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Robust synchronization of multiplicatively perturbed multi-agent systems and an LMI-based approach to the robust stabilization problem

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Abstract

The main topic of this master thesis is robust synchronization of uncertain multi-agent systems using observer based protocols. For a given network where the dynamics of each agent is the same, we consider multiplicative perturbations on the transfer matrix of the agents. These perturbations are assumed to be stable and bounded in $\mathcal{H}_\infty$-norm by some a priori given tolerance. The problem of robust synchronization is to synchronize the network for all perturbations that are bounded by this tolerance. It is shown that a protocol achieves robust synchronization if and only if all controllers in a finite set of observer based controllers robustly stabilize a given, single linear system. A solution to this problem is given for the case of undirected network graphs and heterogeneous perturbations on the agents. Furthermore a similar solution is given for the case of directed graphs and homogeneous perturbations. For both cases, robustly synchronizing protocols are expressed in terms of the solutions of certain algebraic Riccati equations. It will be shown that an upper bound for the guaranteed achievable tolerance of the perturbations is given in terms of the spectral radius of the solutions of these Riccati equations and in terms of the ratio between the second smallest and the largest eigenvalue of the Laplacian matrix.

The second part of this thesis consists of an LMI-based approach to the $\mathcal{H}_\infty$-control problem and an application of this theory to the robust stabilization problem. Necessary and sufficient conditions for the solvability of the $\mathcal{H}_\infty$-control problem are established and are expressed in terms of the solvability of certain linear matrix inequalities (LMI's). An algorithm is provided to compute controllers that solve the $\mathcal{H}_\infty$-control problem for any given tolerance. The connection of the $\mathcal{H}_\infty$-control problem and the robust stabilization problem is made via the small-gain theorem. In the robust stabilization problem also the special well-known cases of additive, coprime factor and multiplicative perturbations are analyzed. In these cases, necessary and sufficient conditions for the solvability of the robust stabilization problem are given in terms of the solvability of algebraic Riccati equalities and inequalities. Furthermore, a condition for the maximum achievable tolerance can be isolated and is expressed in terms of the spectral radius of the solutions of the Riccati (in)equalities.

Contents

Chapter 1

Introduction

A lot of research in the field of networks of systems has been done in the last years, in particular on multi-agent systems. A multi-agent system consists of a network of input-output systems in which the systems are called *agents*. The agents in the network have identical dynamics and the agents are connected by a *network graph*. This graph can either be directed or undirected. The vertices of the network graph represent the agents and the edges represent the interconnection topology of the network. The network graph can be represented in terms of its *Laplacian matrix* and many properties of the multi-agent network can be expressed in terms of the spectrum of the Laplacian matrix.

Each agent in the multi-agent network exchanges information with each of its neighbors. The way the information is exchanged through the network graph is determined by a *protocol*. This protocol acts as a feedback control which acts locally since it collects only information from neighboring agents. In the theory of multi-agent systems often the objective is design a protocol such that a desired dynamics of the entire network is achieved.

The most-well known problem in the framework of multi-agents systems is perhaps the *consensus problem*, see for example [7],[8],[9] and [10]. In the consensus problem the agents may for example represent sensors that exchange information only with their neighbors. The aim is to reach agreement on the values of certain quantities of interest that depend on the states of all agents. A protocol that achieves this aim is said to achieve consensus.

A strongly related problem is the problem of *synchronization* of multi-agent systems, see for example [3],[6],[11] and [14]. In the synchronization problem the dynamics of the agents are identical and the objective is to construct a protocol such that the state of each of the agents converges to a common trajectory. The information that is available for the protocol is the sum of the relative states or the sum of the relative outputs of the agents, depending on the problem. If the sum of the relative states of the agents is known we will show that a *static protocol* is sufficient to achieve synchronization of the network. However sometimes the sum of the relative states is not available but instead the sum of the relative outputs is. Then often it is possible to reconstruct the information on the sum of the relative states using a *dynamic protocol*. Here the dynamic part of the protocol acts as an observer for the sum of the relative states of each agent.

In this thesis *robust synchronization* is considered. In particular we will look at multiplicative perturbations on the agent dynamics. These perturbations can be modeled as perturbations on the nominal transfer matrix of the agents, where the nominal transfer matrix is a model of the input-output behavior of the agents in the absence of uncertainty. The only assumption on the perturbations will be that they are stable. The size of the perturbations is bounded by a given tolerance. If the the perturbations can be different in each agent, but are bounded by the same tolerance, then the perturbations are called *heterogeneous*. On the other hand, if the perturbations are identical for each agent then the perturbations are called *homogeneous*. The problem of robust synchronization is to synchronize the network for all homogeneous or heterogeneous perturbations bounded by a given tolerance.

The problems of robust synchronization of additive and coprime factor perturbations are discussed in [16] and [1] respectively. Furthermore the problem of robust synchronization of multiplicative perturbations using *static* protocols is treated in [2]. In these three references conditions for the existence of robustly synchronizing protocols and an explicit form of these protocols are given. In this thesis we will extend these results to multiplicatively perturbed multi-agent systems using *observer based* protocols. We will introduce a method to compute robustly synchronizing protocols and we will show that these protocols are dependent on the second smallest and largest eigenvalue of the Laplacian matrix. One is often interested in maximizing the permissible tolerance for which there still exists a robustly synchronizing protocol. We establish an upper bound on the guaranteed achievable tolerance which is expressed in terms of the ratio of the second smallest and largest eigenvalue of the Laplacian matrix.

The second subject of this thesis is the robust stabilization problem for a single linear input-output system. In the robust stabilization problem we assume that the dynamics of the system is *uncertain*. We would like to find a controller that stabilizes the system even if the actual system is slightly different from the model we have started with. A way to model these uncertainties is by introducing a stable linear system, which we call the *perturbation system*, in an additional feedback loop around the *nominal* system [17]. It is assumed that the $\mathcal{H}_\infty$-norm of the transfer matrix of this system is bounded by some given tolerance. The goal in the robust stabilization problem is to design a controller that internally stabilizes the closed-loop system for all perturbation systems, whose transfer matrices are bounded by this tolerance. Other well-known ways to model the uncertainties is by additive, coprime factor or multiplicative perturbations of the nominal transfer matrix. However, it will be shown that a realization of these perturbed transfer matrices can be represented as discussed before, i.e. by introducing a perturbation system in an additional feedback loop around the nominal system.

There is a strong connection between the robust stabilization problem and $\mathcal{H}_\infty$-control problem, which is made via the small-gain theorem [17]. It is therefore possible to solve the robust stabilization problem by applying the theory of the $\mathcal{H}_\infty$-control problem. In the latter problem we consider a linear input-output system that is affected by disturbances. The aim is to design a controller that minimizes the effect of the disturbances on certain (additional) outputs of the system. The performance measure that is used for this is the $\mathcal{H}_\infty$-norm of the transfer matrix of the closed-loop system from the disturbance input to these outputs. The objective is then to design a controller such that this closed-loop transfer matrix is bounded by a given tolerance.

A lot of research has already been done on the $\mathcal{H}_\infty$-control problem, see for example [12], [15] and [17]. Another viewpoint is provided in this thesis, which is inspired by [13]. Here an approach based on linear matrix inequalities (LMI's) is used to solve the $\mathcal{H}_\infty$-control problem. We extend these results in several ways. Firstly, we will derive necessary and sufficient conditions for the solvability of a more general $\mathcal{H}_\infty$-control problem providing an alternative proof. These conditions are expressed in terms of the solvability of certain LMI's. Secondly, we apply this theory on the robust stabilization problem and we look at three special well-known types of the robust stabilization problem. For the case of additive, coprime factor and multiplicative perturbations we establish necessary and sufficient conditions for the solvability of the associated robust stabilization problem which can be expressed in terms of the solvability of algebraic Riccati (in)equalities. Finally, we will derive upper bounds on the tolerance for which there still exists a robustly stabilizing controller in each of the three cases. These upper bounds are expressed in terms of the spectral radius of the solutions of these Riccati equations.

The outline of this thesis is as follows. In Chapter 2 the preliminaries for the two main subjects are stated. Here we start with the notation used in the thesis followed by some preliminaries on graphs, systems and linear matrix inequalities. In Chapter 3 the synchronization problem using relative state feedback will be discussed. Also network observers are introduced in this section followed by the synchronization problem using observed based protocols. In Chapter 4 we explain how multiplicative perturbations are modeled and we state an equivalence between robust synchronization and robust stabilization of a single system with multiple feedback controllers for both undirected and directed network graphs. Conditions for the existence of robustly synchronizing protocols and an explicit form of these protocols is given in Chapter 5 for either case. We also establish an upper bound on guaranteed achievable uncertainty radius here.

The results on the $\mathcal{H}_\infty$-control problem and the robust stabilization problem commence at Chapter 6. In this chapter the $\mathcal{H}_\infty$-control problem is formulated and furthermore necessary and sufficient conditions for solvability of the general $\mathcal{H}_\infty$-control problem are derived, where we distinguish between two different cases. In the first case we assume that the feedthrough term from the input to the output is zero in the nominal system, then we consider the more general case that this term is not necessarily zero. For both cases, an algorithm is provided for computing controllers that solve the $\mathcal{H}_\infty$-control problem for any given tolerance. In Chapter 7 we apply the theory of the $\mathcal{H}_\infty$-control problem to the robust stabilization problem. We will establish necessary and sufficient conditions for the existence of a robustly stabilizing controller. Also we look at some special cases of the robust stabilization problem where we deal with additive, coprime factor and multiplicative perturbations. For these cases, we will establish upper bounds for the permissible uncertainty radius.

Chapter 2

Preliminaries

In this chapter we will state some preliminaries. We will start with the notation used in this thesis. Next, we introduce some graph theory for multi-agent networks and we state some well known results about linear input-output systems that we will use in the main results of this thesis. Finally, some preliminaries on linear matrix inequalities are given.

2.1 Notation

We denote by $\mathbb{R}^{m\times n}$ the space of all real $m \times n$ matrices, and similarly by $\mathbb{C}^{m\times n}$ the space of all complex $m \times n$ matrices. For $\lambda \in \mathbb{C}$, we denote by $\mathrm{Re}(\lambda)$ and $\mathrm{Im}(\lambda)$ the real and imaginary part of λ respectively. We denote the $p \times p$ identity matrix by I_p or $I_{p\times p}$ depending on the context. Given a real matrix Q we denote by $\rho(Q)$ the spectral radius of Q i.e. $\rho(Q) := \sqrt{\lambda_{\max}(Q^TQ)}$. A matrix P is called *symmetric positive definite* (short: SPD) if it is symmetric (and thus all eigenvalues are real) and all eigenvalues are positive, we write $P > 0$. For a square matrix A we denote by $\sigma(A)$ the spectrum of A and we call A *Hurwitz* if $\mathrm{Re}(\lambda) < 0$ for each eigenvalue λ of A. For a given real or complex $m \times n$ matrix C, we denote the null space of C by $\ker(C)$, i.e. all $x \in \mathbb{R}^n (x \in \mathbb{C}^n)$ such that $Cx = 0$ and we denote by $\mathrm{im}(C)$ the image of C, i.e. all $w \in \mathbb{R}^m (w \in \mathbb{C}^m)$ that can be written as $w = Cv$ for some $v \in \mathbb{R}^n (v \in \mathbb{C}^n)$. Denote by $R\mathcal{H}_\infty$ the set of all proper and stable rational transfer matrices. If $G \in R\mathcal{H}_\infty$, then $\|G\|_\infty$ will denote its usual infinity norm, i.e. $\|G\|_\infty = \sup_{\mathrm{Re}(\lambda)\geq 0} \|G(\lambda)\|$. The *Kronecker product* of two matrices $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$ is defined by [4]:

$$A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix}.$$

The Kronecker product has the following properties:

$$\begin{aligned} (A \otimes B)(C \otimes D) &= AC \otimes BD, \\ (A \otimes B)^T &= A^T \otimes B^T, \\ A \otimes B + A \otimes C &= A \otimes (B + C). \end{aligned}$$

2.2 Graphs

In this thesis, we consider multi-agent systems whose interconnection structures are described by directed or undirected unweighted graphs. In general a directed graph is a pair $(\mathcal{V}, \mathcal{E})$ where the elements of $\mathcal{V} = \{1, 2, \ldots, p\}$ are called *vertices* and the elements of $\mathcal{E}$ are pairs (i, j), called *directed edges* or *arcs*. An arc from vertice $i \in \mathcal{V}$ to vertice $j \in \mathcal{V}$ where $i \neq j$ is represented by $(i, j) \in \mathcal{E}$. If for every $(i, j) \in \mathcal{E}$ we also have $(j, i) \in \mathcal{E}$ we call the network *undirected*. For a given $i \in \mathcal{V}$ its *neighboring set* $\mathcal{N}_i$ is defined as $\mathcal{N}_i := \{j \in \mathcal{V} \mid (i, j) \in \mathcal{E}\}$. Given a graph $(\mathcal{V}, \mathcal{E})$ its adjacency matrix $A = (a_{ij})$ satisfies $a_{ii} = 0, a_{ij} = 1$ if $(j,i) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. The *Laplacian matrix* $L = (l_{ij})$ of the graph is defined by $l_{ii} = \sum_{j \neq i} a_{ij}, l_{ij} = -a_{ij}, i \neq j$. The Laplacian matrix is a real symmetric positive semi-definite matrix if the graph is undirected. If the graph is directed then the eigenvalues of the Laplacian are in general complex but it can be shown that the eigenvalues all have a nonnegative real part. For any graph the Laplacian matrix has an eigenvalue equal to zero so the rank of the Laplacian matrix is at most $p-1$.

An undirected graph is called *connected* if for every pair of distinct vertices i and j there exists a path from i to j, i.e. there exists a finite set of edges $(i_k, i_{k+1}), k = 1,2,\dots,r-1$ such that $i_1 = i$ and $i_r = j$. An undirected graph is connected if and only if the Lapacian matrix has rank $p-1$. In that case the eigenvalue zero has multiplicity one and all other eigenvalues are real and positive. In this thesis the eigenvalues of the Laplacian are ordered as $0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_p$ for the undirected graph case.

A directed graph is said to contain a *spanning tree* if it contains a node i such that there exists a path from this node to any other node j. A directed graph contains a spanning tree if and only if its Laplacian matrix has rank $p-1$. In this case the eigenvalue $\lambda_1 = 0$ has multiplicity one and all other eigenvalues have a positive real part. The eigenvalues of the Laplacian are then ordered such that $0 = \mathrm{Re}(\lambda_1) < \mathrm{Re}(\lambda_2) \leq \mathrm{Re}(\lambda_3) \leq \cdots \leq \mathrm{Re}(\lambda_p)$.

2.3 Systems

Throughout this section we consider the linear input-output system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \tag{2.1}$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}^m, y \in \mathbb{R}^p$ and the transfer matrix of (2.1) is denoted by $T(s) = C(sI-A)^{-1}B+D$. We call the pair (A,B) *stabilizable* if the matrix $(A - sI \;\; B)$ has rank n for all $s \in \sigma(A)$ with $\mathrm{Re}(s) \geq 0$. Similarly the pair (C,A) is called *detectable* if the matrix

$$\begin{pmatrix} A - sI \\ C \end{pmatrix}$$

has rank n for all $s \in \sigma(A)$ with $\mathrm{Re}(s) \geq 0$. In this thesis we will consider a version of the bounded real lemma where *strict* matrix inequalities appear. Although this not the actual bounded real lemma we still refer to this version as the *bounded real lemma* throughout this thesis.

Lemma 2.3.1 (Bounded real lemma). *Let $\gamma > 0$. The following statements are equivalent.*

i. A is Hurwitz and $\|T\|_\infty < \frac{1}{\gamma}$

ii. $\frac{1}{\gamma^2}I - D^TD > 0$ and there exists $Y > 0$ such that

$$YA + A^TY + (YB + C^TD)(\frac{1}{\gamma^2}I - D^TD)^{-1}(YB + C^TD)^T + C^TC < 0$$

iii. There exists $Y > 0$ such that

$$\begin{pmatrix} YA + A^TY & YB & C^T \\ B^TY & -\frac{1}{\gamma^2}I & D^T \\ C & D & -I \end{pmatrix} < 0$$

For the proof we refer to [13], see also [12]. In the sequel we also use the complex version of Lemma 2.3.1 where the matrices A, B, C, D are complex. It is not hard to show that the result of the lemma also holds for the complex case if replace the transpose by the conjugate transpose in the matrix inequalities and furthermore the matrix Y is now required to be Hermitian.

An important result on the interconnection of two systems that have stable transfer matrices is the small gain theorem [17].

Theorem 2.3.2 (Small gain theorem). *Consider the pair of internally stable systems*

$$\Sigma_1 : \begin{array}{l} \dot{x}_1 = A_1 x_1 + B_1 u_1 \\ y_1 = C_1 x_1 + D_1 u_1 \end{array}$$

$$\Sigma_2 : \begin{array}{l} \dot{x}_2 = A_2 x_2 + B_2 u_2 \\ y_2 = C_2 x_2 + D_2 u_2 \end{array}$$

with transfer matrices T_1, T_2 *respectively. Let* $\gamma > 0$*. The feedback interconnection of* Σ_1 *and* Σ_2 *is well-posed (equivalently:* $I - D_1 D_2$ *is invertible) and internally stable for all* Σ_2 *with transfer matrix* $T_2(s)$ *satisfying* $\|T_2\|_\infty \leq \gamma$ *if and only if* $\|T_1\|_\infty < \frac{1}{\gamma}$*.*

In the main results often algebraic Riccati equations (ARE's) arise. Consider the ARE

$$PA + A^T P - PBB^T P + C^T C = 0. \tag{2.2}$$

One is often interested in finding a stabilizing positive semi-definite matrix P that is a solution to (2.2). Here a solution $P \geq 0$ to (2.2) is called *stabilizing* if the matrix $A - BB^T P$ is Hurwitz. Necessary and sufficient conditions for the existence of an *unique* stabilizing $P \geq 0$ that satisfies ARE (2.2) are given below [5].

Lemma 2.3.3. *The ARE* (2.2) *has an unique stabilizing solution* $P \geq 0$ *if and only if* (A, B) *is stabilizable and* (C, A) *is detectable.*

Lemma 2.3.4. *The ARE* (2.2) *has an unique stabilizing solution* $P > 0$ *if and only if* (A, B) *is stabilizable and* (C, A) *is observable.*

Note that if we replace the triple (A, B, C) by the triple (A^T, C^T, B^T) in the previous lemmas we obtain similar results for the ARE

$$AQ + QA^T - QC^T CQ + BB^T = 0. \tag{2.3}$$

It is easy to see that the stabilizing solution Q to (2.3) in this case means that the matrix $A - QC^T C$ is Hurwitz.

2.4 Linear matrix inequalities

In the theory of linear matrix inequalities (LMI's) Finsler's Lemma is a useful result. Before we can state the lemma we have to define what an annihilator of a matrix is. Let $M \in \mathbb{R}^{n \times m}$ with $m < n$ and rank m. Then there exists $M^\perp \in \mathbb{R}^{(n-m) \times n}$ of rank $n - m$ such that $M^\perp M = 0$. Such $M^\perp$ is called an *annihilator* of M. It can be shown that $N = M^\perp$ is an annihilator of M if and only if $\text{im}(M) = \ker(N)$. Note that there are in fact many annihilators of M. We ignore this and we take any annihilator of M and call it $M^\perp$. The following properties will be used in the sequel.

- Let T be a nonsingular matrix such that TM is defined. Then $(TM)^\perp = M^\perp T^{-1}$.
- Let $0_{q \times m} \in \mathbb{R}^{q \times m}$ be the zero matrix. Then

$$\begin{pmatrix} M \\ 0_{q \times m} \end{pmatrix}^\perp = \begin{pmatrix} M^\perp & 0 \\ 0 & I_{q \times q} \end{pmatrix},$$

$$\begin{pmatrix} M & 0 \\ 0 & I_{q \times q} \end{pmatrix}^\perp = \begin{pmatrix} M^\perp & 0_{(n-m) \times q} \end{pmatrix}$$

In this thesis we will consider a variant of the Finsler's Lemma [13]. We still refer to this variant as the Finsler's Lemma throughout this thesis.

Lemma 2.4.1 (Finsler)**.** *Let $B \in \mathbb{R}^{n\times m}$ have rank m and assume $n > m$. Let $Q \in \mathbb{R}^{n\times n}$ be symmetric. There exists a symmetric positive definite matrix $R \in \mathbb{R}^{m\times m}$ such that $Q - BRB^T < 0$ if and only if $B^{\perp}QB^{\perp T} < 0$.*

It can be observed that without loss of generality R can be chosen of the form $R = rI$ where $r > 0$ is a real scalar. In this report a specific LMI will appear which in general is given by

$$BXC + (BXC)^T + Q < 0, \tag{2.4}$$

where $B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{k\times n}, Q \in \mathbb{R}^{n\times n}$ are given and we assume that B has rank m and C has rank k. We are interested in finding a solution $X \in \mathbb{R}^{m\times k}$ that satisfies the LMI (2.4). The next theorem provides necessary and sufficient conditions for the solvability of this LMI as well as an explicit form of a solution X.

Theorem 2.4.2. *The following statements are equivalent [13].*

i. The LMI (2.4) *has a solution X*

ii. $B^{\perp}QB^{\perp T} < 0$ and $C^{T\perp}QC^{T\perp T} < 0$

If ii. holds then $X = -RB^T\Phi C^T(C\Phi C^T)^{-1}$ where $R > 0$ is such that

$$\Phi := (BRB^T - Q)^{-1} > 0$$

is a solution of (2.4)*.*

Proof. ($i. \Rightarrow ii.$) Trivial. ($ii. \Rightarrow i.$) Assume $B^{\perp}QB^{\perp T} < 0$ and $C^{T\perp}QC^{T\perp T} < 0$. By Finsler's Lemma there exists a matrix $R > 0$ such that

$$BRB^T - Q > 0.$$

Then also

$$\Phi := (BRB^T - Q)^{-1} > 0.$$

Since C has linearly independent rows, also $C\Phi C^T > 0$, so $(C\Phi C)^{-1}$ exists. Define the matrix $X := -RB^T\Phi C^T(C\Phi C^T)^{-1}$. Then X satisfies (2.4) as we will show now. Consider the matrix

$$S := \begin{pmatrix} C^{T\perp} \\ C\Phi \end{pmatrix}$$

then note that S is nonsingular. We will now show that

$$S(BXC + (BXC)^T + Q)S^T < 0. \tag{2.5}$$

First observe that

$$C^{T\perp}(BXC + (BXC)^T + Q)C^{T\perp T} = C^{T\perp}QC^{T\perp T} < 0,$$

$$\begin{aligned} &C\Phi(-BRB^T\Phi C^T(C\Phi C^T)^{-1}C - C^T(C\Phi C^T)^{-1}C\Phi BRB^T + Q)\Phi C^T \\ &= -2C\Phi BRB^T\Phi C^T + C\Phi Q\Phi C^T = -C\Phi BRB^T\Phi C^T - C\Phi C^T \leq -C\Phi C^T < 0, \end{aligned}$$

$$\begin{aligned} &C^{T\perp}\left[-BRB^T\Phi C^T(C\Phi C^T)^{-1}C - C^T(C\Phi C^T)^{-1}C\Phi BRB^T + Q\right]\Phi C^T \\ &= -C^{T\perp}(BRB^T - Q)\Phi C^T = C^{T\perp}C^T = 0. \end{aligned}$$

In fact we have shown now that

$$S(BXC + (BXC)^T + Q)S^T \leq \begin{pmatrix} C^{T\perp}QC^{T\perp T} & 0 \\ 0 & -C\Phi C^T \end{pmatrix} < 0.$$

Since S is nonsingular it follows that X satisfies the LMI (2.4). □

It should be noted that we have given a different proof compared to the one given in [13].

Remark 2.4.3. Note that if $ii.$ in Theorem 2.4.2 holds then a solution X is provided in the previous theorem. Sometimes one is interested in all solutions of (2.4). A characterization of all possible solutions X can be found in [13].

Chapter 3

Synchronization

In this chapter we consider the synchronization problem for multi-agents systems with p agents. Here the dynamics of each agent is given by one and the same linear system, which we call the *nominal* system. The problem of synchronization is to find a protocol such that the network is synchronized, i.e. the states of all agents converge to a common trajectory. We consider network graphs that are directed, and therefore the results derived in this chapter will also hold for undirected network graphs. Important properties of network graph are reflected in the Laplacian matrix, which we will denote by L. With little loss of generality we will assume that the network graph contains a spanning tree if the network graph is directed, equivalently $\mathrm{Re}(\lambda_2) > 0$. In the undirected graph case this is equivalent with saying that the network graph is connected, equivalently $\lambda_2 > 0$.

First we will consider the synchronization problem where the information on the relative states is available. In this case we will show that a *static* protocol is sufficient to achieve synchronization of the network. Then the notion of distributed relative state observers is introduced. Finally, we derive necessary and sufficient conditions for the solvability of the synchronization problem using *observer based* protocols. In this case only the information on the relative outputs of the agents is available for the protocol.

3.1 Relative state feedback

Consider dynamics of the agents given by

$$\dot{x}_i = Ax_i + Bu_i \tag{3.1}$$

for $i = 1, 2, \ldots, p$, where p is the number of agents in the network. Suppose that for all i the relative state of agent i is available for the protocol. A possible structure of a synchronizing protocol is obtained by introducing a suitable feedback matrix on this relative state, i.e. a possible (static) protocol is of the form

$$u_i = F \sum_{j \in \mathcal{N}_i} (x_i - x_j) \tag{3.2}$$

for $i = 1, 2, \ldots, p$. The problem of synchronization is to find F such that in the closed-loop multi-agent system the states of the agents follow a common trajectory as $t \to \infty$, i.e. the relative states of the agents converge to zero.

Definition 3.1.1. The network is said to be *synchronized* by the static protocol if for all $i, j = 1, 2, \ldots, p$ we have $x_i(t) - x_j(t) \to 0$ as $t \to \infty$.

We denote the *aggregate* state vector by $\mathbf{x} = \mathrm{col}(x_1, x_2, \ldots, x_p)$. The full closed-loop dynamics of all the agents, which is obtained by interconnecting (3.1) with (3.2), can then we written as

$$\begin{aligned}\dot{\mathbf{x}} &= (I \otimes A)\mathbf{x} + (I \otimes B)(L \otimes F)\mathbf{x} \\ &= (I \otimes A)\mathbf{x} + (L \otimes BF)\mathbf{x}.\end{aligned}$$

By the Schur decomposition there exists a unitary $p \times p$ matrix U such that $U^T L U = \Lambda_u$ where Λ_u is an complex upper triangular matrix with $0, \lambda_2, \dots, \lambda_p$ on the diagonal and the diagonal elements are ordered such that $0 < \mathrm{Re}(\lambda_2) \leq \mathrm{Re}(\lambda_3) \leq \dots \leq \mathrm{Re}(\lambda_p)$. Next, apply the transformation $\tilde{\mathbf{x}} = (U^T \otimes I)\mathbf{x}$ to get the transformed dynamics

$$\dot{\tilde{\mathbf{x}}} = [I \otimes A + \Lambda_u \otimes BF]\,\tilde{\mathbf{x}}.$$

By using this transformation on the aggregate state we obtain the following result.

Lemma 3.1.2. *Consider agent dynamics* (3.1) *and assume that the network graph is directed and contains a spanning tree. The static protocol* (3.2) *synchronizes the network if and only if the system*

$$x = Ax + Bu \tag{3.3}$$

is internally stabilized by all $p-1$ controllers

$$u = \lambda_i F x \tag{3.4}$$

where $i = 2, 3, \dots, p$ and λ_i is the ith eigenvalue of the Laplacian L.

Proof. Note that $\ker(L) = \mathrm{im}(1_p)$, where $1_p = (1, 1, \dots, 1) \in \mathbb{R}^p$. Let U be an orthogonal matrix such that $LU = \Lambda_u U$ where Λ_u is an upper triangular matrix with $0, \lambda_2, \dots, \lambda_p$ on the diagonal. We have $x_i - x_j \to 0$ if and only if $\mathbf{x}(t) \to \mathrm{im}(1_p \otimes I) = \ker(L \otimes I)$. This holds if and only if $(L \otimes I)\mathbf{x}(t) \to 0$. Since $\mathbf{x} = (U \otimes I)\tilde{\mathbf{x}}$ the latter holds if and only if $(LU \otimes I)\tilde{\mathbf{x}} \to 0$. Since $LU = U\Lambda_u$ and by the fact that U is nonsingular, this holds if and only if $(\Lambda_u \otimes I)\tilde{\mathbf{x}} \to 0$, equivalently $\tilde{x}_i(t) \to 0$ for $i = 2, 3, \dots, p$. The latter holds if and only if the matrix $I_{p-1} \otimes A + \Lambda_2 \otimes BF$ is Hurwitz, where the $(p-1) \times (p-1)$ matrix Λ_2 is equal to Λ_u but without the first column and row. This matrix is an block upper triangular matrix with $A + \lambda_2 BF, \dots, A + \lambda_p BF$ on the diagonal. Clearly $I_{p-1} \otimes A + \Lambda_2 \otimes BF$ is Hurwitz if and only if the matrices $A + \lambda_i BF$ are Hurwitz for $i = 2, 3, \dots, p$. The latter holds if and only if the interconnection of (3.3) with (3.4) is internally stable for $i = 2, 3, \dots, p$. ☐

The latter lemma provides the equivalence between the synchronization problem using static protocols and a stabilization problem for a single system using multiple state feedback controllers. It can be observed that protocol (3.2) synchronizes the network if and only

$$A + \lambda_i BF \tag{3.5}$$

is Hurwitz for $i = 2, 3, \dots, p$. Clearly it is necessary that (A, B) is stabilizable for $A + \lambda_i BF$ to be Hurwitz for $i = 2, 3, \dots, p$. It is less obvious that the stabilizability of (A, B) is also sufficient for the existance of a single F such that $A + \lambda_i BF$ is Hurwitz for $i = 2, 3, \dots, p$. We will give a proof of this now. Assume that (A, B) is stabilizable. Then there exists $Y > 0$ such that

$$YA + A^T Y - 2\,\mathrm{Re}(\lambda_2) YBB^T Y < 0$$

since $\mathrm{Re}(\lambda_2) > 0$. Define now $F = -B^T Y$ then

$$\begin{aligned} Y(A + \lambda_i BF) + (A + \lambda_i BF)^* Y &= Y(A - \lambda_i BB^T Y) + (A - \lambda_i BB^T Y)^* Y \\ &= YA + A^T Y - 2\,\mathrm{Re}(\lambda_i) YBB^T Y \leq YA + A^T Y - 2\,\mathrm{Re}(\lambda_2) YBB^T Y < 0 \end{aligned}$$

and therefore $A + \lambda_i BF$ is Hurwitz for $i = 2, 3, \dots, p$. We can now conclude the following.

Corollary 3.1.3. *Consider agent dynamics* (3.1)*. Assume the network graph is directed and contains a spanning tree. There exists a static protocol* (3.2) *that synchronizes the network if and only if the pair* (A, B) *is stabilizable.*

3.2 Distributed relative state observers

Until now we have looked at the case that the relative states are available for the protocol. Often this is not the case and instead only information on the relative outputs of the agents is available for the protocol, i.e. we assume that the dynamics of agent i is given by

$$\begin{aligned} \dot{x}_i &= Ax_i + Bu_i \\ y_i &= Cx_i + Du_i, \end{aligned} \tag{3.6}$$

where y_i is the output of agent i. We will show that if (C, A) is detectable then it is possible to reconstruct the sum of relative states of agent i using a system, called a *distributed relative state observer*, that takes the relative inputs and outputs of agent i as inputs and yields an output that is an estimate of the sum of the relative states of agent i with respect to its neighbors. The observers, denoted by Ω_i are of the form

$$\dot{w}_i = Pw_i + Q \sum_{j\in\mathcal{N}_i} (u_i - u_j) + R \sum_{j\in\mathcal{N}_i} (y_i - y_j) \tag{3.7}$$

for $i = 1, 2, \dots, p$. Note that the P, Q and R are required to be the same for all agents. Hence the observer is *distributed*. We define the estimation error as $e_i = w_i - \sum_{j\in\mathcal{N}_i} (x_i - x_j)$.

Definition 3.2.1. The system Ω_i given by (3.7) is called a *distributed relative state observer* for agent i if for all initial values $x_j(0), w_j(0), j = 1, 2, \dots, p$ such that $e_i(0) = 0$, for arbitrary input functions $u_1, \dots, u_p$, we have $e_i(t) = 0$ for all $t > 0$.

The above definition says that Ω_i is a distributed relative state observer for agent i then whenever the estimation error is zero at $t = 0$, it remains zero for $t > 0$ for arbitrary input functions. However, if the estimation error is not zero at $t = 0$ there is no guarantee that the estimation error goes to zero as $t \to \infty$. Therefore we are often interested in a distributed relative state observer that is *stable*.

Definition 3.2.2. A distributed relative state observer Ω_i for agent i is called *stable* if for each pair of initial values $x_i(0), w_i(0)$ for $j = 1, 2, \dots, p$ we have $e_i(t) \to 0$ as $t \to \infty$.

We are interested in finding necessary and sufficient conditions for existence of a stable distributed relative state observer. We will first show that the detectability of the pair (C, A) is sufficient for the existence of a stable distributed relative state observer for agent i for $i = 1, 2, \dots, p$. Consider dynamics of the observer Ω_i given by

$$\dot{w}_i = Aw_i + (B - GD) \sum_{j\in\mathcal{N}_i} (u_i - u_j) + G \left[\sum_{j\in\mathcal{N}_i} (y_i - y_j) - Cw_i \right] \tag{3.8}$$

for $i = 1, 2, \dots, p$ where G is a matrix to be determined. It can be observed that error signal

$e_i = w_i - \sum_{j\in\mathcal{N}_i}(x_i - x_j)$ satisfies the differential equation

$$\begin{aligned}
\dot{e}_i &= \dot{w}_i - \sum_{j\in\mathcal{N}_i}(\dot{x}_i - \dot{x}_j)\\
&= Aw_i + (B-GD)\sum_{j\in\mathcal{N}_i}(u_i - u_j) + G\left[\sum_{j\in\mathcal{N}_i}(y_i - y_j) - Cw_i\right] - \sum_{j\in\mathcal{N}_i}(Ax_i + Bu_i - Ax_j + Bu_j)\\
&= Aw_i + (B-GD)\sum_{j\in\mathcal{N}_i}(u_i - u_j) + G\left[\sum_{j\in\mathcal{N}_i}(Cx_i - Cx_j + Du_i - Du_j) - Cw_i\right]\\
&\quad - \sum_{j\in\mathcal{N}_i}(Ax_i + Bu_i - Ax_j + Bu_j)\\
&= Aw_i + B\sum_{j\in\mathcal{N}_i}(u_i - u_j) + GC\left[\sum_{j\in\mathcal{N}_i}(x_i - x_j) - w_i\right] - \sum_{j\in\mathcal{N}_i}(Ax_i + Bu_i - Ax_j + Bu_j)\\
&= Aw_i + GC\left[\sum_{j\in\mathcal{N}_i}(x_i - x_j) - w_i\right] - A\sum_{j\in\mathcal{N}_i}(x_i - x_j)\\
&= (A-GC)\left[w_i - \sum_{j\in\mathcal{N}_i}(x_i - x_j)\right] = (A-GC)e_i.
\end{aligned}$$

for $i = 1, 2, \ldots, p$. Note that this implies that if $e_i(0) = 0$ then, for arbitrary input functions $u_1, \ldots, u_p$, we have $e_i(t) = 0$ for $t > 0$ and $i = 1, 2, \ldots, p$ thus (3.8) is a distributed relative state observer for agent i. Moreover, if the pair (C, A) is detectable then there exists G such that $A - GC$ is Hurwitz from which it follows that $e_i(t) \to 0$ as $t \to \infty$ for $i = 1, 2, \ldots, p$. Hence Ω_i given by (3.8) is also a distributed *stable* relative state observer for agent i for $i = 1, 2, \ldots, p$. Thus the detectability of (C, A) is sufficient for the existence of a distributed stable relative state observer for agent i for $i = 1, 2, \ldots, p$. It can also be proven that this is also a necessary condition but this will not be done in this thesis. We refer to [1] for the necessary part of the proof.

3.3 Observer based synchronization

We will assume now that only the relative outputs are available for the protocol and we consider agent dynamics (3.6). In the previous section we have constructed a distributed (stable) relative state observer for the network. Combining this with the results on relative state feedback we are inspired to use a *dynamic* protocol of the form

$$\begin{aligned}
\dot{w}_i &= Aw_i + (B-GD)\sum_{j\in\mathcal{N}_i}(u_i - u_j) + G\left[\sum_{j\in\mathcal{N}_i}(y_i - y_j) - Cw_i\right]\\
u_i &= Fw_i
\end{aligned}\tag{3.9}$$

for $i = 1, 2, \ldots, p$ to achieve synchronization of the network. Because of the structure of the protocol we will often call (3.9) an *observer based* protocol. Denote the *aggregate state* as $\mathbf{x} = \mathrm{col}(x_1, x_2, \ldots, x_p)$ and likewise define $\mathbf{w}$, $\mathbf{u}$ and $\mathbf{y}$. Then the full dynamics of the multi-agent network is described by

$$\begin{aligned}
\dot{\mathbf{x}} &= (I \otimes A)\mathbf{x} + (I \otimes B)\mathbf{u}\\
\mathbf{y} &= (I \otimes C)\mathbf{x} + (I \otimes D)\mathbf{u}\\
\dot{\mathbf{w}} &= I \otimes (A - GC)\mathbf{w} + L \otimes (B - GD)\mathbf{u} + (L \otimes G)\mathbf{y}\\
\mathbf{u} &= (I \otimes F)\mathbf{w}.
\end{aligned}$$

This leads to the closed-loop system

$$\begin{aligned}\begin{pmatrix}\dot{\mathbf{x}}\\ \dot{\mathbf{w}}\end{pmatrix} &= \begin{pmatrix} I\otimes A & (I\otimes B)(I\otimes F)\\ (L\otimes G)(I\otimes C) & I\otimes(A-GC)+L\otimes(BF-GDF)+(L\otimes G)(I\otimes D)(I\otimes F)\end{pmatrix}\begin{pmatrix}\mathbf{x}\\ \mathbf{w}\end{pmatrix}\\ &= \begin{pmatrix} I\otimes A & I\otimes BF\\ L\otimes GC & I\otimes(A-GC)+L\otimes BF\end{pmatrix}\begin{pmatrix}\mathbf{x}\\ \mathbf{w}\end{pmatrix} \qquad (3.10)\end{aligned}$$

The problem of synchronization is now to find F, G such that the relative states of the agents and the relative states of the dynamic protocol converge to zero as $t \to \infty$.

Definition 3.3.1. The network is said to be *synchronized* by the dynamic protocol if for all $i, j = 1, 2, \dots, p$ we have $x_i(t) - x_j(t) \to 0$ and $w_i(t) - w_j(t) \to 0$ as $t \to \infty$.

As mentioned before, there exists an unitary $p \times p$ matrix U such that $U^T L U = \Lambda_u$ where Λ_u is an upper triangular matrix with $0, \lambda_2, \dots, \lambda_p$ on the diagonal. Next, apply the state transformation

$$\begin{pmatrix}\tilde{\mathbf{x}}\\ \tilde{\mathbf{w}}\end{pmatrix} = \begin{pmatrix} U^T\otimes I & 0\\ 0 & U^T\otimes I\end{pmatrix}\begin{pmatrix}\mathbf{x}\\ \mathbf{w}\end{pmatrix} \qquad (3.11)$$

to (3.10) to get the network dynamics

$$\begin{pmatrix}\dot{\tilde{\mathbf{x}}}\\ \dot{\tilde{\mathbf{w}}}\end{pmatrix} = \begin{pmatrix} I\otimes A & I\otimes BF\\ \Lambda_u\otimes GC & I\otimes(A-GC)+\Lambda_u\otimes BF\end{pmatrix}\begin{pmatrix}\tilde{\mathbf{x}}\\ \tilde{\mathbf{w}}\end{pmatrix}.$$

Similar as in the relative state feedback case there is an equivalence between the synchronization problem and a stabilization problem for a single system with multiple (observer based) controllers.

Lemma 3.3.2. *Consider the network with agent dynamics* (3.6)*. Assume the network graph is directed and contains a spanning tree. Then protocol* (3.9) *synchronizes the network if and only if the system*

$$\begin{aligned}\dot{x} &= Ax + Bu\\ y &= Cx + Du\end{aligned} \qquad (3.12)$$

is internally stabilized by all $p-1$ controllers

$$\begin{aligned}\dot{w} &= Aw + (B-GD)u + G(y-Cw)\\ u &= \lambda_i F w\end{aligned} \qquad (3.13)$$

where $i = 2, 3, \dots, p$ and λ_i is the ith eigenvalue of the Laplacian L.

Proof. Note that $\ker(L) = \operatorname{im}(1_p)$, where $1_p = (1, 1, \dots, 1) \in \mathbb{R}^p$. Let U be an orthogonal matrix such that $LU = \Lambda_u U$ where Λ_u is an upper triangular matrix with $0, \lambda_2, \dots, \lambda_p$ on the diagonal. We have $x_i - x_j \to 0$ if and only if $\mathbf{x}(t) \to \operatorname{im}(1_p \otimes I) = \ker(L \otimes I)$. This holds if and only if $(L \otimes I)\mathbf{x}(t) \to 0$. Since $\mathbf{x} = (U \otimes I)\tilde{\mathbf{x}}$ the latter holds if and only if $(LU \otimes I)\tilde{\mathbf{x}} \to 0$. Since $LU = U\Lambda_u$ and by the fact that U is nonsingular, this holds if and only if $(\Lambda_u \otimes I)\tilde{\mathbf{x}} \to 0$, equivalently $\tilde{x}_i(t) \to 0$ for $i = 2, 3, \dots, p$. The same argument applies to the variables w_i and $\tilde{w}_i$. Define $\bar{\mathbf{x}} = \operatorname{col}(\tilde{x}_2, \dots, \tilde{x}_p)$ and $\bar{\mathbf{w}} = \operatorname{col}(\tilde{w}_2, \dots, \tilde{w}_p)$. Clearly $\tilde{x}_i(t), \tilde{w}_i(t) \to 0$ for $i = 2, 3, \dots, p$ if and only if the system

$$\begin{pmatrix}\dot{\bar{\mathbf{x}}}\\ \dot{\bar{\mathbf{w}}}\end{pmatrix} = \begin{pmatrix} I_{p-1}\otimes A & I_{p-1}\otimes BF\\ \Lambda_2\otimes GC & I_{p-1}\otimes(A-GC)+\Lambda_2\otimes BF\end{pmatrix}\begin{pmatrix}\bar{\mathbf{x}}\\ \bar{\mathbf{w}}\end{pmatrix}$$

is stable. Here Λ_2 is equal to the matrix Λ_u without the first row and column. By applying state transformation $\hat{\mathbf{x}} = \bar{\mathbf{x}}$, $\hat{\mathbf{w}} = (\Lambda_2 \otimes I)\bar{\mathbf{w}}$ we obtain dynamics

$$\begin{pmatrix}\dot{\hat{\mathbf{x}}}\\ \dot{\hat{\mathbf{w}}}\end{pmatrix} = \begin{pmatrix} I_{p-1}\otimes A & \Lambda_2\otimes BF\\ I_{p-1}\otimes GC & I_{p-1}\otimes(A-GC)+\Lambda_2\otimes BF\end{pmatrix}\begin{pmatrix}\hat{\mathbf{x}}\\ \hat{\mathbf{w}}\end{pmatrix}$$

Note that $\bar{\mathbf{x}}, \bar{\mathbf{w}} \to 0$ if and only if $\hat{\mathbf{x}}, \hat{\mathbf{w}} \to 0$. It is not hard to see that $\hat{\mathbf{x}}, \hat{\mathbf{w}} \to 0$ if and only if the system (3.12) in internally stabilized by the $p-1$ controllers (3.13) for $i = 2, 3, \dots, p$. □

Lemma 3.3.2 provides the equivalence between the synchronization problem and a stabilization problem of a single system by multiple feedback controllers. We would like to know for which conditions these closed-systems can be stabilized. More precisely, we would like to find necessary and sufficient conditions for the existence of F, G such that the system (3.12) in internally stabilized by the $p-1$ controllers (3.13) for $i = 2, 3, \dots, p$. These $p-1$ closed-loop systems are internally stable if and only if the matrices

$$\begin{pmatrix} A & \lambda_i BF \\ GC & A + \lambda_i BF - GC \end{pmatrix} \tag{3.14}$$

are Hurwitz for $i = 2, 3, \dots, p$. Observe that by applying a similarity transformation we obtain

$$\begin{aligned} \begin{pmatrix} I & 0 \\ -I & I \end{pmatrix} \begin{pmatrix} A & \lambda_i BF \\ GC & A + \lambda_i BF - GC \end{pmatrix} \begin{pmatrix} I & 0 \\ I & I \end{pmatrix} &= \begin{pmatrix} A & \lambda_i BF \\ -(A - GC) & A - GC \end{pmatrix} \begin{pmatrix} I & 0 \\ I & I \end{pmatrix} \\ &= \begin{pmatrix} A + \lambda_i BF & \lambda_i BF \\ 0 & A - GC \end{pmatrix} \end{aligned}$$

for $i = 2, 3, \dots, p$. So the closed-loop matrices (3.14) are Hurwitz if and only if $A + \lambda_i BF$ and $A - GC$ are Hurwitz for $i = 2, 3, \dots, p$. It is well known that there exists G such that $A - GC$ is Hurwitz if and only if (C, A) is detectable. As shown before in Section 3.1 the stabilizability of (A, B) is necessary and sufficient for the existence of a single F such that $A + \lambda_i BF$ is Hurwitz for $i = 2, 3, \dots, p$. Hence stabilizability and detectability are necessary and sufficient conditions for solvability of the synchronization problem using observed based protocols.

Corollary 3.3.3. *Consider the network with agent dynamics* (3.6)*. Assume the network graph is directed and contains a spanning tree. Then there exists a dynamic protocol* (3.9) *that synchronizes the network if and only if* (A, B) *is stabilizable and* (C, A) *is detectable.*

In this chapter we have covered the synchronization problem using static and dynamic protocols. In the next two chapters we assume that the dynamics of the agents are uncertain, i.e. we consider the problem of *robust* synchronization.

Chapter 4

Robust synchronization

The main topic of this thesis is *robust* synchronization. In the robust synchronization problem we assume that the dynamics of each agent is uncertain. The perturbed system can then be represented by any system that is in a 'ball' around the nominal system, where the nominal system is the system without perturbations. In this thesis we allow multiplicative perturbations of the transfer matrix of the nominal system as will be discussed in Section 4.1. In particular we will consider robust synchronization of multiplicatively perturbed multi-agent systems using *observer based protocols* as the relative state feedback case was already covered in [2].

Like in the synchronization problem there is an equivalence between robust synchronization and robust stabilization of $p-1$ systems. This will be proven in Section 4.2. We will show that this equivalence holds for the case of undirected network graphs and heterogeneous perturbations on the agents. However for the directed graph case, we were only able to prove that this equivalence holds for the case of homogeneous perturbations.

4.1 Multiplicative perturbations

Consider the nominal system described by

$$\Sigma_n : \begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx. \end{aligned} \tag{4.1}$$

Let $T(s) = C(sI - A)^{-1}B$ represent the transfer matrix of this system. In this thesis we deal *multiplicative perturbations* on the nominal dynamics. We assume that the model contains uncertainties and that the exact model is described by the transfer matrix $T_\Delta(s) = T(s)(I + \Delta(s))$ where $\Delta \in R\mathcal{H}_\infty$. If we realize $\Delta(s) = C_\Delta(sI - A_\Delta)^{-1}B_\Delta + D_\Delta$ then we obtain the linear system

$$\Sigma_p : \begin{aligned} \dot{\xi} &= A_\Delta \xi + B_\Delta u \\ y_1 &= C_\Delta \xi + D_\Delta u \end{aligned} \tag{4.2}$$

which we call the *perturbation system.* We denote the transfer matrix of Σ_p by $\Delta(s)$ and we will often write the dynamics (4.2) shortly as $y_1 = \Delta u$. The exact model given by $T_\Delta(s)$ can be represented as in Figure 4.1

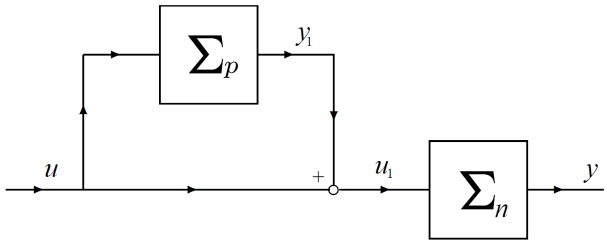

Figure 4.1: A multiplicatively perturbed system [17].

by noting that $Y(s) = T(s)U_1(s) = T(s)(Y_1(s) + U(s)) = T(s)(I + \Delta(s))U(s) = T_\Delta(s)U(s)$. It is an easy exercise to show that the transfer from u to y in Figure 4.1 can also be represented as in Figure 4.2

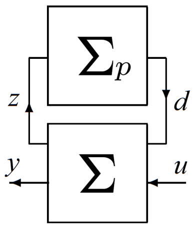

Figure 4.2: A second way to represent a multiplicatively perturbed system [17].

where the dynamics of $\Sigma \times \Sigma_p$ is described by

$$\begin{aligned} \dot{x} &= Ax + Bu + Bd \\ y &= Cx \\ z &= u \\ d &= \Delta z. \end{aligned} \tag{4.3}$$

So the exact model can also be represented by introducing additional inputs and outputs in the nominal system and adding a feedback loop around this system as shown in Figure 4.2.

Now we will turn to the case of multi-agent systems. Unlike in the synchronization problem, we will for simplicity assume that the feedthrough term from u_i to y_i is zero for each agent, i.e. we assume from now on that the nominal dynamics of agent i is of the form

$$\begin{aligned} \dot{x}_i &= Ax_i + Bu_i \\ y_i &= Cx_i \end{aligned} \tag{4.4}$$

for $i = 1, 2, \ldots, p$, where p is again the number of agents in the network. Denote $T(s)$ as the transfer matrix of nominal agent i. Then the exact dynamics of each agent is given by $T_{\Delta_i}(s) = T(s)(I + \Delta_i(s))$ where $\Delta_i \in R\mathcal{H}_\infty$ and $T_{\Delta_i}(s)$ is the transfer matrix of the multiplicative perturbed agent for $i = 1, 2, \ldots, p$. The perturbations $\Delta_i \in R\mathcal{H}_\infty$ are allowed to be distinct for $i = 1, 2, \ldots, p$. In this case we call the perturbations *heterogeneous* since each agent may be perturbed differently. When each agent is perturbed identically, i.e. $\Delta_i = \Delta \in R\mathcal{H}_\infty$ for $i = 1, 2, \ldots, p$ we call the perturbations *homogeneous*. As before, each individual agent can represented by the interconnection in Figure 4.2 and the dynamics of the perturbed agent i is given by

$$\begin{aligned} \dot{x}_i &= Ax_i + Bu_i + Bd_i \\ y_i &= Cx_i \\ z_i &= u_i \\ d_i &= \Delta_i z_i \end{aligned} \tag{4.5}$$

for $i = 1, 2, \ldots, p$.

4.2 Equivalence with robust stabilization

As in the synchronization problem we will derive an equivalence between robust synchronization and robust stabilization of a single system by $p-1$ feedback controllers. However in the case of robust synchronization we have to distinguish between the directed and the undirected network graph case. We will prove that in the *undirected* network graph case the problem of robust synchronization with *heterogeneous* perturbations of the agents is equivalent with a robust stabilization problem of a single system using $p-1$ controllers. However, in the *directed* network graph case we are only able to prove a similar result for the case that *homogeneous* perturbations on the agents are allowed.

In this thesis we consider multiplicatively perturbed agent dynamics (4.5) as discussed in the previous section. Like in the synchronization problem we will consider observer based protocols of the form

$$\begin{aligned} \dot{w}_i &= Aw_i + BF\sum_{j\in\mathcal{N}_i}(w_i - w_j) + G\left[\sum_{j\in\mathcal{N}_i}(y_i - y_j) - Cw_i\right] \\ u_i &= Fw_i. \end{aligned} \tag{4.6}$$

for $i = 1, 2, \ldots, p$. Denote the aggregate state vector by $\mathbf{x} = \mathrm{col}(x_1, x_2, \ldots, x_p)$ and likewise define $\mathbf{w}, \mathbf{z}, \mathbf{d}$. Then the interconnection of the network of perturbed agents (4.5) with dynamic protocols (4.6) yields the closed-loop network dynamics

$$\begin{aligned} \begin{pmatrix}\dot{\mathbf{x}}\\ \dot{\mathbf{w}}\end{pmatrix} &= \begin{pmatrix} I\otimes A & I\otimes BF \\ L\otimes GC & I\otimes(A-GC) + L\otimes BF\end{pmatrix}\begin{pmatrix}\mathbf{x}\\ \mathbf{w}\end{pmatrix} + \begin{pmatrix} I\otimes B\\ 0\end{pmatrix}\mathbf{d} \\ \mathbf{z} &= \begin{pmatrix}0 & I\otimes F\end{pmatrix}\begin{pmatrix}\mathbf{x}\\ \mathbf{w}\end{pmatrix} \\ \mathbf{d} &= \begin{pmatrix}\Delta_1 & & 0\\ & \ddots & \\ 0 & & \Delta_p\end{pmatrix}\mathbf{z}. \end{aligned} \tag{4.7}$$

Let $\gamma > 0$ be a desired tolerance. Then the problem of *robust* synchronization is to find F, G such that the system (4.7) is synchronized for all perturbations $\Delta_i \in R\mathcal{H}_\infty$ that are bounded by $\|\Delta_i\|_\infty \le \gamma$ with $i = 1, 2, \ldots, p$. Formally:

Definition 4.2.1. Given a desired tolerance $\gamma > 0$, we say that the protocol *robustly* synchronizes the network if for all i and for all $\Delta_i \in R\mathcal{H}_\infty$ with $\|\Delta_i\|_\infty \le \gamma$, the network is synchronized. The tolerance γ is called the *uncertainty radius* of the network.

In the remainder of this chapter we assume that the network graph is undirected and in addition we assume it is connected, equivalently $\lambda_2 > 0$. In that case the Laplacian matrix is a real symmetric positive semi-definite matrix so there exists an orthogonal $p\times p$ matrix U such that $U^T LU = \Lambda$ where $\Lambda = \mathrm{diag}(0, \lambda_2, \ldots, \lambda_p)$. With this U we apply the state transformation (3.11) together with the transformations $\tilde{\mathbf{d}} = (U^T\otimes I)\mathbf{d}$ and $\tilde{\mathbf{z}} = (U^T\otimes I)\mathbf{z}$ to obtain the transformed dynamics

$$\begin{aligned} \begin{pmatrix}\dot{\tilde{\mathbf{x}}}\\ \dot{\tilde{\mathbf{w}}}\end{pmatrix} &= \begin{pmatrix} I\otimes A & I\otimes BF \\ \Lambda\otimes GC & I\otimes(A-GC) + \Lambda\otimes BF\end{pmatrix}\begin{pmatrix}\tilde{\mathbf{x}}\\ \tilde{\mathbf{w}}\end{pmatrix} + \begin{pmatrix} I\otimes B\\ 0\end{pmatrix}\tilde{\mathbf{d}} \\ \tilde{\mathbf{z}} &= \begin{pmatrix}0 & I\otimes F\end{pmatrix}\begin{pmatrix}\tilde{\mathbf{x}}\\ \tilde{\mathbf{w}}\end{pmatrix} \\ \tilde{\mathbf{d}} &= (U^T\otimes I)\begin{pmatrix}\Delta_1 & & 0\\ & \ddots & \\ 0 & & \Delta_p\end{pmatrix}(U\otimes I)\tilde{\mathbf{z}}. \end{aligned} \tag{4.8}$$

Note that since the perturbations are assumed to be heterogeneous among the agents, the transfer matrix from $\tilde{\mathbf{z}}$ to $\tilde{\mathbf{d}}$ is in general not block diagonal. By the assumption that the network graph is undirected we are able to prove the equivalence between robust synchronization and robust stabilization of a single system by multiple controllers.

Theorem 4.2.2. *Consider the network with nominal agent dynamics* (4.4)*. Assume the network graph is undirected and connected. Let* $\gamma > 0$*. The following two statements are equivalent:*

1. *The dynamic protocol* (4.6) *synchronizes the network with multiplicatively perturbed agents:*

$$\begin{aligned} \dot{x}_i &= Ax_i + Bu_i + Bd_i, \\ y_i &= Cx_i, \\ z_i &= u_i \\ d_i &= \Delta_i z_i, \quad i = 1, 2, \dots, p, \end{aligned}$$

for all $\Delta_i \in R\mathcal{H}_\infty$ *with* $\|\Delta_i\|_\infty \leq \gamma$*.*

2. *The multiplicatively perturbed linear system*

$$\dot{x} = Ax + Bu + Bd, \quad y = Cx, \quad z = u, \quad d = \Delta z \tag{4.9}$$

is internally stabilized for all $\Delta \in R\mathcal{H}_\infty$ *such that* $\|\Delta\|_\infty \leq \gamma$ *by all* $p - 1$ *controllers*

$$\begin{aligned} \dot{w} &= Aw + Bu + G(y - Cw) \\ u &= \lambda_i F w, \end{aligned} \tag{4.10}$$

where $i = 2, 3, \dots, p$ *and* λ_i *is the* i*th eigenvalue of the Laplacian* L*.*

Proof. As in the proof of Lemma 3.3.2 observe that in (4.7) we have $x_i(t) - x_j(t) \to 0$ and $w_i(t) - w_j(t) \to 0$ for all i, j as $t \to \infty$ if and only if in (4.8) we have $\tilde{x}_i(t) \to 0$ and $\tilde{w}_i(t) \to 0$ for $i = 2, 3, \dots, p$ as $t \to \infty$.

(1. $\Rightarrow$ 2.) Assume that the dynamic protocol (4.6) synchronizes the network for all perturbations Δ_i with $\|\Delta_i\|_\infty \leq \gamma$ for $i = 1, 2, \dots, p$. Consider the system (4.9) and take an arbitrary $\Delta \in R\mathcal{H}_\infty$ such that $\|\Delta\|_\infty \leq \gamma$. We want to show that the closed-loop systems obtained by interconnecting the multiplicatively perturbed linear system (4.9) with controllers (4.10) for $i = 2, 3, \dots, p$ are internally stable. These systems are represented by

$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{w} \end{pmatrix} &= \begin{pmatrix} A & \lambda_i BF \\ GC & A - GC + \lambda_i BF \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} d \\ z &= \begin{pmatrix} 0 & \lambda_i F \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix} \\ d &= \Delta z. \end{aligned} \tag{4.11}$$

for $i = 2, 3, \dots, p$. Perturb each agent i with $\Delta_i = \Delta$ for all i in (4.8). Then we get the dynamics

$$\begin{aligned} \begin{pmatrix} \dot{\tilde{\mathbf{x}}} \\ \dot{\tilde{\mathbf{w}}} \end{pmatrix} &= \begin{pmatrix} I \otimes A & I \otimes BF \\ \Lambda \otimes GC & I \otimes (A - GC) + \Lambda \otimes BF \end{pmatrix} \begin{pmatrix} \tilde{\mathbf{x}} \\ \tilde{\mathbf{w}} \end{pmatrix} + \begin{pmatrix} I \otimes B \\ 0 \end{pmatrix} \tilde{\mathbf{d}} \\ \tilde{\mathbf{z}} &= \begin{pmatrix} 0 & I \otimes F \end{pmatrix} \begin{pmatrix} \tilde{\mathbf{x}} \\ \tilde{\mathbf{w}} \end{pmatrix} \\ \tilde{\mathbf{d}} &= (I \otimes \Delta) \tilde{\mathbf{z}}. \end{aligned} \tag{4.12}$$

Since the network is synchronized, it follows that $\tilde{x}_i \to 0$ and $\tilde{w}_i \to 0$ as $t \to \infty$ for $i = 2, 3, \dots, p$. This implies that for each $i = 2, 3, \dots, p$ for the system

$$\begin{aligned} \begin{pmatrix} \dot{\tilde{x}}_i \\ \dot{\tilde{w}}_i \end{pmatrix} &= \begin{pmatrix} A & BF \\ \lambda_i GC & A - GC + \lambda_i BF \end{pmatrix} \begin{pmatrix} \tilde{x}_i \\ \tilde{w}_i \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} \tilde{d}_i \\ \tilde{z}_i &= \begin{pmatrix} 0 & F \end{pmatrix} \begin{pmatrix} \tilde{x}_i \\ \tilde{w}_i \end{pmatrix} \\ \tilde{d}_i &= \Delta \tilde{z}_i. \end{aligned}$$

we have that $\tilde{x}_i \to 0$ and $\tilde{w}_i \to 0$ as $t \to \infty$. Applying the transformation $\tilde{w}_i = \lambda_i \bar{w}_i$ the above systems are copies of the systems (4.11) which are therefore internally stable for $i = 2, 3, \dots, p$.

(2. ⇒ 1.) Assume that all $p-1$ controllers (4.10) internally stabilize the system (4.9) for all $\Delta \in R\mathcal{H}_\infty$ with $\|\Delta\|_\infty \le \gamma$. By the small-gain theorem the closed-loop systems (4.11) are internally stable and their transfer matrices T_i from d to z satisfy $\|T_i\|_\infty < \frac{1}{\gamma}$ for $i = 2, 3, \dots, p$. We want to show that the dynamic protocol (4.6) synchronizes the perturbed network for all agent perturbation Δ_i with $\|\Delta_i\|_\infty \le \gamma$. Denote

$$\begin{pmatrix} \Delta_{11} & \cdots & \Delta_{1p} \\ \vdots & \ddots & \vdots \\ \Delta_{p1} & \cdots & \Delta_{pp} \end{pmatrix} = (U^T \otimes I) \begin{pmatrix} \Delta_1 & & 0 \\ & \ddots & \\ 0 & & \Delta_p \end{pmatrix} (U \otimes I). \tag{4.13}$$

Since U is orthogonal, the $\mathcal{H}_\infty$-norm of the left hand side is less than or equal to γ. We consider the dynamics of the transformed states $\tilde{x}_2, \tilde{x}_3, \dots, \tilde{x}_p$ and $\tilde{w}_2, \tilde{w}_3, \dots, \tilde{w}_p$. Define $\bar{\mathbf{x}} = \text{col}(\tilde{x}_2, \tilde{x}_3, \dots, \tilde{x}_p)$ and $\bar{\mathbf{w}}, \bar{\mathbf{z}}, \bar{\mathbf{d}}$ likewise. Then we get dynamics similar as in (4.8) given by

$$\begin{pmatrix} \dot{\bar{\mathbf{x}}} \\ \dot{\bar{\mathbf{w}}} \end{pmatrix} = \begin{pmatrix} I_{p-1} \otimes A & I_{p-1} \otimes BF \\ \Lambda_1 \otimes GC & I_{p-1} \otimes (A - GC) + \Lambda_1 \otimes BF \end{pmatrix} \begin{pmatrix} \bar{\mathbf{x}} \\ \bar{\mathbf{w}} \end{pmatrix} + \begin{pmatrix} I_{p-1} \otimes B \\ 0 \end{pmatrix} \bar{\mathbf{d}} \tag{4.14}$$

$$\bar{\mathbf{z}} = \begin{pmatrix} 0 & I_{p-1} \otimes F \end{pmatrix} \begin{pmatrix} \bar{\mathbf{x}} \\ \bar{\mathbf{w}} \end{pmatrix} \tag{4.15}$$

$$\bar{\mathbf{d}} = \begin{pmatrix} \Delta_{22} & \cdots & \Delta_{2p} \\ \vdots & \ddots & \vdots \\ \Delta_{p2} & \cdots & \Delta_{pp} \end{pmatrix} \bar{\mathbf{z}} + \begin{pmatrix} \Delta_{21} \\ \vdots \\ \Delta_{p1} \end{pmatrix} \tilde{z}_1, \tag{4.16}$$

where $\Lambda_1 = \text{diag}(\lambda_2, \lambda_3, \dots, \lambda_p)$. In this system the transfer matrix from $\bar{\mathbf{d}}$ to $\bar{\mathbf{z}}$ is equal to $T := \text{blockdiag}(T_2, T_3, \dots, T_p)$ and therefore $\|T\|_\infty < \frac{1}{\gamma}$. Because

$$\left\| \begin{pmatrix} \Delta_{22} & \cdots & \Delta_{2p} \\ \vdots & \ddots & \vdots \\ \Delta_{p2} & \cdots & \Delta_{pp} \end{pmatrix} \right\|_\infty \le \gamma$$

and since $\tilde{z}_1 = F\tilde{w}_1$ with $\dot{\tilde{w}}_1 = (A - GC)\tilde{w}_1$ stable, it follows that $\bar{\mathbf{x}}(t) \to 0$ and $\bar{\mathbf{w}}(t) \to 0$ as $t \to \infty$. So the dynamic protocol (4.6) synchronizes the network with multiplicative perturbed agents for all $\Delta_i \in R\mathcal{H}_\infty$ with $\|\Delta_i\|_\infty \le \gamma$. □

We will now consider the case that the network graph is directed and contains a spanning tree. Then one can prove a similar result as in Theorem 4.2.2, more precisely:

Proposition 4.2.3. *Consider the network with agent dynamics given by* (4.4)*. Assume the network graph is directed and contains a spanning tree. Let $\gamma > 0$. The following two statements are equivalent:*

1. *The dynamic protocol* (4.6) *synchronizes the network with multiplicatively perturbed agents:*

$$\begin{aligned} \dot{x}_i &= Ax_i + Bu_i + Bd_i, \\ y_i &= Cx_i, \\ z_i &= u_i \\ d_i &= \Delta z_i, \quad i = 1, 2, \dots, p, \end{aligned}$$

   for all $\Delta \in R\mathcal{H}_\infty$ with $\|\Delta\|_\infty \le \gamma$.

2. *The multiplicatively perturbed linear system*

$$\dot{x} = Ax + Bu + Bd, \quad y = Cx, \quad z = u, \quad d = \Delta z$$

is internally stabilized for all $\Delta \in R\mathcal{H}_\infty$ such that $\|\Delta\|_\infty \leq \gamma$ by all $p-1$ controllers

$$\begin{aligned} \dot{w} &= Aw + Bu + G(y - Cw) \\ u &= \lambda_i F w, \end{aligned}$$

where $i = 2, 3, \ldots, p$ and λ_i is the ith eigenvalue of the Laplacian L.

Sketch of the proof. The proof of this proposition is along the same lines as the proof of Theorem 4.2.2. Since the graph is directed there no longer exists an orthogonal matrix that transforms the Laplacian matrix into a diagonal matrix. Yet there still exists an unitary matrix U such that $U^T L U = \Lambda_u$ is a complex upper triangular matrix with $0, \lambda_2, \ldots, \lambda_p$ on the diagonal. The proof from 1. to 2. is now basically the same as before.

However in the proof from 2. to 1. we can no longer prove that the transfer matrix T from $\bar{\mathbf{d}}$ to $\bar{\mathbf{z}}$ is a block diagonal matrix. But, if we assume that the perturbations are homogeneous then the right-hand side of (4.13) remains block diagonal. Then the second term of (4.16) vanishes and the transfer matrix from $\bar{\mathbf{z}}$ to $\bar{\mathbf{d}}$ is block diagonal and then the small gain theorem can still be used. This can be seen by first using the small gain argument on the dynamics of $\tilde{x}_p, \tilde{w}_p, \tilde{z}_p, \tilde{d}_p$ and then prove by induction that the small gain argument also holds for the dynamics of $\tilde{x}_2, \tilde{w}_2, \tilde{z}_2, \tilde{d}_2$. ☐

Remark 4.2.4. Note that there is an fundamental difference between the result on undirected network graphs in Theorem 4.2.2 compared to the result on directed network graphs in Proposition 4.2.3. Whereas for the undirected case it is sufficient to find F, G such that the system (4.9) is robustly stabilized by all controllers (4.10) to achieve robust synchronization of the network for all $\|\Delta_i\|_\infty \leq \gamma, i = 2, 3, \ldots, p$, for the directed graph case the protocol (4.6) will only achieve robust synchronization for the case that $\Delta_i = \Delta, i = 2, \ldots, p$ and $\|\Delta\|_\infty \leq \gamma$ and in general not for heterogeneous perturbations.

Remark 4.2.5. Observe that to achieve robust synchronization with uncertainty radius γ, by the small-gain theorem and Theorem 4.2.2 (or Proposition 4.2.3) it is sufficient to find F, G such any of the controllers (4.10) solves the $\mathcal{H}_\infty$-control problem for the system

$$\begin{aligned} \dot{x} &= Ax + Bu + Bd \\ y &= Cx \\ z &= u \end{aligned}$$

in the sense that the closed-loop system is internally stable and the transfer matrices T_i from d to z satisfy $\|T_i\|_\infty < \frac{1}{\gamma}$ for $i = 2, \ldots, p$. In the next chapter we will focus on finding F and G explicitly such that such that these conditions are satisfied.

Chapter 5

Robustly synchronizing protocols

In this chapter we establish conditions for the existence of robustly synchronizing protocols for a given uncertainty tolerance and we provide a method to explicitly compute these protocols. We will first consider undirected network graphs and list the full proof in this case. Then we look at the more complicated case of directed network graphs. One is interested in maximizing the tolerance for which there still exists a robustly synchronizing protocol. For both the undirected as the directed graph case, we will also provide a maximal upper bound for the guaranteed uncertainty radius. Throughout this chapter we will assume that the pair (A, B) is stabilizable and the pair (C, A) is detectable.

5.1 Undirected network graphs

In this chapter we consider the pair of algebraic Riccati equations (ARE's) given by

$$P_0 A + A^T P_0 - P_0 B B^T P_0 = 0 \tag{5.1}$$

$$A Q_0 + Q_0 A^T - \nu\, Q_0 C^T C Q_0 + B B^T = 0 \tag{5.2}$$

where $\nu > 0$ is a real scalar used as parameter. We let $P_0 \geq 0$ be the maximal solution of (5.1). Also, for any $\nu > 0$ there exists a unique solution $Q_0(\nu) \geq 0$ to (5.2) such that $A - \nu Q_0 C^T C$ is Hurwitz, by the dual version of Lemma 2.3.3. In addition we consider slightly different versions of the ARE's (5.1) and (5.2) given by

$$PA + A^T P - PBB^T P + \sigma I = 0 \tag{5.3}$$

$$AQ + QA^T - \nu\, QC^T CQ + BB^T + \tau I = 0 \tag{5.4}$$

with real scalar parameters $\sigma, \tau > 0$. By Theorem 2.3.4 there exist unique stabilizing positive definite solutions to (5.3) and (5.4) which we will denote by $P(\sigma)$ and $Q(\nu, \tau)$ respectively. It should be noted that $P(\sigma) \downarrow P_0$ and $Q(\nu, \tau) \downarrow Q_0(\nu)$ as $\sigma, \tau \downarrow 0$. The main result of this section is stated below.

Theorem 5.1.1. *Consider the network with p agents, where the network graph is undirected and connected. Let perturbed agent i be given by*

$$\dot{x}_i = Ax_i + Bu_i + Bd_i, \quad y_i = Cx_i, \quad z_i = u_i, \quad d_i = \Delta_i z_i.$$

Let $\nu > 0$. Let P_0 be the maximal real symmetric solution and $Q_0(\nu)$ be the unique stabilizing solution of (5.1) *and* (5.2) *respectively. Let $\eta > 0$ be such that $\eta < \frac{1}{\rho(P_0 Q_0(\nu))}$. Let $P(\sigma), Q(\nu, \tau)$ be the unique solutions of* (5.3) *and* (5.4) *depending on parameters $\sigma, \tau > 0$. Let $\epsilon > 0$ be such that*

$$\eta < \frac{1}{\rho(P(\sigma) Q(\nu, \tau))}$$

for all $\sigma, \tau \in (0, \epsilon)$*. Next let* $\sigma \in (0, \epsilon)$ *and compute* $P(\sigma)$*, then let* $\tau \in (0, \epsilon)$ *be such that*

$$\tau I < \frac{\sigma}{\eta} P(\sigma)^{-2}.$$

Let $\gamma > 0$ *be such that*

$$\gamma \leq 2 \frac{\sqrt{\lambda_2 \lambda_p}}{\lambda_2 + \lambda_p} \sqrt{\frac{\eta}{1+\eta}}.$$

Define

$$N = \frac{2\lambda_2\lambda_p}{(\lambda_2 + \lambda_p)(1+\eta)}, \qquad G = \nu Q(\tau) C^T, \qquad F = -\frac{1}{N} B^T P(\sigma) \left(I - \eta Q(\tau) P(\sigma)\right)^{-1}. \tag{5.5}$$

Then dynamic protocol

$$\begin{aligned} \dot{w}_i &= A w_i + BF \sum_{j \in \mathcal{N}_i} (w_i - w_j) + G \left[\sum_{j \in \mathcal{N}_i} (y_i - y_j) - C w_i \right] \\ u_i &= F w_i \end{aligned} \tag{5.6}$$

synchronizes the network for all perturbations $\Delta_i \in R\mathcal{H}_\infty$*, with* $i = 1, 2, \ldots, p$ *such that* $\|\Delta_i\|_\infty \leq \gamma$*.*

Proof. By Theorem 4.2.2 we should prove that any of the controllers (4.10), with F and G chosen as above, solves the $\mathcal{H}_\infty$-control problem for the systems

$$\begin{aligned} \dot{x} &= Ax + Bu + Bd \\ y &= Cx \\ z &= u \end{aligned}$$

in the sense that the closed-loop systems are internally stable and the transfer matrices T_i from d to z satisfy $\|T_i\|_\infty < \frac{1}{\gamma}$ for $i = 2, \ldots, p$. These systems are represented by

$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{w} \end{pmatrix} &= \begin{pmatrix} A & \lambda_i BF \\ GC & A - GC + \lambda_i BF \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} d \\ z &= \begin{pmatrix} 0 & \lambda_i F \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix}. \end{aligned}$$

By applying the state transformation

$$\begin{pmatrix} \bar{x} \\ \bar{w} \end{pmatrix} = \begin{pmatrix} I & -I \\ 0 & I \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix}$$

we obtain the transformed closed-loop system

$$\begin{aligned} \begin{pmatrix} \dot{\bar{x}} \\ \dot{\bar{w}} \end{pmatrix} &= \begin{pmatrix} A - GC & 0 \\ GC & A + \lambda_i BF \end{pmatrix} \begin{pmatrix} \bar{x} \\ \bar{w} \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} d \\ z &= \begin{pmatrix} 0 & \lambda_i F \end{pmatrix} \begin{pmatrix} \bar{x} \\ \bar{w} \end{pmatrix}. \end{aligned}$$

Note that the transfer matrices T_i do not change after this state transformation. By the bounded real lemma the closed-loop systems are internally stable and transfer matrices T_i satisfy $\|T_i\|_\infty < \frac{1}{\gamma}$ if and only if there exist $Y_i > 0$ such that

$$Y_i \begin{pmatrix} A - GC & 0 \\ GC & A + \lambda_i BF \end{pmatrix} + \begin{pmatrix} A - GC & 0 \\ GC & A + \lambda_i BF \end{pmatrix}^T Y_i + \gamma^2 Y_i \begin{pmatrix} BB^T & 0 \\ 0 & 0 \end{pmatrix} Y_i + \lambda_i^2 \begin{pmatrix} 0 & 0 \\ 0 & F^T F \end{pmatrix} < 0$$

for $i = 2, 3, \ldots, p$. Let Y_i be of the form

$$Y_i = \begin{pmatrix} Y_1 & 0 \\ 0 & Y_2 \end{pmatrix} > 0$$

for $i = 2, 3, \ldots, p$. Then it suffices to prove that there exist $Y_1, Y_2 > 0$ such that

$$\begin{aligned}
&Y_i \begin{pmatrix} A - GC_p & 0 \\ GC & A + \lambda_i BF \end{pmatrix} + \begin{pmatrix} A - GC & 0 \\ GC & A + \lambda_i BF \end{pmatrix}^T Y_i + \gamma^2 Y_i \begin{pmatrix} BB^T & 0 \\ 0 & 0 \end{pmatrix} Y_i + \lambda_i^2 \begin{pmatrix} 0 & 0 \\ 0 & F^T F \end{pmatrix} \\
&= \begin{pmatrix} Y_1(A - GC) & 0 \\ Y_2 GC & Y_2(A + \lambda_i BF) \end{pmatrix} + \begin{pmatrix} (A - GC)^T Y_1 & C^T G^T Y_2 \\ 0 & (A + \lambda_i BF)^T Y_2 \end{pmatrix} \\
&+ \gamma^2 \begin{pmatrix} Y_1 BB^T Y_1 & 0 \\ 0 & 0 \end{pmatrix} + \lambda_i^2 \begin{pmatrix} 0 & 0 \\ 0 & F^T F \end{pmatrix} = \\
&\begin{pmatrix} Y_1(A - GC) + (A - GC)^T Y_1 + \gamma^2 Y_1 BB^T Y_1 & C^T G^T Y_2 \\ Y_2 GC & Y_2(A + \lambda_i BF) + (A + \lambda_i BF)^T Y_2 + \lambda_i^2 F^T F \end{pmatrix} < 0
\end{aligned}$$

for $i = 2, 3, \ldots, p$. For ease of notation we denote $P := P(\sigma), Q := Q(\nu, \tau)$. Recall that we have $G = \nu QC^T, F = -\frac{1}{N} B^T \tilde{P}$ where $\tilde{P} := P(I - \eta QP)^{-1}$. Then the diagonal and the off-diagonal elements of the latter matrix 2×2 block matrix become respectively

$$\begin{aligned}
\Psi_{11} &:= Y_1(A - \nu QC^T C) + (A - \nu QC^T C)^T Y_1 + \gamma^2 Y_1 BB^T Y_1 \\
\Psi_{22} &:= Y_2(A - \frac{\lambda_i}{N} BB^T \tilde{P}) + (A - \frac{\lambda_i}{N} BB^T \tilde{P})^T Y_2 + \left(\frac{\lambda_i}{N}\right)^2 \tilde{P} BB^T \tilde{P} \\
\Psi_{21} &:= \nu Y_2 QC^T C.
\end{aligned}$$

for $i = 2, 3, \ldots, p$. Note here that Ψ_{22} depends on i. It suffices to prove that there exist $Y_1, Y_2 > 0$ such that

$$\begin{pmatrix} \Psi_{11} & \Psi_{21}^T \\ \Psi_{21} & \Psi_{22} \end{pmatrix} < 0$$

for $i = 2, 3, \ldots, p$. The latter inequality holds if and only if

$$\begin{aligned}
&\begin{pmatrix} I & 0 \\ 0 & Q^{-1} Y_2^{-1} \end{pmatrix} \begin{pmatrix} \Psi_{11} & \Psi_{21}^T \\ \Psi_{21} & \Psi_{22} \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & Y_2^{-1} Q^{-1} \end{pmatrix} = \begin{pmatrix} \Psi_{11} & \Psi_{21}^T \\ Q^{-1} Y_2^{-1} \Psi_{21} & Q^{-1} Y_2^{-1} \Psi_{22} \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & Y_2^{-1} Q^{-1} \end{pmatrix} \\
&= \begin{pmatrix} \Psi_{11} & \Psi_{21}^T Y_2^{-1} Q^{-1} \\ Q^{-1} Y_2^{-1} \Psi_{21} & Q^{-1} Y_2^{-1} \Psi_{22} Y_2^{-1} Q^{-1} \end{pmatrix} = \begin{pmatrix} \Psi_{11} & \nu C^T C \\ \nu C^T C & Q^{-1} Y_2^{-1} \Psi_{22} Y_2^{-1} Q^{-1} \end{pmatrix} < 0 \qquad (5.7)
\end{aligned}$$

for $i = 2, 3, \ldots, p$. Define $Y_1 = \frac{1}{\gamma^2} Q^{-1}$. Then the term in the upper left corner of (5.7) satisfies

$$\begin{aligned}
\Psi_{11} &= Y_1(A - \nu QC^T C) + (A - \nu QC^T C)^T Y_1 + \gamma^2 Y_1 BB^T Y_1 \\
&= Y_1 A + A^T Y_1 - \nu Y_1 QC^T C - \nu C^T CQY_1 + \gamma^2 Y_1 BB^T Y_1 \\
&= \frac{1}{\gamma^2} Q^{-1}(AQ + QA - 2\nu QC^T CQ + BB^T) Q^{-1} < -\frac{\nu}{\gamma^2} C^T C.
\end{aligned}$$

Now we will look at the term in the lower right corner of (5.7). This is equal to

$$Q^{-1} Y_2^{-1} \Psi_{22} Y_2^{-1} Q^{-1} = Q^{-1} Y_2^{-1} \left[Y_2(A - \mu_i BB^T \tilde{P}) + (A - \mu_i BB^T \tilde{P})^T Y_2 + \mu_i^2 \tilde{P} BB^T \tilde{P} \right] Y_2^{-1} Q^{-1}$$

for $i = 2, 3, \ldots, p$ where $\mu_i := \frac{\lambda_i}{N}$. Define $Y_2 = \beta \tilde{P}$ where $\beta = \frac{(\lambda_2 + \lambda_p)^2 (1 + \eta)}{4 \lambda_2 \lambda_p}$. Note that $\tilde{P} > 0$ since

$$(P^{-1} - \eta Q)\tilde{P} = (P^{-1} - \eta Q) P (I - \eta QP)^{-1} = (I - \eta QP)(I - \eta QP)^{-1} = I$$

so $\tilde{P}^{-1} = P^{-1} - \eta Q > 0$ because $\eta < \frac{1}{\rho(PQ)}$. Hence $Y_2 > 0$ and substituting this Y_2 gives

$$\begin{aligned}
Q^{-1}Y_2^{-1}\Psi_{22}Y_2^{-1}Q^{-1} &= Q^{-1}Y_2^{-1}\left[Y_2(A - \mu_i BB^T\tilde{P}) + (A - \mu_i BB^T\tilde{P})^T Y_2 + \mu_i^2 \tilde{P}BB^T\tilde{P}\right]Y_2^{-1}Q^{-1} \\
&= \frac{1}{\beta^2}Q^{-1}\tilde{P}^{-1}\left[\beta\tilde{P}A + \beta A^T\tilde{P} - \mu_i(2\beta - \mu_i)\tilde{P}BB^T\tilde{P}\right]\tilde{P}^{-1}Q^{-1} \\
&= \frac{1}{\beta}Q^{-1}\tilde{P}^{-1}\left[\tilde{P}A + A^T\tilde{P} - \mu_i(2 - \frac{\mu_i}{\beta})\tilde{P}BB^T\tilde{P}\right]\tilde{P}^{-1}Q^{-1}
\end{aligned}$$

for $i = 2, 3, \ldots, p$. Define $R := I - \eta PQ$ and observe that $\tilde{P} = R^{-1}P = PR^{-T}$. Then

$$\begin{aligned}
&\tilde{P}A + A^T\tilde{P} - \mu_i(2 - \frac{\mu_i}{\beta})\tilde{P}BB^T\tilde{P} = R^{-1}PA + A^TPR^{-T} - \mu_i(2 - \frac{\mu_i}{\beta})R^{-1}PBB^TPR^{-T} \\
&= R^{-1}\left[PAR^T + RA^TP - \mu_i(2 - \frac{\mu_i}{\beta})PBB^TP\right]R^{-T} \\
&= R^{-1}\left[PA(I - \eta QP) + (I - \eta PQ)A^TP - \mu_i(2 - \frac{\mu_i}{\beta})PBB^TP\right]R^{-T} \\
&= R^{-1}\left[PA + A^TP - \eta P(AQ + QA^T)P - \mu_i(2 - \frac{\mu_i}{\beta})PBB^TP\right]R^{-T} \\
&= R^{-1}\left[PA + A^TP - \eta P(\nu QC^TCQ - BB^T - \tau I)P - \mu_i(2 - \frac{\mu_i}{\beta})PBB^TP\right]R^{-T} \\
&= R^{-1}\left[PA + A^TP - \nu\eta PQC^TCQP + \eta\tau P^2 - (2\mu_i - \frac{\mu_i^2}{\beta} - \eta)PBB^TP\right]R^{-T}
\end{aligned}$$

for $i = 2, 3, \ldots, p$. We will now show that

$$2\mu_i - \frac{\mu_i^2}{\beta} - \eta \geq 1 \tag{5.8}$$

for $i = 2, 3, \ldots, p$ or equivalently

$$\frac{\lambda_i}{N}\left(2 - \frac{\lambda_i}{N\beta}\right) \geq 1 + \eta$$

for $i = 2, 3, \ldots, p$. Recall that

$$\begin{aligned}
N &= \frac{2\lambda_2\lambda_p}{(\lambda_2 + \lambda_p)(1 + \eta)} \\
\beta &= \frac{(\lambda_2 + \lambda_p)^2(1 + \eta)}{4\lambda_2\lambda_p}.
\end{aligned}$$

By construction of N, β we have

$$\begin{aligned}
\frac{\lambda_2}{N}\left(2 - \frac{\lambda_2}{N\beta}\right) &= \frac{(\lambda_2 + \lambda_p)(1 + \eta)}{2\lambda_2\lambda_p}\lambda_2\left(2 - \frac{(\lambda_2 + \lambda_p)(1 + \eta)}{2\lambda_2\lambda_p}\frac{4\lambda_2\lambda_p}{(\lambda_2 + \lambda_p)^2(1 + \eta)}\lambda_2\right) \\
&= \frac{(\lambda_2 + \lambda_p)(1 + \eta)}{2\lambda_p}\left(2 - \frac{1}{2\lambda_2\lambda_p}\frac{4\lambda_2\lambda_p}{(\lambda_2 + \lambda_p)}\lambda_2\right) \\
&= \frac{(\lambda_2 + \lambda_p)(1 + \eta)}{2\lambda_p}\left(2 - \frac{2\lambda_2}{(\lambda_2 + \lambda_p)}\right) \\
&= \frac{1 + \eta}{\lambda_p}\left(\lambda_2 + \lambda_p - \frac{\lambda_2(\lambda_2 + \lambda_p)}{(\lambda_2 + \lambda_p)}\right) = 1 + \eta
\end{aligned}$$

and by symmetry of λ_2 and λ_p in N and β we similarly obtain

$$\frac{\lambda_p}{N}\left(2 - \frac{\lambda_p}{N\beta}\right) = 1 + \eta.$$

Since

$$\frac{\partial^2}{\partial x^2}\left[\frac{x}{N}\left(2-\frac{x}{N\beta}\right)\right] = -\frac{2}{N^2\beta} < 0$$

it is clear that

$$\frac{\lambda_i}{N}\left(2-\frac{\lambda_i}{N\beta}\right) \geq 1+\eta$$

for $i = 2,\dots,p$. We now finally have that

$$\begin{aligned}
&Q^{-1}Y_2^{-1}\Psi_{22}Y_2^{-1}Q^{-1}\\
&= \frac{1}{\beta}Q^{-1}P^{-1}RR^{-1}\left[PA + A^TP - \nu\eta PQC^TCQP + \eta\tau P^2 - (2\mu_i - \frac{\mu_i^2}{\beta} - \eta)PBB^TP\right]R^{-T}R^TP^{-1}Q^{-1}\\
&\leq \frac{1}{\beta}Q^{-1}P^{-1}\left[-\nu\eta PQC^TCQP + \eta\tau P^2 - \sigma I\right]P^{-1}Q^{-1} < -\frac{\nu\eta}{\beta}C^TC
\end{aligned}$$

for $i = 2,3,\dots,p$ since $\tau I < \frac{\sigma}{\eta}P^{-2}$. In fact we have proven now that

$$\begin{pmatrix} \Psi_{11} & \nu C^TC \\ \nu C^TC & Q^{-1}Y_2^{-1}\Psi_{22}Y_2^{-1}Q^{-1}\end{pmatrix} < \nu\begin{pmatrix} -\frac{1}{\gamma^2}C^TC & C^TC \\ C^TC & -\frac{\eta}{\beta}C^TC\end{pmatrix} = \nu\begin{pmatrix} -\frac{1}{\gamma^2} & 1 \\ 1 & -\frac{\eta}{\beta}\end{pmatrix}\otimes C^TC$$

for $i = 2,3,\dots,p$. We will now prove that the latter is negative semi-definite for $i = 2,3,\dots,p$. This holds if and only if the 2×2 matrix on the right-hand side of the equation is negative semi-definite. Since the diagonal elements are negative we only have to prove that the determinant is nonnegative, i.e. $\frac{1}{\gamma^2}\frac{\eta}{\beta} - 1 \geq 0$, equivalently $\frac{1}{\gamma^2}\frac{\eta}{\beta} \geq 1$. We will prove that this holds. We know that γ satisfies

$$2\frac{\sqrt{\lambda_2\lambda_p}}{\lambda_2+\lambda_p}\sqrt{\frac{\eta}{1+\eta}} \geq \gamma$$

and therefore

$$\frac{1}{\gamma^2}\frac{4\lambda_2\lambda_p}{(\lambda_2+\lambda_p)^2}\frac{\eta}{1+\eta} = \frac{1}{\gamma^2}\frac{\eta}{\beta} \geq 1.$$

We conclude that the inequality in (5.7) holds and hence by the bounded real lemma we have that the closed-loop systems are internally stable and $\|T_i\|_\infty < \frac{1}{\gamma}$ for $i = 2,3,\dots,p$. By the small gain theorem and Theorem 4.2.2 it follows that the dynamic protocol (5.6) synchronizes the network for all perturbations $\Delta_i \in R\mathcal{H}_\infty$ such that $\|\Delta_i\|_\infty \leq \gamma$ for $i = 1,2,\dots,p$. □

Theorem 5.1.1 provides a method to compute a robustly synchronizing protocol for a given uncertainty radius. One is of course also interested in the question how large the uncertainty radius can be chosen so that there still exists a robustly synchronizing protocol. This issue is discussed next.

5.1.1 Maximal uncertainty radius

Using the result obtained in Theorem 5.1.1 we will focus on maximizing the uncertainty radius for which there still exists a robustly synchronizing protocol. It can be observed that if η is chosen larger in Theorem 5.1.1 then the upper bound for γ also becomes larger. However η is bounded by $\eta < \frac{1}{\rho(P_0Q_0(\nu))}$. In the next remark we discuss how to maximize this bound.

Remark 5.1.2. Note that in Proposition 5.1.3 and Theorem 5.1.1 the parameter ν is still free to be chosen. It should be observed that the algebraic Riccati equation (5.2) is associated with the linear system

$$\dot{x} = A^T x + C^T u, \qquad x(0) = x_0 \tag{5.9}$$

with the optimal state feedback control

$$u = -CQ_0(\nu)x \tag{5.10}$$

and optimal performance

$$J(x_0,\nu) = x_0^T Q_0(\nu)x_0 = \min_{\substack{u(.)\text{ s.t.}\\ x(t)\to 0 \text{ as } t\to\infty}} \int_0^\infty (x^T BB^T x + \frac{1}{\nu}u^T u)dt. \tag{5.11}$$

Clearly if ν increases and $x_0 \neq 0$ then, for a given optimal u, $J(x_0,\nu)$ decreases and especially $Q_0(\nu)$ wil decrease, i.e. $Q_0(\nu_l) \leq Q_0(\nu_s)$ for $\nu_l > \nu_s$. Denote

$$Q_{\min} = \lim_{\nu\to\infty} Q_0(\nu). \tag{5.12}$$

It is then clear that for all $\nu > 0$ we have $Q_{\min} \leq Q_0(\nu)$. Moreover it can be observed that for any $\eta < \frac{1}{\rho(P_0 Q_{\min})}$ there exists ν sufficiently large such that $\eta < \frac{1}{\rho(P_0 Q_0(\nu))}$. In this way we have maximized the upper bound on η.

The next proposition uses this upper bound for η to provide an upper bound on the maximal achievable uncertainty radius.

Proposition 5.1.3. *Assume the network graph is undirected and connected. Let $P_0 \geq 0$ be the maximal solution of* (5.1) *and $Q_{\min}$ be defined as in* (5.12)*. For every $\gamma > 0$ that satisfies*

$$\gamma < 2\frac{\sqrt{\lambda_2\lambda_p}}{\lambda_2+\lambda_p}\frac{1}{\sqrt{1+\rho(P_0Q_{\min})}} \tag{5.13}$$

There exists dynamic protocol (4.6) *such that the network is synchronized for all perturbations $\Delta_i \in R\mathcal{H}_\infty$ such that $\|\Delta_i\|_\infty \leq \gamma$ for $i = 1,2,\ldots,p$.*

Proof. Assume γ satisfies (5.13) then by Remark 5.1.2 there exists ν sufficiently large such that

$$\gamma < 2\frac{\sqrt{\lambda_2\lambda_p}}{\lambda_2+\lambda_p}\frac{1}{\sqrt{1+\rho(P_0Q_0(\nu))}} < 2\frac{\sqrt{\lambda_2\lambda_p}}{\lambda_2+\lambda_p}\frac{1}{\sqrt{1+\rho(P_0Q_{\min})}}. \tag{5.14}$$

It is easy to see that, because of the relation in (5.14), there exist $\eta < \frac{1}{\rho(P_0Q_0(\nu))}$ such that

$$\gamma \leq 2\frac{\sqrt{\lambda_2\lambda_p}}{\lambda_2+\lambda_p}\frac{1}{\sqrt{1+\frac{1}{\eta}}} < 2\frac{\sqrt{\lambda_2\lambda_p}}{\lambda_2+\lambda_p}\frac{1}{\sqrt{1+\rho(P_0Q_0(\nu))}}.$$

Then for this ν we have that η, γ satisfy the conditions in Theorem 5.1.1 and one can construct a robustly synchronizing protocol by (5.5). □

Remark 5.1.4. Note that the upper bound for the uncertainty radius mentioned in Proposition 5.1.3 is in terms of λ_2, λ_p. In fact, this bound depends only on the ratio $r := \lambda_2/\lambda_p$ since

$$\gamma < 2\frac{\sqrt{\lambda_2\lambda_p}}{\lambda_2+\lambda_p}\frac{1}{\sqrt{1+\rho(P_0Q_{\min})}} = 2\frac{\sqrt{\frac{\lambda_2}{\lambda_p}}}{1+\frac{\lambda_2}{\lambda_p}}\frac{1}{\sqrt{1+\rho(P_0Q_{\min})}} = \frac{2\sqrt{r}}{1+r}\frac{1}{\sqrt{1+\rho(P_0Q_{\min})}}. \tag{5.15}$$

Actually, if we would define $r := \lambda_p/\lambda_2$ then we get exactly the same bound in terms of r as in (5.15). This is due to the symmetry in λ_2, λ_p. Hence the bound in (5.15) holds for either $r = \lambda_2/\lambda_p$ or $r = \lambda_p/\lambda_2$.

Observe that the upper bound for the uncertainty radius is proportional to $2\sqrt{r}/(1+r)$. This quantity is maximal at $r = 1$ and in that case it is equal to 1. For *complete graphs* we indeed have that $\lambda_2 = \lambda_p$ so $r = 1$. Hence, for a complete graph the upper bound for the uncertainty radius is maximal. For *star graphs* we have that $\lambda_2/\lambda_p = 1/p$ and therefore $2\sqrt{r}/(1+r) \sim 2/\sqrt{p}$ for large p. For *line graphs* we have that $\lambda_2 = 2(1 - \cos(\pi/p))$ and $\lambda_p = 2(1 + \cos(\pi/p))$ [16]. Then $2\sqrt{r}/(1+r) \sim \pi/p$ for large p. Similarly it can be shown that for *cycle graphs* we have that $2\sqrt{r}/(1+r) \sim \pi/p$ as for large p. Hence, in a multi-agent system with a line or cycle network graph, the upper bound on the permissible uncertainty radius decreases faster (as the number of agents increases) compared to the case where the network graph is a star graph or a complete graph.

Remark 5.1.5. In the proof of Theorem 5.1.1 we have to satisfy the condition $\frac{1}{\gamma^2}\frac{\eta}{\beta} \geq 1$, equivalently $\gamma \leq \sqrt{\frac{\eta}{\beta}}$. To maximize the upper bound for γ we want to choose β as small as possible, however β has to satisfy (5.8). It can be noted that in Theorem 5.1.1 the constants N and β are already chosen optimally in the sense that N and β solve the optimization problem

$$\begin{aligned} &\min_{N,\beta} \beta \quad \text{s.t.} \\ &\frac{\lambda_i}{N}\left(2 - \frac{\lambda_i}{N\beta}\right) \geq 1 + \eta, \quad i = 2, 3, \dots, p. \end{aligned}$$

for a given $0 < \lambda_2 \leq \cdots \leq \lambda_p$ and $\eta > 0$. The proof is left as an exercise to the reader.

5.2 Directed network graphs

In the previous section we have solved the robust synchronization problem with heterogeneous multiplicative perturbations in the undirected network graph case. We will now consider the case that the graph is directed and contains a spanning tree. Since the network graph is directed the eigenvalues of the Laplacian matrix will in general be complex. Therefore important properties of the multi-agent network will not only be expressed in terms of the real part but also depend on the imaginary part of the eigenvalues. In this subsection we will use the shorthand notation $\kappa_i := \text{Re}(\lambda_i)$. Recall that the eigenvalues λ_i are ordered such that

$$0 < \text{Re}(\lambda_2) \leq \text{Re}(\lambda_3) \leq \cdots \leq \text{Re}(\lambda_p).$$

and therefore $0 < \kappa_2 \leq \kappa_3 \leq \cdots \leq \kappa_p$. We define $\theta_{\max}$ as the maximal argument of all λ_i, i.e.

$$\theta_{\max} := \max_{i=2,3,\dots,p} \text{Arg}(\lambda_i).$$

Here "Arg" denotes the principal value of the argument. Note that $-\pi/2 < \text{Arg}(\lambda_i) < \pi/2$ since $\text{Re}(\lambda_i) > 0$ for $i = 2, 3, \dots, p$. We now state the main theorem for the directed network graphs case. This result is analogous to the undirected graph case except that in this case only homogeneous perturbations of the agents are allowed.

Theorem 5.2.1. *Consider the network with p agents, where the network graph is undirected and connected. Let perturbed agent i be given by*

$$\dot{x}_i = Ax_i + Bu_i + Bd_i, \quad y_i = Cx_i, \quad z_i = u_i, \quad d_i = \Delta z_i.$$

Let $\nu > 0$. Let P_0 be the maximal solution of (5.1) and let $Q_0(\nu)$ be the unique stabilizing solution of (5.2). Let $\eta > 0$ be such that $\eta < \frac{1}{\rho(P_0 Q_0(\nu))}$. Let $P(\sigma), Q(\nu, \tau)$ be the unique stabilizing solutions to (5.3) and (5.4) respectively, depending on the parameters $\sigma, \tau > 0$. Let $\epsilon > 0$ be such that

$$\eta < \frac{1}{\rho(P(\sigma)Q(\tau))}$$

for all $\sigma,\tau \in (0,\epsilon)$. Next let $\sigma \in (0,\epsilon)$ and compute $P(\sigma)$. Then let $\tau \in (0,\epsilon)$ be such that

$$\tau I < \frac{\sigma}{\eta} P(\sigma)^{-2}$$

Let $\gamma > 0$ be such that

$$\gamma \le 2\cos(\theta_{\max}) \frac{\sqrt{\kappa_2 \kappa_p}}{\kappa_2 + \kappa_p} \sqrt{\frac{\eta}{1+\eta}}.$$

Define

$$N = \frac{2\kappa_2\kappa_p}{(\kappa_2+\kappa_p)(1+\eta)}, \qquad G = \nu Q(\tau) C^T, \qquad F = -\frac{1}{N} B^T P(\sigma)(I - \eta Q(\tau) P(\sigma))^{-1}.$$

Then the dynamic protocol (5.6) *synchronizes the network for all perturbations $\Delta \in R\mathcal{H}_\infty$ such that $\|\Delta\|_\infty \le \gamma$.*

Sketch of the proof. The proof follows the same lines as in undirected graph case except that we take the conjugate transpose instead of the transpose in the bounded real lemma. Since the eigenvalues of the Laplacian matrix are in general complex we get a slightly different condition in (5.8). In can be observed that in the directed graph case this condition becomes

$$\mu_i + \mu_i^* - \frac{|\mu_i|^2}{\beta} - \eta \ge 1$$

for $i = 2,3,\ldots,p$ where $\mu_i := \frac{\lambda_i}{N}$ or equivalently

$$\begin{aligned}
&2\frac{\mathrm{Re}\lambda_i}{N} - \frac{(\mathrm{Re}\lambda_i)^2 + (\mathrm{Im}\lambda_i)^2}{N^2\beta} = \frac{\mathrm{Re}\lambda_i}{N}\left[2 - \frac{(\mathrm{Re}\lambda_i)^2 + (\mathrm{Im}\lambda_i)^2}{(\mathrm{Re}\lambda_i)^2}\frac{\mathrm{Re}\lambda_i}{N\beta}\right] \\
&= \frac{\mathrm{Re}\lambda_i}{N}\left[2 - \left(1 + \frac{(\mathrm{Im}\lambda_i)^2}{(\mathrm{Re}\lambda_i)^2}\right)\frac{\mathrm{Re}\lambda_i}{N\beta}\right] = \frac{\mathrm{Re}\lambda_i}{N}\left[2 - \left(1 + \tan^2(\mathrm{Arg}(\lambda_i))\right)\frac{\mathrm{Re}\lambda_i}{N\beta}\right] \\
&= \frac{\mathrm{Re}\lambda_i}{N}\left[2 - \sec^2(\mathrm{Arg}(\lambda_i))\frac{\mathrm{Re}\lambda_i}{N\beta}\right] \ge 1 + \eta
\end{aligned}$$

for $i = 2,3,\ldots,p$. Here $\sec(x)$ is the secant function defined by $\sec(x) = 1/\cos(x)$. The latter inequality certainly holds if one can prove that

$$\frac{\mathrm{Re}\lambda_i}{N}\left[2 - \sec^2(\theta_{\max})\frac{\mathrm{Re}\lambda_i}{N\beta}\right] \ge 1 + \eta$$

for $i = 2,3,\ldots,p$. This is what we will prove next. This time β is chosen slightly different and it is now given by

$$\beta = \frac{\sec^2(\theta_{\max})(\kappa_2+\kappa_p)^2(1+\eta)}{4\kappa_2\kappa_p}.$$

By construction of N, β we have

$$\begin{aligned}
&\frac{\mathrm{Re}\lambda_2}{N}\left[2 - \sec^2(\theta_{\max})\frac{\mathrm{Re}\lambda_2}{N\beta}\right] = \frac{\kappa_2}{N}\left[2 - \sec^2(\theta_{\max})\frac{\kappa_2}{N\beta}\right] \\
&= \frac{(\kappa_2+\kappa_p)(1+\eta)}{2\kappa_2\kappa_p}\kappa_2\left(2 - \frac{(\kappa_2+\kappa_p)(1+\eta)}{2\kappa_2\kappa_p}\frac{4\kappa_2\kappa_p}{\sec^2(\theta_{\max})(\kappa_2+\kappa_p)^2(1+\eta)}\kappa_2\sec^2(\theta_{\max})\right) \\
&= \frac{(\kappa_2+\kappa_p)(1+\eta)}{2\kappa_p}\left(2 - \frac{1}{2\kappa_2\kappa_p}\frac{4\kappa_2\kappa_p}{(\kappa_2+\kappa_p)}\kappa_2\right) \\
&= \frac{(\kappa_2+\kappa_p)(1+\eta)}{2\kappa_p}\left(2 - \frac{2\kappa_2}{(\kappa_2+\kappa_p)}\right) \\
&= \frac{1+\eta}{\kappa_p}\left(\kappa_2+\kappa_p - \frac{\kappa_2(\kappa_2+\kappa_p)}{(\kappa_2+\kappa_p)}\right) = 1+\eta
\end{aligned}$$

and by symmetry one can also show that

$$\frac{\mathrm{Re}\lambda_p}{N}\left[2-\sec^2(\theta_{\max})\frac{\mathrm{Re}\lambda_p}{N\beta}\right]=\frac{\kappa_p}{N}\left[2-\sec^2(\theta_{\max})\frac{\kappa_p}{N\beta}\right]=1+\eta.$$

Since $0\le\theta_{\max}<\pi/2$ we have that

$$\frac{\partial^2}{\partial x^2}\left[\frac{x}{N}\left(2-\sec^2(\theta_{\max})\frac{x}{N\beta}\right)\right]=-\frac{2\sec^2(\theta_{\max})}{N^2\beta}<0.$$

It is now clear that

$$\frac{\mathrm{Re}\lambda_i}{N}\left[2-\sec^2(\mathrm{Arg}(\lambda_i))\frac{\mathrm{Re}\lambda_i}{N\beta}\right]\ge\frac{\mathrm{Re}\lambda_i}{N}\left[2-\sec^2(\theta_{\max})\frac{\mathrm{Re}\lambda_i}{N\beta}\right]\ge 1+\eta$$

for $i=2,\dots,p$. The proof is along on the same lines as in the undirected case until one has to prove that

$$\frac{1}{\gamma^2}\frac{\eta}{\beta}-1\ge 0.$$

equivalently $\frac{1}{\gamma^2}\frac{\eta}{\beta}\ge 1$. We will prove that this holds. We have

$$2\cos(\theta_{\max})\frac{\sqrt{\kappa_2\kappa_p}}{\kappa_2+\kappa_p}\sqrt{\frac{\eta}{1+\eta}}\ge\gamma\quad\Leftrightarrow\quad\frac{1}{\gamma^2}\frac{4\kappa_2\kappa_p}{(\kappa_2+\kappa_p)^2}\frac{\eta\cos^2(\theta_{\max})}{1+\eta}\ge 1$$

and therefore

$$\frac{1}{\gamma^2}\frac{4\kappa_2\kappa_p}{(\kappa_2+\kappa_p)^2}\frac{\eta\cos^2(\theta_{\max})}{1+\eta}=\frac{1}{\gamma^2}\frac{4\kappa_2\kappa_p}{(\kappa_2+\kappa_p)^2}\frac{\eta}{\sec^2(\theta_{\max})(1+\eta)}=\frac{1}{\gamma^2}\frac{\eta}{\beta}\ge 1.$$

By the bounded real lemma, the small-gain theorem and Proposition 4.2.3 the result follows now. □

5.2.1 Maximal uncertainty radius

It can be observed that similar results as derived in Section 5.1.1 also hold for the case of directed network graphs and homogeneous perturbations. This time the maximal upper bound on the uncertainty radius is expressed in terms of the real part and the maximum argument of the eigenvalues of the Laplacian matrix. The proof of the proposition follows the same lines as the proof of Proposition 5.1.3.

Proposition 5.2.2. *Assume the network graph is directed and contains a spanning tree. Let $P_0\ge 0$ be the maximal solution of* (5.1) *and $Q_{\min}$ be defined as in* (5.12)*. For every $\gamma>0$ that satisfies*

$$\gamma<2\frac{\sqrt{\kappa_2\kappa_p}}{\kappa_2+\kappa_p}\frac{\cos(\theta_{\max})}{\sqrt{1+\rho(P_0Q_{\min})}}\tag{5.16}$$

There exists dynamic protocol (4.6) *such that the network is synchronized for all perturbations $\Delta_i=\Delta\in R\mathcal{H}_\infty$ such that $\|\Delta\|_\infty\le\gamma$ for $i=1,2,\dots,p$.*

Remark 5.2.3. Unlike in Remark 5.1.5 for undirected graphs, in the case of directed network graphs we are unable to prove that N,β are chosen optimally in Theorem 5.2.1, i.e. N,β are not the solutions of the optimization problem

$$\begin{aligned}&\min_{N,\beta}\beta\quad\text{s.t.}\\&\frac{\kappa_i}{N}\left(2-\sec^2(\theta_i)\frac{\kappa_i}{N\beta}\right)\ge 1+\eta,\quad i=2,3,\dots,p.\end{aligned}$$

where $\theta_i := \mathrm{Arg}(\lambda_i)$ and $0 < \lambda_2 \leq \cdots \leq \lambda_p$ and $\eta > 0$ are given. However it makes sense to express N and β in terms of the real part and the maximum argument of the eigenvalues of the Laplacian since it can be shown that the individual (N, β depend on i) optimal solutions are given by

$$N_i = \frac{\kappa_i}{1+\eta}, \qquad \beta_i = \sec^2(\theta_i)(1+\eta).$$

By the condition $\gamma \leq \sqrt{\frac{\eta}{\beta_i}}$ for $i = 2, 3, \ldots, p$ it follows that γ must at least satisfy

$$\gamma \leq \cos(\theta_{\max})\sqrt{\frac{\eta}{1+\eta}}$$

so this upper bound on the tolerance depends on the maximum argument of the eigenvalues of the Laplacian.

Chapter 6

An LMI-based approach to the $\mathcal{H}_\infty$-control problem

In this chapter we will provide an LMI-based approach to the $\mathcal{H}_\infty$-control problem. In the $\mathcal{H}_\infty$-control problem we consider a linear input-output system that is affected by disturbances. We would like to construct a controller that minimizes the effect of these disturbances on certain (additional) outputs of the system. More explicitly, in the $\mathcal{H}_\infty$-control problem the aim is to construct a controller such that the $\mathcal{H}_\infty$-norm of the closed-loop transfer matrix is smaller than a given tolerance. We will derive necessary and sufficient conditions for the solvability of the $\mathcal{H}_\infty$-control problem for an a priori given tolerance. It will be shown that these conditions can be expressed in terms of the solvability of multiple linear matrix inequalities (LMI's). Also a characterization will be given for controllers that solve the $\mathcal{H}_\infty$-control problem.

We will start this chapter with introducing the setting of the general $\mathcal{H}_\infty$-control problem. Then in Section 6.2 the main result will be given where we first consider the case that in the nominal system the feedthrough term from the input to the output is zero and then we will extend the result to the case that this term does not vanish.

6.1 Formulation of the $\mathcal{H}_\infty$-control problem

We will now formulate the general $\mathcal{H}_\infty$-control problem. Suppose we are given a linear input-output system Σ. We assume that the dynamics of the system is affected by disturbances. The system Σ is represented by:

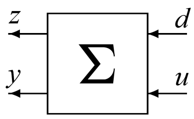

Figure 6.1: Representation of a system with disturbances [17].

The system as in Figure 6.1 has two inputs, the control input u and the input d representing a disturbance input acting on the system. We want to minimize the dependence of the output z on the input d. The output y represents the measurement we make on the system. This measurement will be used to choose our input u, which in turn is the tool we have to minimize the effect of d on z [17]. The transfer from y to u is determined by a controller we call Γ.

We assume Σ to be a finite-dimensional linear time-invariant system so the dynamics of Σ is in general described by

$$\begin{aligned} \dot{x} &= Ax + Bu + Ed \\ y &= C_1 x + D_{11} u + D_{12} d \\ z &= C_2 x + D_{21} u + D_{22} d \end{aligned} \tag{6.1}$$

with $x \in \mathbb{R}^n, u \in \mathbb{R}^m, y \in \mathbb{R}^p, d \in \mathbb{R}^{q_1}, z \in \mathbb{R}^{q_2}$ and all matrices are of appropriate dimensions. We assume that the dynamics of controller Γ is given by

$$\begin{aligned} \dot{w} &= Kw + Ly \\ u &= Mw + Ny \end{aligned} \tag{6.2}$$

with $w \in \mathbb{R}^{n_c}$. The dimension n_c and the matrices K, L, M, N are to be determined. We denote by T the transfer matrix of $\Sigma \times \Gamma$ from d to z and we assume that controller Γ is chosen such that T is stable. As mentioned before we want to minimize the dependence of the output z on the input d. The performance measure we will use for this is the $\mathcal{H}_\infty$-norm of T. The aim in the $\mathcal{H}_\infty$-control problem is then to design a controller such that the $\mathcal{H}_\infty$-norm of T is bounded by some a priori given tolerance.

Definition 6.1.1. Let $\gamma > 0$ be a desired tolerance. We say that controller Γ solves the $\mathcal{H}_\infty$-control problem with tolerance γ if the closed-loop system $\Sigma \times \Gamma$ is internally stable and its transfer matrix satisfies $||T||_\infty < \gamma$.

One could be interested in necessary and sufficient conditions for the existence of a controller that solves the $\mathcal{H}_\infty$-control problem for given γ. This is the subject of the next section.

6.2 Solution to the $\mathcal{H}_\infty$-control problem

In this section we establish necessary and sufficient conditions for the solvability of the general $\mathcal{H}_\infty$-control problem. We show that these conditions can be expressed in term of the solvability of certain LMI's. We will first consider the case that there is no feedthrough term from u to y, equivalently $D_{11} = 0$. In this case the well-posedness of the interconnection $\Sigma \times \Gamma$ is guaranteed. Then in Section 6.2.2 we look at the general case that D_{11} is not necessarily equal to the zero matrix and we explain how to deal with the well-posedness condition of the interconnection $\Sigma \times \Gamma$.

6.2.1 The case that $D_{11} = 0$

Throughout this section we assume $D_{11} = 0$. In addition we assume, without loss of generality, that

$$\begin{pmatrix} B \\ D_{21} \end{pmatrix}, \qquad \begin{pmatrix} C_1 & D_{12} \end{pmatrix} \tag{6.3}$$

have full column rank and full row rank respectively. If this does not hold then the number of inputs or outputs respectively can be reduced so that the latter rank condition does hold. We denote by T the transfer matrix from d to z of the interconnected system $\Sigma \times \Gamma$. The closed-loop equations for $\Sigma \times \Gamma$ are given by

$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{w} \end{pmatrix} &= \begin{pmatrix} A + BNC_1 & BM \\ LC_1 & K \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix} + \begin{pmatrix} E + BND_{12} \\ LD_{12} \end{pmatrix} d \\ z &= \begin{pmatrix} C_2 + D_{21}NC_1 & D_{21}M \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix} + (D_{22} + D_{21}ND_{12})d. \end{aligned}$$

Observe that we can factorize the matrices given above as

$$\begin{pmatrix} A + BNC_1 & BM \\ LC_1 & K \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} B & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} N & M \\ L & K \end{pmatrix} \begin{pmatrix} C_1 & 0 \\ 0 & I \end{pmatrix}$$
$$\begin{pmatrix} E + BND_{12} \\ LD_{12} \end{pmatrix} = \begin{pmatrix} E \\ 0 \end{pmatrix} + \begin{pmatrix} B & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} N & M \\ L & K \end{pmatrix} \begin{pmatrix} D_{12} \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} C_2 + D_{21}NC_1 & D_{21}M \end{pmatrix} = \begin{pmatrix} C_2 & 0 \end{pmatrix} + \begin{pmatrix} D_{21} & 0 \end{pmatrix} \begin{pmatrix} N & M \\ L & K \end{pmatrix} \begin{pmatrix} C_1 & 0 \\ 0 & I \end{pmatrix}$$
$$D_{22} + D_{21}ND_{12} = D_{22} + \begin{pmatrix} D_{21} & 0 \end{pmatrix} \begin{pmatrix} N & M \\ L & K \end{pmatrix} \begin{pmatrix} D_{12} \\ 0 \end{pmatrix}.$$

Define now the so called extended matrices

$$\begin{aligned} A_e &:= \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}, & B_e &:= \begin{pmatrix} B & 0 \\ 0 & I \end{pmatrix}, & C_{1,e} &:= \begin{pmatrix} C_1 & 0 \\ 0 & I \end{pmatrix}, \\ E_e &:= \begin{pmatrix} E \\ 0 \end{pmatrix}, & D_{1,e} &:= \begin{pmatrix} D_{12} \\ 0 \end{pmatrix}, & C_{2,e} &:= \begin{pmatrix} C_2 & 0 \end{pmatrix}, \\ D_{2,e} &:= \begin{pmatrix} D_{21} & 0 \end{pmatrix}, & G &:= \begin{pmatrix} N & M \\ L & K \end{pmatrix}, & x_e &:= \begin{pmatrix} x \\ w \end{pmatrix}. \end{aligned} \tag{6.4}$$

Then a shorthand notation for the closed-loop dynamics of $\Sigma \times \Gamma$ is given by

$$\begin{aligned} \dot{x}_e &= (A_e + B_e G C_{1,e}) x_e + (E_e + B_e G D_{1,e}) d \\ z &= (C_{2,e} + D_{2,e} G C_{1,e}) x_e + (D_{22} + D_{2,e} G D_{1,e}) d \end{aligned} \tag{6.5}$$

which can be interpreted as the interconnection of system

$$\begin{aligned} \dot{x}_e &= A_e x_e + B_e u_e + E_e d \\ y_e &= C_{1,e} x_e + D_{1,e} d \\ z &= C_{2,e} x_e + D_{2,e} u_e + D_{22} d \end{aligned}$$

with static output feedback

$$u_e = G y_e.$$

The following theorem provides necessary and sufficient conditions for the existence of a controller Γ that solves the $\mathcal{H}_\infty$-control problem with tolerance $1/\gamma$. For the next theorem we refer to [13]. However, we will establish an alternative proof.

Theorem 6.2.1. *Let $\gamma > 0$ and $D_{11} = 0$. Let n_c be a nonnegative integer. Consider the system Σ and let the matrices $A_e, B_e, C_{1,e}, C_{2,e}, D_{1,e}, D_{2,e}, E_e, G$ be given as in* (6.4). *Let T be the transfer matrix from d to z of $\Sigma \times \Gamma$. Then the following three statements are equivalent [13].*

1. *There exists a controller Γ with state dimension n_c that internally stabilizes the system $\Sigma \times \Gamma$ and yields $\|T\|_\infty < \frac{1}{\gamma}$.*

2. *There exist $(n + n_c) \times (n + n_c)$ matrices $X_e, Y_e > 0$ such that*

   2.a
$$\begin{pmatrix} B_e \\ D_{2,e} \end{pmatrix}^{\perp} \begin{pmatrix} A_e X_e + X_e A_e^T + E_e E_e^T & X_e C_{2,e}^T + E_e D_{22}^T \\ C_{2,e} X_e + D_{22} E_e^T & -\frac{1}{\gamma^2} I + D_{22} D_{22}^T \end{pmatrix} \begin{pmatrix} B_e \\ D_{2,e} \end{pmatrix}^{\perp T} < 0$$

   2.b
$$\begin{pmatrix} C_{1,e}^T \\ D_{1,e}^T \end{pmatrix}^{\perp} \begin{pmatrix} Y_e A_e + A_e^T Y_e + C_{2,e}^T C_{2,e} & Y_e E_e + C_{2,e}^T D_{22} \\ E_e^T Y_e + D_{22}^T C_{2,e} & -\frac{1}{\gamma^2} I + D_{22}^T D_{22} \end{pmatrix} \begin{pmatrix} C_{1,e}^T \\ D_{1,e}^T \end{pmatrix}^{\perp T} < 0$$

2.c

$$X_e Y_e = \frac{1}{\gamma^2} I.$$

3. There exist $n \times n$ matrices $X, Y > 0$ such that

3.a

$$\begin{pmatrix} B \\ D_{21} \end{pmatrix}^{\perp} \begin{pmatrix} AX + XA^T + EE^T & XC_2^T + ED_{22}^T \\ C_2 X + D_{22} E^T & -\frac{1}{\gamma^2} I + D_{22} D_{22}^T \end{pmatrix} \begin{pmatrix} B \\ D_{21} \end{pmatrix}^{\perp T} < 0$$

3.b

$$\begin{pmatrix} C_1^T \\ D_{12}^T \end{pmatrix}^{\perp} \begin{pmatrix} YA + A^T Y + C_2^T C_2 & YE + C_2^T D_{22} \\ E^T Y + D_{22}^T C_2 & -\frac{1}{\gamma^2} I + D_{22}^T D_{22} \end{pmatrix} \begin{pmatrix} C_1^T \\ D_{12}^T \end{pmatrix}^{\perp T} < 0$$

3.c

$$Y - \frac{1}{\gamma^2} X^{-1} \geq 0$$

3.d

$$\operatorname{rank}(Y - \frac{1}{\gamma^2} X^{-1}) \leq n_c.$$

If 2. holds then a suitable controller is given by

$$G = \begin{pmatrix} N & M \\ L & K \end{pmatrix} = -R\Psi^T \Phi \Lambda^T (\Lambda \Phi \Lambda^T)^{-1}$$

where R is an arbitrary positive definite matrix such that

$$\Phi := (\Psi R \Psi^T - \Theta)^{-1} > 0$$

with

$$\Theta := \begin{pmatrix} Y_e A_e + A_e^T Y_e & Y_e E_e & C_{2,e}^T \\ E_e^T Y_e & -\frac{1}{\gamma^2} I & D_{22}^T \\ C_{2,e} & D_{22} & -I \end{pmatrix}$$
$$\Psi := \begin{pmatrix} Y_e B_e \\ 0 \\ D_{2,e} \end{pmatrix}, \qquad \Lambda := \begin{pmatrix} C_{1,e} & D_{1,e} & 0 \end{pmatrix}. \tag{6.6}$$

Proof. (1.) $\Leftrightarrow$ (2.). We have seen that the closed-loop system $\Sigma \times \Gamma$ can be written as

$$\begin{aligned} \dot{x}_e &= (A_e + B_e G C_{1,e}) x_e + (E_e + B_e G D_{1,e}) d \\ z &= (C_{2,e} + D_{2,e} G C_{1,e}) x_e + (D_{22} + D_{2,e} G D_{1,e}) d. \end{aligned}$$

By the bounded real lemma we have that $A_e + B_e G C_{1,e}$ is Hurwitz and $\|T\| < \frac{1}{\gamma}$ if and only if there exists $Y_e > 0$ such that

$$\begin{pmatrix} Y_e(A_e + B_e G C_{1,e}) + (A_e + B_e G C_{1,e})^T Y & Y_e(E_e + B_e G D_{1,e}) & (C_{2,e} + D_{2,e} G C_{1,e})^T \\ (E_e + B_e G D_{1,e})^T Y_e & -\frac{1}{\gamma^2} I & (D_{22} + D_{2,e} G D_{1,e})^T \\ C_{2,e} + D_{2,e} G C_{1,e} & D_{22} + D_{2,e} G D_{1,e} & -I \end{pmatrix} < 0$$

The latter LMI can be written as

$$\Psi G \Lambda + (\Psi G \Lambda)^T + \Theta < 0 \tag{6.7}$$

where Θ, Ψ, Λ are given by (6.6). Since Λ has full row rank and Ψ has full column rank, by Theorem 2.4.2 there exists G and $Y_e > 0$ such that (6.7) holds if and only if there exists $Y_e > 0$ such that

$$\Psi^{\perp} \Theta \Psi^{\perp T} < 0$$
$$\Lambda^{T\perp} \Theta \Lambda^{T\perp T} < 0.$$

Note that

$$\Psi = \begin{pmatrix} Y_e & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{pmatrix} \begin{pmatrix} B_E \\ D_{2,e} \\ 0 \end{pmatrix}, \qquad \Lambda^T = \begin{pmatrix} C_{1,e}^T \\ D_{1,e}^T \\ 0 \end{pmatrix}$$

and therefore possible annihilators of Ψ, Λ^T are given by

$$\Psi^\perp = \begin{pmatrix} \begin{pmatrix} B_e \\ D_{2,e} \end{pmatrix}^\perp & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} Y_e^{-1} & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{pmatrix}, \qquad \Lambda^{T\perp} = \begin{pmatrix} \begin{pmatrix} C_{1,e}^T \\ D_{1,e}^T \end{pmatrix}^\perp & 0 \\ 0 & I \end{pmatrix}.$$

Then

$$\begin{aligned}
&\Psi^\perp \Theta \Psi^{\perp T} \\
&= \begin{pmatrix} \begin{pmatrix} B_e \\ D_{2,e} \end{pmatrix}^\perp & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} Y_e^{-1} & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{pmatrix} \begin{pmatrix} Y_e A_e + A_e^T Y_e & Y_e E_e & C_{2,e}^T \\ E_e^T Y_e & -\frac{1}{\gamma^2} I & D_{22}^T \\ C_{2,e} & D_{22} & -I \end{pmatrix} \begin{pmatrix} Y_e^{-1} & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{pmatrix} \begin{pmatrix} \begin{pmatrix} B_e \\ D_{2,e} \end{pmatrix}^{\perp T} & 0 \\ 0 & I \end{pmatrix} \\
&= \begin{pmatrix} \begin{pmatrix} B_e \\ D_{2,e} \end{pmatrix}^\perp & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A_e + Y_e^{-1} A_e^T Y_e & E_e & Y_e^{-1} C_{2,e}^T \\ C_{2,e} & D_{22} & -I \\ E_e^T Y_e & -\frac{1}{\gamma^2} I & D_{22}^T \end{pmatrix} \begin{pmatrix} Y_e^{-1} & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{pmatrix} \begin{pmatrix} \begin{pmatrix} B_e \\ D_{2,e} \end{pmatrix}^{\perp T} & 0 \\ 0 & I \end{pmatrix} \\
&= \begin{pmatrix} \begin{pmatrix} B_e \\ D_{2,e} \end{pmatrix}^\perp & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A_e Y_e^{-1} + Y_e^{-1} A_e^T & Y_e^{-1} C_{2,e}^T & E_e \\ C_{2,e} Y_e^{-1} & -I & D_{22} \\ E_e^T & D_{22}^T & -\frac{1}{\gamma^2} I \end{pmatrix} \begin{pmatrix} \begin{pmatrix} B_e \\ D_{2,e} \end{pmatrix}^{\perp T} & 0 \\ 0 & I \end{pmatrix} \\
&= \begin{pmatrix} \begin{pmatrix} B_e \\ D_{2,e} \end{pmatrix}^\perp \begin{pmatrix} A_e Y_e^{-1} + Y_e^{-1} A_e^T & Y_e^{-1} C_{2,e}^T \\ C_{2,e} Y_e^{-1} & -I \end{pmatrix} \begin{pmatrix} Bv \\ D_{2,e} \end{pmatrix}^{\perp T} & \begin{pmatrix} B_e \\ D_{2,e} \end{pmatrix}^\perp \begin{pmatrix} E_e \\ D_{22} \end{pmatrix} \\ \begin{pmatrix} E_e^T & D_{22}^T \end{pmatrix} \begin{pmatrix} B_e \\ D_{2,e} \end{pmatrix}^{\perp T} & -\frac{1}{\gamma^2} I \end{pmatrix} < 0
\end{aligned}$$

if and only if

$$\begin{pmatrix} B_e \\ D_{2,e} \end{pmatrix}^\perp \begin{pmatrix} A_e Y_e^{-1} + Y_e^{-1} A_e^T + \gamma^2 E_e E_e^T & Y_e^{-1} C_{2,e}^T + \gamma^2 E_e D_{22}^T \\ C_{2,e} Y_e^{-1} + \gamma^2 D_{22} E_e^T & -I + \gamma^2 D_{22} D_{22}^T \end{pmatrix} \begin{pmatrix} B_e \\ D_{2,e} \end{pmatrix}^{\perp T} < 0.$$

Define $X_e := \frac{1}{\gamma^2} Y_e^{-1}$ then the latter LMI is equivalent with

$$\begin{pmatrix} B_e \\ D_{2,e} \end{pmatrix}^\perp \begin{pmatrix} A_e X_e + X_e A_e^T + E_e E_e^T & X_e C_{2,e}^T + E_e D_{22}^T \\ C_{2,e} X_e + D_{22} E_e^T & -\frac{1}{\gamma^2} I + D_{22} D_{22}^T \end{pmatrix} \begin{pmatrix} B_e \\ D_{2,e} \end{pmatrix}^{\perp T} < 0.$$

Similarly we have that

$$\begin{aligned}
\Lambda^{T\perp} \Theta \Lambda^{T\perp T} &= \begin{pmatrix} \begin{pmatrix} C_{1,e}^T \\ D_{1,e}^T \end{pmatrix}^\perp & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} Y_e A_e + A_e^T Y_e & Y_e E_e & C_{2,e}^T \\ E_e^T Y_e & -\frac{1}{\gamma^2} I & D_{22}^T \\ C_{2,e} & D_{22} & -I \end{pmatrix} \begin{pmatrix} \begin{pmatrix} C_{1,e}^T \\ D_{1,e}^T \end{pmatrix}^{\perp T} & 0 \\ 0 & I \end{pmatrix} \\
&= \begin{pmatrix} \begin{pmatrix} C_{1,e}^T \\ D_{1,e}^T \end{pmatrix}^\perp \begin{pmatrix} Y_e A_e + A_e^T Y_e & Y_e E_e \\ E_e^T Y_e & -\frac{1}{\gamma^2} I \end{pmatrix} \begin{pmatrix} C_{1,e}^T \\ D_{1,e}^T \end{pmatrix}^{\perp T} & \begin{pmatrix} C_{1,e}^T \\ D_{1,e}^T \end{pmatrix}^\perp \begin{pmatrix} C_{2,e}^T \\ D_{22}^T \end{pmatrix} \\ \begin{pmatrix} C_{2,e} & D_{22} \end{pmatrix} \begin{pmatrix} C_{1,e}^T \\ D_{1,e}^T \end{pmatrix}^{\perp T} & -I \end{pmatrix} < 0
\end{aligned}$$

if and only if

$$\begin{pmatrix} C_{1,e}^T \\ D_{1,e}^T \end{pmatrix}^\perp \begin{pmatrix} Y_e A_e + A_e^T Y_e + C_{2,e}^T C_{2,e} & Y_e E_e + C_{2,e}^T D_{22} \\ E_e^T Y_e + D_{22}^T C_{2,e} & -\frac{1}{\gamma^2} I + D_{22}^T D_{22} \end{pmatrix} \begin{pmatrix} C_{1,e}^T \\ D_{1,e}^T \end{pmatrix}^{\perp T} < 0.$$

If (2.) holds then (6.7) has a solution G. By Theorem 2.4.2 a possible solution G is given by $G = -R\Psi^T\Phi\Lambda^T(\Lambda\Phi\Lambda^T)^{-1}$ where $R > 0$ is such that $\Phi := (\Psi R\Psi^T - \Theta)^{-1} > 0$. We have proven now that (1.) $\Leftrightarrow$ (2.). Next, we will prove (2.) $\Rightarrow$ (3.). Let X_e, Y_e be partitioned in

$$X_e = \begin{pmatrix} X & X_{pc} \\ X_{pc}^T & X_c \end{pmatrix} > 0, \qquad Y = \begin{pmatrix} Y & Y_{pc} \\ Y_{pc}^T & Y_c \end{pmatrix} > 0$$

then $X > 0, Y > 0$. It can be observed that

$$\begin{pmatrix} B_e \\ D_{2,e} \end{pmatrix} = \begin{pmatrix} B & 0 \\ 0 & I_{n_c} \\ D_{21} & 0 \end{pmatrix} = \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{pmatrix} \begin{pmatrix} B & 0 \\ D_{21} & 0 \\ 0 & I_{n_c} \end{pmatrix}$$

$$\begin{pmatrix} B_e \\ D_{2,e} \end{pmatrix}^{\perp} = \begin{pmatrix} \begin{pmatrix} B \\ D_{21} \end{pmatrix}^{\perp} & 0 \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{pmatrix}.$$

From condition $2.a$ it follows that

$$\begin{aligned} &\begin{pmatrix} B_e \\ D_{2,e} \end{pmatrix}^{\perp} \begin{pmatrix} A_eX_e + X_eA_e^T + E_eE_e^T & X_eC_{2,e}^T + E_eD_{22}^T \\ C_{2,e}X_e + D_{22}E_e^T & -\frac{1}{\gamma^2}I + D_{22}D_{22}^T \end{pmatrix} \begin{pmatrix} B_e \\ D_{2,e} \end{pmatrix}^{\perp T} \\ &= \begin{pmatrix} B_e \\ D_{2,e} \end{pmatrix}^{\perp} \begin{pmatrix} AX + XA^T + EE^T & AX_{pc} & XC_2^T + ED_{22}^T \\ X_{pc}^TA^T & 0 & X_{pc}^TC_2^T \\ C_2X + D_{22}E^T & C_2X_{pc} & -\frac{1}{\gamma^2}I + D_{22}D_{22}^T \end{pmatrix} \begin{pmatrix} B_e \\ D_{2,e} \end{pmatrix}^{\perp T} \\ &= \begin{pmatrix} B \\ D_{21} \end{pmatrix}^{\perp} \begin{pmatrix} AX + XA^T + EE^T & XC_2^T + ED_{22}^T \\ C_2X + D_{22}E^T & -\frac{1}{\gamma^2}I + D_{22}D_{22}^T \end{pmatrix} \begin{pmatrix} B \\ D_{21} \end{pmatrix}^{\perp T} < 0. \end{aligned} \tag{6.8}$$

Similarly one can prove that

$$\begin{aligned} &\begin{pmatrix} C_{1,e}^T \\ D_{1,e}^T \end{pmatrix}^{\perp} \begin{pmatrix} Y_eA_e + A_e^TY_e + C_{2,e}^TC_{2,e} & Y_eE_e + C_{2,e}^TD_{22} \\ E_e^TY_e + D_{22}^TC_{2,e} & -\frac{1}{\gamma^2}I + D_{22}^TD_{22} \end{pmatrix} \begin{pmatrix} C_{1,e}^T \\ D_{1,e}^T \end{pmatrix}^{\perp T} \\ &= \begin{pmatrix} C_1^T \\ D_{12}^T \end{pmatrix}^{\perp} \begin{pmatrix} YA + A^TY + C_2^TC_2 & YE^T + C_2^TD_{22} \\ E^TY + D_{22}^TC_2 & -\frac{1}{\gamma^2}I + D_{22}^TD_{22} \end{pmatrix} \begin{pmatrix} C_1^T \\ D_{12}^T \end{pmatrix}^{\perp T} < 0. \end{aligned} \tag{6.9}$$

Since $X_e = \gamma^2Y_e^{-1}$ this implies that

$$X = \gamma^2(Y - Y_{pc}Y_c^{-1}Y_{pc}^T)^{-1}$$

hence

$$Y - \frac{1}{\gamma^2}X^{-1} = Y_{pc}Y_c^{-1}Y_{pc}^T \geq 0.$$

Finally, because of the latter equation and since Y_c is a $n_c \times n_c$ matrix we have

$$\text{rank}(Y - \frac{1}{\gamma^2}X^{-1}) = \text{rank}(Y_{pc}Y_c^{-1}Y_{pc}^T) \leq n_c.$$

(3.) $\Rightarrow$ (2.). We have that $Y - \frac{1}{\gamma^2}X^{-1} \geq 0$ and $\text{rank}(Y - \frac{1}{\gamma^2}X^{-1}) \leq n_c$. Therefore there exists $n_c \times n_c$ matrix $Y_c > 0$ and a matrix Y_{pc} such that

$$Y - \frac{1}{\gamma^2}X^{-1} = Y_{pc}Y_c^{-1}Y_{pc}^T.$$

Define

$$Y_e = \begin{pmatrix} Y & Y_{pc} \\ Y_{pc}^T & Y_c \end{pmatrix}, \qquad X_e = \frac{1}{\gamma^2}Y_e^{-1}$$

then $X_e, Y_e > 0$ because the Schur complement of Y_e satisfies $Y - Y_{pc}Y_c^{-1}Y_{pc}^T = Y - Y + \frac{1}{\gamma^2}X^{-1} = \frac{1}{\gamma^2}X^{-1} > 0$. It can be observed that conditions $2.a$ and $2.b$ both hold for this X_e, Y_e because of the equalities in (6.8) and (6.9). $\square$

Note that condition 3. of Theorem 6.2.1 is very useful in determining the dimension n_c of a controller Γ that solves the $\mathcal{H}_\infty$-control problem with tolerance $1/\gamma$. Since n_c appears only in condition 3.d, n_c can be chosen after X, Y are determined. With this observation and using the construction of the proof of the theorem we can state an algorithm that computes a suitable controller Γ that solves the $\mathcal{H}_\infty$-control problem with tolerance $1/\gamma$.

Algorithm 6.2.2. Given a desired tolerance $\gamma > 0$.

1) Solve $3.a, 3.b$ and $3.c$ in Theorem 6.2.1 for $X, Y > 0$. If there do not exists such X, Y then stop, since then there exists no controller Γ that solves the $\mathcal{H}_\infty$-control problem with tolerance $1/\gamma$, else go to step 2).

2) Define $n_c = \text{rank}(Y - \frac{1}{\gamma^2}X^{-1})$.

3) Let $Y_{pc} \in \mathbb{R}^{n \times n_c}$ and $Y_c \in \mathbb{R}^{n_c \times n_c}$ be such that $Y_c > 0$ and

$$Y - \frac{1}{\gamma^2}X^{-1} = Y_{pc}Y_c^{-1}Y_{pc}^T.$$

4) Define

$$Y_e = \begin{pmatrix} Y & Y_{pc} \\ Y_{pc}^T & Y_c \end{pmatrix}.$$

5) Define $G = -R\Psi^T\Phi\Lambda^T(\Lambda\Phi\Lambda^T)^{-1}$ where $R > 0$ is such that $\Phi := (\Psi R \Psi^T - \Theta)^{-1} > 0$ and Θ, Ψ, Λ are given by (6.6).

6) Partition G as

$$G = \begin{pmatrix} N & M \\ L & K \end{pmatrix}.$$

In this algorithm the most difficult step is the first one. Here three LMI's have to be solved with unknowns $X, Y > 0$. In general it is unclear for which conditions there exist such $X, Y > 0$ that satisfy the LMI's for a given γ. Numerical techniques have to be considered in finding a possible solution to condition $3.a, 3.b$ and $3.c$ in Theorem 6.2.1. For some special cases of the $\mathcal{H}_\infty$-control problem solutions to the LMI's will turn out to exist, and closed form solutions for these solution can be given. We will come back to this in Chapter 7.

In this section we have discussed the case that $D_{11} = 0$. We are also interested in the case that this term does not vanish. It turns outs that the procedure of computing a controller that solves the $\mathcal{H}_\infty$-control problem essentially remains the same. In this case however, a transformation on the controller matrices has to be made. In addition, an extra condition to guarantee well-posedness of the interconnected system $\Sigma \times \Gamma$ has to be introduced.

6.2.2 The case that $D_{11} \neq 0$

Consider the closed-loop system as in Figure 7.2 with Σ, Γ given as before. We again assume that the matrices in (6.3) have full column rank and full row rank respectively. In this subsection, we will extend the result of [13] to the case that D_{11} is no longer assumed to be zero. Then an extra condition has to be considered, since we want the closed-loop system $\Sigma \times \Gamma$ to be *well-posed*. It can be seen that well-posedness is assured if and only if $I - D_{11}N$ is invertible (and thus also $I - ND_{11}$ is invertible), see [17]. Note that this condition is automatically satisfied if $D_{11} = 0$ or $N = 0$. We will now write down the full dynamics of $\Sigma \times \Gamma$. Assume that N is chosen such that $I - D_{11}N$ is invertible. Then by interconnecting Σ and Γ we obtain following equations for u, y:

$$\begin{aligned} y &= (I - D_{11}N)^{-1}(C_1x + D_{11}Mw + D_{12}d) \\ u &= (I - ND_{11})^{-1}(Mw + NC_1x + ND_{12}d). \end{aligned}$$

In order to simplify the notation we define $S_1 := I - D_{11}N, S_2 := I - ND_{11}$. The closed-loop dynamics of $\Sigma \times \Gamma$ can then be written as

$$\begin{aligned}\begin{pmatrix}\dot{x}\\ \dot{w}\end{pmatrix} &= \begin{pmatrix}A + BS_2^{-1}NC_1 & BS_2^{-1}M\\ LS_1^{-1}C_1 & K + LS_1^{-1}D_{11}M\end{pmatrix}\begin{pmatrix}x\\ w\end{pmatrix} + \begin{pmatrix}E + BS_2^{-1}ND_{12}\\ LS_1^{-1}D_{12}\end{pmatrix}d\\ z &= \begin{pmatrix}C_2 + D_{21}S_2^{-1}NC_1 & D_{21}S_2^{-1}M\end{pmatrix}\begin{pmatrix}x\\ w\end{pmatrix} + (D_{22} + D_{21}S_2^{-1}ND_{12})d.\end{aligned} \tag{6.10}$$

Observe that the matrices in (6.10) can be partitioned in

$$\begin{pmatrix}A + BS_2^{-1}NC_1 & BS_2^{-1}M\\ LS_1^{-1}C_1 & K + LS_1^{-1}D_{11}M\end{pmatrix} = \begin{pmatrix}A & 0\\ 0 & 0\end{pmatrix} + \begin{pmatrix}B & 0\\ 0 & I_{n_c}\end{pmatrix}\begin{pmatrix}S_2^{-1}N & S_2^{-1}M\\ LS_1^{-1} & K + LS_1^{-1}D_{11}M\end{pmatrix}\begin{pmatrix}C_1 & 0\\ 0 & I_{n_c}\end{pmatrix},$$

$$\begin{pmatrix}E + BS_2^{-1}ND_{12}\\ LS_1^{-1}D_{12}\end{pmatrix} = \begin{pmatrix}E\\ 0\end{pmatrix} + \begin{pmatrix}B & 0\\ 0 & I_{n_c}\end{pmatrix}\begin{pmatrix}S_2^{-1}N & S_2^{-1}M\\ LS_1^{-1} & K + LS_1^{-1}D_{11}M\end{pmatrix}\begin{pmatrix}D_{12}\\ 0\end{pmatrix},$$

$$\begin{pmatrix}C_2 + D_{21}S_2^{-1}NC_1 & D_{21}S_2^{-1}M\end{pmatrix} = \begin{pmatrix}C_2 & 0\end{pmatrix} + \begin{pmatrix}D_{21} & 0\end{pmatrix}\begin{pmatrix}S_2^{-1}N & S_2^{-1}M\\ LS_1^{-1} & K + LS_1^{-1}D_{11}M\end{pmatrix}\begin{pmatrix}C_1 & 0\\ 0 & I_{n_c}\end{pmatrix},$$

$$D_{22} + D_{21}S_2^{-1}ND_{12} = D_{22} + \begin{pmatrix}D_{21} & 0\end{pmatrix}\begin{pmatrix}S_2^{-1}N & S_2^{-1}M\\ LS_1^{-1} & K + LS_1^{-1}D_{11}M\end{pmatrix}\begin{pmatrix}D_{12}\\ 0\end{pmatrix}.$$

Similarly as in (6.4) we define the extended matrices $A_e, B_e, C_{1,e}, C_{2,e}, D_{1,e}, D_{2,e}, E_e, x_e$:

$$\begin{aligned}&A_e := \begin{pmatrix}A & 0\\ 0 & 0\end{pmatrix}, \quad B_e := \begin{pmatrix}B & 0\\ 0 & I\end{pmatrix}, \quad C_{1,e} := \begin{pmatrix}C_1 & 0\\ 0 & I\end{pmatrix},\\ &E_e := \begin{pmatrix}E\\ 0\end{pmatrix}, \quad D_{1,e} := \begin{pmatrix}D_{12}\\ 0\end{pmatrix}, \quad C_{2,e} := \begin{pmatrix}C_2 & 0\end{pmatrix},\\ &D_{2,e} := \begin{pmatrix}D_{21} & 0\end{pmatrix}, \quad x_e := \begin{pmatrix}x\\ w\end{pmatrix}.\end{aligned}$$

We define the controller matrix $\bar{G}$ as

$$\begin{aligned}&\bar{G} := \begin{pmatrix}\bar{N} & \bar{M}\\ \bar{L} & \bar{K}\end{pmatrix}, \text{ where}\\ &\bar{N} := S_2^{-1}N, \quad \bar{M} := S_2^{-1}M, \quad \bar{L} := LS_1^{-1}, \quad \bar{K} := K + LS_1^{-1}D_{11}M.\end{aligned} \tag{6.11}$$

The closed-loop system $\Sigma \times \Gamma$ can be interpreted as the interconnection of the system

$$\begin{aligned}\dot{x}_e &= A_e x_e + B_e u_e + E_e d\\ y_e &= C_{1,e}x_e + D_{1,e}d\\ z &= C_{2,e}x_e + D_{2,e}u_e + D_{22}d\end{aligned}$$

with output feedback

$$u_e = \bar{G}y_e$$

which yields the closed-loop dynamics

$$\begin{aligned}\dot{x}_e &= (A_e + B_e\bar{G}C_{1,e})x_e + (E_e + B_e\bar{G}D_{1,e})d\\ z &= (C_{2,e} + D_{2,e}\bar{G}C_{1,e})x_e + (D_{22} + D_{2,e}\bar{G}D_{1,e})d.\end{aligned}$$

Here $\bar{G}$ is the unknown, like G is the unknown in (6.5). This motivates us to use the same algorithm as before. The only technicalities are that the controller matrix $\bar{G}$ is transformed and the well-posedness must be guaranteed. Suppose now that we have constructed some $\bar{G}$ and therefore $\bar{K}, \bar{L}, \bar{M}, \bar{N}$ using Algorithm 6.2.2. We will show that if the matrix $I + D_{11}\bar{N}$ is invertible then one is able to construct a K, L, M, N such that the system $\Sigma \times \Gamma$ is well-posed and the transfer matrix satisfies $\|T\|_\infty < 1/\gamma$. To see this we formulate the following lemma.

Lemma 6.2.3. *Let $D_{11}, \bar{N}$ be matrices such that $I + D_{11}\bar{N}$ is defined and invertible. Then*

$$V := I - \bar{N}(I + D_{11}\bar{N})^{-1}D_{11} \tag{6.12}$$

is invertible.

Proof. Let v be any vector such that $Vv = 0$ then

$$\bar{N}(I + D_{11}\bar{N})^{-1}D_{11}v = v$$

Define $u = D_{11}v$ and $w = (I + D_{11}\bar{N})^{-1}u$. Then the above implies that

$$D_{11}\bar{N}w = D_{11}\bar{N}(I + D_{11}\bar{N})^{-1}u = D_{11}\bar{N}(I + D_{11}\bar{N})^{-1}D_{11}v = D_{11}v = u = (I + D_{11}\bar{N})w.$$

This implies that $w = 0$ and therefore subsequently also $u = 0$ and $v = 0$. Hence V is invertible. □

Assume now that $I + D_{11}\bar{N}$ is invertible and define $N = \bar{N}(I + D_{11}\bar{N})^{-1}$ then by the previous lemma it follows that $I - ND_{11}$ is invertible. Moreover this $N, \bar{N}$ satisfies the relation given in (6.11) since

$$\begin{aligned} N = \bar{N}(I + D_{11}\bar{N})^{-1} \quad &\Leftrightarrow \quad N(I + D_{11}\bar{N}) = \bar{N} \\ \Leftrightarrow N = (I - ND_{11})\bar{N} \quad &\Leftrightarrow \quad \bar{N} = (I - ND_{11})^{-1}N = S_2^{-1}N. \end{aligned}$$

Since N is determined then also S_1, S_2 are defined. Then K, L, M can be easily defined as well. To satisfy the relations given in (6.11) we define

$$M := S_2\bar{M}, \quad L := \bar{L}S_1, \quad K := \bar{K} - \bar{L}D_{11}M,$$

in this order. By using the above observations it is easy to see that the following result holds.

Corollary 6.2.4. *There exists a controller Γ such that the system $\Sigma \times \Gamma$, given by the interconnection of* (6.1) *with* (6.2)*, is well-posed and internally stable and the transfer matrix satisfies $\|T\|_\infty < \frac{1}{\gamma}$ if and only if there exists a controller $\bar{\Gamma}$ with dynamics* (6.14) *such that $I + D_{11}\bar{N}$ is invertible and the closed-loop system given by the interconnection of*

$$\begin{aligned} \dot{x} &= Ax + Bu + Ed \\ y &= C_1x + D_{12}d \\ z &= C_2x + D_{21}u + D_{22}d \end{aligned} \tag{6.13}$$

with the controller

$$\begin{aligned} \dot{w} &= \bar{K}w + \bar{L}y \\ u &= \bar{M}w + \bar{N}y \end{aligned} \tag{6.14}$$

is internally stable and satisfies $\|\bar{T}\|_\infty < \frac{1}{\gamma}$, where $\bar{T}$ is the transfer matrix from d to z of the interconnection of (6.13), (6.14).

The above corollary actually says that if we want to solve the $\mathcal{H}_\infty$-control problem with $D_{11} \neq 0$ we can reduce the problem to the case where $D_{11} = 0$. The only additional problem is to ensure well-posedness of the closed-loop system, therefore an extra condition is introduced. However, if for example one can find a strictly proper controller (6.14), i.e. $\bar{N} = 0$ such that the closed-loop system (6.13), (6.14) is internally stable and satisfies $\|\bar{T}\|_\infty < \frac{1}{\gamma}$, then well-posedness of the original system is guaranteed automatically. Because of the result in Corollary 6.2.4 we only have to make a slight modification of Algorithm 6.2.2 to cover the case when $D_{11} \neq 0$.

Algorithm 6.2.5. Given a desired tolerance $\gamma > 0$.

1) Solve $3.a, 3.b$ and $3.c$ in Theorem 6.2.1 for $X, Y > 0$. If there do not exists such X, Y then stop, since the $\mathcal{H}_\infty$-control problem with tolerance $1/\gamma$ does not have a solution, else go to step 2).

2) Define $n_c = \text{rank}(Y - \frac{1}{\gamma^2} X^{-1})$.

3) Let $Y_{pc}, Y_c > 0$ be such that
$$Y - \frac{1}{\gamma^2} X^{-1} = Y_{pc} Y_c^{-1} Y_{pc}^T.$$

4) Define
$$Y_e = \begin{pmatrix} Y & Y_{pc} \\ Y_{pc}^T & Y_c \end{pmatrix}.$$

5) Define $\bar{G} = -R\Psi^T \Phi \Lambda^T (\Lambda \Phi \Lambda^T)^{-1}$ where $R > 0$ is such that $\Phi := (\Psi R \Psi^T - \Theta)^{-1} > 0$ and Θ, Ψ, Λ are given by (6.6).

6) Partition $\bar{G}$ as
$$\bar{G} = \begin{pmatrix} \bar{N} & \bar{M} \\ \bar{L} & \bar{K} \end{pmatrix}.$$
Check if $I + D_{11}\bar{N}$ is invertible. If not, go to step 5) but use different $R > 0$, else go to step 7).

7) Define K, L, M, N in the following order:
$$N := \bar{N}(I + D_{11}\bar{N})^{-1}, \qquad M := (I - ND_{11})\bar{M},$$
$$L := \bar{L}(I - D_{11}N), \qquad K := \bar{K} - \bar{L}D_{11}M.$$

Note that in step 6) the matrix $I + D_{11}\bar{N}$ will often be invertible in practice, if this is not the case then there is quite some freedom in the choice of $\bar{G}$ in step 5) by choosing (another) R, see also Remark 2.4.3. Therefore from a practical point of view we can always ensure that $I + D_{11}\bar{N}$ is invertible in the algorithm. An additional benefit is that we do not have repeat step 1) again which is computationally the most demanding step of the algorithm.

Observe that in step 1) a solution has to be found for conditions $3.a, 3.b$ and $3.c$ in Theorem 6.2.1. Since γ occurs in each of these linear matrix inequalities it is in general hard to find algebraically an upper bound on the tolerance γ such there still exists a solution to condition $3.a, 3.b$ and $3.c$. It will be shown in Chapter 7 that for some specific cases it possible to explicitly compute an upper bound on the permissible tolerance γ.

In the next chapter we introduce the robust stabilization problem and we connect this problem with the $\mathcal{H}_\infty$-control problem. Here we will also discuss the robust stabilization problem where we consider additively, coprime factor and multiplicatively perturbed systems. We will show that these robust stabilization problems can be solved by applying the theory we have developed in this chapter.

Chapter 7

Application of the $\mathcal{H}_\infty$-control problem to robust stabilization

This chapter deals with the robust stabilization problem. In the robust stabilization problem we assume that the dynamics of a linear input-output system is *uncertain*. The uncertain system can then be given by any system that is in some 'ball' around the *nominal* system. A way to represent the uncertain system is by an interconnection of the nominal system with some linear stable system, which we call the *perturbation system* [17]. Here we measure the magnitude of the uncertainty in terms of the $\mathcal{H}_\infty$-norm of the transfer matrix of this perturbation system. A given upper bound on this norm we call the tolerance. The aim in the robust stabilization problem is to achieve stabilization of the closed-loop system for every perturbation system which has a transfer matrix that is bounded by this tolerance.

There is a strong connection between the robust stabilization problem and the $\mathcal{H}_\infty$-control problem. The connection is made via small-gain theorem which we will discuss in Section 7.1 [17]. We will also consider some specific cases of the robust stabilization problem where we deal with additive, coprime factor and multiplicative perturbations in Section 7.2.1, Section 7.2.2 and Section 7.2.3 respectively. These robust stabilization problems are solved by applying special cases of the $\mathcal{H}_\infty$-control problem.

7.1 The robust stabilization problem

Suppose we are given a linear input-output system Σ_n, which we call the *nominal* system, with input u and output y. We assume that the dynamics of the system contains uncertainties. A way to model these uncertainties is by introducing a perturbation system that is fed back into the nominal system [17]. See Figure 7.1.

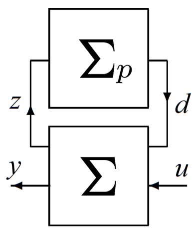

Figure 7.1: Modeling uncertainty as an interconnection of two systems [17].

Here Σ_p is a linear system that represents the uncertainty and if the transfer matrix of Σ_p, which we will denote by $\Delta(s)$, is zero then we obtain the nominal system. We assume that the perturbation system is *stable* and we measure the magnitude of the uncertainty in terms of the $\mathcal{H}_\infty$-norm of the transfer matrix of Σ_p. We assume that this quantity is bounded by $\|\Delta\|_\infty \leq \gamma$ where we call γ the *tolerance*. The perturbation system Σ_p has dynamics

$$\begin{aligned} \dot{\xi} &= A_\Delta \xi + B_\Delta z \\ d &= C_\Delta \xi + D_\Delta z \end{aligned}$$

where $\xi \in \mathbb{R}^{q_3}$. For this system we often use the shorthand notation $d = \Delta z$. As before, the dynamics for the system Σ is given by

$$\begin{aligned} \dot{x} &= Ax + Bu + Ed \\ y &= C_1 x + D_{11} u + D_{12} d \\ z &= C_2 x + D_{21} u + D_{22} d \end{aligned} \tag{7.1}$$

with $x \in \mathbb{R}^n, u \in \mathbb{R}^m, y \in \mathbb{R}^p, d \in \mathbb{R}^{q_1}, z \in \mathbb{R}^{q_2}$. The goal is now to find a dynamic feedback controller Γ that internally stabilizes the system $\Sigma \times \Gamma \times \Sigma_p$ for all systems Σ_p with transfer matrix bounded by an a priori given tolerance. The full closed-loop system can be represented as in Figure 7.2,

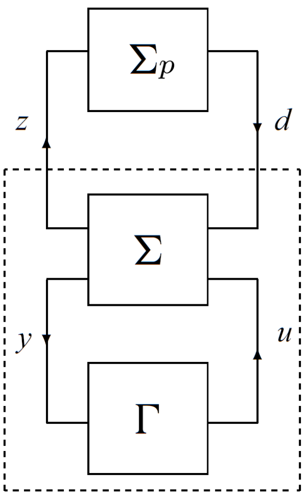

Figure 7.2: The setting of the robust stabilization problem [17].

where the dynamics of the controller Γ is given by

$$\begin{aligned} \dot{w} &= Kw + Ly \\ u &= Mw + Ny \end{aligned}$$

with $w \in \mathbb{R}^{n_c}$. In the robust stabilization problem the objective is to find, for a given tolerance γ, a nonnegative integer n_c and matrices K, L, M, N such that the closed-loop system $\Sigma \times \Gamma \times \Sigma_p$ as in Figure 7.2 is well-posed and internally stable for all $\Delta \in R\mathcal{H}_\infty$ such that $\|\Delta\|_\infty \leq \gamma$. A controller Γ that achieves this is goal is said to solve the robust stabilization problem with uncertainty radius γ. Formally:

Definition 7.1.1. Given a desired tolerance $\gamma > 0$. We say that controller Γ solves the robust stabilization problem with uncertainty radius γ if the closed-loop system $\Sigma \times \Gamma \times \Sigma_p$ is well-posed and internally stable for all $\Delta \in R\mathcal{H}_\infty$ such that $\|\Delta\|_\infty \leq \gamma$.

There is a strong connection between the $\mathcal{H}_\infty$-control problem and the robust stabilization problem. This is discussed in the next subsection.

7.1.1 Connection with the $\mathcal{H}_\infty$-control problem

Assume that the controller Γ is chosen such that $\Sigma \times \Gamma$ is internally stable. Then, by the small gain theorem, the interconnection of $\Sigma \times \Gamma$ with the perturbation system Σ_p is well-posed and internally stable for all Δ such that $\|\Delta\|_\infty \leq \gamma$ if and only if the transfer matrix T from d to z of $\Sigma \times \Gamma$ satisfies $\|T\|_\infty < \frac{1}{\gamma}$. In the previous chapter we have derived necessary and sufficient conditions for the existence of a controller such that the latter holds. Therefore it follows that from Theorem 6.2.1 that there exists a controller Γ that solves the robust stabilization problem with uncertainty radius γ if and only if one of the conditions of Theorem 6.2.1 is met.

Corollary 7.1.2. *Let $\gamma > 0$ and $D_{11} = 0$. There exists controller Γ that solves the robust stabilization problem with uncertainty radius γ if and only if one of the conditions* 1., 2. *or* 3. *of Theorem 6.2.1 is satisfied.*

The latter result can be easily extended to the case that D_{11} is not necessarily zero. However, in this case, one must also take the well-posedness condition mentioned in Corollary 6.2.4 into account. In fact, it follows now from Corollary 7.1.2 that Algorithm 6.2.5 can be used to construct a controller that solves the robust stabilization problem with uncertainty radius γ. In the next section we will discuss some special cases of the the robust stabilization problem and we explain how we can apply the algorithm to compute robustly stabilizing controllers.

7.2 Special cases of the robust stabilization problem

In this section we consider three special types of perturbations of the nominal system. For simplicity we consider in all three cases that the dynamics of the nominal system Σ_n is of the form

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx. \end{aligned} \tag{7.2}$$

and the controller Γ is of the usual form (6.2). Since in this case the feedthrough term from u to y is zero, we can use Algorithm 6.2.2 to construct a controller Γ that solves the robust stabilization problem with uncertainty radius γ. In this section we will focus on the first step in the algorithm as the rest of the steps is then straightforward. More precisely, we will focus on finding $X, Y > 0$ such that conditions $3.a, 3.b$ and $3.c$ of Theorem 6.2.1 are satisfied. Recall that condition $3.d$ is satisfied once the second step of Algorithm 6.2.2 has been executed.

The three types of perturbations we will consider are *additive*, *coprime* and *multiplicative* perturbations. In each of the three cases the perturbations can be modeled as perturbations of the transfer matrix of the nominal system. We will show that the closed-loop system of the perturbed system can be represented as in Figure 7.2. By applying the small-gain theorem and Theorem 6.2.1 to each case we show that the linear matrix inequalities in this theorem can be rewritten in terms of algebraic Riccati (in)equalities. We will also show that a condition for the tolerance γ can be isolated and an explicit upper bound for γ can be given, which is in terms of the spectral radius of the solutions of these algebraic Riccati (in)equalities.

7.2.1 Additive perturbations

Consider the nominal system (7.2) with transfer matrix $T_n(s)$. In the case of additive perturbations we assume that the transfer matrix of the exact model is given by $T(s) = T_n(s) + \Delta(s)$ where $\Delta \in R\mathcal{H}_\infty$, see [16], [17]. Hence a stable transfer matrix is *added* to the nominal transfer matrix in the case *additive* perturbations. The additively perturbed system can represented as the interconnection $\Sigma \times \Sigma_p$ where the dynamics is described by

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + d \\ z &= u \\ d &= \Delta z. \end{aligned} \tag{7.3}$$

It is easy to prove that the transfer matrix from u to y of the latter system is indeed equal to $T(s) = T_n(s) + \Delta(s)$. We will now formulate necessary and sufficient conditions for the existence of a controller that solves the robust stabilization problem with uncertainty radius γ in the case of additive perturbations.

Theorem 7.2.1. *Let $\gamma > 0$. Consider system $\Sigma \times \Sigma_p$ given by (7.3). There exists a controller Γ that solves the robust stabilization problem with uncertainty radius γ if and only if there exists $P, Q > 0$ such that*

$$PA + A^T P - PBB^T P < 0 \tag{7.4}$$

$$AQ + QA^T - QC^T CQ < 0 \tag{7.5}$$

$$\gamma \leq \frac{1}{\sqrt{\rho(PQ)}}. \tag{7.6}$$

Proof. By the small-gain theorem the system $\Sigma \times \Gamma \times \Sigma_p$ is internally stable for all $\Delta \in R\mathcal{H}_\infty$ with $\|\Delta\|_\infty \leq \gamma$ if and only if $\Sigma \times \Gamma$ is internally stable and the transfer matrix T from d to z satisfies $\|T\|_\infty < \frac{1}{\gamma}$. By Theorem 6.2.1 the latter holds if and only if there exists $X, Y > 0$ such that conditions $3.a, 3.b, 3.c$ and $3.d$ of Theorem 6.2.1 hold. Clearly there always exists $n_c (\leq n_p)$ such that $3.d$ holds. Using the notation of (6.1) then in the case of additive perturbations we have

$$E = 0, \quad C_1 = C, \quad C_2 = 0, \quad D_{11} = 0, \quad D_{12} = I, \quad D_{21} = I, \quad D_{22} = 0.$$

Note that

$$\begin{pmatrix} B \\ D_{21} \end{pmatrix}^\perp = \begin{pmatrix} B \\ I \end{pmatrix}^\perp = \begin{pmatrix} I & -B \end{pmatrix}, \qquad \begin{pmatrix} C_1^T \\ D_{12}^T \end{pmatrix}^\perp = \begin{pmatrix} C^T \\ I \end{pmatrix}^\perp = \begin{pmatrix} I & -C^T \end{pmatrix}.$$

Let $P, Q > 0$ be such that $X = \frac{1}{\gamma^2} P^{-1}, Y = \frac{1}{\gamma^2} Q^{-1}$ then condition 3.a becomes

$$\begin{aligned}
&\begin{pmatrix} B \\ D_{21} \end{pmatrix}^\perp \begin{pmatrix} AX + XA^T + EE^T & XC_2^T + ED_{22}^T \\ C_2 X + D_{22} E^T & -\frac{1}{\gamma^2} I + D_{22} D_{22}^T \end{pmatrix} \begin{pmatrix} B \\ D_{21} \end{pmatrix}^{\perp T} \\
&= \begin{pmatrix} I & -B \end{pmatrix} \begin{pmatrix} AX + XA^T & 0 \\ 0 & -\frac{1}{\gamma^2} I \end{pmatrix} \begin{pmatrix} I \\ -B^T \end{pmatrix} \\
&= AX + XA^T - \frac{1}{\gamma^2} BB^T \\
&= \frac{1}{\gamma^2} P^{-1} (PA + A^T P - PBB^T P) P^{-1} < 0
\end{aligned}$$

and condition 3.b becomes

$$\begin{aligned}
&\begin{pmatrix} C_1^T \\ D_{12}^T \end{pmatrix}^\perp \begin{pmatrix} YA + A^T Y + C_2^T C_2 & YE + C_2^T D_{22} \\ E^T Y + D_{22}^T C_2 & -\frac{1}{\gamma^2} I + D_{22}^T D_{22} \end{pmatrix} \begin{pmatrix} C_1^T \\ D_{12}^T \end{pmatrix}^{\perp T} \\
&= \begin{pmatrix} I & -C^T \end{pmatrix} \begin{pmatrix} YA + A^T Y & 0 \\ 0 & -\frac{1}{\gamma^2} I \end{pmatrix} \begin{pmatrix} I \\ -C \end{pmatrix} \\
&= YA + A^T Y - \frac{1}{\gamma^2} C^T C \\
&= \frac{1}{\gamma^2} Q^{-1} (AQ + QA^T - QC^T CQ) Q^{-1} < 0.
\end{aligned}$$

Condition 3.c holds if and only if

$$Y - \frac{1}{\gamma^2} X^{-1} = \frac{1}{\gamma^2} Q^{-1} - P \geq 0 \quad \Leftrightarrow \quad \frac{1}{\gamma^2} Q^{-1} \geq P \quad \Leftrightarrow \quad \frac{1}{\gamma^2} I \geq Q^{\frac{1}{2}} P Q^{\frac{1}{2}} \quad \Leftrightarrow \quad \gamma \leq \frac{1}{\sqrt{\rho(PQ)}}.$$

Hence there exists $X, Y > 0$ such that 3.a,3.b and 3.c hold if and only if there exists $P, Q > 0$ such that (7.4), (7.5) and (7.6) hold. □

Note that the above theorem we have separated a condition for γ from the other two conditions. Therefore it is possible to first solve (7.4) and (7.5) and then choose a γ such that (7.6) holds. Clearly there exists $P, Q > 0$ such that (7.4) and (7.5) holds if and only if (A, B) is stabilizable and (C, A) is detectable.

7.2.2 Coprime factor perturbations

Consider again the nominal system Σ_n given by (7.2). In the case of coprime factor perturbations the perturbations can again be modeled as a perturbation of the transfer matrix, however the derivation is more involved compared to additive perturbations. For more details we refer to [1] and [17]. There it is assumed that the nominal system is stabilizable and detectable. In that case there exists a unique positive semi-definite $Q \geq 0$ such that

$$AQ + QA^T - QC^TCQ + BB^T = 0 \tag{7.7}$$

and $A - QC^TC$ is Hurwitz, see also the dual version Lemma 2.3.3. It can be proven that a coprime factor perturbed system can be modeled as a system $\Sigma \times \Sigma_p$ with dynamics

$$\begin{aligned} \dot{x} &= Ax + Bu + QC^T d \\ y &= Cx + d \\ z &= \begin{pmatrix} C \\ 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ I \end{pmatrix} u + \begin{pmatrix} I \\ 0 \end{pmatrix} d \\ d &= \Delta z. \end{aligned} \tag{7.8}$$

The solution to the robust stabilization problem where we deal with coprime factor perturbations, which is the main result of this subsection, is given below.

Theorem 7.2.2. *Let $\gamma > 0$ and assume that (A, B) is stabilizable and (C, A) is detectable. Consider the system $\Sigma \times \Sigma_p$ given by (7.8). There exists a controller Γ that solves the robust stabilization problem with uncertainty radius γ if and only if there exists $P > 0$ such that*

$$PA + A^TP - PBB^TP + C^TC < 0 \tag{7.9}$$

$$\gamma < \frac{1}{\sqrt{1 + \rho(PQ)}} \tag{7.10}$$

where $Q \geq 0$ is the unique stabilizing solution of (7.7).

Proof. By the small-gain theorem the system $\Sigma \times \Gamma \times \Sigma_p$ is internally stable for all $\Delta \in R\mathcal{H}_\infty$ with $\|\Delta\|_\infty \leq \gamma$ if and only if $\Sigma \times \Gamma$ is internally stable and the transfer matrix T from d to z satisfies $\|T\|_\infty < \frac{1}{\gamma}$. By Theorem 6.2.1 the latter holds if and only if there exists $X, Y > 0$ such that conditions $3.a, 3.b, 3.c$ and $3.d$ of Theorem 6.2.1 hold. Clearly there always exists $n_c (\leq n_p)$ such that $3.d$ holds. Using the notation of (6.1) then in the case of coprime factor perturbations we have

$$E = QC^T, \quad C_1 = C, \quad D_{11} = 0, \quad D_{12} = I, \quad C_2 = \begin{pmatrix} C \\ 0 \end{pmatrix}, \quad D_{21} = \begin{pmatrix} 0 \\ I \end{pmatrix}, \quad D_{22} = \begin{pmatrix} I \\ 0 \end{pmatrix}$$

where $Q \geq 0$ is the unique solution of (7.7). Observe that

$$\begin{pmatrix} B \\ D_{21} \end{pmatrix}^\perp = \begin{pmatrix} B \\ 0 \\ I \end{pmatrix}^\perp = \begin{pmatrix} I & 0 & -B \\ 0 & I & 0 \end{pmatrix}, \qquad \begin{pmatrix} C_1^T \\ D_{12}^T \end{pmatrix}^\perp = \begin{pmatrix} C^T \\ I \end{pmatrix}^\perp = \begin{pmatrix} I & -C^T \end{pmatrix}.$$

Then condition 3.a becomes

$$\begin{aligned}
&\begin{pmatrix} B \\ D_{21} \end{pmatrix}^{\perp} \begin{pmatrix} AX + XA^T + EE^T & XC_2^T + ED_{22}^T \\ C_2X + D_{22}E^T & -\frac{1}{\gamma^2}I + D_{22}D_{22}^T \end{pmatrix} \begin{pmatrix} B \\ D_{21} \end{pmatrix}^{\perp T} \\
&= \begin{pmatrix} I & 0 & -B \\ 0 & I & 0 \end{pmatrix} \begin{pmatrix} AX + XA^T + QC^TCQ & XC^T + QC^T & 0 \\ CX + CQ & -\frac{1-\gamma^2}{\gamma^2}I & 0 \\ 0 & 0 & -\frac{1}{\gamma^2}I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & I \\ -B^T & 0 \end{pmatrix} \\
&= \begin{pmatrix} AX + XA^T + QC^TCQ & XC^T + QC^T & \frac{1}{\gamma^2}B \\ CX + CQ & -\frac{1-\gamma^2}{\gamma^2}I & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & I \\ -B^T & 0 \end{pmatrix} \\
&= \begin{pmatrix} AX + XA^T + QC^TCQ - \frac{1}{\gamma^2}BB^T & XC^T + QC^T \\ CX + CQ & -\frac{1-\gamma^2}{\gamma^2}I \end{pmatrix} < 0.
\end{aligned}$$

Define $Z \coloneqq X + Q > 0$. The latter LMI holds if and only if $\gamma < 1$ and

$$\begin{aligned}
&AX + XA^T + QC^TCQ - \frac{1}{\gamma^2}BB^T + \frac{\gamma^2}{1-\gamma^2}(XC^T + QC^T)(CX + CQ) \\
&= AX + XA^T + QC^TCQ - \frac{1}{\gamma^2}BB^T + \frac{\gamma^2}{1-\gamma^2}ZC^TCZ \\
&= AX + XA^T + AQ + QA^T + BB^T - \frac{1}{\gamma^2}BB^T + \frac{\gamma^2}{1-\gamma^2}ZC^TCZ \\
&= AZ + ZA^T - \frac{1-\gamma^2}{\gamma^2}BB^T + \frac{\gamma^2}{1-\gamma^2}ZC^TCZ < 0.
\end{aligned}$$

Let $\tilde{Z} \coloneqq \frac{\gamma^2}{1-\gamma^2}Z$ (note $\tilde{Z} > 0$ since $\gamma < 1$) then $Z = \frac{1-\gamma^2}{\gamma^2}\tilde{Z}$ and the above LMI is equivalent with

$$\begin{aligned}
&A\tilde{Z} + \tilde{Z}A^T - BB^T + \tilde{Z}C^TC\tilde{Z} \\
&= \tilde{Z}\left[\tilde{Z}^{-1}A + A^T\tilde{Z}^{-1} - \tilde{Z}^{-1}BB^T\tilde{Z}^{-1} + C^TC\right]\tilde{Z} < 0.
\end{aligned}$$

Finally define $P \coloneqq \tilde{Z}^{-1}$ then the latter holds if and only

$$PA + A^TP - PBB^TP + C^TC < 0.$$

Note that $\tilde{Z} = \frac{\gamma^2}{1-\gamma^2}(X + Q) = P^{-1} > 0$. There exists $X > 0$ such that this holds if and only if

$$\begin{aligned}
&X = \frac{1-\gamma^2}{\gamma^2}P^{-1} - Q > 0, \quad \Leftrightarrow \quad \frac{1-\gamma^2}{\gamma^2}I > P^{\frac{1}{2}}QP^{\frac{1}{2}} \\
&\Leftrightarrow \quad \frac{1-\gamma^2}{\gamma^2} = \frac{1}{\gamma^2} - 1 > \rho(PQ) \quad \Leftrightarrow \quad \gamma < \frac{1}{\sqrt{1+\rho(PQ)}}.
\end{aligned}$$

Hence there exists $X > 0$ such that 3.a holds if and only if there exists $P > 0$ such that (7.9) and (7.10) hold.

Condition $3.b$ becomes

$$\begin{aligned}
&\begin{pmatrix} C_1^T \\ D_{12}^T \end{pmatrix}^{\perp} \begin{pmatrix} YA + A^TY + C_2^TC_2 & YE + C_2^TD_{22} \\ E^TY + D_{22}^TC_2 & -\frac{1}{\gamma^2}I + D_{22}^TD_{22} \end{pmatrix} \begin{pmatrix} C_1^T \\ D_{12}^T \end{pmatrix}^{\perp T} \\
&= \begin{pmatrix} I & -C^T \end{pmatrix} \begin{pmatrix} YA + A^TY + C^TC & YQC^T + C^T \\ CQY + C & I - \frac{1}{\gamma^2}I \end{pmatrix} \begin{pmatrix} I \\ -C \end{pmatrix} \\
&= \begin{pmatrix} YA + A^TY - C^TCQY & YQC^T + \frac{1}{\gamma^2}C^T \end{pmatrix} \begin{pmatrix} I \\ -C \end{pmatrix} \\
&= YA + A^TY - C^TCQY - YQC^TC - \frac{1}{\gamma^2}C^TC \\
&= Y(A - QC^TC) + (A - QC^TC)^TY - \frac{1}{\gamma^2}C^TC < 0.
\end{aligned}$$

Clearly there exists an arbitrarily large $Y > 0$ satisfying the latter inequality since $A - QC^TC$ is Hurwitz. Hence the condition

$$Y - \frac{1}{\gamma^2}X^{-1} \geq 0$$

can always be satisfied once X has been chosen. We conclude that there exists $X, Y > 0$ such that $3.a, 3.b$ and $3.c$ hold if and only if there exists $P > 0$ such that (7.9) and (7.10) hold, where $Q \geq 0$ is the unique stabilizing solution to (7.7). □

In this subsection we have derived necessary and sufficient conditions for the existence of a robustly stabilizing controller in the case of coprime factor perturbations. Observe that these conditions are expressed in terms of a Riccati equality and a Riccati inequality instead of two Riccati inequalities as in the case additive and, as we will see next, multiplicative perturbations. This is due to the modeling of coprime factor perturbations where a Riccati equation is involved.

7.2.3 Multiplicative perturbations

As with additive and coprime factor perturbations we can model multiplicative perturbations as perturbations of the transfer matrix of the nominal system (7.2). For the details we refer to Section 4.1. There it is shown that the dynamics of a multiplicative perturbed system $\Sigma \times \Sigma_p$ are described by

$$\begin{aligned}
\dot{x} &= Ax + Bu + Bd \\
y &= Cx \\
z &= u \\
d &= \Delta u.
\end{aligned} \tag{7.11}$$

We will derive necessary and sufficient conditions for the existence of a controller Γ that solves the $\mathcal{H}_\infty$-control problem with uncertainty radius γ. We will first do this for the case that $0 < \gamma < 1$, subsequently we will consider the less interesting case that $\gamma \geq 1$. We will see that there is a strong connection between the results derived in Chapter 5 and these necessary and sufficient conditions. More specifically, we will see that the same Riccati (in)equalities and the same bound on the tolerance appear in both results.

Theorem 7.2.3. *Let $0 < \gamma < 1$. Consider the system $\Sigma \times \Sigma_p$ given by (7.11). There exists controller Γ that solves the robust stabilization problem with uncertainty radius γ if and only if there exists $P, Q > 0$ and real scalar $\nu > 0$ such that*

$$PA + A^TP - PBB^TP < 0 \tag{7.12}$$

$$AQ + QA^T - \nu QC^TCQ + BB^T < 0 \tag{7.13}$$

$$\gamma \leq \frac{1}{\sqrt{1 + \rho(PQ)}} \tag{7.14}$$

Proof. By the small-gain theorem the system $\Sigma \times \Gamma \times \Sigma_p$ is internally stable for all $\Delta \in R\mathcal{H}_\infty$ with $\|\Delta\|_\infty \leq \gamma$ if and only if $\Sigma \times \Gamma$ is internally stable and the transfer matrix T from d to z satisfies $\|T\|_\infty < \frac{1}{\gamma}$. By Theorem 6.2.1 the latter holds if and only if there exists $X, Y > 0$ such that conditions $3.a, 3.b, 3.c$ and $3.d$ of Theorem 6.2.1 hold. It is clear that there exists a nonnegative integer n_c such that $3.d$ holds. Using the notation of (6.1) then in the case of multiplicative perturbations we have

$$E = B, \quad C_1 = C, \quad C_2 = 0, \quad D_{11} = 0, \quad D_{12} = 0, \quad D_{21} = I, \quad D_{22} = 0.$$

Note that

$$\begin{pmatrix} B \\ D_{21} \end{pmatrix}^{\perp} = \begin{pmatrix} B \\ I \end{pmatrix}^{\perp} = \begin{pmatrix} I & -B \end{pmatrix}, \qquad \begin{pmatrix} C_1^T \\ D_{12}^T \end{pmatrix}^{\perp} = \begin{pmatrix} C^T \\ 0 \end{pmatrix}^{\perp} = \begin{pmatrix} C^{T\perp} & 0 \\ 0 & I \end{pmatrix}.$$

Since $\gamma < 1$ there exists $P > 0$ such that $X = \frac{1-\gamma^2}{\gamma^2} P^{-1}$. Rewriting condition $3.a$ of Theorem 6.2.1 yields

$$\begin{aligned}
&\begin{pmatrix} B \\ D_{21} \end{pmatrix}^{\perp} \begin{pmatrix} AX + XA^T + EE^T & 0 \\ 0 & -\frac{1}{\gamma^2} I \end{pmatrix} \begin{pmatrix} B \\ D_{21} \end{pmatrix}^{\perp T} \\
&= \begin{pmatrix} I & -B \end{pmatrix} \begin{pmatrix} AX + XA^T + EE^T & 0 \\ 0 & -\frac{1}{\gamma^2} I \end{pmatrix} \begin{pmatrix} I \\ -B^T \end{pmatrix} \\
&= \begin{pmatrix} AX + XA^T + BB^T & \frac{1}{\gamma^2} B \end{pmatrix} \begin{pmatrix} I \\ -B^T \end{pmatrix} \\
&= AX + XA^T - \frac{1-\gamma^2}{\gamma^2} BB^T \\
&= \frac{1-\gamma^2}{\gamma^2} P^{-1} (PA + A^T P - PBB^T P) P^{-1} < 0.
\end{aligned}$$

Rewriting $3.b$ gives

$$\begin{aligned}
&\begin{pmatrix} C_1^T \\ D_{12}^T \end{pmatrix}^{\perp} \begin{pmatrix} YA + A^T Y + C_2^T C_2 & YE \\ E^T Y & -\frac{1}{\gamma^2} I \end{pmatrix} \begin{pmatrix} C_1^T \\ D_{12}^T \end{pmatrix}^{\perp T} \\
&= \begin{pmatrix} C^{T\perp} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} YA + A^T Y & YB \\ B^T Y & -\frac{1}{\gamma^2} I \end{pmatrix} \begin{pmatrix} C^{T\perp T} & 0 \\ 0 & I \end{pmatrix} \\
&= \begin{pmatrix} C^{T\perp}(YA + A^T Y) & C^{T\perp} YB \\ B^T Y & -\frac{1}{\gamma^2} I \end{pmatrix} \begin{pmatrix} C^{T\perp T} & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} C^{T\perp}(YA + A^T Y) C^{T\perp} & C^{T\perp} YB \\ B^T Y C^{T\perp T} & -\frac{1}{\gamma^2} I \end{pmatrix} < 0.
\end{aligned}$$

Let $Q > 0$ be such that $Y = \frac{1}{\gamma^2} Q^{-1}$. Then the above holds if and only if

$$\begin{aligned}
&C^{T\perp}(YA + A^T Y + \gamma^2 YBB^T Y) C^{T\perp T} \\
&= \frac{1}{\gamma^2} C^{T\perp} Q^{-1} (AQ + QA^T + BB^T) Q^{-1} C^{T\perp T} < 0.
\end{aligned}$$

Multiplying the equation by γ^2 and by Finsler's Lemma the latter holds if and only if there exists a positive real scalar $\nu > 0$ such that

$$Q^{-1}(AQ + QA^T + BB^T) Q^{-1} + \nu C^T C = Q^{-1} \left[AQ + QA^T - \nu QC^T CQ + BB^T \right] Q^{-1} < 0.$$

Condition $3.c$ holds if and only if

$$\begin{aligned}
&Y - \frac{1}{\gamma^2} X^{-1} = \frac{1}{\gamma^2} Q^{-1} - \frac{1}{1-\gamma^2} P \geq 0 \quad \Leftrightarrow \quad \frac{1-\gamma^2}{\gamma^2} Q^{-1} \geq P \quad \Leftrightarrow \quad \frac{1-\gamma^2}{\gamma^2} I \geq Q^{\frac{1}{2}} P Q^{\frac{1}{2}} \\
&\Leftrightarrow \quad \frac{1-\gamma^2}{\gamma^2} \geq \rho(Q^{\frac{1}{2}} P Q^{\frac{1}{2}}) \quad \Leftrightarrow \quad \frac{1}{\gamma^2} \geq \rho(PQ) + 1 \quad \Leftrightarrow \quad \gamma \leq \frac{1}{\sqrt{1 + \rho(PQ)}}.
\end{aligned}$$

Hence there exists $X, Y > 0$ such that $3.a, 3.b$ and $3.c$ hold if and only if there exists $P, Q > 0$ such that (7.12), (7.13) and (7.14) hold. □

Remark 7.2.4. As we have shown with robust synchronization of multiplicative perturbed multi-agent systems there is an extra freedom in the scalar ν. It is clear that if (A, B) is stabilizable and (C, A) detectable then for any $\nu > 0$ there exists $P, Q > 0$ that satisfy (7.13) and (7.13). It can be observed that if Q is a solution to (7.13) for a given $\nu = \nu_s$ then it is also a solution to the same inequality for a larger $\nu = \nu_l \geq \nu_s$. Therefore the inequality (7.13) has a greater than or equal number of solutions Q for $\nu = \nu_l$ compared to $\nu = \nu_s$. Therefore in general if ν is chosen larger then the solution Q can be chosen smaller and thus $\rho(PQ)$ is smaller which implies a higher upper bound on the tolerance γ. See also Remark 5.1.2.

Remark 7.2.5. Observe that in Proposition 5.1.3 and Theorem 7.2.3 the same type of algebraic Riccati (in)equalities appear as well as the same type of bound on the tolerance γ. If we would assume that the network graph in Theorem 5.1.1 is complete then $\lambda_2 = \lambda_p$ and from Theorem 4.2.2 it follows, by the small-gain theorem, that the robust synchronization problem is equivalent with a $\mathcal{H}_\infty$-control problem. If we ignore the technicalities of the inequality and equality signs in the Riccati (in)equalities and the upper bound for the tolerance we can prove, by using Proposition 5.1.3, that the protocol given in Theorem 5.1.1 achieves the maximum possible uncertainty radius. In fact, if $\lambda_2 = \lambda_p$, it similarly can be shown that the protocols given in [16] and [1] achieve the maximum possible uncertainty radius for the case of additive and coprime factor perturbation respectively.

In Theorem 7.2.3 it is assumed that $\gamma < 1$. The case that $\gamma \geq 1$ is not a very interesting one and we will state the result without giving the proof.

Corollary 7.2.6. *Let $\gamma \geq 1$. Consider the system $\Sigma \times \Sigma_p$ given by (7.11). There exists a controller Γ that solves the $\mathcal{H}_\infty$-control problem with uncertainty radius γ if and only if A is Hurwitz.*

7.3 Final remarks

In Section 7.2 we have shown that for additive, coprime factor and multiplicative perturbations, the solvability of the problem of robust stabilization is equivalent with the solvability of a pair of algebraic Riccati (in)equalities and a separate condition for the uncertainty radius. These pairs of Riccati (in)equalities have a solution if and only if (A, B) is stabilizable and (C, A) is detectable. The condition on the tolerance is given in terms of the spectral radius of these pairs of solutions in all three cases. This makes it possible to choose the desired tolerance *after* solving the Riccati (in)equalities, which is not possible for the LMI's in the general robust stabilization problem.

The results obtained here for additive, coprime factor and multiplicative perturbations hold for the case that the feedthrough term from u to y is zero. One has to be careful when introducing a nonzero feedthrough term in the dynamics of the nominal system Σ_n. As shown in Section 6.2.2 the feedthrough term in the disturbed system Σ can be eliminated by a controller transformation. However, if a feedthrough term is added in Σ_n, the dynamics in Σ in general also changes since the perturbations are modeled as perturbations of the transfer matrix of the nominal system.

It can be shown that in the case of additive perturbations, the addition of a feedthrough term in the nominal system Σ_n also results in the addition of a feedthrough term in the dynamics of Σ. Then the feedthrough term can be fully eliminated by a transformation on the controller matrices. However, in the case of multiplicative and coprime factor perturbations this is no longer true. In that case not only a feedthrough term from the input to the output added in Σ but also other extra terms in the system Σ appear and the robust stabilization problem becomes more difficult to solve, see also [17].

As a final remark, by using Theorem 6.2.1 one can find necessary and sufficient conditions for the solvability of the $\mathcal{H}_\infty$-control problem using output feedback as well. In that case $n_c = 0$ and thus $Y = \frac{1}{\gamma^2} X^{-1}$ must be satisfied in condition 3.d of the theorem. It can be observed that for additive, coprime factor and multiplicative perturbations the necessary and sufficient conditions for the existence of an output feedback controller is the existence of a single $X > 0$ such that two LMI's (in terms of X) are satisfied. A topic for future research is to derive explicitly the form of these LMI's in each of the three cases.

Chapter 8

Conclusions

In this thesis we have extended the existing theory of robust synchronization of multiplicative perturbed multi-agent systems. We looked at the synchronization problem with directed network graphs where the feedthrough term from the input to the output is not assumed to be zero. We introduced distributed relative state observers which have been used to solve synchronization and robust synchronization problem. It is shown that the (robust) synchronization problem is equivalent with a (robust) stabilization problem of a single system with multiple observer based controllers. We have derived necessary and sufficient conditions for the solvability of the synchronization problem for directed graphs that contain a spanning tree using static and observer based protocols.

We have solved the problem of robust synchronization of multiplicatively perturbed multi-agent systems for undirected network graphs. In this case we allow heterogeneous perturbations of the agents. Also a method to compute a protocol that solves the latter problem is given. It is shown that these robustly synchronizing protocols are expressed in terms of the solutions of certain algebraic Riccati equations. We have also solved the similar problem of robust synchronization of multiplicative perturbed multi-agent systems for directed network graphs. However in this case only homogeneous perturbations on the agents are allowed. In both the directed and the undirected case we have computed an upper bound on the guaranteed uncertainty radius. This upper bound is expressed in terms of the spectral radius of the solutions of certain Riccati equations and in terms of the ratio between the second smallest and largest eigenvalues of the Laplacian matrix.

In Chapter 6 an approach to the $\mathcal{H}_\infty$-control problem based on linear matrix inequalities (LMI's) is discussed. In the book of Skelton [13], necessary and sufficient conditions are provided for the existence of a controller that solves the $\mathcal{H}_\infty$-control problem for any given tolerance. These conditions are expressed in terms of solvability of certain LMI's. In this thesis, we have provided an alternative proof to this result and we have established an algorithm that can be used to compute controllers that solve the $\mathcal{H}_\infty$-control problem for this tolerance. Furthermore, we have extended the result to the case where the feedthrough term from the input to the output of the system is not necessarily zero.

It is shown that the results derived for the $\mathcal{H}_\infty$-control problem can be applied in the robust stabilization problem. Also, the algorithm that is used to compute controllers that solve the $\mathcal{H}_\infty$-control problem can also be used to compute controllers that solve that robust stabilization problem. We have also looked at some well-known special cases of the robust stabilization problem. For case of additive, coprime factor and multiplicative perturbations it is proven that the existence of a robustly stabilizing controller is equivalent with the solvability of certain algebraic Riccati (in)equalities. In each of the three cases, a condition for the uncertainty radius γ can be expressed in the solutions of these algebraic Riccati (in)equalities. This result makes it possible to explicitly compute the maximum upper bound on the uncertainty radius for which there still exists a robustly stabilizing controller. To the author's best knowledge these results for additively, coprime factor and multiplicatively perturbed systems were not established before.

We have found a solution to the problem of robust synchronization of multiplicative perturbed multi-agent systems with undirected network graphs and heterogeneous perturbations. However, the case of heterogeneous perturbations and directed graphs is still to be solved. Also in this thesis we have considered linear perturbations. For future research, one might also allow nonlinear perturbations that, for example, are bounded in $\mathcal{L}_2$-norm. For the robust stabilization problem, we have analyzed the special cases of additive, coprime factor and multiplicative perturbations. However we have assumed that the feedthrough term from the input to the output of the nominal system is zero. One might be interested in the case that this term does not vanish. For future research also an interesting case is the $\mathcal{H}_\infty$-control problem with output feedback. To solve the latter problem the results derived in this thesis can be very useful.

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