0715717,-4.5476911,-4.0376774,-3.5449828,-3.0651769,-2.5845735,
-2.1202109,-1.6700493,-1.2587190,-0.8722722,-0.5432713))

axSp0<-matrix(c(-4.0377526,-7.0472863,-7.8084569,-7.8705387,-7.0718650,
-6.7078716,-6.5677050,-6.3929851,-6.1434235,-5.7908761,-5.3698493,
-4.9169433,-4.4736126,-4.0146644,-3.5598531,-3.0741099,-2.5892952,
-2.1248484,-1.6803884,-1.3162565,-0.9820019,-0.7994526))

bxFr0<-matrix(c(0.018999468,0.002487043,-0.021784967,-0.042511233,-0.072073503,
-0.084095236,-0.078960703,-0.077866709,-0.077269523,-0.077889586,-0.077569432,
-0.073616444,-0.069680416,-0.067733288,-0.066590662,-0.066298696,-0.067012126,
-0.072002227,-0.078522026,-0.087280083,-0.097364769,-0.085364884))

bxIt0<-matrix(c(0.032289081,0.035862239,-0.001940854,-0.023415684,-0.060117511,
-0.070153246,-0.069775473,-0.066352166,-0.060752788,-0.057022893,-0.055459738,
-0.057250568,-0.061176510,-0.065298853,-0.068763875,-0.071014077,-0.072195413,
-0.073072379,-0.076162026,-0.080623174,-0.084541973,-0.093062117))

bxNl0<-matrix(c(-0.04302736,-0.01118951,-0.01817998,-0.05665866,-0.09322012,
-0.10169975,-0.10560301,-0.11307681,-0.11360109,-0.11269554,-0.11127331,
-0.10938798,-0.11028248,-0.11612502,-0.12343997,-0.13049967,-0.13397399,
-0.13655291,-0.13730473,-0.13994884,-0.13995933,-0.14229995))

bxSp0<-matrix(c(0.068279430,0.062791663,0.027400835,0.004667998,-0.039003640,
-0.043425871,-0.039574027,-0.045351841,-0.047184050,-0.046836019,-0.048817537,
-0.047109770,-0.046351554,-0.043669972,-0.044386793,-0.042521023,-0.044459490,
-0.050355624,-0.054622835,-0.068391077,-0.071474630,-0.099604175))

kt0<-matrix(c(7.40949748,7.41436507,6.03046873,5.96851690,4.80975839,
4.84939442,5.20722696,4.98497227,3.69874914,3.72958124,3.69211355,
3.32729469,3.75251136,3.76883279,2.93804667,3.03878679,2.46353419,
2.45216004,2.60520422,2.70914455,2.24236740,2.26143834,1.78683537,
1.90558517,1.36385587,1.33162406,1.00199450,0.69997304,0.52461533,
0.18600004,0.07589075,-0.36349586,-0.86846513,-0.70437471,-1.33302423,
-1.41806269,-1.76774358,-2.18012275,-2.25392576,-2.35732159,-2.26456353,
-2.29282274,-2.72337936,-2.91075772,-3.44151938,-3.55190979,-3.90242528,
-4.60050064,-4.96259322,-5.31812248,-5.71358489,-5.97810921,-6.32681777,
-6.63185842,-7.77560706,-7.97659746,-8.61263396))

theta0<-c(axFr0,axIt0,axNl0,axSp0,bxFr0[1:21],bxIt0[1:21],bxNl0[1:21],
bxSp0[1:21],kt0[1:56])

```

```
ll.model<-optim(theta0,loglik,hessian=T,x=FrItNlSp,control=list(maxit=100000))

LogLikelihood<-ll.model$value
LogLikelihood

Convergence<-ll.model$convergence
Convergence
```

## 8.6 Model restricted to $\alpha_x$ , $\beta_x$ and $\kappa_t$ females and males

### Females

```

Fr<-read.csv(file="C:\\france.csv",head=TRUE,sep=";")
attach(Fr)
It<-read.csv(file="C:\\italy.csv",head=TRUE,sep=";")
attach(It)
Nl<-read.csv(file="C:\\netherlands.csv",head=TRUE,sep=";")
attach(Nl)
Sp<-read.csv(file="C:\\spain.csv",head=TRUE,sep=";")
attach(Sp)

loglik<-function(theta,x){
  ax<-theta[1:22]
  bx<-theta[23:43]
  bx<-c(bx,1-sum(bx))
  kt<-theta[44:99]
  kt<-c(kt,-sum(kt))

  one<-rep(1,length(kt))
  a<-ax%*%t(one)

  DxtFr<-matrix(DxtFFr/100,nrow=22)
  DxtIt<-matrix(DxtFIt/100,nrow=22)
  DxtNl<-matrix(DxtFNl/100,nrow=22)
  DxtSp<-matrix(DxtFSp/100,nrow=22)
  Dxt<-DxtFr+DxtIt+DxtNl+DxtSp

  ExtFr<-matrix(ExtFFr/100,nrow=22)
  ExtIt<-matrix(ExtFIt/100,nrow=22)
  ExtNl<-matrix(ExtFNl/100,nrow=22)
  ExtSp<-matrix(ExtFSp/100,nrow=22)
  Ext<-ExtFr+ExtIt+ExtNl+ExtSp

  ll<-sum(colSums(Dxt%*%t(a+bx%*%t(kt))-Ext%*%t(exp(a+bx%*%t(kt)))))

  return(-ll)
}

ax0<-matrix(c(-4.3517278,-7.3669846,-8.2695969,-8.3589322,-7.8488373,
-7.6199908,-7.4371085,-7.1578999,-6.7985991,-6.3987215,-5.9710319,
-5.5606902,-5.1629180,-4.7130945,-4.2115066,-3.6433957,-3.0445857,
-2.4545040,-1.9057266,-1.4365057,-1.0455664,-0.7226119))

bx0<-matrix(c(-0.01622118,-0.01543498,-0.04352775,-0.05994233,-0.07548065,
-0.07006364,-0.06728030,-0.07167931,-0.07624606,-0.08150322,-0.08415775,
-0.08354450,-0.08241646,-0.07925217,-0.07686681,-0.07568832,-0.07865122,
-0.08527881,-0.09322626,-0.10220721,-0.10798104,-0.11335002))

kt0<-matrix(c(16.48737995,16.84870592,14.26318778,13.84279648,11.68136711,
11.25201975,11.46321156,11.22273195,8.84240701,8.70632957,8.07913921,
6.97680186,7.76896295,7.67782531,5.75266823,5.81408353,4.79619821,
4.83821587,4.74330831,4.92167085,3.83772473,3.71101735,3.08128055,
3.02066929,1.77686450,1.56337691,0.90459042,0.06401831,-0.39377744,

```

```

-1.11216629,-1.08010924,-1.99431313,-2.85678880,-2.53688978,-3.85104335,
-4.01216302,-4.39348755,-5.22766646,-5.50719788,-5.70599950,-6.01108393,
-6.14689579,-6.76579107,-6.80206237,-7.57841914,-7.74492840,-8.20689046,
-9.12394240,-9.66363487,-10.00986156,-10.61447212,-11.19573287,-11.49092283,
-11.76331021,-13.64918434,-13.78825257,-14.71156612))

theta0<-c(ax0,bx0[1:21],kt0[1:56])
ll.model<-optim(theta0,loglik,hessian=T,x=FrItNlSp,control=list(maxit=100000))

LogLikelihood<-ll.model$value
LogLikelihood

Convergence<-ll.model$convergence
Convergence

Males

Fr<-read.csv(file="C:\\france.csv",head=TRUE,sep=";")
attach(Fr)
It<-read.csv(file="C:\\italy.csv",head=TRUE,sep=";")
attach(It)
Nl<-read.csv(file="C:\\netherlands.csv",head=TRUE,sep=";")
attach(Nl)
Sp<-read.csv(file="C:\\spain.csv",head=TRUE,sep=";")
attach(Sp)

loglik<-function(theta,x){
  ax<-theta[1:22]
  bx<-theta[23:43]
  bx<-c(bx,1-sum(bx))
  kt<-theta[44:99]
  kt<-c(kt,-sum(kt))

  one<-rep(1,length(kt))
  a<-ax%*%t(one)

  DxtFr<-matrix(DxtMFr/100,nrow=22)
  DxtIt<-matrix(DxtMIT/100,nrow=22)
  DxtNl<-matrix(DxtMNl/100,nrow=22)
  DxtSp<-matrix(DxtMSp/100,nrow=22)
  Dxt<-DxtFr+DxtIt+DxtNl+DxtSp

  ExtFr<-matrix(ExtMFr/100,nrow=22)
  ExtIt<-matrix(ExtMIT/100,nrow=22)
  ExtNl<-matrix(ExtMNl/100,nrow=22)
  ExtSp<-matrix(ExtMSp/100,nrow=22)
  Ext<-ExtFr+ExtIt+ExtNl+ExtSp

  ll<-sum(colSums(Dxt%*%t(a+bx%*%t(kt))-Ext%*%t(exp(a+bx%*%t(kt)))))

  return(-ll)
}

ax0<-matrix(c(-4.1221507,-7.1963862,-7.9593439,-7.9523302,-7.0124908,
-6.6835114,-6.6238091,-6.4607874,-6.1758586,-5.7667060,-5.3036172,

```

```

-4.8379047,-4.3917915,-3.9512193,-3.5144188,-3.0569006,-2.5849625,
-2.1094486,-1.6534537,-1.2462158,-0.9055158,-0.6015493))

bx0<-matrix(c(0.0371364406,0.0309653061,0.0005653156,-0.0223959902,-0.0601398549,
-0.0694324426,-0.0666929963,-0.0671010981,-0.0657233285,-0.0645070863,-0.0639985882,
-0.0620777844,-0.0614830316,-0.0621078732,-0.0638668926,-0.0648770850,-0.0662205702,
-0.0693403281,-0.0735109054,-0.0809975395,-0.0876408156,-0.0965528519))

kt0<-matrix(c(11.3992269,11.4067155,9.2776442,9.1823337,7.3996283,
7.4606068,8.0111184,7.6691881,5.6903833,5.7378173,5.6801747,
5.1189149,5.7730944,5.7982043,4.5200718,4.6750566,3.7900526,
3.7725539,4.0080065,4.1679147,3.4497960,3.4791359,2.7489775,
2.9316695,2.0982398,2.0486524,1.5415300,1.0768816,0.8071005,
0.2861539,0.1167550,-0.5592244,-1.3361002,-1.0836534,-2.0508065,
-2.1816349,-2.7196055,-3.3540350,-3.4675781,-3.6266486,-3.4839439,
-3.5274196,-4.1898144,-4.4780888,-5.2946452,-5.4644766,-6.0037312,
-7.0776933,-7.6347588,-8.1817269,-8.7901306,-9.1970911,-9.7335658,
-10.2028591,-11.9624724,-12.2716884,-13.2502061))

theta0<-c(ax0,bx0[1:21],kt0[1:56])
ll.model<-optim(theta0,loglik,hessian=T,x=FrItNlSp,control=list(maxit=100000))

LogLikelihood<-ll.model$value
LogLikelihood

Convergence<-ll.model$convergence
Convergence

```