university of groningen

faculty of mathematics and natural sciences

Fault detection and isolation in water distribution networks

Bachelor Thesis Applied Mathematics

June 2014

Student:

H.J. van Waarde

Supervisors:

Prof. dr. H.L. Trentelman

F. de Roo, MSc

Prof. dr. J. Top

Abstract

This thesis deals with fault detection and isolation in water distribution networks. Inflows of contaminated water in water distribution networks are regarded as faults. The contaminant concentration in a water distribution network can be modelled by a linear time-invariant system, where the input of the system represents inflows of contaminated water, and the output corresponds to sensor measurements. We consider fault detection and isolation by residual generation. This thesis provides a necessary and sufficient condition for the existence of a bank of residual generators that induces a diagonal transfer matrix from faults to residuals. Furthermore, we present a graph theoretic characterization of this condition. This characterization allows us to verify the existence of a bank of residual generators in a more intuitive way, namely by analyzing the structure of the water distribution network under consideration.

Contents

Acknowledgements

I would like to dedicate a few words to express my gratitude towards my primary supervisors: Prof. dr. Harry Trentelman and Froukje de Roo, MSc. I would like to thank professor Trentelman for his guidance during this project. Thank you for the clear explanation of the subject, your feedback on my work and the occasional jokes. I greatly appreciate your enthusiasm for mathematics.
I would like to thank Froukje for her support during the project. You always had time to answer my questions and discuss our results. Thank you for the large amount of feedback on my thesis. I wish you good luck with your PhD and much happiness in marriage.

Chapter 1

Introduction

This thesis deals with fault detection and isolation in water distribution networks. Fault detection is concerned with indicating whether faults occur, while the goal of fault isolation is to locate the occuring faults. A water distribution network (WDN) is a branching pipeline system that provides industries and houses with purified drinking water. Inflows of contaminated water in water distribution networks are seen as faults.

Nowadays the quality of water in a water distribution network is examined by taking water samples from the network that are analyzed in laboratories [1]. This current process has a number of drawbacks. First of all, it is time-consuming. Secondly, it only provides information about the water quality at the times the samples are taken from the network. A possible solution to these problems is to install contaminant sensors in water distribution networks. These sensors allow monitoring of the water quality on a continuous basis. This makes it possible to quickly detect contamination in water distribution networks. Contaminant sensors are still under development [8, 11]. In this thesis we will not discuss different types of sensors and their functioning, instead we concentrate on a mathematical model of the contaminant concentration in water distribution networks, and a condition under which fault detection and isolation is possible.

The contaminant concentration in a WDN can be modelled by a linear time-invariant system, where the input of the system represents influx of contaminants in the network, and the output corresponds to sensor measurements. The derivation of this linear time-invariant system is based on the 1-dimensional mass balance principle. In general, the sensor measurements are sensitive to multiple faults, i.e. a sensor detects contaminants that have flowed into the network at multiple different locations. To detect and isolate faults, residual generation is considered in this report. In this thesis we consider a residual generator in the form of a linear time-invariant system that has the measurement vector as input, and a residual vector as output. We are interested in a residual vector with the property that the transfer matrix from fault vector to residual vector is diagonal with nonzero diagonal elements. In general, one residual generator cannot produce a residual vector with the property that the transfer matrix from faults to residuals is diagonal. Hence, we use a set of residual generators, called a bank of residual generators. Residual generators act as filters, in the sense that the components of the input (measurement) vector of the generator are influenced by multiple faults, while each component of the output (residual) vector is only sensitive to one fault.

Note that a bank of residual generators that induces a diagonal transfer matrix does not always exists. For example, in a water distribution network that contains a larger number of contaminant influxes than contaminant sensors, it is impossible to detect and isolate all faults. The goal of this thesis is to provide a necessary and sufficient condition for the existence of a bank of residual generators that induces a diagonal transfer matrix from faults to residuals. As our first contribution, we prove that there exists a bank of residual generators with the aforementioned property if and only if the transfer matrix from the fault vector to the measurement vector has full column rank. This proof is inspired by ideas from the article *"Oberserver-based Fault Detection and Isolation for Structured Systems"* [4]. The main difference between the article and this thesis is that [4] deals with structured systems, while the linear system associated with a water distribution network considered in this thesis is not structured.

Verifying the rank condition for the existence of a bank of residual generators requires setting up a mathematical model in the form of a linear time-invariant system, and the calculation of the transfer

matrix and its rank. In order to simplify this condition, we present our second contribution: a graph-theoretic characterization of the rank of the transfer matrix from the fault vector to the measurement vector. We first describe how to associate a graph with a water distribution network. Subsequently, we prove that the maximum number of independent paths from fault vertices to measurement vertices in this graph is equal to the rank of the transfer matrix from the fault vector to the measurement vector. This result allows us to verify the existence of a bank of residual generators in a more intuitive way, namely by analyzing the structure of the water distribution network under consideration. It also gives insight in how to modify the sensor locations in a water distribution network if it is not possible to detect and isolate all faults in a current network configuration. The proof of the graph theoretic characterization is based on ideas from the article *"A Graph Theoretic Characterization for the Rank of the Transfer Matrix of a Structured System"* [10]. The relevance of the graph theoretic characterization discussed in this thesis lies in the fact that this characterization holds for a specific type of non-structured linear system, associated with a water distribution network, while previous results were only known for structured systems [10].
The thesis is arranged in the following way: In Chapter 2 we derive a model of the evolution of the contaminant concentration in water distribution networks. Chapter 3 contains a more extensive formulation of the problem of fault detection and isolation by residual generation. In Chapter 4 we give a necessary and sufficient condition for the existence of a bank of residual generators. Chapter 5 provides a graph theoretic characterization of the rank of the transfer matrix from faults to measurements. In Chapter 6 we present an example to illustrate the results gained in Chapter 4 and 5. We conclude this thesis by discussing our main results in Chapter 7.

Chapter 2

Mathematical model of water distribution networks

In this chapter we model the evolution of the contaminant concentration in a water distribution network. In order to do this, the network is divided into n compartments. Compartment i is assumed to have volume V_i (in m^3), and contains water with average concentration $x_i(t)$ (in g/m^3) of contaminant, for $i = 1, 2, ..., n$. The constant flow rate through compartment i is denoted by a_i (in m^3/s), for $i = 1, 2, ..., n$. We assume that the flow direction of water is fixed, and that the network does not contain loops. The three elements of a water distribution network considered in this report are pipes, junctions and conflux junctions. We will model the evolution of the contaminant concentration in each of the three elements, using the 1-dimensional mass balance principle. The mass balance principle states that the mass that enters a system must either leave the system or accumulate within the system [6].

2.1 Pipe

First, we consider a pipe (Figure 2.1). In this case the flow rate in each compartment is the same,

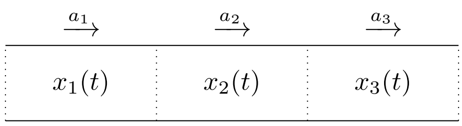

Figure 2.1: Pipe.

therefore we take $a := a_1 = a_2 = a_3$. Applying the 1-dimensional mass balance principle, the amount of contaminant (in g) in compartment i at time $t+\Delta t$ is equal to the amount of contaminant in compartment i at time t, minus the amount that leaves compartment i during the time span Δt, plus the amount of contaminant that enters compartment i during the time span Δt. The amount of contaminant (in g) in compartment 1 is given by $V_1 x_1(t)$. Therefore, the amount of contaminant in compartment 1 at time $t + \Delta t$ satisfies the equation

$$V_1 x_1(t + \Delta t) = V_1 x_1(t) - a_1 x_1(t) \Delta t,$$

where $a_1 x_1(t) \Delta t$ is the amount of contaminant that leaves compartment 1 during the time span Δt. After rearranging terms and dividing by $V_1 \Delta t$, this results in

$$\frac{x_1(t + \Delta t) - x_1(t)}{\Delta t} = -\frac{a_1 x_1(t)}{V_1},$$

and when we take the limit $\Delta t \to 0$, the differential equation describing the evolution of the average contaminant concentration in compartment 1 is given by:

$$\dot{x}_1(t) = -\frac{a_1}{V_1} x_1(t).$$

In a similar way, the amounts of contaminant in compartments 2 and 3 satisfy

$$V_2 x_2(t + \Delta t) = V_2 x_2(t) + a_1 x_1(t)\Delta t - a_2 x_2(t)\Delta t$$
$$V_3 x_3(t + \Delta t) = V_3 x_3(t) + a_2 x_2(t)\Delta t - a_3 x_3(t)\Delta t,$$

where $a_1 x_1(t)\Delta t$ and $a_2 x_2(t)\Delta t$ are the amounts of contaminant that enter compartments 2 and 3 respectively. Once again, we rearrange terms, and take the limit $\Delta t \to 0$, which yields two differential equations that describe the evolution of the average contaminant concentration in compartments 2 and 3. Combining the three previously described differential equations, we obtain a mathematical model of the evolution of the contaminant concentration in a pipe. This model is given in equation (2.1).

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{pmatrix} = \begin{pmatrix} -\frac{a}{V_1} & 0 & 0 \\ \frac{a}{V_2} & -\frac{a}{V_2} & 0 \\ 0 & \frac{a}{V_3} & -\frac{a}{V_3} \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}. \tag{2.1}$$

Note that we made use of the fact that $a = a_1 = a_2 = a_3$. Equation (2.1) has the form

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t),$$

where $\mathbf{x}(t)$ is the vector containing the contaminant concentrations in compartments 1,2 and 3, and A is a lower triangular matrix with negative diagonal entries, and non-negative off-diagonal elements.

2.2 Junction

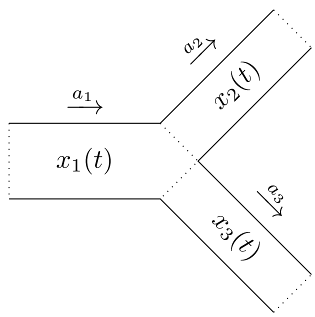

Figure 2.2: Junction.

In the case of a junction (Figure 2.2), the flow rate in compartment 1 is equal to the sum of the flow rates in compartments 2 and 3, i.e. $a_1 = a_2 + a_3$. The amount of contaminant in each compartment at time $t + \Delta t$ can be described by the 1-dimensional mass balance principle, as explained in Section 2.1, and is given by:

$$V_1 x_1(t + \Delta t) = V_1 x_1(t) - a_1 x_1(t)\Delta t$$
$$V_2 x_2(t + \Delta t) = V_2 x_2(t) + a_2 x_1(t)\Delta t - a_2 x_2(t)\Delta t$$
$$V_3 x_3(t + \Delta t) = V_3 x_3(t) + a_3 x_1(t)\Delta t - a_3 x_3(t)\Delta t.$$

After taking the limit $\Delta t \to 0$, we obtain three differential equations that describe the evolution of the average contaminant concentration in compartments 1, 2 and 3:

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{pmatrix} = \begin{pmatrix} -\frac{a_1}{V_1} & 0 & 0 \\ \frac{a_2}{V_2} & -\frac{a_2}{V_2} & 0 \\ \frac{a_3}{V_3} & 0 & -\frac{a_3}{V_3} \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}, \tag{2.2}$$

where $a_1 = a_2 + a_3$. Once again, we notice that the concentration vector $\mathbf{x}(t)$ satisfies:

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t),$$

where A is a lower triangular matrix with negative diagonal entries, and non-negative off-diagonal elements.

2.3 Conflux junction

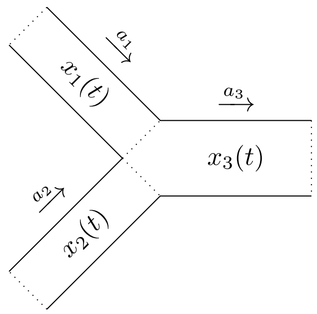

Figure 2.3: Conflux junction.

In conflux junctions (Figure 2.3), the flow rate a_3 is equal to the sum of a_1 and a_2, i.e. $a_3 = a_1 + a_2$. The amounts of contaminant in compartments 1, 2 and 3 at time $t + \Delta t$ satisfy

$$\begin{aligned}
V_1 x_1(t+\Delta t) &= V_1 x_1(t) - a_1 x_1(t)\Delta t \\
V_2 x_2(t+\Delta t) &= V_2 x_2(t) - a_2 x_2(t)\Delta t \\
V_3 x_3(t+\Delta t) &= V_3 x_3(t) - a_3 x_3(t)\Delta t + a_1 x_1(t)\Delta t + a_2 x_2(t)\Delta t.
\end{aligned}$$

Following the same steps as described in the previous sections, we obtain three differential equations that describe the evolution of the average contaminant concentration in compartments 1, 2 and 3:

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{pmatrix} = \begin{pmatrix} -\frac{a_1}{V_1} & 0 & 0 \\ 0 & -\frac{a_2}{V_2} & 0 \\ \frac{a_1}{V_3} & \frac{a_2}{V_3} & -\frac{a_3}{V_3} \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}, \tag{2.3}$$

where $a_3 = a_1 + a_2$. Equation (2.3) can be written as:

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t),$$

where A is a lower triangular matrix with negative diagonal entries, and non-negative off-diagonal elements.

2.4 Complete model

Consider a water distribution network that consists of a finite combination of the three main elements described in the previous sections. We divide the network into n compartments. The concentration vector $\mathbf{x}(t) \in \mathbb{R}^n$ satisfies

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t),$$

where A is a lower triangular matrix with diagonal entries $-\frac{a_1}{V_1}, -\frac{a_2}{V_2}, ..., -\frac{a_n}{V_n}$, and non-negative off-diagonal elements. To model the influx of contaminant into the system, suppose that contaminated water enters the system at compartments $i_1, i_2, ..., i_m$, where $m \leq n$. The contaminant concentration of the inflow into compartment i_k is denoted by $f_k(t)$ (in g/m^3), and the flow rate of the inflow is given by b_{i_k} (in m^3/s) for $k = 1, 2, ..., m$. The model under influence of the faults $f_1(t), f_2(t), ..., f_m(t)$ is given by

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{f}(t),$$

where $\mathbf{f}(t) = (f_1(t), f_2(t), ..., f_m(t))^T$ is the fault vector, and B is an $n \times m$ matrix with $\frac{b_{i_k}}{V_{i_k}}$ in position (i_k, k) for $k = 1, 2, ..., m$ and zeroes elsewhere. Furthermore, in order to incorporate the measurements of sensors in our model, let $y_l(t)$ denote the measured amount of contaminant in compartment j_l. We assume that there exist a total of p fully accurate sensors, where $p \leq n$. Now, by taking $\mathbf{y}(t) =$

$(y_1(t), y_2(t), ..., y_p(t))^T$, the output equation is given by $\mathbf{y}(t) = C\mathbf{x}(t)$, where C is a $p \times n$ matrix with ones at the positions (l, j_l) for $l = 1, 2, ..., p$, and zeroes elsewhere. This yields the complete model:

$$\begin{aligned}\dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + B\mathbf{f}(t) \\ \mathbf{y}(t) &= C\mathbf{x}(t),\end{aligned} \tag{2.4}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$, $\mathbf{f}(t) \in \mathbb{R}^m$, and $\mathbf{y}(t) \in \mathbb{R}^p$. In this report we often refer to system (2.4) as system Σ, or simply (A, B, C). A detailed example of a water distribution network, and its associated LTI sytem is given in Example 1.
The transfer matrix from the fault vector $\mathbf{f}$ to the measurement vector $\mathbf{y}$ of system Σ is given by

$$T_{fy}(s) = C(sI - A)^{-1}B.$$

A very important notion in this thesis is the rank of the transfer matrix $T_{fy}(s)$. The rank of a rational matrix is defined in the following way.

Definition 1. *A rational matrix $T(s)$ has rank r if there exists an rth-order minor of $T(s)$ that equals a nonzero rational function, while every (r+1)th-order minor, if defined, is equal to the zero function.*

Example 1. In this example we consider a water distribution network that is divided into 13 compartments, as displayed in Figure 2.4. The faults f_1, f_2, f_3 indicate at which compartments contaminated water is able to flow into the network, and the measurements y_1, y_2, y_3 indicate in which compartments the contaminant concentrations are measured. The flow velocities $a_1, a_2, ..., a_{13}$ are displayed above every compartment. In this particular example, the compartments have volumes equal to 1, i.e. $V_1 = V_2 = ... = V_{13} = 1$.

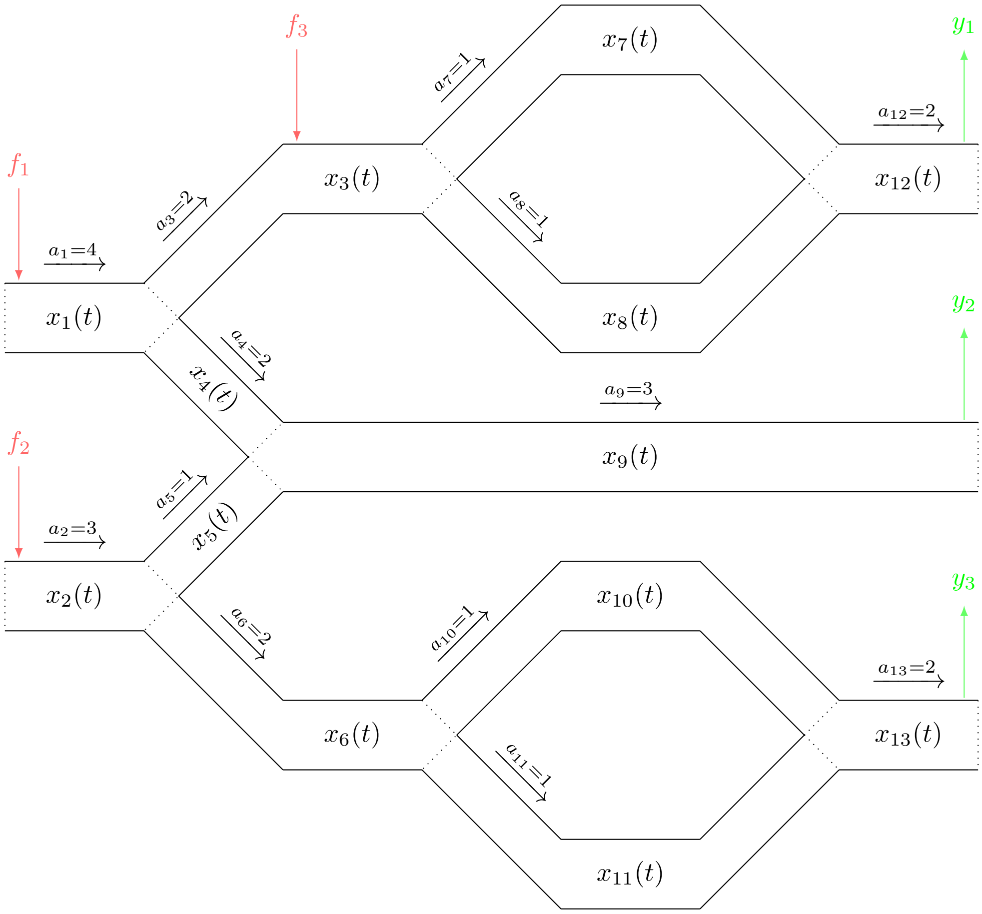

Figure 2.4: Schematic depiction of a water distribution network.

The evolution of the contaminant concentration in each compartment is modelled by:

$$\dot{\mathbf{x}}(t) = \begin{pmatrix} -4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -2 \end{pmatrix} \mathbf{x}(t) + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mathbf{f}(t) \tag{2.5}$$

$$\mathbf{y}(t) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \mathbf{x}(t),$$

where $\mathbf{x}(t) = \begin{pmatrix} x_1(t) & x_2(t) & ... & x_{13}(t) \end{pmatrix}^T$, $\mathbf{f}(t) = \begin{pmatrix} f_1(t) & f_2(t) & f_3(t) \end{pmatrix}^T$, and $\mathbf{y}(t) = \begin{pmatrix} y_1(t) & y_2(t) & y_3(t) \end{pmatrix}^T$.

Chapter 3

Problem formulation

Fault detection is concerned with indicating whether faults occur in a system, while the goal of fault isolation is to locate the faults occuring. The inflows of contaminated water in water distribution networks are regarded as faults. Hence, in the context of water distribution networks, fault detection and isolation has the following meaning: the detection of contaminated water in water distribution networks is called fault detection, while the determination of the exact inflow locations of contaminated water is called fault isolation. The goal of this thesis is to provide a necessary and sufficient condition under which fault detection and isolation by residual generation is possible in water distribution networks.

We are interested in a residual vector $\mathbf{r} \in \mathbb{R}^m$ that is sensitive to faults, and has the property that the transfer matrix from the fault vector $\mathbf{f} \in \mathbb{R}^m$ to the residual vector $\mathbf{r} \in \mathbb{R}^m$ is diagonal, with nonzero diagonal elements. The meaning of the diagonal transfer matrix is as follows: all elements of the ith row of the transfer matrix are zero, except the diagonal element. Hence, if the ith component of the residual vector is nonzero, this can solely be caused by the fault f_i. Therefore, we know f_i has occured if the ith residual is nonzero. In other words: we are able to detect and isolate faults using the residual vector.

In this thesis, we consider residual generators in the form of linear time-invariant systems that have the measurement vector $\mathbf{y}$ as input, and a residual vector $\mathbf{r}$ as output. In general, one residual generator cannot produce a residual vector with the property that the transfer matrix from faults to residuals is diagonal. Hence, an m-tuple of residual generators is used. This m-tuple of residual generators is called a bank of residual generators. The ith residual generator of this bank of residual generators produces a single residual r_i that is only sensitive to the fault f_i for $i = 1, 2, ..., m$. Combining the m residuals from the bank of m residual generators produces a residual vector $\mathbf{r} = \begin{pmatrix} r_1 & r_2 & \dots & r_m \end{pmatrix}^T$ that has the property that the transfer matrix from the fault vector $\mathbf{f}$ to the residual vector $\mathbf{r}$ is diagonal with nonzero diagonal elements. This is the case by construction, as residual r_i is only sensitive to the fault f_i for $i = 1, 2, ..., m$.

The goal of this thesis is to provide a necessary and sufficient condition for the existence of a bank of residual generators that induces a diagonal transfer matrix with nonzero diagonal elements from the fault vector to the residual vector. In the next chapter we will give a more precise, mathematical definition of a residual generator, and provide the necessary and sufficient condition for the existence of a bank of residual generators. In this thesis we will sometimes speak of "the existence of a bank of residual generators", instead of "the existence of a bank of residual generators that induces a diagonal transfer matrix from faults to residuals". This should not lead to any confusion.

Chapter 4

Fault detection and isolation by residual generation

In this chapter we describe a necessary and sufficient condition for the existence of a bank of residual generators that induces a diagonal transfer matrix from faults to residuals. First, some basic notions on rational matrices are recalled in Section 4.1. Subsequently, in Section 4.2 we provide a necessary and sufficient condition for the existence of a single residual generator that induces an upper triangular transfer matrix from faults to residuals. In Section 4.3 this result is applied to prove that there exists a bank of residual generators with the property that the transfer matrix from faults to residuals is diagonal if and only if the transfer matrix from faults to measurements has full column rank.

Note that the $p \times m$ transfer matrix $T_{fy}(s)$ cannot have full column rank if the number of measurements p is less than the number of faults m. Hence, there does not exist a bank of residual generators that induces a diagonal transfer matrix if $p < m$. In the case that $p > m$, the matrix $T_{fy}(s)$ has full column rank if and only if there exists an $m \times m$ submatrix of $T_{fy}(s)$ with full column rank. Therefore, in this chapter we assume without loss of generality that $p = m$, i.e. the number of faults equals the number of measurements.

4.1 Preliminaries

Definition 2. *A rational function $f(s) = \frac{p(s)}{q(s)}$ is called proper if the degree of $q(s)$ is greater than or equal to the degree of $p(s)$, where $p(s)$ and $q(s)$ are polynomials. A rational matrix is called proper if all its entries are proper [9].*

Definition 3. *A rational matrix $Z(s)$ is called bicausal if $Z(s)$ is proper, and $Z^{-1}(s)$ exists and is proper [3].*

Lemma 1. *Let $W(s)$ be an $m \times m$ proper, rational matrix, and let $T(s) = C(sI - A)^{-1}B$ be an $m \times m$ proper transfer matrix. There exists a constant $m \times n$ matrix F and a constant non-singular $m \times m$ matrix G such that*

$$T(s)W(s) = C(sI - A - BF)^{-1}BG$$

if and only if $W(s)$ is bicausal, and for every polynomial vector $u(s)$ such that $(sI - A)^{-1}Bu(s)$ is polynomial, the vector $W^{-1}(s)u(s)$ is polynomial as well [5].

4.2 Single residual generator

We consider a residual generator in the form of a linear time-invariant system that has the measurement vector $\mathbf{y}$ as input, and a residual vector $\mathbf{r}$ as output. A residual generator Ω satisfies the equations

$$\begin{aligned} \dot{\hat{\mathbf{x}}}(t) &= (A - KC)\hat{\mathbf{x}}(t) + K\mathbf{y}(t) \\ \mathbf{r}(t) &= Q(\mathbf{y}(t) - C\hat{\mathbf{x}}(t)), \end{aligned} \tag{4.1}$$

where $\hat{\mathbf{x}}(t) \in \mathbb{R}^n$ is the state vector of the generator, and $\mathbf{r}(t) \in \mathbb{R}^m$ is the residual vector. K is an $n \times m$ matrix and Q is a non-singular $m \times m$ matrix. The first equation of (4.1) has the form of a state observer. Hence, this type of residual generation is called observer-based residual generation [4]. Figure 4.1 displays the interconnection of the linear time-invariant system Σ associated with a water distribution network, and the residual generator Ω.

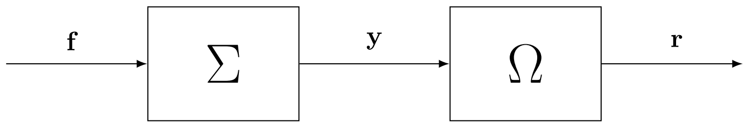

Figure 4.1: Interconnection of the LTI system Σ associated with a WDN, and the residual generator Ω.

The residual generator functions as a filter, in the sense that the input of the generator is a measurement vector with components that are sensitive to multiple faults, while the output of the generator is a residual vector that is suitable for fault isolation. As previously mentioned, the ideal residual vector has the property that the transfer matrix from faults to residuals is diagonal. However, in general a single residual generator cannot produce a residual vector with this property. More precisely, in general there does not exist a matrix K and non-singular matrix Q such that the transfer matrix from faults to residuals is diagonal. However, it turns out that under a certain condition, it is possible to find a matrix K and non-singular matrix Q such that the transfer matrix from faults to residuals is upper triangular. The goal of this section is to provide a necessary and sufficient condition for the existence of a residual generator that induces an upper triangular transfer matrix from faults to residuals. This result will be used in Section 4.3 to provide a necessary and sufficient condition for the existence of a bank of residual generators that induces a diagonal transfer matrix from faults to residuals. We first introduce some notation, after which the main result of this section is given in Theorem 2.
The error $\mathbf{e}(t)$ is defined as $\mathbf{e}(t) := \mathbf{x}(t) - \hat{\mathbf{x}}(t)$, and satisfies the equation

$$\begin{aligned}\dot{\mathbf{e}}(t) &= A\mathbf{x}(t) + B\mathbf{f}(t) - (A - KC)\hat{\mathbf{x}}(t) - K\mathbf{y}(t)\\ &= (A - KC)\mathbf{e}(t) + B\mathbf{f}(t).\end{aligned}$$

The linear time-invariant system with the error $\mathbf{e}(t)$ as state, the fault vector $\mathbf{f}(t)$ as input and the residual vector $\mathbf{r}(t)$ as output is given by:

$$\begin{aligned}\dot{\mathbf{e}}(t) &= (A - KC)\mathbf{e}(t) + B\mathbf{f}(t)\\ \mathbf{r}(t) &= QC\mathbf{e}(t).\end{aligned} \tag{4.2}$$

Therefore, the transfer matrix from the fault vector $\mathbf{f}$ to the residual vector $\mathbf{r}$ satisfies:

$$T_{fr}(s) = QC(sI - A + KC)^{-1}B. \tag{4.3}$$

Theorem 2. *Let $T_{fr}(s)$ be the transfer matrix as described in Equation (4.3). There exists an $n \times m$ matrix K and a non-singular $m \times m$ matrix Q such that $A - KC$ is stable, and*

$$T_{fr}(s) = \begin{pmatrix} t_{11}(s) & t_{12}(s) & \dots & t_{1m}(s) \\ 0 & t_{22}(s) & \dots & t_{2m}(s) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & t_{mm}(s) \end{pmatrix},$$

where $t_{ii}(s) \neq 0$ and $t_{ij}(s)$ is proper for $i, j = 1, 2, ..., m$ if and only if the rank of $T_{fy}(s)$ is equal to m.

Before we prove Theorem 2, a few remarks are in place. Note that $T_{fy}(s)$ is the transfer matrix from the fault vector to measurement vector, while $T_{fr}(s)$ denotes the transfer matrix from the fault vector to residual vector, these matrices should not be confused. The stability of the matrix $A - KC$ is crucial for fault isolation, as can be seen in the following way. From (4.2) it follows that the residual vector satisfies the equation [9]

$$\mathbf{r}(t) = QCe^{(A-KC)t}\mathbf{e}_0 + \int_0^t QCe^{(A-KC)(t-\tau)}B\mathbf{f}(\tau)d\tau, \tag{4.4}$$

where $\mathbf{e}_0$ is the error at time $t = 0$. The vector $\mathbf{e}_0$ is in general nonzero, hence if $A - KC$ is unstable the term $QCe^{(A-KC)t}\mathbf{e}_0$ becomes very large as time increases. This is undesirable for fault detection and isolation as in this case components of the residual vector can be nonzero, while no faults occur. Therefore, it is important that the matrix $A - KC$ is stable. Note that even if $A - KC$ is stable, it is not possible to immediately detect and isolate faults at time $t = 0$ using the residual vector, as the term $QCe^{(A-KC)t}\mathbf{e}_0$ must first damp out.
The following relation between the matrices $T_{fy}(s)$ and $T_{fr}(s)$ holds:

$$\begin{aligned}
Q\left(I - C(sI - A + KC)^{-1}K\right)T_{fy}(s) &= Q\left(I - C(sI - A + KC)^{-1}K\right)C(sI - A)^{-1}B \\
&= QC\left[(sI - A)^{-1}B - (sI - A + KC)^{-1}KC(sI - A)^{-1}B\right] \\
&= QC\left[(I - (sI - A + KC)^{-1}KC\right](sI - A)^{-1}B \\
&= QC\left[(sI - A + KC)^{-1}(sI - A + KC - KC)\right](sI - A)^{-1}B \\
&= QC\left[(sI - A + KC)^{-1}(sI - A)\right](sI - A)^{-1}B \\
&= QC(sI - A + KC)^{-1}B.
\end{aligned}$$

In other words,

$$Q\left(I - C(sI - A + KC)^{-1}K\right)T_{fy}(s) = T_{fr}(s).$$

We will use this relation to prove the necessity of the rank condition in Theorem 2. The proof of sufficiency of the rank condition in Theorem 2 consists of two steps: given that $T_{fy}(s)$ has full column rank, we first prove the existence of an $m \times m$ bicausal matrix $Z(s)$ such that $Z(s)T_{fy}(s)$ has the desired upper triangular structure. Subsequently we show that there exist matrices K and Q such that $Z(s)T_{fy}(s) = QC(sI - A + KC)^{-1}B = T_{fr}(s)$. It then follows that the matrix $T_{fr}(s)$ is upper triangular. We will formulate these two steps in the following two lemmas, after which the proof of Theorem 2 follows.

Lemma 3. *Let $T(s)$ be an $m \times m$, proper, rational non-singular matrix. There exists a bicausal matrix $Z(s)$ such that $Z(s)T(s) = H(s)$, where*

$$H(s) = \begin{pmatrix} \pi^{-n_1}(s) & h_{12}(s) & \dots & h_{1m}(s) \\ 0 & \pi^{-n_2}(s) & \dots & h_{2m}(s) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \pi^{-n_m}(s) \end{pmatrix}.$$

Here $\pi(s) = s + a$, with $a \in \mathbb{R}$. The rational function $h_{ij}(s) = \gamma(s)\pi^{-n_{ij}}(s)$ is proper for $1 \leq i < j \leq m$, with $\gamma(s)$ a polynomial. The integers n_i and n_{ij} are positive for $1 \leq i < j \leq m$, and $h_{ij}(s) = 0$ for $1 \leq j < i \leq m$ [7].

$H(s)$ is called the π-Hermite form of $T(s)$. The π-Hermite form is uniquely determined by the constant a, and the matrix $T(s)$ [3, 7]. The proof of Lemma 3 can be found in [7].

Lemma 4. *Let $Z(s)$ be an $m \times m$ proper, rational matrix, and let $T_{fy}(s) = C(sI - A)^{-1}B$ be an $m \times m$ proper transfer matrix. There exists a constant matrix K and a constant, non-singular matrix Q such that*

$$Z(s)T_{fy}(s) = QC(sI - A + KC)^{-1}B,$$

if and only if $Z(s)$ is bicausal, and for all polynomial row vectors $v(s)$ such that $v(s)C(sI - A)^{-1}$ is polynomial, the vector $v(s)Z^{-1}(s)$ is polynomial as well.

Notice the similarities between Lemma 4 and Lemma 1. In Lemma 4 the transfer matrix is multiplied from the left by a bicausal matrix, while in Lemma 1 the transfer matrix is multiplied from the right by a bicausal matrix. In the proof of Lemma 4 we will use Lemma 1.

Proof of Lemma 4.
Sufficiency: We assume that $Z(s)$ is bicausal, and for all polynomial row vectors $v(s)$ such that $v(s)C(sI - A)^{-1}$ is polynomial, the vector $v(s)Z^{-1}(s)$ is polynomial as well. As $Z(s)$ is bicausal, $Z^T(s)$ is bicausal as well. Furthermore, if $v^T(s)$ is a polynomial vector with the property that $(sI - A^T)^{-1}C^Tv^T(s)$ is polynomial, it follows that $Z^{-T}(s)v^T(s)$ is polynomial. Let $\bar{A} := A^T$, $\bar{B} := C^T$, $\bar{C} := B^T$, $W(s) :=$

$Z^T(s)$ and $u(s) := v^T(s)$. Note that this means that $W(s)$ is bicausal, and if $u(s)$ is a polynomial vector with the property that $(sI-\bar{A})^{-1}\bar{B}u(s)$ is polynomial, it follows that $W^{-1}(s)u(s)$ is polynomial. Hence, Lemma [5] states that there exists a constant matrix F and a constant non-singular matrix G such that

$$\bar{C}(sI-\bar{A})^{-1}\bar{B}\cdot W(s) = \bar{C}(sI-\bar{A}-\bar{B}F)^{-1}\bar{B}G.$$

By transposition this equation becomes:

$$W^T(s)\cdot \bar{B}^T(sI-\bar{A}^T)^{-1}\bar{C}^T = G^T\bar{B}^T(sI-\bar{A}^T-F^T\bar{B}^T)^{-1}\bar{C}^T \tag{4.5}$$

As $\bar{A}^T = A$, $\bar{B}^T = C$, $\bar{C}^T = B$ and $W^T(s) = Z(s)$, Equation (4.5) can be rewritten in the following way:

$$Z(s)\cdot C(sI-A)^{-1}B = G^TC(sI-A-F^TC)^{-1}B$$

We define the non-singular constant matrix $Q := G^T$ and the constant matrix $K := -F^T$. This yields:

$$Z(s)\cdot C(sI-A)^{-1}B = QC(sI-A+KC)^{-1}B$$

Therefore there exists a constant matrix K and a constant non-singular matrix Q such that

$$Z(s)T_{fy}(s) = QC(sI-A+KC)^{-1}B.$$

Necessity:
As $Z(s)T_{fy}(s) = QC(sI-A+KC)^{-1}B$, the following equality holds:

$$T_{fy}^T(s)Z^T(s) = B^T(sI-A^T+C^TK^T)^{-1}C^TQ^T.$$

Using the same reasoning as in the first part of the proof, it follows from Lemma 1 that $Z^T(s)$ is bicausal, and for every polynomial vector $v^T(s)$ with the property that $(sI-A^T)^{-1}C^Tv^T(s)$ is polynomial, the vector $Z^{-T}(s)v^T(s)$ is polynomial as well. It follows that $Z(s)$ is bicausal, and for all polynomial row vectors $v(s)$ with the property that $v(s)C(sI-A)^{-1}$ is polynomial, the vector $v(s)Z^{-1}(s)$ is polynomial as well.

□

With the aforementioned lemmas in mind, we are now able to prove Theorem 2.

Proof of Theorem 2.
Necessity: We assume that there exists a constant matrix K and constant non-singular matrix Q such that

$$T_{fr}(s) = \begin{pmatrix} t_{11}(s) & t_{12}(s) & \dots & t_{1m}(s) \\ 0 & t_{22}(s) & \dots & t_{2m}(s) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & t_{mm}(s) \end{pmatrix},$$

where $t_{ii}(s) \neq 0$ and $t_{ij}(s)$ is proper for $i,j = 1,2,...,m$. It follows that $\det[T_{fr}(s)] = t_{11}(s)\cdot t_{22}(s)\cdot ...\cdot t_{mm}(s) \neq 0$. Hence, the $m\times m$ matrix $T_{fr}(s)$ has rank m. We have previously seen that the following equality holds:

$$Q\left(I - C(sI-A+KC)^{-1}K\right)T_{fy}(s) = QC(sI-A+KC)^{-1}B = T_{fr}(s). \tag{4.6}$$

We now assume that the rank of the $m\times m$ matrix $T_{fy}(s)$ is less than m. It follows from Equation (4.6) that the rank of $T_{fr}(s)$ is less than m. This is a contradiction, hence the rank of $T_{fy}(s)$ is equal to m.
Sufficiency: We assume that the rank of $T_{fy}(s)$ is equal to m. Let a be a positive constant. It follows from Lemma 3 that there exists a bicausal matrix $Z(s)$ such that $Z(s)T_{fy}(s) = H(s)$ has the desired upper triangular structure. We now want to prove that there exist matrices K and Q such that $Z(s)T_{fy}(s) = QC(sI-A+KC)^{-1}B = T_{fr}(s)$. Therefore, we will verify the conditions of Lemma 4. We already know that $Z(s)$ is bicausal. Let $v(s)$ be a polynomial row vector with the property that $v(s)C(sI-A)^{-1}$ is polynomial. It follows that $v(s)C(sI-A)^{-1}B = v(s)Z^{-1}(s)H(s)$ is polynomial. The matrix $H(s)$ is proper as all its elements are proper (see Lemma 4). Furthermore, $H(s)$ is non-singular, as its determinant is the product of its diagonal entries, which is nonzero as a rational function.

We now assume that $v(s)Z^{-1}(s)$ is not polynomial. This means that the product $v(s)Z^{-1}(s)H(s)$ is not polynomial, because $H(s)$ is proper and non-singular. This is a contradiction, hence $v(s)Z^{-1}(s)$ is polynomial. By Lemma 4 there exists a constant matrix K and a constant, non-singular matrix Q such that

$$Z(s)T_{fy}(s) = QC(sI - A + KC)^{-1}B = T_{fr}(s).$$

But as $Z(s)T_{fy}(s) = H(s)$, the following equality holds:

$$T_{fr}(s) = H(s) = \begin{pmatrix} \pi^{-n_1}(s) & h_{12}(s) & \dots & h_{1m}(s) \\ 0 & \pi^{-n_2}(s) & \dots & h_{2m}(s) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \pi^{-n_m}(s) \end{pmatrix},$$

where $\pi(s) = s + a$, and $h_{ij}(s) = \gamma(s)\pi^{-n_{ij}}(s)$ is a proper, rational function for $1 \leq i < j \leq m$, with $\gamma(s)$ a polynomial. The integers n_i and n_{ij} are positive for $1 \leq i < j \leq m$, and $h_{ij}(s) = 0$ for $1 \leq j < i \leq m$. Hence, there exists a constant $n \times m$ matrix K and a constant non-singular $m \times m$ matrix Q such that

$$T_{fr}(s) = \begin{pmatrix} t_{11}(s) & t_{12}(s) & \dots & t_{1m}(s) \\ 0 & t_{22}(s) & \dots & t_{2m}(s) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & t_{mm}(s) \end{pmatrix},$$

where $t_{ii}(s) \neq 0$ and $t_{ij}(s)$ is proper for $i, j = 1, 2, ..., m$. The stability of the matrix $A - KC$ can be proven in a similar way as in [3]. □

4.3 Bank of residual generators

In this section we consider a set of m residual generators, called a bank of residual generators. The goal of this section is to provide a necessary and sufficient condition for the existence of a bank of residual generators that induces a diagonal transfer matrix from the fault vector $\mathbf{f}$ to the residual vector $\mathbf{r}$. We will prove that there exists a bank of residual generators if and only if the transfer matrix from the fault vector $\mathbf{f}$ to the measurement vector $\mathbf{y}$ has full column rank. In the previous section we have proven that there exists a residual generator that induces an upper triangular transfer matrix from faults to residuals if and only if the transfer matrix from faults to measurements has full column rank. Note that because of this upper triangular structure, the mth residual is sensitive to the mth fault, but insensitive to all other faults. This observation will be of key importance in this section. We proceed in the following manner: first we will define m systems $\Sigma^1, \Sigma^2, ..., \Sigma^m$, where the components of the fault vector of system Σ^i are permutated in such a way that the ith fault $f_i(t)$ is moved to the mth position of the fault vector for $i = 1, 2, ..., m$. For each of the m systems, a residual generator is designed. This produces m residual vectors $\mathbf{r}^1, \mathbf{r}^2, ..., \mathbf{r}^m$. Subsequently, we define the vector $\mathbf{r}$ as the vector composed of the mth entries of the vectors $\mathbf{r}^1, \mathbf{r}^2, ..., \mathbf{r}^m$. Finally, we prove that this residual vector $\mathbf{r}$ can be generated in such a way that the transfer matrix from $\mathbf{f}$ to $\mathbf{r}$ is diagonal with nonzero diagonal elements if and only if the transfer matrix from $\mathbf{f}$ to $\mathbf{y}$ has full column rank.

Let $\mathbf{f}(t) = \begin{pmatrix} f_1(t) & f_2(t) & \dots & f_m(t) \end{pmatrix}^T$ denote the fault vector. We introduce for $i = 1, 2, ..., m$ the system Σ^i, given by

$$\begin{aligned} \dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + B^i\mathbf{f}^i(t) \\ \mathbf{y}(t) &= C\mathbf{x}(t), \end{aligned}$$

where $\mathbf{f}^i(t) := \begin{pmatrix} f_1(t) & \dots & f_{i-1}(t) & f_{i+1}(t) & \dots & f_m(t) & f_i(t) \end{pmatrix}^T$ is the vector formed by moving the ith component of the fault vector to the mth position, and B^i is an $n \times m$ matrix formed by switching the columns of B in the following way: $B^i := \begin{pmatrix} \mathbf{b}_1 & \dots & \mathbf{b}_{i-1} & \mathbf{b}_{i+1} & \dots & \mathbf{b}_m & \mathbf{b}_i \end{pmatrix}$, where $\mathbf{b}_j$ is the jth column of B for $j = 1, 2, ..., m$. Note that

$$\begin{aligned} B^i\mathbf{f}^i(t) &= \begin{pmatrix} \mathbf{b}_1 & \dots & \mathbf{b}_{i-1} & \mathbf{b}_{i+1} & \dots & \mathbf{b}_m & \mathbf{b}_i \end{pmatrix} \begin{pmatrix} f_1(t) & \dots & f_{i-1}(t) & f_{i+1}(t) & \dots & f_m(t) & f_i(t) \end{pmatrix}^T \\ &= \left(\mathbf{b}_1 f_1(t) + \dots + \mathbf{b}_{i-1} f_{i-1}(t) + \mathbf{b}_{i+1} f_{i+1}(t) + \dots + \mathbf{b}_m f_m(t) + \mathbf{b}_i f_i(t)\right) \\ &= \left(\mathbf{b}_1 f_1(t) + \dots + \mathbf{b}_2 f_2(t) + \dots + \mathbf{b}_m f_m(t)\right) \\ &= B\mathbf{f}(t). \end{aligned}$$

Hence, the only difference between the system Σ and the systems $\Sigma^1, \Sigma^2, ..., \Sigma^m$ is that the fault vector in each of the systems $\Sigma^1, \Sigma^2, ..., \Sigma^m$ is permutated. For $i = 1, 2, ..., m$, the ith residual generator of the bank of m residual generators has the form

$$\begin{aligned} \dot{\hat{\mathbf{x}}}^i(t) &= (A - K^i C)\hat{\mathbf{x}}^i(t) + K^i \mathbf{y}(t) \\ \mathbf{r}^i(t) &= Q^i \left(\mathbf{y}(t) - C\hat{\mathbf{x}}^i(t)\right), \end{aligned} \tag{4.7}$$

where $\hat{\mathbf{x}}^i(t) \in \mathbb{R}^n$ is the state vector of the ith residual generator, $\mathbf{r}^i(t) = \begin{pmatrix} r_1^i(t) & r_2^i(t) & \dots & r_m^i(t) \end{pmatrix}^T$ is the ith residual vector, K^i is an $n \times m$ matrix and Q^i is a non-singular $m \times m$ matrix. For $i = 1, 2, ..., m$, the ith error $\mathbf{e}^i(t) := \mathbf{x}(t) - \hat{\mathbf{x}}^i(t)$ satisfies

$$\begin{aligned} \dot{\mathbf{e}}^i(t) &= (A - K^i C)\mathbf{e}^i(t) + B^i \mathbf{f}^i(t) \\ \mathbf{r}^i(t) &= Q^i C \mathbf{e}^i(t). \end{aligned}$$

Hence, for $i = 1, 2, ..., m$ the transfer matrix from $\mathbf{f}^i$ to $\mathbf{r}^i$ is given by

$$T_{fr}^i(s) = Q^i C(sI - A + K^i C)^{-1} B^i. \tag{4.8}$$

Note that it follows from Equation (4.8) that for $i = 1, 2, ..., m$:

$$r_m^i(s) = Q_m^i C(sI - A + K^i C)^{-1} B^i \mathbf{f}^i(s), \tag{4.9}$$

where $r_m^i(s)$ is the mth component of the vector $\mathbf{r}^i(s)$ and Q_m^i is the mth row of Q^i. Here $\mathbf{r}^i(s)$ and $\mathbf{f}^i(s)$ are the Laplace transforms of $\mathbf{r}^i(t)$ and $\mathbf{f}^i(t)$ respectively. As $B^i \mathbf{f}^i(s) = B\mathbf{f}(s)$, Equation (4.9) can be rewritten as

$$r_m^i(s) = Q_m^i C(sI - A + K^i C)^{-1} B\mathbf{f}(s), \tag{4.10}$$

for $i = 1, 2, ..., m$. Note that the m residual generators produce a total of m residual vectors. We now define the residual vector $\mathbf{r}$ as follows: $\mathbf{r}(t) := \begin{pmatrix} r_m^1(t) & r_m^2(t) & ... & r_m^m(t) \end{pmatrix}^T$, i.e. the residual vector $\mathbf{r}$ contains the bottom entries of the vectors $\mathbf{r}^1, \mathbf{r}^2, ..., \mathbf{r}^m$. Let $T_{fr}(s)$ denote the transfer matrix from $\mathbf{f}$ to $\mathbf{r}$. From the definition of $\mathbf{r}$, and from Equation (4.10) it follows that $T_{fr}(s)$ is given by:

$$T_{fr}(s) = \begin{pmatrix} Q_m^1 C(sI - A + K^1 C)^{-1} B \\ Q_m^2 C(sI - A + K^2 C)^{-1} B \\ \vdots \\ Q_m^m C(sI - A + K^m C)^{-1} B \end{pmatrix}, \tag{4.11}$$

where Q_m^i is the mth row of Q^i for $i = 1, 2, ..., m$.

Theorem 5. *Let $T_{fr}(s)$ be the transfer matrix as described in Equation (4.11). For $i = 1, 2, ..., m$ there exists a constant $n \times m$ matrix K^i and constant $1 \times m$ matrix Q_m^i such that $A - K^i C$ is stable, and*

$$T_{fr}(s) = \begin{pmatrix} t_{11}(s) & 0 & \dots & 0 \\ 0 & t_{22}(s) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & t_{mm}(s) \end{pmatrix},$$

where $t_{jj}(s)$ is nonzero and proper for $j = 1, 2, ..., m$ if and only if the rank of $T_{fy}(s)$ is equal to m.

Note that we are able to detect and isolate faults using a residual vector with the property that the transfer matrix from faults to residuals is diagonal. This can be seen in the following way: the fault f_i only influences the residual r_i for $i = 1, 2, ..., m$. Hence, if the residual r_i has a value above a certain threshold, this means that the fault f_i occurs. As we have previously seen, this type of reasoning is only possible after the initial errors of the residual generators have damped out.

Proof of Theorem 5.
Necessity: As $T_{fr}(s)$ is diagonal with nonzero diagonal elements, the rank of $T_{fr}(s)$ is equal to m. Similar to the proof of Theorem 2, the following equality holds

$$\begin{pmatrix} Q_m^1\left(I-C(sI-A+K^1C)^{-1}K^1\right) \\ Q_m^2\left(I-C(sI-A+K^2C)^{-1}K^2\right) \\ \vdots \\ Q_m^m\left(I-C(sI-A+K^mC)^{-1}K^m\right) \end{pmatrix} C(sI-A)^{-1}B = \begin{pmatrix} Q_m^1C(sI-A+K^1C)^{-1}B \\ Q_m^2C(sI-A+K^2C)^{-1}B \\ \vdots \\ Q_m^mC(sI-A+K^mC)^{-1}B \end{pmatrix} = T_{fr}(s).$$

Hence, if the rank of $T_{fy}(s) = C(sI-A)^{-1}B$ is less than m, the rank of $T_{fr}(s)$ is less than m, which is a contradiction. Therefore, the rank of $T_{fy}(s)$ is equal to m.
Sufficiency: The transfer matrix of system Σ^i from $\mathbf{f}^i$ to $\mathbf{y}$ is given by

$$T_{fy}^i(s) = \begin{pmatrix}\mathbf{t}_1(s) & \dots & \mathbf{t}_{i-1}(s) & \mathbf{t}_{i+1}(s) & \dots & \mathbf{t}_m(s) & \mathbf{t}_i(s)\end{pmatrix},$$

where $\mathbf{t}_j(s)$ is the jth column of $T_{fy}(s)$ for $j = 1, 2, ..., m$. As the rank of $T_{fy}(s)$ is equal to m, the rank of $T_{fy}^i(s)$ equals m for $i = 1, 2, ..., m$. It follows from Theorem 2 that there exist matrices K^i and Q^i such that $A - K^iC$ is stable, and $T_{fr}^i(s)$ is upper triangular, where $T_{fr}^i(s)$ is the transfer matrix from $\mathbf{f}^i$ to $\mathbf{r}^i$ for $i = 1, 2, ..., m$. Notice that because of this triangular structure, the bottom entry $r_m^i(s)$ of the residual vector is only influenced by the fault $f_i(s)$ for $i = 1, 2, ..., m$. Hence,

$$r_m^i(s) = Q_m^iC(sI-A+K^iC)^{-1}B^i\mathbf{f}^i(s) = \begin{pmatrix}0 & \dots & 0 & t_{ii}(s)\end{pmatrix}\mathbf{f}^i(s),$$

where $t_{ii}(s)$ is nonzero and proper and Q_m^i is the mth row of Q^i for $i = 1, 2, ..., m$. As $B^i\mathbf{f}^i(s) = B\mathbf{f}(s)$ we conclude that

$$r_m^i(s) = Q_m^iC(sI-A+K^iC)^{-1}B\mathbf{f}(s) = \begin{pmatrix}0 & \dots & 0 & t_{ii}(s) & 0 & \dots & 0\end{pmatrix}\mathbf{f}(s).$$

It follows from the definition of $\mathbf{r}(s)$ that

$$\mathbf{r}(s) = \begin{pmatrix} r_m^1(s) \\ r_m^2(s) \\ \vdots \\ r_m^m(s) \end{pmatrix} = \begin{pmatrix} t_{11}(s) & 0 & \dots & 0 \\ 0 & t_{22}(s) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & t_{mm}(s) \end{pmatrix}\mathbf{f}(s).$$

Therefore the transfer matrix $T_{fr}(s)$ satisfies the equation

$$T_{fr}(s) = \begin{pmatrix} Q_m^1C(sI-A+K^1C)^{-1}B \\ Q_m^2C(sI-A+K^2C)^{-1}B \\ \vdots \\ Q_m^mC(sI-A+K^mC)^{-1}B \end{pmatrix} = \begin{pmatrix} t_{11}(s) & 0 & \dots & 0 \\ 0 & t_{22}(s) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & t_{mm}(s) \end{pmatrix}$$

where $t_{jj}(s)$ is nonzero and proper for $j = 1, 2, ..., m$, which proves the theorem. □

Chapter 5

A graph theoretic characterization of the rank of the transfer matrix

In the previous chapter we have seen that the rank of the transfer matrix from $\mathbf{f}$ to $\mathbf{y}$ is of key importance when verifying the existence of a bank of residual generators. In this chapter, we describe the graph associated with a water distribution network, and the relation between this graph and the rank of the transfer matrix from $\mathbf{f}$ to $\mathbf{y}$. This relation provides a second approach to determine the existence of a bank of residual generators. We first state some necessary graph theoretic preliminaries in Section 5.1, whereupon we state Theorem 11, which is the main result of this chapter.

5.1 Preliminaries

We consider a directed graph $G = (V, E)$ that consists of the set of vertices V, and set of directed edges E. The graph $G = (V, E)$ is called weighted if every edge of G has been assigned a weight [2].
A path is a sequence of connected edges. We say there is a path between the vertices $v_1 \in V$ and $v_k \in V$ if there exist vertices $v_2, v_3, .., v_{k-1} \in V$ such that $(v_i, v_{i+1}) \in E$ for $i = 1, 2, ..., k-1$. Two paths are called (vertex) disjoint if they have no vertex in common [10].
A path between the vertices v_1 and v_k is closed if $v_1 = v_k$. Furthermore, a path that consists of the edges (v_i, v_{i+1}) for $i = 1, 2, .., k-1$ is called simple if the vertices $v_1, v_2, ..., v_{k-1}$ are distinct. A closed, simple path is called a cycle. A spanning cycle family of a graph G is a collection of cycles, with the property that each vertex in V appears in exactly one cycle [10].
A graph $G_s = (V_s, E_s)$ is a subgraph of the graph $G = (V, E)$ if the sets V_s and E_s are subsets of V and E respectively [2].
Let $w(i, j)$ denote the weight associated with the edge (i, j) of the graph G. If G_s is a subgraph of G, then $w(G_s)$ is defined as:

$$w(G_s) := \prod w(i, j),$$

where the product is taken over all edges (i, j) in G_s [2].

5.1.1 System graph and Coates graph

In this section we describe two types of weighted, directed graphs. The first type is the graph $G_\Sigma = (V_\Sigma, E_\Sigma)$, associated with the system Σ (see Equation (2.4)). This graph is composed of the set of vertices V_Σ, and the set of edges E_Σ. The set of vertices is given by $V_\Sigma = X \cup F \cup Y$, where $X := \{x_1, x_2, ..., x_n\}$ is the set of state vertices, $F := \{f_1, f_2, ..., f_m\}$ is the fault vertices set, and $Y := \{y_1, y_2, ..., y_p\}$ is the set of measurement vertices. The set of edges is defined as $E_\Sigma := E_x \cup E_f \cup E_y$, where $E_x := \{(x_j, x_i) | a_{i,j} \neq 0\}$, $E_f := \{(f_j, x_i) | b_{i,j} \neq 0\}$, and $E_y := \{(x_j, y_i) | c_{i,j} \neq 0\}$. Here $a_{i,j}$, $b_{i,j}$ and $c_{i,j}$ denote the (i, j)th entries of the matrices A, B and C respectively. Furthermore, the edges of the graph G_Σ are weighted in the following way: every edge $(x_j, x_i) \in E_x$ contains the weight $a_{i,j}$. Similarly, each edge $(f_j, x_i) \in E_f$ is weighted by $b_{i,j}$, and $(x_j, y_i) \in E_y$ is weighted by $c_{i,j}$.

Example 2. Consider the system from Example 1 (see Equation 2.5). The system graph associated with this system is displayed in Figure 5.1.

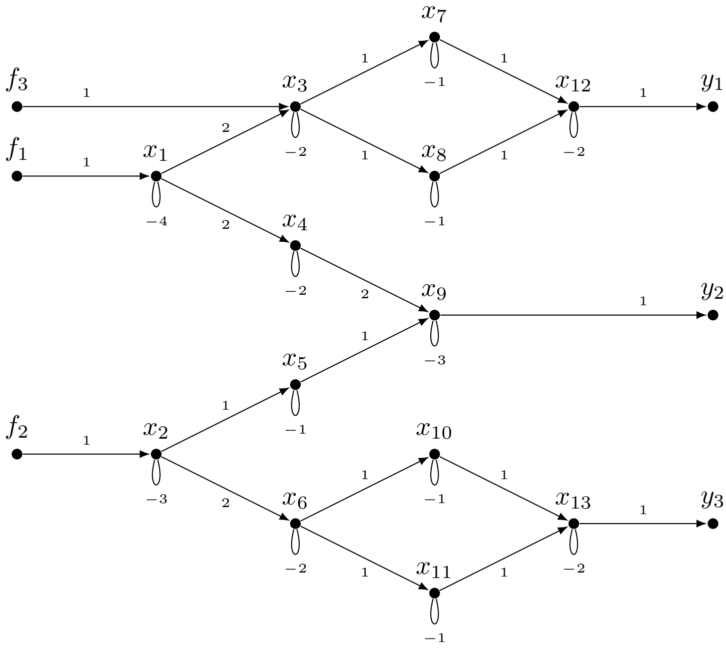

Figure 5.1: Weighted directed graph associated with the LTI system as given in Example 1.

The second type of weighted, directed graph we consider is called the Coates graph. Let $M \in \mathbb{R}^{q \times q}$ be a real matrix. The Coates graph of M, denoted by $G_M = (V_M, E_M)$, consists of the set V_M of q vertices, and the set E_M of edges. Let the set of vertices be denoted by $V_M = \{v_1, v_2, ..., v_q\}$. The set of edges is defined as $E_M := \{(v_j, v_i) | m_{i,j} \neq 0\}$, where $m_{i,j}$ indicates the (i, j)th element of the matrix M. Similar to the graph G_Σ, every edge (v_j, v_i) of the Coates graph is weighted by $m_{i,j}$. There exists a relation between the determinant of a real square matrix M and the Coates graph of M. This result is formulated in Theorem 6.

Theorem 6. *Let G_M be the Coates graph associated with the real, $q \times q$ matrix M. Then*

$$det(M) = (-1)^q \sum_{\mathcal{C}} (-1)^{N_c} w(c),$$

where $\mathcal{C}$ is the set of all spanning cycle families of G_M, $c \in \mathcal{C}$ is a spanning cycle family of G_M and N_c denotes the number of cycles in c. If G_M contains no spanning cycle families, it holds that $det(M) = 0$ [2].

In order to clarify Theorem 6, we consider the following example.

Example 3. We consider the real, square matrix M:

$$M = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

The Coates graph $G_M = (V_M, E_M)$ associated with the matrix M consists of the set of vertices $V_M = \{v_1, v_2\}$, and the set of edges $E_M = \{(v_1, v_1), (v_2, v_2), (v_1, v_2), (v_2, v_1)\}$, where the edges (v_1, v_1) and (v_2, v_2) are weighted by the number 2, and the edges (v_1, v_2) and (v_2, v_1) are weighted by 1. The Coates graph G_M is displayed in Figure 5.2.
The determinant of M equals 3. On the other hand we see that there are 2 spanning cycle families in G_M, hence

$$(-1)^q \sum_{\mathcal{C}} (-1)^{N_c} w(c) = (-1)^2 \left[(-1)^1 \cdot 1 \cdot 1 + (-1)^2 \cdot 2 \cdot 2\right] = 3.$$

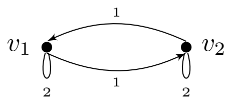

Figure 5.2: Coates graph of the matrix M.

The subsequent two corrolaries follow immediately from Theorem 6.

Corollary 7. *The determinant of M is nonzero if there exists exactly one spanning cycle family of the graph G_M.*

Corollary 8. *If the determinant of M is nonzero, there exists at least one spanning cycle family of the graph G_M.*

5.1.2 Relation between Coates graph and System graph

In this section we consider a system Σ with an equal number of faults and measurements ($m = p$), and the associated graph G_Σ. Moreover, we define the real, $(n+m) \times (n+m)$ matrix M as follows:

$$M := \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}.$$

Recall that the graph G_Σ is composed of

$$\begin{aligned} V_\Sigma &= \{x_1, x_2, ..., x_n, f_1, f_2, ..., f_m, y_1, y_2, ..., y_m\} \\ E_\Sigma &= \{(x_j, x_i) | a_{i,j} \neq 0 \text{ for } 1 \leq i, j \leq n\} \\ &\quad \cup \{(f_j, x_i) | b_{i,j} \neq 0 \text{ for } 1 \leq i \leq n, 1 \leq j \leq m\} \\ &\quad \cup \{(x_j, y_i) | c_{i,j} \neq 0 \text{ for } 1 \leq i \leq m, 1 \leq j \leq n\}. \end{aligned}$$

For this particular matrix M, the Coates graph G_M consists of

$$\begin{aligned} V_M &= \{v_1, v_2, ..., v_n, v_{n+1}, ..., v_{n+m}\} \\ E_M &= \{(v_j, v_i) | a_{i,j} \neq 0 \text{ for } 1 \leq i, j \leq n\} \\ &\quad \cup \{(v_{j+n}, v_i) | b_{i,j} \neq 0 \text{ for } 1 \leq i \leq n, 1 \leq j \leq m\} \\ &\quad \cup \{(v_j, v_{i+n}) | c_{i,j} \neq 0 \text{ for } 1 \leq i \leq m, 1 \leq j \leq n\}. \end{aligned}$$

The Coates graph G_M can be obtained from the graph G_Σ in the following way:

- Define the vertices $v_i := x_i$ for $i = 1, 2, ..., n$.
- Merge the vertices f_j and y_j, and rename this vertex to v_{j+n} for $j = 1, 2, ..., m$.

Example 4. In this example we consider the simple linear time-invariant system Σ:

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} f_1(t)$$
$$y_1(t) = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}.$$

The matrix M is defined as

$$M := \begin{pmatrix} A & B \\ C & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

The system graph $G_\Sigma = (V_\Sigma, E_\Sigma)$ consists of the set of vertices $V_\Sigma = \{x_1, x_2\} \cup \{f_1\} \cup \{y_1\}$, and the set of edges $E_\Sigma = \{(f_1, x_1), (x_1, x_2), (x_1, x_1), (x_2, x_2), (x_2, y_1)\}$, where the edges (x_1, x_1) and (x_2, x_2) contain the weight -1, and the edges (f_1, x_1), (x_1, x_2) and (x_2, y_1) contain the weight 1. The Coates graph $G_M = (V_M, E_M)$ consists of the the set of vertices $V_M = \{v_1, v_2, v_3\}$ and the set of edges $E_M =$

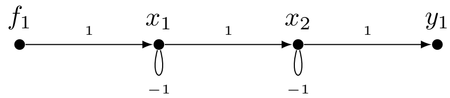

Figure 5.3: Graph G_Σ associated with the system Σ.

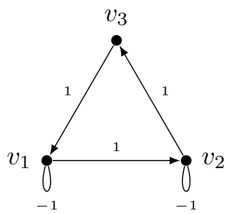

Figure 5.4: Coates graph G_M associated with the matrix M.

$\{(v_3, v_1), (v_1, v_2), (v_2, v_3), (v_2, v_2), (v_1, v_1)\}$, where the edges (v_3, v_1), (v_1, v_2) and (v_2, v_3) are weighted by the number 1, and the edges (v_2, v_2) and (v_1, v_1) contain the weight -1. Figures 5.3 and 5.4 display the graphs G_Σ and G_M respectively.
Notice that the Coates graph G_M can be obtained from G_Σ in the following way:

- Define the vertices $v_1 := x_1$ and $v_2 := x_2$.
- Merge the vertex f_1 with y_1 and rename this vertex to v_3.

There exists a relation between the number of vertices contained in a cycle in the Coates graph G_M, and the number of disjoint paths from F to Y in the system graph G_Σ. This relation is formulated in Lemma 9.

Lemma 9. *If there exists a cycle in G_M that contains exactly k vertices of the set $\{v_{n+1}, v_{n+2}, ..., v_{n+m}\}$, then the graph G_Σ contains k disjoint paths from F to Y [10].*

Proof. Without loss of generality we may assume that the k vertices are the vertices $v_{n+1}, v_{n+2}, ..., v_{n+k}$. Furthermore, we may assume that in the aforementioned cycle, there are paths between v_{i+n} and v_{i+n+1} for $i = 1, 2, ..., k-1$, and between v_{n+k} and v_{n+1} that contain no other vertices from the set $\{v_{n+1}, v_{n+2}, ..., v_{n+m}\}$.
Now suppose that there is no disjoint collection of paths from f_i to y_{i+1} for $i = 1, 2, ..., k-1$ and from f_k to y_1. This implies that an arbitrary closed path that connects the vertices $v_{n+1}, v_{n+2}, ..., v_{n+k}$ passes twice through a vertex in V_M. But this means that there does not exist a cycle that contains the vertices $v_{n+1}, v_{n+2}, ..., v_{n+k}$. This is a contradiction, and therefore there exists a collection of paths between f_i and y_{i+1} for $i = 1, 2, ..., k-1$ and between f_k and y_1 that is disjoint. Hence, there are k disjoint paths from F to Y in G_Σ. □

Corollary 10. *If there exists a cycle family in G_M that contains all vertices of the set $\{v_{n+1}, v_{n+2}, ..., v_{n+m}\}$, then there are m disjoint paths from F to Y in G_Σ [10].*

5.2 A graph theoretic characterization of the rank of the transfer matrix

We now arrive at the main result of this chapter. Recall that in the previous chapter we have proven that there exists a bank of residual generators that generates a residual vector with the property that the transfer matrix from faults to residuals is diagonal if and only if the transfer matrix from $\mathbf{f}$ to $\mathbf{y}$ has full column rank. Our following theorem provides a graph theoretic characterization of the rank of the transfer matrix from $\mathbf{f}$ to $\mathbf{y}$. This theorem allows us to verify the existence of a bank of residual generators by analyzing the system graph G_Σ, associated with a given water distribution network.

Theorem 11. *Given a linear time-invariant system Σ associated with a water distribution network. Let G_Σ denote the system graph and let $T_{fy}(s)$ be the transfer matrix from faults to measurements. The maximum number of disjoint paths from the set of fault vertices F to the set of measurement vertices Y in the graph G_Σ is equal to the rank of $T_{fy}(s)$.*

Example 5. Once again, we consider the water distribution network as described in Example 1. The transfer matrix of system (2.5), evaluated in $s = 0$ is given by

$$T_{fy}(0) = -\frac{1}{36}\begin{pmatrix} 9 & 0 & 18 \\ 6 & 4 & 0 \\ 0 & 12 & 0 \end{pmatrix},$$

and $\operatorname{rank}[T_{fy}(0)] = 3$. Hence, there exists a third-order minor of $T_{fy}(s)$ that equals a nonzero rational function. It follows that $\operatorname{rank}[T_{fy}(s)] = 3$. We observe that there are also 3 disjoint paths from F to Y in the graph associated with system (2.5). These disjoint paths are indicated by dashed arrows in Figure 5.5.

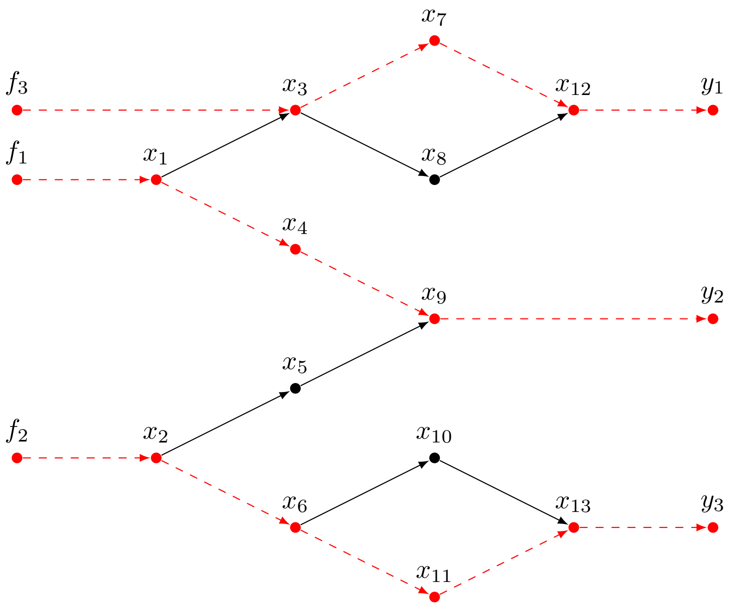

Figure 5.5: Disjoint paths in the weighted directed graph associated with system (2.5).

Proof of Theorem 11. Let l denote the maximum number of disjoint paths from F to Y in the graph G_Σ, and let r be the rank of $T_{fy}(s)$. This proof consists of two parts. First we prove that $l \geq r$, subsequently we prove the converse: $l \leq r$.

Proof ($\mathbf{l \geq r}$):

The first part of the proof of Theorem 11 is similar to the proof given by J.W. van der Woude in [10]. As the rank of $T_{fy}(s)$ is r, there exists an rth-order minor of $T_{fy}(s)$ that equals a nonzero rational function. Without loss of generality, we assume that this minor has the form

$$\det\left[\bar{C}(sI - A)^{-1}\bar{B}\right],$$

where $\bar{B}$ is composed of r columns of B, and $\bar{C}$ is composed of r rows of C. As the rth-order minor is a nonzero rational function, there exists an $\bar{s} \in \mathbb{R}$ such that

$$\det\left[\bar{C}(\bar{s}I - A)^{-1}\bar{B}\right] = q \in \mathbb{R} \setminus \{0\},$$

and

$$\det\left[A - \bar{s}I\right] = p \in \mathbb{R} \setminus \{0\}.$$

We define the square matrix $M(s)$ as

$$M(s) := \begin{pmatrix} A - sI & \bar{B} \\ \bar{C} & 0 \end{pmatrix},$$

and note that for all $s \notin \sigma(A)$

$$M(s) = \begin{pmatrix} I & 0 \\ \bar{C}(A-sI)^{-1} & I \end{pmatrix} \begin{pmatrix} A - sI & 0 \\ 0 & \bar{C}(sI-A)^{-1}\bar{B} \end{pmatrix} \begin{pmatrix} I & (A-sI)^{-1}\bar{B} \\ 0 & I \end{pmatrix}.$$

The determinant of $M(\bar{s})$ is nonzero because

$$\begin{aligned} \det\left[M(\bar{s})\right] &= 1 \cdot \det\left[\begin{pmatrix} A - \bar{s}I & 0 \\ 0 & \bar{C}(\bar{s}I-A)^{-1}\bar{B} \end{pmatrix}\right] \cdot 1 \\ &= \det\left[A - \bar{s}I\right] \cdot \det\left[\bar{C}(\bar{s}I-A)^{-1}\bar{B}\right] = p \cdot q \neq 0. \end{aligned}$$

Hence, the Coates graph of $M(\bar{s})$ contains a spanning cycle family (Corollary 8). It follows from Corollary 10 that there are r disjoint paths from the fault vertices to the measurement vertices in the graph associated with the system $\left(A - \bar{s}I, \bar{B}, \bar{C}\right)$. Note that the matrix $-\bar{s}I$ only induces n self-loops in this graph. These self-loops do not influence the number of disjoint paths. This means that the number of disjoint paths from the fault vertices to the measurement vertices in the graph associated with the system $\left(A, \bar{B}, \bar{C}\right)$ is also equal to r. The graph associated with the system $\left(A, \bar{B}, \bar{C}\right)$ is a subgraph of G_Σ, as $\bar{B}$ consists of r columns of B, and $\bar{C}$ is composed of r rows of C. Therefore, the maximum number of disjoint paths from F to Y in the graph G_Σ is greater than or equal to r. This proves that $l \geq r$.

Proof ($\mathbf{l \leq r}$):
We introduce the submatrix B_l that consists of l columns of B containing the weights of the first edge of each of the l disjoint paths. In a similar fashion, let C_l be the submatrix that consists of the l rows of C that contain the weights of the last edge of each path. It is obvious that $\det[C_l(sI-A)^{-1}B_l]$ is an lth-order minor of $T_{fy}(s)$. If we prove that this minor equals a nonzero rational function, it follows immediately that $l \leq r$. Let $M(s) := \begin{pmatrix} A - sI & B_l \\ C_l & 0 \end{pmatrix}$. In the first part of the proof we have shown that for $s \notin \sigma(A)$

$$\det[M(s)] = \det\left[A - sI\right] \cdot \det\left[C_l(sI-A)^{-1}B_l\right]. \tag{5.1}$$

Hence, the determinant of $[C_l(sI-A)^{-1}B_l]$ is nonzero if and only if the determinant of $M(s)$ is nonzero. Let $\alpha \notin \sigma(A)$ be a sufficiently large, real constant. It follows from Theorem 6 that

$$\det[M(\alpha)] = (-1)^{n+l} \sum_{\mathcal{C}} (-1)^{N_c} w(c), \tag{5.2}$$

where $\mathcal{C}$ is the set of all spanning cycle families of the Coates graph $G_{M(\alpha)}$, $c \in \mathcal{C}$ is a spanning cycle family of $G_{M(\alpha)}$ and N_c denotes the number of cycles in c. As there are l disjoint paths from F to Y in G_Σ, it is obvious that $G_{M(\alpha)}$ contains at least one spanning cycle family. If there is exactly one cycle family that spans $G_{M(\alpha)}$, the proof is complete due to Corollary 7. However, in general this will not be the case. Therefore, we continue the proof for the case that there exist at least two spanning cycle families in $G_{M(\alpha)}$. Notice that a spanning cycle family in $G_{M(\alpha)}$ consists of l disjoint cycles that each pass through one vertex of the set $\{v_{n+1}, v_{n+2}, ..., v_{n+l}\}$, and a number of self-loops spanning the remaining vertices that are not contained in these l cycles. Furthermore, we observe that only the self-loops around $v_1, v_2, ..., v_n$ contain weights that depend on α. The final observation we need is that every non-diagonal element of $M(\alpha)$ is either zero, or positive.
We now take a closer look at the summands in equation (5.2). Let $\bar{c}$ be a spanning cycle family of $G_{M(\alpha)}$ that contains p self-loops and $k + 2l$ edges. The term $(-1)^{N_{\bar{c}}} w(\bar{c})$ is given by

$$\begin{aligned} & (-1)^{N_{\bar{c}}} w(\bar{c}) \\ = \;& (-1)^{p+l} \left(b_1 b_2 \cdot ... \cdot b_l \cdot \frac{a_{i_1}}{V_{j_1}} \frac{a_{i_2}}{V_{j_2}} \cdot ... \cdot \frac{a_{i_{k-p}}}{V_{j_{k-p}}} \cdot c_1 c_2 \cdot ... \cdot c_l \left(-\frac{a_{i_{k-p+1}}}{V_{i_{k-p+1}}} - \alpha \right) \cdot ... \cdot \left(-\frac{a_{i_k}}{V_{i_k}} - \alpha \right) \right) \\ = \;& (-1)^{2p+l} \left(b_1 b_2 \cdot ... \cdot b_l \cdot \frac{a_{i_1}}{V_{j_1}} \frac{a_{i_2}}{V_{j_2}} \cdot ... \cdot \frac{a_{i_{k-p}}}{V_{j_{k-p}}} \left(\frac{a_{i_{k-p+1}}}{V_{i_{k-p+1}}} + \alpha \right) \cdot ... \cdot \left(\frac{a_{i_k}}{V_{i_k}} + \alpha \right) \right), \end{aligned} \tag{5.3}$$

where a_i denotes the flow velocity in compartment i, V_j is the volume of compartment j, and b_i and c_i denote the entries of B_l and C_l respectively.
Without loss of generality, we may assume that $\bar{c}$ contains the largest number of self-loops (that is, every spanning cycle family in $G_{M(\alpha)}$ contains p or less self-loops). As every summand in equation (5.2) has the form as described in equation (5.3), we can rewrite $\sum_{\mathcal{C}}(-1)^{N_c}w(c)$ in the following way

$$\sum_{\mathcal{C}}(-1)^{N_c}w(c) = a_p\alpha^p + a_{p-1}\alpha^{p-1} + ... + a_1\alpha + a_0 =: P(\alpha), \tag{5.4}$$

where $a_0, a_1, ..., a_p$ are real constants. Note that the expression α^p only occurs in terms corresponding to spanning cycle families with exactly p self-loops. These terms have the form

$$(-1)^{2p+l}\left(b_1b_2 \cdot ... \cdot b_l \cdot \frac{a_{i_1}}{V_{j_1}}\frac{a_{i_2}}{V_{j_2}} \cdot ... \cdot \frac{a_{i_{k-p}}}{V_{j_{k-p}}}\right)\alpha^p.$$

Because $a_1, a_2, ..., a_n$ and $V_1, V_2, ..., V_n$ are positive, every arbitrary product of these constants is positive as well. Aditionally, all terms containing α^p are of equal sign. Hence, the constant a_p is the sum of nonzero constants with equal signs, and is therefore nonzero.

By taking α sufficiently large, the term $a_p\alpha^p$ dominates the other terms in equation (5.4). Consequently, $P(\alpha)$ is nonzero, which yields

$$\det[M(\alpha)] = (-1)^{n+l}P(\alpha) \neq 0.$$

It follows from equation (5.1) that det $\left[C_l(\alpha I - A)^{-1}B_l\right]$ is nonzero.
Finally, we conclude that det $\left[C_l(sI - A)^{-1}B_l\right]$ equals a nonzero rational function, which proves the theorem. □

Chapter 6

Example

In this example we show an application of the graph theoretic characterization given in the previous chapter. This characterization will be used to choose the sensor locations in a water distribution network in such a way that fault detection and isolation is possible. Consider the water distribution network as displayed in Figure 6.1. This network consists of 18 compartments. The contaminant concentration is each compartment is indicated by $x_1(t), x_2(t), ..., x_{18}(t)$, and the flow velocity of water in each compartment is indicated by $a_1, a_2, ..., a_{18}$.

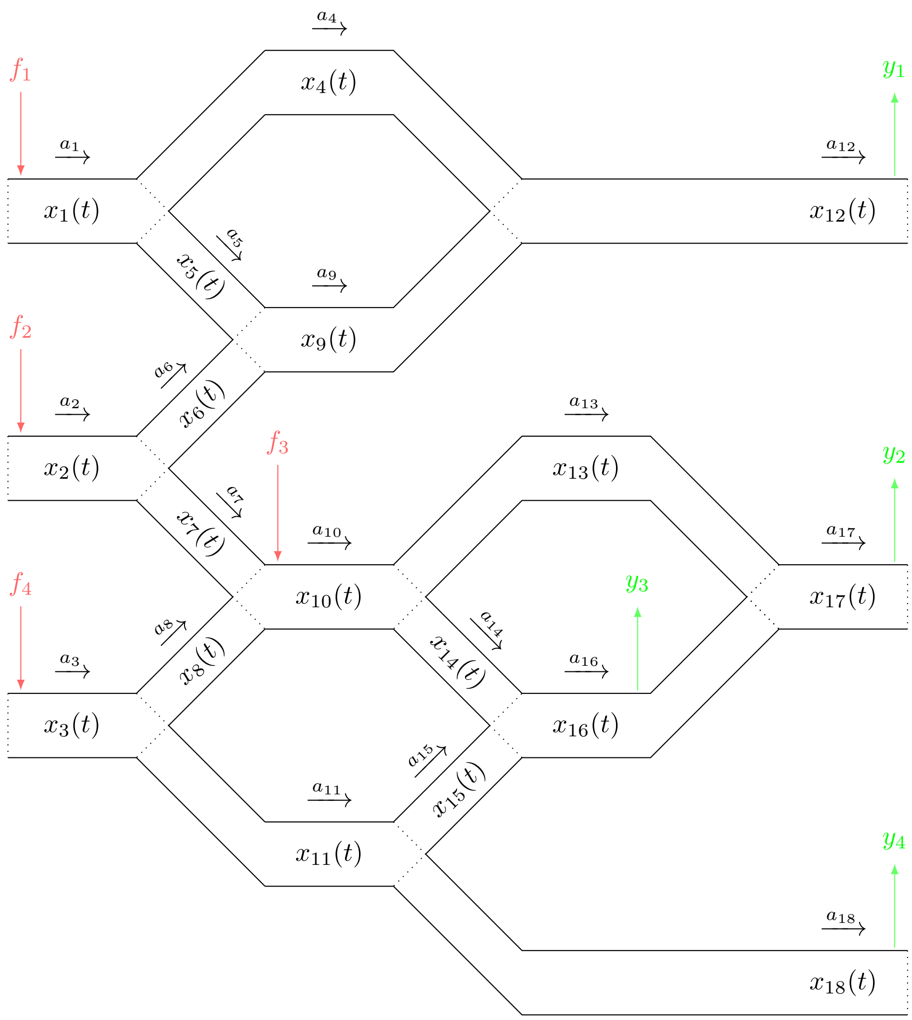

Figure 6.1: Schematic depiction of a water distribution network.

We want to know whether fault detection and isolation by residual generation is possible for this particular network. This means that we want to verify the existence of a bank of residual generators that induces a diagonal transfer matrix from the fault vector $\mathbf{f}(t) = \begin{pmatrix} f_1(t) & f_2(t) & f_3(t) & f_4(t) \end{pmatrix}^T$ to the residual vector $\mathbf{r}(t) = \begin{pmatrix} r_1(t) & r_2(t) & r_3(t) & r_4(t) \end{pmatrix}^T$. We have proven in Chapter 4 that there exists a bank of residual generators if and only if the transfer matrix from the fault vector $\mathbf{f}$ to the measurement vector $\mathbf{y}(t) = \begin{pmatrix} y_1(t) & y_2(t) & y_3(t) & y_4(t) \end{pmatrix}^T$ has full column rank. In this example we will not directly compute the rank of this transfer matrix. Instead, we use the result from Chapter 5. In Chapter 5 we have proven that the rank of the transfer matrix from $\mathbf{f}$ to $\mathbf{y}$ is equal to the maximum number of disjoint paths from fault vertices to measurement vertices in the graph associated with the water distribution network. The graph associated with the WDN in Figure 6.1 is displayed in Figure 6.2. We have omitted the weights of the edges in this graph, as they do not influence the maximum number of disjoint paths.

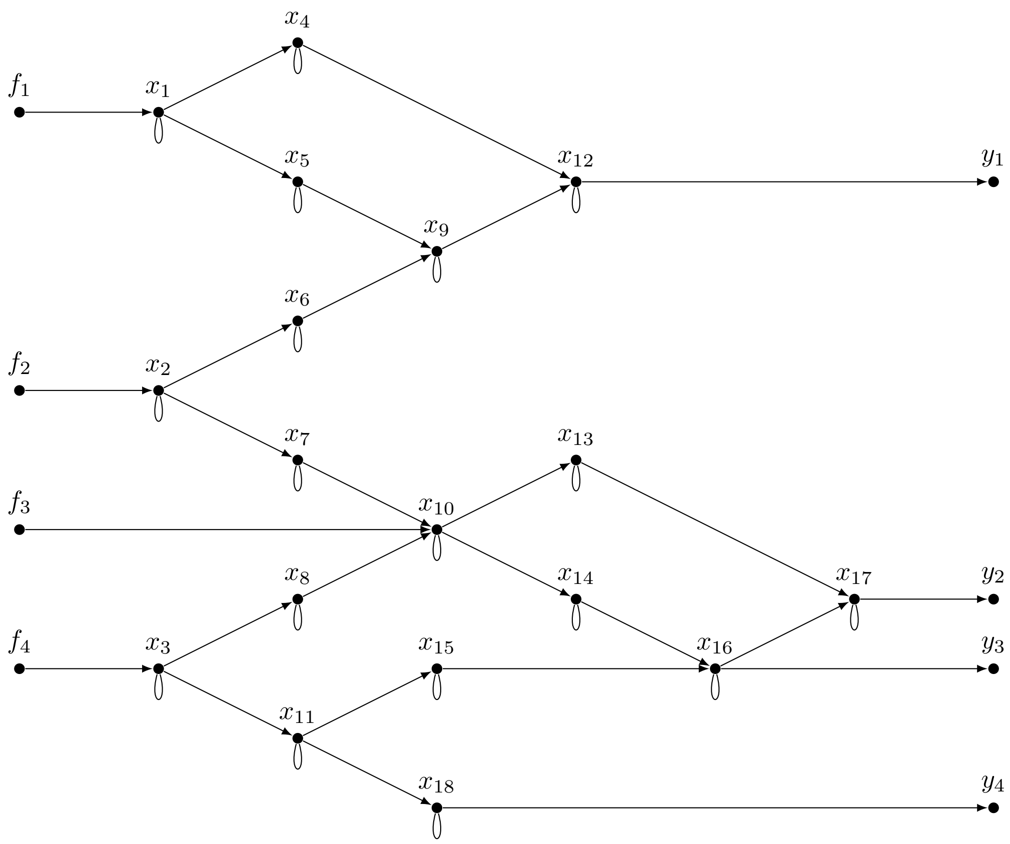

Figure 6.2: Directed graph associated with the WDN in Figure 6.1.

Note that there exists a maximum of 3 disjoint paths from fault vertices $F = \{f_1, f_2, f_3, f_4\}$ to measurement vertices $Y = \{y_1, y_2, y_3, y_4\}$. This is due to the fact that the path from f_2 to one of the vertices in Y intersects with either a path that starts in f_1, or a path that starts in f_3. A set of 3 disjoint paths from F to Y is displayed in Figure 6.3. Note that it is impossible to add a fourth path from f_2 to a measurement vertex that is disjoint from the existing three paths. A path from f_2 to a measurement vertex will either intersect in x_{12} or in x_{10} with one of the existing paths. In general, A path from f_2 to a measurement vertex will either intersect the path from f_1 to y_1 in x_9 or x_{12}, or intersect a path from f_3 to a measurement vertex in x_{10}. Hence there exists a maximum of three disjoint paths from F to Y.

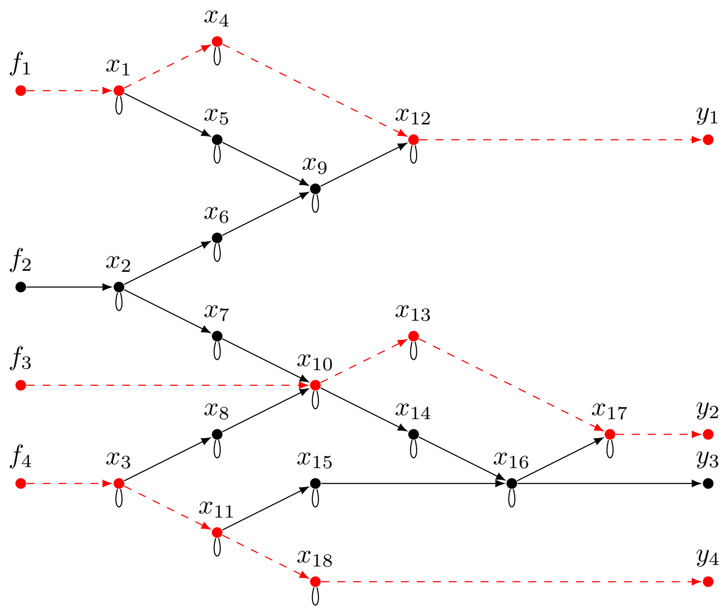

Figure 6.3: 3 disjoint paths in directed graph.

From Theorem 11 it follows that the rank of the transfer matrix from $\mathbf{f}$ to $\mathbf{y}$ is equal to 3. We conclude from Theorem 5 that there does not exist a bank of residual generators that induces a diagonal transfer matrix from faults to residuals. Hence, fault detection and isolation by residual generation is not possible in this network configuration. We now want to modify the sensor locations in such a way that fault detection and isolation is possible. Therefore, we have to move the measurement vertices in such a way that there are 4 disjoint paths from F to Y. There are multiple ways of doing this, but we choose to preserve the location of y_1, y_2 and y_4, and move y_3. We relocate y_3 in such a way that there is a path of length one from x_9 to y_3 (see Figure 6.4). This means that instead of measuring the contaminant concentration in compartment 16, we measure the contaminant concentration in compartment 9. Clearly, there are 4 disjoint paths from F to Y in the graph as displayed in Figure 6.4.

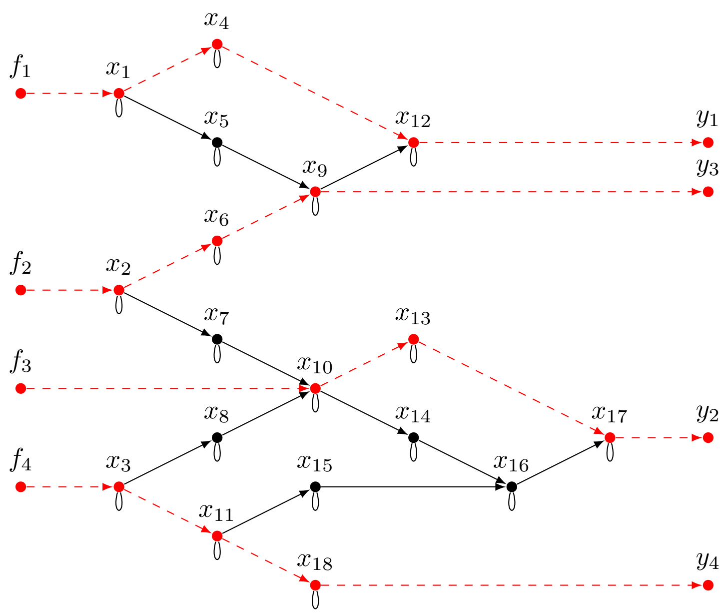

Figure 6.4: 4 disjoint paths in directed graph.

The water distribution network corresponding to the graph in Figure 6.4 is displayed in Figure 6.5. Notice that in this network, we measure the concentration in compartment 9 instead of the concentration in compartment 16 (indicated by y_3). As there are 4 disjoint paths from F to Y in the graph associated with the water distribution network in Figure 6.5, we conclude from Theorem 11 that the rank of the transfer matrix from $\mathbf{f}$ to $\mathbf{y}$ is 4. Finally, it follows from Theorem 5 that there exists a bank of residual generators that induces a diagonal transfer matrix from faults to residuals. In other words: fault detection and isolation by residual generation is possible for the water distribution network as depicted in Figure 6.5.

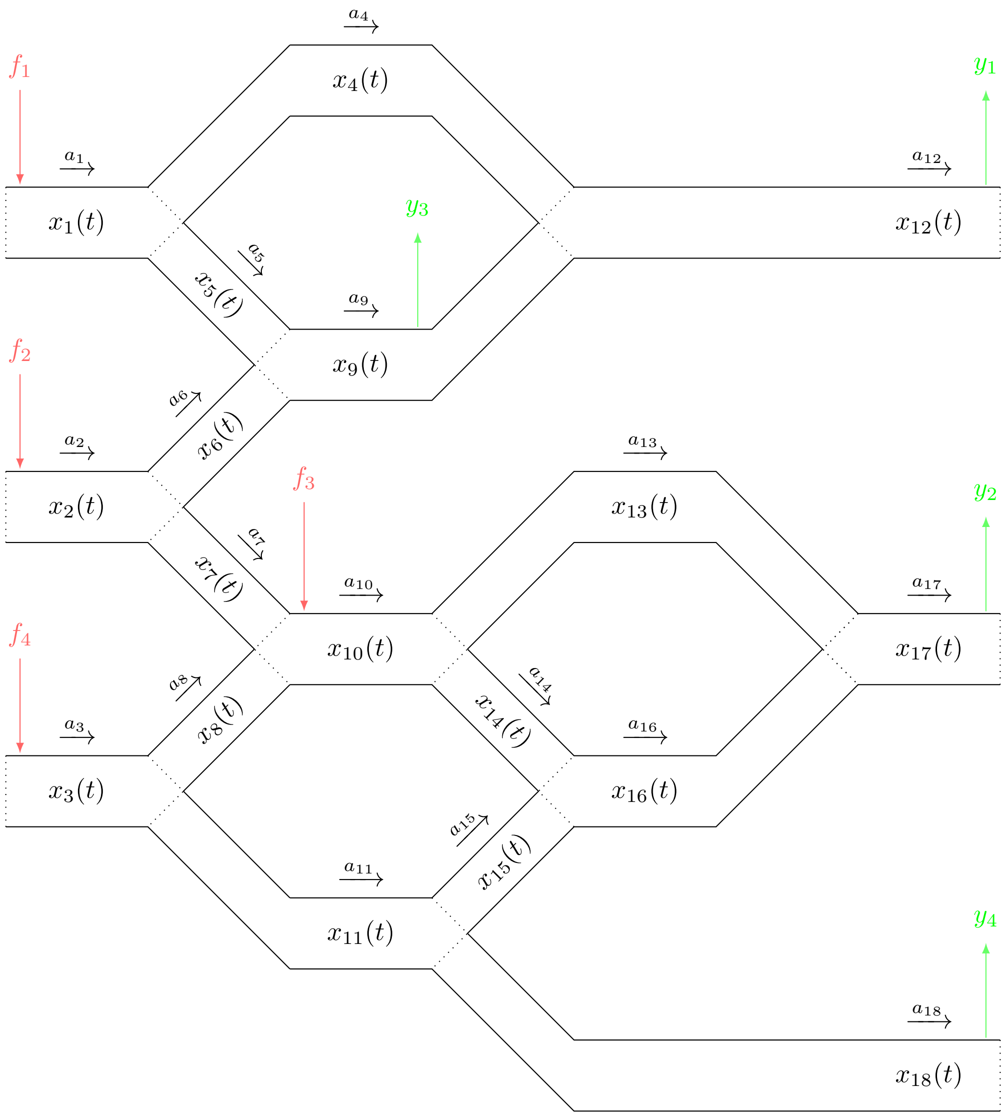

Figure 6.5: Schematic depiction of a water distribution network.

Chapter 7

Conclusions

The goal of this thesis was to provide a necessary and sufficient condition for the existence of a bank of residual generators that induces a diagonal transfer matrix from the fault vector to the measurement vector. This thesis has two main contributions. In the first place we have proven that there exists a bank of residual generators with the aforementioned property if and only if the transfer matrix from faults to measurements has full column rank. This proof is inspired by the ideas from the article *"Oberserver-based Fault Detection and Isolation for Structured Systems"* [4].

As our second contribution, we have shown that the rank of the transfer matrix from faults to measurements is equal to the maximum number of disjoint paths from fault vertices to measurement vertices in the graph associated with a water distribution network. The graph theoretic characterization of the rank of the transfer matrix provides a different approach to determine the existence of a bank of residual generators that induces a diagonal transfer matrix from faults to residuals. Additionally, this characterization gives insight in how to modify the sensor locations if fault detection and isolation is not possible in a current network configuration. The proof of this graph theoretic characterization is based on ideas from the article *"A Graph Theoretic Characterization for the Rank of the Transfer Matrix of a Structured System"* [10]. The relevance of the graph theoretic characterization discussed in this thesis lies in the fact that this characterization holds for a specific type of non-structured linear system, associated with a water distribution network, while previous results were only known for structured systems [10].

Note that in Section 4.2 we have not explicitly proven the stability of the matrix $A - KC$. In future work an explicit proof should be provided, inspired by the ideas from [3].

In this thesis we have only given a condition for the existence of a bank of residual generators. Future research will focus on a technique to compute residual generators that generate a residual vector with the property that the transfer matrix from faults to residuals is diagonal.

Furthermore, the results gained in the report are based on a simplistic model of the contaminant concentration in water distribution networks. For instance, we neglected the influence of an inflow on the flow velocity in a pipe. Moreover, the flows in water distribution networks are assumed to be laminar, even though in practice turbulences will occur near junctions. Hence, future research should focus on finding a more accurate model of the evolution of the contaminant concentration in water distribution networks.

Bibliography

[1] E.J. Blokker and H. Beverloo. *Verandering van waterkwaliteit in het distributienet: metingen in Rosmalen in vermaasde en vertakte netten.* KWR Watercycle Research Institute, 2008.

[2] W.K. Chen. *Applied graph theory.* North-Holland series in applied mathematics and mechanics. North-Holland Pub. Co., 1971.

[3] C. Commault and J.M. Dion. Transfer matrix approach to the triangular block decoupling problem. *Automatica*, 19(5):533 – 542, 1983.

[4] C. Commault, J.M. Dion, O. Sename, and R. Motyeian. Observer-based fault detection and isolation for structured systems. *IEEE Trans. Automat. Contr.*, 47(12):2074–2079, 2002.

[5] M.L.J. Hautus and M. Heymann. Linear feedback, an algebraic approach. *SIAM J. Control*, 16(1):83 – 105, 1978.

[6] D.M. Himmelblau and J.B. Riggs. *Basic Principles and Calculations in Chemical Engineering.* Prentice Hall International Series in the Physical and Chemical Engineering Sciences. Pearson Education, 2012.

[7] A.S. Morse. System invariants under feedback and cascade control. In *Mathematical Systems Theory*, volume 131 of *Lecture Notes in Economics and Mathematical Systems*, pages 61–74. Springer Berlin Heidelberg, 1976.

[8] R. Janke W.E. Hart J. Berry C. Phillips R. Murray, T. Haxton. *Sensor network design for drinking water contamination warning systems – a compendium research results and case studies using the TEVA-SPOT software.* United States Environmental Protection Agency, 2010,EPA/600/R-09/141.

[9] H.L. Trentelman, A. Stoorvogel, and M. Hautus. *Control Theory for Linear Systems.* Communications and Control Engineering. Springer Verlag, London, UK, 2001.

[10] J.W. van der Woude. *A Graph Theoretic Characterization for the Rank of the Transfer Matrix of a Structured System.* Report OS / Centrum voor Wiskunde en Informatica. Stichting Mathematisch Centrum. Centre for Mathematics and Computer Science, 1988.

[11] M. Woutersen. *Development and Validation of an On-line Water Toxicity Sensor Based on Genetically Modified, Luminescent Bacteria.* 2013.