university of groningen

## Multidimensional Residues

## applied to real integrals

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Master Thesis in Mathematics
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#### Abstract

Summary In this master thesis we investigate the possibilities of extending Cauchy's theorem to several complex variables. Most of the problems we meet during generalizing are of a topological nature. Local residues of a form $h /\left(f_{1} \cdots f_{n}\right) d z$ over $\mathbb{C}^{n}$ are defined as integrals over local cycles around the intersection points of $n$ hyperplanes $f_{1}=\ldots=f_{n}=0$. In turns out that only cycles which are separable can be replaced as a sum of local cycles.


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## Contents

1 Preliminaries ..... 3
2 Fundamental groups ..... 5
2.1 Fundamental groups ..... 5
2.2 Fundamental polygons ..... 8
2.3 Seifert - van Kampen Theorem ..... 10
3 Homology ..... 13
3.1 Simplicial and singular homology ..... 13
3.2 Mayer-Vietoris exact sequence ..... 15
4 Holomorphic and Analytic functions ..... 17
4.1 Holomorphic functions ..... 17
4.2 Analytic continuation ..... 20
5 Holomorphic forms ..... 23
5.1 Complex differential forms ..... 23
5.2 Stokes and Cauchy ..... 26
6 Calculating residues ..... 29
6.1 Introduction ..... 29
6.2 Local residues ..... 30
6.3 Residues on local intersections ..... 35
6.4 Separating cycles ..... 35
7 Applications to integrals over $\mathbb{R}^{2}$ ..... 39
7.1 A class of functions over $\mathbb{R}^{2}$ ..... 39
7.2 Real trigonometric integrals ..... 42
A Some computations... ..... 45

## Introduction

The Cauchy residue theory is one of the most important theories in the complex analysis of one variable. The applications of this theory extend beyond the boundaries of mathematics into fields such as physics, mechanics, etc. Unfortunately, this theory only holds for functions of one variable. Not only from a mathematical perspective but also from a practical point of view, it would be interesting to have such a theory for two or more variables.

Halfway through the first half of the previous century mathematicians such as Friedrich Hartogs started to work on a theory of complex functions of $n$ variables. He discovered that when $n>1$ each isolated singularity is removable. Moreover, shortly after the Second World War a large number of obscurities on analytic continuation was clarified. A major difference from the one-variable theory became evident: while for any connected open set $D$ in $\mathbb{C}$ we can find a function that will be nowhere continue analytically on the boundary of $D$, that is not the case for any connectod open set $D^{\prime}$ in $\mathbb{C}^{n}$, $n>1$ (See section 4.2).

Naturally the analogues of contour integrals will be harder to handle: when $n=2$ an integral surrounding a point should be over a three-dimensional manifold (since we are working in four real dimensions), while iterating contour (line) integrals over two separate complex variables results in a double integral over a two-dimensional surface. This means that the residue calculus will be quite different from the residue calculus in one dimension.

This thesis contains two parts. The first part, i.e. chapters 1 through 5, is a summary of the knowledge acquired in order to be able to start my investigation. The remaining chapters are the results of this research.

The goal of this research is to investigate what is known about the existence and the applications of a theorem like Cauchy's theorem, but in several variables. Does there exist a residue theorem for functions from $\mathbb{C}^{n}$ to $\mathbb{C}$ ? Is it still easy to evaluate such integrals? Which difficulties do we meet? (See chapter 6)

In this thesis we will follow closely the theory expounded in the book of A.K.Tsikh [15], but we add some material for a broader understanding. Furthermore, we discuss local residues on local intersections and display some examples of trigonometric integrals. (See chapter 7)

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## Chapter 1

## Preliminaries

Let us first recall [9] that $\mathbb{C}^{n} \simeq \mathbb{R}^{2 n}$ by the isomorphism $\varphi(x, y)=x+i y$, with $x, y \in \mathbb{R}^{n}$ and $i$ the imaginary unit. Now, $\mathbb{C}^{n}$ forms a vector space of complex vectors $z=x+i y$ over $\mathbb{R}$. We denote $\bar{z}=x-i y$ the complex conjugate of $z$ and define the real and imaginairy part of $z$ resp. as $\operatorname{Re}(z):=x \in \mathbb{R}^{n}, \operatorname{Im}(z):=y \in \mathbb{R}^{n}$, furthermore we define the inner product to be $\langle z, w\rangle:=\sum_{j=1}^{n} z_{j} \bar{w}_{j}$ which defines the (Euclidian- or 2-) norm on $\mathbb{C}^{n}$ by: $\|z\|_{2}=\sqrt{\langle z, z\rangle}$.

The normed complex vector space is also a metric space, with the metric $d(z, w)=\|z-w\|_{2}$. With this notion ahead, it makes sense to talk about open and closed subsets of $\mathbb{C}^{n}$, about continuity of functions defined on this vector space and about convergence of sequences and series. Since every metric space is a topological space, $\mathbb{C}^{n}$ is also a topological space with open sets generated by open balls $B_{r}(z)=\left\{w \in \mathbb{C}^{n}: d(z, w)<r\right\}, r>0$. A set $\mathcal{U}_{a}$ is called a neighborhood of the point $a$ if there exists an open ball $B_{r}(a)$ which is contained in $\mathcal{U}_{a}$.

Let $K \subset \mathbb{C}^{n}$, then $K$ is called convex if for every pair of vectors $k_{1}, k_{2} \in K$ and for all $t \in[0,1]$ also the vector $(1-t) k_{1}+t k_{2}$ is in $K$. A set $K$ is called logarithmically convex if the image of the mapping $K \rightarrow M: z \mapsto \ln |z|=\left(\ln \left|z_{1}\right|, \ldots, \ln \left|z_{n}\right|\right)$ is convex and $K$ is called circular or Reinhardt if for every $k \in K$ also $k e^{i \theta} \in K$, for $\theta \in[0,2 \pi)$. Note that in one complex variable, a logarithmically convex Reinhardt domain is simply an annulus centered at the origin. The closure of $V$, will be denoted by $\bar{V}$. For example, the closure $\mathbb{C}$ is $\overline{\mathbb{C}}:=\mathbb{C} \cup\{\infty\}$.

Let us also recall that functions on the complex plane may be multiple valued, i.e. $f(z) \subset \mathbb{C}$. This phenomenon also occurs on $\mathbb{C}^{n}$. If $f(z)$ is a multiple valued function over a domain $D \in \mathbb{C}^{n}$, then $\mathcal{B}_{f}(z)$ is called a branch of $f$ if $\mathcal{B}_{f}(z)$ is single valued and continuous over $D$ and has one of the values of $f(z)$ [11]. For instance, the function $f(z)=z^{\frac{1}{2}}$, where we choose $f(1)=1$, is double valued throughout the negative real axis. One takes two copies of the complex plane and cuts both plains open on the negative real axis. Now one glues the two copies together such that the function $f(z)$ moves over the two sheets in a continuous way. The glued sheets are a Riemann manifold as in figure 1.1.


Figure 1.1: Construction of the Riemann surface of $f(z)=z^{\frac{1}{2}}$

## Chapter 2

## Fundamental groups

In this chapter we recall, following [10], some topological aspects of subspaces in $\mathbb{C}^{n}$ by looking at fundamental groups, chains and loops. We start with describing homotopy and continue with the study of simplices, which will be used to define homology. This will be the basis for chapter 6 .

### 2.1 Fundamental groups

Assume that $X$ and $Y$ are two topological spaces. We define the following equivalence relation on the set of continuous maps from $X$ to $Y$ as follows.

Definition 1. Let $f, g: X \rightarrow Y$ continuous, then $f$ and $g$ are called homotopy equivalent (write $f \simeq g)$ if there exists a continuous map $F: X \times[0,1] \rightarrow Y$ such that $F(x, 0)=f$ and $F(x, 1)=g$. The map $F$ is called a homotopy between $f$ and $g c f$. [10].

This is an equivalence relation. Indeed, $F(x, t)=f(x)$ shows that $f \sim f$ (reflexivity) and if $f \simeq g$ by $F(x, t)$ then $g \simeq f$ by $F(x, 1-t)$ (symmetry). To show that also transitivity holds, let $F_{1}(x, t)$ be the homotopy of $f \simeq g$ and $F_{2}(x, t)$ be the homotopy of $g \simeq h$. We then have that $f \simeq h$ by the homotopy

$$
F(x, t)= \begin{cases}F_{1}(x, 2 t) & : 0 \leq t \leq \frac{1}{2}  \tag{2.1}\\ F_{2}(x, 2 t-1) & : \frac{1}{2} \leq t \leq 1\end{cases}
$$

One can think of homotopy equivalence of two maps as meaning that one can transform the first map into the second map in a continuous way. The next proposition is meant as an illustration.

Proposition 1. Every pair of continuous maps $f, g: X \rightarrow Y$ is homotopy equivalent if $Y$ is convex.
Proof. The homotopy is $F(x, t)=(1-t) f(x)+t g(x)$.
Note that $F$ is indeed a correct homotopy, since $F$ is continuous, because $f$ and $g$ are continuous, and the image of $F$ is a subset of $Y$, because $Y$ is convex.

Using homotopy equivalent maps, we define homotopy equivalence in the next definition. Here $I d_{X}$ will be the identity mapping from $X$ to itself.

Definition 2. $X, Y$ are called homotopy equivalent if there exists continuous maps $f: X \rightarrow Y$ and $g: Y \rightarrow X$, such that $g \circ f \simeq I d_{X}$ and $f \circ g \simeq I d_{Y}$. Then $g$ is called the homotopy equivalent inverse of $f$.

Homotopy equivalence is a weaker notion than homeomorphy.

Theorem 1. If $X$ and $Y$ are homeomorphic, then they are homotopy equivalent.
Proof. If two spaces are homeomorphic, then there exists a homeomorphism $\phi$ which is invertible and continuous. Let $f=\phi$ and $g=\phi^{-1}$, then $f \circ g=\operatorname{Id}_{Y}$.

Not every pair of spaces that is homotopy equivalent is also homeomorphic. Indeed, a solid disk is homotopy equivalent to a point, see also proposition 1, but a point is obviously not homeomorphic to a disk. Homotopy equivalence is not necessarily dimension preserving, homeomorphy is.

Another way of classifying topological spaces is by means of paths and loops. This will result in two types of connectedness. Let us first give formal definitions of paths and loops.

Definition 3. Suppose $x_{1}, x_{2}$ are points in the space $X$. A path from $x_{1}$ to $x_{2}$ is a continuous map $u:[0,1] \rightarrow X$ such that $u(0)=x_{1}$ and $u(1)=x_{2}$. A loop in $X$ based at $x_{1}$ is a path from $x_{1}$ to $x_{1}$.

We call a topological space $X$ path-connected if and only if there exists for every two points $x_{1}, x_{2} \in X$ a path $u$ from $x_{1}$ to $x_{2}$ and $X$ is called simply connected if every loop in $X$ is contractible. The next theorem shows that a homotopy preserves these two properties.

Theorem 2. Let $X$ and $Y$ be homotopy equivalent spaces, then

1. if $X$ is path-connected then so is $Y$;
2. if $X$ is simply connected then so is $Y$.


Figure 2.1: Two homotopy equivalent spaces $X$ and $Y$ and some mappings.

Proof. Let $X$ and $Y$ be homotopy equivalent then by the previous definition there exists two maps $f$ and $g$ such that $g \circ f \simeq \operatorname{Id}_{X}$ and $f \circ g \simeq \operatorname{Id}_{Y}$. Also, let $F(x, 0)=f \circ g, F(x, 1)=\operatorname{Id}_{X}, G(y, 0)=g \circ f$ and $G(y, 1)=\operatorname{Id}_{Y}$. See also figure 2.1.

1. It's clear that the image of $f$ is path connected. Thus, it is enough to show that any point of $Y$ can be connected to a point of $f(X)$. Let $f \circ g$ homotopic to $\mathrm{id}_{Y}$, via the homotopy $h: Y \times I \rightarrow Y$. Let $y \in Y$, then $y^{\prime}=f(g(y)) \in f(X)$ and $\gamma(t)=h(y, t)$, a path from $y^{\prime}$ to $y$.
2. If $X$ is simply connected then $X$ is homotopy equivalent with a point. Therefore, also $Y$ is homotopy equivalent to a point and hence it is simply connected.

To be specific, also paths and loops can be homotopy equivalent, because they are maps and definition 1 applies. Two paths $u_{1}, u_{2}$ both from $x_{1}$ to $x_{2}$ are homotopy equivalent if there exists a map $U:[0,1]^{2} \rightarrow X$ with $U(x, 0)=u_{1}(x)$ and $U(0,1)=u_{2}(x)$ for every $x \in X$. We write $u_{1} \sim u_{2}$. This is also an equivalence relation, since homotopy equivalence is an equivalence relation. The equivalence class of a path (or a loop) $u$ will be denoted as $[u]$ and is called the homotopy class of $u$.

Definition 4. The set $\pi_{1}\left(X, x_{0}\right)$ of all homotopy classes of loops $u:[0,1] \rightarrow X$ at basepoint $x_{0}$ is called the fundamental group of $X$ at basepoint $x_{0}$.

Let $u, v$ be two loops with $u(1)=v(0)$, then the composition of $u$ and $v$ is given by

$$
u * v:=\left\{\begin{array}{lll}
u(2 x) & \text { if } \quad 0 \leq x \leq \frac{1}{2}  \tag{2.2}\\
v(2 x-1) & \text { if } \quad \frac{1}{2} \leq x \leq 1
\end{array}\right.
$$

The word 'group' in the definition is permitted because of the following theorem.
Theorem 3. The fundamental group $\pi_{1}\left(X, x_{0}\right)$ is a group with respect to the group law $*$.
For a formal and detailed proof, we refer to [7]. Here, we just try to convince the reader that $\pi_{1}\left(X, x_{0}\right)$ is indeed a group.

The identity of the group is the equivalence class of the constant loop $u:[0,1] \rightarrow\left\{x_{0}\right\}$ and the inverse of a class of loops $[u]$ is given by $[u]^{-1}=\left[u^{-1}\right]=\{v(x): v \sim u(1-x)\}$. The product of two loops can be seen as walking a first loop at double speed and afterwards also the second loop at double speed. One can imagine that this is again a loop. The group operation $*$ is therefore closed. Also, * is associative, because walking two loops and then a third is equivalent to walking the first loop and then the last two.

It is natural to ask about the dependence of $\pi_{1}\left(X, x_{0}\right)$ on the choice of the basepoint $x_{0}$. In order to answer this question, we recall the change of basepoint map $\beta_{h}: \pi_{1}\left(X, x_{1}\right) \rightarrow \pi_{1}\left(X, x_{0}\right)$ by $\beta_{h}[p]=\left[h * p * h^{-1}\right]$. Here $h$ is a path in $X$ from $x_{1}$ to $x_{0}$.

Theorem 4. The map $\beta_{h}: \pi_{1}\left(X, x_{1}\right) \rightarrow \pi_{1}\left(X, x_{0}\right)$ is an isomorphism if $h$ is a path in $X$ from $x_{1}$ to $x_{0}$ and $X$ is path connected.

Proof. $\beta_{h}[u * v]=\left[h * u * v * h^{-1}\right]=\left[h * u * h^{-1} * h * v * h^{-1}\right]=\beta_{h}[u] \circ \beta_{h}[v]$. The inverse of $\beta_{h}[u]$ is $\beta_{h^{-1}}[u]$ since $\beta_{h}[u] \circ \beta_{h^{-1}}[u]=\beta_{h}\left[h^{-1} * u * h\right]=[u]$. Similarly $\beta_{h^{-1}}[u] \circ \beta_{h}[u]=[u]$.

Thus the fundamental group $\pi_{1}\left(X, x_{0}\right)$ is isomorphic for all base points $x_{0} \in X$, if $X$ is pathconnected. If $X$ is not path-connected, then the fundamental groups are isomorphic for all base points which can be reached by a path $h$ in $X$.

We give three examples of spaces and calculate their fundamental groups. Note that the fundamental group is in general not commutative.

Example 1. The Euclidean space $\mathbb{C}^{n}$ (or any convex subset of $\mathbb{C}^{n}$ ). All loops in the (convex subspace) of the Euclidean space are contractible to the basepoint. This means that the fundamental group is the group with the constant map. Such a group is called trivial.

Example 2. The circle. Each homotopy class consists of all loops which wind around the circle a given number of times (which can be positive or negative, depending on the direction of winding). The product of a loop which winds around $m$ times and another that winds around $n$ times is a loop which winds around $m+n$ times. So the fundamental group of the circle is isomorphic to $(\mathbb{Z},+)$, the additive group of integers. We can write down explicitly the generator $\varphi_{n}(s)=(\cos (\pi n s), \sin (\pi n s))$. To prove that $\varphi: \mathbb{Z} \rightarrow \pi_{1}\left(X, x_{0}\right)$ is indeed an isomorphism, needs quite some work and a detailed proof can be found, for example, in [7].

Let us consider an example where this theory is applied to complex analysis.
Theorem 5. Fundamental theorem of algebra Every nonconstant polynomial with coefficients in $\mathbb{C}$ has at least one root in $\mathbb{C}$.

Proof. Let us assume that there is a polynomial $p(z)=z^{n}+a_{1} z^{n-1}+\cdots+a_{n}$ without roots in $\mathbb{C}$. Define a homotopy $p_{t}(z)=t p(z)+(1-t) z^{n}$ for $t \in[0,1]$. Now,

$$
\begin{equation*}
\frac{p_{t}(z)}{z^{n}}=t\left(1+a_{1} \frac{1}{z}+\ldots+a_{n} \frac{1}{z^{n}}\right)+1-t=1+t\left(a_{1} \frac{1}{z}+\ldots+a_{n} \frac{1}{z^{n}}\right) . \tag{2.3}
\end{equation*}
$$

The terms between the parenthesis tend to zero as $z$ goes to infinity. Therefore $p_{t}(z)$ is never zero on a circle $|z|=r>0$. Now the function

$$
\begin{equation*}
f_{r}(s)=\frac{p_{t}\left(r e^{2 \pi i s}\right)}{\left|p_{t}\left(r e^{2 \pi i s}\right)\right|} \tag{2.4}
\end{equation*}
$$

defines a loop in the unit circle $S^{1} \subset \mathbb{C}$ with basepoint 1 and $\left[f_{r}\right] \in \pi_{1}\left(S^{1}, 1\right)$. This shows that for the complex polynomial $p(z)$ of degree $n$, there is a circle of sufficiently large radius in $\mathbb{C}$ such that both $p(z) /|p(z)|$ and $z^{n} /\left|z^{n}\right|$ are homotopic mappings from $\{|z|=r\}$ to the unit circle. This implies that $p(z) /|p(z)|$ has also degree $n$.

Now define $f_{t}=p(t z) /|p(t z)|$, which is a homotopy from the constant map $p(0) /|p(0)|$ to $p(z) /|p(z)|$. Under the homotopy, the degree should not change, so if we assume $p(z)$ to have no zeros, it has to have degree 0 , which is a constant map.

Example 3. Lemniscate ( $\infty$-figure, rose or bouquet of circles). This figure contains two fundamental loops. The left- and the right part of the $\infty$-figure. Therefore, the fundamental group is the free group generated by those two paths. In general, the fundamental group of a rose is free, with one generator for each petal.

Note that every $n$-dimensional torus of genus $m$ is homotopy equivalent to a rose with $n \cdot m$ petals, so it has the same fundamental group. Suppose we have two path-connected spaces $X$ and $Y$, the fundamental group $\pi_{1}(X \times Y)$ is isomorphic to $\pi_{1}(X) \times \pi_{1}(Y)$. One can see this from the fact that a loop $u$ in $X \times Y$ based at $\left(x_{0}, y_{0}\right)$ is a pair of loops $v$ in $X$ based in $x_{0}$ and $w$ in $Y$ based at $y_{0}$. Similarly, a homotopy $F(X \times Y, t)$ is a pair of homotopies $G(X, t)$ and $H(Y, t)$, so we obtain a bijection $\pi_{1}\left(X \times Y,\left(x_{0}, y_{0}\right)\right) \sim \pi_{1}\left(X, x_{0}\right) \times \pi_{1}\left(Y, y_{0}\right),[u] \mapsto([v],[w])$. Indeed, this is only an isomorphism when both $X$ and $Y$ are path-connected.

### 2.2 Fundamental polygons

We can construct for every closed surface an even-sided polygon, called the fundamental polygon, and visa verse by identification of the edges of the polygon. This construction can be represented as a
string of length $2 n$ of $n$ distinct symbols where each symbol appears twice with exponent either +1 or -1 . The exponent -1 signifies that the corresponding edge has the orientation opposing the one of the fundamental polygon. First some examples.


Figure 2.2: Examples of fundamental polygons.
(a) If we cut open the sphere from the north pole to the south pole, we can deform it to a square. Let us call the path from the north pole $\left(z_{0}\right)$ to the equator $A$ and from the equator to the south pole $B$. If we walk the paths $A, B, B$ in opposite direction and finally $A$ in opposite direction, we have that $A B B^{-1} A^{-1} \sim z_{0}$. Now the fundamental group is the group generated by loops, and so there is just one loop $A B B^{-1} A^{-1}$, which is a contractible loop with basepoint $z_{0}$. The fundamental group is trivial.
(b) If we cut open the torus first longitudinal and second latitudinal, we obtain a square. Let $A$ be the path along the first cut and $B$ the path along the second cut. Both $A$ and $B$ are loops, so the group is generated by $A$ and $B$ with relation $A B A^{-1} B^{-1} \sim z_{0}$. Hence the fundamental group is isomorphic to $\mathbb{Z} \times \mathbb{Z}$ and one can see from this relation that the fundamental group is Abelian.
(c) It is not easiliy seen without figure 2.3 how the transformation from the Klein Bottle to the square is done. Again, by setting $A$ to be the first cut and $B$ to be the second, we have the group relation $A B A B^{-1} \sim z_{0}$. As one can see, the fundamental group is not Abelian.


Figure 2.3: From Klein bottle to square.
For the set of polygons, the symbols of the edges of the polygon may be understood to be the generators of the fundamental group. Then, the polygon, written in terms of group elements, becomes a constraint on the free group generated by the edges, giving a group presentation with one constraint.

Thus, for example, given the complex plane $\mathbb{C}$, let the group element $A$ act on the plane as $A(x+$ $i y)=x+1+i y$ while $B(x+i y)=x+i(y+1)$. Then $A, B$ generate the lattice $\Gamma=\mathbb{Z}^{2}$. The torus is given by the quotient space $T=\mathbb{C} / \mathbb{Z}^{2}$. For the torus, the constraint on the free group in two letters is given by $A B A^{-1} B^{-1}=z_{0}$.

### 2.3 Seifert - van Kampen Theorem

The Seifert - van Kampen theorem (abbreviated Van Kampen's theorem) gives a method for computing the fundamental groups of spaces that can be decomposed into simpler spaces whose fundamental groups are already known. By systematic use of this theorem one can compute the fundamental groups of a very large number of spaces. The free product is in this context very important, because van Kampen's theorem states that the fundamental group of the union of two path-connected topological spaces is always an amalgamated free product of the fundamental groups of the spaces.

The free product $\star_{j=1}^{n} G_{\alpha_{j}}$ (abbreviated $\star_{\alpha} G_{\alpha}$ ) of groups $G_{\alpha_{j}}$ is the set of words $g_{1} g_{2} \cdots g_{m}$ of finite length $m \geq 0$, where each letter $g_{k}$ belongs to a group $G_{\alpha_{j}}$, not the identity, and successive letters $g_{k}, g_{k+1}$ are from different groups. We also allow the empty word, which is the identity of the free product. To show that the free product is a group is quite tedious, but not difficult. One can imagine that the free product would be a group if the group operation was just the juxtaposition $\left(w_{1} \cdot w_{2}=w_{1} w_{2}\right)$ of words. The compulsary simplifications of successive letters form the same group, as is treaded in the proof of Hatcher [7].

Before we state Van Kampen's theorem, we will come up with two important remarks.
Remark 1. Because successive letters from the same group in a word generates a unique simplified word and any sequence of simplifications in any order produces the same reduced word the group operation is associative.

Remark 2. A collection of homomorphisms $\varphi_{\alpha}: G_{\alpha} \rightarrow H$ extends uniquely to a homomorphism $\varphi: \star_{\alpha} G_{\alpha} \rightarrow H: \varphi\left(g_{1} \cdots g_{m}\right)=\varphi_{\alpha_{1}}\left(g_{1}\right) \cdots \varphi_{\alpha_{m}}\left(g_{m}\right)$.

Suppose a space $X$ can be decomposed as a union of two path-connected open subsets $A$ and $B$. Each of these subsets contain the basepoint $x_{0}$. Denote $i_{\alpha \beta}: \pi_{1}(A \cap B) \rightarrow \pi_{1}(A)$ and $i_{\beta \alpha}: \pi_{1}(A \cap$ $B) \rightarrow \pi_{1}(B)$, which are homomorphisms induced by resp. the inclusions $A \cap B \hookrightarrow A$ and $A \cap B \hookrightarrow$ $B$. Likewise $j_{\alpha}: \pi_{1}(A) \rightarrow \pi_{1}(X)$ and $j_{\beta}: \pi_{1}(B) \rightarrow \pi_{1}(X)$, also by inclusion maps $A \hookrightarrow X$ and $B \hookrightarrow X$. The homomorphisms $j$ extend naturally to a homomorphism $\Psi: \pi_{1}(A) \star \pi_{1}(B) \rightarrow \pi_{1}(X)$. Since $j_{\alpha} i_{\alpha \beta}=j_{\beta} i_{\beta \alpha}$, the kernel of $\Psi$ contains of elements $i_{\alpha \beta}(\omega) i_{\beta \alpha}(\omega)^{-1}$, for $\omega \in \pi_{1}(A \cap B)$.

Theorem 6. Seifert-van Kampen theorem Under the assumptions as stated above, $\Psi$ is surjective, the kernel of $\Psi$ is the normal subgroup $N$ and $\Psi$ induces an isomorphism

$$
\begin{equation*}
\Phi: \pi_{1}(X) \rightarrow\left(\pi_{1}(A) \star \pi_{1}(B)\right) / N \tag{2.5}
\end{equation*}
$$



Figure 2.4: Loops in $A, B$ and $A \cap B$.

Proof. First, pick a loop $u \subset X$ with basepoint $x_{0}$ in $X$. Start in $x_{0}$ and follow the loop until you are again in $A \cap B$ and call this walk $u_{1}$. Since $A \cap B$ is path connected, we can walk back to the basepoint with a path $v$ to have a loop in just $A$ or just $B$. Now we can walk back with $v^{-1}$ and continue this


Figure 2.5: Decomposition of the Sphere and the Torus
precedure until you have walked the entire loop $u$. In other words, $u=u_{1} v v^{-1} u_{2} v v^{-1} \cdots u_{k}$, where $u_{j}$ is in just $A$ or just $B$. Hence, $\Psi$ is surjective.

Second, we will show that $N=\operatorname{ker}(\Psi)$. Obviously, the kernel of $\Psi$ is a normal subgroup. To show that the normal subgroup is the kernel, let $[v] \in \pi_{1}(A \cap B)$, then

$$
\Psi\left(i_{\alpha \beta}(v) \star i_{\beta \alpha}(v)^{-1}\right)=\Psi\left(v \star v^{-1}\right)=\left[v \star v^{-1}\right]=x_{0}
$$

and hence $N \subset \operatorname{ker}(\Psi)$.
The statement that $\Psi$ is an isomorphism is a direct consequence of the first isomorphism theorem.

Let us consider the following example.
Example 4. The fundamental group of $X=S^{1}$ is euqal to $\mathbb{Z}$, since we have one class of noncontractible loops. A more interesting case is the fundamental group of $X=S^{k}$ for $k>1$. Assume $A$ and $B$ to be two hemispheres, as in figure $2.5(a)$, and $A \cap B$ to be the 'equatorial' $(k-1)$-sphere. Since the $k$-hemisphere is contractible, $A$ and $B$ have a trivial fundamental group. Now Van Kampen's theorem tells us that $\pi_{1}(X) \sim \pi_{1}(A) \star \pi_{1}(B) / \operatorname{kern}(\Psi)$ and so also $\pi_{1}(X)$ is trivial.

Let us consider an other example where we need the use of the fundamental polygon. We calculate the fundamental groups of the torus.

Since $A \cap B$ must be path-connected, on can't split up the torus into two tubes. In stead of calculating the fundamental group of the torus $X$, we calculate the fundamental group of a tube with a ring on it $X^{\prime}$ as in figure 2.5 b . We will call this space the 'tubering', since $\pi_{1}(X) \sim \pi_{1}\left(X^{\prime}\right)$.

Example 5. We split up the tubering in a tube $A$ and an ring $B$ as in figure 2.5 (b). Now Van Kampen's theorem tells us that $\pi_{1}\left(X^{\prime}\right) \sim \pi_{1}(A) \star \pi_{1}(B) / \operatorname{kern}(\Psi)$. The kern $(\Psi)$ is the set of contractible loops and therefore from the fundamental polygon, the free group generated by $\left\langle a b a^{-1} b^{-1}\right\rangle$. Here a is a loop in $A$ and $b$ a loop in B. Now, $\pi_{1}\left(X^{\prime}\right) \sim\langle a, b\rangle /\left\langle a b a^{-1} b^{-1}\right\rangle$. Indeed, this is the abelization of $\langle a, b\rangle$ and hence the fundamental group is isomorphic to $\mathbb{Z}^{2}$.

## Chapter 3

## Homology

In this chapter we discuss the theory of homology like in Hatcher's book [7].

### 3.1 Simplicial and singular homology

Recall from chapter 2 that the sphere, the torus and the Klein bottle can each be obtained from a square by identifying opposite edges. The idea of simplicial homology is to generalize constructions like these to any number of dimensions. The $n$ dimensional analog of the triangle is the $n$-simplex.

Definition 5. Let $\left\{e_{0}, \ldots, e_{n}\right\}$ be a basis of $\mathbb{R}^{n+1}$, then the standard $n$-simplex is defined as the closed polyhedron with the orthogonal basis vectors as vertices. In general an n-simplex is a polyhedron with $n+1$ linear independent vectors $v_{i}$ as their vertices denoted as $\left[v_{0}, \ldots, v_{n}\right]$. Also points and line segments are considered to be simplices. We denote the set of $n$-simplices as $\Delta^{n}$.

If we delete one of the $n+1$ vertices of an $n$-simplex $\left[v_{0}, \ldots, v_{n}\right]$, the the remaining $n$ vertices span an $(n-1)$-simplex, called a face of the $n$-simplex.

Definition 6. Let $K$ be a set of simplices. Then $K$ is called a simplicial complex if every face of $\sigma_{j}$ is also a simplex in $K$ and $\sigma_{j} \cap \sigma_{k}$ is a lateral face of both $\sigma_{j}$ and $\sigma_{k}$. $K$ is said to be an $n$-complex if any $k$-simplex contained in $K$ satisfies $k \leq n$.

The boundary of a simplex can be calculated as follows. The boundary of a simplex $A_{n}=$ $\left(a^{1}, \ldots, a^{n}\right)$ is defined as

$$
\begin{equation*}
\partial\left(A_{n}\right)=\sum_{r=0}^{n}(-1)^{r}\left(a^{1}, \ldots, a^{r-1}, a^{r+1}, \ldots, a^{n}\right) \tag{3.1}
\end{equation*}
$$

In what follows, let $F^{r}:\left(a_{1}, \ldots, a_{n}\right) \mapsto\left(a^{1}, \ldots, a^{r-1}, a^{r+1}, \ldots, a^{n}\right)$. One can easily verify that $F^{r} \circ F^{s}=F^{s} \circ F^{r+1}$ if $s<r$.

Proposition 2. For a $n$-simplex $A_{n}(X), \partial^{2}\left(A_{n}\right):=\partial\left(\partial\left(A_{n}\right)\right)=0$.

(a) A 2-simplex in $\mathbb{C}^{3}$

(b) A simplicial complex

(c) Not a simplical complex

Figure 3.1: Example of a 2 -simplex, a simplical 3-complex in $\mathbb{C}^{3}$ and of a complex that is not simplicial.

Proof. For $n>1$, we can explicitly derive the boundary of a boundary.

$$
\begin{align*}
\partial^{2} & =\partial\left[\sum_{r=0}^{n}(-1)^{r} F^{r}\right]  \tag{3.2}\\
& =\sum_{s=0}^{n-1}(-1)^{s} F^{s}\left[\sum_{r=0}^{n}(-1)^{r} F^{r}\right]  \tag{3.3}\\
& =\sum_{s=0}^{n-1} \sum_{r=0}^{n}(-1)^{r+s} F^{r} \circ F^{s}  \tag{3.4}\\
& =\sum_{s<r}^{n}(-1)^{r+s} F^{r} \circ F^{s}+\sum_{s \geq r}^{n}(-1)^{r+s} F^{r} \circ F^{s}  \tag{3.5}\\
& =\sum_{s<r}^{n}(-1)^{r+s} F^{s} \circ F^{r-1}+\sum_{s \geq r}^{n}(-1)^{r+s} F^{r} \circ F^{s} \tag{3.6}
\end{align*}
$$

Since every term $F^{i} F^{j}$ appears twice but with opposite sign, this equals 0 . This proves the proposition.

For a topological space $X$, we denote $C_{n}(X)$ to be the free abelian group generated by the $n$ simplices from the simplicial complex $K$ in $X$. The boundary mapping as in 3.1 together with the groups $C_{n}(X)$ is called the simplicial chain complex $\left(C_{n}, \partial\right)$. Note the two important properties $\partial^{2}=0$ (as proven in proposition 2) and $\partial: C_{n} \rightarrow C_{n-1}$. Now, the image of $\partial$ is the group of boundaries and the kernel of $\partial$ is the group of cycles. The group of boundaries is a subset of the group of cycles. This leads us to the simplicial homology groups.

Definition 7. Let $C_{n}$ a singular complex then

1. the group $B_{n}\left(C_{n}\right)$ is the set of boundaries and hence the image of $\partial_{n}$;
2. the group $Z_{n}\left(C_{n}\right)$ is the set of $n$-cycles and hence the kernel of $\partial_{n-1}$;
3. the quotient group $H_{n}\left(C_{n}\right)=Z_{n}\left(C_{n}\right) / B_{n}\left(C_{n}\right)$ is called the nth homology group.

The fundamental groups of a topological space $X$ are related to its first simplicial homology group, because a loop is also a simplical 1-cycle. It turns out, by Hurewicz theorem, that $\pi_{1}\left(X, x_{0}\right) \simeq$ $H_{1}(X)$, if $X$ is connected. The required isomorphism between $H_{1}(X)$ and $\pi_{1}\left(X, x_{0}\right)$ is the abelization of $\pi_{1}\left(X, x_{0}\right)$. The proof of this theorem is beyond the purpose of this thesis.

This idea can be extended to a more general theory. We can use a general covering of a topological space $X$ in stead of simplexes. Direct sums of such covers form abelian groups. In general, a chain complex is defined as a sequence of abelian groups $A_{n}$ and connecting homomorphisms $d_{n}: A_{n} \rightarrow A_{n-1}$ with the relation $d_{n} \circ d_{n-1}=0$. Just analogous to the simplical complex, we define homology groups to be $\operatorname{im}\left(d_{n-1}\right) / \operatorname{ker}\left(d_{n}\right)$. Note that these connecting homomorphisms $d_{n}$ not need to be boundaries.

### 3.2 Mayer-Vietoris exact sequence

Let $X$ be a topological space which can be written as the union of two open subspaces $X=A \cup B$. We are concerned with the question what relation exists between the three subspaces $A, B$ and $A \cap B$. The answer can be found in the exact sequence of Mayer-Vietoris.

Let us give the definition of an exact sequence.
Definition 8. A diagram $A \xrightarrow{f} B \xrightarrow{g} C$ of abelian groups $A, B$ and $C$ and connecting homomorphisms $f$ and $g$ is short exact if $k e r(g)=\operatorname{im}(f)$ and $f$ is injective and $g$ is surjective. A diagram $D \xrightarrow{d_{1}} E \xrightarrow{d_{2}} F$ of chain complexes and chain maps is short exact if the resulting $D_{k} \rightarrow E_{k} \rightarrow F_{k}$ is short exact for every $k \in \mathbb{Z}$.

Now, let us look at the following inclusion maps

$$
\begin{align*}
& i_{*}: H_{n}(A \cap B) \hookrightarrow H_{n}(A)  \tag{3.7}\\
& j_{*}:  \tag{3.8}\\
& k_{*}(A \cap B) \hookrightarrow H_{n}(B)  \tag{3.9}\\
& k_{*}: H_{n}(A) \hookrightarrow H_{n}(X)  \tag{3.10}\\
& l_{*}: H_{n}(B) \hookrightarrow H_{n}(X)
\end{align*}
$$

and the following homomorphisms.

$$
\begin{array}{lll}
\varphi: H_{n}(A \cap B) \rightarrow H_{n}(A) \oplus H_{n}(B) & \varphi(z) \mapsto\left(i_{*}(z), j_{*}(z)\right) \\
\psi: H_{n}(A) \oplus H_{n}(B) \rightarrow H_{n}(X) & \psi(u, v) \mapsto k_{*}(u)-l_{*}(v) \tag{3.12}
\end{array}
$$

Without proving, we state one of the most important results in algebraic topology and homology theory.

Theorem 7. The Mayer-Vietoris exact sequence. The following sequence is exact.

$$
\begin{equation*}
\cdots \xrightarrow{\partial_{n+1}} H_{n}(A \cap B) \xrightarrow{\varphi} H_{n}(A) \oplus H_{n}(B) \xrightarrow{\psi} H_{n}(X) \xrightarrow{\partial_{n}} H_{n-1}(A \cap B) \xrightarrow{\varphi} \cdots \tag{3.13}
\end{equation*}
$$

Here $\partial_{q}$ is the boundary operator as defined before.
We conclude this section with an example. We want to calculate the homology class of the $k$ sphere with the Mayer-Vietoris exact sequence. With our intuitive concept of the homology groups - the $n$-th homology of $X$ equals the number of "holes of dimension $n$ " in $X$ - we expect that the homology will be $\mathbb{Z}$ for the $k$-th homology group of the $k$-sphere and otherwise zero.


Figure 3.2: Decomposition of the Sphere and the Torus

Example 6. Take $X=S^{k}$ for $k>0$, and assume $A$ and $B$ to be two hemispheres, as in figure 2.5, and $A \cup B$ to be the 'equatorial' $(k-1)$-sphere. Since the $k$-hemisphere is contractible, $A$ and $B$ both have a trivial homology. The exact sequence will be given by

$$
\begin{equation*}
0 \rightarrow H_{n}\left(S^{k}\right) \xrightarrow{\partial_{n}} H_{n-1}\left(S^{k-1}\right) \rightarrow 0 \tag{3.14}
\end{equation*}
$$

Now, the statement that $H_{n}\left(S^{k}\right) \sim \mathbb{Z}$ for $n=k$ and zero otherwise can be proven by induction to the dimension $k$. Let us first look at the 1 -sphere (the circle). The first homology group $H_{1}\left(S^{1}\right)$, containing one singular cycle, is isomorphic to the additive group $\mathbb{Z}$, since it is isomorphic to it's fundamental group (see example 1). The first homology group of the ball $H_{1}\left(S^{2}\right)$ is zero, becease it's surface is simply connected. Now assume for some $m, H_{n}\left(S^{m}\right) \sim \mathbb{Z}$ if $m=n$ and zero otherwise. By sequence 3.14, there is an isomorphism $\partial_{n+1}: H_{n+1}\left(S^{m+1}\right) \rightarrow H_{n}\left(S^{m}\right)$. This homology group is the required $\mathbb{Z}$ if $m=n$ and zero otherwise from the induction hypothesis.

Let us consider an other example. We calculate the homology groups of the torus. Note that, in contrast to Seifert-Van Kampen, Mayer-Vietoris does not require $A \cap B$ to be path-connected.
Example 7. Take $X=T^{2}$. We split up the torus in two tubes $A$ and $B$. Now, $H_{k}(A)=H_{k}(B)=$ $H_{k}(A \cap B)=H_{k}\left(S^{1}\right)=\mathbb{Z}$ if $k=0$ or 1 and zero otherwise (see previous example). Let us set up the Mayer-Vietors sequence.

$$
\begin{equation*}
0 \rightarrow H_{2}\left(T^{2}\right) \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow 0 \tag{3.15}
\end{equation*}
$$

Now, $H_{2}\left(T^{2}\right)$ is either 0 or $\mathbb{Z}$. From the first part of the sequence,

$$
\begin{equation*}
0 \xrightarrow{\psi} H_{2}\left(T^{2}\right) \xrightarrow{\partial_{2}} \mathbb{Z} \tag{3.16}
\end{equation*}
$$

we see that the kernel of the left map is 0 , so the kernel of the right map is $\mathbb{Z}$, so the image of the left map is also $\mathbb{Z}$, so $H_{2}\left(T^{2}\right)$ cannot be 0 . Hence, $H_{2}\left(T^{2}\right)=\mathbb{Z}$.

## Chapter 4

## Holomorphic and Analytic functions

In this chapter we study the local properties of functions from $\mathbb{C}^{n}$ tot $\mathbb{C}$, which can be deduced from the classical theory of complex functions in one complex variable. We discuss differentiability, integration, Cauchy's integral formula for polydiscs and power series. Subsequently, we discuss analytic continuation, which is in higher dimensions quite different from analytic continuation in one dimension. The information in this chapter relies mainly on the lecture notes of Korevaar and Wiegerinck [8].

### 4.1 Holomorphic functions

A complex valued function is a mapping $f: \Omega \subset \mathbb{C}^{n} \rightarrow \mathbb{C}$ which is of the form

$$
\begin{equation*}
f(z)=f\left(z_{1}, z_{2}, \ldots, z_{n}\right)=u\left(x_{1}, y_{1}, \ldots, x_{n}, y_{n}\right)+i v\left(x_{1}, y_{1}, \ldots, x_{n}, y_{n}\right) \tag{4.1}
\end{equation*}
$$

These functions map $n$-tuples of complex numbers onto the complex plane. Here $u, v$ are functions from $\mathbb{R}^{2 n}$ to $\mathbb{R}$. The derivative of a complex function with respect to the $j$ th variable is, by definition, given by the limit

$$
\begin{equation*}
\frac{\partial f}{\partial z_{j}}=\lim _{\Delta z \rightarrow 0} \frac{f\left(z+\Delta z e_{j}\right)-f(z)}{\Delta z} \tag{4.2}
\end{equation*}
$$

if it exists. Here $\Delta z=\Delta x+i \Delta y \in \mathbb{C}$ and $e_{j}$ is the $j$ th basis vector of $\mathbb{C}^{n}$. For $\mathbb{R}$-differentiable functions, this limit exits and is unique, i.e.

$$
\begin{equation*}
\lim _{\Delta x \rightarrow 0} \frac{u\left(x+\Delta x e_{j}, y\right)-u(x, y)}{\Delta x}+i \frac{v\left(x+\Delta x e_{j}, y\right)-v(x, y)}{\Delta x}=\frac{\partial u}{\partial x_{j}}+i \frac{\partial v}{\partial x_{j}} \tag{4.3}
\end{equation*}
$$

and

$$
\begin{equation*}
\lim _{\Delta y \rightarrow 0} \frac{u\left(x, y+\Delta y e_{j}\right)-u(x, y)}{i \Delta y}+i \frac{v\left(x, y+\Delta y e_{j}\right)-v(x, y)}{i \Delta y}=-i \frac{\partial u}{\partial y_{j}}+\frac{\partial v}{\partial y_{j}} \tag{4.4}
\end{equation*}
$$

must be equal.
Definition 9. Cauchy-Riemann Equations A function $f(x+i y)=u(x, y)+i v(x, y): \mathbb{C}^{n} \rightarrow \mathbb{C}$ is said to be complex differentiable if it satisfies the Cauchy-Riemann equations:

$$
\begin{equation*}
\frac{\partial u}{\partial x_{j}}=\frac{\partial v}{\partial y_{j}} \quad \text { and } \quad \frac{\partial v}{\partial x_{j}}=-\frac{\partial u}{\partial y_{j}}, \quad j=1, \ldots, n . \tag{4.5}
\end{equation*}
$$

A function $f: \Omega \rightarrow \mathbb{C}$ is said to be holomorphic if it is complex differentiable in a neighborhood of each point in $\Omega$ and we write $f \in \mathcal{O}(\Omega)$.

In practical calculations, we do not split up the function in a real and an imaginary part. If $f$ is complex differentiable, we write $\frac{\partial f}{\partial x_{j}}=\frac{\partial u}{\partial x_{j}}+i \frac{\partial v}{\partial x_{j}}$ and $\frac{1}{i} \frac{\partial f}{\partial y_{j}}=\frac{1}{i}\left(\frac{\partial u}{\partial y_{j}}+i \frac{\partial v}{\partial y_{j}}\right)=\frac{\partial u}{\partial x_{j}}-\frac{1}{i} \frac{\partial v}{\partial x_{j}}$. Analogous to $2 x=z+\bar{z}$ and $2 i y=z-\bar{z}$, we write

$$
\begin{equation*}
2 \frac{\partial f}{\partial x_{j}}=\frac{\partial f}{\partial z_{j}}+\frac{\partial f}{\partial \bar{z}_{j}} \quad \text { and } \quad 2 i \frac{\partial f}{\partial y_{j}}=\frac{\partial f}{\partial z_{j}}-\frac{\partial f}{\partial \bar{z}_{j}} \tag{4.6}
\end{equation*}
$$

Now we write derivatives in terms of the Wirtinger differential operators which are given by

$$
\begin{array}{rlr}
\frac{\partial f}{\partial z_{j}}:=\frac{1}{2}\left(\frac{\partial f}{\partial x_{j}}+\frac{1}{i} \frac{\partial f}{\partial y_{j}}\right) & j=1, \ldots, n \\
\frac{\partial f}{\partial \bar{z}_{j}}:=\frac{1}{2}\left(\frac{\partial f}{\partial x_{j}}-\frac{1}{i} \frac{\partial f}{\partial y_{j}}\right) & j=1, \ldots, n \tag{4.8}
\end{array}
$$

These operators look like partial derivatives with respect to $z_{j}$ and $\bar{z}_{j}$ which, actually, they are not. On the other hand, in calculations they do behave like partial derivatives. This justifies this notation. Hence, the Cauchy-Riemann equations can stated as follows.

Let $f: \Omega \rightarrow \mathbb{C}$, where $\Omega \subset \mathbb{C}^{n}$, and suppose for all $z \in \Omega$ and $j=1, \ldots, n \frac{\partial f}{\partial \bar{z}_{j}}=0$, then $f$ is holomorphic at $\Omega$ and we write $f \in \mathcal{O}(\Omega)$.

Assume $\alpha$ to be a $n$-tuple of natural numbers, which is called a multi-index. Then,

$$
\begin{equation*}
|\alpha|=\alpha_{1}+\ldots+\alpha_{n} \quad \alpha!=\alpha_{1}!\cdot \ldots \cdot \alpha_{n}!\quad z^{\alpha}=z_{1}^{\alpha_{1}} \cdot \ldots \cdot z_{n}^{\alpha_{n}} \tag{4.9}
\end{equation*}
$$

Definition 10. Assume $\alpha$ and $\beta$ to be multi indexes, then for $\alpha=\left(\alpha_{1}, \ldots, \alpha_{n}\right)$ and $\beta=\left(\beta_{1}, \ldots, \beta_{m}\right)$ resp. the operators $D^{\alpha}, D^{\bar{\beta}}$ and $D^{\alpha \bar{\beta}}$ are defined as

$$
\begin{equation*}
D^{\alpha}=\frac{\partial^{|\alpha|}}{\partial z^{\alpha}} \quad D^{\bar{\beta}}=\frac{\partial^{|\beta|}}{\partial \bar{z}^{\beta}} \quad D^{\alpha, \bar{\beta}}=\frac{\partial^{|\alpha|+|\beta|}}{\partial z^{\alpha} \partial \bar{z}^{\beta}} \tag{4.10}
\end{equation*}
$$

For a function $f: \Omega \subset \mathbb{C}^{n} \rightarrow \mathbb{C} \in \mathcal{O}(\Omega)$ we have the Taylor expansion of a function $f$ in a point $a$ is given by

$$
\begin{equation*}
T_{f(a)}(z)=\sum_{\alpha \geq 0} \frac{(z-a)^{\alpha}}{\alpha!} D^{\alpha} f(a)=\sum_{\alpha \geq 0} c_{\alpha}(z-a)^{\alpha} \tag{4.11}
\end{equation*}
$$

This is the complex version of the Taylor Series ${ }^{1}$ for real valued functions over $\mathbb{R}^{n}$. Of course, this is only true if $D^{\bar{\alpha}} f(a)=0$ and thus if $f$ is holomorphic in a neighborhood of $a$. Now it is easily seen that also in several complex variables every holomorphic function is analytic and vice versa.

Like we have Cauchy's integral formula for every continuous function over $\mathbb{C}$ there is a similar definition for $\mathbb{C}^{n}$.

Theorem 8. Multi dimensional Cauchy integral formula Let $f(z) \in \mathcal{C}\left(\mathbb{C}^{n}\right)$ and $T($ a, r) be the torus around a point $a$ with radii $r_{j}$ then the following equality holds.

$$
\begin{equation*}
f(z)=\frac{1}{(2 \pi i)^{n}} \oint_{T(a, r)} \frac{f(\zeta)}{\left(\zeta_{1}-z_{1}\right) \ldots\left(\zeta_{n}-z_{n}\right)} d \zeta_{1} \ldots d \zeta_{n} \tag{4.12}
\end{equation*}
$$

The following proof uses induction to the dimension.

[^0]Proof. For $n=1$ we have the classical theorem of Cauchy. Now suppose that formula 4.12 holds for $n$. Let $a^{\prime}=\left(a_{0}, a\right), r^{\prime}=\left(r_{0}, r\right), z^{\prime}=\left(z_{0}, z\right)$ and $\zeta^{\prime}=\left(\zeta_{0}, \zeta\right)$ all in $\mathbb{C}^{n+1}$. Now let us fix $z=w$ so we can apply Cauchy's theorem in one variable with respect to $z_{0}$.

$$
\begin{equation*}
f\left(z_{0}, w\right)=g\left(z_{0}\right)=\frac{1}{2 \pi i} \oint_{C\left(a_{0}, r_{0}\right)} \frac{g\left(\zeta_{0}\right)}{\zeta_{0}-z_{0}} d \zeta_{0}=\frac{1}{2 \pi i} \oint_{C\left(a_{0}, r_{0}\right)} \frac{f\left(\zeta_{0}, w\right)}{\zeta_{0}-z_{0}} d \zeta_{0} \tag{4.13}
\end{equation*}
$$

If we fix $\zeta_{0}=w_{0}$, then we define $h(w)=f\left(w_{0}, w\right)$ to have by the induction hypothesis

$$
\begin{equation*}
h(w)=\frac{1}{(2 \pi i)^{n}} \oint_{T(a, r)} \frac{f\left(w_{0}, \zeta\right)}{\left(\zeta_{1}-z_{1}\right) \ldots\left(\zeta_{n}-z_{n}\right)} d \zeta_{1} \ldots d \zeta_{n} \tag{4.14}
\end{equation*}
$$

By substitution of (4.14) in (4.13) we have

$$
\begin{equation*}
f\left(z^{\prime}\right)=\frac{1}{(2 \pi i)^{n+1}} \oint_{C\left(a_{0}, r_{0}\right)} \frac{d \zeta_{0}}{\zeta_{0}-z_{0}} \oint_{T(a, r)} \frac{f\left(\zeta_{0}, \zeta\right)}{\left(\zeta_{1}-z_{1}\right) \ldots\left(\zeta_{n}-z_{n}\right)} d \zeta_{1} \ldots d \zeta_{n} \tag{4.15}
\end{equation*}
$$

Which is by Fubini the required

$$
\begin{equation*}
f\left(z^{\prime}\right)=\frac{1}{(2 \pi i)^{n+1}} \oint_{T\left(a^{\prime}, r^{\prime}\right)} \frac{f\left(\zeta^{\prime}\right)}{\left(\zeta_{0}-z_{0}\right) \ldots\left(\zeta_{n}-z_{n}\right)} d \zeta_{0} \ldots d \zeta_{n} \tag{4.16}
\end{equation*}
$$

For functions that are just smooth in stead of holomorphic, there is an extended version of the Cauchy integral theorem. We state it without proof.
Theorem 9. Extended Cauchy integral formula Let $f(z) \in \mathcal{C}^{\infty}(D)$ be a smooth function over a closed disk centered at a point a with radius $r$ then the following equality holds.

$$
\begin{equation*}
f(z)=\frac{1}{2 \pi i} \oint_{\partial D} \frac{f(\zeta)}{\zeta-z} d \zeta+\frac{1}{2 \pi i} \oint_{\partial D} \frac{\partial f(\zeta)}{\partial \bar{w}}(w) \frac{d w d \bar{w}}{\zeta-z} d \zeta \tag{4.17}
\end{equation*}
$$

Note that if $f$ is holomorphic then $\frac{\partial f}{\partial \bar{w}}=0$, and so the second term vanishes and we recover the standard Cauchy integral formula. The proof of theorem 8 relies on Green's Formula.

We conclude this section with Osgood's lemma.
Theorem 10. If a complex valued function $f$ is continuous in an open set $\Omega \subset \mathbb{C}^{n}$, and $f$ is holomorphic in each variable separately, then $f$ is holomorphic in $D$.

We will construct a power series for $f$ to show that it is analytic and thus holomorphic.
Proof. For a fixed point $a$ in a closed polydisc $D(w, r) \subset \Omega$ the series expansion

$$
\begin{equation*}
\frac{1}{z-a}=\sum_{\alpha \geq 0} \frac{(a-w)^{\alpha}}{(z-w)^{\alpha+1}} \tag{4.18}
\end{equation*}
$$

is absolutely uniformly convergent for all points $\zeta$ on $\partial D=T(w, r)$. Now, with the multidimensional Cauchy integral formula we find for $f(z)=\sum_{\alpha \geq 0} c_{\alpha}(z-a)^{\alpha}$

$$
\begin{equation*}
c_{\alpha}=\frac{1}{(2 \pi i)^{n}} \oint_{T(w, r)} \frac{f(z) d z}{(z-w)^{\alpha+1}} \tag{4.19}
\end{equation*}
$$

### 4.2 Analytic continuation

Recall that for any $D \subset \mathbb{C}$ of the complex plane there exists a complex function $f$ that is analytic on $D$ such that there exists no complex function $g$ analytic on $D^{\prime} \subset \mathbb{C}$ with the property on a open subset $D \subset D^{\prime}$ and $f=\left.g\right|_{D}$ where $D \neq D^{\prime}$. For an analytic function over a domain $\Omega \subset \mathbb{C}^{n}$, such a function does exist. This phenomena is called analytic continuation.

Let $f$ be an analytic function over a neighborhood $U$ around the point $a \in \mathbb{C}^{n}$. A triple $(a, U, f)$ is called a function element at $a$. We write down a equivalence relation between such triples $(a, U, f) \sim$ $(b, V, g)$ if and only if $a=b$ and $f=g$ on a neighborhood $W \subset U \cap V$. The class of function elements at $a$ is called the germ of the function $f$ at $a$ and is denoted by $[f]_{a}$. One can show that every element of the same germ has the same power series.

Now, the functions holomorphic in a point $a$ modulo their equivalence relation forms a commutative ring $\mathcal{O}_{a}$ with additive operation defined as $[f]_{a}+[g]_{a}=[f+g]_{a}$ and multiplication by $[f]_{a} *[g]_{a}=[f g]_{a}$ as one would expect.

Definition 11. Consider a curve $\gamma:[0,1] \rightarrow \mathbb{C}$ and fix $k+1$ points $\gamma\left(t_{0}\right), \ldots, \gamma\left(t_{k}\right)$. Let $f$ be an analytic function defined on a neighborhood $U$ of a point $z$. An analytic continuation of the function element $(z, U, f)$ along a curve $\gamma$ is a collection of function elements $\left\{\left(\gamma\left(t_{j}\right), U_{t_{j}}, f_{t_{j}}\right)\right\}_{j=0}^{k}$ such that

1. $f_{t_{0}}=f, U_{t_{0}}=U$ and $\gamma\left(t_{0}\right)=z$;
2. For each $j$ the function $f_{t_{j}}$ is analytic on $U_{t_{j}}$;
3. For each pair of successive neighborhoods $U_{t_{j}} \cap U_{t_{j+1}}$ is connected and not empty.
4. For each pair of successive neighborhoods $U_{t_{j}} \cap U_{t_{j+1}}$ yields $\left.f_{t_{j}}\right|_{U_{t_{j}}}=f_{t_{j+1}} \mid U_{t_{j+1}}$.

The definition of analytic continuation along a curve is a bit technical, but the basic idea is that one starts with an analytic function defined around a point, and one extends that function along a curve via analytic functions defined on small overlapping neighborhoods covering that curve.

The continuation along a curve is unique if it exists. This is a very powerful statement, because it tells us that you can 'reconstruct' a function on the entire domain if you know it on just a small compact subdomain. It is stated in the next theorem.

Theorem 11. Let $V$ be the connected domain of two analytic functions $f$ and $g$ such that for all $z \in U$ yields $f(z)=g(z)$ for $U \subset V$, then $f=g$ for all $z \in V$.

We will prove that the set $S=\{z \in V: f(z)=g(z)\}$ is both open and closed in $V$.
Proof. That $S$ is closed, follows from the continuity of $f$ and $g$. Now we will take a look at the Taylor series of $f$ and $g$, so we consider the same set $S$ but rewrite it as

$$
S=\left\{z \in V: f^{(k)}=g^{(k)}, k \geq 0\right\} .
$$

Assume $w$ to be an element of $S$. Then, because the Taylor series of $f$ and $g$ at $w$ have non-zero radius of convergence, the open disk $B_{r}(w)$ also lies in $S$ for some positive real $r$. (In fact, $r$ can be anything less than the distance from $w$ to the boundary of $V$ ). This shows that $S$ is open. Since $S$ is open and closed in $V$, we have shown that $S=V$.

The main consequence of this theorem in the context of this thesis is that analytic functions can't have isolated zeros and poles in $\mathbb{C}^{n}, n \geq 2$.

Let $f$ be holomorphic on a "punctured polydisc" $\Delta=D_{n}(a, r) \backslash\{a\}$. Then $f$ has an analytic extension to $D_{n}(a, r)$ and has no irremovable pole at $a$. As a direct consequence, holomorphic functions can also not have isolated zeros since an isolated zero of $f$ would be a irremovable isolated singularity for $1 / f$.

## Chapter 5

## Holomorphic forms

In this chapter we will go a little deeper into complex functions. We discuss differential forms and the Weierstrass theorem analogous to Korevaar and Wiegerinck [8]. We discuss the relation between cohomology and Stokes Theorem as in Shabath [12].

### 5.1 Complex differential forms

In this section we will consider functions and differential forms on complex manifolds.
Definition 12. Let $M$ be a Hausdorff space, then $M$ is a manifold if $M$ can be covered by a collection of domains $\mathcal{U}=\left\{U_{\alpha}\right\}_{\alpha \in A}$, A an arbitrary set of indices, and $U_{\alpha} \subset M$ that are homeomorphic to open balls in $\mathbb{R}^{n}$. Each homeomorphism $\varphi_{\alpha}: U_{\alpha} \rightarrow \mathbb{R}^{n}$ is called a chart and the collection of all charts is called an atlas of the covering.

If for every pair of domains $\left(U_{1}, U_{2}\right)$ of charts $\left(\varphi_{\alpha}, \varphi_{\beta}\right)$ with non empty intersection $\varphi_{\alpha \beta}=$ $\left.\varphi_{\alpha} \circ \varphi_{\beta}^{-1}\right|_{\varphi_{\beta}\left(V_{1} \cup V_{2}\right)}$ is biholomorphic, i.e. $\varphi_{\alpha \beta}$ and $\varphi_{\alpha \beta}^{-1}=\varphi_{\beta \alpha}$ are holomorphic, then $M$ is called a holomorphic manifold.

Let us, by way of example, prove the theorem that the complex projective space $C P^{n}$ is a holomorphic manifold.

Theorem 12. The complex projective space is defined as the set of equivalent classes corresponding to the equivalence relation $z \sim z^{\prime}$ in $\mathbb{C}^{n+1} \backslash\{0\}$ if $z=\lambda z^{\prime}$ for some nonzero complex number $\lambda$. We will show that this is indeed a holomorphic manifold.

Proof. Consider the subsets $U_{j}=\left\{\left(z_{0}, \ldots, z_{n}\right) \mid z_{j} \neq 0\right\}$ of $C P^{n}$ for $j \in\{0, \ldots, n\}$. These subsets cover the whole projective space. Now, set

$$
\begin{equation*}
\varphi_{j}:\left(z_{0}, \ldots, z_{n}\right) \mapsto\left(\frac{z_{0}}{z_{j}}, \ldots, \frac{z_{j-1}}{z_{j}}, \frac{z_{j+1}}{z_{j}}, \ldots, \frac{z_{n}}{z_{j}}\right) \tag{5.1}
\end{equation*}
$$

and

$$
\begin{equation*}
\varphi_{j}^{-1}:\left(\zeta_{0}, \ldots, \zeta_{n-1}\right) \mapsto\left(\zeta_{0}, \ldots, \zeta_{j-1}, 1, \zeta_{j+1}, \ldots, \zeta_{n-1}\right) \tag{5.2}
\end{equation*}
$$

All, $\varphi_{j}$ 's are clearly biholomorphic on $U_{j}$ and so the complex projective space is a holomorphic manifold with the above atlas.

On holomorphic manifolds we consider complex differentiable forms. We use differential forms as an approach to define integrands over curves, surfaces, volumes, and higher dimensional manifolds.

So we identify (again) $\mathbb{C}^{n}$ with $\mathbb{R}^{2 n}$ and a function is said to be $\mathbb{R}$-differentiable iff $f$ is differentiable with respect to $x$ and $y$. Now define $d z_{j}$ and $d \bar{z}_{j}$ such that $d x_{j}=\frac{1}{2}\left(d z_{j}+d \bar{z}_{j}\right)$ and $d y_{j}=\frac{1}{2 i}\left(d z_{j}-\right.$ $\left.d \bar{z}_{j}\right)$.

Definition 13. Let $\Omega \subset \mathbb{C}^{n}$ be a holomorphic manifold and $f: \Omega \rightarrow \mathbb{C}$ be a $\mathbb{R}$-differentiable function. The exterior derivative of $f$ is given by $d f=\sum_{j=1}^{n} \frac{\partial f}{\partial z_{j}} d z_{j}+\frac{\partial f}{\partial \bar{z}_{j}} d \bar{z}_{j}$.

We can apply the exterior derivative $d$ multiple on differential forms, and in general we define the ( $p, q$ )-form to be

$$
\begin{equation*}
\omega=\sum_{J, K} \omega_{J, K}(z) d z_{J} \wedge d \bar{z}_{K}, \quad J=\left\{j_{1} \ldots j_{p}\right\} \quad K=\left\{k_{1} \ldots k_{q}\right\} \tag{5.3}
\end{equation*}
$$

The space of these $(p, q)$-forms is denoted by $\Omega^{p, q}$, which is defined as

$$
\Omega^{p, q}=\bigwedge_{j=1}^{p} \Omega^{1,0} \wedge \bigwedge_{j=1}^{q} \Omega^{0,1}
$$

Now, we can define for a $n$-dimensional $(p, q)$-form three operators if both $p, q \leq n$.

$$
\begin{align*}
& \partial: \quad \Omega^{p, q} \rightarrow \Omega^{p+1, q} \quad \partial: \omega \mapsto \sum_{J, K} \sum_{l=1}^{n} \frac{\partial f_{J, K}}{\partial z_{l}} d z_{l} \wedge d z_{J} \wedge d \bar{z}_{K}  \tag{5.4}\\
& \bar{\partial}: \quad \Omega^{p, q} \rightarrow \Omega^{p, q+1} \quad \bar{\partial}: \omega \mapsto \sum_{J, K} \sum_{l=1}^{n} \frac{\partial f_{J, K}}{\partial \bar{z}_{l}} d \bar{z}_{l} \wedge d z_{J} \wedge d \bar{z}_{K}  \tag{5.5}\\
& d: \quad \Omega^{p, q} \rightarrow \Omega^{p+1, q}+\Omega^{p, q+1} \quad d: \omega \mapsto \partial \omega+\bar{\partial} \omega \tag{5.6}
\end{align*}
$$

where, $\partial$ and $\bar{\partial}$ are called the Dolbeault operators and $d$ is the exterior derivative of a $(p, q)$-form.
In general, for a $n$-dimensional $(n, 0)$ form $\omega=f(z) d z=f\left(z_{1}, \ldots, z_{n}\right) d z_{1} \wedge \ldots \wedge d z_{n}$, with $f(z)$ holomorphic in $\Omega \subset \mathbb{C}^{n}$, then $d \omega$ can be calculated as follows.

$$
\begin{align*}
d \omega & =d f\left(z_{1}, \ldots, z_{l}\right) \wedge d z_{1} \wedge \ldots \wedge d z_{l}  \tag{5.7}\\
& =\sum_{j=1}^{l} \frac{\partial f}{\partial z_{j}} d z_{j} \wedge d z_{1} \wedge \ldots \wedge d z_{l}+\sum_{j=1}^{l} \frac{\partial f}{\partial \bar{z}_{j}} d \bar{z}_{j} \wedge d z_{1} \wedge \ldots \wedge d z_{l} \tag{5.8}
\end{align*}
$$

The left sum equals zero, because each term can be written as $(-1)^{l-1} \frac{\partial f}{\partial z_{j}} d w_{j} \wedge d z_{j} \wedge d z_{j}$ where $d w_{j}=$ $d z$ with $d z_{j}$ omitted. The right sum is also zero, because $f$ is holomorphic and all the derivatives with respect to $\bar{z}$ vanish.

Thus the exterior derevative of a holomorphic ( $n, 0$ )-form is zero.

$$
\begin{equation*}
\oint_{\partial M} \omega=\oint_{M} d \omega=0 \tag{5.9}
\end{equation*}
$$

The $d$-operator is characterized by three important properties. These properties are stated in the next lemma.

Lemma 1. Let $X$ be a complex manifold and $\omega \in \Omega^{p, q}(X), \eta \in \Omega^{s, t}(X)$ then:

1. $d(\alpha \omega+\eta)=\alpha d(\omega)+d(\eta)$ (Linearity),
2. $d^{2}=\partial(\partial \omega)=\bar{\partial}(\bar{\partial} \omega)=0$ and $\partial(\bar{\partial} \omega)=-\bar{\partial}(\partial \omega)$ (Idempotency),
3. $\partial(\omega \wedge \eta)=\partial \omega \wedge \eta+(-1)^{p+q} \omega \wedge \partial \eta$ and $\bar{\partial}(\omega \wedge \eta)=\bar{\partial} \omega \wedge \eta+(-1)^{p+q} \omega \wedge \bar{\partial} \eta$ (Leibniz rule).

Proof. The proof follows from direct calculations. Since the second statement is of the most interest in this thesis, we will only prove this statement. The first and third statement are left for the reader. The definition of $d$ implies

$$
\begin{equation*}
d^{2}=\partial(\partial+\bar{\partial})+\bar{\partial}(\partial+\bar{\partial})=\partial^{2}+\partial \bar{\partial}+\bar{\partial} \partial+\bar{\partial}^{2} \tag{5.10}
\end{equation*}
$$

so let us calculate each therm.

$$
\begin{align*}
\partial(\partial \omega) & =\partial\left(\sum_{J, K} \sum_{l=1}^{n} \frac{\partial f_{J, K}}{\partial z_{l}} d z_{l} \wedge d z_{J} \wedge d \bar{z}_{K}\right)  \tag{5.11}\\
& =\sum_{J, K} \sum_{l=1}^{n} \sum_{k=1}^{n} \frac{\partial^{2} f_{J, K}}{\partial z_{l} \partial z_{k}} d z_{k} \wedge d z_{l} \wedge d z_{J} \wedge d \bar{z}_{K}=0 \tag{5.12}
\end{align*}
$$

since if $k \neq l, \frac{\partial^{2} f_{J, K}}{\partial z_{l} \partial z_{k}} d z_{k} \wedge d z_{l}=-\frac{\partial^{2} f_{J, K}}{\partial z_{k} \partial z_{l}} d z_{l} \wedge d z_{k}$ and sum up to zero. If $k=l \frac{\partial^{2} f_{J, K}}{\partial z_{l}^{2}} d z_{k} \wedge d z_{k}=0$. The proof of $\bar{\partial}^{2}=0$ goes similarly.

Let us calculate $\partial(\bar{\partial}(\omega))$.

$$
\begin{align*}
\partial(\bar{\partial} \omega) & =\partial\left(\sum_{J, K} \sum_{l=1}^{n} \frac{\partial f_{J, K}}{\partial \bar{z}_{l}} d \bar{z}_{l} \wedge d z_{J} \wedge d \bar{z}_{K}\right)  \tag{5.14}\\
& =\sum_{J, K} \sum_{l=1}^{n} \sum_{k=1}^{n} \frac{\partial^{2} f_{J, K}}{\partial \bar{z}_{l} \partial z_{k}} d z_{k} \wedge d \bar{z}_{l} \wedge d z_{J} \wedge d \bar{z}_{K}=0 \tag{5.15}
\end{align*}
$$

If we change the order of $d z_{k} \wedge d \bar{z}_{l}$ once, we can bring the minus sign in front of the summation and so

$$
\begin{align*}
& =-\sum_{J, K} \sum_{l=1}^{n} \sum_{k=1}^{n} \frac{\partial^{2} f_{J, K}}{\partial \bar{z}_{l} \partial z_{k}} d \bar{z}_{l} \wedge d z_{k} \wedge d z_{J} \wedge d \bar{z}_{K}=0  \tag{5.16}\\
& =-\bar{\partial}\left(\sum_{J, K} \sum_{k=1}^{n} \frac{\partial f_{J, K}}{\partial z_{k}} d z_{k} \wedge d z_{J} \wedge d \bar{z}_{K}\right)  \tag{5.17}\\
& =-\bar{\partial}(\partial \omega) \tag{5.18}
\end{align*}
$$

Thus $\partial(\bar{\partial} \omega)=-\bar{\partial}(\partial \omega)$. Since $\partial^{2}=0, \partial \bar{\partial}=-\bar{\partial} \partial$ and $\bar{\partial}^{2}=0$, also $d^{2}=0$.
Let us look at the following important definitions.

Definition 14. Let $\omega \in \Omega^{p, q}$, then

1. $\omega$ is called closed (resp. $\bar{\partial}$-closed) if $d \omega=0$ (resp. $\bar{\partial} \omega=0$ );
2. $\omega$ is called exact (resp. $\bar{\partial}$-exact) if there exists a form $\sigma$ such that $d \sigma=\omega$ (resp. $\bar{\partial} \sigma=\omega$ ).

Let $\omega \in \Omega^{p, q}(X)$ for a complex manifold $X \subset \mathbb{C}^{n}$. From the notion that $\bar{\partial}^{2}=0$, the $\bar{\partial}$-operator is a connecting homomorphism and therefore the next cochain complex is an exact sequence.

$$
\begin{equation*}
\ldots \xrightarrow{\bar{d}} \Omega^{p, q-1} \xrightarrow{\bar{o}} \Omega^{p, q} \xrightarrow{\bar{\jmath}} \Omega^{p, q+1} \xrightarrow{\overline{\mathrm{o}}} \ldots \tag{5.19}
\end{equation*}
$$

The theory of cohomology is dual to the theory of homology. Cohomological chains and boundaries are called cochains and coboundaries. In the cochaincomplex the connecting homomorphisms work in the opposite direction, i.e. homomorphisms $\partial_{k}: A^{k} \rightarrow A^{k+1}$. Note the use of superscript in stead of subscript.

At first glance cohomology seems completely dual to homology, and therefore seemingly redundant. But in fact it has more structure. Since you multiply (wedge) differential forms together, cohomology becomes a ring. This is still true in more general approaches such as singular cohomology. On the homology side, one has an intersection pairing, but this is harder to describe and only available for really "nice" spaces.

### 5.2 Stokes and Cauchy

We recall Stokes' formula for a form $\omega$, the exterior derivative $d$, a manifold $M$ and the boundary of the manifold $\partial M$, namely

$$
\begin{equation*}
\oint_{\partial M} \omega=\int_{M} d w . \tag{5.20}
\end{equation*}
$$

Let us recall what this means for integration by chains. For a simplicial chain complex $S(X)$ over a manifold $M \subset X$ and a form $\omega \in \Omega^{p, q}$ and a chain $\sigma=\sum_{j=1}^{k} \lambda_{j} \sigma_{n}^{j} \in S(X), \lambda_{j} \in \mathbb{C}, \sigma_{n}^{j} \in S_{n}(X)$, the chain integral is defined by

$$
\begin{equation*}
\int_{\sigma} \omega=\sum_{j=1}^{k} \lambda_{j} \int_{\sigma_{j}} \omega \tag{5.21}
\end{equation*}
$$

By Stokes' theorem, this is equal to

$$
\begin{equation*}
\int_{\sigma} \omega=\sum_{j=1}^{k} \lambda_{j} \int_{\partial \sigma_{j}} d \omega . \tag{5.22}
\end{equation*}
$$

For holomorphic forms $d \omega=0$, integrals over these chains are all zero. The interesting case is when a function is not holomorphic on the whole manifold. Let us conclude this section with two important corollaries of Stokes' formula concerning non holomorphic forms.
Corollary 1. The integral of a closed form $\omega(d \omega=0)$ over an exact cycle $\sigma=\partial \sigma^{\prime}$ is equal to zero.
Proof. The proof follows from direct calculations. First we apply that the cycle $\sigma$ is exact, next we use Stokes' formula and last we use that the form $\omega$ is closed.

$$
\begin{equation*}
\int_{\sigma} \omega=\int_{\partial \sigma^{\prime}} \omega=\int_{\sigma^{\prime}} d \omega=0 \tag{5.23}
\end{equation*}
$$

Corollary 2. The integral of an exact form $\omega=d \omega^{\prime}$ over a closed cycle $\sigma(\partial \sigma=0)$ is equal to zero.
Proof. This proof follows also from direct calculations, but now we first we apply that the the form $\omega$ is exact, next we use Stokes' formula and last we use that the boundary $\sigma$ is closed.

$$
\begin{equation*}
\int_{\sigma} \omega=\int_{\sigma} d \omega^{\prime}=\int_{\partial \sigma} \omega^{\prime}=0 \tag{5.24}
\end{equation*}
$$

Since every closed form is exact, we can split up each form $\omega=\tilde{\omega}+d \omega^{\prime}$ in a exact part $\omega^{\prime}$ and a not exact part $\tilde{\omega}$. The same is valid for the cycle $\sigma=\tilde{\sigma}+\partial \sigma^{\prime}$. Combining the results of the previous corollaries, we have that the integral of a form $\omega$ over a cycle $\sigma$ in fact depends only on the not exact parts of the form and the cycle.

$$
\begin{align*}
\int_{\sigma} \omega & =\int_{\tilde{\sigma}+\partial \sigma^{\prime}} \tilde{\omega}+d \omega^{\prime}  \tag{5.25}\\
& =\int_{\tilde{\sigma}} \tilde{\omega}+\int_{\sigma} d \omega^{\prime}+\int_{\partial \sigma^{\prime}} \tilde{\omega}+\int_{\partial \sigma^{\prime}} d \omega^{\prime}  \tag{5.26}\\
& =\int_{\tilde{\sigma}} \tilde{\omega} \tag{5.27}
\end{align*}
$$

This is what is called the cohomology class of the form and the homology class of the cycle. We will apply this result to residues in the next chapter.

## Chapter 6

## Calculating residues

In the next sections we will expound the theory, developed by Tsikh [15], which is needed to prove and understand the theory of residues in $\mathbb{C}^{n}$. We present also our extension regarding residues on local intersections in section 6.3.

### 6.1 Introduction

We recall two equivalent definitions of a residue of $f \in \mathcal{O}\left(A \backslash\left\{z_{0}\right\}\right)$ at an isolated singularity $z_{0}$ in the complex plane. Let us rewrite $f(z)$ as its Laurent series in the neighborhood of $z_{0}$, namely $f(z)=\sum_{k \in \mathbb{Z}} a_{k}\left(z-z_{0}\right)^{k}$, then the residue of $f$ at $z_{0}$ is defined either as

$$
\begin{equation*}
\operatorname{res}_{z_{0}}(f):=\frac{1}{2 \pi i} \oint_{C\left(z_{0}, \epsilon\right)} f(z) d z \tag{6.1}
\end{equation*}
$$

or as the definition by its Laurent series $\sum_{k \in \mathbb{Z}} a_{k}\left(z-z_{0}\right)^{k}$

$$
\begin{equation*}
\operatorname{res}_{z_{0}}(f):=a_{-1} \tag{6.2}
\end{equation*}
$$

To illustrate one of the problems of generalizing the residue theorem for higher dimensions, we elaborate the example given by Tsikh ([15] p. 13).

Consider the quotient function $f: D \rightarrow \mathbb{C}$, with $D=\mathbb{C}^{2} \backslash(\{z=0\} \cup\{w=0\} \cup\{z=w\})$.

$$
\begin{equation*}
f(z, w)=\frac{h(z, w)}{z w(z-w)} \tag{6.3}
\end{equation*}
$$

where $h(z, w)$ is holomorphic and nonzero in the neighborhood of the origin. We can expand the function $f$ over two distinct cycles

$$
\begin{align*}
& \Gamma_{1}=\left\{|z|=\epsilon_{1},|w|=\epsilon_{2}>\epsilon_{1}\right\}  \tag{6.4}\\
& \Gamma_{2}=\left\{|z|=\delta_{1},|w|=\delta_{2}<\delta_{1}\right\} \tag{6.5}
\end{align*}
$$

where $\epsilon_{1,2}$ and $\delta_{1,2}$ are small positive real numbers.
Now the Laurent series of $f$ with respect to respectively $\Gamma_{1}$ and $\Gamma_{2}$ are

$$
\begin{equation*}
\left.f\right|_{\Gamma_{1}}=-\frac{h}{z w^{2}} \sum_{k \geq 0}\left(\frac{z}{w}\right)^{k} \quad \text { and }\left.\quad f\right|_{\Gamma_{2}}=\frac{h}{z^{2} w} \sum_{k \geq 0}\left(\frac{w}{z}\right)^{k} \tag{6.6}
\end{equation*}
$$

By term by term integration we have

$$
\begin{equation*}
\frac{1}{(2 \pi i)^{2}} \int_{\Gamma_{1}} f d z \wedge d w=-\frac{\partial h}{\partial w}(0,0) \quad \frac{1}{(2 \pi i)^{2}} \int_{\Gamma_{2}} f d z \wedge d w=\frac{\partial h}{\partial z}(0,0) \tag{6.7}
\end{equation*}
$$

From this example we see that, in contrary to the single dimensional case, a point can have different Laurent series which approximate $f$ each on a 'different' neighborhood of that point, i.e. two domains are different when they are not homotopy equivalent in $D$. Such a neighborhood is called a canonical domain. In our case we find two canonical domains $\mathcal{U}_{1}=\{|z|<|w|\}$ and $\mathcal{U}_{2}=\{|z|>|w|\}$. We will see later on that there is also a third canonical domain, namely $\mathcal{U}_{3}=\{|z|=|w|\}$.

Since the residue will be calculated only on a canonical domain, and thus we have only local convergence of the Laurent series, we speak of local residues.

### 6.2 Local residues

As in a one-dimensional setting we would like $f$ to have an isolated zero in the denominator around which we can integrate. From section 4.2 we know that a function $f: D \subset \mathbb{C}^{n} \rightarrow \mathbb{C}$ has never an isolated singularity, but a mapping $f: \mathbb{C}^{n} \rightarrow \mathbb{C}^{n}$ might have one. Now, consider the following definition.

Definition 15. The residue of a function $h$ associated with the mapping $f=\left(f_{1}, \ldots, f_{n}\right)$, holomorphic in the neighborhood of an isolated zero a of the mapping $f$, is given by

$$
\begin{equation*}
\operatorname{res}_{a}(h)=\frac{1}{(2 \pi i)^{n}} \int_{\Gamma_{a}} \frac{h d z}{f_{1} \cdots f_{n}} \tag{6.8}
\end{equation*}
$$

where $\Gamma_{a}=\left\{z \in \mathcal{U}_{a}:\left|f_{j}(z)\right|=\epsilon_{j}, j=1, \ldots, n\right\}$.
In fact, the components of the mapping $f$ form a factorization of the denominator of the integrand.
When we use this formula, we have to be aware of that we might have different possibilities in choosing such a mapping $f$ with different outcomes. In the case of our example we have three choices of the mapping $f$. The choice $f^{I}(z, w)=(z, w(z-w)), f^{I I}(z, w)=(w, z(z-w))$ and $f^{I I I}(z, w)=(z w,(z-w))$. For this reason we speak of a local residue associated with a holomorphic mapping. The value of a local residue at a point $a$ depends on the choice of the factorization of the denominator, i.e. the chosen associated mapping.

We will denote $I_{a}(f)$ to be the ideal in $\mathcal{O}_{a}$ generated by the components $\left[f_{1}\right], \ldots,\left[f_{n}\right]$ of the mapping $f$. Now we can prove the following proposition.

Proposition 3. If $h \in I_{a}(f)$, then $\operatorname{res}_{a}(h)=0$.
Proof. If $h$ is in the ideal $I_{a}(f)$, it can be rewritten as $h=q_{1} f_{1}+\cdots+q_{n} f_{n}$, for $q_{j} \in \mathcal{O}_{a}$.

$$
\begin{equation*}
\operatorname{res}_{a}(h)=\sum_{k=1}^{n} \frac{1}{(2 \pi i)^{n}} \int_{\Gamma_{a}} \frac{q_{k} d z}{f_{1} \cdots[k] \cdots f_{n}} \tag{6.9}
\end{equation*}
$$

Let us rewrite $\Gamma_{a}=(-1)^{k-1} \partial c_{k}$, where $c_{k}=\left\{z \in \mathcal{U}_{a} \quad:\left|f_{j}(z)\right|=\epsilon_{j}, j \neq k ;\left|f_{k}(z)\right| \leq \epsilon_{k}\right\}$. This tells us that $\Gamma_{a}$ is a boundary (exact and closed) and therefore homologous to zero. By Stokes' theorem each term equals 0 .

From now we will assume $h$ not to be in $I_{a}(f)$.
Lemma 2. Continuity principle of local residues Let $f_{t}: \overline{\mathcal{U}}_{a} \rightarrow \mathbb{C}^{n}$ be a family of holomorphic mappings continuously depending on a parameter $t$ with a finite set of isolated zeros $\left\{P_{t}\right\}$. We assume that for $t=t_{0}$ the mapping $f_{t_{0}}$ has an isolated zero at the the point a and for $t$ close to $t_{0} f$ has no zeros on the boundary $\partial \overline{\mathcal{U}}_{a}$. Let $h_{t}(z)$ be a function holomorphic in $z$ and continuous in $t$. Now,

$$
\begin{equation*}
\lim _{t \rightarrow t_{0}} \sum_{\left\{P_{t}\right\}} \operatorname{res}_{P_{t}} f_{t}\left(h_{t}\right)=\operatorname{res}_{a} f_{t_{0}}\left(h_{t_{0}}\right) \tag{6.10}
\end{equation*}
$$



Figure 6.1: Multiple poles
For a proof of this lemma we refer to [15]. The main idea of the proof relies on the fact that we present the local residue as an integral over the boundary of the neighborhood of the singularity. As long as you stay in the neighborhood, the integral over the boundary will remain the same and therefore the residue will be the same. This lemma will be needed in the proof of the following important proposition.
Proposition 4. Let $f$ be a holomorphic mapping, then for every holomorphic mapping $g$ and matrix $A$ with elements holomorphic in a neighborhood of a such that $g=A f$,

$$
\begin{equation*}
\operatorname{res}_{a}(h)=\operatorname{res}_{a}(h \cdot \operatorname{det} A) \tag{6.11}
\end{equation*}
$$

To prove this result, we need two lemmas.
Lemma 3. The Jacobian $J_{g}(a)$ and $J_{f}(a)$ resp. of the functions $f, g$ in the point a are related in the following way if $f$ and $g$ satisfy the conditions of Proposition 4.

$$
\begin{equation*}
J_{g}(a)=\operatorname{det} A(a) \cdot J_{f}(a) \tag{6.12}
\end{equation*}
$$

Proof. The proof follows from direct computations. De Jacobian matrix of $g$ in the point $a$ is given by $\frac{\partial\left(g_{1}, \ldots g_{n}\right)}{\partial\left(z_{1}, \ldots z_{n}\right)}(a)=\frac{\partial A}{\partial z}(a) f(a)+A(a) \frac{\partial\left(f_{1}, \ldots f_{n}\right)}{\partial\left(z_{1}, \ldots z_{n}\right)}(a)=A(a) \frac{\partial\left(f_{1}, \ldots f_{n}\right)}{\partial\left(z_{1}, \ldots z_{n}\right)}(a)$, since $f(a)=0$. The lemma follows from taking the determinant on both sides.

Lemma 4. If $f$ and $g$ are holomorphic mappings also satisfying the conditions of Proposition 4, then for a point $p$ in the domain of $f$ and $g$ such that $f(p) \neq 0$ and $g(p)=0$, we have $\operatorname{det}(A) \in I_{p}(f)$ and hence $\operatorname{res}_{p}(h \operatorname{det} A)=0$.

Proof. From the relation $g=A f$ we have that $g_{j}=\sum_{k=1}^{n} a_{j k} f_{k}$. Since $f(p) \neq 0$, there is at least one component nonzero. Without loss of generality let us assume this to be the first component $f_{1}$, then

$$
\begin{equation*}
a_{j 1}=\frac{g_{j}}{f_{1}}-\sum_{k=2}^{n} a_{j k} \frac{f_{k}}{f_{1}} \tag{6.13}
\end{equation*}
$$

Furthermore, let $\left|A_{i j}\right|$ be the determinant of the matrix $A$ without the $i$ th row and the $j$ th column, then we can rewrite

$$
\begin{align*}
\operatorname{det}(A) & =\sum_{k=1}^{n} a_{k 1}\left|A_{k 1}\right|=\sum_{k=1}^{n}\left[\frac{g_{k}}{f_{1}}-\sum_{q=2}^{n} a_{k q} \frac{f_{q}}{f_{1}}\right]\left|A_{k 1}\right|  \tag{6.14}\\
& =\sum_{k=1}^{n} \frac{\left|A_{k 1}\right|}{f_{1}} g_{k}-\sum_{k=1}^{n} \sum_{q=2}^{n} a_{k q}\left|A_{k 1}\right| \frac{f_{k}}{f_{1}} \tag{6.15}
\end{align*}
$$

The first summation is obviously in $I_{p}(g)$, and the second summation is equal to zero, because every term is canceled out.

Proof. (of Proposition 5) There are three possibilities.

1. Assume the Jacobian $J_{f}(a)$ of $f$ in $a$ and $\operatorname{det}(A)$ to be nonzero. Now we apply Cauchy's formula on $h / J_{f}$ and on $h \operatorname{det}(A) / J_{g}$ to find

$$
\begin{align*}
\operatorname{res}_{a}(h) & =\frac{1}{(2 \pi i)^{n}} \int_{\Gamma_{a}} \frac{h \cdot J_{f}}{J_{f} \cdot f} d z=\frac{h(a)}{J_{f}(a)}  \tag{6.16}\\
\operatorname{res}_{a}(h \cdot \operatorname{det} A) & =\frac{1}{(2 \pi i)^{n}} \int_{\Gamma_{a}^{\prime}} \frac{h \cdot \operatorname{det}(A) \cdot J_{g}}{J_{g} \cdot g} d z=\frac{h(a) \operatorname{det}(A)(a)}{J_{g}(a)} . \tag{6.17}
\end{align*}
$$

Now, from lemma 3 we see that (6.16) equals (6.17) and hence $\operatorname{res}_{a}(h)=\operatorname{res}_{a}(h \cdot \operatorname{det} A)$.
2. Let $J_{f}(a)=0$, but $\operatorname{det}(A) \neq 0$. We construct a sequence ${ }^{1}\left\{\zeta^{(k)}\right\} \subset \mathbb{C}^{n}$ with $\zeta^{(k)} \rightarrow 0$ such that $w=f(z)-\zeta^{(k)}$ has only simple zeros at $\left\{P_{\nu}^{(k)}, \nu=1, \ldots, \mu\right\}$. Now,

$$
\begin{equation*}
\operatorname{res}_{a}(h)=\lim _{k \rightarrow \infty} \sum_{\nu=1}^{\mu} \operatorname{res}_{P_{\nu}^{(k)}} f-\zeta^{(k)}(h) \tag{6.18}
\end{equation*}
$$

since the continuity principle of local residues, and

$$
\begin{equation*}
\operatorname{res}_{a} f(h)=\lim _{k \rightarrow \infty} \sum_{\nu=1}^{\mu} \operatorname{res}_{P_{\nu}^{(k)}} A\left(f-\zeta^{(k)}\right)(h \cdot \operatorname{det} A)=\operatorname{res}_{a} A f(h \cdot \operatorname{det} A) \tag{6.19}
\end{equation*}
$$

which is justified by case 1 .
3. Let us assume $\operatorname{det} A(a)=0$. We write down a sequence of nonsingular matrices $\left\{A_{t}\right\}, t \in$ $[0,1]$ and $A_{t} \rightarrow A$. Now we use the second case $g_{t}=A_{t} f$. We have proven in lemma 4 that ${\underset{p}{t}}^{\operatorname{res}_{g_{t}}}\left(h \operatorname{det} A_{t}\right)=0$ for all $p_{t} \neq a$. Now we complete the proof with the following computation.

$$
\begin{align*}
\underset{a}{\operatorname{res}_{g}(h \operatorname{det} A)} & =\lim _{t \rightarrow 0} \sum_{\left\{p_{t}\right\}} \operatorname{res}_{p_{t}}\left(h \operatorname{det} A_{t}\right)  \tag{6.20}\\
& =\lim _{t \rightarrow 0} \operatorname{res}_{a}\left(h \operatorname{det} A_{t}\right)=\lim _{t \rightarrow 0} \operatorname{res}_{a}(h)  \tag{6.21}\\
& =\operatorname{res}_{a}(h) \tag{6.22}
\end{align*}
$$

[^1]Transformation formula 6.11 gives one of the two important tools for calculating local residues. In practical calculations one would always use $g=\left(\left(z_{1}-a_{1}\right)^{k_{1}}, \ldots,\left(z_{n}-a_{n}\right)^{k_{n}}\right)$ for some integers $k_{1}, \ldots, k_{n}$. Hilbert's Nullstellensatz shows us that for every holomorphic mapping $f$ we can find a holomorphic matrix $A$ and an integer $r$ such that $g=A f$ and $k_{1}=\ldots=k_{n}=r$.
Theorem 13. (Hilbert's Nullstellensatz) Let I be an ideal in the polynomial ring $\mathbb{C}\left[z_{1}, \ldots, z_{n}\right]$ of $n$ complex variables. Furthermore let $V(I)=\left\{z \in \mathbb{C}^{n}: \forall p \in I, p(z)=0\right\}$. If p satisfies $p(a)=0$ for all $a \in V(I)$, then there is an integer $r>0$ such that $p^{r} \in I$.

Proof. Suppose that $p^{r} \notin I$ for all $r>0$; in particular, $I$ is strictly smaller than $\mathbb{C}\left[z_{1}, \ldots, z_{n}\right]$ and $p \neq 0$. Consider the ring

$$
\begin{equation*}
R=\mathbb{C}\left[z_{1}, \ldots, z_{n}, 1 / p\right] \subset \mathbb{C}\left(z_{1}, \ldots, z_{n}\right) \tag{6.23}
\end{equation*}
$$

The $R$-ideal $R I$ is strictly smaller than $R$, since $R I=\bigcup_{r=0}^{\infty} p^{-r} I$ does not contain the unit element. Let $w$ be an indeterminate over $\mathbb{C}\left[z_{1}, \ldots, z_{n}\right]$, and let $J$ be the inverse image of $R I$ under the homomorphism $\phi: \mathbb{C}\left[z_{1}, \ldots, z_{n}, w\right] \rightarrow R$ acting as the identity on $\mathbb{C}\left[z_{1}, \ldots, z_{n}\right]$ and sending $w$ to $1 / f$. Then $J$ is strictly smaller than $\mathbb{C}\left[z_{1}, \ldots, z_{n}, w\right]$, so since $V(I)$ is not empty for a proper ideal $I$, there exists an element $\left(a_{1}, \ldots, a_{n}, b\right) \in \mathbb{C}^{n+1}$ such that $q\left(a_{1}, \ldots, a_{n}, b\right)=0$ for all $q \in J$. In particular, we see that $q\left(a_{1}, \ldots, a_{n}\right)=0$ for all $q \in I$. Our assumption on $p$ therefore implies $p\left(a_{1}, \ldots, a_{n}\right)=0$. However, $J$ also contains the element $1-w p$ since $\phi$ sends this element to zero. Hence

$$
\begin{equation*}
0=(1-w p)\left(a_{1}, \ldots, a_{n}, b\right)=1-b p\left(a_{1}, \ldots, a_{n}\right)=1 \tag{6.24}
\end{equation*}
$$

which is a contradiction.
The relation $g=A f$ can be written as a system of equations $\left(z_{j}-a_{j}\right)^{k_{j}}=\sum_{l=1}^{n} a_{j l} f_{l}$ for each $j=1 \ldots n$. In fact, every integer $k_{j}$ can be minimized. This leads to the definition of algebraic multiplicity.
Definition 16. The (algebraic) multiplicity $\mu_{a}(f)$ of a holomorphic mapping $f$ with an isolated zero $a$ is defined as a multi index of integers $\mu_{a}(f)=\left(\kappa_{1}, \ldots, \kappa_{n}\right)$ with property

$$
\begin{equation*}
\kappa_{j}=\min _{k \in \mathbb{N}}\left\{\left(z_{j}-a_{j}\right)^{k}=\sum_{l=1}^{n} a_{j l} f_{l}: \exists a_{j l} \in \mathcal{O}\left(\mathcal{U}_{a}\right)\right\} \tag{6.25}
\end{equation*}
$$

where $a_{j l}$ is the holomorphic element on the $j^{\text {th }}$ row and the $l^{\text {th }}$ column of the matrix $A$ such that $g=A f$.

We highlight that this is a different definition of multiplicity than is given in the book of Tsikh, since he speaks about geometric multiplicity. From now on, if we write multiplicity, the algebraic multiplicity is meant.

The next theorem shows us that the $j^{\text {th }}$ integer of $\mu_{a}(f)$ will appear in the residue as the derivative of $h$ with respect to $z_{j}$.

Theorem 14. Let $f$ be a holomorphic mapping, $f(a)=0$ for some isolated zero a and choose a multi index $k$ and a holomorphic matrix $A$ such that $(z-a)^{k}=A f$ then for a holomorphic function $h$

$$
\begin{equation*}
\operatorname{res}_{a}(h)=\frac{1}{(k-1)!} D^{k-1}[h \cdot \operatorname{det}(A)](a) \tag{6.26}
\end{equation*}
$$

Proof. Recall from chapter 4 that the holomorphic function $h$ can be represented as a Taylor series as follows.

$$
\begin{equation*}
h=\sum_{\alpha \geq 0} \frac{D^{\alpha} h(a)}{\alpha!}(z-a)^{\alpha} \tag{6.27}
\end{equation*}
$$

This leads to the following result.

$$
\begin{align*}
\operatorname{res}_{a}(f) & =\frac{1}{(2 \pi i)^{n}} \int_{\left|(z-a)^{k}\right|=\epsilon} \frac{\operatorname{det}(A) \cdot h d z}{(z-a)^{k}}  \tag{6.28}\\
& =\frac{1}{(2 \pi i)^{n}} \int_{|(z-a)|=\epsilon} \sum_{\alpha \geq 0} \frac{(z-a)^{\alpha-k}}{\alpha!} D^{\alpha}[\operatorname{det}(A) \cdot h](a) d z \tag{6.29}
\end{align*}
$$

All the integrals vanish except for $\alpha=k-1$.

$$
\begin{equation*}
\frac{1}{(k-1)!} D^{(k-1)}[\operatorname{det}(A) \cdot h](a) \cdot \frac{1}{(2 \pi i)^{n}} \int_{|(z-a)|=\epsilon} \frac{d z}{z-a}=\frac{1}{(k-1)!} D^{k-1}[h \cdot \operatorname{det}(A)](a) . \tag{6.30}
\end{equation*}
$$

The previous theorem is a guide to the connection between the multiplicity of a zero and the derivatives of $h$ that appear in the residue.

Now let us calculate the residues of the example at the beginning of this chapter. Recall that we want to calculate the residue of $h / f$, with $f(z, w)=z w(z-w)$, in $a=(0,0)$ associated with a holomorphic mapping $f$, such that $f_{1} \cdot f_{2}=f$. We have three possible choices for the mapping $f$.
Choice 1. Let us first consider $f^{I}(z, w)=(z, w(z-w))$. Since the multiplicity of $z$ is one and of $w$ is two, we solve $g=A f^{I}$ :

$$
\binom{z}{w^{2}}=A\binom{z}{w(z-w)} \Rightarrow A=\left(\begin{array}{cc}
1 & 0  \tag{6.31}\\
w & -1
\end{array}\right)
$$

By using proposition 4 and theorem 14 we find

$$
\begin{equation*}
\operatorname{res}_{a} f_{f^{I}}(h)=\operatorname{res}_{a}(h \cdot \operatorname{det} A)=\frac{1}{(2 \pi i)^{2}} \int_{|z|=\epsilon} \int_{|w|^{2}=\delta} \frac{-h(z, w)}{z w^{2}} d w d z=-\frac{\partial h}{\partial w}(0,0) \tag{6.32}
\end{equation*}
$$

A few comments are in order. One sees that the multiplicity of $(z, w)$ has influence on the derivatives of $h$, as in the one-dimensional case. Also note that for an other choice of $A$ and $g$ the same residue appears.

Let us also calculate the residue with respect to the holomorphic mapping $f^{I I}=(w, z(z-w))$. Since the multiplicity of $\mu_{(0,0)}\left(f^{I I}\right)=(2,1)$, we expect a derivative with respect to $z$. Let us verify.
Choice 2. Let us next consider $f^{I I}(z, w)=(w, z(z-w))$. Since the multiplicity of $w$ is one and of $z$ is two, we solve $g=A f^{I I}$ :

$$
\binom{w}{z^{2}}=A\binom{w}{z(z-w)} \Rightarrow A=\left(\begin{array}{ll}
1 & 0  \tag{6.33}\\
z & 1
\end{array}\right)
$$

By using proposition 4 and theorem 14 we find

$$
\begin{equation*}
\operatorname{res}_{a} f^{I I}(h)=\operatorname{res}_{a}(h \cdot \operatorname{det} A)=\frac{1}{(2 \pi i)^{2}} \int_{|z|^{2}=\epsilon} \int_{|w|=\delta} \frac{h(z, w)}{w z^{2}} d z d w=\frac{\partial h}{\partial z}(0,0) \tag{6.34}
\end{equation*}
$$

This example shows that the second cycle $\Gamma_{2}=\{|w|<|z|\}$ is homotopy equivalent to $\Gamma_{2}=\{|w|=$ $\epsilon,|z|=\delta, \epsilon<\delta\}$.

In order to be complete, now also follows the third choice.
Choice 3. We finally consider $f^{I I I}(z, w)=(z w, z-w)$. Since now the multiplicity of $z$ and $w$ are now both two, we solve $g=A f^{I I I}$ :

$$
\binom{z^{2}}{w^{2}}=A\binom{z w}{z-w} \Rightarrow A=\left(\begin{array}{cc}
1 & z  \tag{6.35}\\
1 & -w
\end{array}\right)
$$

By using proposition 4 and theorem 14 we find
$\operatorname{res}_{a} f_{\text {III }}(h)=\operatorname{res}_{a}(h \cdot \operatorname{det} A)=\frac{1}{(2 \pi i)^{2}} \int_{|z|^{2}=\epsilon} \int_{|w|^{2}=\delta} \frac{-h(z, w)(z+w)}{z^{2} w^{2}} d z d w=-\frac{\partial h}{\partial z}(0,0)-\frac{\partial h}{\partial w}(0,0)$.

### 6.3 Residues on local intersections

In the previous section we have seen that in order to calculate a local residue of a differential form $\omega$, you have to find a mapping $\mathbb{C}^{n} \rightarrow \mathbb{C}^{n}$ with an isolated zero. Consider the following differential form.

$$
\begin{equation*}
\omega=\frac{h(z, w)}{z^{2}-w^{2}(w+1)} d z d w \tag{6.37}
\end{equation*}
$$

Locally, and only locally, this form can be rewritten such that it has an isolated zero at the origin.

$$
\begin{equation*}
\omega=\frac{h(z, w)}{(z+i w \sqrt{w+1})(z-i w \sqrt{w+1})} d z d w \tag{6.38}
\end{equation*}
$$

Let us be careful, because of the multivaluedness of $\sqrt{w+1}$. We pick the positive branch $(\sqrt{-1}=i)$ and we consider the mapping $f(z, w)=(z+i w \sqrt{w+1}, z-i w \sqrt{w+1})$. One can easily see that if we would have chosen the negative branch, we would have the same mapping $f$ except that the components of the mapping would be switched.

Since now the multiplicity of $z$ and $w$ are now both one, we solve $g=A f$ :

$$
\binom{z}{w}=A\binom{z+i w \sqrt{w+1}}{z-i w \sqrt{w+1}} \Rightarrow A=\left(\begin{array}{cc}
\frac{1}{2} & \frac{1}{2}  \tag{6.39}\\
\frac{1}{2 i \sqrt{w+1}} & -\frac{1}{2 i \sqrt{w+1}}
\end{array}\right)
$$

By using proposition 4 and theorem 14 we find

$$
\begin{equation*}
\underset{(0,0)}{\operatorname{res}} f(h)=\underset{(0,0)}{\operatorname{res}} g(h \cdot \operatorname{det} A)=-\left.\frac{h(z, w)}{2 i \sqrt{w+1}}\right|_{(z, w)=(0,0)}=\frac{h(0,0)}{8 i \pi^{2}} \tag{6.40}
\end{equation*}
$$

### 6.4 Separating cycles

The theory of separating cycles is used to investigate when it is possible to replace a cycle by the sum of the simplest contours, namely small tori "around" singular points (poles) of the integrand. In one complex dimension, it is not hard to see which poles are inside and which poles are outside the contour. In higher dimensions it is sometimes pretty hard. Let us be more precise.

Let $X$ be a complex analytic manifold of dimension $n$, and let $\omega$ be a quotient differential form of degree $n$ on $X$. We assume that the polar set of $\omega$ can split into $n$ hyper surfaces $F_{1}, \ldots, F_{n}$. Here $F_{j}=\left\{f_{j}=0\right\}$. Let us denote $F=\cup_{j=1}^{n} F_{j}$ and $Z$ to be the set of isolated intersection points of these surfaces. Consider the following integral

$$
\begin{equation*}
(2 \pi i)^{-n} \int_{\Gamma} \omega \tag{6.41}
\end{equation*}
$$

with $\Gamma$ a cycle of complex dimension $n$ in $X \backslash F$. For each point $a \in Z$ there is an associated local cycle

$$
\begin{equation*}
\Gamma_{a}=\left\{z \in \mathcal{U}_{a}:\left|f_{j}(z)\right|=\epsilon, j=1, \ldots, n\right\} \tag{6.42}
\end{equation*}
$$

If it is possible to write

$$
\begin{equation*}
\Gamma \sim \sum_{a \in Z} n_{a} \Gamma_{a}, \quad n_{a} \text { are integers } \tag{6.43}
\end{equation*}
$$

meaning that $\Gamma$ is homotopy equivalent in $X \backslash F$ to a sum of local cycles, then we can rewrite this according to Stokes' formula to

$$
\begin{equation*}
(2 \pi i)^{-n} \int_{\Gamma} \omega=\sum_{a \in Z} n_{a} \underset{a}{\operatorname{res}}(\omega) \tag{6.44}
\end{equation*}
$$

Definition 17. Let $\Gamma$ be a n-dimensional cycle on $X \backslash F$ not homologous to zero. We call the cycle $\Gamma$ a separating cycle if and only if $\Gamma \sim 0$ in $X \backslash\left(F_{1} \cup \cdots[k] \cdots \cup F_{n}\right)$ for all $k=1, \ldots, n$.

One can see this as every hypersurface $F_{j}$ is an obstacle for $\Gamma$ to be contractible outside the union of hypersurfaces. The next corollary shows us what this means for cycles in $\mathbb{C}$. We omit the proof.

Corollary 3. Let $F_{1}=\left\{a_{1}, \ldots, a_{m}\right\}$ the discrete polar set of points in a convex domain $X \subset \mathbb{C}$. Every cycle with at least one point from $F_{1}$ inside is a separating cycle.

Theorem 15. Let $X \subset \mathbb{C}^{n}$ be a complex manifold of dimension $n$ such that $H_{3}(X)$ is trivial, and let $F_{1}, \ldots, F_{n}$ be an arbitrary system of hyper surfaces in $X$ such that $Z=F_{1} \cap \ldots \cap F_{n}$ is discrete. For $a \Gamma$ to be separating it is necessary and sufficient that it can be represented in the form of equation (6.43).

Proof. First we prove that any cycle $\Gamma$ representable in the form (6.43) is separating.
Let us write

$$
\begin{equation*}
c_{a}^{(k)}=\left\{z \in \mathcal{U}_{a}:\left|f_{j}(z)\right|=\epsilon_{j}, j \neq k,\left|f_{k}(z)\right| \leq \epsilon_{k}\right\} \tag{6.45}
\end{equation*}
$$

for a $(n+1)$-chain in $X \backslash\left(F_{1} \cup \cdots[k] \cdots \cup F_{n}\right)$, then we have for each local cycle $\Gamma_{a}=(-1)^{k-1} \partial c_{a}^{(k)}$. This means that $\Gamma_{a} \sim 0$ in $X \backslash\left(F_{1} \cup \cdots[k] \cdots \cup F_{n}\right)$ and so is $\Gamma$, since it is homologous to a linear combination of separating cycles.

The proof of the fact that every separating cycle $\Gamma$ is representable in the form (6.43) is more involved. We will prove it only for $n=2$, since the proof for $n>2$ is quite similar, although it has some technical difficulties. The complete proof can be found in [15].

Let $A=X \backslash F_{2}$ and $B=X \backslash F_{1}$. We apply the Mayer-Vietors exact sequence to the open subspaces $X \backslash Z(=A \cup B), A \cap B, A$ and $B$.

$$
\begin{equation*}
H_{3}(X \backslash Z) \xrightarrow{\partial_{*}} H_{2}(A \cap B) \xrightarrow{i_{*}} H_{2}(A) \oplus H_{2}(B) . \tag{6.46}
\end{equation*}
$$

Here $i_{*}=\left(i_{1 *},-i_{2 *}\right)$ a pair of inclusion mappings resp. from $A \cap B$ to $A$ and to $B$. The mapping $\partial_{*}$ is a connecting homomorphism which adds to every 3-cycle in $X \backslash Z$ a certain sum $c^{(1)}+c^{(2)}$ of chains $c^{(1)} \subset A, c^{(2)} \subset B$.

Let us pick a 3-cycle $h$ in $X \backslash Z$. By definition of the connecting homomorphism $\partial c^{(1)}+\partial c^{(2)}=$ $\partial \partial_{*} h=0$ and hence the boundary $\partial c^{(1)}$ is a cycle in $A \cap B$ which is of the homology class of the image of $\partial_{*}[h]$. Now, for a 2-cycle $\Gamma \in X \backslash\left(F_{1} \cup F_{2}\right)$ we have that $i_{*}[\Gamma]=0$ and since the sequence is exact, there is a 3-cycle $h$ in $X \backslash Z$ such that

$$
\begin{equation*}
\partial_{*}[h]=[\Gamma] . \tag{6.47}
\end{equation*}
$$

Since $H_{3}(X)$ is trivial, $H_{3}(X \backslash Z)$ is generated by a family of 3 -spheres around the points in $Z$, or equivalently by a family of boundaries of the polyhedra $\Pi_{a}=\left\{z \in \mathcal{U}_{a}:\left|f_{j}\right| \leq \epsilon_{j}, j=1,2\right\}$. Thus,

$$
\begin{equation*}
h \sim \sum_{a \in Z} n_{a} h_{a} \tag{6.48}
\end{equation*}
$$

where, $h_{a}=\partial \Pi_{a}$. Now from (6.47) and (6.48) it follows that

$$
\begin{equation*}
[\Gamma]=\partial_{*}\left[\sum_{a \in Z} n_{a} h_{a}\right]=\sum_{a \in Z} n_{a}\left[\partial_{*} h_{a}\right] \tag{6.49}
\end{equation*}
$$

We already observed that $\Gamma_{a}=(-1)^{k-1} \partial c_{a}^{(k)}$, where $c_{a}^{(k)}$ is defined as in (6.45).

$$
\begin{equation*}
[\Gamma]=\sum_{a \in Z} n_{a}\left[\partial c_{a}^{(1)}\right]=\sum_{a \in Z} n_{a}\left[\Gamma_{a}\right] \tag{6.50}
\end{equation*}
$$

which proves the theorem.
This is the main result of the thesis. Only cycles which are separable can be replaced by local cycles around the intersection points of the hypersurfaces and therefore only integrals around separable cycles can be used to calculate real integrals by local residues.

## Chapter 7

## Applications to integrals over $\mathbb{R}^{2}$

In complex analysis over $\mathbb{C}$ we can use contour integrals to compute real integrals. In several variables this is possible when the cycles we want to integrate around are separable. The goal of this chapter is to give an overview of which type of integrals can be solved by local residues, and which can not.

### 7.1 A class of functions over $\mathbb{R}^{2}$

For rational functions $f(x)=P(x) / Q(x)$, one can solve the principal value of the integral $\int_{-\infty}^{\infty} f(x) d x$ by complexification of de domain of $f$ and apply Cauchy's residue theorem. There exists a class of functions $G(x, y)$ of two real variables for which we can also calculate the integral over the whole real plane by using local residues. This class is stated in the next theorem.

Theorem 16. Let $I=\iint_{\mathbb{R}^{2}} G(x, y) d x d y$ where

$$
\begin{equation*}
G(x, y)=\frac{h(x, y)}{\Pi_{j=1}^{m}\left[\left(a_{j} x+b_{j} y+c_{j}\right)^{2}+r_{j}^{2}\right]^{q_{j}}}, \tag{7.1}
\end{equation*}
$$

where $h$ is a polynomial and $a_{j}, b_{j}, c_{j}$ and $r_{j}$ are real numbers, $r_{j}>0$ and $q_{j}$ are positive integers and for each $j$ at least one of $a_{j}$ or $b_{j}$ is nonzero. Without loss of generality we assume that $a_{j} \geq$ $0, r_{j}>0$ and if $a_{j}>0$ assume $b_{j}>0$. Consider the differential form $\omega=G(z, w) d z \wedge d w$ be the differential form over $\mathbb{C}^{2}$. Now, the denominator of $\omega$ can be factorized as $f=f_{+} \cdot f_{-}$, where $f_{ \pm}=\Pi_{j=1}^{m}\left[a_{j} z+b_{j} w+c_{j} \pm i r_{j}\right]^{q_{j}}$. Let $\mathbb{H}_{ \pm}=\{\zeta \in \mathbb{C}: \operatorname{Im}(\zeta) \gtrless 0\}$, each half of the plane. Then

$$
\begin{equation*}
I=(2 \pi)^{2} \sum_{a \in \mathbb{C} \times \mathbb{H}_{-}} \operatorname{res}_{a}\left(f_{+}, f_{-}\right)(h)=-(2 \pi)^{2} \sum_{a \in \mathbb{C} \times \mathbb{H}_{+}} \operatorname{res}_{a}\left(f_{+}, f_{-}\right)(h) . \tag{7.2}
\end{equation*}
$$

For proving this theorem we need the following lemma.
Lemma 5. The divisors $F_{ \pm}=\left\{f_{ \pm}=0\right\}$ can only intersect the chain $c_{ \pm}=\overline{\mathbb{H}}_{ \pm} \times \overline{\mathbb{R}}$ at the point $(\infty, \infty)$. I.e. $F_{ \pm} \cap\left(c_{ \pm} \backslash\{(\infty, \infty)\}\right)=\emptyset$.

Proof. Let $\varphi_{j \pm}=a_{j} z+b_{j} w+c_{j} \pm i r_{j}$ on $c_{ \pm}$, then $\operatorname{Im} \varphi_{+}$is strictly positive and $\operatorname{Im} \varphi_{-}$is strictly negative since we assumed that $a_{j} \geq 0, r_{j}>0$.

We pass to a Riemann sphere. We write down the chart $z^{\prime}=1 / z$ and $w^{\prime}=1 / w$, now the origin of our new coordinates corresponds to infinity $(\infty, \infty)$ of the old ones. If both $a_{j}$ and $b_{j}$ are nonzero, then the divisor $F_{j \pm}=\left\{\varphi_{j \pm}=0\right\}$ is just the origin of the Riemann sphere. If $a_{j}=0$ and $b_{j}>0$, then
$\operatorname{Im}\left(w^{\prime}\right)=\operatorname{Im}\left(-b_{j} /\left(c_{j} \pm i r_{j}\right)\right) \neq 0$. And if $a_{j}>0$ and $b_{j}=0$, then $\operatorname{Im}\left(z^{\prime}\right)=\operatorname{Im}\left(-a_{j} /\left(c_{j} \pm i r_{j}\right)\right) \neq$ 0 . On the charts $(z, 1 / w)$ and $(1 / z, w)$, we have, as in the cases $a_{j}=0$ or $b_{j}=0$, empty intersections of $F_{ \pm}$and $c_{ \pm}$. This proves the lemma.

Proof. (Theorem 16) Let us define a change of variables $\Phi: \mathbb{H}_{+} \times \mathbb{H}_{+} \rightarrow \mathbb{D} \times \mathbb{D}$, which maps the upper half planes of both coordinates $z$ and $w$ onto the unit disk, that is

$$
\begin{gather*}
(\zeta, \omega)=\Phi(z, w)=\left(\frac{i-z}{i+z}, \frac{i-w}{i+w}\right)  \tag{7.3}\\
(z, w)=\Phi^{-1}(\zeta, \omega)=\left(\frac{\zeta-1}{\zeta+1}, \frac{\omega-1}{\omega+1}\right) \tag{7.4}
\end{gather*}
$$

The integral to be calculated is

$$
\begin{equation*}
I=\iint_{\Gamma} \frac{\tilde{h}(\zeta, \omega)}{\tilde{f}_{+} \tilde{f}_{-}} d \zeta d \omega \tag{7.5}
\end{equation*}
$$

The chains $c_{ \pm}$will change under this transformation to

$$
\begin{equation*}
\gamma_{+}=\{|\zeta| \leq 1,|\omega|=1\}, \quad \gamma_{-}=\{|\zeta| \geq 1,|\omega|=1\} \tag{7.6}
\end{equation*}
$$

and the point $(\infty, \infty)$ into $(-1,-1)$. According to lemma 5, the divisor $\tilde{F}_{-}=\left\{\tilde{f}_{-}=0\right\}$ can only intersect the chain $\gamma_{-}$at $(-1,-1)$.

Let $R(\zeta)$ be the resultant of the polynomial $f_{1}$ and $f_{2}$, i.e. for two polynomials $f_{1}, f_{2}$ we find two polynomials $L(\zeta, \omega), S(\zeta, \omega)$ such that $R(\zeta)=L f_{1}+S f_{2}$ only depends on $\zeta$. The following follows from direct calculations.

$$
\begin{align*}
\frac{h}{f_{1} f_{2}} & =\frac{h\left(L f_{1}+S f_{2}\right)}{f_{1} f_{2}\left(L f_{1}+S f_{2}\right)}  \tag{7.7}\\
& =\frac{h L f_{1}}{f_{1} f_{2} R}+\frac{h S f_{2}}{f_{1} f_{2} R}  \tag{7.8}\\
& =\frac{h L}{R f_{2}}+\frac{h S}{R f_{1}} \tag{7.9}
\end{align*}
$$

Since the second fraction is holomorphic in $\omega$, the integral reduces to

$$
\begin{equation*}
I=\int_{|\zeta|=1} d \zeta \int_{|\omega|=1} \frac{\tilde{h} L}{R \tilde{f}_{-}} d \omega . \tag{7.10}
\end{equation*}
$$

Now, with the residue theory over $\mathbb{C}$, the inner integral is just a polynomial $p(\omega)$. Since the original integral converges absolutely by assumption of the theorem, there are no poles of $p$ on the circle $|\zeta|=1$. It follows that $p(\zeta)$ has no poles in the annulus $\{1 \leq|\zeta| \leq 1+\epsilon\}$ for sufficiently small $\epsilon$. Thus,

$$
\begin{equation*}
I=\int_{\Gamma(\epsilon)} \frac{\tilde{h} L}{R \tilde{f}_{-}} d \zeta \wedge d \omega \tag{7.11}
\end{equation*}
$$

Let $F_{\infty}$ be the line at infinity, then since $\Gamma(\epsilon)=\{|\zeta|=1+\epsilon,|\omega|=1\}$ is a separating cycle for the divisors $F_{1}=\{R(\zeta)=0\} \cup F_{\infty}$ and $F_{2}=\left\{\tilde{f}_{-}=0\right\}$ we have proven the theorem.

Example 8. We will evaluate the integral

$$
\begin{equation*}
I=\iint_{\mathbb{R}^{2}} \frac{1}{\left((x+2 y)^{2}+4\right)\left((x+5 y)^{2}+9\right)^{3}} d x d y \tag{7.12}
\end{equation*}
$$

Note that this integral converges because the order of the nominator is greater than the order of the denominator. After complexification of the domain of the integrand, we find divisors $f_{ \pm}=\{(z+2 w \pm$ $\left.2 i)(z+5 w \pm 3 i)^{3}=0\right\}$. We find the poles $a_{1}=\left(\frac{16}{3} i,-\frac{5}{3} i\right)$ and $a_{2}=\left(-\frac{16}{3} i, \frac{5}{3} i\right)$. Only $a_{2}$ is on the right upper halve plane $(\{\operatorname{Im} w>0\})$. This is a fourth order pole with respect to both $z$ and $w$.

$$
\begin{equation*}
\operatorname{res}_{\left(-\frac{16}{3} i, \frac{5}{3} i\right)}^{\operatorname{res}} f_{+}, f_{-}(1)=-\frac{1}{15552}, \tag{7.13}
\end{equation*}
$$

we have $I=\frac{\pi^{2}}{3888}$.
In some cases it is not possible to compute the surface integral of a real valued quotient function. For example

$$
\begin{equation*}
I=\iint_{\mathbb{R}^{2}} \frac{d x d y}{\left(x^{2}+y^{2}+R\right)^{2}}, \quad \quad R>0 \tag{7.14}
\end{equation*}
$$

In this case also the dominated convergence theorem cannot help us out.
We choose a function

$$
\begin{equation*}
f_{t}(x, y)=\frac{1}{\left(x^{2}+y^{2}+R\right)\left(\left(1+\frac{1}{t}\right) x^{2}+y^{2}+R\right)} \tag{7.15}
\end{equation*}
$$

This function is dominated by $f_{\infty}(x, y)=\frac{1}{\left(x^{2}+y^{2}+R\right)^{2}}$ everywhere. Note $\lim _{t \rightarrow \infty} f_{t}=f_{\infty}$. If we evaluate the integrals $\int_{\mathbb{R}^{2}} f_{t} d x d y$ with 'polar' coordinates ${ }^{1}$, i.e.

$$
\begin{equation*}
x=\sqrt{r} \cos (\theta), \quad y=\sqrt{r} \sin (\theta) \tag{7.16}
\end{equation*}
$$

and integrate first with respect to $r$ and than with respect to $\theta$, we find $\frac{2 \pi}{R}(\sqrt{t(t+1)}-t)$. Taking the limit of $t$ to infinity we find the integral $I=\pi / R$, which is exactly what we would achieve by evaluating integral $I$ by polar coordinates.

Now let us extend the domain of the function $f_{t}$ to $\mathbb{C}^{2}$ such that for $t>0$ we have two isolated zeroes $(0, \pm i \sqrt{R})$ of the mapping $f_{t}=\left(z^{2}+w^{2}+R,\left(1+\frac{1}{t}\right) z^{2}+w^{2}+R\right)$. Let us solve $g_{t}=A f_{t}$.

$$
\begin{gather*}
\binom{z^{2}}{w-i \sqrt{R}}=A\binom{z^{2}+w^{2}+R}{\left(1+\frac{1}{t}\right) z^{2}+w^{2}+R} \Rightarrow A=-\frac{t}{w+i \sqrt{R}}\left(\begin{array}{cc}
w+i \sqrt{R} & -w-i \sqrt{R} \\
-1-\frac{1}{t}
\end{array}\right)  \tag{7.18}\\
\operatorname{det} A=\frac{t^{2}}{(w+i \sqrt{R})^{2}}(w+i \sqrt{R})\left(1-1-\frac{1}{t}\right)=-\frac{t}{w+i \sqrt{R}} .  \tag{7.17}\\
\operatorname{res} f_{(0, i \sqrt{R})} f_{t}(1)=\underset{(0, i \sqrt{R})}{\operatorname{res}} g_{t}(\operatorname{det} A)=\frac{1}{(2 \pi i)^{2}} \int_{|z|^{2}=\epsilon} \int_{|w-i \sqrt{R}|=\delta} \frac{-t}{z^{2}\left(w^{2}+R\right)} d z d w=0 \tag{7.19}
\end{gather*}
$$

This result can be checked by calculating the Laurent series around $(0, \pm i \sqrt{R})$ of $f_{t}(x, y)$. The series is given by

$$
\begin{equation*}
\sum_{n \geq 0} \sum_{m \geq 0}(-1)^{n} \frac{(1+1 / t)^{m}}{\left(y^{2}+R\right)^{n+2}} x^{2 n} \tag{7.20}
\end{equation*}
$$

As one sees, this is an even power series and, since de nominator of $f_{t}$ is independent of $z$, the residue is zero indeed.

[^2]Remark 3. Note that in stead of solving $f=A g$, one can solve $g=A^{-1} f$, which is more easy to calculate in practice. Since $\operatorname{det}(A)=1 / \operatorname{det}\left(A^{-1}\right)$ one even doesn't need to invert $A$. However, one should check whether $A$ has only entries holomorphic in $a$.

Also if we would have chosen an other function which dominates the integrand, the residue theorem will not lead to a correct answer.

Let us consider for example the function $f_{t}(z, w)=\frac{1}{\left(z^{2}+(w-2 t)^{2}+R\right)\left(z^{2}+w^{2}+R\right)}$. Let us solve $g_{t}=A f_{t}$ for this $f_{t}$.

$$
\binom{z-\lambda}{w-t}=A\binom{z^{2}+(w-2 t)^{2}+R}{z^{2}+w^{2}+R} \Rightarrow A=-\frac{1}{4 t(z+\lambda)}\left(\begin{array}{cc}
w-3 t & -w-t  \tag{7.21}\\
-z-\lambda & z+\lambda
\end{array}\right)
$$

Here $-\lambda^{2}=t^{2}+R$. Now calculating the residue,

$$
\begin{equation*}
\underset{(t, \lambda)}{\operatorname{res}} f_{t}(1)=\underset{(t, \lambda)}{\operatorname{res}} g_{t}(\operatorname{det} A)=\frac{1}{(2 \pi i)^{2}} \int_{|z-\lambda|=\epsilon} \int_{|w-t|=\delta} \frac{-1}{4 t\left(z^{2}-\lambda^{2}\right)(w-t)} d z d w=-\frac{1}{8 \lambda t} . \tag{7.22}
\end{equation*}
$$

Indeed, now we have a result, but this residue tends to infinity when $t$ goes to zero.

### 7.2 Real trigonometric integrals

In complex analysis it is well known that we can transform real trigonometric integrals into complex valued integrals by the transformation $\sin (\theta)=\frac{1}{2 i}\left(e^{i \theta}-e^{-i \theta}\right)=\frac{1}{2 i}\left(z-z^{-1}\right)$ and $\cos (\theta)=\frac{1}{2}\left(z+z^{-1}\right)$. Likewise we could set for $j \in\{1, \ldots, n\}$

$$
\begin{equation*}
\cos \left(\theta_{j}\right)=\frac{1}{2}\left(z_{j}+z_{j}^{-1}\right), \quad \sin \left(\theta_{j}\right)=\frac{1}{2 i}\left(z_{j}-z_{j}^{-1}\right), \quad d \theta_{j}=\frac{d z_{j}}{i z_{j}} \tag{7.23}
\end{equation*}
$$

but this does not necessarily lead to a solution. For example, suppose we want to evaluate

$$
\begin{equation*}
I=\iint_{[-\pi, \pi]^{2}} \frac{d \theta d \varphi}{4+\cos ^{2}(\theta) \sin ^{2}(\varphi)} \tag{7.24}
\end{equation*}
$$

Note that we can factorize the denominator as $(2+i \cos (\theta) \sin (\varphi))(2-i \cos (\theta) \sin (\varphi))$. After transformation, the integral will be

$$
\begin{equation*}
I=-\int_{|w|=1} \int_{|z|=1} \frac{16 z w d z d w}{\left(8 z w+\left(w^{2}-1\right)\left(z^{2}+1\right)\right)\left(8 z w-\left(w^{2}-1\right)\left(z^{2}+1\right)\right)} \tag{7.25}
\end{equation*}
$$

One can easily see that there are four isolated zeros $a^{(1)}=(-i, 0), a^{(2)}=(i, 0), a^{(3)}=(0,-1)$ and $a^{(4)}=(0,1)$ which are all inside the torus $\Gamma=\{(z, w):|z|=|w|=1\}$.

Let $X=\mathbb{C}^{2}$ and $F_{ \pm}=\left\{8 z w \pm\left(w^{2}-1\right)\left(z^{2}+1\right)=0\right\}$. We will show that $\Gamma=\{|z|=|w|=1\}$ is not a separating cycle.

The next remark is useful in calculating Laurent series.
Remark 4. For $|a|<|b|$

$$
\begin{gather*}
\frac{1}{a-b}=\frac{1}{b} \frac{1}{1-\frac{a}{b}}=\frac{1}{b} \sum_{m \geq 0}\left(\frac{a}{b}\right)^{m}=\sum_{m \geq 0} \frac{a^{m}}{b^{m+1}},  \tag{7.26}\\
\frac{1}{a+b}=\frac{1}{b} \frac{1}{1--\frac{a}{b}}=\frac{1}{b} \sum_{m \geq 0}\left(-\frac{a}{b}\right)^{m}=\sum_{m \geq 0} \frac{(-a)^{m}}{b^{m+1}} . \tag{7.27}
\end{gather*}
$$

Clearly $\Gamma$ is not homologous zero in $X \backslash\left(F_{+} \cup F_{-}\right)$, since there are four isolated zeros within $\Gamma$. Let us look at the Laurent series of $1 / f_{-}$, with remark 4 in mind.

$$
\begin{equation*}
\frac{1}{f_{-}}=\frac{1}{8 z w-\left(w^{2}-1\right)\left(z^{2}+1\right)}=\sum_{m \geq 0} \frac{(8 z w)^{m}}{\left(w^{2}-1\right)^{m+1}\left(z^{2}+1\right)^{m+1}} \tag{7.28}
\end{equation*}
$$

Now it is easily seen that $\int_{\Gamma} 1 / f_{-}$is nonzero in $X \backslash F_{-}$, and hence $\Gamma$ is not a separating cycle. Analogous we find that $\int_{\Gamma} 1 / f_{+}$is nonzero in $X \backslash F_{+}$.

## Appendix A

## Some computations...

Theorem 17. Let $t>0$, then

$$
\begin{equation*}
\iint_{\mathbb{R}^{2}} \frac{d x d y}{\left(x^{2}+y^{2}+R\right)\left((1+1 / t) x^{2}+y^{2}+R\right)}=\frac{2 \pi}{R}(\sqrt{t(t+1)}-t) \tag{A.1}
\end{equation*}
$$

Proof. First we apply the change of coordinates $x=\sqrt{r} \cos (\theta)$ and $y=\sqrt{r} \sin (\theta)$, with Jacobian

$$
\left|\begin{array}{cc}
\frac{1}{2 \sqrt{r}} \cos (\theta) & -\sqrt{r} \sin (\theta)  \tag{A.2}\\
\frac{1}{2 \sqrt{r}} \sin (\theta) & \sqrt{r} \cos (\theta)
\end{array}\right|=\frac{1}{2} \cos ^{2}(\theta)+\frac{1}{2} \sin ^{2}(\theta)=\frac{1}{2}
$$

on the integral of (A.1).

$$
\begin{equation*}
\frac{1}{2} \int_{0}^{2 \pi} \int_{0}^{\infty} \frac{d r d \theta}{(r+R)\left(\left(1+\cos ^{2}(\theta) / t\right) r+R\right)} \tag{A.3}
\end{equation*}
$$

Since we first integrate with respect to $r$, we set $a=1+\cos ^{2}(\theta) / t$ as a constant and we take a partial fraction decomposition.

$$
\begin{equation*}
\frac{1}{(r+R)(a r+R)}=\frac{(a-1)^{-1}}{r(r+R)}-\frac{(a-1)^{-1}}{r(a r+R)} \tag{A.4}
\end{equation*}
$$

This leaves a single integral over $\theta$.

$$
\begin{gather*}
\frac{1}{2} \int \frac{\log (r)-\log (r+R)}{(a-1) R}-\left.\frac{\log (r)-\log (a r+R)}{(a-1) R}\right|_{0} ^{\infty} d \theta  \tag{A.5}\\
\left.\frac{1}{2} \int \frac{\log (a r+R)-\log (r+R)}{(a-1) R}\right|_{r=0} ^{\infty} d \theta  \tag{A.6}\\
\frac{1}{2} \int \frac{\log (a)}{(a-1) R} d \theta \tag{A.7}
\end{gather*}
$$

Putting in back $a(\theta)=1+\cos ^{2}(\theta) / t$ gives the following integral.

$$
\begin{equation*}
\frac{1}{2 R} \int_{0}^{2 \pi} \frac{t \log \left(1+\cos ^{2}(\theta) / t\right)}{\cos ^{2}(\theta)} d \theta \tag{A.8}
\end{equation*}
$$

One can verify by differentiating that this is equal to

$$
\begin{equation*}
\frac{1}{2 R}\left|2 \sqrt{t(t+1)} \arctan \left(\sqrt{\frac{t}{t+1}} \tan (\theta)\right)+t \tan (\theta) \log \left(1+\cos ^{2}(\theta) / t\right)-2 t \theta\right|_{\theta=0}^{2 \pi} \tag{A.9}
\end{equation*}
$$

which gives us the required

$$
\begin{equation*}
\frac{1}{2 R}(4 \pi(\sqrt{t(t+1)}-1)-0)=\frac{2 \pi}{R}(\sqrt{t(t+1)}-t) \tag{A.10}
\end{equation*}
$$

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[^0]:    ${ }^{1}$ If we allow $\alpha_{j}$ to be negative, we have the Laurent series.

[^1]:    ${ }^{1}$ The proof of the existence of such a sequence is given in [15] and involves the Martinelli-Bochner integral representation for holomorphic functions.

[^2]:    ${ }^{1}$ See appendix A for the complete computation.

