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Symmetries of phase space and optical states

Bachelor thesis in Physics and Mathematics
June 2013

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Abstract

Coherent states are quantum mechanical states with properties close to the classical description. Before coherent states are considered there will be some theory about canonical transformations and Poisson brackets. Transformations that leave the Poisson bracket invariant are symplectic matrices and form for n dimensions the symplectic group $\mathrm{Sp}(2n, \mathbb{R})$. Sp(2,$\mathbb{R}$) is isomorphic to SU(1,1), which has various representations. Coherent states can be created from the vacuum state by a displacement operator, which is in the super-Lie algebra of SU(1,1). Coherent states have minimal uncertainty and can be transformed to squeezed states. Squeezed states are states with one of its standard deviations smaller while the minimal uncertainty relation still holds. Squeezing can be done by the squeezing operator, which is in SU(1,1). Squeezed states can for example be used as qubit states or to amplify measurement signals without amplifying the noise.

Contents

Chapter 1

Introduction

In this bachelor thesis we are going to discuss some topics of quantum optics. Before considering quantum optics, we start with some theory about phase space. Here will consider canonical transformations and Poisson brackets and some properties of both. Next we will consider coherent states. Those type of states are of interest since they are quantum mechanic states with properties close to classical mechanical states. Coherent states have a circular area of uncertainty, which turns out to be an area of minimum uncertainty. After considering the coherent state we will give an introduction to some symmetry groups which will later on show to be useful. In the fourth chapter we will focus on squeezed states. Those type of states can be created out of coherent states by the use of a squeeze operator. Squeezed states can produce a variance smaller than the variance of the coherent states. This nice property can for example be used by measuring very weak signals. In theory the signal can then be amplified without amplifying the uncertainty. The last chapter we will therefore spend on some applications of the squeezed state.

Chapter 2

Phase space

In classical mechanics a physical system is described by states which are points of its phase space. Phase space is a space in which all possible states of a system are represented. Each possible state of the system corresponds to an unique point in phase space. Usually the phase space consists of all possible values of position variables, q_i, and momentum variables, p_i. The Hamiltonian mathematical function or operator can be used to describe the state of a system. In classical mechanics, the Hamiltonian is a function of coordinates and momenta of bodies in the system, and can be used to derive the equations of motion for the system. In quantum mechanics, the Hamiltonian is an operator corresponding to the total energy of the system. Hamiltonian formulations of classical mechanics serves as a point of departure for both statistical mechanics and quantum mechanics.[1]

2.1 Canonical transformations

The Hamiltonian is a function dependent on the coordinates q_i and momenta p_i, so in general we have $H = H(q_1, q_2, \ldots, q_n, p_1, p_2, \ldots, p_n, t)$. There is one type of solutions of the Hamilton's equations that is trivial, namely if all coordinates q_i are cyclic. (A coordinate is cyclic if it doesn't explicitly show up in the Lagrangian and thus the generalized momentum becomes a conserved quantity. The generalized momentum is defined as the momentum expressed in the coordinates selected such that number of independent coordinates is minimal. Such coordinates are called generalized coordinates). The number of cyclic coordinates can depend upon the choice of generalized coordinates. For each problem there may be one particular choice for which all coordinates are cyclic. The obvious generalized set of coordinates will normally not be cyclic, so we have to find a specific procedure for transforming from one set of variables to some other set of variables that may be more suitable. In the Hamiltonian formulation the coordinates and the momenta are both independent variables. We need to do a simultaneous transformation of the independent coordinates and momenta, q_i, p_i to a new set Q_i, P_i, with invertible equations of transformation

$$Q_i = Q_i(q, p, t), \qquad P_i = P_i(q, p, t). \tag{2.1}$$

Equations 2.1 define a point transformation of phase space. Transformations are only of interest if the new coordinates Q_i, P_i are canonical coordinates. Canonical coordinates is defined as the set of the generalized coordinates together with their conjugate momenta $p_i = \partial L \partial \dot{q}_i$. This requirement will be satisfied if there exists some function $K(Q, P, t)$ such that the equations of motion in the new coordinates are in the Hamiltonian form

$$\dot{Q}_i = \frac{\partial K}{\partial P_i}, \qquad \dot{P}_i = -\frac{\partial K}{\partial Q_i} \tag{2.2}$$

The function K is the Hamiltonian of the new set. Equations 2.2 must be the form of the equations of motion in the new coordinates and momenta no matter what the particular form of H is. We then have that the transformations are problem-independent. As is shown in a most classical mechanics textbooks (like [1] and [2]), Q_i and P_i must satisfy Hamilton's principle to be canonical coordinates. Hamilton's principle can be stated as

$$\delta \int_{t_1}^{t_2} (P_i \dot{Q}_i - K(Q, P, t))dt = 0, \tag{2.3}$$

where summation over i is implied. For the old coordinates we have a similar principle:

$$\delta \int_{t_1}^{t_2} (p_i \dot{q}_i - H(q, p, t))dt = 0, \tag{2.4}$$

The simultaneous validity gives a relation of the integrands of the form

$$\lambda[p_i \dot{q}_i - H(q, p, t)] = P_i \dot{Q}_i - K(Q, P, t) + \frac{dF}{dt}, \tag{2.5}$$

where F is any function of the phase space coordinates with continuous second derivatives, and λ is a constant independent of the canonical coordinates and the time. With the aid of a suitable scale transformation, it will always be possible to restrict our attention to transformations of canonical coordinates for which $\lambda = 1$. When we simply speak of a canonical transformation we assume $\lambda = 1$. If $\lambda \neq 1$ we speak of an extended canonical transformation. Thus, for canonical transformations we are now left with the relation:

$$p_i \dot{q}_i - H(q, p, t) = P_i \dot{Q}_i - K(Q, P, t) + \frac{dF}{dt}. \tag{2.6}$$

The function F is useful for specifying the exact form of the canonical transformation only when half of the variables (except time) are from the old set and half from the new. It then acts, as it were, as a bridge between the two sets of canonical variables and is called the generating function of the transformation. To show how the generating functions can specify the equations of transformation, we are going to treat an example that is called a basic canonical transformation. Suppose F is given as

$$F = F_1(q, Q, t). \tag{2.7}$$

Then equation 2.6 becomes

$$\begin{aligned} p_i \dot{q}_i - H(q, p, t) &= P_i \dot{Q}_i - K(Q, P, t) + \frac{dF_1}{dt} \\ &= P_i \dot{Q}_i - K(Q, P, t) + \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial q_1}\dot{q}_i + \frac{\partial F_1}{\partial Q_1}\dot{Q}_i. \end{aligned} \tag{2.8}$$

Generating Function	Generating Function Derivatives	Trivial Special Case
$F = F_1(q, Q, t)$	$p_i = \frac{\partial F_1}{\partial q_i}$ $\quad P_i = -\frac{\partial F_1}{\partial Q_i}$	$F_1 = q_i Q_i$, $Q_i = p_i$, $P_i = -q_i$
$F = F_2(q, P, t) - Q_i P_i$	$p_i = \frac{\partial F_2}{\partial q_i}$ $\quad Q_i = \frac{\partial F_2}{\partial P_i}$	$F_2 = q_i P_i$, $Q_i = q_i$, $P_i = p_i$
$F = F_3(p, Q, t) + q_i p_i$	$q_i = -\frac{\partial F_3}{\partial p_i}$ $\quad P_i = -\frac{\partial F_3}{\partial Q_i}$	$F_3 = p_i Q_i$, $Q_i = -q_i$, $P_i = -p_i$
$F = F_4(p, P, t) + q_i p_i - Q_i P_i$	$q_i = -\frac{\partial F_4}{\partial p_i}$ $\quad Q_i = \frac{\partial F_4}{\partial P_i}$	$F_4 = p_i P_i$, $Q_i = p_i$, $P_i = -q_i$

Figure 2.1: Properties of the four basic canonical transformations.[2]

Since the old and new coordinates, q_i and Q_i, are mutually independent the coefficients $\dot{q}$ and $\dot{Q}$ needs to vanish, so

$$p_i = \frac{\partial F_1}{\partial q_1}, \tag{2.9}$$

$$P_i = -\frac{\partial F_1}{\partial Q_1}, \tag{2.10}$$

leaving finally

$$K = H + \frac{\partial F_1}{\partial Q_1}, \tag{2.11}$$

Assuming the equations 2.9 can be inverted, they could then be solved for the n Q_i's in terms of q_i, p_i and t. Once the relations between the Q_i's and the old canonical variables (q, p) have been established, equations 2.10 can be used to give the n P_i's as functions of q_i, p_i and t. Finally equation 2.11 gives the relation between the new Hamiltonian K and the old Hamiltonian H. This procedure described shows how, starting from a given generating function F_1 the equations of the canonical transformations can be obtained. Usually the process can be reversed and we can derive an appropriate generating function from given equations of transformation. The corresponding procedures for the remaining three basic types of generating functions are obvious and the general results are displayed in figure 2.1.
Not all transformations can be expressed in terms of the four basic types. Some transformations are just not suitable for descriptions in terms of these or other elementary forms of generating functions. Furthermore it is possible, and for some canonical transformations necessary, to use a generating function that is a mixture of four types. It is then a mixture in the sense that different coordinates can use different types of the generating function. For this reasons we define F as a unspecified function of $2n$-coordinates and momenta with continuous second derivatives. In formula:

$$F := F(q_1, \ldots, q_n, p_1, \ldots, p_n, Q_1, \ldots, Q_n, P_1, \ldots, P_n) \tag{2.12}$$

An instructive transformation is provided by the generating function of the first type, $F_1(q, Q, t)$, of the form

$$F_1 = q_i Q_i, \tag{2.13}$$

which gives the transformation equations (from 2.9 and 2.10)

$$p_i = \frac{\partial F_1}{\partial q_1} = Q_i, \tag{2.14}$$

$$P_i = -\frac{\partial F_1}{\partial Q_1} = -q_i. \tag{2.15}$$

The transformation interchanges the momenta and the coordinates! This simple example should emphasize the independent status of generalized coordinates and momenta. They are both needed to describe the motion of the system in the Hamiltonian formulation. The distinction between them is basically one of naming.The names can be shift around with at most no more than a change in sign. A transformation that leaves some of the (q, p) pairs unchanged is a canonical transformation of a "mixed" form.[1]

2.2 Symplectic transformations

As an introduction to symplectic transformations we start considering canonical transformations (transformations that preserve the form of Hamilton's equations) in which time does not appear in the equations of the transformation. Those type of canonical transformations are called restricted canonical transformations and give the following equations of transformation:

$$Q_i = Q_i(q, p) \tag{2.16}$$

$$P_i = P_i(q, p), \tag{2.17}$$

with the inverses

$$q_i = q_i(Q, P) \tag{2.18}$$

$$p_i = p_i(Q, P). \tag{2.19}$$

In a restricted canonical transformation the Hamiltonian does not change. By equation 2.2 the transformation will be canonical if

$$\dot{Q}_i = \frac{\partial K}{\partial P_i} = \frac{\partial H}{\partial P_i}. \tag{2.20}$$

We have

$$\dot{Q}_i = \frac{\partial Q_i}{\partial q_j}\dot{q}_j + \frac{\partial Q_i}{\partial p_j}\dot{p}_j = \frac{\partial Q_i}{\partial q_j}\frac{\partial H}{\partial p_j} - \frac{\partial Q_i}{\partial p_j}\frac{\partial H}{\partial q_j} := \{Q, H\}, \tag{2.21}$$

where $\{Q, H\}$ represents the Poisson bracket, which will be explained in the next section. Furthermore we have

$$\frac{\partial H}{\partial P_i} = \frac{\partial H}{\partial q_j}\frac{\partial q_j}{\partial P_i} + \frac{\partial H}{\partial p_j}\frac{\partial p_j}{\partial P_i}, \tag{2.22}$$

so the transformation is canonical, only if

$$\frac{\partial Q_i}{\partial q_j} = \frac{\partial p_j}{P_i}, \qquad \frac{\partial Q_i}{\partial p_j} = -\frac{\partial q_j}{P_i}. \tag{2.23}$$

In a similar way we can compare $\dot{P}_i$ with the partial of H with respect to Q_j to get the conditions

$$\frac{\partial P_i}{\partial q_j} = -\frac{\partial p_j}{Q_i}, \qquad \frac{\partial P_i}{\partial p_j} = \frac{\partial q_j}{Q_i}. \tag{2.24}$$

The algebraic manipulation that leads to equations 2.23 and 2.24 can be preformed in a compact manner if we make use of the symplectic notation for the Hamiltonian formulation. If η is a column matrix with the $2n$ elements q_i, p_i then Hamilton's equations can be written as

$$\dot{\eta} = J\frac{\partial H}{\partial \eta}, \tag{2.25}$$

where J is the matrix

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \tag{2.26}$$

with I the unit n-by-n matrix.[3, 4, 5] Similarly the equations of transformation of a canonical transformation from q_i, p_i to Q_i, P_i take the form

$$\zeta = \zeta(\eta), \tag{2.27}$$

where ζ is a column matrix with the $2n$ elements Q_i, P_i. The equations of motion can be found by looking at the time derivative of a typical element of ζ:

$$\dot{\zeta}_i = \frac{\partial \zeta_i}{\eta_j}\dot{\eta}_j, \qquad i, j = 1, \ldots, 2n \tag{2.28}$$

In matrix notation this gives

$$\dot{\zeta} = M\dot{\eta} \tag{2.29}$$

with M the Jacobian matrix of the transformation:

$$M_{ij} = \frac{\partial \zeta_i}{\partial \eta_j}. \tag{2.30}$$

Combining equations 2.25 and 2.28 gives

$$\dot{\zeta} = MJ\frac{\partial H}{\partial \eta}. \tag{2.31}$$

By the inverse transformation we get

$$\frac{\partial H}{\partial \eta_i} = \frac{\partial H}{\partial \zeta_j}\frac{\partial \zeta_j}{\partial \eta_i}, \tag{2.32}$$

or, in matrix notation

$$\frac{\partial H}{\partial \eta} = M^T\frac{\partial H}{\partial \zeta}. \tag{2.33}$$

Making use of equations 2.31 and 2.33 we get the form of the equations of motion for any set of variables ζ transforming, independently of time, from the canonical set η:

$$\dot{\zeta} = MJM^T\frac{\partial H}{\partial \zeta}. \tag{2.34}$$

From the generator formalism we know that for a restricted canonical transformation the old Hamiltonian expressed in terms of the new variables is the new Hamiltonian, so

$$\dot{\zeta} = J\frac{\partial H}{\partial \zeta}. \tag{2.35}$$

The transformation will therefore be canonical if M satisfies the condition

$$MJM^T = J, \tag{2.36}$$

This condition is called the symplectic condition. The matrix M satisfying the condition is called the symplectic matrix.[1]
The symplectic matrices for a system with n degrees of freedom form the symmetry group Sp($2n$, $\mathbb{R}$).
For a classical system with just one degree of freedom we have the group of symplectic matrices Sp(2,$\mathbb{R}$). The general form of a Sp(2,$\mathbb{R}$)-matrix is given by

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{with} \quad ad - bc = 1. \tag{2.37}$$

[6, 5]

2.3 Poisson brackets

The Poisson bracket of two functions u, v with respect to the canonical variables (q, p) is defined as

$$\{u, v\}_{q,p} = \frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial v}{\partial q_i}. \tag{2.38}$$

Since we have a typical symplectic structure, as in Hamilton's equations (where q is coupled with p and p with $-q$) the Poisson bracket lends itself to being written in matrix form:

$$\{u, v\}_\eta = \frac{\partial u}{\partial \eta}^T J \frac{\partial v}{\partial \eta}. \tag{2.39}$$

The transpose sign is omitted a lot of times, but indicates the fact that the first matrix must be treated as a single-row matrix. It follows from the definition that Poisson brackets of canonical variables are given by

$$\{q_j, q_k\}_{q,p} = 0 = \{p_j, p_k\}_{q,p} \tag{2.40}$$

and

$$\{q_j, p_k\}_{q,p} = \delta_{jk} = -\{p_j, q_k\}_{q,p}. \tag{2.41}$$

We can introduce a square matrix Poisson bracket, $\{\eta, \eta\}$, with element jk given by $\{n_j, n_k\}$. Equations 2.40 and 2.41 can then be summarized as

$$\{\eta, \eta\}_\eta = J \tag{2.42}$$

If we take for u and v the members of the transformed variables (Q, P) represented by ζ, we get the set of all Poisson brackets formed out of (Q, P) by the matrix

$$\{\zeta, \zeta\}_\eta = \frac{\partial \zeta}{\partial \eta}^T J \frac{\partial \zeta}{\partial \eta} = M^T J M, \tag{2.43}$$

where we use that the partial derivatives define the Jacobian matrix M of the transformation. If the transformation from η to ζ is canonical the symplectic condition holds, so we get

$$\{\zeta, \zeta\}_\eta = J, \tag{2.44}$$

and conversely we have that a transformation is canonical if 2.44 holds. We are now going to show that all Poisson brackets are invariant under canonical transformation. We consider the Poisson bracket of two functions u, v with respect to the set of coordinates represented by η. The partial derivative of v with respect to η can be written as

$$\frac{\partial v}{\partial \eta} = M^T \frac{\partial v}{\partial \zeta} \tag{2.45}$$

and in a similar way

$$\frac{\partial u}{\partial \eta}^T = (M^T \frac{\partial u}{\partial \zeta})^T = \frac{\partial u}{\partial \zeta}^T M. \tag{2.46}$$

Now we can write the Poisson bracket

$$\{u, v\}_\eta = \frac{\partial u}{\partial \eta}^T J \frac{\partial v}{\partial \eta} = \frac{\partial u}{\partial \zeta}^T M J M^T \frac{\partial v}{\partial \zeta}. \tag{2.47}$$

If the transformation is canonical, the symplectic condition holds and we have

$$\{u, v\}_\eta = \frac{\partial u}{\partial \zeta}^T J \frac{\partial v}{\partial \zeta} = \{u, v\}_\zeta. \tag{2.48}$$

So the Poisson bracket has the same value when evaluated with respect to any canonical set of variables, thus all Poisson brackets are canonical invariants. This means we can leave away the subscripts of the Poisson brackets.

Another important canonical invariant is the magnitude of a volume element in phase space. A canonical transformation from η to ζ transforms the $2n$-dimensional phase space with coordinates η_i to another phase space with coordinates ζ_i, i.e. volume element

$$d\eta = dq_1 dq_2 \dots dq_n dp_1 dp_2 \dots dp_n \tag{2.49}$$

transforms to the new volume element

$$d\zeta = dQ_1 dQ_2 \dots dQ_n dP_1 dP_2 \dots dP_n. \tag{2.50}$$

The sizes of this two volume elements are related by the absolute value of the Jacobian determinant $det(M)$. Thus,

$$d\zeta = |\det(M)| d\eta. \tag{2.51}$$

In the two dimensional transformation this equation becomes

$$dQdP = \begin{vmatrix} \frac{\partial q}{\partial Q} & \frac{\partial q}{\partial P} \\ \frac{\partial p}{\partial Q} & \frac{\partial p}{\partial P} \end{vmatrix} dqdp = \{q, p\} dqdp. \tag{2.52}$$

But if we take the determinant of both sides of the symplectic condition we get

$$\det(M)^2 \det(J) = \det(J), \tag{2.53}$$

which means that in a real canonical transformation the Jacobian determinant is ± 1. The absolute value is thus always unity, such that

$$d\zeta = d\eta, \tag{2.54}$$

proving the canonical invariance of the volume element in phase space. It follows that the volume of any arbitrary region in phase space is a canonical invariant.

Chapter 3

Coherent states

Coherent states are superpositions of quantum states which have many features analogous to those of their classical counterparts.This are features like properties and dynamical behavior. Coherent states can be viewed as formally close to the classical description and are defined as the eigenvectors of the annihilation operator. They allow for a classical interpretation in a host of quantum situations, but coherent states are strictly quantum states which saturate the Heisenberg inequality ($\sigma_x \sigma_p = \frac{1}{2}\hbar$, where σ represents the standard deviation of the subscripted quantity) which will be shown later on.[7] Before considering coherent states we start with considering monochromatic light from a classical point of view.

3.1 Classical point of view

The sinusoidal electric field strength of monochromatic light can be expressed as a sum of two complex time-varying quantities,

$$E(t) = \frac{1}{2}[a(t) + a^*(t)], \tag{3.1}$$

where $a^*(t)$ stands for the complex conjugate of $a(t)$. The quantities $a(t)$ and $a^*(t)$ are phasors that rotate in the complex plane as time progresses. The complex time-varying quantities can be described in a complex amplitude $a = x + iy$ and a time dependent factor $e^{-i\omega t}$. The electric field strength can now be written as

$$E(t) = x \cos \omega t + y \sin \omega t. \tag{3.2}$$

where

$$x = \frac{a + a^*}{2} \quad \text{and} \quad y = \frac{a - a^*}{2i} \tag{3.3}$$

Since the sine and cosine differ in phase by 90 degrees the components x and y are called quadrature components. They represent, respectively, the real and imaginary parts of the complex amplitude a. The phasor $a(t)$ can be represented either in terms of its x and y projections (cartesian coordinates), or in terms of its magnitude and initial phase ϕ (polar coordinates). The phasor rotates with an angular frequency ω of the optical field. [8]

3.2 Quantum-mechanical point of view

To represent single-mode monochromatic light viewed from a quantum-mechanical point of view the quantities $E(t), a(t), a^*(t), x$, and y must be converted into operators in a Hilbert space. This conversion follows from common principles in quantum mechanics. It follows from the Schrödinger equation that the annihilation operator $a(t)$, and its hermitian conjugate the creation operator $a^\dagger(t)$, obey the boson commutation relation

$$[a(t), a^\dagger(t)] = a(t)a^\dagger(t) - a^\dagger(t)a(t) = 1. \tag{3.4}$$

This means that x and y do not commute with each other, too. They obey the commutation relation.

$$[x, y] = \frac{i}{2}. \tag{3.5}$$

This relation implies a Heisenberg uncertainty relation of the form

$$\sigma_x \sigma_y \geq \frac{1}{4}. \tag{3.6}$$

So the state of minimum uncertainty obeys the equality $\sigma_x\sigma_y = \frac{1}{4}$. The mean values $\bar{x}$ and $\bar{y}$ and their uncertainties σ_x and σ_y together create an area of uncertainty, which is typical for the quantum description (see figure 3.1). Just like for the harmonic oscillator, the average energy in the quantum mode is $\hbar\omega(\bar{n} + \frac{1}{2})$. Here $\hbar\omega$ comes from the energy per photon and $n = a^\dagger a$ is the photon-number operator, which tells how much photons there are in a specific state. As can be seen, the average energy is not equal to zero for $\bar{a} = 0$. The energy for $\bar{a} = 0$, $\frac{1}{2}\hbar\omega$, represent vacuum fluctuations. Since x and y have standard deviations $|a|$ has a standard deviation, $\sigma_{|a|}$. So the photon-number will have an uncertainty. In polar coordinates a phasor has an amplitude standard deviation $\sigma_{|a|}$, a phase-angle uncertainty σ_ϕ and a mean magnitude $\bar{a}$. Using the approximate relationship $n \approx |a|^2$ we get

$$\Delta n \approx 2|a|\Delta|a| \tag{3.7}$$

or, by defining $\sigma_n \equiv \Delta_n$,

$$\sigma_n = 2\bar{n}^{1/2}\sigma_{|a|}. \tag{3.8}$$

The azimuthal uncertainty σ_ϕ can be expressed as the ratio of the arc-length uncertainty to $\bar{n}^{1/2}$. Note that there is no defined operator for the phase. You can compare this to σ_t. The uncertainty in t can be described by the energy-time uncertainty principle $\sigma_E\sigma_t \geq \frac{\hbar}{2}$ [9], while there is no explicit time-operator.

3.3 Coherent State

Now we consider coherent states. A coherent state is represented by a phasor of mean magnitude $\bar{a} = \alpha$ and a surrounding circular area of uncertainty. Coherent states are defined as the eigenfunctions of the lowering operator a_- of the harmonic oscillator, so

$$a_- \mid \alpha\rangle = \alpha \mid \alpha\rangle, \tag{3.9}$$

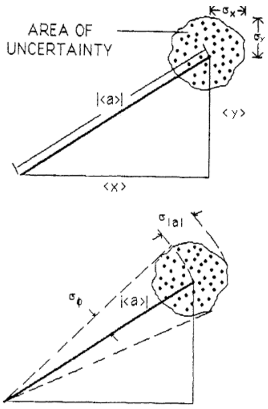

Figure 3.1: Cartesian and polar coordinate representations of the uncertainty area associated with a quantum-mechanical field.[8]

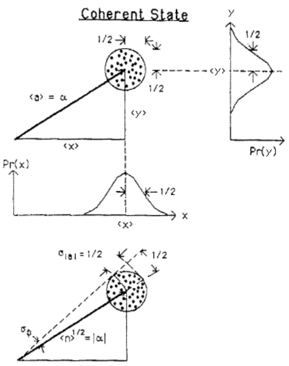

Figure 3.2: Quadrature-component and number-phase uncertainties for the coherent state.[8]

where α can be any complex number. The probability density $P(x)$ of finding the value x is Gaussian, with mean $\bar{x}$ and standard deviation $\sigma_x = \frac{1}{2}$. This is the same for the value y, since the area of uncertainty is circular, so $\sigma_x\sigma_y = \frac{1}{4}$, which means that the coherent state is a minimum-uncertainty relation. [8]
As told earlier the minimum-uncertainty is also reached in the Heisenberg uncertainty relation $\sigma_x\sigma_p = \frac{1}{2}\hbar$. For the stationary states of the harmonic oscillator ($| \, n\rangle = \psi_n(x)$) holds, in general, $\sigma_x\sigma_p = \frac{2n+1}{2}\hbar$. So for stationary states of the Harmonic oscillator only $n = 0$ has minimum uncertainty. Stationary states are solutions of the time-independent Schrödinger equation:

$$\hat{H}\psi = E_\psi\psi. \tag{3.10}$$

Here E_ψ is a real number which corresponds with the eigenvalue of ψ. $\hat{H}$ is the harmonic oscillator Hamiltonian operator.
Although coherent states are no stationary states, they are linear combinations

of stationary states

$$| \alpha\rangle = \sum_{n=0}^{\infty} c_n | n\rangle \tag{3.11}$$

which also minimize the uncertainty product. Before showing that coherent states are indeed minimum uncertainty states, we will first find out what the coefficients c_n exactly are. We need to use that $\psi_n = \frac{1}{\sqrt{n!}}(a_+)^n\psi_0$. Then we have

$$c_n = \langle \psi_n | \alpha\rangle = \frac{1}{\sqrt{n!}}\alpha^n\langle \psi_0 | \alpha\rangle = \frac{\alpha^n}{\sqrt{n!}}c_0 \tag{3.12}$$

c_0 is determined by normalizing α:

$$1 = \sum_{n=0}^{\infty} | c_n |^2 = | c_0 |^2 \sum_{n=0}^{\infty} \frac{| \alpha |^{2n}}{n!} = | c_0 |^2 e^{|\alpha|^2} \tag{3.13}$$

So the coherent state becomes

$$| \alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} | n\rangle. \tag{3.14}$$

Now we are going to show that coherent states indeed minimize the uncertainty limit. We recall that $\sigma_a^2 = \langle a^2\rangle - \langle a\rangle^2$. Since x and p can be expressed in terms of the raising and lowering operators:

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-); \qquad p = i\sqrt{\frac{\hbar m\omega}{2}}(a_+ - a_-) \tag{3.15}$$

So for the Heisenberg relation we need to get the values of $\langle x^2\rangle_\alpha, \langle x\rangle_\alpha^2, \langle p^2\rangle_\alpha$ and $\langle p\rangle_\alpha^2$:

- $\langle x^2\rangle_\alpha = \langle \alpha | x^2 | \alpha\rangle = \frac{\hbar}{2m\omega}[1 + (\alpha + \alpha^*)^2]$
- $\langle x\rangle_\alpha^2 = (\langle \alpha | x | \alpha\rangle)^2 = \frac{\hbar}{2m\omega}(\alpha + \alpha^*)^2$
- $\langle p^2\rangle_\alpha = \langle \alpha | p^2 | \alpha\rangle = \frac{\hbar m\omega}{2}[1 - (\alpha - \alpha^*)^2]$
- $\langle p\rangle_\alpha^2 = (\langle \alpha | p | \alpha\rangle)^2 = -\frac{\hbar m\omega}{2}(\alpha - \alpha^*)^2$

Combining this altogether we get

$$\sigma_x\sigma_p = \sqrt{\frac{\hbar}{2m\omega}}\sqrt{\frac{\hbar m\omega}{2}} = \frac{\hbar}{2} \qquad \square \tag{3.16}$$

Another really nice property of coherent states is that they stay coherent, and continue to minimize the uncertainty product.This can be seen be putting in the time dependence we have for an harmonic oscillator, $| n\rangle \to e^{-iE_nt/\hbar} | n\rangle$. The time-dependent state becomes

$$\begin{aligned} | \alpha(t)\rangle &= \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} e^{-|\alpha|^2/2} e^{-i(n+\frac{1}{2})\omega t} | n\rangle \\ &= e^{-i\omega t/2} \sum_{n=0}^{\infty} \frac{(\alpha e^{-i\omega t})^n}{\sqrt{n!}} e^{-|\alpha|^2/2} | n\rangle. \end{aligned} \tag{3.17}$$

If we compare this to equation 3.14 we see that this is still a coherent state. Apart from the overall phase factor $e^{-i\omega t/2}$, which does not affect its status as an eigenfunction of a_-, $\mid \alpha(t)\rangle$ is the same as $\mid \alpha\rangle$, but with eigenvalue $\alpha(t) = e^{-i\omega t}\alpha$. [9] The eigenvalue behaves like a classical field. To show this we first get an alternative expression for the coherent state α.

$$\begin{aligned} \mid \alpha\rangle &= e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{(\alpha a_+)^n}{\sqrt{n!}} \mid 0\rangle \\ &= e^{-|\alpha|^2/2} e^{\alpha a_+} \mid 0\rangle \end{aligned} \tag{3.18}$$

Now it is clear that we have $\langle\alpha \mid a_- \mid \alpha\rangle = \alpha$ and $\langle\alpha \mid a_+ \mid \alpha\rangle = \alpha^*$. Using these equations we get

$$E_{effective}(\alpha, t) = \langle e^{-i\omega t}\alpha \mid \hat{A}(t) \mid e^{-i\omega t}\alpha\rangle = \sqrt{\frac{\hbar}{2\epsilon_0\omega V}}[\alpha\epsilon e^{-i\omega t+ik\cdot r} + \alpha^*\epsilon e^{i\omega t-ik\cdot r}], \tag{3.19}$$

where $\hat{A}$ stands for the magnetic vector potential. Comparing this to the classical electric field of monochromatic light as given in equation 3.1 we conclude that the field expectation values in a coherent state behave as a monochromatic classical field. [10]

Furthermore for the coherent state the probability density $P(n)$ of the photon-number is a Poisson distribution. This can be seen by looking at equation 3.8:

$$\sigma_n = 2\bar{n}^{1/2}\sigma_{|a|}. \tag{3.20}$$

For a coherent state we have that the area of uncertainty is circular and thus $\sigma_{|a|} = \frac{1}{2}$, giving

$$\sigma_n = 2\bar{n}^{1/2}\frac{1}{2} = \bar{n}^{1/2} \tag{3.21}$$

so the photon-number variance σ_n^2 is equal to the photon-number mean $\bar{n}$, in accordance with the Poisson distribution. Since the area of uncertainty is circular we have, besides an amplitude standard deviation of $\frac{1}{2}$ and an azimuthal arc-length uncertainty of $\frac{1}{2}$, leads to the number-phase equality

$$\sigma_n\sigma_\phi = \frac{1}{2}. \tag{3.22}$$

[8]
The state with $\alpha = 0$, and thus $\bar{x} = \bar{y} = 0$, is given by

$$a_- \mid \psi_0\rangle = 0. \tag{3.23}$$

This is also a coherent state, with eigenvalue $\alpha = 0$. This state is known as the vacuum state. [9, 10]
Unlike the classical electric field $E(t)$ the quantum electric field is always uncertain. Each value of α in the uncertainty circle traces out a sinusoidal time function, of appropriate magnitude and phase, determined by its projection on the x-axis (the real part). For coherent states, including the vacuum state, the noise about the mean is phase independent.

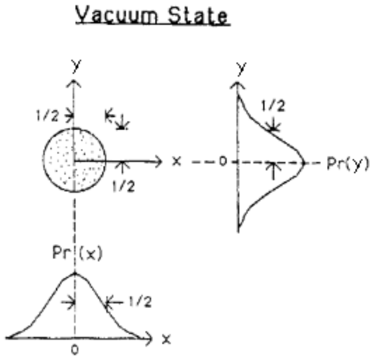

Figure 3.3: Quadrature-component uncertainties for the vacuum state.[8]

3.4 Displacement operator

Coherent states can be generated by acting with the displacement operator on the vacuum

$$\mid \alpha\rangle = D(\alpha) \mid 0\rangle \tag{3.24}$$

where $D(\alpha)$ is defined as

$$D(\alpha) := \exp[\alpha a^\dagger - \alpha^* a] \tag{3.25}$$

which also can be written as

$$D(\alpha) = e^{\alpha a^\dagger} e^{-\alpha^* a} e^{-|\alpha|^2/2} \tag{3.26}$$

The displacement operator is a unitary operator, since

$$D^\dagger(\alpha) = D^{-1}(\alpha) = D(-\alpha). \tag{3.27}$$

It can be shown that

$$D^\dagger(\alpha) a D(\alpha) = a + \alpha \tag{3.28}$$

$$D^\dagger(\alpha) a^\dagger D(\alpha) = a^\dagger + \alpha^* \tag{3.29}$$

To discuss the displacement operator in more detail we first have to learn something about some symmetry groups.

3.5 The Husimi function

The coherent state can also be used to represent the Husimi function. The Husimi function is one of the simplest distributions of quasiprobability in phase space. If the normal probability function quantum mechanical state ψ, $\rho_t(\mathbf{x}) = |\psi(\mathbf{x}, t)|^2$, is computed a lot information about the quantum mechanical state will be lost. Meanwhile, the Husimi function can be used to encode the full quantum information, so there is no loss of information about a state.[11]
The Husimi function can be defined directly in terms of coherent states $\mid \alpha\rangle$. If we have the density operator $\hat{\rho} = |\psi\rangle\langle\psi|$, then we have the Husimi function

$$\mathcal{H}_\psi(\alpha) = \langle\alpha \mid \hat{\rho} \mid \alpha\rangle = \mathrm{tr}\hat{\rho} \mid \alpha\rangle\langle\alpha \mid = \mid \langle\alpha \mid \psi\rangle \mid^2 . \tag{3.30}$$

The Husimi function can also be defined in terms of Wigner functions. Therefore we first define the Wigner function for a $2L$-dimensional phase space:

$$W(x) = \frac{1}{(\pi\hbar)^L} \mathrm{tr} \hat{\rho} \hat{R}_x, \tag{3.31}$$

where $\hat{R}_x$ the operator for the reflection through the point $x = (p_1, \ldots, p_L, q_1, \ldots, q_L)$:

$$\hat{R}_x = \frac{1}{2^L} \int dQ \mid q - \frac{Q}{2} \rangle \langle q - \frac{Q}{2} \mid e^{ip \cdot Q/\hbar}. \tag{3.32}$$

Now the Husimi function is defined as

$$\mathcal{H}(\alpha) = \frac{1}{(\pi\hbar)^L} \int dx W(x) \exp \frac{-(x-\alpha)^2}{\hbar}. \tag{3.33}$$

The Husimi function is a way to represent a state as a function on phase space, whereas a wave function is a function in position or momentum only. [12]

Chapter 4

Symmetry groups

After we have seen some properties of coherent states we now will look which transformations will let coherent states remain coherent states. Since the uncertainty is a disk in phase space we can say in other words that where looking for transformations that keep the open disc in phase space invariant. To get these transformations we have to use some group theory. We first consider a number of various groups which will be useful.

4.1 SU(2)

The group SU(2) is the special unitary group of 2 dimensions. The group contains all complex unitary 2-by-2 matrices with determinant one. In formula:

$$SU(2) = \{U \in GL(2, \mathbb{C}) \mid \det(U) = 1, U^\dagger = U^{-1}\} \tag{4.1}$$

Here $GL(2, \mathbb{C})$ is the general linear group. A SU(2) matrix can be written in the general form

$$\begin{pmatrix} \alpha_1 + i\alpha_2 & \beta_1 + i\beta_2 \\ -\beta_1 + i\beta_2 & \alpha_1 - i\alpha_2 \end{pmatrix} \tag{4.2}$$

with α_1, α_2, β_1 and β_2 four real numbers which confirm the relation

$$\alpha_1^2 + \alpha_2^2 + \beta_1^2 + \beta_2^2 = 1 \tag{4.3}$$

The number of independent parameters is three. The relation between the four real ones defines the surface of a three-dimensional sphere embedded in four dimensional Euclidean space.
The Lie algebra of SU(2) consists of the three generators J_0, J_1 and J_2 and is defined by the commutation relations

$$[J_1, J_2] = iJ_0, \quad [J_0, J_1] = iJ_2, \quad [J_2, J_0] = iJ_1 \tag{4.4}$$

The generators of SU(2) can be represented by a set of three linearly independent, traceless 2-by-2 anti-Hermitian matrices which are proportional to the Pauli-matrices via

$$J_k = \frac{\sigma_i}{2}. \tag{4.5}$$

Here σ_k are the Pauli matrices

$$\sigma_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_0 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.6}$$

Since the generators do not commute with one another SU(2) is a non-Abelian group. The group has three parameters, too. These are given by

$$U_\theta = e^{-\theta_k J_k} \quad \text{where} \quad k = 0, 1, 2 \tag{4.7}$$

Every element can now be written in the form $U = \exp(\sum_{k=1}^{3} \theta_k J_k)$. In terms of the Pauli matrices we get

$$U = e^{i \sum_{k=1}^{3} \frac{\theta_k}{2} \sigma_1} = \cos\frac{\theta}{2} + i \sin\frac{\theta}{2} \hat{n} \cdot \vec{\sigma}, \tag{4.8}$$

where $\theta_i = \theta n_1$ and $\hat{n} = (n_1, n_2, n_3)$. The angle θ can run over the interval $[-2\pi, 2\pi]$. Now the parameter space can be seen as a filled sphere of radius 2π, with all the points on the surface identified with each other. [13, 3, 14]

We now make a change of basis and define the ladder operators $J_\pm$ by

$$J_\pm = J_1 \pm i J_2 \tag{4.9}$$

and the Casimir operator

$$J^2 = J_1^2 + J_2^2 + J_0^3. \tag{4.10}$$

A Casimir operator is a quadratic operator that commutes with all elements of the Lie algebra, in this case every J_k. We know from quantum mechanical applications that J^2 has eigenvalues $j(j+1)$ with $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$.
The commutation relations become

$$[J^2, J_\pm] = 0, \quad [J_0, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = 2J_0 \tag{4.11}$$

Since J^2 and J_0 commute, they can be diagonalized simultaneously. The eigenvalue of J_0 gives the well-known m which runs over $2j+1$ values from $-j$ to j. [13]

4.2 SU(1,1)

The next group we are going to consider is the group SU(1,1). The group SU(1,1) is another special unitary group. SU(1,1) consists of all non-singular 2-by-2 matrices which leave the matrix $g_1 = \text{diag}(1, -1)$ invariant. This leaves us with the definition

$$SU(1,1) = \{U \in GL(2, \mathbb{C}) \mid \det(U) = 1, U^\dagger = gU^{-1}g^{-1}\}, \tag{4.12}$$

where $g = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. An SU(1,1) matrix can be written in the general form

$$\begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix} \tag{4.13}$$

with α and β two complex numbers which confirm the relation

$$|\alpha|^2 - |\beta|^2 = 1 \tag{4.14}$$

[13, 3]

The Lie algebra of SU(1,1) consists of the three generators K_0, K_1 and K_2 and satisfy the commutation relations

$$[K_1, K_2] = -iK_0, \quad [K_0, K_1] = iK_2, \quad [K_2, K_0] = iK_1 \tag{4.15}$$

[6, 15]
The generators of SU(1,1) can be represented, just like SU(2), as matrices proportional to the Pauli matrices (4.6). They are proportional via

$$K_1 = \frac{i}{2}\sigma_2 \quad K_2 = -\frac{-i}{2}\sigma_1 \quad K_0 = \frac{1}{2}\sigma_3 \tag{4.16}$$

[13, 3]
Just like for SU(2), we can also choose a different basis

$$K_\pm = K_1 \pm iK_2 \tag{4.17}$$

Now the commutation relations become

$$[K_0, K_\pm] = \pm K_\pm, \quad [K_+, K_-] = -2K_0, \tag{4.18}$$

where we have a difference in sign compared to SU(2).
For SU(1,1) the Casimir operator becomes

$$C = K_0^2 - K_1^2 - K_2^2 \tag{4.19}$$

The eigenvalue of C is equal to $k(k-1)$. The parameter k is called the Bargmann index and is a positive real number. For the representations of interest the states $| k, m\rangle$ diagonalize the operator K_0:

$$K_0 \mid k, m\rangle = (k + m) \mid k, m\rangle, \tag{4.20}$$

where m can be any nonnegative integer.
All states can be obtained from the lowest state $| k, 0\rangle$ by the action of K_+ according to

$$| k, m\rangle = \sqrt{\frac{\Gamma(2k)}{m!\Gamma(2k+m)}}(K_+)^m \mid k, 0\rangle \tag{4.21}$$

[14]

4.3 Möbius Transformations

A Möbius transformation is a transformation of the form

$$f(z) = \frac{az + b}{cz + d}, \tag{4.22}$$

with a, b, c and $d \in \mathbb{C}$ and $ad \neq bc$. These kind of transformations are complex maps, which are useful in many applications. [16]
We define

$$f(\infty) = \begin{cases} \frac{a}{c} & \text{if } c \neq 0 \\ \infty & \text{if c=0} \end{cases} \quad \text{and} \quad f(-\frac{d}{c}) = \infty \text{ if } c \neq 0 \tag{4.23}$$

[17]
Since the derivative of $f(z)$,

$$f'(z) = \frac{ad - bc}{(cz + d)^2}, \tag{4.24}$$

does not vanish, the Möbius transformation $f(z)$ is conformal at every point except its pole $z = -d/c$. This means that the map preserves angles.
For $c = 0$ we clearly have a linear transformation. For $c \neq 0$ we can show the decomposition by writing

$$\frac{az + b}{cz + d} = \frac{\frac{a}{b}(cz + d) - \frac{ad}{c} + b}{cz + d} = \frac{a}{c} + \frac{b - \frac{ad}{c}}{cz + d}. \tag{4.25}$$

We now can see that the Möbius transformation can be expressed as a linear combination $w_1 = cz + d$, followed by an inversion $w_2 = 1/w_1$ and thereafter again a linear transformation $w = (b - ad/c)w_2 + a/c$.[17, 18] This results into the following properties of Möbius transformations:

- $f(z)$ can be expressed as the composition of a finite sequence of translations, magnifications, rotations and inversions.
- $f(z)$ is a $1 - 1$ map of the extended complex plane ($\mathbb{C} \cup \{\infty\}$) onto itself.
- $f(z)$ maps the class of circles and lines into itself.
- $f(z)$ is conformal at every point except its origin.

[18] The first property may need some explanation. It states that $f(z)$ is a finite sequence of translations, magnifications, rotations and inversions. The four operations are in formula given as

$$\begin{aligned} \text{translations}: &\quad z \mapsto z + b, \quad b \in \mathbb{C} \\ \text{magnifications}: &\quad z \mapsto az, \quad a \in \mathbb{C}\ \{0\} \\ \text{rotations}: &\quad z \mapsto (\cos\theta + i\sin\theta)z = e^{i\theta}z, \quad \theta \in \mathbb{R} \\ \text{inversions}: &\quad z \mapsto \frac{1}{z} \end{aligned} \tag{4.26}$$

We have shown that a Möbius transformation can be expressed as a linear combination, followed by an inversion and then a linear transformation. These three operations all can be expressed in combinations of the four basic maps given in 4.26.
Furthermore note that our goal is to try to find transformations that leave a disc in phase space invariant. So the third property is interesting for us especially. The possibilities for this property are as follows. A line or circle that doesn't pass through the pole $z = -d/c$ of the Möbius transformation, gets mapped

into a circle. If a line or circle does pass through the pole, it gets mapped to a unbounded figure, its image is a straight line. We can think of a line as a circle that happens to go through infinity. [17, 18]
Next we are going to show that the Möbius transformation describes a group. We start by calculating the inverse of an arbitrary Möbius transformation given by

$$f(z) = \frac{az+b}{cz+d} \qquad (ad \neq bc). \tag{4.27}$$

The inverse can easily be calculated by expressing z in terms of w, giving

$$z = f^{-1}(w) = \frac{dw-b}{-cw+a}. \tag{4.28}$$

So the inverse of any Möbius transformation is again a Möbius transformation. Moreover, if we take the composition of two Möbius transformations,

$$f_1(z) = \frac{a_1 z + b_1}{c_1 z + d_1} \quad \text{and} \quad u = f_2(w) = \frac{a_2 w + b_2}{c_2 w + d_2}, \tag{4.29}$$

we have that

$$u = f_2(f_1(z)) = \frac{(a_2 a_1 + b_2 c_1)z + (a_2 b_1 + b_2 d_1}{(c_2 a_1 z + d_2 c_1)z + (c_2 b_1 + d_2 d_1)}. \tag{4.30}$$

. This is again a Möbius transformation. The last part we have to concern about is the identity element of the group. However if we take $f_1^{-1}(f_1)$ we definitely get the identity function $I(z) = z$. [18] We have that the collection of all Möbius transformations form a group denoted by PGL(2,$\mathbb{C}$), the projective general linear group. The map

$$\phi : \mathrm{GL}(2, \mathbb{C}) \mapsto \mathrm{PGL}(2, \mathbb{C}); \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto f(z) \tag{4.31}$$

is a group homomorphism where GL(2,$\mathbb{C}$) is the general linear group, is the set of 2-by-2 invertible matrices.A matrix is in the kernel of this homomorphism when

$$\frac{az+b}{cz+d} = z \qquad \text{for all} z \in \mathbb{C} \cup \{\infty\}. \tag{4.32}$$

This occurs if and only if $a = d$ and $b = c = 0$, so $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \lambda I$ for some scalar $\lambda \in \mathbb{C}\ \{0\}$. This shows that a M "obius transformations is unaltered when we multiply each coefficients a, b, c, d by a non-zero scalar λ. We can always choose λ so that the determinant $ad - bc = 1$. Then the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is in the special linear group SL(2,$\mathbb{C}$), defined as all 2-by-2 matrices with $a, b, c, d \in \mathbb{C}$ and $ad - bc = 1$. Now

$$\phi : \mathrm{SL}(2, \mathbb{C}) \mapsto \mathrm{PGL}(2, \mathbb{C}); \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto f(z) \tag{4.33}$$

is a group homomorphism whose kernel consists of the two matrices $\pm I$. Consequently, the Möbius group is the quotient SL(2,$\mathbb{C}$)/$\mathbb{Z}_2$. So we have found PGL(2,$\mathbb{C}$ = SL(2,$\mathbb{C}$)/$\mathbb{Z}_2$.[19]

Note that we also have found that PGL(2,ℂ) =PSL(2,ℂ). Since Möbius transformations were unaltered when multiplied by a non-zero scalar λ all matrices in PGL(2,ℂ) can be set to matrices with determinant 1, which exactly defines the projective special linear group PSL(2,ℂ). It turns out that for all projective general linear groups PGL(n, F) holds that PGL(n, F) equals PSL(n, F) if and only if every element of F has an nth root in F. So, for example, we have that PSL(2,ℂ) = PGL(2,ℂ), but PSL(2,ℝ) < PGL(2,ℝ).[20]

4.4 A closer look at SU(1,1)

Now we are going use the acquired knowledge of symmetry groups in single mode optics. The radiation field can be described by the bosonic operators a and $a^\dagger$. We obtain a realization of the $su(1,1)$ algebra if we form the quadratic combinations

$$K_+ = \frac{1}{2\sqrt{2}}(a^\dagger)^2, \quad K_- = \frac{1}{2\sqrt{2}}a^2, \quad K_0 = \frac{1}{4}(1 + 2a^\dagger a) \tag{4.34}$$

In this case the Casimir operator reduces identically to

$$C = k(k-1) = -\frac{3}{16} \tag{4.35}$$

so we have $k = \frac{1}{4}$ or $k = \frac{3}{4}$. [21, 6] The action of the operators is relatively simple. If we start from the vacuum state we note that

$$K_- \mid 0\rangle = 0 \tag{4.36}$$

and

$$K_+ \mid 0\rangle = \frac{1}{2} \mid 2\rangle. \tag{4.37}$$

By repeated application of the raising operator K_+ we obtain an infinite sequence of states

$$(K_+)^m \mid 0\rangle = \frac{\sqrt{(2m)!}}{2\sqrt{2}} \mid 2m\rangle. \tag{4.38}$$

Each state is an eigenstate of K_0

$$K_0 \mid 2n\rangle = \frac{1+4n}{4} \mid 2n\rangle. \tag{4.39}$$

Now recall equation 4.21:

$$\mid k, m\rangle = \sqrt{\frac{\Gamma(2k)}{m!\Gamma(2k+m)}}(K_+)^m \mid k, 0\rangle \tag{4.40}$$

and fill in 4.38 to obtain

$$\mid k, m\rangle = \sqrt{\frac{\Gamma(2k)}{m!\Gamma(2k+m)}}\frac{\sqrt{(2m)!}}{2\sqrt{2}} \mid 2m\rangle = \sqrt{\frac{(2m)!\Gamma(2k)}{m!8\Gamma(2k+m)}} \mid 2m\rangle \tag{4.41}$$

So we get an infinite sequence of states $\mid 0\rangle, \mid 2\rangle, \ldots, \mid 2n\rangle, \ldots$ that forms a representation of the algebra where the spectrum K_0 is bounded below by the value $\frac{1}{4}$. In the same way we can get an infinite sequence of states $| 1\rangle, | 3\rangle, \ldots, | 2n+1\rangle, \ldots$ where the spectrum starts at $\frac{3}{4}$. So we have that states with even $2m$ form a basis for the unitary representation with $k = \frac{1}{4}$, while the states with odd n form a basis for the case $k = \frac{3}{4}$. The two infinite towers are called singleton representations since they involve only one harmonic oscillator. The two different singleton seem a bit strange, since the one-dimensional oscillator has no particular symmetry of its own. It happens to come from the fact that the Hamiltonian

$$H = a^\dagger a + \frac{1}{2} = 2K_0 \tag{4.42}$$

is itself a member of the algebra. The algebra relates states of different energy; such an algebra is called a spectrum generating algebra. Each representation contains all of its states of a given parity: all states in the singleton $| 0\rangle, | 2\rangle, \ldots, | 2n\rangle, \ldots$ have even parity and all states in the singleton $| 1\rangle, | 3\rangle, \ldots, | 2n+1\rangle, \ldots$ have odd parity. The singleton representations is of course not the only representation of the su(1,1) algebra, some other representations can be found in [21].

In terms of $su(1,1)$ algebra canonical transformations of are generated by the vector fields

$$\{-q\frac{\partial}{\partial p} + p\frac{\partial}{\partial q} = 2iK_0, \quad -q\frac{\partial}{\partial p} - p\frac{\partial}{\partial q} = 2iK_1, \quad -q\frac{\partial}{\partial q} + p\frac{\partial}{\partial p} = 2iK_2\} \tag{4.43}$$

[6, 5] These operators clearly have the same commutation relation, where the Poisson bracket is used as the product, as the su(1,1) algebra. We are now going to show that Sp(2,$\mathbb{R}$) and SU(1,1) do not only share the same Lie algebra: they are isomorphic, too! Two groups are isomorph if there is an isomorphism between them. An isomorphism is a bijective homomorphism, so in other words we have to find a map which gives a one-one correspondence and is structure-preserving (which means that for a map $\phi : G \mapsto G'$ and all group elements $g_{i,j} \in G$ we want to have $\phi(g_i \circ g_j) = \phi(g_1) \cdot \phi(g_2)$[22]). Consider the matrix

$$T = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \tag{4.44}$$

T is a unitary matrix since we have

$$T^{-1} = \frac{1}{\frac{1}{2}+\frac{1}{2}}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} = T^\dagger \tag{4.45}$$

and

$$\det(T) = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} - \frac{-i}{\sqrt{2}}\frac{-i}{\sqrt{2}} = 1 \tag{4.46}$$

Recall that a matrix in Sp(2,$\mathbb{R}$) can be written in the general form

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{with} \quad ad - bc = 1. \tag{4.47}$$

This is exactly the same form as the defining matrix of SL(2,$\mathbb{R}$). If we do a similarity transformations on M using T we get

$$TMT^\dagger = \frac{1}{2}\begin{pmatrix} a+d+i(b-c) & b+c+i(a-d) \\ b+c+i(d-a) & a+d-i(b-c) \end{pmatrix} \tag{4.48}$$

Let us now define

$$\alpha := \frac{1}{2}[a + d + i(b - c)] \quad \text{and} \quad \beta := \frac{1}{2}[b + c + i(a - d)] \tag{4.49}$$

Then we get the following matrix

$$TMT^{\dagger} = \begin{pmatrix} \alpha & \beta \\ \beta^{*} & \alpha^{*} \end{pmatrix} := U_M \tag{4.50}$$

with determinant

$$\begin{aligned} \det(U_M) &= \mid \alpha \mid^2 - \mid \beta \mid^2 \\ &= \frac{(a+d)^2}{4} + \frac{(b-c)^2}{4} - [\frac{(b+c)^2}{4} + \frac{(a-d)^2}{4} \\ &= \frac{1}{4}(4ad - 4bc) \\ &= ad - bc = 1 \end{aligned} \tag{4.51}$$

So $U_M \in$ SU(1,1) and we can conclude that T is a bijective map. The only thing left to show is that the map is homomorphic, this means that we have to show that

$$U_{MM'} = U_M U_{M'} \tag{4.52}$$

So we want to have

$$TMM'T^{\dagger} = TMT^{\dagger}TM'T^{\dagger} \tag{4.53}$$

However this is clearly the case since T is a unitary matrix and thus $T^{\dagger}T = T^{-1}T = I$ We now have shown that T is an isomorphism between Sp(2,$\mathbb{R}$) and SU(1,1), so Sp(2,$\mathbb{R}$) $\cong$ SU(1,1).

Furthermore we have by section 4.3 that the group of Möbius transformations can be represented by matrices of the form:

$$\frac{az + b}{cz + d} \sim \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{4.54}$$

As we have $ad \neq bc$ the matrix is invertible. If we now choose $c = b^*$ and $d = a^*$ we get

$$\frac{az + b}{b^* z + a^*} \sim \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{4.55}$$

Since a common vector is unimportant in the transformation, we can associate it with the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{with} \quad |a|^2 - |b|^2 = 1, \tag{4.56}$$

but this is exactly the general form of the SU(1,1) matrices given in equation 4.13. So we have found that all SU(1,1) matrices are in fact Möbius transformations!

4.5 The super-Lie algebra of SU(1,1)

Recall the infinite towers of states defined by the equations 4.38 and 4.39, thus $| \, 0\rangle, | \, 2\rangle, \ldots, | \, 2n\rangle, \ldots$ where the spectrum K_0 is bounded below by the value $\frac{1}{4}$ and $| \, 1\rangle, | \, 3\rangle, \ldots, | \, 2n+1\rangle, \ldots$ where the spectrum starts at $\frac{3}{4}$. In a sense these singleton irreps are the simplest unitary irreps of the $su(1,1)$ algebra, but they are actually part of a representation of a super algebra, which is an algebra closed under both commutators and anti-commutators. The states of the two singleton representation, $| \, 2n\rangle$ and $| \, 2n+1\rangle$, can be related to one another by the application of the creation operator $a^\dagger$. In this way we can extend the Lie algebra to include the operators a and $a^\dagger$ that relate the two singleton representations. The commutator of operators a and $a^\dagger$ is not in the Lie algebra, but the anti-commutator, where we have a plus sign instead of the minus sign, is:

$$[a, a^\dagger]_- := aa^\dagger + a^\dagger a = 1 + 2a^\dagger a = 4K_0. \tag{4.57}$$

Furthermore we have

$$[K^+, a] = -\frac{1}{\sqrt{2}}a^\dagger, \qquad [K_0, a] = -\frac{1}{2}a. \tag{4.58}$$

We have to extend the Lie-algebra operation to include both commutators and anti-commutators to obtain a super-Lie algebra. [21] Since the displacement operator is defined as

$$D(\alpha) := \exp[\alpha a^\dagger - \alpha^* a] \tag{4.59}$$

[6] it is not a function with generators of SU(1,1) in the exponent. The displacement operator is not an element of SU(1,1), however is is an element of the super-Lie group. The super-Lie algebra defines a unique double cover of SU(1,1), and thus of SP(2,$\mathbb{R}$). This double cover is probably the metaplectic group and is denoted as Mp(2,$\mathbb{R}$).[23, 24, 25]

Chapter 5

Squeezed states

Now let us go back to quantum optics. Coherent states can be transformed into squeezed states, which means that one of its standard deviations will be made smaller while keeping the minimal uncertainty property. States can be squeezed in various ways. We will first treat quadrature squeezed states, then photon-number squeezed states and we will end this chapter with a section about the squeeze operator.

5.1 Quadrature squeezed states

Let's first consider the quadrature squeezed state. A state is quadrature squeezed, by definition, if any of its quadratures has a standard deviation that falls below the coherent-state value of $\frac{1}{2}$. If the uncertainty in quadrature component is squeezed below $\frac{1}{2}$, the uncertainty in the other quadrature need to be stretched above $\frac{1}{2}$, because by the Heisenberg uncertainty relation the product must at least have a value of $\frac{1}{4}$.
A field in a minimum uncertainty state can be quadrature squeezed by multiplying its x-component by the factor e^{-r} and its y-component by the factore^{r}. The positive quantity r is called the squeeze parameter. It happens to be convenient to include a phase factor $e^{i\xi}$ in one of the quadratures. The resulting electric field becomes

$$E_s(t) = xe^{-r}e^{i\xi}\cos\omega t + ye^{r}\sin\omega t \tag{5.1}$$

The x-component uncertainty σ_x is squeezed to $e^{-r}\sigma_x$ and simultaneously the y-component uncertainty σ_y is stretched to $e^{r}\sigma_y$. In this way the vacuum states for example becomes the squeezed vacuum state. They both are minimum uncertainty states. The squeezed vacuum states in no longer truly a vacuum state, since the mean photon number is no longer zero:

$$\bar{n} = \sinh^2 r > 0. \tag{5.2}$$

Furthermore its photon-number statistics are super-Poissonian, since its variance is twice the Bose-Einstein (geometric) distribution,

$$\sigma_n^2 = 2(\bar{n} + \bar{n}^2). \tag{5.3}$$

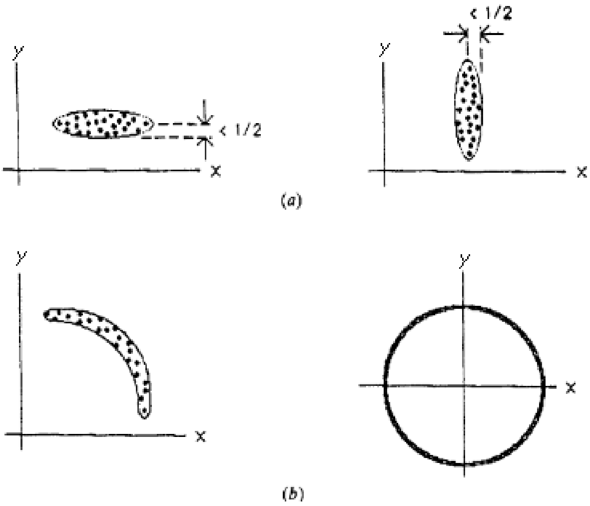

Figure 5.1: (*a*) quadrature squeezed states, (*b*) photon-number squeezed states.[8]

A coherent state in general can be similarly transformed into a squeezed state, then we have the state $SD \mid 0\rangle$. Here the factor $e^{i\xi}$ is used. By changing the angle ξ relative to the angle of α, the angle θ between the major axis of the ellipse and the phasor α is controlled.
The mean photon number

$$\bar{n} = |\alpha|^2 + \sinh^2 r \tag{5.4}$$

has a coherent part $|\alpha|^2$ and a squeeze part $\sinh^2 r$. For $|\alpha|^2 \geq e^{2r}$ its variance is

$$\sigma_n^2 = \bar{n}(e^{2r}\cos^2\theta + e^{-2r}\sin^2\theta). \tag{5.5}$$

Depending on the angle θ the squeezed coherent state can exhibit either super-Poisson or sub-Poisson photon statistics. The variance is largest when the major axis of the ellipse aligns with the phasor, so for θ an even integer multiple of $\pi/2$. This position gives a large uncertainty in the radial direction, thus a large photon-number variance. For θ with an odd integer multiple of $\pi/2$ the minor ellipse axis is aligning with the phasor. This gives a small uncertainty in the radial direction and thus a small photon-number variance, , see figures 5.2 and 5.3. The electric field uncertainty falls off to a minimum periodically. The noise is reduced below the coherent-state value at certain preferred values of the phase, but is increased at other values of the phase.

5.2 Photon-number squeezed state

Now let us turn to photon-number squeezing. A state is photon-number squeezed, by definition, if its photon-number uncertainty σ_n falls below the coherent-state value of $\bar{n}^{1/2}$. If the uncertainty in n is squeezed, the uncertainty in ϕ needs

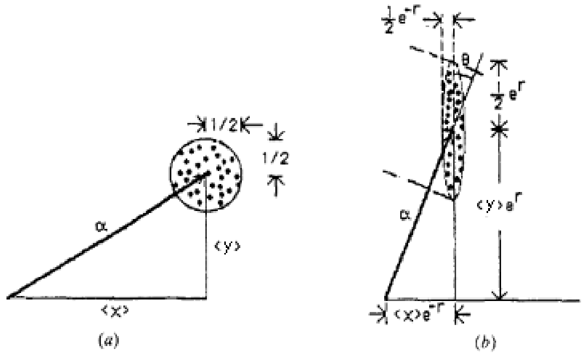

Figure 5.2: Comparison of quadrature-component uncertainties for a coherent state (a) and a quadrature squeezed state $DS \mid 0\rangle (b)$. [8]

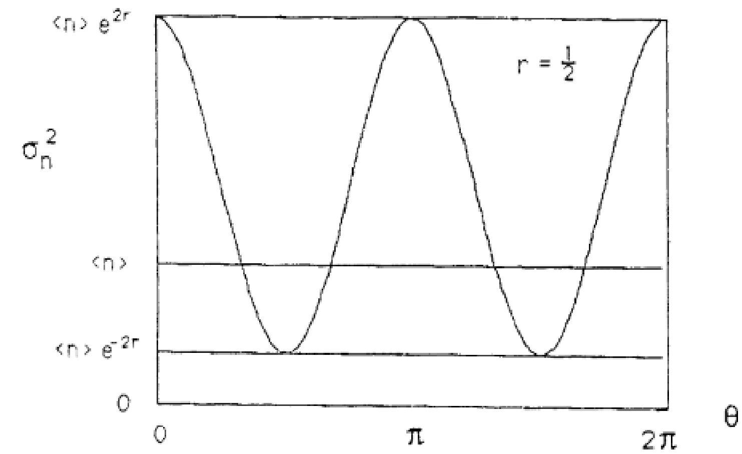

Figure 5.3: Dependence of the squeezed coherent state photon-number variance, σ_n^2, on the angle θ. The maxima represent phase squeezed states and the minima represent photon-number squeezed states.[8]

to be stretched. An example of a photon-number squeezed state is the number state with the properties

$$\sigma_x = (\frac{n}{2} + \frac{1}{4})^{1/2}; \qquad \sigma_y = (\frac{n}{2} + \frac{1}{4})^{1/2} \tag{5.6}$$

$$\sigma_{|a|} \approx 1; \qquad \sigma_\phi = \infty \tag{5.7}$$

$$\sigma_a / \bar{a} = \bar{n}^{1/2} \tag{5.8}$$

$$\sigma_n = 0 \tag{5.9}$$

Its quadrature uncertainties are symmetrical, but large. The state is not a minimum-uncertainty state. In polar coordinates the phase is totally uncertain though its magnitude is rather restricted. The uncertainty area becomes a ring. (See figure 5.1.) The mean photon-number variance is equal to zero for this state, so the state is photon-number squeezed since $\sigma_n < \bar{n}$. The electric-field for the number state is phase-independent. [8]

5.3 Squeeze operator

It is possible to transform a vacuum state into squeezed state by the use of a squeeze operator S. We define the squeeze operator

$$S(\epsilon) := \exp[\frac{\epsilon^*}{2}a^2 - \frac{\epsilon}{2}a^{\dagger 2}] = \exp(\epsilon^* K_- - \epsilon K_+) \tag{5.10}$$

where $\epsilon = re^{2i\phi}$.[6] The squeeze operator obeys the relation

$$S^\dagger(\epsilon) = S^{-1} = S(-\epsilon) \tag{5.11}$$

and is thus unitary. We can prove that

$$S^\dagger(\epsilon) a S(\epsilon) = a\cosh(r) - a^\dagger e^{-2i\theta}\sinh(r) \tag{5.12}$$

$$S^\dagger(\epsilon) a^\dagger S(\epsilon) = a^\dagger \cosh(r) - a e^{-2i\theta}\sinh(r) \tag{5.13}$$

We define the field quadrature components

$$X_1 = a + a^\dagger \tag{5.14}$$

$$X_2 = -i(a - a^\dagger). \tag{5.15}$$

They obey the commutation relation

$$[X_1, X_2] = 2i \tag{5.16}$$

So we get

$$S^\dagger(\epsilon)(Y_1 + iY_2)S(\epsilon) = e^{-r}Y_1 + iY_2e^r \tag{5.17}$$

where we define $Y_1 + iY_2 := (X_1 + iX_2)e^{-i\theta}$ (see figure 5.4).

Moreover, the squeeze operator is an element of SU(1,1). Since the squeeze operator is defined as given in equation 5.10, it is a function with generators of SU(1,1) in the exponent and thus an element of SU(1,1). This means that all properties we have found for SU(1,1) (and thus Sp(2,ℝ) holds for the squeezing

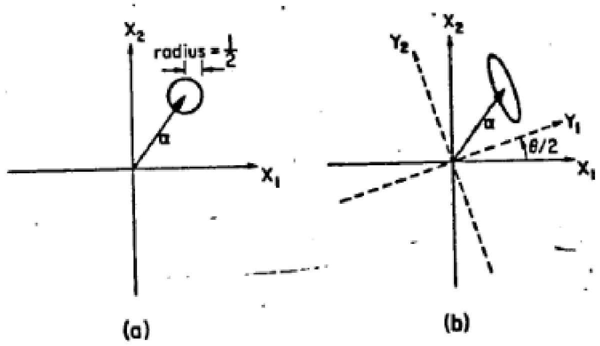

Figure 5.4: (a) Uncertainty circle in complex-amplitude plane for coherent state $\mid \alpha\rangle$. (b) Uncertainty ellipse in complex-amplitude plane for squeezed state $\mid \alpha, re^{i\theta}\rangle$ ([26]

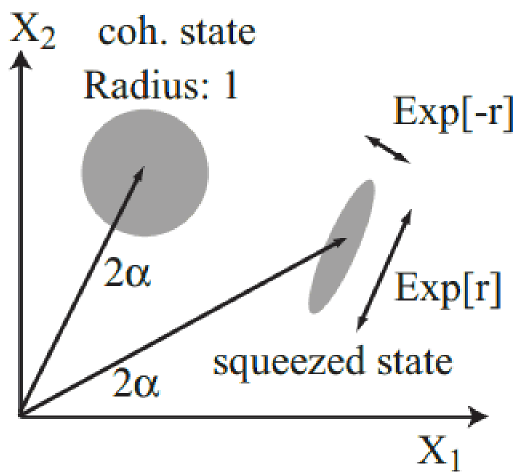

Figure 5.5: The coherent state and the squeezed state.[27]

operator. So we have for instance that the squeezing operator is a canonical invariant. Further more the surface element given by the uncertainty area will be invariant under the squeeze operator.

We can write the squeeze operator in another form. If we define θ and ϕ such that

$$\epsilon = -\frac{1}{2}\theta e{-}i\phi, \tag{5.18}$$

we can define

$$\zeta = -\tanh(\frac{\theta}{2}e{-}i\phi. \tag{5.19}$$

The range of the parameters is given by

$$\theta \in (-\infty, \infty), \qquad \phi \in (0, 2\pi), \qquad |\zeta| \in (0, 1). \tag{5.20}$$

We can use the disentangling therom of SU(1,1) Lie algebra [28] to write the squeeze operator as

$$S(\zeta) = \exp(\zeta K_+)\exp(\ln[1 - |\zeta|^2])\exp(-\zeta^* K_-) \tag{5.21}$$

Using this notation we can express the squeezed state as

$$\mid k, \zeta\rangle = (1 - |\zeta|^2)^k \sum_{m=0}^{\infty} \sqrt{\frac{\Gamma(m + 2k)}{m!\Gamma(2k)}}\zeta^m \mid k, m\rangle \tag{5.22}$$

[15] We sum over all m, but we know from 4.41 that $\mid k, m\rangle$ will never leave its own singleton. So the action of the squeeze state will never relate both singleton to one another, as we would have expected since $S \in SU(1, 1)$. Recall that the displacement operator D is not in the Lie algebra. However, D is in the super-Lie algebra which include the operators a and $a^\dagger$ that relate the two singletons. We thus have that the squeezed vacuum state will have even parity, since the vacuum state has even parity, and the (squeezed) coherent state can have parity odd or even.

Chapter 6

Generation and applications of squeezed states

We have learned some things about the squeezing operator. The squeezing operator works nice in theorem, but in practice we can not just put the operator in front of a coherent state. So now it its time to tell how squeezed states can be generated and what the utility of squeezed states can be.

6.1 Generation of quadrature squeezed states

Quadrature states can be generated by separating the field into its x and y components, and then stretching one and squeezing the other. We need a phase shift to accomplish the separation. The phase shit can be achieved by a non-linear minimum as can be seen in figure 6.1.
If the wave and its conjugate are multiplied by $\mu = \cosh r$ and $\nu = \sinh r$, respectively, and then added together the net result is a quadrature state with squeeze parameter r. Since $\cosh r$ and $\sinh r$ have a trigonometric relation, we have

$$\mu^2 - \nu^2 = 1. \tag{6.1}$$

The superposition can be viewed schematically as in figure 6.2.
Quadrature-squeezed can also be generated by the use of a phase-conjugate mirror. The experiment now uses a degenerate four-wave mixing. A phase-conjugate mirror of the type used in the process multiplies by a constant and conjugates fields reflected from it, so $a \mapsto \mu a^\dagger$. However, transmitted fields are simply multiplied by a constant, thus $a \mapsto \nu a$. Any open port in such a system admits vacuum fluctuations which do not weaken the squeezing properties of the result, which is the squeezed coherent state. This process is illustrated in 6.3. [8]

6.2 Squeezed states as qubits

In quantum computing, a qubit is a unit of quantum information: the quantum analogue of the classical bit. A pure qubit state is a linear superposition of the

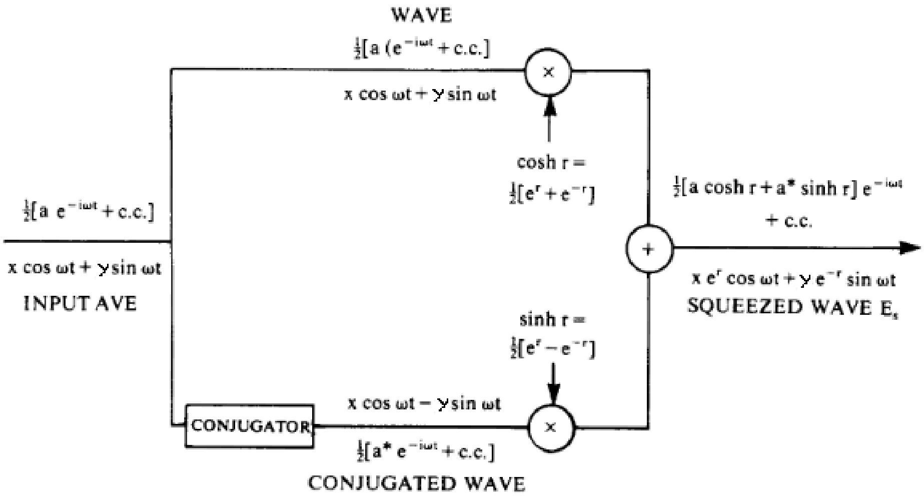

Figure 6.1: Generation of quadrature-squeezed light by the use of a non-linear medium providing phase conjugation)[8]

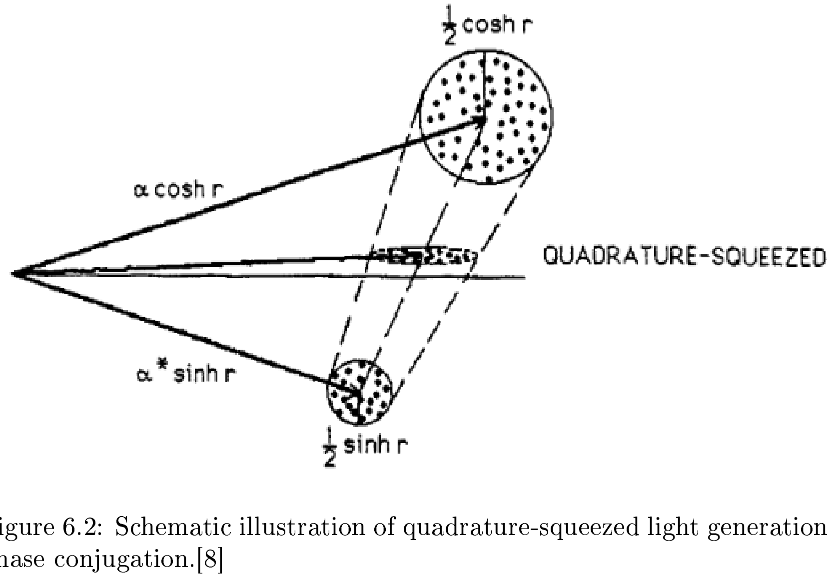

Figure 6.2: Schematic illustration of quadrature-squeezed light generation using phase conjugation.[8]

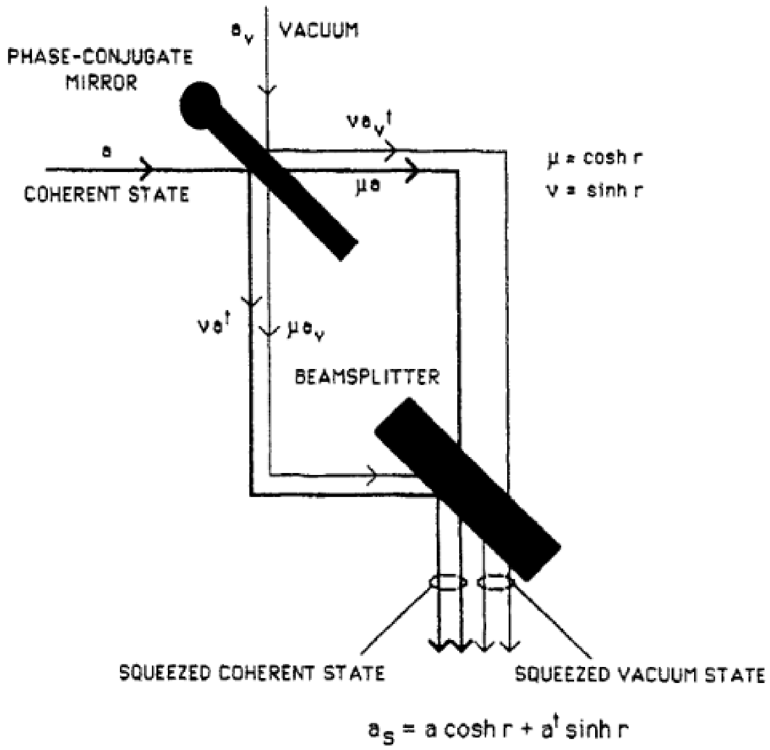

Figure 6.3: Generation of quadrature-squeezed light using a phase-conjugate mirror. [8]

basis states. Thus the qubit can be represented as a linear combination of $|0\rangle$ and $|1\rangle$:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, \tag{6.2}$$

where α and β are complex numbers. The probability to measure $|0\rangle$ is $|\alpha|^2$, the probability to measure $|1\rangle$ is $|\beta|^2$, so we have the restriction

$$|\alpha|^2 + |\beta|^2 = 1 \tag{6.3}$$

An important distinguishing feature between a qubit and a classical bit is that multiple qubits can be entangled. Entanglement allows a set of qubits to express higher correlation than is possible in classical systems. Entanglement also allows multiple states to be acted on simultaneously, while classical bits can only have one value at a time. Entanglement is a necessary ingredient of any quantum computation that cannot be done efficiently on a classical computer.We define the Bell state as a maximally entangled quantum state of two qubits.[29]

In a lot of experiment people tried to encode logical qubits into the physical higher dimensional coherent states $| \alpha\rangle$ and $| -\alpha\rangle$. It was shown that such schemes could allow efficient computation with only simple linear linear operations in-line. For large amplitudes α the two states are basically orthogonal, but for small α they have a finite overlap. Still, this scheme is capable of fault-tolerant quantum computing with certain advantages over traditional photon-encoded schemes, as can be seen in [30].In contrast to the usual two cubit case, all four Bell states can be distinguished. This gives higher efficiency for, per example, quantum teleportation. Furthermore it has been shown that simple coherent states attain the channel capacity, the tightest upper bound on the rate of information that can be reliably transmitted over a communications channel, though the receiver have to apply collective decoding with a quantum computer for coherent state signals to extract he maximum information.
To realize a coherent state-based computing scheme as described here, we need access to resources of arbitrary qubits $a \mid \alpha\rangle + b \mid -\alpha\rangle$.This generation can be accomplished by two-photon subtraction of squeezed vacuum with an added displacement. The displacement before the second photon detector can be adjusted, according to which the conditional output state is prepared in any arbitrary superposition of one-photon and two-photon subtracted states as long α is small. The generated states are therefore superpositions of squeezed vacuum and single-photon subtracted squeezed vacuum, which is equivalent to a squeezed photon.An extensive explanation of the experiment can be found in [31]
The conditional output state ends up in a superposition of single-photon subtracted squeezed vacuum and normal squeezed vacuum. The exact parameters of the superposition depend on the phase and amplitude of the displacement beam. If the displacement beam is strong (weak), the output will be close to a squeezed vacuum (photon). The output states as a whole can also be considered as qubits of squeezed states, being superpositions of the two orthogonal states $S|0\rangle$ and $S|1\rangle$.[32]

6.3 Squeezed states and amplification of measured signals

Squeezed states can for example be used to increase the sensitivity of a gravitational wave detector. The effect of the gravitational radiation is so weak that the expected value of the measurement id of the same order of magnitude as quantum mechanical uncertainties. If squeezed states are used is should be possible to detect measurement values less than the quantum uncertainty.
To get this nice results we use the fact that uncertainties in the quadratures of the generated photon field become unequal. One is less than that of a coherent state and the other is greater. Now we can use a typical form of amplifier. The most common form of an amplifier produces phase-insensitive amplification (PIA), which amplifies both quadratures of the field equally. There also can be realized phase-sensitive amplification (PSA). Is PSA on quadrature is amplified while the other is de-amplified. In PIA, the quotient between the signal-to-noise ratio of the input field and that of the output, amplified field, approaches two for large values of the gain, while the quotient for PSA can be unity independent of the gain. In a suitable PSA the noise is added preferentially to the quadrature not carrying information. This leaves the information of the quadrature carrying information essentially noise free.
Interferometric techniques are used to detect very weak forces such as the gravitational wave. They experience limitations on sensitivity due to quantum noise arising from photon counting and radiation pressure fluctuations. These sources of noise can be interpreted as arising from the beating of the input laser with the vacuum fluctuations entering the unused port of the interferometer. These two different noise sources are from fluctuations in the two different quadrature phases of the vacuum entering the unused port. It has been suggested that injecting a squeezed state into the unused port will reduce one or other of two sources of noise depending on which quadrature is squeezed.[33, 34]

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