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On duality transformations of the electromagnetic field tensor in non-linear modifications of classical electrodynamics

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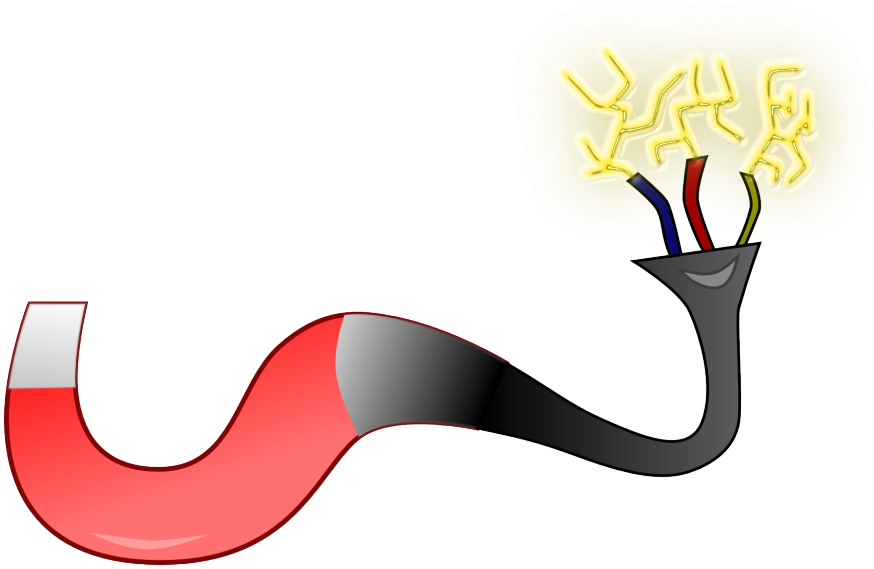

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Abstract

A review of $SO(2)$ duality transformations in classical electrodynamics is given, followed by an investigation of constraints on general theories that possess a similar duality invariance. As an example, Born-Infeld theory is shown to possess such a symmetry. The general constraints on a theory of one field are the indepence of the Lagrangian density on the potential and a first-order partial differential equation of the Lagrangian density in the field strength tensor.

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Chapter 1

Introduction

It is very well known that throughout the history of physics, symmetries and conservation laws have played an enormous role.[1] In classical mechanics this can be seen by the conservation of linear and angular momentum and energy and the formulation of the theory that is independent of location and orientation in time and space. These principles are now being considered fundamental properties of Nature, as opposed to being nice properties of a theory. Any physical theory that does not possess such symmetries or conservations are nowadays looked at with great scepticism. As modern physics developed, notably with Quantum Mechanics and Special and General Relativity, new theories all possessed similar symmetries and conservation laws and even new such properties were introduced. For example, we can think of Lorentz invariance in Relativity and probability current conservation in Quantum Mechanics. In many ways we may even assert that these modern theories were developed with these symmetry and conservation properties as a requirement.

Currently, many theoretical models exist in an attempt to describe Nature in the most precise way. Some of these try this by introducing entirely new concepts, but a great deal of them – if not most – try to accomplish this by modifying and unifying existing theories that have proven to be quite successful in the past and are able to explain fundamental experimental facts.

In this thesis, we will discuss modifications of the classical theory of Electrodynamics. In the upcoming sections we will do a quick recap of the necessary theory and discuss an important symmetry in the classical theory: the hodge duality of the electromagnetic field tensor. The importance of this duality may not be clear at first, but we will make an attempt to make this clear.[2] We will then discuss a particular way to modify this theory, keeping in mind what we have discussed above: it is generally desirable to retain nice properties in modifications of an existing theory. In this case we will focus on retaining the duality of the electromagnetic field tensor.

[1] The role of symmetries of the action and the conservation law are connected via Noether's first theorem, which states that any (differentiable) symmetry of the action is associated with a conservation law.

[2] The classic duality symmetry can be researched in two cases: one with no charges and one with both magnetic and electric charge. In the latter case, the duality transformation is the mechanism that makes the statement "there are no magnetic monopoles" possible.

1.1 Setting the stage: spacetime

In order to work in a precise and systematic way we will introduce a concise mathematical framework in which we will investigate the properties of the electromagnetic fields. The framework is a model for the union of space and time we live in, and we will call it spacetime. For a more precise treatment of these matters, I refer the reader to the book "Analysis, Manifolds and Physics" [3], which describes exactly what the title advertises. Readers familiar with spacetime and vector calculus may skip this and the following section and continue with the next chapter.

We shall start of with a concise definition:

Definition 1. Spacetime *is a hyperbolic 4-dimensional differentiable pseudo-Euclidean Riemannian manifold M with metric g, that has index 1.*

For the reader that is not comfortable with calculus on manifolds, a concise summary of the basic concepts is given in appendix A. Here, we only note that we can have tensors with components up (contravariant) and down (covariant), and that we have a metric g (which is a 2-covariant tensor), with two components that is symmetric (i.e. invariant under the exchange of two indices) and non-degenerate. This metric induces an isomorphism between the spaces of contravariant and covariant vector fields at a point. This last statement can be read as: *we may use the metric to raise and lower indices.*

Most of the time we will work in flat spacetime or *Minkowski Space*. This space is isometric to $\mathbb{R}^4$ with the metric:

$$[g_{\mu\nu}] = [\eta_{\mu\nu}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \tag{1.1}$$

This metric makes raising or lowering indices particularly easy.

1.2 Vector calculus

Our model of spacetime will at each point have local coordinates, given by a chart at that point. We will denote these coordinates by $x = (x^0, \mathbf{x}) = (ct, x, y, z)$, where c is the well-known speed of light. Also, at each point, the tangens space of contravariant vectors will be isomorphic to $\mathbb{R}^4$. Again, we will take $v = (v^0, \mathbf{v}) = (cv_t, v_x, v_y, v_z)$. This will enable us to do vector calculus in the regular sense of the word on local regions on the manifold. A 4-vector field is then a mapping $x \mapsto v_x$, and in a chart, we can take partial derivatives with respect to the coordinate functions. We will define a regular vector field to be a mapping from M to $\mathbb{R}^3$ and a scalar field to be a mapping from M to $\mathbb{R}$. Although this is mathematically not the most pure way to do it, it will suffice for our purposes, as we will soon not need this notation anymore. Note that we will regard the triple of the last three coordinates of a tangent vector as a vector in $\mathbb{R}^3$. This will simplify the math greatly.

On fields in $\mathbb{R}^3$, we have the well-known operator ∇. This operator works

on scalar fields as follows:

$$\nabla\Phi = \left(\frac{\partial\Phi}{\partial x}, \frac{\partial\Phi}{\partial y}, \frac{\partial\Phi}{\partial z}\right)^T \tag{1.2}$$

On vector fields, we have the divergence and curl operations:

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}, \tag{1.3}$$

$$\nabla \times \mathbf{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}, \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}, \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)^T. \tag{1.4}$$

Lastly, we will introduce the Laplacian operator ∇^2:

$$\begin{aligned} \nabla^2\Phi &= \nabla \cdot (\nabla\Phi) \\ \nabla^2\mathbf{A} &= \left(\nabla^2 A_x, \nabla^2 A_y, \nabla^2 A_z\right)^T \end{aligned} \tag{1.5}$$

We will use the following identities:

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0 \tag{1.6a}$$

$$\nabla \times (\nabla\Phi) = 0 \tag{1.6b}$$

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A} \tag{1.6c}$$

Chapter 2

Classical electromagnetism

2.1 Maxwell's Equations

Perhaps the most famous set of equations was published by James Clerk Maxwell in his paper *A Dynamical Theory of the Electromagnetic Field* in 1865, in The Philosophical Transactions of the Royal Society of London. They were twenty equations describing the general behaviour of the electric field. Modern vector notation now allows us to rewrite these into four simple equations. This was made possible due to the work of Oliver Heaviside in 1884. [2] Without further adue, I will present them to you, perhaps as a reminder:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{2.1a}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{2.1b}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{2.1c}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t}. \tag{2.1d}$$

Here, $\mathbf{E}$ and $\mathbf{B}$ are vector fields $M \to \mathbb{R}^3$, that are called the *electric field* and *magnetic induction* respectively. The symbols μ_0 and ϵ_0 are constants of nature called the *magnetic* and *electric permeability of the vacuum* respectively. These constants combine to form the speed of light in the following way:

$$c = \frac{1}{\epsilon_0 \mu_0}. \tag{2.2}$$

The scalar field ρ is called the *charge density* and the vector field $\mathbf{j}$ is the *current density*. These equations may be supplemented with the *continuity equation*:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0. \tag{2.3}$$

These equations describe the behaviour of electromagnetism in full, when there is no matter present in the region.

We will immediately put these equations into a form that is more natural to our model of spacetime. We will show that these equations are compatible with special relativity, which plays in Minkowski Space. First, we note that

equation 2.1c means that the field $\mathbf{B}$ is a so called solenoidal field. Under not so restrictive conditions, which we assume to be satisfied, this implies that there exists a vector field $\mathbf{A}$, called the *vector potential* such that:

$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{2.4}$$

This field will then automatically (see equation 1.6a) satisfy equation 2.1c. Also, equation 2.1b now implies:

$$0 = \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = \nabla \times \mathbf{E} + \nabla \times \frac{\partial \mathbf{A}}{\partial t} = \nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right).$$

Therefore, the field $\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}$ is *conservative*, which implies the existence of a scalar field Φ, called the *scalar electric potential*, such that:

$$\mathbf{E} = -\nabla \Phi - \frac{\partial \mathbf{A}}{\partial t} \tag{2.5}$$

So, equations 2.1b and 2.1c are equivalent with the existence of the vector and scalar potential. Now, equation 2.1a is then equivalent to:

$$\begin{aligned} c^2 \mu_0 \rho = \frac{\rho}{\epsilon_0} = \nabla \cdot \mathbf{E} &= \nabla \cdot \left(-\nabla \Phi - \frac{\partial \mathbf{A}}{\partial t} \right) \\ &= -\nabla \cdot (\nabla \Phi) - \nabla \cdot \frac{\partial \mathbf{A}}{\partial t} \\ &= -\nabla^2 \Phi - \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) + \frac{1}{c^2} \frac{\partial}{\partial t} \frac{\partial \Phi}{\partial t} - \frac{1}{c^2} \frac{\partial}{\partial t} \frac{\partial \Phi}{\partial t} \\ &= \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \Phi - \nabla^2 \Phi - \frac{\partial}{\partial t} \left(\frac{1}{c^2} \frac{\partial \Phi}{\partial t} + \nabla \cdot \mathbf{A} \right) \\ &= \Box \Phi - \frac{\partial}{\partial t} \left(\frac{1}{c^2} \frac{\partial \Phi}{\partial t} + \nabla \cdot \mathbf{A} \right) \end{aligned} \tag{2.6}$$

and equation 2.1d is equivalent to:

$$\begin{aligned} \mu_0 \mathbf{j} = \nabla \times \mathbf{B} - \epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} &= \nabla \times (\nabla \times \mathbf{A}) - \frac{1}{c^2} \frac{\partial}{\partial t} \left(-\nabla \Phi - \frac{\partial \mathbf{A}}{\partial t} \right) \\ &= \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} + \frac{1}{c^2} \nabla \frac{\partial \Phi}{\partial t} + \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} \\ &= \Box \mathbf{A} + \nabla \left(\frac{1}{c^2} \frac{\partial \Phi}{\partial t} + \nabla \cdot \mathbf{A} \right). \end{aligned} \tag{2.7}$$

Here we have used $\Box$ as the symbol for the d'Alembertian operator:

$$\Box := \partial^\mu \partial_\mu = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \tag{2.8}$$

Note that these equations are very similar. In fact, this motivates us to define the following contravariant vector fields on M: $j = (c\rho, \mathbf{j})$ and $A = (\frac{\Phi}{c}, \mathbf{A})$. We now see that we can rewrite the equations as follows:

$$\begin{cases} j^0 = \dfrac{1}{\mu_0} \left(\Box \dfrac{\Phi}{c} - \dfrac{\partial}{c \partial t} \left[\dfrac{\partial}{c \partial t} \dfrac{\Phi}{c} + \nabla \cdot \mathbf{A} \right] \right) \\ j^i = \dfrac{1}{\mu_0} \left(\Box \mathbf{A} + \nabla \left[\dfrac{\partial}{c \partial t} \dfrac{\Phi}{c} + \nabla \cdot \mathbf{A} \right] \right) \end{cases} \tag{2.9}$$

or in better suiting notation:

$$j^\mu = \frac{1}{\mu_0}\left(\Box A^\mu - \partial^\mu(\partial_\nu A^\nu)\right) \tag{2.10}$$

Note that with these contravariant vectors defined, we may easily write down the continuity equation in a compact form:

$$\partial_\mu j^\mu = 0 \tag{2.11}$$

2.2 The Field Strength Tensor

We have now shown that Maxwell's Equations are entirely equivalent to the existince of a contravariant tensor field A, such that:

$$j^\mu = \frac{1}{\mu_0}\left(\Box A^\mu - \partial^\mu(\partial_\nu A^\nu)\right).$$

We have, however no easy way of getting the components of getting the electric field and magnetic induction. Initially for this purpose, we will define a particular tensor, the *field strength tensor*:

$$F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{2.12}$$

Note that this is the exterior derivative of the 1-form $A_\mu = \eta_{\mu\nu}A^\nu$ and thus is a 2-form, or an antisymmetric 2-covariant tensor field. We quickly see:

$$\begin{aligned}
\mathbf{B} = \nabla \times \mathbf{A} &= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}, \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}, \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \\
&= \left(\partial_2 A^3 - \partial_3 A^2, \partial_3 A^1 - \partial_1 A^3, \partial_1 A^2 - \partial_2 A^1\right) \\
&= (\partial_3 A_2 - \partial_2 A_3, \partial_1 A_3 - \partial_3 A_1, \partial_2 A_1 - \partial_1 A_2,) \\
&= (F_{32}, F_{13}, F_{21}) = (-F_{23}, -F_{31}, -F_{12}) \\
\frac{\mathbf{E}}{c} = -\frac{\partial \mathbf{A}}{c\partial t} - \nabla\frac{\Phi}{c} &= -\partial_0 \mathbf{A} - \nabla A^0 \\
&= \left(-\partial_0 A^1 - \partial_1 A^0, -\partial_0 A^2 - \partial_2 A^0, -\partial_0 A^3 - \partial_3 A^0\right) \\
&= (\partial_0 A_1 - \partial_1 A_0, \partial_0 A_2 - \partial_2 A_0, \partial_0 A_3 - \partial_3 A_0) \\
&= (F_{01}, F_{02}, F_{03}) = (-F_{10}, -F_{20}, -F_{30}),
\end{aligned}$$

such that:

$$[F_{\mu\nu}] = \begin{pmatrix} 0 & \frac{E_x}{c} & \frac{E_y}{c} & \frac{E_z}{c} \\ -\frac{E_x}{c} & 0 & -B_z & B_y \\ -\frac{E_y}{c} & B_z & 0 & -B_x \\ -\frac{E_z}{c} & -B_y & B_x & 0 \end{pmatrix} \tag{2.13}$$

Furthermore, we immediately see that:

$$\begin{aligned}
-\partial_\nu F^{\mu\nu} = \partial_\nu F^{\nu\mu} &= \partial_\nu\left(\partial^\nu A^\mu - \partial^\mu A^\nu\right) \\
&= \partial_\nu \partial^\nu A^\mu - \partial^\mu \partial_\nu A^\nu = \Box A^\mu - \partial^\mu \partial_\nu A^\nu
\end{aligned} \tag{2.14}$$

We can now finally write down the full set of Maxwell's Equations in a very concise way[1]:

$$\partial_{[\lambda}F_{\mu\nu]} = 0 \iff \begin{cases} \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t} \end{cases} \tag{2.15a}$$

$$\partial_\nu F^{\mu\nu} = -\Box A^\mu + \partial^\mu(\partial_\nu A^\nu) = -\mu_0 j^\mu \iff \begin{cases} \nabla \cdot \mathbf{E} = \dfrac{\rho}{\epsilon_0} \\ \nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \epsilon_0\mu_0 \dfrac{\partial \mathbf{E}}{\partial t} \end{cases} \tag{2.15b}$$

Advanced readers may notice that equation 2.15a is automatically satisfied for any antisymmetric 2-covariant tensor (2-form) defined as the exterior derivative of a 1-form, so in particular for the field strength tensor.

2.3 Lagrangian Formalism

In modern theories, Lagrangian formalisms are used extensively to define a new theory. We will deduce a formalism for the classical electromagnetic theory that can be classified as a field theory. The field of which we will define the behaviour will be the covariant vector potential A_μ, instead of the classical fields $\mathbf{E}$ and $\mathbf{B}$, since we can define these two in terms of the vector potential.

The field theory formalism uses a Lagrangian density $\mathcal{L}$ defined as a function on the components and first derivatives of the field. Then an action is defined as:

$$S := \int \mathcal{L}(A_\mu, \partial_\mu A_\nu)\ d^4x, \tag{2.16}$$

where the integration is over an arbtirary region. Next, we vary the field randomly (with the field value fixed on the border) and determine for which field the action takes on an extreme value. This yields the Euler-Lagrange equations for the field:

$$\partial_\nu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)}\right] = \frac{\partial \mathcal{L}}{\partial A_\mu} \tag{2.17}$$

For the theory of electromagnetism we propose the following Lagrangian density:

$$\mathcal{L} := -\frac{1}{4\mu_0} F^{\mu\nu}F_{\mu\nu} - A_\mu j^\nu \tag{2.18}$$

[1] In the first equation, we use antisymmetric notation for typographic convenience. This notation means: $\partial_{[i_1}F_{i_2i_3]} = \frac{1}{3!}\epsilon^{j_1j_2j_3}_{i_1i_2i_3}\partial_{j_1}F_{j_2j_3}$. This type of identity is called a *Bianchi identity* and it is fully equivalent with the fact that F is actually the outer derivative of a potential 1-form. As is noted further on, the equation following more directly from Maxwell's Equations is in the form of cyclic sums, and may therefore arguably better be displayed here, but this notation is more concise, is more frequently used in technical texts and stresses the way F is (or better, must be) defined. We therefore prefer to use this notation.

Taking the Euler-Lagrange equations 2.17 then yields:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial(\partial_\beta A_\alpha)} &= \frac{\partial}{\partial A_{\alpha,\beta}}\left[-\frac{1}{4\mu_0}F_{\rho\sigma}g^{\rho\mu}g^{\sigma\nu}F_{\mu\nu} - A_\mu j^\nu\right] \\
&= -\frac{1}{4\mu_0}\frac{\partial}{\partial A_{\alpha,\beta}}\left[(A_{\sigma,\rho} - A_{\rho,\sigma})g^{\rho\mu}g^{\sigma\nu}(A_{\nu,\mu} - A_{\mu,\nu})\right] \\
&= -\frac{1}{4\mu_0}\frac{\partial}{\partial A_{\alpha,\beta}}\left[A_{\sigma,\rho} - A_{\rho,\sigma}\right]g^{\rho\mu}g^{\sigma\nu}(A_{\nu,\mu} - A_{\mu,\nu}) \\
&\quad - \frac{1}{4\mu_0}\frac{\partial}{\partial A_{\alpha,\beta}}\left[A_{\nu,\mu} - A_{\mu,\nu}\right]g^{\rho\mu}g^{\sigma\nu}(A_{\sigma,\rho} - A_{\rho,\sigma}) \\
&= -\frac{1}{2\mu_0}\left[F^{\beta\alpha} - F^{\alpha\beta}\right] \\
&= \frac{1}{\mu_0}F^{\alpha\beta} = -\frac{1}{\mu_0}F^{\beta\alpha} \\
\frac{\partial \mathcal{L}}{\partial A_\alpha} &= \frac{\partial}{\partial A_\alpha}\left[-\frac{1}{4\mu_0}F^{\mu\nu}F_{\mu\nu} - A_\mu j^\mu\right] = -j^\alpha \\
&\implies \quad \boxed{\mu_0 j^\mu = \partial_\nu F^{\nu\mu}}
\end{aligned} \tag{2.19}$$

This is indeed equation 2.15b that we derived earlier and since equation 2.15a is automatically satisfied, we see that the field derived from this Lagrangian density is entirely equivalent to the classical theory.

The continuity equation is actually also satisfied by noting that the Lagrangian density and therefore the action is invariant under a so-called gauge transformation of the potential 4-vector. Noether's Theorem then yields a conserved current, which is the charge density 4-vector.

In the next chapter we will explore a Lagrangian density known as the Born-Infeld Lagrangian, where we will see that the field derived from this modified theory shares features with the classical theory.

2.4 Hodge duality

Now, we come to the crux of the case: the hodge duality of the electromagnetic field tensor. Formally, we may define an operation on antisymmetric tensor fields (actually on the whole exterior algebra of p-forms on a manifold), called the *Hodge star* ($\star$). For our purposes, this operation will bijectively and linearly map an antisymmetric 2-covariant tensor field into another antisymmetric 2-covariant tensor field. It can be defined by the following relation:

$$\star F_{\mu\nu} = \frac{1}{2}\tau_{\rho\sigma\mu\nu}F^{\rho\sigma}. \tag{2.20}$$

The definition given here, actually defines the *dual field strength tensor*. Using this definition and assuming a flat metric, we see:

$$[\star F_{\mu\nu}] = \begin{bmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & -\frac{E_z}{c} & \frac{E_y}{c} \\ B_y & \frac{E_z}{c} & 0 & -\frac{E_x}{c} \\ B_z & -\frac{E_y}{c} & \frac{E_x}{c} & 0 \end{bmatrix} \quad [\star F^{\mu\nu}] = \begin{bmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -\frac{E_z}{c} & \frac{E_y}{c} \\ -B_y & \frac{E_z}{c} & 0 & -\frac{E_x}{c} \\ -B_z & -\frac{E_y}{c} & \frac{E_x}{c} & 0 \end{bmatrix} \tag{2.21}$$

Immediately we may note the correspondence between the electric and the magnetic fields:

$$B_i \longrightarrow \frac{E_i}{c} \qquad \frac{E_i}{c} \longrightarrow -B_i \tag{2.22}$$

In fact, this correspondence goes deeper. Let us summarize the equations of motion of the field strength tensor:

$$\partial_{[\lambda} F_{\mu\nu]} = 0 \tag{2.23a}$$
$$\partial_\mu F^{\mu\nu} = \mu_0 j^\nu. \tag{2.23b}$$

We shall now consider the corresponding expressions for $\star F$. First we look at the equivalent of equation 2.23a:

$$\begin{aligned}
\partial_0 \star F_{12} + \partial_1 \star F_{20} + \partial_2 \star F_{01} &= -\frac{1}{c^2}\frac{\partial E_z}{\partial t} + \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \\
&= -\frac{1}{c^2}\frac{\partial E_z}{\partial t} + (\nabla \times \mathbf{B})_z \\
&= \mu_0 \mathbf{j}_z = \mu_0 j^3 \\
\partial_0 \star F_{13} + \partial_1 \star F_{30} + \partial_3 \star F_{01} &= -\mu_0 \mathbf{j}_y = -\mu_0 j^2 \\
\partial_0 \star F_{23} + \partial_2 \star F_{30} + \partial_3 \star F_{02} &= \mu_0 \mathbf{j}_x = \mu_0 j^1 \\
\partial_1 \star F_{23} + \partial_2 \star F_{31} + \partial_3 \star F_{12} &= -\frac{1}{c}\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right) \\
&= -\frac{1}{c}\nabla \cdot \mathbf{E} \\
&= -\frac{1}{c^2\epsilon_0} c\rho = -\mu_0 j^0
\end{aligned}$$

This already looks promising. To finish it off, we will define another tensor: the dual current tensor $\star j$. This is a totally antisymmetric 3-covariant tensor, derived from j^μ. We define it as follows:

$$\star j_{\mu\nu\rho} = \epsilon_{\lambda\mu\nu\rho} j^\lambda. \tag{2.24}$$

This tensor has 4 independent components, which we can calculate explicitly:

$$\begin{aligned}
\star j_{012} &= \epsilon_{3012} j^3 = \quad -j^3, \\
\star j_{013} &= \epsilon_{2013} j^2 = \quad\ \ j^2, \\
\star j_{023} &= \epsilon_{1023} j^1 = \quad -j^1, \\
\star j_{123} &= \epsilon_{0123} j^0 = \quad\ \ j^0.
\end{aligned} \tag{2.25}$$

And now the components fall into place! We can summarize the above relations by the following expression:

$$\boxed{\partial_\mu \star F_{\nu\rho} + \partial_\nu \star F_{\rho\mu} + \partial_\rho \star F_{\mu\nu} = -\mu_0 \star j_{\mu\nu\rho}} \tag{2.26}$$

Now for the equivalent of equation 2.23b:

$$\partial_\mu \star F^{\mu 0} = \partial_i \star F^{i0} = \nabla \cdot \mathbf{B} = 0$$
$$\partial_\mu \star F^{\mu 1} = \frac{1}{c}\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} + \frac{\partial B_x}{\partial t}\right) = \frac{1}{c}\left[(\nabla \times \mathbf{E})_x + \frac{\partial B_x}{\partial t}\right] = 0$$
$$\partial_\mu \star F^{\mu 2} = \frac{1}{c}\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} + \frac{\partial B_y}{\partial t}\right) = \frac{1}{c}\left[(\nabla \times \mathbf{E})_y + \frac{\partial B_y}{\partial t}\right] = 0$$
$$\partial_\mu \star F^{\mu 3} = \frac{1}{c}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} + \frac{\partial B_z}{\partial t}\right) = \frac{1}{c}\left[(\nabla \times \mathbf{E})_z + \frac{\partial B_z}{\partial t}\right] = 0$$

And from this we can conclude:

$$\boxed{\delta_\mu \star F^{\mu\nu} = 0} \tag{2.27}$$

In order to appreciate this symmetry in its full glory I will place the equations of motion of the field strength tensor and its dual next to each other[2].

$$\partial_{[\nu} F_{\rho\sigma]} = 0 \tag{2.28a}$$
$$\partial_\mu F^{\mu\nu} = \mu_0 j^\nu \tag{2.28b}$$
$$\partial_{[\lambda} \star F_{\mu\nu]} = -\frac{1}{3}\mu_0 \star j_{\lambda\mu\nu} \tag{2.28c}$$
$$\partial_\mu \star F^{\mu\nu} = 0 \tag{2.28d}$$

In particular, we note that in the case of the free Maxwell equations (where $j^\nu = 0$), we can rotate the equations into each other. Let us take an action of $SO(2)$, which describes a two-dimensional rotation. We can then make the following transformation [6]:

$$\frac{\mathbf{E}}{c} \longrightarrow \frac{\mathbf{E}'}{c} = \cos\alpha \frac{\mathbf{E}}{c} - \sin\alpha \mathbf{B}, \tag{2.29a}$$
$$\mathbf{B} \longrightarrow \mathbf{B}' = \sin\alpha \frac{\mathbf{E}}{c} + \cos\alpha \mathbf{B}, \tag{2.29b}$$

or in terms of the field strength tensor:

$$F_{\mu\nu} \longrightarrow \overline{F}_{\mu\nu} = \cos\alpha F_{\mu\nu} + \sin\alpha \star F_{\mu\nu}. \tag{2.30}$$

When there are no sources present, this rotation will transform solutions of Maxwell's equations in other solutions.

2.5 Magnetic monopoles

If we look closely at equations 2.28, we see that Maxwell's theory almost begs for another current density to exist. This current density would represent magnetic charges. If we write k^μ for this current, the "complete" set of Maxwell's

[2]The factor $\frac{1}{3}$ in the third equation seems to be distorting the symmetric pattern in the equations of motion, but recall that the original equations of motion where given in the form of cyclic sums instead of antisymmetric notation. This rewriting generates the multiplying factor. Therefore, the cyclic sum notation would be more accurate to display the symmetry here, but also typographically inconvenient and makes the equations appear unclear.

equations would be:

$$\begin{cases} \partial_\mu F^{\mu\nu} = \mu_0 j^\nu \\ \partial_\mu \star F^{\mu\nu} = \dfrac{1}{c} k^\nu. \end{cases} \tag{2.31}$$

However, we may again observe that an $SO(2)$ duality transformation may be made, transforming F into $\star F$ and vice versa, while at the same time transforming j into k and vice versa. Experimental observations indicate that the ratio between j and k is always constant, which allows us to fully transform the description of electromagnetics into a form where only electric charge and no magnetic charge exists. This makes it fully possible to state that there are no magnetic monopoles in nature. [7]

Chapter 3

The Born-Infeld Lagrangian

The classical formulation of electromagnetics in terms of Maxwell's Equations has been very succesful and is widely applied even today at the macroscopic level. However, modern theories that describe Nature at a microscopic level pose some problems. These theories are generally formulated as a field theory, based on a Lagrangian formalism. When we use the classical Lagrangian density 2.18, this leads to significant problems. These problems are mainly related to the fact that the self-energy of a point-charge will diverge to infinity. Furthermore, singularities occur in the field strengths in already the simplest case of the presence of one electron. In order to overcome this problem, Max Born and Leopold Infeld proposed another Lagrangian density, which in essence limits the field strength. This was based on the idea that physical quantities should not tend to infinity (what they called the *principal of finiteness*) and was developed analogously to the theory of Special Relativity, where a limit on velocity was imposed. [1] This new Lagrangian took the following form[1]:

$$\mathcal{L} = \left(\sqrt{-\hat{g}} - \sqrt{-\det(g_{\mu\nu} + F_{\mu\nu})} \right), \tag{3.1}$$

where $\hat{g}$ denotes the determinant of the matrix representation of the metric and F is again a totally antisymmetric 2-covariant tensor. Born and Infeld show in their paper that this may be rewritten as:

$$\begin{aligned} \mathcal{L} = \sqrt{-\hat{g}} \left(\sqrt{1 + \frac{1}{2} F_{\mu\nu} F^{\mu\nu} - \frac{1}{16} (F_{\rho\sigma} \star F^{\rho\sigma})^2} - 1 \right) \\ = \sqrt{-\hat{g}} \left(\sqrt{1 + \Phi - \Gamma^2} - 1 \right) = \sqrt{-\hat{g}}\, L, \end{aligned} \tag{3.2}$$

where $\Phi = \frac{1}{2} F_{\mu\nu} F^{\mu\nu}$ and $\Gamma = \frac{1}{4} F_{\mu\nu} \star F^{\mu\nu}$. What is to be noted, is that they – in analogy to the Maxwell theory – assume the existence of a vector potential A_μ, such that:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{3.3}$$

This then gives us the first field equation:

$$\partial_{[\lambda} F_{\mu\nu]} = 0. \tag{3.4}$$

[1] The most general case involves the presence of a scale factor. For tha sake of simplicity we have taken this factor to be 1, so it drops out.

The second field equation can be derived from the Euler-Lagrange equations by noting that:

$$\frac{\partial \mathcal{L}}{\partial A_{\mu,\nu}} = -2\frac{\partial \mathcal{L}}{\partial F_{\mu\nu}} \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial A_{\mu}} = 0,$$

such that when we define[2]:

$$\begin{aligned} G^{\mu\nu} = -\frac{1}{\sqrt{2}}\frac{\partial L}{\partial F_{\mu\nu}} &= -\frac{1}{\sqrt{2}}\left(\frac{\partial L}{\partial \Phi}\frac{\partial \Phi}{\partial F_{\mu\nu}} + \frac{\partial L}{\partial \Gamma}\frac{\partial \Gamma}{\partial F_{\mu\nu}}\right) \\ &= -\frac{1}{\sqrt{2}}\left(\frac{\partial L}{\partial \Phi}2F^{\mu\nu} + \frac{\partial L}{\partial \Gamma} \star F^{\mu\nu}\right) = \frac{F^{\mu\nu} - \Gamma \star F^{\mu\nu}}{\sqrt{2}\sqrt{1+\Phi-\Gamma^2}} \end{aligned} \tag{3.5}$$

we get:

$$\partial_\nu G^{\mu\nu} = 0. \tag{3.6}$$

The components of this G (a rescaling of what Born and Infeld called p) represent the electric displacement field $\mathbf{D}$ and the magnetic field $\mathbf{H}$ in Born-Infeld theory. The rescaling is necessary for the constitutive relation between G and F to hold and expresses that the "real" invariant objects are a scaling of what Born and Infeld viewed as $\mathbf{D}$ and $\mathbf{H}$.

We will now investigate a duality rotation similar to the one in equation 2.30. Let us define two new ("rotated") fields:

$$\widetilde{F}_{\mu\nu} = \cos\alpha\, F_{\mu\nu} + \sin\alpha \star G_{\mu\nu} \tag{3.7a}$$

$$\widetilde{G}^{\mu\nu} = \cos\alpha\, G^{\mu\nu} + \sin\alpha \star F^{\mu\nu} \tag{3.7b}$$

As we shall see in the next chapter, the equations of motion will automatically be invariant under this duality rotation due to the fact that $\frac{\partial \mathcal{L}}{\partial A_\mu} = 0$. It would then at first sight seem that the duality invariance is satisfied, but there is still one relation that must remain invariant: the definition of G in terms of F, equation 3.5. In fact, the requirement for this relation to be invariant under this rotation can be stated in terms of infinitesimal rotations. The infinitesimal transformations generating 3.7 are:

$$\begin{cases} \delta F_{\mu\nu} = \star G_{\mu\nu}, \\ \delta G_{\mu\nu} = \star F_{\mu\nu}. \end{cases} \tag{3.8}$$

The constitutive relation 3.5 between the two fields F and G will then continue to hold if and only if:

$$\star F^{\mu\nu} = \delta G^{\mu\nu} = \delta F_{\rho\sigma}\frac{\partial G^{\mu\nu}}{\partial F_{\rho\sigma}}. \tag{3.9}$$

It can be shown (see appendix B) that this requirement is indeed satisfied by the Born-Infeld Lagrangian. So, the Born-Infeld theory posesses, just like the free Maxwell equations an $SO(2)$ duality symmetry.

This already gives us an example of a non-linear theory (the theory is called non-linear because of the non-linear relation between F and G, which arises

[2]Note that this definition of G changes with a factor of $\frac{1}{2\sqrt{2}}$ from the G we define in chapter 4. The implications of that chapter do hold. In general we can observe that the implications of that chapter hold for any tensor that is defined as the product of a constant with the partial derivatives of L with respect to the components of F.

from the fact that the Lagrangian density L is not quadratic in F), which possesses a duality symmetry. We also observe that we may actually write down any Lagrangian density we want, and that this will give us a theory of the electromagnetic fields. The significance of such a theory is determined by the properties it possesses and one of the nice properties of Maxwell's Theory is the duality invariance. We will therefore try to categorize the types of Lagrangian densities for which this property holds, as a tool to find theories that are likely to be "good".

Chapter 4

General duality rotations

In the preceeding chapters, we have seen two examples of a hodge duality rotation symmetry. Here, the Lagrangian density was generally a function of the potential, the field strength tensor and the metric. We were able to deduce the equations of motion using the Euler-Lagrange equations and to determine the duality symmetry[1] from this. In both cases we found a symmetry of $SO(2)$. We will now take a look at general Lagrangians that are functions of the potential, the field strength tensor and the metric and deduce criteria for the $SO(2)$ duality symmetry to hold. This is also a good point to note that whereas duality symmetry is a property of a theory, it is in general not a symmetry of the Lagrangian, but of the equations of motion! This means that is not a symmetry in the usual sense of the word, and therefore in general has not the usual consequences such as the existence of a concerved current, etc. It is however a very nice property as it simplifies some math greatly and is supported by experimental evidence. Therefore, "good" theories can be required to possess this symmetry and we will now try to derive when a theory complies with this.

4.1 The general field equations

To this end, let A_μ denote the 4-vector potential and let $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\nu$. We denote the Lagrangian density by:

$$\mathcal{L}(A_\mu, \partial_\mu A_\nu, g_{\mu\nu}) = \sqrt{-\hat{g}} L(A_\mu, F_{\mu\nu}, g_{\mu\nu}). \tag{4.1}$$

Furthermore, we assume the metric does not depend on the 4-vector potential and hence not on the field strength tensor.

The first field equations will now be, due to the definition of F: $\partial_{[\nu} F_{\rho\sigma]} = 0$. The second field equation we can derive from the Euler-Lagrange equations. We first note:

$$\frac{\partial \mathcal{L}}{\partial A_{\mu,\nu}} = \frac{\partial \mathcal{L}}{\partial F_{\rho\sigma}} \frac{\partial F_{\rho\sigma}}{\partial A_{\mu,\nu}} = \frac{\partial \mathcal{L}}{\partial F_{\rho\sigma}} \left(\delta^\nu_\rho \delta^\mu_\sigma - \delta^\mu_\rho \delta^\nu_\sigma\right) = -2 \frac{\partial \mathcal{L}}{\partial F_{\mu\nu}}. \tag{4.2}$$

[1] The terms *hodge duality rotation symmetry* and *duality symmetry* will be interchanged freely throughout this chapter

We therefore define[2]:

$$G^{\mu\nu} = -2\frac{\partial \mathcal{L}}{\partial F_{\mu\nu}} \tag{4.3}$$

and find the second field equation:

$$\boxed{\partial_\nu G^{\mu\nu} = \partial_\nu \frac{\partial \mathcal{L}}{\partial A_{\mu,\nu}} = \frac{\partial \mathcal{L}}{\partial A_\mu}.} \tag{4.4}$$

4.2 Criteria from the field equations

At this point, we can define the rotated tensors and ask ourself when both field equations are satisfied for the second field equations to. To this point, consider an action in $SO(2)$, which can be parametrized by an arbitrary $\alpha \in \mathbb{R}$. We shall assume that the action is not the identity, which is equivalent to $\sin(\alpha) \neq 0$. The rotated fields are then:

$$\widetilde{F}_{\mu\nu} = \cos\alpha F_{\mu\nu} + \sin\alpha \star G_{\mu\nu}, \tag{4.5a}$$

$$\widetilde{G}^{\mu\nu} = \cos\alpha G^{\mu\nu} + \sin\alpha \star F^{\mu\nu}. \tag{4.5b}$$

The definition of $\widetilde{G}$ implies:

$$\begin{aligned} \partial_\nu \widetilde{G}^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial A_\mu} &\iff \cos\alpha\, \partial_\nu G^{\mu\nu} + \sin\alpha\, \partial_\nu \star F^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial A_\mu} \\ &\iff \sin\alpha\, \partial_\nu \star F^{\mu\nu} = (1-\cos\alpha)\frac{\partial \mathcal{L}}{\partial A_\mu} \\ &\iff \partial_\nu \star F^{\mu\nu} = \tan\frac{\alpha}{2}\frac{\partial \mathcal{L}}{\partial A_\mu} \end{aligned} \tag{4.6}$$

For this to hold for arbitrary $\alpha \in \mathbb{R}$, we must demand that $\frac{\partial \mathcal{L}}{\partial A_\mu} = 0$. So, our first requirement for invariance of the field equations under a duality rotation is that the Lagrangian is independent of the 4-vector potential. This then implies a second condition:

$$\partial_\nu \widetilde{G}^{\mu\nu} = 0 \iff \partial_\nu \star F^{\mu\nu} = 0. \tag{4.7}$$

This condition is actually equivalent to the Bianchi identity $\partial_{[\nu} F_{\rho\sigma]} = 0$, so this condition is automatically satisfied for the field strength tensor with this definition.

Then, to derive the third and last condition, we notice:

$$\begin{aligned} \partial_{[\nu} \widetilde{F}_{\rho\sigma]} = 0 &\iff \cos\alpha\, \partial_{[\nu} F_{\rho\sigma]} + \sin\alpha\, \partial_{[\nu} \star G_{\rho\sigma]} = 0 \\ &\iff \partial_{[\nu} \star G_{\rho\sigma]} = 0. \end{aligned} \tag{4.8}$$

Again, this condition is equivalent to $\partial_\nu G^{\mu\nu} = 0$, which is the second field equation. So, the only real criterium for the field equations to be invariant under duality rotations is that the Lagrangian must not depend on A_μ. This is however not the whole story. We will also require the relation between G and F given by equation 4.3 to be invariant. We will study this next.

[2]The factor -2 can in general be replaced by a constant k. Under this replacement the equations of motion will take on a different form, but the requirements for duality invariance will remain the same.

4.3 Interdepence of $G^{\mu\nu}$ and $F_{\mu\nu}$

In order for the relation between G and F to be invariant under a duality rotation, we will study infinitesimal rotations. This method yields criteria on the Lagrangian for duality symmetry to hold. It was considered by Gaillard and Zumino in a broader context in 1981 [5] and reviewed and extended later by Gibbons and Rasheed [6, §2].

We consider an infinitesimal rotation of $F_{\mu\nu}$ and $G_{\nu\nu}$:

$$\begin{cases} \delta F_{\mu\nu} = \star G_{\mu\nu}, \\ \delta G_{\mu\nu} = \star F_{\mu\nu}. \end{cases} \tag{4.9}$$

We then require equation 4.3 to be invariant under this rotation, which is equivalent to:

$$\star F^{\mu\nu} = \delta G^{\mu\nu} = \delta F_{\rho\sigma} \frac{\partial G^{\mu\nu}}{\partial F_{\rho\sigma}} = \star G_{\rho\sigma} \frac{\partial G^{\mu\nu}}{\partial F_{\rho\sigma}}. \tag{4.10}$$

This equation states that under an infinitesimal rotation of the form 4.9 F transforms in the same way as G. We can rewrite this in the form of a second order partial differential equation of $\mathcal{L}$:

$$\frac{1}{2}\tau^{\mu\nu\rho\sigma} F_{\rho\sigma} = \frac{1}{2}\tau_{\rho\sigma\alpha\beta} G^{\alpha\beta} \frac{\partial}{\partial F_{\rho\sigma}} \left(-2 \frac{\partial \mathcal{L}}{\partial F_{\mu\nu}} \right). \tag{4.11}$$

Gibbons and Rasheed then integrate and rewrite this equation to obtain the equivalent condition[3]:

$$F_{\mu\nu} \star F^{\mu\nu} = 2 G_{\mu\nu} \star G^{\mu\nu} + C, \tag{4.12}$$

where C is an arbitrary constant of integration.

Now, this condition is necessary and sufficient for the duality invariance of the field equations under infinitesimal rotations, since the relation 4.3 is invariant, and the field equation 4.4 is derived from this equation. Because of this, the field equations are entirely invariant under general $SO(2)$ duality rotations under this condition.

4.4 Further generalizations

We may wonder whether the $SO(2)$ duality rotation symmetry may be extended to a larger symmetry group. Gaillard and Zumino have considered field theories with n tensor fields and a number of scalar or vector fields and have shown that the maximal duality symmetry group will then be the symplectic group $Sp(2n, \mathbb{R})$. The real symmetry group of a particular theory must then be a subgroup of $Sp(2, \mathbb{R})$, the maximal compact subgroup being $U(n)$ [5].

[3] The expression given here differs from the one given by Gibbons and Rasheed by a factor 2 in front of the $G_{\mu\nu} \star G^{\mu\nu}$ term. This is due to the fact that we adopt the convention that the components $F_{\mu\nu}$ and $F_{\nu\mu}$ are not independent when taking partial derivatives.

Chapter 5

Conclusion

We have seen that Maxwell Electrodynamics posesses a symmetry in the equations of motion, in both the case of no charges and with an extension to both electric and magnetic charges. Even though this theory can describe electric and magnetic phenomena with great accuracy, certain problems arise in the microscopic case. To overcome this, non-linear theories may be written down. However, since the duality symmetry of Maxwell Electrodynamics is such a striking feature, one may have the wish to retain this nice property. We have therefore researched under what conditions a Lagrangian formalism may lead to a similar symmetry in the equations of motion. We also may note that this method has also been used in a more general case. As Gaillard an Zumino note [5], there have been instances in which the assumption of duality invariance was used to search for a correct theory for supergravity and supersymmetric theories (e.g. [4]).

Then we first showed that indeed a non-linear theory of electrodynamics exists that possesses a $SO(2)$ duality symmetry, namely Born-Infeld Theory. In the more general case, where we wish to write down a Lagrangian for one field as an exterior derivative of one potential that possesses such a symmetry, we found that:

1. The Lagrangian must not depend on the potential, and
2. The Lagrangian must satisfy a set of second order partial differential equations in the field strength tensor.

This greatly limits the set of possible Lagrangians and may therefore be a valuable tool for finding a correct theory of extended electrodynamics.

A generalization to the case of any number n of fields, coupled with a set of vector or scalar fields may be made. Here we can research the most general symmetry and it turns out that the most general duality symmetry such a theory can possess is $Sp(2n, \mathbb{R})$.

The concept of duality symmetry may be further investigated by looking at more objects that are important in classical electrodynamics such as the Stress-Energy Tensor and the Poynting vector. Demanding invariance or certain transformation properties of these objects under a duality transformation may provide further restrictions on "correct" theories.

Appendix A

Tensor Calculus

Many readers may not have a thorough grip on the concepts of calculus on manifolds, wo we will give a basic summary here, although we will not go too deep into the mathematical details. The information of this chapter was derived from [3].

A.1 Manifolds

An n-dimensionsal manifold is a set M with an associated topology $T = \{U_i | U_i \subseteq M\}$, where the U_i give a concept of open sets.[1] We then equip this "topological space" with an atlas: a collection of charts. These charts are pairs (U_i, ψ_i), where the U_i are open sets, and the ψ_i are homeomorphisms from U_i to a subset $\psi_i(U_i)$ of $\mathbb{R}^n$. The ψ_i being homeomorphisms says nothing else that they are bijections, continuous and have a continuous inverse. They may be viewed as local coordinates on a patch of spacetime. In order to perform calculus on these manifolds, we require that the coordinate transformations are diffeomorphisms. That is, the mappings $\psi_i \circ \psi_j^{-1} : \psi_i(U_i) \cap \psi_j(U_j) \to \psi_i(U_i) \cap \psi_j(U_j)$ are differentiable bijections. Differentiability of a function $f : M \to \mathbb{R}$ is now defined as the requirement that for any chart (U_i, ψ_i), the composition $f \circ \psi_i^{-1}$ is differentiable in the usual sense. We will denote the vector space of infinetely differentiable functions to $\mathbb{R}$ by $C^\infty(M)$.

A.2 Tangent and cotangent vectors

With each point $x \in M$ we can now associate a vector space $T_x(M)$: the *tangent space at* x. This is the space of linear functions $v_x : C^\infty(M) \to \mathbb{R}$. The vector space structure is induced by the regular structure on $\mathbb{R}$. At all points this space is isomorphic to $\mathbb{R}^n$. At this point some mathematical scrutiny is needed to define *vector fields* on M. We can define a fiber bundle structure on the tangent spaces at each point. A vector field is then a cross section on this fibre

[1] The topological space is most often required to be *Hausdorff seperated*, which means that any two distinct points have disjoint neighbourhoods (sets that contain an open set which contains the point). This greatly simplifies the conditions under which certain theorems are true. We will henceforth adopt this convention.

bundle. However, for our purposes it suffices to say that a vector field is a mapping $x \mapsto v_x$, associating a tangent vector with each point on the manifold.

We can also map the tangent vectors to $\mathbb{R}$. This gives us another vector space: the cotangent space at $x \in M$, $T_x^*(M)$. This is the set of linear mappings $\omega_x : T_x(M) \to \mathbb{R}$. So, let $x \in M$, $v_x \in T_x(M)$ and $\omega_x \in T_x^*(M)$. Then: $\omega_x(v_x) \in \mathbb{R}$ and we will define $v_x(\omega_x) = \omega_x(v_x)$. In a similar way to with tangent vectors, we can define a covector field to be a mapping $x \mapsto \omega_x$, associating a cotangent vector with each point on the manifold.

If $(e_1, \dots e_n)$ is a basis for $T_x(M)$, we can induce a basis on $T_x^*(M)$ by letting $\theta^i \in T_x^*(M)$ be the unique mapping so that $\theta^i(e_j) = \delta^i_j$. The basis $(\theta^1, \dots, \theta^n)$ is then called the dual basis. We will at this point introduce the Einstein summation convention: if two indices occur twice in an expression, one as an upper index and one as a lowered index, we will implicitly assume summation over these indices. Let $v_x \in T_x(M)$ and $\omega_x \in T_x^*(M)$ with the dual bases as above. Then we can write out these vectors in terms of their components: $v_x = v_x^i e_i$ and $\omega_x = \omega_{xj}\theta^j$. Most importantly, this gives:

$$\omega_x(v_x) = \omega_{xj}\theta^j(v_x^i e_i) = v_x^i \omega_{xj}\delta_i^j = v_x^i \omega_{xi}.$$

The elements of $T_x(M)$ are called **contravariant vectors** and the elements of $T_x^*(M)$ are called **covariant vectors**. This terminology comes from the transformation rules for their components. Let us introduce a change of basis $(e_i) \to (e_i')$, induced by a real $n \times n$ matrix A, whose components we will denote by a^i_j, where the upper index denotes the row, and the lower one denotes the column. We will take: $e_i' = a_i^k e_k$. This gives us: $v_x = v_x^i e_i = v_x^{i\prime} e_{i\prime} = v_x^{i\prime} a_i^k e_k$ and thus $v_x^k = a_i^k v_x^{i\prime}$. If we let V and $V\prime$ be the real (column) vectors with the coordinates in the first and primed basis as components respectively, we see: $V\prime = A^{-1}V$. We similarly get: $\Omega\prime = \Omega A$.

A.3 Tensors

Now, we introduce the last of the fundamental objects we will need: **tensors**. Let $x \in M$. A p-covariant and q-contravariant tensor at x is then a multilinear (i.e. it is linear in all its arguments) mapping elements from the cartesian product $\prod^p T_x(M) \prod^q T_x^*(M)$ into $\mathbb{R}$. The space of such tensors will be denoted by $\otimes^p T_x^*(M) \otimes^q T_x(M)$. This is again a vector space with the regular vector space structure for mappings to $\mathbb{R}$. We will illustrate this with tensors in $\otimes^2 T_x^*(M)$. Let (e_i) be a basis for $T_x(M)$ and (θ_i) be its dual basis. We then define $\theta_i \otimes \theta_j$ as the unique tensor such that for $v_x, w_x \in T_x(M)$: $\theta_i \otimes \theta_j(v_x, w_x) = v_x^i w_x^j$. If then $\omega \in \otimes^2 T_x^*(M)$, we have $\omega(v_x, w_x) = v_x^i w_x^j \omega(e_i, e_j) = \omega(e_i, e_j)(\theta_i \otimes \theta_j(v_x, w_x))$. Clearly the tensors $\theta_i \otimes \theta_j$ are linearly independent and any tensor ω is a linear combination of them. This gives $\omega = \omega_{ij}\theta_j \otimes \theta_j$, where $\omega_{ij} = \omega(e_i, e_j)$. Again in the same way, we can introduce tensor fields as mappings from points in the manifold to tensors at that point.

Symmetries play a central role in the theory of tensors. In particularly, we say that a tensor is antisymmetric (resp. symmetric) in two indices if there occurs a sign change (resp. if the sign remains the same) in the components under exchange of two indices. Covariant tensors that are totally antisymmetric (i.e. in all indices) are so important that they get a special name: p-forms. In fact, almost all tensors (e.g. F and G) we encounter in this thesis are 2-forms.

A.4 Metrics

Now we can come to the meaning of our definition of spacetime: it is a 4-dimensional manifold. Furthermore, there is a metric associated with it. This is a 2-covariant tensor field that introduces a notion of distance. It has the properties that is continuous, symmetric ($g_x(v_x, w_x) = g_x(w_x, v_x)$) and it is non-degenerate: for each $x \in M$, $g_x(v_x, w_x) = 0$ for all $v_x \in T_x(M)$ if and only if $w_x = 0$. Note that we do not require that $v_x \in T_x(M) \setminus \{0\} \iff g_x(v, v) > 0$. That is, we do **not** require the metric to be positive-definite. A manifold associated with such a structure is a *Riemannian manifold*. If the metric is positive-definite, the manifold is said to be proper; otherwise it is called a *Pseudo-Riemannian manifold*. So, in our case we have a metric that may be negative for some vectors. Since the metric is symmetric, we may express it in terms of $\theta^i\theta^j := \frac{1}{2}(\theta^i \otimes \theta^j + \theta^j \otimes \theta^i$. The components of g are then determined by: $g = g_{ij}\theta^i\theta^j$, where $g_{ij} = g(e_i, e_j)$. The metric induces a quadratic form at each $x \in M$ on $T_x(M)$. This is given by $g_x(v_x, v_x)$. By Sylvester's Law of Inertia we may for each point $x \in M$ choose a basis of $T_x(M)$ such that this form is given by:

$$g_x(v_x, v_x) = \sum_{i=1}^{p}(v_x^i) - \sum_{j=p+1}^{n}(v_x^j).$$

Though this is a very convenient basis to work in, we will require to work in another basis – the natural basis –, since we require partial derivatives with respect to the coordinate functions to work easily. Most of the time we will work in flat spacetime, or *Minkowski Space*. In this chapter the natural basis suffices to put the quadratic form into the above form. Therefore, in Minkowski Space we will denote the components of the metric by a symmetric matrix η:

$$[\eta_{\mu\nu}] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{A.1}$$

which puts the metric in the following form: $g(v, w) = \eta_{\mu\nu}v^\mu w^\nu$. Also, the metric induces an isomorphism between the tangent and cotangent spaces. This isomorphism is given by the mapping: $v_x \mapsto v_x^* := g_x(v_x, \cdot)$, where v_x^* is a covariant vector and its components in an arbitrary basis are given by: $v_{x\mu} = g_{\mu\nu}v_x^\nu$. Similarly: $v_x^\mu = g^{\mu\nu}v_{x\nu}$, where $g^{\mu\nu}$ is the inverse matrix of the matrix representation of g_x. Because of this, we say we can *lower* or *raise indices* with the metric. Not that in the natural basis defined above, the indices are raised and lowered by η.

A.5 Hodge dual

Within a n-dimensional manifold with an associated metric g, we may define an isomorphism between the spaces of p-forms and $(n-p)$-forms. For this we will define the *constant volume element* τ:

$$\tau_{i_1\ldots i_n} = \sqrt{|\hat{g}|}\,\epsilon_{i_1\ldots i_n}, \tag{A.2}$$

where $|\hat{g}|$ is the absolute value of the determinant of the metric in matrix form. Then we define the hodge dual ($\star$) of a p-form β to be the $(n-p)$-form with components:

$$\star\,\beta_{i_{p+1}\ldots i_n} = \frac{1}{p!}\tau_{i_1\ldots i_p i_{p+1}\ldots i_n}\beta^{i_1\ldots i_p}. \tag{A.3}$$

As a special case that occurs often in this thesis, we note that for a 2-form F in a 4-dimensional manifold:

$$\star\,F_{\mu\nu} = \frac{1}{2}\tau_{\alpha\beta\mu\nu}F^{\alpha\beta}. \tag{A.4}$$

Appendix B

Proof of Born-Infeld duality

We consider the Lagrangian:

$$L = \sqrt{1 + \Phi - \Gamma^2} - 1, \qquad \Phi = \frac{1}{2} F_{\mu\nu} F^{\mu\nu}, \quad \Gamma = \frac{1}{4} F_{\mu\nu} \star F^{\mu\nu}, \tag{B.1}$$

and define the following quantity:

$$G^{\mu\nu} = -\frac{1}{\sqrt{2}} \frac{\partial L}{\partial F_{\mu\nu}} = \frac{F^{\mu\nu} - \Gamma \star F^{\mu\nu}}{\sqrt{2}\sqrt{1 + \Phi - \Gamma^2}}. \tag{B.2}$$

Born-Infeld Theory concerns itself with the dynamics of the two fields F and G.[1]

Let us consider an infinitesimal transformation:

$$\begin{cases} \delta F^{\mu\nu} = \star G^{\mu\nu}, \\ \delta G^{\mu\nu} = \star F^{\mu\nu}. \end{cases} \tag{B.3}$$

In order to proof the invariance of the constitutive relation B.2 under this transformation, we first note that:

$$\begin{aligned} G_{\rho\sigma} \star G^{\rho\sigma} &= \frac{1}{2} \frac{F_{\rho\sigma} - \Gamma \star F_{\rho\sigma}}{\sqrt{1 + \Phi - \Gamma^2}} \frac{\star F^{\rho\sigma} + \Gamma F^{\rho\sigma}}{\sqrt{1 + \Phi - \Gamma^2}} \\ &= \frac{1}{2} \frac{F_{\rho\sigma} \star F^{\rho\sigma} - \Gamma \star F_{\rho\sigma} \star F^{\rho\sigma} + \Gamma F_{\rho\sigma} F^{\rho\sigma} - \Gamma^2 \star F_{\rho\sigma} F^{\rho\sigma}}{1 + \Phi - \Gamma^2} \\ &= \frac{1}{2} \frac{4\Gamma + 4\Gamma\Phi - 4\Gamma^3}{1 + \Phi - \Gamma^2} \\ &= 2\Gamma = \frac{1}{2} F_{\rho\sigma} \star F^{\rho\sigma}. \end{aligned} \tag{B.4}$$

So, we have the equality:

$$F_{\rho\sigma} \star F^{\rho\sigma} = 2 G_{\rho\sigma} \star G^{\rho\sigma} = 2 G^{\rho\sigma} \star G_{\rho\sigma} \tag{B.5}$$

[1] F represents $\mathbf{E}$ and $\mathbf{B}$, whereas G represents electric displacement $\mathbf{D}$ and the magnetic field $\mathbf{H}$.

This follows from the fact that $\star F_{\rho\sigma} \star F^{\rho\sigma} = -F_{\rho\sigma}F^{\rho\sigma}$ and $\star F_{\rho\sigma}F^{\rho\sigma} = F_{\rho\sigma} \star F^{\rho\sigma}$. Now we can differentiate this equation with respect to $F_{\mu\nu}$ to get:

$$\begin{aligned}
\delta G^{\mu\nu} = \star F^{\mu\nu} &= \frac{\partial}{\partial F_{\mu\nu}} \left(\frac{1}{4} F_{\rho\sigma} \star F^{\rho\sigma} \right) \\
&= \frac{\partial}{\partial F_{\mu\nu}} \left(\frac{1}{2} G^{\rho\sigma} \star G_{\rho\sigma} \right) \\
&= \frac{1}{4} \tau_{\alpha\beta\rho\sigma} \frac{\partial}{\partial F_{\mu\nu}} \left(G^{\rho\sigma} G^{\alpha\beta} \right) \\
&= \star G_{\rho\sigma} \frac{\partial G^{\rho\sigma}}{\partial F_{\mu\nu}} \\
&= \star G^{\rho\sigma} \frac{-1}{\sqrt{2}} \frac{\partial}{\partial F_{\mu\nu}} \left(\frac{\partial L}{\partial F_{\rho\sigma}} \right) \\
&= \star G^{\rho\sigma} \frac{-1}{\sqrt{2}} \frac{\partial}{\partial F_{\rho\sigma}} \left(\frac{\partial L}{\partial F_{\mu\nu}} \right) \\
&= \star G_{\rho\sigma} \frac{\partial G^{\mu\nu}}{\partial F_{\rho\sigma}}.
\end{aligned} \tag{B.6}$$

The seventh equality follows from the fact that we may exchange the order in which partial derivatives are taken, under the assumption that L is of class C^2 in F. This finally lets us conclude:

$$\delta G^{\mu\nu} = \star F^{\mu\nu} = \star G_{\rho\sigma} \frac{\partial G^{\mu\nu}}{\partial F_{\rho\sigma}} = \delta F_{\rho\sigma} \frac{\partial G^{\mu\nu}}{\partial F_{\rho\sigma}}. \tag{B.7}$$

This equation expresses that G transforms in the same way as F and thus the constitutive relation is invariant under rotations.

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