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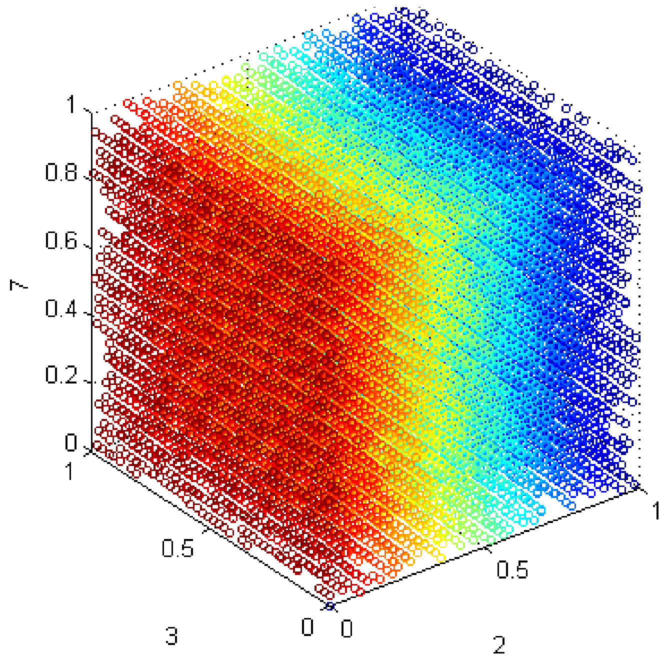
Gelfand's Question and Poncelet's Closure Theorem



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Abstract

In this thesis the underlying systems of Poncelet's Closure theorem and Gelfand's question are shown to be closely related to rotations on a circle. Also, solutions to Gelfand's questions are given: It is shown that in Gelfand's system no rows of equal digits can appear, and that the row '23456789' can not reappear after the first row.

Introduction

Part of the beauty of mathematics is in discovering that two very dissimilar problems have, in fact, the same underlying mechanisms. This paper will describe the relations between two problems that have this property. These problems are Poncelet's closure theorem and Gelfand's question. The first of these is a theorem about tangents to conic sections, and the second is a series of questions about the first digit of powers of integers. The relations between both problems is not a new discovery, it is in fact the subject of a paper by J. L. King, which can be found in [1]. This paper was the starting point of this present work. The aim of this thesis is to make the relations given by King more explicit, and to actually answer the questions posed by Gelfand.

Poncelet's Closure Theorem

To state Poncelet's theorem we will first require the notion of a Poncelet transverse.

Definition 1. *For any two smooth conic sections C_1 and C_2 in the real projective plane ($\mathbb{RP}^2$), starting position $x_0 \in C_1$ and starting line segment $l_0 = [x_0\ x_1]$ tangent to C_2, the* **Poncelet transverse** *is the union of line segments $l_i = [x_i\ x_{i+1}]$ tangent to C_2.*

A Poncelet transverse closes up if for some n, $x_n = x_0$, thus forming an n-gon. This leads us to stating Poncelet's Theorem.

Theorem 1. *If a Poncelet transverse of C_1 and C_2 closes up in n steps for a certain starting point that is not in $C_1 \cap C_2$, it closes up in n steps for all starting points $x \in C_1$.*

plaatje

Gelfand's question

To properly define the system from which Gelfand's question arises, we require a couple

Definition 2. *For each $x \in \mathbb{R}_{>0}$ we can take $a_i \in \{0, 1, ...9\}$ such that $x = \sum_{n=k}^{\infty} a_n 10^{-n}$, with $a_k \neq 0$. This is called a* **decimal expansion** *of x.*

Because $0.999... = 1.00...$ this decimal expansion is not always unique. If in these cases we only allow the commonly preferred expansion ending in $a_i = 0$ for $i > N$, we obtain a unique expansion for each $x \in \mathbb{R}_{>0}$.

This allows us to define a 'take the first digit'-operator on the multiplication group $(\mathbb{R}_{>0}, \cdot, 1)$, which is given by:

$$\begin{array}{rccc} \langle\langle x \rangle\rangle : & \mathbb{R}_{>0} & \to & \{1, 2, ..., 9\} \\ & x & \mapsto & a_k \end{array}$$

An example of this is $\langle\langle 10 \rangle\rangle = 1$, or $\langle\langle \pi \rangle\rangle = 3$

This operator can be made explicit by viewing it as a two step process: First divide by the highest power of ten, to shift the decimal point to the first digit, and then floor the result. This gives us a direct expression

$$\langle\langle x\rangle\rangle = \lfloor x \cdot 10^{-\lfloor \log_{10} x \rfloor} \rfloor.$$

Gelfand's question arises from a system defined by taking the first decimal digits of n^{th} powers of integers. At the first step the numbers $2, 3, ..., 9$ are considered. Then, for step two the first decimal digits of $2^2, 3^2, ...9^2$ are considered, and continuing for cubes and n^{th} powers. This can be shown in a matrix defined by $(y_{i,j}) = \langle\langle (y_{i,1})^j \rangle\rangle$, which is shown beneath here for the first 10 values.

1	2	3	4	5	6	7	8	9
2	4	9	1	2	3	4	6	8
3	8	2	6	1	2	3	5	7
4	1	8	2	6	1	2	4	6
5	3	2	1	3	7	1	3	5
6	6	7	4	1	4	1	2	5
7	1	2	1	7	2	8	2	4
8	2	6	6	3	1	5	1	4
9	5	2	2	2	1	4	1	3
10	1	5	1	9	6	2	1	3

An extension of this table up to a hundred rows can be found in appendix C on page 21. Regarding this table, Gelfand's question is actually a series of questions, most of which are related to the frequencies and ergodicity in this system.

1. Will a '9' ever occur in the column of 2^n?
2. Will the row '23456789' ever occur again? If so, will it have a frequency?
3. Will a row of the same numbers appear?
4. Will the decimal expansion of an 8-digit prime ever occur?

The first of these is rather easy to solve: Yes it does, for the first time at $n = 53$. The fourth one is less obvious, but knowing that 23456789 and 21443183 are prime numbers, gives us rows 1 and 11 as examples.

1 Rotations on circles

One of the central mechanisms in this thesis will be rotations on a circle. The operation on the circle group can be described in many ways, rotations over an angle or multiplication on the complex plane. A description analogue to rotation is by parametrizing the circle by the interval $\mathbb{K}$=[0,1), using ths isomorphism $t \mapsto (\cos 2\pi t, \sin 2\pi t)$. On this interval we can define the operation $\oplus$ as addition modulo 1. This makes a rotation of $2\pi\alpha$ equal to adding α modulo one.

An n-torus is the Cartesian product of n circles, and thus can be viewed as a hypercube $[0,1)\times \ldots \times[0,1)$. Therefore n simultaneous rotations on circles can be viewed as rotations on a torus. An example is rotating by $\frac{2\pi}{3}$ on one circle, while rotating by π on another. This can be seen as the mapping

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \oplus \frac{1}{3} \\ y \oplus \frac{1}{2} \end{pmatrix}$$

on $\mathbb{K} \times \mathbb{K}$, which ends up at the starting position after 6 iterations.

According to a theorem by L.Kronecker[3] the orbit of simultaneous rotations over $2\pi x_1, \ldots, 2\pi x_n$ on an n-torus is uniformly distributed over the closure of the orbit. This is the entire torus if and only if $1, x_1, \ldots, x_n$ are rationally independent.

Let E be an elliptic curve over $\mathbb{R}$ defined by: $\eta^2 = f(\lambda)$ with $f(\lambda)$ a third degree polynomial with exactly one root, α. Then the group E($\mathbb{R}$) is isomorphic to the circle group,$\mathbb{K} = [0, 1)$, by the isomorphism φ:

$$\begin{array}{ll} \mathrm{E}(\mathbb{R}) & \to \mathbb{K} \\ (\eta, \lambda) & \mapsto \varphi(\eta, \lambda) \end{array}$$

$$\varphi(\eta, \lambda) = \begin{cases} 1 - g(\lambda) & \text{if} \quad \eta < 0 \\ g(\lambda) & \text{if} \quad \eta \geq 0 \end{cases}, g(\lambda) = \frac{\int\limits_{\lambda}^{\infty} \frac{\mathrm{d}t}{\sqrt{f(t)}}}{2\int\limits_{\alpha}^{\infty} \frac{\mathrm{d}t}{\sqrt{f(t)}}}$$

A proof of this statement can be found in [2].

This isomorphism can be visualized by putting the point $(\alpha, 0)$ at the leftmost point of a circle and then ‘folding’ the elliptic curve around it, putting the point at infinity ($\mathcal{O}$) at the rightmost point of the circle.

2 Gelfand's question and rotations on a circle

In this section the relation between Gelfand's system and rotations on a circle are shown. This relation is then used to answer Gelfand's questions.

2.1 Isomorphism to Rotations

As the first digits of x and $10 \cdot x$ are the same We can define the equivalence relation $\sim$ on $\mathbb{R}_{>0}$ as $x \sim y \Leftrightarrow (\exists n \in \mathbb{Z}$ such that $x = y \cdot 10^n)$. As $10^{\mathbb{Z}}$ is a subgroup of $\mathbb{R}_{>0}$, we can define the factorgroup $\mathbb{A} = \mathbb{R}_{>0}/10^{\mathbb{Z}}$ as the set of all equivalence classes $\widehat{x} = \{y \in \mathbb{R}_{>0} | y \sim x\}$. This construction is similar to modular arithmetic, except using the multiplication operator instead of addition.

If we view the interval $\mathbb{K} = [0, 1)$, with addition modulo 1 ($\oplus$), as a circle we can construct an isomorphism between $\mathbb{A}$ and $\mathbb{K}$ by beginning with the (surjective) mapping:

$$\begin{array}{rccl} \psi : & \mathbb{R}_{>0} & \longrightarrow & \mathbb{K} = [0,1) \\ & x & \longmapsto & \log_{10} x \pmod 1 \end{array} \tag{1}$$

This is a group homomorphism because

$$\psi(xy) = \log_{10} xy \pmod 1 = \log_{10} x \oplus \log_{10} y$$

Using the first isomorphism theorem and the fact that $10^{\mathbb{Z}}$ is the kernel of ψ $\mathbb{K}$ is isomorphic $\mathbb{A}$.

Due to the constructions of both the 'take the first digit'-operator and the group $\mathbb{A}$ we know $x \sim y \Rightarrow \langle\langle x \rangle\rangle = \langle\langle y \rangle\rangle$. As $k \in \mathbb{K}$ implies that $10^k \in [1, 10)$ we can simplify $\langle\langle 10^k \rangle\rangle$ to simply $\lfloor 10^k \rfloor$.

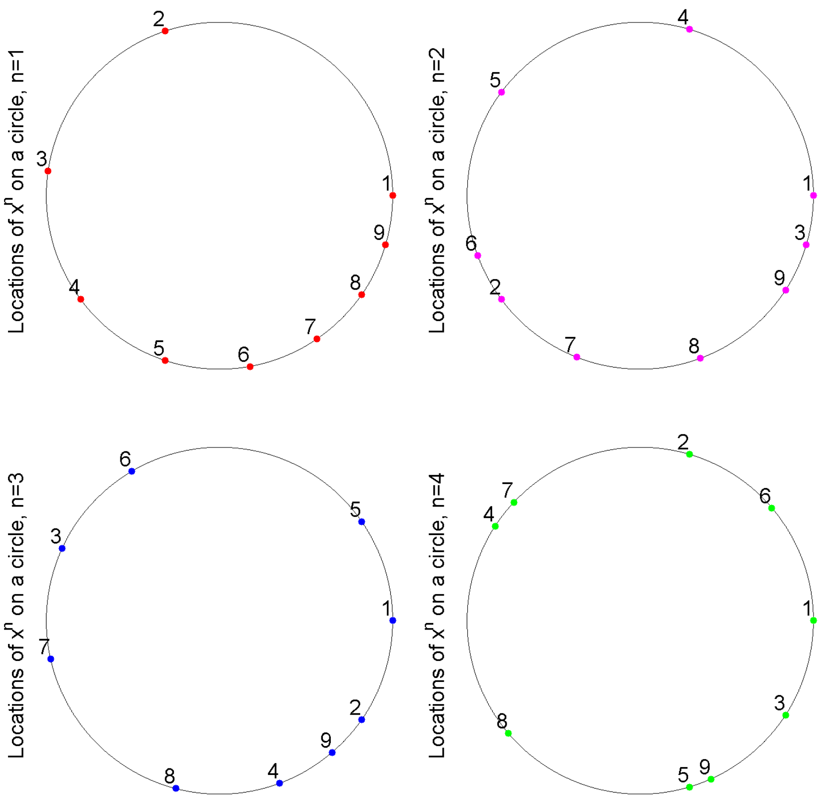
$$\begin{tikzcd}
\mathbb{R}_{>0} \arrow[r, "x \mapsto x^n"] \arrow[d, "x \mapsto \widehat{x}"'] & \mathbb{R}_{>0} \arrow[dr, "x \mapsto \langle\langle x \rangle\rangle"] & \\
\mathbb{A} \arrow[r, "\widehat{x} \mapsto \widehat{x^n}"] \arrow[d, "\widehat{x} \mapsto \psi(x)"'] & \mathbb{A} \arrow[r, "\widehat{x} \mapsto \langle\langle x \rangle\rangle"] & \{1, 2, ..., 9\} \\
\mathbb{K} \arrow[r, "k \mapsto n \cdot k \bmod 1"] & \mathbb{K} \arrow[u, "k \mapsto \widehat{10^k}"] \arrow[ur, "k \mapsto \lfloor 10^k \rfloor"'] &
\end{tikzcd}$$

This means that multiplication in $\mathbb{A}$ corresponds to rotation (addition modulo 1) on a circle, therefore making Gelfand's system correspond to rotations on an n-torus. This can be visualized as rotating numbered points around a circle, we will show this for n =1, 2, 3, 4:

2.2 Results on Gelfand's Question

Using the isomorphism from the last section, calculations on Gelfand's System can be easily executed, as for each entry only the $\log_{10} x \pmod 1$, has to be stored. We can do this in an 'underlying' array for Gelfand's system. We call the n^{th} element of the column starting with $\log_{10} \alpha$: $x_{\alpha,n}$

$$\begin{aligned} x_{\alpha,n} &= n \log_{10} \alpha && \pmod 1 \\ &= x_{\alpha,n-1} + \log_{10} \alpha && \pmod 1; \\ x_{\alpha,1} &= \log_{10} \alpha && \pmod 1. \end{aligned}$$



The array generated by simultaneously taking 2,3,...,9 for α is directly related to Gelfand's system by taking

$$\begin{aligned} y_{\alpha,n} &= \langle\langle \alpha^n \rangle\rangle = \lfloor 10^{x_{\alpha,n}} \rfloor \\ \Rightarrow y_{\alpha,n} = k &\Leftrightarrow \log_{10} k \leq x_{\alpha,n} < \log_{10} k + 1. \end{aligned}$$

It is important to take caution when performing numerical calculations using this system. An error ϵ in $x_{\alpha,1}$ will proliferate to an error of $n \cdot \epsilon$ in $x_{\alpha,n}$. As the $x_{\alpha,i} \in [0,)$ this error can quickly cause problems. The questions asked about Gelfand's system were:

1. Will a '9' ever occur in the collumn of 2^n?
2. Will the row '23456789' ever occur again? If so, will it have a frequency?
3. Will a row of the same numbers appear?
4. Will the decimal expansion of an 8-digit prime ever occur?

2.2.1 Question 1

Question 1 can be easily answered by using Kronecker's Theorem (in section 1) and noting that $\log_{10} 2$ and 1 are rationally independent. This implies that the orbit of rotation over an angle $2\pi \log_{10} 2$ on the circle is uniformly distributed, which implies that the probability that $x_{2,n}$ is in a certain interval is equal to the relative length of the interval.

$$y_{2,n} = 9 \iff \log_{10} 9 \leq x_{2,n} < 1$$

Therefore the probability of $y_{2,n} = 9$ is equal to $1 - \log_{10} 9 \approx 0.0458$. This means that for close to 4.6 % of the integers $n > 0$, $\langle\langle 2^n \rangle\rangle = 9$. Checking with Matlab showed that in the first 1,000 values 45 start with a 9, and in the first 1,000,000 values, 45757 start with a 9. This suggests that indeed:

$$\lim_{n\to\infty} \frac{\{\#k | y_{2,k} = 9, k < n\}}{n} = 1 - \log_{10} 9.$$

Generalizing this result by taking $\alpha \in [2, 3, \ldots, 9]$, easily shows that these rotations are also uniformly distributed. Among other things, this means that the columns of the system independently satisfy the famous Benford's Law[4].

2.2.2 Questions 2,3 and 4

The other three questions are about the relations between the different columns. Each column is uniformly distributed on the circle, but this does *not* imply that the system is uniformly distributed on the 8-torus. For example, if we consider the columns starting with 2 and 4, some simple calculations reveal that these are not independent.

$$\begin{aligned}
y_{2,n} = 2 \quad &\Longrightarrow \log_{10} 2 \leq x_{2,n} < \log_{10} 3 \\
&\Longrightarrow 2\log_{10} 2 \leq 2x_{2,n} < 2\log_{10} 3 \\
&\Longrightarrow \log_{10} 4 \leq 2n\log_{10} 2 < \log_{10} 9 \\
&\Longrightarrow \log_{10} 4 \leq n\log_{10} 4 < \log_{10} 9 \\
&\Longrightarrow 4 \leq y_{4,n} < 9
\end{aligned}$$

This is caused by the rational relation $\log_{10} 4 = 2\log_{10} 2$. Not only the logarithms 4 and 2 have a rational relations.

$$\left.\begin{array}{rl}
2^2 & = 4 \\
2 \cdot 5 & = 10 \\
2 \cdot 3 & = 6 \\
2^3 & = 8 \\
3^2 & = 9
\end{array}\right\} \Rightarrow \left\{\begin{array}{ll}
\log_{10} 4 & = 2\log_{10} 2 \\
\log_{10} 5 & = -\log_{10} 2 \\
\log_{10} 6 & = \log_{10} 2 \cdot \log_{10} 3 \\
\log_{10} 8 & = 3\log_{10} 2 \\
\log_{10} 9 & = 2\log_{10} 3
\end{array}\right.$$

How these rotations behave on a torus can be seen in appendix A on page 18.

Suppose $\alpha, \beta, \gamma \in \{2, 3, ..., 9\}$, and $\gamma = \alpha\beta$. We know $\alpha^n = (y_{\alpha,n} + \epsilon) \cdot 10^k$, with ϵ smaller than 1. The same is true for β and γ. Having $\epsilon = 0$ means $\alpha^n = y_{\alpha,n} \cdot 10^k$. Using the unique prime factorization of both sides makes it clear that for $n > 3$ this can not happen, so we consider the first 3 rows separately.

$$\begin{aligned}
\gamma^n &= \alpha^n \beta^n \\
&= (y_{\alpha,n} + \epsilon_1)10^{k_1}(y_{\beta,n} + \epsilon_2)10^{k_2}.
\end{aligned}$$

This gives us two conditions for γ^n:

$$\begin{aligned}
\gamma^n \quad &\in 10^{(k_1+k_2)} \cdot \Big(y_{\alpha,n} \cdot y_{\beta,n}, (y_{\alpha,n} + 1) \cdot (y_{\beta,n} + 1)\Big) \\
&\text{And} \\
\gamma^n \quad &\in 10^{k_3} \cdot (y_{\gamma,n}, y_{\gamma,n} + 1)
\end{aligned}$$

This implies that checking if $y_{\alpha,n}$, $y_{\beta,n}$ and $y_{\gamma,n}$ can simultaneously appear in rows α, β and γ respectively comes down to checking whether:

$$(y_{\alpha,n} \cdot y_{\beta,n}, (y_{\alpha,n}+1) \cdot (y_{\beta,n}+1)) \cap 10^{(k_3-k_1-k_2)} \cdot (y_{\gamma,n}, y_{\gamma,n}+1) \neq \emptyset$$

The size of k_3 is also bounded.

$$\begin{aligned} k_1 + k_2 &\leq k_3 & &< k_1 + k_2 + \log_{10}((y_{\alpha,n}+1) \cdot (y_{\beta,n}+1)) \\ 0 &\leq k_3 - k_2 - k_1 & &< \log_{10}(y_{\alpha,n}+1) + \log_{10}(y_{\beta,n}+1)) \end{aligned}$$

Therefore checking whether a triplet $y_{\alpha,n}, y_{\beta,n}, y_{\gamma,n}$ can appear as a row in the system can be checked using a simple algorithm.

2.2.3 Example

If, for example, we take $\alpha = 2$, $\beta = 3$, and $\gamma = 6$ and assume that the values $y_{2,n}$ and $y_{3,n}$ are 3 and 4 respectively, we can check what values for $y_{6,n}$ can be attained.

$$0 \leq k_3 - k_2 - k_1 < \log_{10}(5) + \log_{10}(4)) < 2$$

Therefore we have to check two possible cases.

$$\begin{array}{rcl} [12, 20) & \cap & [y_{6,n}, y_{6,n}+1) \neq \emptyset \\ & \text{OR} & \\ [12, 20) & \cap & 10 \cdot [y_{6,n}, y_{6,n}+1) \neq \emptyset \end{array}$$

This condition leaves us with only $y_{6,n} = 1$.

2.2.4 Answers

If row $[y_{2,n}, \ldots, y_{9,n}]$ is attainable, this means a set of $[2^n, \ldots, 9^n]$ exists in the intersections corresponding to all rational relations. As these are intersections of open sets, we know that there exists an open interval in $\mathbb{R}^8_>0$ that maps to $[y_{2,n}, \ldots, y_{9,n}]$

If we view the logarithms of this, we get an open interval in $\mathbb{K}^8$. Using the theorem by L. Kronecker[3], the probability that for a certain n the simultaneous rotations will be in an interval is relative to the size of the interval. As a non-empty open set always has length greater then zero, this implies that every attainable value, will also be attained.

Suppose $\langle\langle 2^n \rangle\rangle = 2$ and $\langle\langle 5^n \rangle\rangle = 5$. This means that there is an $\epsilon \in [0, 1)$ such that $2^n \sim 2 + \epsilon$. As $\widehat{5^n}$ is the inverse of $\widehat{2^n}$ in $\mathbb{A}$, we know $5^n \sim \frac{10}{2+\epsilon}$. Due to the construction of the equivalence relation, we know $\langle\langle 5^n \rangle\rangle = \langle\langle \frac{10}{2+\epsilon} \rangle\rangle = 5$. This means that $\frac{10}{2+\epsilon} \geq 5$, which can only happen if $\epsilon = 0$. Therefore $2^n \sim 2$, or in other words, $2^{n-1} = 10^k$, which can only happen if $n = 1$. A result of this is that the row '23456789' can only appear at row number 1.

A row of equal numbers can only appear if all $x_{\alpha,n}$ are in the same interval $[\log_{10} k, \log_{10} k + 1)$ for some k and n. Using the condition $n \log_{10} 4 \pmod 1 = 2n \log_{10} 2 \pmod 1$, we know that this interval has to be close to 0, leaving only $k = 1$ and $k = 9$. Now using the condition $n \log_{10} 5 \pmod 1 = -n \log_{10} 2$

(mod 1) shows that $n \log_{10} 2 \pmod 1$ and $n \log_{10} 5 \pmod 1$ can not both be in $[0, \log_{10} 2)$ or $[\log_{10} 9, 1)$ unless both are zero. Therefore, only row zero can have all the same numbers (the row with the first digits of α^0).

Generalizing this for all possibilities, a Matlab program (appendix B) was written to check for all $9^8 = 43046721$ possible 8-tuples whether they were attainable in Gelfand's system. Only 153999 different 8-tuples were attainable for $n > 3$. As this already contained row 2 and 3, only row 1 needs to be added. This gives us a total of 154000 different rows, one of which can only appear in the first row.

The last question about prime numbers was solved using Matlab, by intersecting the set of possible values with the set of 8-digit prime numbers. This nets us 9890 prime numbers that can (and will) appear.

3 Poncelet's Theorem and rotations on a circle

Poncelet's Theorem can be seen as a statement about a system. After all, for each pair (x_i, l_i), we can find (x_{i+1}, l_{i+1}). Using elementary operations as rotations, translations and scaling, all of which preserve tangents, we can reduce each conic section C_1 in $\mathbb{RP}^2$ to the unit circle. Therefore one can consider the Poncelet system as finding the (ordered) points on this circle such that two consecutive points define a line tangent to another conic section.

3.1 Concentric Circles

An interesting example of a pair of conic sections would be two concentric circles. As we can 'shrink' the outer circle to the unit circle, the only parameter of the system is the ratio of the radii of C_2 and C_1, which we will call k. We can parametrize the outer circle, C_1 by

$$C_1 := \begin{cases} x &= \frac{1-s^2}{1+s^2} \\ y &= \frac{2x}{1+s^2} \end{cases} \quad , s \in \mathbb{R}$$

Using rotational symmetry, it is evident that each chord tangent to C_2 will have the same length. Therefore we can always take $(1, 0)$ as the starting point x_0. Every chord from x_0 to x_1 is in the form:

$$T(\tau) = x_0 + \tau\,(x_1 - x_0) \qquad , \tau \in [0, 1]$$

To be tangent to C_2 means that there is only one point on the chord with distance k from the origin. Using elementary geometry, this point has to be the middle of the chord, the point with $\tau = \frac{1}{2}$.

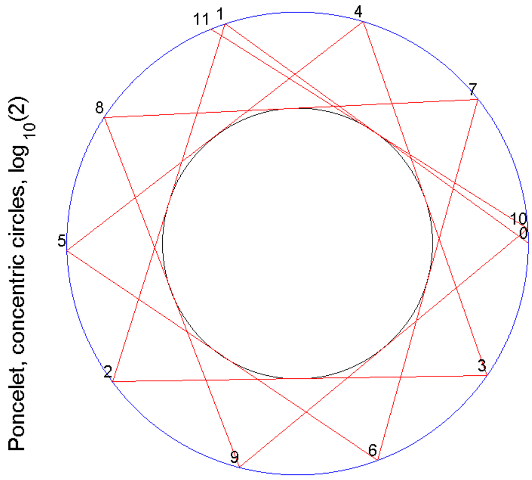
$$\begin{aligned} |T(\tfrac{1}{2})| &= \sqrt{\left(\frac{1}{2} + \frac{1}{2}\frac{1-s_1^2}{1+s_1^2}\right)^2 + \left(\frac{1}{2}\frac{2s_1}{1+s_1^2}\right)^2} \\ &= \frac{1}{2}\sqrt{1 + 2\frac{1-s_1^2}{1+s_1^2} + \left(\frac{1-s_1^2}{1+s_1^2}\right)^2 + \left(\frac{2s_1}{1+s_1^2}\right)^2} \\ &= \frac{1}{\sqrt{2}}\sqrt{1 + \frac{1-s_1^2}{1+s_1^2}} \\ &= \sqrt{\frac{1}{1+s_1^2}} \qquad = k \end{aligned}$$

Therefore, the upper tangent to C_2 from x_0 ends for $s_1 = \sqrt{\frac{1}{k^2} - 1}$, which makes

$$x_1 = \begin{pmatrix} \frac{1-\frac{1}{k^2}+1}{1+\frac{1}{k^2}-1} \\ \frac{2\sqrt{\frac{1}{k^2}-1}}{1+\frac{1}{k^2}-1} \end{pmatrix} = \begin{pmatrix} 2k^2 - 1 \\ 2k\sqrt{1-k^2} \end{pmatrix}$$

Applying the rotational symmetry built in in this system allows take this new point as a starting point and applying the same step again, therefore taking an equal step. This shows that the Poncelet system for this example is the same as rotating on the circle

Using this, a link between Gelfand's problem and Poncelet's Theorem can be found: In section 2 Gelfand's system for starting value 10^α, the behaviour of



$\langle\langle(10^\alpha)^n\rangle\rangle$ is shown to correspond to repeated addition modulo one of α on $\mathbb{K}$, or a rotation of $2\pi\alpha$ on a circle.

Using the results above, if we take $k = |\cos(\pi\alpha)|$, taking one step ends up at:

$$x_1 = \begin{pmatrix} 2\cos^2(\pi\alpha) - 1 \\ 2\cos(\pi\alpha)\sqrt{1-\cos^2(\pi\alpha)} \end{pmatrix} = \begin{pmatrix} \cos(2\pi\alpha) \\ \sin(2\pi\alpha) \end{pmatrix}$$

Taking $\alpha = \log_{10}(2)$ shows the relation between the first column of Gelfand's system and a specific case of Poncelet's system.

3.2 A more general case

If we start with two conics, and apply an isomorphism, to change C_1 into the unit circle, the system will be isomorphic to the system with:

$$\begin{aligned} C_1 &: x^2 + y^2 - 1 = 0 \\ C_2 &: Ax^2 + Bxy + Cy^2 + Dx + Ey + F = f_{C_2}(x, y) = 0 \end{aligned}$$

In this thesis the fact that Poncelet's system is isomorphic to rotations on a circle, is shown, not proven. Therefore a far more specific case is used. This is done because using the general case, with it's 8 variables (6 for the conic section and 2 for x_0, l_0), will be horribly incomprehensible. Almost all cases can shown to be isomorphic to rotations using nearly the same procedure as follows: finding an elliptic curve describing the conditions for x_{i+1}, l_{i+1} and then using the isomorphism from section 1. This is illustrated by the following example.

The system that is used is again a system with 2 circles, but C_2 is shifted from the centre this time.

$$\begin{aligned} C_1 &: x^2 + y^2 = 1 \\ C_2 &: (x-k)^2 + (y-l)^2 = l^2. \end{aligned}$$

The radius of C_2 is chosen to be l to guarantee a point $x = (1, 0) \in C_1$ for which we can easily find a tangent to C_2, namely $l_0 : y = 0$.

For a line $l : y = cx + d$ to be tangent to circle C_2 we require the intersection of l and C_2 to contain only one point. As all coefficients of the problem are real, this means that the expression describing the intersection needs to have a double root.

$$\begin{aligned} & x^2 - 2kx + k^2 + (cx+d)^2 - 2l(cx+d) \\ = \; & (1+c^2)x^2 + (2dc - 2k - 2lc)x + k^2 + d^2 - 2ld \end{aligned}$$

As this expression is quadratic in x we can find this double root by calculating the discriminant.

$$(l^2 - k^2)c^2 - d^2 + 2lkc - 2kcd + 2ld = 0$$

Using this relation we can parametrize c and d by a single variable t that links both variables by $d = tc$.

$$\begin{aligned} 0 &= (l^2 - k^2)c^2 - (tc)^2 + 2lkc - 2ktc^2 + 2ltc \\ &= (-t^2 - 2kt + l^2 - k^2)c^2 + (2lt + 2kl)c \end{aligned}$$

$$c(t) = \frac{2l(k+t)}{(k+t)^2 - l^2}, d(t) = \frac{2lt(k+t)}{(k+t)^2 - l^2}$$

The case that $c = 0$ can be seen as an extension of $c(t)$ if we allow $t = \infty$, as on the projective plane.

If we pick a point $x_i = (x(s), y(s)) = \left(\frac{2s}{1+s^2}, \frac{1-s^2}{1+s^2}\right) \in C_1$ we want to be able to find a line tangent to C_2, after which we can easily find the second intersection with C_1, to find the point x_{i+1}. After this a new tangent can be found by repeating the process, thus forming the Poncelet system.

Having a tangent through x_i amounts to $y(s) = c(t)x(s) + d(t)$, thus:

$$0 = \frac{2l(k+t)}{(k+t)^2 - l^2} \cdot \frac{2s}{1+s^2} + \frac{2lt(k+t)}{(k+t)^2 - l^2} - \frac{1-s^2}{1+s^2}$$

Multiplying both sides by $((k+t)^2 - l^2)(1+s^2)$ results in:

$$\begin{aligned} 0 = \; & 4ls(k+t) + 2lt(k+t)(1+s^2) + (s^2-1)((k+t)^2 - l^2) \\ = \; & (2l(1+s^2) + s^2 - 1)t^2 + 2(2ls + lk(1+s^2) + k(s^2-1))t \\ & +4lsk + (s^2-1)(k^2 - l^2) \\ = \; & a_2(s)t^2 + a_1(s)t + a_0(s) \end{aligned}$$

We can now complete the square, in the way it is done while deriving the quadratic formula.

$$\begin{aligned} \left(a_2(s)t + \tfrac{1}{2}a_1(s)\right)^2 &= a_2(s)^2t^2 + a_2(s)a_1(s)t + \tfrac{1}{4}a_1(s)^2 \\ &= \tfrac{1}{4}a_1(s)^2 + a_2(s)\left(a_2(s)t^2 + a_1(s)t + a_0(s)\right) - a_2(s)a_0(s) \\ &= \tfrac{1}{4}a_1(s)^2 - a_2(s)a_0(s) \\ &= (1+s^2)(s^2k^2 + s^2 + 2ls^2 - 4sk + k^2 - 2l + 1)l^2 \end{aligned}$$

If we then pick $\rho = \frac{1}{l}\left(a_2(s)t + \frac{1}{2}a_1(s)\right)$ as a variable instead of t, we obtain an equation in ρ and s, namely:

$$\rho^2 = (1+s^2)(s^2k^2 + s^2 + 2ls^2 - 4sk + k^2 - 2l + 1).$$

3.2.1 Isomorphism to an elliptic curve

We defined the problem to allow a certain solution, $x = (1, 0)$, $l : y = 0$, for which we can find a possible point on the curve in the (ρ, s)-plane

$$\begin{array}{llll} x & = (1,0) & \Rightarrow s & = 1 \\ c, d & = 0 & \Rightarrow t & = -k \\ s & = 1, t = -k & \Rightarrow \rho & = \frac{1}{l}\left(-a_2(1)k + \frac{1}{2}a_1(1)\right) \\ & & & = \frac{1}{l}\left(-4kl + (2l + 2lk)\right) \\ & & & = -2(k-1) \end{array}$$

As we now have a point in the ρ, s-plane that lies on the genus one curve defined earlier, we know this curve is isomorphic to an elliptic curve, which was found by using MAGMA. This elliptic curve is given by:

```
Elliptic Curve defined by y^2 + (2*k^2 - 4*k - 2*l + 2)/(k^2 - 2*k + 1)*x*y +
    (2*k^3*l - 4*k^2*l + 2*k*l - 2*l^3)/(k^6 - 6*k^5 + 15*k^4 - 20*k^3 + 15*k^2
    - 6*k + 1)*y = x^3 + (-2*k^4 + 6*k^3 + 2*k^2*l - 8*k^2 - 4*k*l + 6*k + 2*l^2
    + 2*l - 2)/(k^4 - 4*k^3 + 6*k^2 - 4*k + 1)*x^2 + (k^6 - 2*k^5*l - 4*k^5 -
    k^4*l^2 + 8*k^4*l + 6*k^4 + 2*k^3*l^2 - 12*k^3*l - 4*k^3 + 2*k^2*l^3 -
    2*k^2*l^2 + 8*k^2*l + k^2 - 4*k*l^3 + 2*k*l^2 - 2*k*l + l^4 + 2*l^3 -
    l^2)/(k^8 - 8*k^7 + 28*k^6 - 56*k^5 + 70*k^4 - 56*k^3 + 28*k^2 - 8*k + 1)*x
```

Using the isomorphism ψ defined by:

```
Mapping from: CrvHyp: C to CrvEll: E
with equations :
(-k^4 + 3*k^3 - k^2*l - 4*k^2 + 2*k*l + 3*k + l^2 - l - 1)/(k^4 - 4*k^3 + 6*k^2
    - 4*k + 1)*$.1^3 + (k^4 - k^3 + k^2*l - 2*k*l - k - 3*l^2 + l + 1)/(k^4 -
    4*k^3 + 6*k^2 - 4*k + 1)*$.1^2*$.3 - 1/(k - 1)*$.1*$.2 + (-k^4 + k^3 + k^2*l
    - 2*k*l + k + 3*l^2 + l - 1)/(k^4 - 4*k^3 + 6*k^2 - 4*k + 1)*$.1*$.3^2 +
    1/(k - 1)*$.2*$.3 + (k^4 - 3*k^3 - k^2*l + 4*k^2 + 2*k*l - 3*k - l^2 - l +
    1)/(k^4 - 4*k^3 + 6*k^2 - 4*k + 1)*$.3^3
(2*k^4 - 6*k^3 + 2*k^2*l + 8*k^2 - 4*k*l - 6*k - 2*l^2 + 2*l + 2)/(k^4 - 4*k^3 +
    6*k^2 - 4*k + 1)*$.1^3 + (-4*k^3 + 8*k^2 - 4*k + 4*l^2)/(k^4 - 4*k^3 + 6*k^2
    - 4*k + 1)*$.1^2*$.3 + 2/(k - 1)*$.1*$.2 + (2*k^4 - 6*k^3 - 2*k^2*l + 8*k^2
    + 4*k*l - 6*k - 2*l^2 - 2*l + 2)/(k^4 - 4*k^3 + 6*k^2 - 4*k + 1)*$.1*$.3^2
-$.1^3 + 3*$.1^2*$.3 - 3*$.1*$.3^2 + $.3^3
and inverse
$.2*$.3
(4*k - 4)*$.1^3*$.3 + (-4*k^4 + 12*k^3 + 4*k^2*l - 16*k^2 - 8*k*l + 12*k + 4*l^2
    + 4*l - 4)/(k^3 - 3*k^2 + 3*k - 1)*$.1^2*$.3^2 + (-4*k^2 + 8*k + 4*l - 4)/(k
    - 1)*$.1*$.2*$.3^2 + (-2*k + 2)*$.2^2*$.3^2
2*$.1*$.3 + $.2*$.3
```

This elliptic curve can be further simplified by bringing it into Weierstrass form,

$$\begin{aligned} y^2 + Axy + By = &\quad x^3 + Cx^2 + Dx \\ (y + \tfrac{1}{2}Ax + \tfrac{1}{2}B)^2 = &\quad x^3 + Cx^2 + Dx + (\tfrac{1}{2}Ax + \tfrac{1}{2}B)^2 \end{aligned}$$

We can now substitute the square with α^2, and shift the x-coordinate to cancel out the term with x^2.

$$\begin{aligned} \alpha^2 = &\quad x^3 + (C + \tfrac{1}{4}A^2)x^2 + (D + \tfrac{1}{2}AB)x + \tfrac{1}{4}B^2 \\ = &\quad (\beta - \tfrac{1}{3}(C + \tfrac{1}{4}A^2))^3 + (C + \tfrac{1}{4}A^2)(\beta - \tfrac{1}{3}(C + \tfrac{1}{4}A^2))^2 \\ &\quad + (D + \tfrac{1}{2}AB)(\xi - \tfrac{1}{3}(C + \tfrac{1}{4}A^2)) + \tfrac{1}{4}B^2 \\ = &\quad \beta^3 + \left(-\tfrac{1}{3}C^2 + D - \tfrac{1}{48}A^4 + \tfrac{1}{2}AB - \tfrac{1}{6}CA^2\right)\beta \\ &\quad + \tfrac{1}{864}A^6 - \tfrac{1}{6}ABC - \tfrac{1}{3}DC + \tfrac{1}{18}C^2A^2 + \tfrac{1}{72}CA^4 \\ &\quad - \tfrac{1}{24}A^3B + \tfrac{2}{27}C^3 + \tfrac{1}{4}B^2 - \tfrac{1}{12}DA^2 \end{aligned}$$

Substituting A, B, C and D allows us to return to the old variables k and l.

$$\alpha^2 = \beta^3 - \frac{1}{3}\frac{(k^4-k^2+1-3l^2)\beta}{(k-1)^4} - \frac{1}{27}\frac{(k^2+1)(-9l^2+2k^4-5k^2+2)}{(k-1)^6}$$

If we don't take $k = 1$, we can simplify this further by scaling the variables:

$$\eta = \frac{\alpha}{(k-1)^3}, \qquad \xi = \frac{\beta}{(k-1)^2}$$

Applying these substitutions and multiplying by $(k-1)^6$, yields:

$$\eta^2 = \xi^3 - \tfrac{1}{3}(k^4-k^2+1-3l^2)\xi - \tfrac{1}{27}(k^2+1)(-9l^2+2k^4-5k^2+2)$$

This has a single solution ($\eta^2 = 0$ for $\xi = -\frac{1}{3}(1+k^2)$) in $\mathbb{RP}^2$ if and only if $l > |\frac{1}{2}(k^2-1)|$. If this condition holds the other two roots are imaginary. The last transformation will be shifting the curve by $\xi = (\lambda - \frac{1}{3}(1+k^2))$ such that this solution will be at the origin.

$$\eta^2 = \lambda^3 - (1+k^2)\lambda^2 + (k^2+l^2)\lambda$$

3.2.2 Isomorphism to a circle

If the curve has a single root, we can use section 1to show that the group of real points on this curve isomorphic to the rotation group on the circle by the isomorphism:

$$\varphi(\eta,\lambda) = \begin{cases} 1-g(\lambda) & \text{if} \quad \eta < 0 \\ g(\lambda) & \text{if} \quad \eta \geq 0 \end{cases}, g(\lambda) = \frac{\int\limits_{\lambda}^{\infty} \frac{\mathrm{d}t}{\sqrt{t^3-(1+k^2)t^2+(k^2+l^2)t}}}{2\int\limits_{0}^{\infty} \frac{\mathrm{d}t}{\sqrt{t^3-(1+k^2)t^2+(k^2+l^2)t}}}$$

The condition on k and l to have a single root is $l > |\frac{1}{2}(k^2-1)|$.

$$\begin{array}{rrl} x_i = & (a,b) & \in C_1 \\ l_i = & cx+d & \text{through } x_i\text{, tangent to } C_2 \\ & \downarrow & a = \frac{2s}{1+s^2}, b = \frac{1-s^2}{1+s^2} \\ & & c(t) = \frac{2l(k+t)}{(k+t)^2-l^2}, d(t) = \frac{2lt(k+t)}{(k+t)^2-l^2} \\ & (s,t) & \\ & \downarrow & \rho = (2t+k)(1+s^2) + \frac{1}{l}(k+t)(s^2-1) + 2s \\ & (\rho,s) & \\ & \downarrow & \psi \\ & (y,x) & \\ & \downarrow & \eta = \frac{y}{(k-1)^3} + \frac{(k-1)^2-l}{(k-1)^5}x + \frac{kl(k-1)^2-l^3}{(k-1)^9} \\ & & \xi = \frac{x}{(k-1)^2} - \frac{1}{3}\frac{k^4-2k^3+2k^2-2k-3l^2+1}{(k-1)^6} \\ & (\eta,\xi) & \\ & \downarrow & \lambda = \xi + \frac{1}{3}(1+k^2) \\ & (\eta,\lambda) & \\ & \downarrow & \varphi \\ & \mathbb{K} & \end{array}$$

3.2.3 Explicit example

Finding the next point (x_{i+1}, l_{i+1}) from (x_i, l_i) is a two step process:

- Find the other point on the intersection of C_1 and l_i, call this x_{i+1}
- Find the other tangent to C_2 through x_{i+1}, call this l_{i+1}

In this example we will take $k = 2$ and $l = 2$, which satisfies the condition to have one root of the elliptic curve.

By construction we can take $(x_0, l_0) = \big((1, 0), y = 0\big)$. This means $s = 1$ and $t = -2$, which leads to $\rho = -2$. The isomorphism ψ maps this point to $\mathcal{O}$, the point at infinity of the (y, x)-plane. Transforming to (η, λ) will keep the point at $\mathcal{O}$.

An easy calculation reveals that $x_1 = (-1, 0)$, this makes $s = -1$. To have l_1 through this point, means $d_1(t) = c_1(t)$, therefore having $t = -2$ or $t = 1$. The line with $t = -2$ is l_0, so l_1 corresponds to having $t = 1$. Having s and t leads to $\rho = 6$. Again using ψ and transforming yields:

$$(\eta, \lambda) = (2, 1)$$

Continuing these steps, we can calculate:

$$\begin{array}{rrl} & (x_2, l_2) & = \left(\left(-\frac{119}{169}, \frac{120}{169}\right), y = -\frac{5560}{18921}x - \frac{80}{159}\right) \\ \Rightarrow & (s, t) & = \left(\frac{-7}{17}, \frac{-238}{139}\right) \\ \Rightarrow & (\rho, s) & = \left(-\frac{754}{289}, \frac{-7}{17}\right) \\ \Rightarrow & (y, x) & = \left(\frac{35}{64}, -\frac{15}{16}\right) \\ \Rightarrow & (\eta, \lambda) & = \left(-\frac{161}{49}, \frac{49}{16}\right) \end{array}$$

For these points we can calculate the φ which reveals that adding on the elliptic curve is indeed isomorphic to addition modulo 1

$$\begin{array}{lrl} (x_0, l_0) \sim & \varphi(\mathcal{O}) & = 0 \\ (x_1, l_1) \sim & \varphi(2, 1) & \approx 0.3822382958 \\ (x_2, l_2) \sim & \varphi\left(-\frac{161}{49}, \frac{49}{16}\right) & \approx 0.7644765858 \end{array}$$

Note that the rows increase by the same number both times. This illustrates that indeed Poncelet's system is very closely related to rotations on a circle.

3.2.4 Rotation number for free k and l

By construction we can take $(x_0, l_0) = \big((1, 0), y = 0\big)$. In section 3.2.1, we found the corresponding (ρ, s) to be $(-2(k-1), 1)$. Using the isomorphism ψ, the corresponding point on the elliptic curve on the (y, x)-plane, is the point $\mathcal{O}$, which corresponds with the point $\mathcal{O}$ in the (η, λ)-plane.

Using the algorithm, it is easy to find $x_1 = (-1, 0)$. The line $l_1 : y = c_1 x + d_1$ is tangent to C_2 and, because it goes through $x_1 = (-1, 0)$, we know $d_1 = c_1$. As both are functions of t, we can use this to calculate t:

$$c_1(t) = d_1(t) \Rightarrow \frac{2l(k+t)}{(k+t)^2 - l^2} = \frac{2lt(k+t)}{(k+t)^2 - l^2} \Rightarrow t = -k \text{ or } t = 1$$

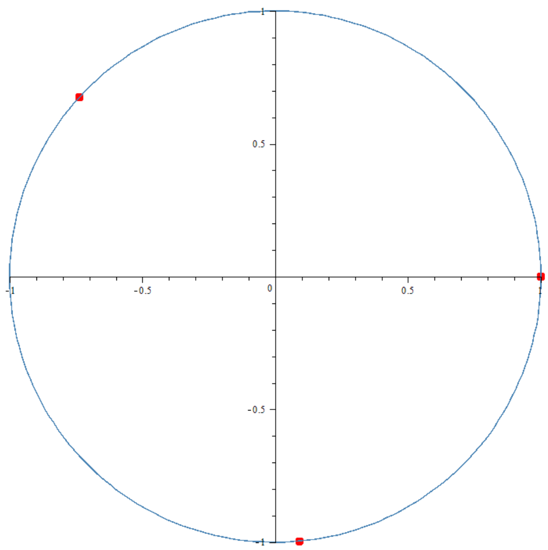


Figure 1: Plot of $\varphi((O))$, $\varphi(2,1)$ and $\varphi\left(-\frac{161}{49}, \frac{49}{16}\right)$(counterclockwise from the rightmost point) on a circle.

The first of these options is l_0, therefore the second corresponds to l_1. This means we have an $(s,t) = (-1,1)$ for (x_1, l_1), implying: $(\rho, s) = (2(k+1), -1)$. Using the transformations to the (η, λ)-plane:

$$(\eta, \lambda) = \left(\frac{l}{(k-1)^6}, \frac{k^6 - 4k^5 + 7k^4 - 8k^3 + 6k^2 - 4k + 3}{3(k-1)^4} \right)$$

The last section made it very convincing that $\varphi(\eta, \lambda)$ will be the rotation number of the system. After calculating this it should be easy to deduce the behaviour of the system: if this number is rational, the transverse will close up in a finite amount of steps, if it is irrational it will never.

Conclusion

In conclusion, both the systems for Poncelet's Closure Theorem and Gelfand's question were shown to be closely related to the circle group. Furthermore, using these results, all of Gelfand's questions were answered.

References

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3. http://www.math.lsa.umich.edu/∼rauch/558/Kronecker.pdf

4. F. Benford "The law of anomalous numbers", Proceedings of the American Philosophical Society, Vol. 78, No. 4, March 31, 1938.

A Plots of Gelfand's system on an 8-torus

Gelfand's system was found to be isomorphic to fixed rotations on an 8-torus, which can be viewed as addition modulo one on an 8-cube. If we take $x = \log_{10} 2$, $y = \log_{10} 3$, and $z = \log_{10} 7$, the entire shape in 8 dimensions will be the closure of the orbit defined by:

$$\begin{aligned} \tau_n &= \tau_{n-1} \oplus (x, y, 2x, -x, xy, z, 3x, 2y) \\ \tau_0 &= (0, 0, 0, 0, 0, 0, 0, 0) \end{aligned}$$

As we can not visualize an 8-cube, we plot the intersections obtained by only looking at 3 starting values, which already hints at the complexity of the attainable shape in the 8-cube.

The plots are made by taking the first 1000 iterations (additions modulo one of the logarithms), for starting values α, β, γ, and plotting them in a 3-dimensional cube.

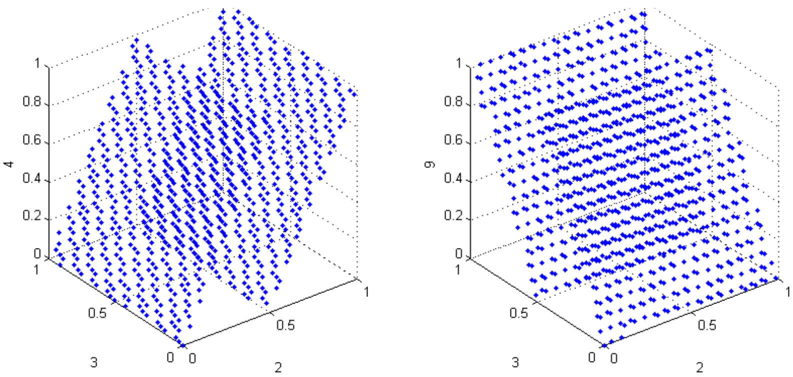

Figure 2: This illustrates that the relation between 2 and 4 is the equivalent to the one between 3 and 9.

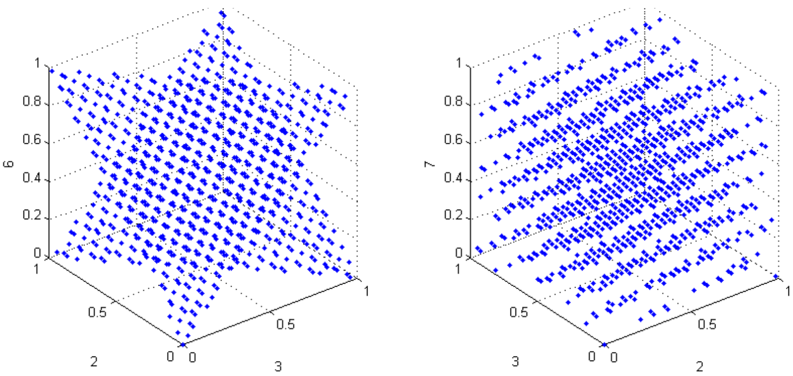

Figure 3: The left plot illustrates that any two of $2, 3, 6$ are independent, but all together they are dependant. The right plot suggests the independence of $2, 3, 7$.

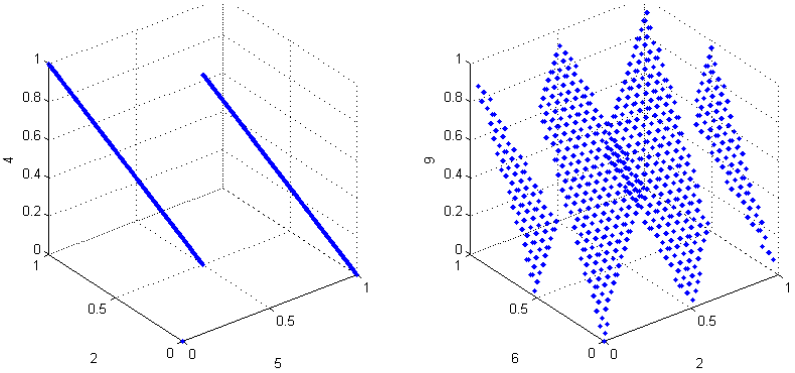

Figure 4: The left plot illustrates the strong dependence between 2, 4, 5: Only a line can be attained. The right plot illustrates the dependence of 2, 6, 9, which shows that not only the relations stated in 2.2.2 determine the shape.

The last two plots attempt to show some more of the complexity, by taking the colour of the points as a function of a fourth rotation.

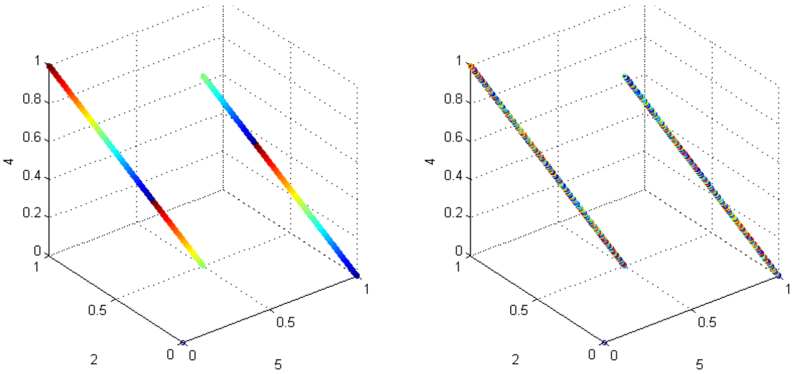

Figure 5: The left image has the colour as a function of the rotation over $\log_{10} 8$. To show the structure this creates, the right image has it's colour defined by rotation over $\log_{10} 3$, which is clearly not as structured.

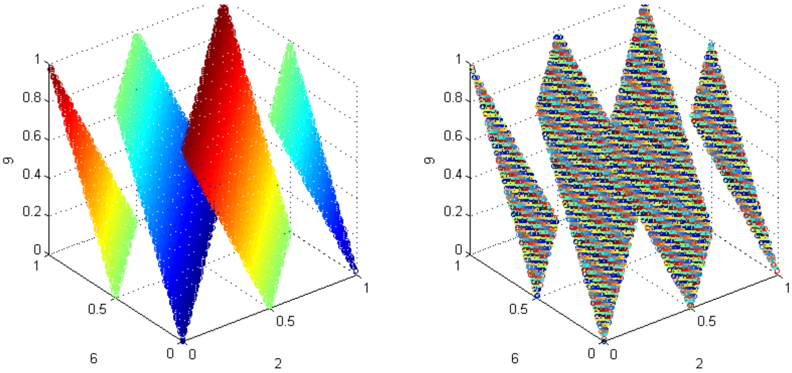

Figure 6: A similar case: The left image has the colour as a function of the rotation over $\log_{10} 3$, the right $\log_{10} 7$

B Matlab Code

This Matlab code uses the relations between the logarithms to remove rows that can no be attained in Gelfand's system.

```
k=allcomb(1:9,1:9,1:9,1:9,1:9,1:9,1:9,1:9);
for n=1:(9^8)
    if ((k(n,3)>= (k(n,1)+1)^2             ) || (k(n,3)+1<=k(n,1)^2)           )
    && ((10*k(n,3)>= (k(n,1)+1)^2          ) || (10*(k(n,3)+1)<=k(n,1)^2)      )
            k(n,:)=[0,0,0,0,0,0,0,0];
    end
    if ((k(n,7)>= (k(n,1)+1)^3             ) || (k(n,7)+1<=k(n,1)^3)           )
    && ((10*k(n,7)>= (k(n,1)+1)^3          ) || (10*(k(n,7)+1)<=k(n,1)^3)      )
    && ((100*k(n,7)>= (k(n,1)+1)^3         ) || (100*(k(n,7)+1)<=k(n,1)^3)     )
            k(n,:)=[0,0,0,0,0,0,0,0];
    end
    if ((k(n,8)>= (k(n,2)+1)^2             ) || (k(n,8)+1<=k(n,2)^2)           )
    && ((10*k(n,8)>= (k(n,2)+1)^2          ) || (10*(k(n,8)+1)<=k(n,2)^2)      )
            k(n,:)=[0,0,0,0,0,0,0,0];
    end
    if ((k(n,5)>= (k(n,1)+1)*(k(n,2)+1)    ) || (k(n,5)+1<=k(n,1)*k(n,2)      ))
    && ((10*k(n,5)>= (k(n,1)+1)*(k(n,2)+1) ) || (10*(k(n,5)+1)<= k(n,1)*k(n,2)))
            k(n,:)=[0,0,0,0,0,0,0,0];
    end
    if (10<=(k(n,4)*k(n,1)) || 10>=((k(n,4)+1)*(k(n,1)+1)))
            k(n,:)=[0,0,0,0,0,0,0,0];
    end

end
X2=k(any(k,2),:) ;
clearvars k n
```

C Gelfand Table

1	2	3	4	5	6	7	8	9
2	4	9	1	2	3	4	6	8
3	8	2	6	1	2	3	5	7
4	1	8	2	6	1	2	4	6
5	3	2	1	3	7	1	3	5
6	6	7	4	1	4	1	2	5
7	1	2	1	7	2	8	2	4
8	2	6	6	3	1	5	1	4
9	5	1	2	1	1	4	1	3
10	1	5	1	9	6	2	1	3
11	2	1	4	4	3	1	8	3
12	4	5	1	2	2	1	6	2
13	8	1	6	1	1	9	5	2
14	1	4	2	6	7	6	4	2
15	3	1	1	3	4	4	3	2
16	6	4	4	1	2	3	2	1
17	1	1	1	7	1	2	2	1
18	2	3	6	3	1	1	1	1
19	5	1	2	1	6	1	1	1
20	1	3	1	9	3	7	1	1
21	2	1	4	4	2	5	9	1
22	4	3	1	2	1	3	7	9
23	8	9	7	1	7	2	5	8
24	1	2	2	5	4	1	4	7
25	3	8	1	2	2	1	3	7
26	6	2	4	1	1	9	3	6
27	1	7	1	7	1	6	2	5
28	2	2	7	3	6	4	1	5
29	5	6	2	1	3	3	1	4
30	1	2	1	9	2	2	1	4
31	2	6	4	4	1	1	9	3
32	4	1	1	2	7	1	7	3
33	8	5	7	1	4	7	6	3
34	1	1	2	5	2	5	5	2
35	3	5	1	2	1	3	4	2
36	6	1	4	1	1	2	3	2
37	1	4	1	7	6	1	2	2
38	2	1	7	3	3	1	2	1
39	5	4	3	1	2	9	1	1
40	1	1	1	9	1	6	1	1
41	2	3	4	4	8	4	1	1
42	4	1	1	2	4	3	8	1
43	8	3	7	1	2	2	6	1
44	1	9	3	5	1	1	5	9
45	3	2	1	2	1	1	4	8
46	7	8	4	1	6	7	3	7
47	1	2	1	7	3	5	2	7
48	2	7	7	3	2	3	2	6
49	5	2	3	1	1	2	1	5
50	1	7	1	8	8	1	1	5
51	2	2	5	4	4	1	1	4
52	4	6	2	2	2	8	9	4
53	9	1	8	1	1	6	7	3
54	1	5	3	5	1	4	5	3
55	3	1	1	2	6	3	4	3
56	7	5	5	1	3	2	3	2
57	1	1	2	6	2	1	2	2
58	2	4	8	3	1	1	2	2
59	5	1	3	1	8	7	1	1
60	1	4	1	8	4	5	1	1
61	2	1	5	4	2	3	1	1
62	4	3	2	2	1	2	9	1
63	9	1	8	1	1	1	7	1
64	1	3	3	5	6	1	6	1
65	3	1	1	2	3	8	5	1
66	7	3	5	1	2	5	4	9
67	1	9	2	6	1	4	3	8
68	2	2	8	3	8	2	2	7
69	5	8	3	1	4	2	2	6
70	1	2	1	8	2	1	1	6
71	2	7	5	4	1	1	1	5
72	4	2	2	2	1	7	1	5
73	9	6	8	1	6	4	8	4
74	1	2	3	5	3	3	6	4
75	3	6	1	2	2	2	5	3
76	7	1	5	1	1	1	4	3
77	1	5	2	6	8	1	3	2
78	3	1	9	3	4	8	2	2
79	6	4	3	1	2	5	2	2
80	1	1	1	8	1	4	1	2
81	2	4	5	4	1	2	1	1
82	4	1	2	2	6	1	1	1
83	9	3	9	1	3	1	9	1
84	1	1	3	5	2	9	7	1
85	3	3	1	2	1	6	5	1
86	7	1	5	1	8	4	4	1
87	1	3	2	6	5	3	3	1
88	3	9	9	3	3	2	2	9
89	6	2	3	1	1	1	2	8
90	1	8	1	8	1	1	1	7
91	2	2	6	4	6	8	1	6
92	4	7	2	2	3	5	1	6
93	9	2	9	1	2	3	9	5
94	1	7	3	5	1	2	7	4
95	3	2	1	2	8	1	6	4
96	7	6	6	1	5	1	4	4
97	1	1	2	6	3	9	3	3
98	3	5	1	3	1	6	3	3
99	6	1	4	1	1	4	2	2
100	1	5	1	7	6	3	2	2