Majorana particles in physics and mathematics

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Abstract

In 1937, the Italian physicist Ettore Majorana showed that there exist real solutions to the Dirac equation. This suggests the existence of the Majorana fermion, a neutral fermion that is equal to its antiparticle. Up until now, no Majorana fermions have been found. Recent developments in solid state physics have led to evidence that so-called Majorana zero modes can exist in superconductors. Sometimes these quasiparticles are also confusingly named "Majorana fermions". These modes or quasiparticles show some resemblance with the real Majorana fermions, however they are two completely different physical phenomena. This article mathematically describes the differences between these two concepts by the use of different Clifford algebras. For the description of the Majorana spinor a Clifford algebra is used that satisfies a pseudo-Euclidean metric, applicable in a selection of space-time dimensions. The Majorana zero mode is described by a Clifford algebra that satisfies a purely Euclidean metric in the abstract space of zero modes. Furthermore the statistics of both entities is described, where Fermi-Dirac statistics applies to the Majorana fermion and non-Abelian anyonic statistics applies to the Majorana zero modes.

Contents

Introduction

The goal of this thesis is to mathematically describe the difference between the Majorana fermion, as introduced by Ettore Majorana, and the confusingly equally named "Majorana fermion", as been observed in nanowires coupled to semiconductors.
In one of his few articles[1], the enigmatic Italian physicist Ettore Majorana published a theory[2] in which he concluded that a neutral fermion has to be equal to its own antiparticle. As a candidate of such a Majorana fermion he suggested the neutrino. Very much has been written about Ettore Majorana and his mysterious disappearance in 1938. This article will not focus on this aspect of Majorana.
The second "Majorana fermion", misleadingly named so, was claimed to be found by the group of Leo Kouwenhoven in May 2012 [26]. This second "Majorana" is actually a zero mode in a one-dimensional semiconductor quantum wire. This quasiparticle is chargeless and has no magnetic dipole moment. There are also several proposals for creating Majorana modes in two-dimensional topological insulators. There is little doubt that the Majorana quasiparticle exists and that its existence will be proven more rigorously in the future. However for the real Majorana fermion there is no such a certainty [10].
Besides the fact that a fermion and a zero mode are totally different concepts, there are several unneglegible differences between the fermion and the zero mode. To get a clear mathematical description of this difference we use Clifford algebras. The true Majorana spinor, and hence also its corresponding particle after quantization, can exist in a selection of spacetime signatures. To realize a Majorana spinor in a certain spacetime, a signature-depending Clifford algebra has to be constructed, consisting out of so-called Γ-matrices. For a more detailed study of this signature-dependence, we refer the reader to section 3.4 and its references.
In describing the Majorana zero mode, also a Clifford algebra can be used. However, this Clifford algebra is very different. Whereas the Majorana spinor corresponds to

[1] In total Majorana has published nine articles in the years 1928-1937

[2] The theory was presented in the article "Teoria simmetrica dell'elettrone e del positrone" (Symmetrical theory of the electron and positron) in 1937 [22]

a Clifford algebra whose dimensions are dictated by the number of spacetime dimensions with non-Euclidean metric, the creation and annihilation operators for the Majorana zero modes act on the Hilbert space of the zero modes of a one-dimensional chain with Euclidean metric. Importantly, the statistics of both particles are different. The Majorana fermion obeys Fermi-Dirac statistics, whereas the Majorana zero mode is a non-Abelian anyon. For a more elaborate explanation of this concept, we refer the reader to section 4.3.1.

Chapter 1

The relation between the Dirac equation and the Clifford algebra

Although Clifford algebra[1] was already introduced by the English mathematician W.K. Clifford[2] in 1882 [34], physicists were not very much interested in it until Dirac posed his relativistic wave equation for the electron. After the publication of the article “The Quantum Theory of the Electron. Part I” on the first of February 1928 [9] and its sequel one month later, the interest of theoretical physicists for Clifford algebra grew exponentially [12]. To see the link between the Dirac equation and Clifford algebra, we follow Dirac’s lines of thought in finding the Dirac equation.

1.1 Constructing the Dirac equation

Let us first derive two other equations, which Dirac used for finding his relativistic wave equation for the electron.

1.1.1 The Schrödinger equation

In classical mechanics, we have the following non-relativistic energy relation for a particle

$$E = E_{kin} + E_{pot} = \frac{\mathbf{p}^2}{2m} + V. \tag{1.1}$$

If we now go to quantum mechanics, we substitute the momentum operator for $\mathbf{p}$ and the energy operator W for E. These operators both act now on a wave function Ψ [12].

$$\mathbf{p} \to -i\hbar\nabla, \qquad E \to i\hbar\frac{\partial}{\partial t} =: W. \tag{1.2}$$

[1]For a self-contained mathematical introduction of the Clifford algebra, see Chapter 3.
[2]William Kingdon Clifford 1845-1879 [20].

Substituting (1.2) into (1.1) gives the well-known Schrödinger[3] equation[4]

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi + V\Psi. \tag{1.3}$$

1.1.2 The Klein-Gordon equation

The Klein-Gordon equation can be derived in a similar manner. We start now with the fundamental energy-momentum relation in the relativistic case for a free particle

$$\frac{E^2}{c^2} - \mathbf{p}^2 = m^2 \cdot c^2 = p^\mu p_\mu = p_\mu p_\nu \eta^{\mu,\nu} = {p_0}^2 - \mathbf{p}^2. \tag{1.4}$$

Since $\eta^{\mu,\nu}$ is defined as

$$\eta^{\mu,\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

We now have to generalize our (1.2)-substitution to covariant notation.

$$p^0 = \frac{i\hbar}{c}\partial^0 = \frac{i\hbar}{c}\frac{\partial}{\partial x_0} = \frac{i\hbar}{c}\frac{\partial}{\partial t}, \tag{1.5}$$

$$p^i = i\hbar\partial^i = i\hbar\frac{\partial}{\partial x_i} \text{ for } i\text{=1,2,3}. \tag{1.6}$$

If we now again use the substitution from (1.2) and apply the operators on the wavefunction Ψ, we get

$$p^\mu p_\mu \Psi = m^2c^2\Psi, \tag{1.7}$$

$$(p_0p^0 - p_ip^i)\Psi = m^2c^2\Psi, \tag{1.8}$$

$$({p_0}^2 - \mathbf{p}^2)\Psi = m^2c^2\Psi, \tag{1.9}$$

$$(\frac{i\hbar}{c}\partial^0)(\frac{i\hbar}{c}\partial_0) - (i\hbar\partial^i)(i\hbar\partial_i)\Psi = m^2c^2\Psi, \tag{1.10}$$

$$(-\frac{\hbar^2}{c^2}\partial^0\partial_0 + \hbar^2\partial^i\partial_i)\Psi = m^2c^2\Psi, \tag{1.11}$$

$$-\hbar^2\left(\frac{1}{c^2}\partial^0\partial_0 - \partial^i\partial_i\right)\Psi = m^2c^2\Psi. \tag{1.12}$$

[3] Erwin Rudolf Josef Alexander Schödinger 1887-1961 [25]

[4] There is certainly no guarantee that Schrödinger himself derived his equation in this way. In [36] D. Ward explains the way Schrödinger found his equation

If we now use natural units, i.e. setting $\hbar = c = 1$, and denoting the Laplacian as $\partial^\mu \partial_\mu = \Box$ we find the Klein-Gordon equation.

$$(-\partial^0 \partial_0 + \partial^i \partial_i)\Psi = m^2 \Psi, \tag{1.13}$$

$$(\partial^\mu \partial_\mu + m^2)\Psi = 0, \tag{1.14}$$

$$(\Box + m^2)\Psi = 0. \tag{1.15}$$

The question which immediately arises is how to interpret this equation. For the Schrödinger equation, we know it describes the time evolution of a wave function of a non-relativistic quantum mechanical system. However, for the Klein-Gordon equation the situation is a bit more precarious. A detailed explanation of the Klein-Gordon equation in the book of A. Das [7] shows that we cannot see the Klein-Gordon equation as a quantum mechanical description for a single relativistic particle. The fact that the Klein-Gordon equation is second-order in the time derivatives, contrary to the Schrödinger equation which is first-order in time derivatives, leads to the possibility of negative energy solutions. This property does not have to be critical, one can account for this solutions with antiparticles. Also the Dirac equation has negative energy solutions. However, this second-order time derivative leads to negative probability densities.

However, the Klein-Gordon equation has a clear meaning in quantum field theory, i.e. interpreted as a field equation for a scalar field ϕ. The negative energy solutions correspond now to antiparticles having a positive energy. In this field interpretation it can be shown that the Klein-Gordon equation is relativistic [33], i.e. invariant under Lorentz transformations.

1.1.3 The Dirac equation

The second-order time derivatives in the Klein-Gordon equation caused the appearance of negative energy solutions and negative probability densities, when interpreting it as a relativistic wave equation for a single particle. Hence the English physicist Paul Dirac[5] decided to construct a new wave equation starting from the Schrödinger equation. That is, an equation linear in temporal derivatives of the form

$$(H - W)\Psi = 0, \tag{1.16}$$

[5] Paul Adrien Maurice Dirac 1902-1984 [12].

where H is the Hamiltonian and $W = i\hbar\frac{\partial}{\partial t}$. Furthermore, the equation should be Lorentz invariant and in the relativistic limit, the equation should recover the relativistic energy relation (1.4).

Using Dirac's original notation, we start with the ansatz that the Hamiltonian is linear in the time derivatives. Lorentz invariance requires now that the Hamiltonian is also linear in the spatial derivatives. This leads to the following ansatz

$$(p_0 + \alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 + \beta)\Psi = 0, \tag{1.17}$$

where $p_\mu = \big(p_0, p_1, p_2, p_3\big)^\top = \big(\frac{i\hbar}{c}\frac{\partial}{\partial x^0}, i\hbar\frac{\partial}{\partial x^1}, i\hbar\frac{\partial}{\partial x^2}, i\hbar\frac{\partial}{\partial x^3}\big)^\top$.
Because we assume that (1.17) is linear in p_μ, we see that our α's and β can be chosen in such a way that they are indepedent of p_μ. Therefore α_i and β commute with x_i and t, and the fact that we are considering a free particle implies that our α's and β are actually independent of x_i and t. This in turn implies their commutation with p_μ. If our α's would just be numbers, we see that the four vector $\big(1, \alpha_1, \alpha_2, \alpha_3\big)^\top$ would define some direction and the equation would not be Lorentz invariant. So what are our α's then?

The following step that Dirac used, in order to determine the α's, is transforming equation (1.17) to a form similar to (1.15) and comparing the terms. This can be done as follows. Start with (1.17) and conveniently multiply this with a certain term as follows.

$$\begin{aligned} 0 =& (-p_0 + \alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 + \beta)(p_0 + \alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 + \beta)\Psi & (1.18)\\ =& [-{p_0}^2 + \Sigma{\alpha_1}^2{p_1}^2 + \Sigma(\alpha_1\alpha_2 + \alpha_2\alpha_1)p_1p_2 + \beta^2 + \Sigma(\alpha_1\beta + \beta\alpha_1)p_1]\Psi. & (1.19)\end{aligned}$$

The Σ denotes here the cyclic permutations of the suffixes 1,2,3. Comparing with (1.15), we see that the two expressions are equal if and only if

$$\begin{aligned} {\alpha_r}^2 &= 1, & \alpha_r\alpha_s + \alpha_s\alpha_r &= 0, & (r \neq s),\\ \beta^2 &= m^2c^2, & \alpha_r\beta + \beta\alpha_r &= 0, & \text{where } r,\ s = 1,\ 2,\ 3. \end{aligned}$$

To simplify this set of expressions, write $\beta = \alpha_4 mc$. Then we get the following anticommutator

$$\{\alpha_\mu, \alpha_\nu\} = \alpha_\mu\alpha_\nu + \alpha_\nu\alpha_\mu = 2\delta_{\mu,\nu}, \qquad \mu,\ \nu = 1, 2, 3, 4$$

The three Pauli matrices σ_1,σ_2,σ_3 satisfy these conditions, where σ_1,σ_2,σ_3 are given as

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

However, we have to represent four terms, namely α_i and β. Therefore we have to construct four 4×4-matrices to get a suitable matrix representation. A direct consequence of this is that our Ψ has to be four-dimensional. We call Ψ, a object with four complex components, a *Dirac spinor*. The representation which Dirac firstly introduced is the so-called chiral or Weyl representation. For this representation also a "σ_0"-matrix" is constructed, defined as

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1}.$$

Definition 1.1.1 Weyl or chiral representation

$$\alpha_i = \begin{pmatrix} -\sigma_i & \mathbf{0} \\ \mathbf{0} & \sigma_i \end{pmatrix}, \; i = 1, 2, 3, \qquad \alpha_4 = \begin{pmatrix} \mathbf{0} & \sigma_0 \\ \sigma_0 & \mathbf{0} \end{pmatrix}. \tag{1.20}$$

In this definition all entries of the matrices are itself again 2×2-matrices, so that the α-matrices are 4×4-matrices. We can subsequently substitute these α_μ in (1.17), implying that Ψ must be four-dimensional.

For reasons explained in [6] it is more convenient to introduce another family of matrices, the γ-matrices, deduced from the α matrices. The reason lies in the fact that the γ-matrices help to have a simple representation for the chiral projection operators, which project out the positive or negative chirality parts of the four-dimensional Dirac spinor Ψ. The definition of the γ-matrices is as follows.

Definition 1.1.2 γ-matrices

$$\gamma_0 = \alpha_4, \qquad \gamma_i = \alpha_4 \alpha_i. \tag{1.21}$$

With this definition, we can rewrite (1.17) by multiplying it with α_4.

$$\begin{aligned}
0 &= \alpha_4(p_0 + \alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 + \beta)\Psi, \\
0 &= (\alpha_4 p_0 + \alpha_4\alpha_1 p_1 + \alpha_4\alpha_2 p_2 + \alpha_4\alpha_3 p_3 + \alpha_4\beta)\Psi, \\
0 &= (\alpha_4 p_0 + \alpha_4\alpha_1 p_1 + \alpha_4\alpha_2 p_2 + \alpha_4\alpha_3 p_3 + {\alpha_4}^2 mc)\Psi, \\
0 &= (\gamma_0 p_0 + \gamma_1 p_1 + \gamma_2 p_2 + \gamma_3 p_3 + mc)\Psi, \\
0 &= (i\gamma_\mu \partial^\mu - m)\Psi, \qquad (1.22)
\end{aligned}$$

where in the last step we switched to natural units ($\hbar = c = 1$) and used the Einstein summation convention. Note that γ_0 is Hermitian, whereas γ_i is antihermitian, since

$$(\gamma_0)^\dagger = (\alpha_4)^\dagger = \alpha_4 = \gamma_0$$

and

$$(\gamma_i)^\dagger = (\alpha_4\alpha_i)^\dagger = (\alpha_i)^\dagger(\alpha_4)^\dagger = \alpha_i\alpha_4 = -\alpha_4\alpha_i = -\gamma_i.$$

The hermiticity properties can be summarized by the relation

$$(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0. \qquad (1.23)$$

One can easily check from the anticommutation relation for α_μ, that γ_μ has the following anticommutation relation.

$$\{\gamma_\mu, \gamma_\nu\} = \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\eta_{\mu,\nu}. \qquad (1.24)$$

So with equation (1.24) we have thus found a representation of the so-called Clifford algebra in Minkowski spacetime, i.e. four-dimensional spacetime in which $\eta^{\mu,\nu}$ determines the metric, where $\eta^{\mu,\nu}$ is defined as in (1.5).

Chapter 2

The Majorana Fermion

2.1 Weyl, Dirac and Majorana Spinors

To provide a rigorous description of the different types of spinors, we must firstly make ourselves comfortable with the concept of the Lorentz group and the Lorentz algebra[1]. The Lorentz algebra will lead us to spinor representations. After categorizing certain types of spinors and showing their key properties, we conclude this chapter by *quantizing* these spinor fields.

2.1.1 Lorentz group and algebra

Let us start with the definition of the Lorentz group. This is an example of a more general family of groups, the *Lie groups*[2].

Definition 2.1.1 Lorentz group
The Lorentz group is the group of all linear transformations, boosts, rotations and inversions which preserve the spacetime interval $c^2\tau^2 = x_0^2 - \boldsymbol{x}^2$.

If we exclude the parity operations (inversions) $x_0 \to x_0$, $\boldsymbol{x} \to -\boldsymbol{x}$, we obtain the *proper Lorentz group.* A boost is a different name for a pure Lorentz transformation, i.e. a Lorentz transformation of the general form;

$$x^{0\prime} = \gamma x^0 + \gamma \boldsymbol{\beta} \cdot \boldsymbol{x}, \tag{2.1}$$

$$\boldsymbol{x}' = \gamma \boldsymbol{\beta} x^0 + \frac{\gamma^2}{1+\gamma} \boldsymbol{\beta}(\boldsymbol{\beta} \cdot \boldsymbol{x}) + \boldsymbol{x}, \tag{2.2}$$

[1]For a more mathematical description of an algebra see Chapter 3
[2]For the formal definition of a Lie group, see [2]

where $x^0 = ct$ and $\beta = \frac{\boldsymbol{v}}{c}$, $\gamma = \frac{1}{\sqrt{1-\boldsymbol{\beta}^2}}$. In other words a boost is just a coordinate transformation between two inertial frames with a relative speed $\boldsymbol{\beta} = \frac{\boldsymbol{v}}{c}$ to each other. For later use we also introduce the concept of rapidity ϕ. Rapidity is an alternative way to describe the speed of an object, defined as

$$\phi = \operatorname{arctanh} \beta = \operatorname{arctanh} \frac{v}{c}. \tag{2.3}$$

Using $\gamma = \cosh \phi$, $\gamma\beta = \sinh \phi$ and $\hat{\boldsymbol{\beta}} = \frac{\boldsymbol{\beta}}{\beta}$, our boost will then get the form

$$x^{0\prime} = x^0 \cosh \phi + \hat{\boldsymbol{\beta}} \cdot \boldsymbol{x} \sinh \phi, \tag{2.4}$$

$$\boldsymbol{x}' = \hat{\boldsymbol{\beta}} x^0 \sinh \phi + \hat{\boldsymbol{\beta}}(\hat{\boldsymbol{\beta}} \cdot \boldsymbol{x})(\cosh \phi - 1) + \boldsymbol{x}. \tag{2.5}$$

Notice that this parametrization is only possible if $\beta = |\boldsymbol{\beta}| < 1$, the domain of $\operatorname{arctanh} x$ is $x \in (-1, 1)$. The group of all symmetries of Minkowski spacetime is called the *Poincaré* or *inhomogeneous Lorentz group*. The Poincaré group can be seen as the Lorentz group extended with spacetime translations. From the Lorentz group we can derive the Lorentz algebra. To acquire the Lorentz algebra from its group, we just find the infinitesimal generators J_i of the group. This set of infinitesimal generators forms a basis for the Lie algebra. Since we have 3 rotations (around the $\hat{x}_1$-, $\hat{x}_2$- or $\hat{x}_3$-direction) and 3 boosts (along the $\hat{x}_1$-, $\hat{x}_2$- or $\hat{x}_3$-direction), which span the Lorentz group, we need the infinitesimal forms of these 6 elements to find the Lie algebra. The infinitesimal generators of the Lie algebra depending on one variable α have the following form in a general representation D

$$-iJ = \left. \frac{dD(\alpha)}{d\alpha} \right|_{\alpha=0}. \tag{2.6}$$

The general form of a Lorentz transformation which is a rotation around the $\hat{x}_1$-axis is

$$R_1(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{pmatrix}.$$

Formula (2.6) now gives us the corresponding infinitesimal generator of this transformation;

$$J_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}.$$

In a similar way we can derive J_2 and J_3, resulting in

$$J_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \qquad J_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

If we for example look at a pure Lorentz transformation in the $\hat{x}_1$-direction, we see that equation (2.4) will get the following form, written out in components

$$B_1(\phi) = \begin{pmatrix} \cosh\phi & \sinh\phi & 0 & 0 \\ \sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The corresponding infinitesimal generator will have the form

$$K_1 = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Again we can also derive the other infinitesimal generators K_2 and K_3, resulting in

$$K_2 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad K_3 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}.$$

Since an algebra is a vector space equipped with an extra product[3], mapping again to the algebra, there must be a product $L : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$, which takes two elements from our Lorentz algebra and maps them again into one single element of the Lorentz algebra. In the Lorentz algebra this product is just simply the commutator of two

[3]besides the "usual" multiplication of the vector space

matrices. One can show that the following commutation relations hold in the Lorentz algebra.

$$\begin{aligned} [J_i, J_j] &= i\epsilon_{ijk}J_k, \\ [J_i, K_j] &= i\epsilon_{ijk}K_k, \\ [K_i, J_j] &= -i\epsilon_{ijk}J_k. \end{aligned} \tag{2.7}$$

Note that the infinitesimal rotation generators form an invariant set under the commutator. Hence the infinitesimal rotation generators form a subalgebra, $so(3)$. We can simplify these commutation relations by introducing the linear combinations

$$J_{\pm r} = \frac{1}{2}(J_r \pm iK_r). \tag{2.8}$$

Explicit calculations show that with these entities we get the following commutation relations

$$\begin{aligned} [J_{+i}, J_{+j}] &= i\epsilon_{ijk}J_{+k}, \\ [J_{-i}, J_{-j}] &= i\epsilon_{ijk}J_{-k}, \\ [J_{+i}, J_{-j}] &= 0.(!) \end{aligned} \tag{2.9}$$

We learn from this last commutation relation that J_+ and J_- satisfy seperately an $su(2)$ algebra. Apparently the Lorentz algebra $so(3,1)$ is isomorphic to $su(2) \otimes su(2)$.

2.1.2 Spinors

The special unitary group $SU(2)$ is the set of all unitary matrices, endowed with the normal matrix multiplication as the group multiplication. One can give a general form of a matrix U in this group by using the following parametrisation

$$U = \begin{bmatrix} \alpha_1 + i\alpha_2 & \beta_1 + i\beta_2 \\ -\beta_1 + \beta_2 & \alpha_1 - i\alpha_2 \end{bmatrix}, \qquad \alpha_1^2 + \alpha_2^2 + \beta_1^2 + \beta_2^2 = 1.$$

Reparametrizing this expression and using the Pauli spin matrices gives

$$U(x_1, x_2, x_3) = \quad \alpha_1 \mathbb{1} + i\mathbf{x} \cdot \boldsymbol{\sigma}/2,$$

where $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ and $x_1 = 2\beta_2$, $x_2 = 2\beta_1$, $x_3 = 2\alpha_2$. Due to the Pauli matrices, it follows that $SU(2)$ can be used to describe the spin of a particle. In $SU(2)$ we know that we have invariant subspaces labelled by j, which is the orbital angular momentum number. The quantum number m_j from the projection operator L_z can run from $-j$ to j. So for each j we have a set of $(2j+1)$ wavefunctions $(|j,\ m_j\rangle)$, i.e. a $(2j+1)$-dimensional subset or irreducible representation.

So it follows that in our $su(2) \otimes su(2)$ algebra any element can be represented by the notation $(j_+,\ j_-)$. Consequently each pair $(j_+,\ j_-)$ will correspond to a $(2j_+ + 1)(2j_- + 1)$-dimensional invariant subspace, since in each state $|j_+\ m_+\rangle\,|j_-\ m_-\rangle$ both m_+ as m_- can run from $-j_+$ to j_+ and from $-j_-$ to j_- respectively. The first four combinations $(j_+,\ j_-)$ are used the most and they have acquired special names

$$\begin{aligned}
(0,\ 0) &= \text{ scalar or singlet,} \\
\left(\frac{1}{2},\ 0\right) &= \text{ left-handed Weyl spinor,} \\
\left(0, \frac{1}{2}\right) &= \text{ right handed Weyl spinor,} \\
\left(\frac{1}{2},\ \frac{1}{2}\right) &= \text{ vector.}
\end{aligned}$$

To see what the $\left(\frac{1}{2},\ 0\right)$ means, let us look at the basis wave functions which span the invariant subspace of $(j_+ = \frac{1}{2},\ j_- = 0)$; $\left|\frac{1}{2}\ \frac{1}{2}\right\rangle|0\ 0\rangle$ and $\left|\frac{1}{2},\ -\frac{1}{2}\right\rangle|0\ 0\rangle$. So we expect Weyl spinors to describe spin-$\frac{1}{2}$ particles. Let us denote these states by Ψ_α respectively, with $\alpha = 1, 2$. If we act with J_{+i} on Ψ_α, we get $\frac{1}{2}\sigma_i$. Acting with J_{-i} results in 0. Combining these two outcomes we get

$$J_i = \frac{1}{2}\sigma_i, \qquad\qquad iK_i = \frac{1}{2}\sigma_i.$$

We could do the same for the $(j_+ = 0,\ j_- = \frac{1}{2})$. We use the so called van der Waerden notation to write down this right handed Weyl spinor. In the van der Waerden notation we "dot" the indices of the right handed Weyl spinors. In this way we can already see from the indices whether we are talking about a left- or right-handed Weyl spinor. Hence we denote the two basis wave functions of $(j_+ = 0,\ j_- = \frac{1}{2})$ as $\xi^{\dagger\dot{\alpha}}$. However we then obtain a minus sign in the iK_{-i}-expression

$$J_i = \frac{1}{2}\sigma_i, \qquad\qquad iK_i = -\frac{1}{2}\sigma_i.$$

We call the two dimensional spinors χ_c and $\xi^{\dagger\dot{c}}$ left- and right-handed Weyl spinors respectively. The Weyl spinors can be seen as the building blocks for Dirac and

Majorana spinors. The Dirac spinor is simply the combination of these two two-dimensional spinors as a four-dimensional entity

$$\boldsymbol{\Psi_{Dir}} = \begin{pmatrix} \chi_c \\ \xi^{\dagger\dot{c}} \end{pmatrix}. \tag{2.10}$$

A different reason, other than the one of chapter 1, why the Dirac spinor has to be four-dimensional is because of parity. Since velocity $\boldsymbol{v}$ changes sign under parity, so does K_i. Angular momentum which corresponds to the infinitesimal generator J_i is an axial vector and hence doesn't change sign under parity operations. The consequence of this is that under parity: $(j_+ = 0,\ j_- = \frac{1}{2}) \leftrightarrow (j_+ = \frac{1}{2},\ j_- = 0)$, i.e. a left-handed Weyl spinor turns into a right-handed Weyl spinor and vice versa. More mathematically, one could say that the Dirac spinor lies in the $(j_+ = 0,\ j_- = \frac{1}{2}) \oplus (j_+ = \frac{1}{2},\ j_- = 0)$ representation [38]. There is an operator which whom we can project out the left- and right handed parts of the Dirac spinor. For this we need the so-called γ^5-matrix. It is defined in the following way

$$\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3.$$

This matrix has the obvious properties

$$\{\gamma^\mu, \gamma^5\} = 0, \qquad\qquad (\gamma^5)^2 = +1.$$

Since $(\gamma^5)^2 = +1$, we see that γ^5 can only have two eigenvalues namely ± 1. In the chiral representation, one can calculate that γ^5 has the form

$$\gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

So if we now define the following Lorentz invariant projection operators

$$P_\pm = \frac{1}{2}(1 \pm \gamma^5),$$

we see that we exactly project out the Weyl spinors.

The Majorana spinor is an even simpler construction. It is actually composed of only one Weyl spinor. As P. B. Pal explains in [27], after its theoretical discovery scientists were not very interested in the Majorana spinor. Neutrino's introduced by Pauli could be Majorana particles, however everyone assumed that neutrino's were Weyl particles, i.e. described by a Weyl spinor. Weyl spinors are elegant solutions of the Dirac equation, provided that the particle is massless. The absence of the mass term will prohibit mixing between left-handed and right-handed spinors. When in the second half of the twentieth century people started studying the consequences of a massive neutrino, the interest in Majorana spinors grew since it described a massive fermion which is its own antiparticle. Despite the simpler nature of the Majorana spinor, scientists are so accustomed to Dirac spinors that working with Majorana spinors is a bit uncomfortable. However the Majorana spinor is actually a more constrained, simpler solution of the Dirac equation.

2.1.3 Majorana spinors

The Majorana spinor is constructed out of one Weyl spinor in the following way. Start with a left-handed Weyl spinor Ψ_c, now define the right-handed part of the Majorana spinor simply as the Hermitian conjugate of Ψ_c, $\Psi^{\dagger\dot{c}}$. We have now created a Majorana spinor

$$\boldsymbol{\Psi_{Maj}} = \begin{pmatrix} \Psi_c \\ \Psi^{\dagger\dot{c}} \end{pmatrix}. \tag{2.11}$$

There is also another approach from which we more directly see how the Majorana spinors arise from the Dirac equation. Look again at the Dirac equation

$$(i\gamma_\mu \partial^\mu - m)\Psi = 0.$$

If we could find a representation of the Clifford algebra in terms of *purely complex* gamma matrices, then $i\gamma_\mu$ would be real. So then this equation could have a real solution, Ψ_{Maj}. But this real solution Ψ_{Maj} would imply $\Psi^\dagger_{Maj} = \Psi_{Maj}$. So indeed the Maj-subscript is well placed. Ettore Majorana found such a purely imaginary representation of the gamma-matrices, namely

$$\begin{aligned} \tilde{\gamma}^0 &= \sigma_2 \otimes \sigma_1 \\ \tilde{\gamma}^1 &= i\sigma_1 \otimes \mathbb{1} \\ \tilde{\gamma}^2 &= i\sigma_3 \otimes \mathbb{1} \\ \tilde{\gamma}^3 &= i\sigma_2 \otimes \sigma_2 \end{aligned}$$

One could write this Kronecker product out to get the following imaginary matrices.

$$\tilde{\gamma}^0 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix},$$

$$\tilde{\gamma}^1 = \begin{pmatrix} 0 & 0 & i & o \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix},$$

$$\tilde{\gamma}^2 = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \end{pmatrix},$$

$$\tilde{\gamma}^3 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}.$$

One can easily check that these matrices satisfy the Clifford algebra from (1.24). To see the key feature of the Majorana spinor we must first familiarize ourselves with the concept of charge conjugation.

2.1.4 Charge conjugation

Besides the continuous symmetries of a dynamical system, such as Lorentz invariance or translational invariance. One can also look at discrete symmetries of a dynamical system. The three most familiar discrete symmetries are charge conjugation ($\mathcal{C}$), parity ($\mathcal{P}$), which we already met, and time reversal ($\mathcal{T}$)[4]. Since this article deals with Majorana fermions it will only focus on charge conjugation and its corresponding symmetry.

Assume we have Dirac fermions minimally coupled to the photons of an electromagnetic field. Minimally coupled means that in the interaction all multipoles are ignored, except for the first, i.e. the monopole or the overall charge. To account for this coupling we must add an interaction term to our Lagrangian, namely $eA_\mu\Psi^\dagger\gamma^0\Psi$, resulting in the Lagrangian

$$\mathcal{L} = \overline{\Psi}(i\gamma^\mu\partial\!\!\!/ - m)\Psi + eA_\mu\gamma^\mu\overline{\Psi} = \overline{\Psi}(i\gamma^\mu D_\mu - m)\Psi. \tag{2.12}$$

[4]These three symmetries are united in the so called $\mathcal{CPT}$-theorem. See for example Mann [23]

Here we have defined the covariant derivative $D_\mu = \partial_\mu - ieA_\mu$. Deriving the Euler-Lagrange equations from this formalisms gives us a different version of the Dirac equation

$$[i\gamma^\mu(\partial_\mu - ieA_\mu) - m]\Psi = 0. \tag{2.13}$$

Taking the complex conjugate of (2.13) gives us

$$[-i\gamma^{\mu*}(\partial_\mu + ieA_\mu) - m]\Psi^* = 0. \tag{2.14}$$

Since the γ^μ satisfy (1.24), we see by complex conjugating (1.24) $-\gamma^{\mu*}$ must satisfy also (1.24). Hence the $-\gamma^{\mu*}$ can be acquired by applying a basis transformation on γ^μ, call this transformation matrix $C\gamma^0$. Thus

$$-\gamma^{\mu*} = (C\gamma^0)^{-1}\gamma^\mu(C\gamma^0). \tag{2.15}$$

This is the definining property of the charge conjugation matrix. If we furthermore define $\Psi^C := C\overline{\Psi}^\top = C\gamma^{0\top}\Psi^*$, insert (2.15) in (2.14) and multiply this from the left by $C\gamma^0$, we get

$$[i\gamma^\mu(\partial_\mu + ieA_\mu) - m]\Psi^C = 0. \tag{2.16}$$

So if Ψ satisfies the Dirac equation (2.13), then the charge conjugate field Ψ^C with the same mass but opposite charge satisfies (2.16). We can also rewrite the defining equation (2.15) in a different form.
Note that if we complex conjugate equation (1.23) we have

$$\begin{aligned}(\gamma^\mu)^\dagger &= \gamma^0\gamma^\mu\gamma^0\\ (\gamma^\mu)^{\dagger*} &= \gamma^{0*}\gamma^{\mu*}\gamma^{0*} = (\gamma^\mu)^\top\end{aligned}$$

Assuming that γ^0 is real.

$$(\gamma^\mu)^\top = \gamma^0\gamma^{\mu*}\gamma^0$$

We can use this expression for deriving the following relation between $\gamma^{\mu\top}$ and γ^μ

$$\begin{aligned}-\gamma^\mu &= C\gamma^0\gamma^{\mu*}\gamma^0C^{-1},\\ &= C\gamma^{\mu\top}C^{-1}.\end{aligned}$$

To see the signature property of the Majorana spinor, let us calculate the charge conjugate of both the Majorana spinor and the Dirac spinor. Define again

$$\boldsymbol{\Psi_{Maj}} = \begin{pmatrix}\Psi_a\\ \Psi^{\dagger\dot{a}}\end{pmatrix}, \qquad\qquad \boldsymbol{\Psi_{Dir}} = \begin{pmatrix}\chi_a\\ \xi^{\dagger\dot{a}}\end{pmatrix}.$$

Then we want to calculate $\Psi^C = C\overline{\Psi}^\top = C\left(\Psi^\dagger\gamma^0\right)^\top$. However by using van der Waerden notation we can be more precise in the spinor index structure.

Intermezzo I: Manipulating spinors in van der Waerden notation

If we have a certain four vector x^μ we can lower the index in the following way: $x^\mu = g^{\mu\nu}x_\nu$. Evenso for x_μ, $x_\mu = g_{\mu\nu}x^\nu$, where $g_{\mu\nu}$ is a Lorentz invariant metric. For raising and lowering the indices in the van der Waerden notation we use a similar Lorentz invariant symbol, namely ϵ_{ab}, the two-dimensional Levi-Cevita symbol, defined in the following way

$$\epsilon_{12} = \epsilon^{21} = -\epsilon^{12} = -\epsilon_{21} = -1.$$

Consequently we raise and lower spinor indices of two dimensional spinors in the following way

$$\Psi_a = \epsilon_{ab}\Psi^b, \qquad \Psi^b = \epsilon^{ba}\Psi_a,$$
$$\Psi^{\dot{a}} = \epsilon^{\dot{a}\dot{b}}\Psi_{\dot{b}}, \qquad \Psi_{\dot{b}} = \epsilon_{\dot{b}\dot{a}}\Psi^{\dot{a}}.$$

We can also define the charge conjugate more precise by writing down explicitly the spinor index structure. Write

$$\Psi^C = C\overline{\Psi}^\top = C(\Psi^\dagger\tau)^\top,$$

here $\tau = \begin{pmatrix} 0 & \delta^{\dot{a}}_{\dot{c}} \\ \delta^c_a & 0 \end{pmatrix}$, where we have substituted τ for γ^0 since τ does have a correct spinor index structure.

The charge conjugation matrix can be written explicitly for four dimensional spinors as

$$C := \begin{pmatrix} \epsilon_{ac} & 0 \\ 0 & \epsilon^{\dot{a}\dot{c}} \end{pmatrix}.$$

One can check with this explicit form that the charge conjugation matrix satisfies certain properties such as $C^\top = C^\dagger = C^{-1} = -C$ and the relation between the gamma matrix γ^μ and its transpose. Let us now finally see what happens when we take the charge conjugate of both the Majorana as Dirac spinors.

$$
\begin{aligned}
\boldsymbol{\Psi_{Maj}}^{C} &= C(\boldsymbol{\Psi_{Maj}}^{\dagger}\tau)^{\top}, \\
&= \begin{pmatrix} \epsilon_{ac} & 0 \\ 0 & \epsilon^{\dot{a}\dot{c}} \end{pmatrix} \left\{ \begin{pmatrix} \Psi_a \\ \Psi^{\dagger\dot{a}} \end{pmatrix}^{\dagger} \begin{pmatrix} 0 & \delta^{\dot{a}}_{\dot{c}} \\ \delta^{c}_{a} & 0 \end{pmatrix} \right\}^{\top}, \\
&= \begin{pmatrix} \epsilon_{ac} & 0 \\ 0 & \epsilon^{\dot{a}\dot{c}} \end{pmatrix} \left\{ \begin{pmatrix} \Psi^{\dagger}_{\dot{a}} & \Psi^{a} \end{pmatrix} \begin{pmatrix} 0 & \delta^{\dot{a}}_{\dot{c}} \\ \delta^{c}_{a} & 0 \end{pmatrix} \right\}^{\top}, \\
&= \begin{pmatrix} \epsilon_{ac} & 0 \\ 0 & \epsilon^{\dot{a}\dot{c}} \end{pmatrix} \begin{pmatrix} \Psi^{c} & \Psi^{\dagger}_{\dot{c}} \end{pmatrix}^{\top}, \\
&= \begin{pmatrix} \epsilon_{ac} & 0 \\ 0 & \epsilon^{\dot{a}\dot{c}} \end{pmatrix} \begin{pmatrix} \Psi^{c} \\ \Psi^{\dagger}_{\dot{c}} \end{pmatrix}, \\
&= \begin{pmatrix} \Psi_a \\ \Psi^{\dagger\dot{a}} \end{pmatrix}, \\
&= \boldsymbol{\Psi_{Maj}}.
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{\Psi_{Dir}}^{C} &= C(\boldsymbol{\Psi_{Dir}}^{\dagger}\tau)^{\top} \\
&= \begin{pmatrix} \epsilon_{ac} & 0 \\ 0 & \epsilon^{\dot{a}\dot{c}} \end{pmatrix} \left\{ \begin{pmatrix} \chi_a \\ \xi^{\dagger\dot{a}} \end{pmatrix}^{\dagger} \begin{pmatrix} 0 & \delta^{\dot{a}}_{\dot{c}} \\ \delta^{c}_{a} & 0 \end{pmatrix} \right\}^{\top} {}^{5} \\
&= \begin{pmatrix} \epsilon_{ac} & 0 \\ 0 & \epsilon^{\dot{a}\dot{c}} \end{pmatrix} \left\{ \begin{pmatrix} \chi^{\dagger}_{\dot{a}} & \xi^{a} \end{pmatrix} \begin{pmatrix} 0 & \delta^{\dot{a}}_{\dot{c}} \\ \delta^{c}_{a} & 0 \end{pmatrix} \right\}^{\top} \\
&= \begin{pmatrix} \epsilon_{ac} & 0 \\ 0 & \epsilon^{\dot{a}\dot{c}} \end{pmatrix} \begin{pmatrix} \xi^{c} & \chi^{\dagger}_{\dot{c}} \end{pmatrix}^{\top} \\
&= \begin{pmatrix} \epsilon_{ac} & 0 \\ 0 & \epsilon^{\dot{a}\dot{c}} \end{pmatrix} \begin{pmatrix} \xi^{c} \\ \chi^{\dagger}_{\dot{c}} \end{pmatrix} \\
&= \begin{pmatrix} \xi_a \\ \chi^{\dagger\dot{a}} \end{pmatrix} \\
&\neq \boldsymbol{\Psi_{Dir}}.
\end{aligned}
$$

As above calculation shows, the key feature of the Majorana spinor is that it is equal to its own charge conjugate. In contrast to the Dirac spinor, where the left- and right-handed fields switch roles.

2.2 Canonical quantization

After quantization of both spinor fields, the Dirac spinor gives rise to electrons and positrons, whereas the Majorana spinor gives rise to only one particle; the Majorana fermion.[6] Very briefly this follows for the *Dirac spinor* from inserting a test solution

$$\Psi(x) = u(\mathbf{p})e^{ipx} + v(\mathbf{p})e^{-ipx}$$

into the Dirac equation (1.22), resulting in the following solution;

$$\boldsymbol{\Psi_{Dirac}}(x) = \sum_{s=\pm} \int \frac{d^3p}{(2\pi)^3 2\omega} [b_s(\mathbf{p})u_s(\mathbf{p})e^{ipx} + d^{\dagger}_s(\mathbf{p})v_s(\mathbf{p})e^{-ipx}].$$

Here the $b_s(\mathbf{p})$ and $d^{\dagger}_s(\mathbf{p})$ can be interpreted as the annihilation and creation operators respectively, which appear just as integration coefficients from solving the Dirac equation with the test equation above. Their hermitian conjugates $b^{\dagger}_s(\mathbf{p})$ and $d_s(\mathbf{p})$ are also creation and annihilation operators respectively. The action of the creation

[6]this will be a very concise description of the quantization, for more information on quantization see [30].

annihiliation operators on the vacuum state $|0\rangle$ can be summarized as follows

$$
\begin{aligned}
b_s(\mathbf{p})\,|0\rangle &= 0, & d_s(\mathbf{p})\,|0\rangle &= 0,\\
b_s^\dagger(\mathbf{p})\,|0\rangle &= |b(\mathbf{p})\rangle\,, & d_s^\dagger(\mathbf{p})\,|0\rangle &= |d(\mathbf{p})\rangle\,.
\end{aligned}
$$

where for example denotes $|d(\mathbf{p})\rangle$ a "d"-particle with momentum $\mathbf{p}$.
We can now incorporate quantum mechanics in our theory by "quantizing" our spinor fields. That is imposing the following quantum mechanical anticommutation relations

$$
\begin{aligned}
\{\boldsymbol{\Psi}_{\boldsymbol{Dir,\alpha}}(\mathbf{x},t), \boldsymbol{\Psi}_{\boldsymbol{Dir,\beta}}(\mathbf{y},t)\} &= 0,\\
\left\{\boldsymbol{\Psi}_{\boldsymbol{Dir,\alpha}}(\mathbf{x},t), \overline{\boldsymbol{\Psi}_{\boldsymbol{Dir,\beta}}}(\mathbf{x},t)\right\} &= (\gamma^0)_{\alpha\beta}\delta^3(\mathbf{x}-\mathbf{y}).
\end{aligned}
$$

This results, by (omitted) explicit calculation[7], in the following anticommutation relations for the Dirac creation and annihilation operators

$$
\begin{aligned}
\{b_s(\mathbf{p}), b_{s'}(\mathbf{p}')\} &= 0,\\
\{d_s(\mathbf{p}), d_{s'}(\mathbf{p}')\} &= 0,\\
\left\{b_s(\mathbf{p}), d_{s'}^\dagger(\mathbf{p}')\right\} &= 0,
\end{aligned}
$$

By hermitian conjugating these expressions

$$
\begin{aligned}
\left\{b_s^\dagger(\mathbf{p}), b_{s'}^\dagger(\mathbf{p}')\right\} &= 0,\\
\left\{d_s^\dagger(\mathbf{p}), d_{s'}^\dagger(\mathbf{p}')\right\} &= 0,\\
\left\{b_s^\dagger(\mathbf{p}), d_{s'}(\mathbf{p}')\right\} &= 0,
\end{aligned}
$$

Explicit calculation gives us

$$
\{b_s(\mathbf{p}), d_{s'}(\mathbf{p}')\} = 0,
$$

But also the non vanishing relations

$$
\begin{aligned}
\left\{b_s(\mathbf{p}), b_{s'}^\dagger(\mathbf{p}')\right\} &= (2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}')2\omega\delta_{ss'},\\
\left\{d_s^\dagger(\mathbf{p}), d_{s'}(\mathbf{p}')\right\} &= (2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}')2\omega\delta_{ss'}.
\end{aligned}
$$

For the Majorana spinor the situation is different. We do arrive in a similar manner as in the case of the Dirac spinor at the solution (2.2). However, we now have to impose the Majorana reality condition (${\boldsymbol{\Psi}_{\boldsymbol{Maj}}}^C = \boldsymbol{\Psi}_{\boldsymbol{Maj}}$) on the solution. Using this condition gives us $d_s(\mathbf{p}) = b_s(\mathbf{p})$. Inserting this in (2.2) gives us the Majorana field as

$$
\boldsymbol{\Psi}_{\boldsymbol{Maj}}(x) = \sum_{s=\pm}\int \frac{d^3p}{(2\pi)^3 2\omega}[b_s(\mathbf{p})u_s(\mathbf{p})e^{ipx} + b_s^\dagger(\mathbf{p})v_s(\mathbf{p})e^{-ipx}].
$$

[7]See again [30].

If we now again apply quantum mechanical anticommutation relations,

$$\begin{aligned}\{\boldsymbol{\Psi_{\alpha,Maj}}(\mathbf{x},t),\boldsymbol{\Psi_{\beta,Maj}}(\mathbf{y},t)\} &= (C\gamma^0)_{\alpha\beta}\delta^3(\mathbf{x}-\mathbf{y}),\\ \left\{\boldsymbol{\Psi_{\alpha,Maj}},\overline{\boldsymbol{\Psi_{\beta,Maj}}}(\mathbf{y},t)\right\} &= (\gamma^0)_{\alpha\beta}\delta^3(\mathbf{x}-\mathbf{y}),\end{aligned}$$

we find the following anticommutators for the creation and annihilation operators.

$$\begin{aligned}\{b_s(\mathbf{p}),b_{s'}(\mathbf{p}')\} =&0,\\ \left\{b_s(\mathbf{p}),b^\dagger_{s'}(\mathbf{p}')\right\} =&(2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}')2\omega\delta_{ss'}.\end{aligned}$$

Chapter 3

Mathematical definition of the Clifford algebra

3.1 Preliminaries

Since this thesis deals with Clifford *algebras*, let us start with the mathematical definition of an algebra over a field $\mathcal{K}$.

Definition 3.1.1 Algebra ($\mathcal{A}$) over a field $\mathcal{K}$
An algebra $\mathcal{A}$ over a field $\mathcal{K}$ (for example $\mathbb{C}$ or $\mathbb{R}$) is a vector space $\mathcal{V}$ over $\mathcal{K}$, together with a binary operation $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$, called multiplication. Let $(\mathbf{a}, \mathbf{b}) \in \mathcal{A} \times \mathcal{A}$ be mapped to $\mathbf{ab} \in \mathcal{A}$. The binary operation must satisfy three properties.

1. *Left distributivity;* $(\alpha\mathbf{a} + \beta\mathbf{b})\mathbf{c} = \alpha\mathbf{ac} + \beta\mathbf{bc}$,

2. *Right distributivity;* $\mathbf{a}(\beta\mathbf{b} + \gamma\mathbf{c}) = \beta\mathbf{ab} = \gamma\mathbf{ac}$,

3. *Scalar compatibility,* $(\lambda\mathbf{a})\mathbf{b} = \mathbf{a}\lambda\mathbf{b} = \lambda(\mathbf{ab})$.

Here the multiplication is simply represented by the juxtaposition **ab**. As explained in [32], Clifford[1] introduced his "geometric algebra" (a.k.a. Clifford algebra) in 1878. The Clifford algebra arose from two earlier constructed algebraic structures, Hamilton's[2] quaternion ring and Grassman's[3] exterior algebra. For didactical purposes we will not follow Clifford's lines of thought. Throughout this whole section we will assume that the so-called characteristic of the field is not equal to 2. The characteristic

[1] William Kingdon Clifford (1845-1879)
[2] Sir William Rowan Hamilton (1805-1865)
[3] Hermann Günter Grassmann (1809-1877)

is the smallest number p such that $\underbrace{1+1...+1}_{\text{p times}} = 0$. It is in other words the smallest generator of the kernel of the map κ, where κ is defined as $\kappa : \mathbb{Z} \to \mathcal{K}_0 \subset \mathcal{K}$, with

$$\begin{aligned}
\kappa(n) &= \underbrace{1+1...+1}_{\text{n times}} \in \mathcal{K}_0 \\
\kappa(0) &= 0 \in \mathcal{K}_0 \\
\kappa(-n) &= -\underbrace{1+1...+1}_{\text{n times}} \in \mathcal{K}_0.
\end{aligned}$$

Here $\mathcal{K}_0$ is the so called prime subfield, it is defined as

$$\mathcal{K}_0 = \bigcap_{\mathcal{K}' \subset \mathcal{K}} \mathcal{K}',$$

where $\mathcal{K}'$ is a general subfield of $\mathcal{K}$. So the prime subfield is the intersection of all those subfields. A field in which the kernel for κ is trivially 0, is said to have characteristic 0. For example the field of real numbers $\mathbb{R}$ has characteristic 0. The reason for this assumption is that in the case that char $\mathcal{K} = 2$, very fundamental theorems are not applicable. We will start with the most algebraic definition of the Clifford algebra right away[4]. For the purpose of this article the algebraic definition is directly given [4]. Since a so-called *quadratic form* is used in the definition of a Clifford algebra, let's first define the quadratic form.

Definition 3.1.2 Quadratic Form

A quadratic form on a vector space $\mathcal{V}$ over a field $\mathcal{K}$ is a map q: $\mathcal{V} \to \mathcal{K}$, such that

1. $q(\alpha v) = \alpha^2 q(v)$, $\forall \alpha \in \mathcal{K},\ v \in \mathcal{V}$.
2. *the map $(v, w) \mapsto q(v + w) - q(v) - q(w)$ is linear in both v and w.*

We can now define a corresponding bilinear form to this map, the so-called *polarization*. Here we need the *char* $\mathcal{K} \neq 2$-assumption, this is to ensure that the quadratic form is induced by a symmetric bilinear form. We always have a symmetric bilinear form ($\beta = \beta_q$) associated to a quadratic form q. This is realized in the following way

$$\beta_q(v, w) := \frac{1}{2}(q(v + w) - q(v) - q(w)).$$

Or the other way around:

$$q(\mathbf{x}) = \beta(\mathbf{x}, \mathbf{x}).$$

[4] Many texts provide introductory explanations in two or three spatial dimensions, see for example [19]

To see why a quadratic form is induced by a symmetric bilinear form, look at the so called polarization identity.

$$\beta(x+y, x+y) - \beta(x,x) - \beta(y,y) = \beta(x,y) + \beta(y,x),$$

If β is symmetric, the right hand side reduces to $2\beta(x,y)$, resulting in

$$\beta(x+y, x+y) - \beta(x,x) - \beta(y,y) = q(x+y) - q(x) - q(y) = 2\beta(x,y).$$

Here we see clearly why we need the *char* $\mathcal{K} \neq 2$-assumption for a good definition of the symmetric bilinear β in terms of the quadratic form q. If the characteristic of the field would be 2, the right hand side would vanish and a definition of the quadratic form in terms of a symmetric bilinear form is not possible. However the quadratic form can then be defined in terms of a non symmetric bilinear form [5].

Intermezzo II: *char*$\mathcal{K}$ = **2-case**

To give a more concrete idea of the *char*$\mathcal{K} \neq 2$-condition on the field, for a proper definition of the polarization, let us see what happens when *char*$\mathcal{K} = 2$. For example, take the 2-dimensional finite field $\mathbb{F}_2^2$ over $\mathbb{F}_2 = \{0,1\}$. Now take the quadratic form $q : \mathbb{F}_2^2 \to \mathbb{F}_2,\ \mathbf{x} = (x_1, x_2) \to x_1 x_2$. Then there is no symmetric bilinear form β, such that $Q(\mathbf{x}) = \beta(\mathbf{x}, \mathbf{x})$.
Since let there be a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then we would have
$\beta(\mathbf{x}, \mathbf{x}) = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = a{x_1}^2 + (b+c)x_1x_2 + d{x_2}^2 = x_1x_2$. Concluding from this, $a = 0$, $d = 0$ and $b + c = 1$. Since b and c lie in $\mathbb{F}_2$, this gives us two possibilities. $(b,c) = (0,1)$ or $(b,c) = (1,0)$. But this obviously means that our bilinear form β is not symmetric.

So from now on we assume that this bilinear form is symmetric, i.e. $\beta_q = (v,w) = \beta_q(w,v)\ \forall v, w \in \mathcal{V}$. Another concept that we need is the tensor and the corresponding tensor algebra.

Definition 3.1.3 Mixed tensor of type (r, s)
Let $\mathcal{V}$ be a vector space with dual space $\mathcal{V}^*$. Then a tensor of type (r,s) is a multilinear mapping

$$\mathbf{T_s^r} : \underbrace{V^* \times V^* ... \times V^*}_{r\ times} \times \underbrace{V \times V ... \times V}_{s\ times} \to \mathbb{R}. \tag{3.1}$$

The set of all tensors with fixed dimensions $(r,\ s)$ is a vector space, denoted by T_s^r. If we now define the following space $\bigoplus_{(r,s)} T_s^r$ and equip this vector space with an additional product which maps two elements in this vector space again onto the vector space we obtain an algebra. This additional product is the tensor product and the corresponding algebra is the so called tensor algebra $\mathcal{T}$.

Definition 3.1.4 Tensor product
Let $\mathbf{T^r_s}$ *be an (r, s)-tensor and* $\mathbf{U^k_l}$ *be an (k, l)-tensor, then their product is* $\mathbf{T^r_s} \otimes \mathbf{U^k_l}$ *which is a* $(r+k, s+l)$*-tensor, which operates on* $(V^*)^{r+k} \times V^{s+l}$*, defined by*

$$\mathbf{T} \otimes \mathbf{U}(\theta^{\mathbf{1}}, ..., \theta^{\mathbf{r+k}}, \mathbf{u_1}, ..., \mathbf{u_{s+l}}) = \mathbf{U}(\theta^{\mathbf{1}}, ..., \theta^{\mathbf{r}}, \mathbf{u_1}, ..., \mathbf{u_s})\mathbf{T}(\theta^{\mathbf{r+1}}, ..., \theta^{\mathbf{r+k}}, \mathbf{u_{s+1}}, ..., \mathbf{u_{s+l}})$$

Definition 3.1.5 Tensor Algebra $\mathcal{T}$
A tensor algebra $\mathcal{T}$ *over a field* $\mathcal{K}$ *is the vector space* $T = \bigoplus_{(r,s)} T^r_s$ *endowed with the tensor product, which serves as the multiplication* $\mathcal{T} \times \mathcal{T} \to \mathcal{T}$*.*

Lastly one should know the ideal $\mathcal{I}_q = (v \otimes v - q(v)\mathbb{1})$ with $v \in \mathcal{V}$. This is the ideal in the tensor algebra generated by the set

$$\{(v \otimes v) - q(v)\mathbb{1}\},$$

where $v \in T$. We are now ready to define the Clifford algebra.

Definition 3.1.6 Clifford Algebra $\mathcal{C}\ell(\mathcal{V}, q)$ over a field $\mathcal{K}$
A Clifford algebra $\mathcal{C}\ell$ *over a field* $\mathcal{K}$ *is a vector space* $\mathcal{V}$ *over the field* $\mathcal{K}$ *endowed with a quadratic form* q*, defined by*

$$\mathcal{C}\ell(\mathcal{V}) := \mathcal{T}(\mathcal{V})/\mathcal{I}_q(\mathcal{V}).$$

The so-called Clifford product *serves as the multiplication* $\mathcal{C}\ell(\mathcal{V}) \times \mathcal{C}\ell(\mathcal{V})$*, defined as* $(\overline{A}, \overline{B}) \mapsto \overline{AB} := \overline{A \otimes B} = A \otimes B + \mathcal{I}_q$*.*

Hence the Clifford algebra is a quotient algebra. Due to the division by the ideal$(v \otimes v - q(v)\mathbb{1})$ every square of an element in $\mathcal{V}$ will be an element of the field $\mathcal{K}$, namely $q(v)$. The Clifford product is now the tensor product in $\mathcal{T}(\mathcal{V})/\mathcal{I}_q(\mathcal{V})$. The associativity and linearity of the Clifford product is inherited from the tensor product. As noted above, every squared element out of $\mathcal{V}$ will be a scalar, by

$$\overline{v}^2 = \overline{v \otimes v} = q(v).$$ [5]

[5] The $\mathbb{1}$ here is to make the element $q(v)$ an element of the tensor algebra, since q itself maps to the field $\mathcal{K}$.

We can also recover the already found expression for the Clifford algebra, by evaluating $q(v+w)$.

$$q(v+w) = (v+w)^2 = v^2 + vw + wv + w^2 = {}^6 q(v) + vw + wv + q(w).$$

By now using the expression of β_q we find

$$vw + wv = 2\beta_q(v,w). \tag{3.2}$$

By choosing the Minkowski metric $\beta_q(v,w) = \eta(v,w)$, we find back our equation (1.24). Another nice thing to see is that if we set $\beta_q(v,w) = 0$, we obtain the Grassmann algebra which inspired Clifford.

3.2 Properties of the Clifford algebra

There is a huge amount of properties and theory of Clifford algebras that can be found in numerous both mathematical and physical articles and books. This article deals with the properties needed for a better understanding of the Majorana spinors and the most fundamental notions of a Clifford algebra.

A convenient basis for the Clifford algebra is the following basis

Theorem 3.2.1 Basis for a Clifford algebra
Let $\mathbf{e_1}, ..., \mathbf{e_N}$ be a basis for the vector space $\mathcal{V}$, then the vectors

$$\mathbf{1}, \mathbf{e_i},\ \mathbf{e_i e_j}\ (\mathbf{i} < \mathbf{j}),\ \mathbf{e_i e_j e_k}\ (\mathbf{i} < \mathbf{j} < \mathbf{k}),\ \mathbf{e_1 ... e_N}\ (\mathbf{1} < ... < \mathbf{N})$$

form a basis for the Clifford algebra $\mathcal{C}\ell(\mathcal{V}, q)$.

If the vector space $\mathcal{V}$ is N-dimensional, this means that we can choose a basiselement for the Clifford algebra, consisting out of k vector space basis elements, in $\binom{N}{k}$ ways. In total this gives us thus a basis for the Clifford algebra consisting out of

$$\sum_{k=0}^{N} \binom{N}{k} = \sum_{k=0}^{N} \binom{N}{k} 1^k 1^{N-k} = (1+1)^N = 2^N \text{ basis vectors.}$$

[6] We have nowhere assumed commutativity.

Note that a different basis can by antisymmetrization of the previous basis in the following way

$$
\begin{aligned}
\mathbf{1} &\rightarrow \mathbf{1} \\
\mathbf{e_i} &\rightarrow \mathbf{e_i} \\
\mathbf{e_i e_j} &\rightarrow \frac{1}{2}(\mathbf{e_i e_j} - \mathbf{e_j e_i}) \\
&... \\
\mathbf{e_1 e_2 ... e_k} &\rightarrow \frac{1}{k!}(\sum_{\sigma \in S_n} sign(\sigma) \cdot \sigma(\mathbf{e_1 e_2 ... e_k}) := \mathbf{e_{[1} e_2 ... e_{k]}} \\
&... \\
\mathbf{e_1 e_2 ... e_n} &\rightarrow \frac{1}{n!}(\sum_{\sigma \in S_n} sign(\sigma) \cdot \sigma(\mathbf{e_1 e_2 ... e_n})) =: \mathbf{e_{[1} e_2 ... e_{n]}} =: \mathbf{e_*}
\end{aligned}
$$

It would be very useful if we could define an orthogonal basis on the space ($\mathcal{V}$,q) for our Clifford algebra. Let us first define what we exactly mean with an orthogonal basis of a Clifford algebra $\mathcal{C}\ell(\mathcal{V}, q)$.

Definition 3.2.1 Orthogonal basis of a Clifford algebra
A basis $\{\mathbf{e_1}, ..., \mathbf{e_n}\}$ *is said to be* orthogonal *if*

$$q(\mathbf{e_i} + \mathbf{e_j}) = \mathbf{q}(\mathbf{e_i}) + \mathbf{q}(\mathbf{e_j}),\ \forall\ \mathbf{i} \neq \mathbf{j}.$$

If we have in addition have $q(\mathbf{e_i}) \in \{\mathbf{-1}, \mathbf{0}, \mathbf{1}\}$ *the basis is called* orthonormal.

We can guarantee the existence of an orthogonal basis when *char* $\mathcal{K} \neq 2$.

Theorem 3.2.2 Existence of orthogonal basis for (V, q)
If char $\mathcal{K} \neq 2$*, then there exists a orthogonal basis for (V,q).*

Proof 1 *For the proof of this theorem, see for example [21] or [5].*

3.2.1 Graded algebras

Let us start with the definition of a graded algebra.

Definition 3.2.2 Graded algebra
An algebra $\mathcal{A}$ *is said to be* $\mathbb{Z}$*-graded if there is a decomposition of the underlying vector space* $\mathcal{A} = \oplus_{p \in \mathbb{Z}} \mathcal{A}^p$ *such that* $\mathcal{A}^p \mathcal{A}^q \subset \mathcal{A}^{p+q}$.

Examples of $\mathbb{Z}$-graded algebras are the tensor algebra and the exterior algebra. The Clifford algebra is also graded algebra, more specifically it is a $\mathbb{Z}_2$-graded algebra (a superalgebra.) This means that the Clifford algebra decomposes as a direct sum of an *even* and an *odd* part. The terms even and odd can be identified with the number of basis elements of the vector space which are used to construct a basis element of the Clifford algebra, i.e.

$$\begin{aligned} \mathcal{C}\ell &= \mathcal{C}\ell^- \oplus \mathcal{C}\ell^+, \\ &= \Big(\bigoplus_{k\,odd} \mathcal{C}\ell^k\Big) \oplus \Big(\bigoplus_{k\,even} \mathcal{C}\ell^k\Big). \end{aligned}$$

For example $\mathbf{e_1e_2e_3}$ is an odd basisvector of $\mathcal{C}\ell^3$ and $\mathbf{e_1e_2e_3e_4}$ is an even basisvector of $\mathcal{C}\ell^4$. This decomposition is actually defined by the following function, which is the so-called *main involution.*

Definition 3.2.3 Involution
An involution is a function f which is its own inverse, i.e. $f(f(x)) = x$ for x a general element of the domain of f.

The main involution is the linear mapping ω: $\mathbf{v} \rightarrow -\mathbf{v}$ where $v \in \mathcal{V}$, this linear map extends to an algebra isomorphism ω: $\mathcal{C}\ell \rightarrow \mathcal{C}\ell$. Since ω^2 gives the identity, we can conclude that ω has two eigenvalues namely $\{\pm 1\}$. Now, the eigenvectors corresponding to the negative eigenvalue form the subspace $\mathcal{C}\ell^-$, the "positive" eigenvectors form the subspace $\mathcal{C}\ell^+$, which is even a subalgebra. A subalgebra is a subset of an algebra, closed under all operations.

3.2.2 Signature of the Clifford algebra

As we know we can represent a bilinear form as a matrix, $q(\mathbf{v}) = \mathbf{v}^\top A\mathbf{v}$. Since we assume char $\mathcal{K} \neq 2$, we know that the bilinear form is symmetric. Hence, also the matrix is symmetric and therefore the matrix is diagonalizable. We are now ready to define the signature.

Definition 3.2.4 *Signature of a quadratic form*
The signature of a quadratic form is the triple (s,t,u) where s, t and u represent the number of positive, negative and vanishing diagonal entries of the quadratic form in the orthogonal basis respectively.

According to Sylvester's law of inertia[7] the signature of a quadratic form is invariant under basis transformations. Thus we can talk without any obscurity about the signature of a quadratic form. As a second assumption for the rest of this article, after the char $\mathcal{K} \neq 2$-assumption, we from now on assume that the field on which we construct our (real) Clifford algebra is $\mathbb{R}$. We will not treat complex Clifford algebras in this article. Furthermore we will from now on focus on a certain type of vector spaces $\mathcal{V}$ to build our Clifford algebra on, the so-called *pseudo-Euclidean spaces*. This focussing is done by constraining our quadratic form even further

Theorem 3.2.3 Non-degenerate bilinear form
A quadratic form β is called non-degenerate if its kernel *or* radical *is zero. The kernel is defined as*

$$Ker\beta := \{\mathbf{v} \in \mathcal{V} | \beta(\mathbf{v}, \mathbf{w}) = 0 \; \forall \mathbf{w} \in \mathcal{V}\}.$$

As a consequence of this non-degeneracy we know that there are no basis vectors $\mathbf{e}_i \neq \mathbf{0}$ with $q(\mathbf{e}_i) = 0$ anymore. If we now assume our bilinear form is non-degenerate and symmetric, then we know according to a fundamental result of linear algebra that there exists an orthonormal basis. This means according to definition 3.2.1 that we can use this orthonormal basis to write the bilinear form of the vector space $\mathcal{V}$ over the field $\mathbb{R}$ as a diagonal matrix with entries $q(\mathbf{e_i}) = \beta(\mathbf{e_i}, \mathbf{e_i}) \in \{-\mathbf{1}, \mathbf{1}\}$. [8] Consequently, our vector space is $d = t + s$-dimensional, denoted by $\mathbb{R}^{t,s}$,[9] and our quadratic form q obtains the form

$$q(\mathbf{v}) = \beta_q(\mathbf{v}, \mathbf{v}) = {v_1}^2 + ... + {v_t}^2 - {v_{t+1}}^2 + ... - {v_d}^2. \tag{3.3}$$

Here the similarity with the pseudo-Euclidean spacetime interval cannot be missed. Equation (3.3) is just expression for the spacetime interval, invariant under Lorentz transformations, in a Minkowski space with t time dimensions and s space dimensions.

3.3 Dirac algebra

To get really acquainted with the interlocking of the mathematical definition and the physical application of a Clifford algebra, let us study in more detail the Clifford algebra of Chapter 1. If we take theorem 3.2.1 and use the gamma matrices from Chapter 1 as basis elements, then we can find a basis for the Clifford algebra derived

[7] See for example [13]

[8] We start to see appearing the Minkowski metric again, only now in a more general form.

[9] Mathematicians use often the opposite sign convention, but anticipating on the fact that we will apply this theory to Minkowski spacetime this sign convention was chosen

by Dirac, the so-called Dirac algebra. The Dirac algebra consists out of the following basis elements, listed below.

- $\mathbb{1}_{4\times4}$ identity matrix, of course we have the identity matrix $\mathbb{1}_{4\times4}$ in our basis.
- Secondly, we have the four gamma matrices γ^μ, $\mu = 0, 1, 2, 3$.
- Thirdly, we have the products of two gamma matrices. $\gamma^\mu\gamma^\nu$. Due to the anticommutativity and due to , six independent combinations
- The products of three gamma matrices provide us with four, new independent basis elements. One can see this by noting that we can omit four times a gamma matrix in the product.
- Finally, we have only one element which is product of four gamma matrices. This matrix is given a special name $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \frac{i}{4!}\epsilon^{\mu\nu\rho\sigma}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_{sigma}$

Table 3.1 summarizes the basis for the Dirac algebra. We already know from Chapter

Basis element	Number of elements
$\mathbf{1}$	1
γ^μ	4
$\gamma^\mu\gamma^\nu$	6
$\gamma^\mu\gamma^\nu\gamma^\rho$	4
$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$	1

Table 3.1: the Dirac basis of the Clifford algebra

1 that we have the anticommutativity relation (1.24) and so we see that the γ-matrices form an orthonormal basis.

3.4 Clifford algebras in different dimensions

We will now develop our most general view on Clifford algebras and the gamma matrices, adapted from [35][10].

Up until now we have looked at various concepts of quantum field theory in Minkowskian spacetime, i.e. a spacetime with three spacelike dimensions and one timelike dimension, with metric signature (+,-,-,-). This metric signature corresponds to the number of positive (corresponding to timelike dimensions) and negative (corresponding

[10]For a more complete description of gamma matrices in arbitrary dimensions, see chapter 3 from [35] and its references.

to spacelike) dimensions of the spacetime. Familiarized with the concepts of Weyl-, Dirac and Majorana spinors in Minkowski spacetime, we can use our mathematical knowledge of Clifford algebras in various dimensions to study different spacetime dimensions and their admitted spinors.
Let us first look at a generic way to construct our gamma matrices forming a Clifford algebra on a spacetime with signature (t, s), where from now on t denotes the number of temporal dimensions (associated with a plus in the metric η) and s denotes the number of spatial dimensions (associated with a minus). Thus we want our gamma matrices to satisfy

$$\Gamma_a \Gamma_b + \Gamma_b \Gamma_a = 2\eta_{ab} \tag{3.4}$$

The gamma matrices for a spacetime with d time dimensions, which means we have the signature $(t, s) = (d, 0)$, can be constructed using the Kronecker product as follows

$$\begin{aligned} \Gamma_1 &= \sigma_1 \otimes \mathbb{1} \otimes \mathbb{1} \otimes ... \\ \Gamma_2 &= \sigma_2 \otimes \mathbb{1} \otimes \mathbb{1} \otimes ... \\ \Gamma_3 &= \sigma_3 \otimes \sigma_1 \otimes \mathbb{1} \otimes ... \\ \Gamma_4 &= \sigma_3 \otimes \sigma_2 \otimes \mathbb{1} \otimes ... \\ \Gamma_5 &= \sigma_3 \otimes \sigma_3 \otimes \sigma_1 \otimes ... \\ ... &= ... \end{aligned} \tag{3.5}$$

Here $\otimes$ denotes the Kronecker product between matrices.
For even dimensions this is a representation of dimension $2^{\frac{d}{2}}$. For odd dimensions one simply ends with a tensor product of σ_3's. The dimension of the representation will be $2^{\frac{d-1}{2}}$. So for example when $d = 5$ the last σ_1 is not needed in Γ_5. We thus have 5 4×4 Γ-matrices[11]. We know that the σ-matrices are Hermitian and that $(\sigma_i)^2 = \mathbb{1}$. We can combine this with the following properties of the Kronecker tensor product concerning hermicity and multiplication [31].

$$(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger$$

and

$$(A \otimes B)(C \otimes D) = AC \otimes BD.$$

From above we see that in this way all of our Γ-matrices are Hermitian and that they square to one. To incorporate s spacelike dimensions in this construction, we

[11] $2^{\frac{5-1}{2}} = 4$

simply multiply the first s matrices with i, the complex unit. This gives us after the multiplication by i

$$\Gamma_s^\dagger = -\Gamma_s, \qquad\qquad \Gamma_t^\dagger = \Gamma_t.$$

In general the following relation can be derived

$$\Gamma_a^\dagger = (-)^t A \Gamma_a A^{-1}, \quad A = \Gamma_1 ... \Gamma_s. \tag{3.6}$$

Since we have

$$\Gamma_s{}^2 = -1 \quad \text{and} \qquad\qquad \Gamma_s{}^{-1}\Gamma_s = 1 \rightarrow \Gamma_s^{-1} = -\Gamma_s = \Gamma_s^\dagger.$$

Now assume that a corresponds to a timelike dimension. Then we have, due to the Clifford algebra (3.4),

$$\Gamma_a \Gamma_i = \Gamma_i \Gamma_a, \ \ \forall i \in [1, .., s].$$

Thus,

$$\begin{aligned}
(-1)^s A \Gamma_a A^{-1} &= (-1)^s \left(\Gamma_1 ... \Gamma_s\right) \Gamma_a \left(\Gamma_1 ... \Gamma_s\right)^{-1} \\
&= (-1)^s (-1)^s \Gamma_1 ... \Gamma_s \Gamma_a \Gamma_s ... \Gamma_1 \\
&= (-1)^s (-1)^s (-1)^s \Gamma_a \Gamma_1 ... \Gamma_s \Gamma_s ... \Gamma_1 \\
&= (-1)^s (-1)^s (-1)^s (-1)^s \Gamma_a \\
&= \Gamma_a \\
&= \Gamma_a^\dagger
\end{aligned}$$

If a corresponds to a spacelike dimension, we would get

$$\Gamma_a \Gamma_i = \Gamma_i \Gamma_a, \forall i \neq a.$$

However we would once have $i = a$, so that

$$\Gamma_a \Gamma_i = \Gamma_i \Gamma_a = \left(\Gamma_a\right)^2 = -\mathbb{1}_{4\times 4}.$$

So starting in the same way,

$$\begin{aligned}
(-1)^s A \Gamma_a A^{-1} &= (-)^s \left(\Gamma_1 ... \Gamma_s\right) \Gamma_a \left(\Gamma_1 ... \Gamma_s\right)^{-1}, \\
&= (-1)^s (-1)^s \Gamma_1 ... \Gamma_s \Gamma_a \Gamma_s ... \Gamma_1.
\end{aligned}$$

Here we now try to swap the Γ_a-matrix to the right, however we will meet the same matrix at some point resulting in ${\Gamma_a}^2 = -\mathbb{1}$ somewhere. So continuing,

$$
\begin{aligned}
&= (-1)^s(-1)^s\Gamma_1...\Gamma_a...\Gamma_s\Gamma_a\Gamma_s...\Gamma_a...\Gamma_1 \\
&= (-1)^s(-1)^s(-1)^{s-a}\Gamma_1...\Gamma_a\Gamma_a...\Gamma_s\Gamma_s...\Gamma_a...\Gamma_1 \\
&= (-1)^s(-1)^s(-1)^{s-a}(-1)\Gamma_1...\mathbb{1}...\Gamma_s\Gamma_s...\Gamma_a...\Gamma_1 \\
&= (-1)^s(-1)^s(-1)^{s-a}(-1)(-1)^{s-a}\Gamma_1...\mathbb{1}\Gamma_a...\Gamma_1 \\
&= (-1)^s(-1)^s(-1)^{s-a}(-1)(-1)^{s-a}\Gamma_1...\Gamma_{a-1}\Gamma_a\Gamma_{a-1}...\Gamma_1 \\
&= (-1)^s(-1)^s(-1)^{s-a}(-1)(-1)^{s-a}(-1)^{a-1}\Gamma_1...\Gamma_{a-1}\Gamma_{a-1}...\Gamma_1\Gamma_a \\
&= (-1)^s(-1)^s(-1)^{s-a}(-1)(-1)^{s-a}(-1)^{a-1}(-1)^{a-1}\Gamma_a \\
&= (-1)\Gamma_a \\
&= {\Gamma_a}^\dagger.
\end{aligned}
$$

One preserves the Clifford algebraic property of the Γ-matrices by applying the following transformation

$$\Gamma' = U^{-1}\Gamma U$$

To still satisfy (3.6) U must be a unitary matrix. As we have seen earlier in chapter 3, we can construct an antisymmetrized matrix. We can distinguish symmetric and antisymmetric tensor parts of a tensor [12].

Symmetric

A tensor is called symmetric if $\tau_\sigma T = T$. τ_σ is here a permutation operator working on T applying a permutation σ. We can apply a so-called symmetrizer on T to get the symmetric part of T.

$$Sym\ T = \frac{1}{r!}\sum_{\sigma\in S_r}\tau_\sigma T$$

Here S_r denotes the permutation group of r elements with $r \in [1, ..., d]$.

Antisymmetric

A tensor is antisymmetric if $\tau_\sigma T = \text{sign}(\sigma)T$ We can also apply an antisymmetrizer

$$Antisym\ T = \frac{1}{r!}\sum_{\sigma\in S_r} sign(\sigma)\tau_\sigma T$$

Here S_r denotes the permutation group of r elements with $r \in [1, ..., d]$.
The complete Clifford algebra consists out of the $\mathbb{1}$-identity matrix, the d generating elements Γ_μ, plus all the independent products of the matrices. It turns out that due

to (3.4) symmetric products reduce to products containing fewer Γ-matrices, thus the new elements must be antisymmetric products [11].
In *even dimensions* a complete set of new Γ-matrices can be formed $\left\{\Gamma^{(n)}\right\}$ of dimensions $2^{\frac{d}{2}} \times 2^{\frac{d}{2}}$ with $n = 1, ..., d$ and

$$\mathbb{1} \text{ and } \Gamma^{(n)} = \Gamma_{a_1 ... a_n} := \Gamma_{[a_1}\Gamma_{a_2}...\Gamma_{a_n]}. \tag{3.7}$$

This is a set of 2^d matrices.

Basis for even dimensions

$$\left\{\mathbb{1}, \Gamma_{a_1}, ..., \Gamma_{[a_1}\Gamma_{a_2]}, ..., \Gamma_{[a_1}\Gamma_{a_2}...\Gamma_{a_d]}\right\}$$

The last matrix $\Gamma^{(d)}$ of this collection becomes a special name, Γ_*.

$$\Gamma_* = (-i)^{\frac{d}{2}+s}\Gamma_1...\Gamma_d, \qquad \Gamma_*\Gamma_* = 1 \tag{3.8}$$

where the normalization. Due to the $(-i)^s$-factor, Γ_* is independent of the spacetime signature; the i-factors from the time-matrices are cancelled. In the representation (3.5), Γ_* is given as

$$\Gamma_* = \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_3 ...$$

In odd dimensions the set of initial Γ-matrices can be extended by Γ_* since we see from (3.8) that Γ_* commutes with all Γ_a and thus we can use it as the next Γ-matrix in the odd dimension.

Basis for odd dimensions

$$\left\{\mathbb{1}, \Gamma_{a_1}, ..., \Gamma_{[a_1}\Gamma_{a_2]}, ..., \Gamma_{[a_1}\Gamma_{a_2}...\Gamma_{a_d]}, \pm\Gamma_*\right\}$$

So we find two sets of matrices which satisfy (3.4) and hence can both serve as a basis. The groups are not equivalent and lead to different representations of the Lorentz group.
Now that we have constructed a general basis for the Clifford algebra. Let us turn to the different kinds of spinors. To classify certain categories of irreducible spinors, one can work with two projections. The two most important concepts we need are chirality and charge conjugation.
The realization of the concept of chirality in arbitrary spacetime dimensions is actually

just the generalization $\gamma^5 \to \Gamma_*$. This leads to the following definition of left- and right-handed spinors

$$\lambda_L = \frac{1}{2}\left(1+\Gamma_*\right)\lambda, \qquad\qquad \lambda_R = \frac{1}{2}\left(1-\Gamma_*\right)\lambda.$$

The existence of a charge conjugation $\mathcal{C}$ matrix is proven in for example [17] or [28]. This matrix satisfies

$$\mathcal{C}^\top = -\epsilon\mathcal{C}, \qquad\qquad \Gamma_a^\top = -\eta\mathcal{C}\Gamma_a\mathcal{C}^{-1}. \tag{3.9}$$

Here η and ϵ can have the values ± 1, depending on the spacetime dimensions. In even dimensions ($d = 2k$) η can both be $+1$ or -1.

For even dimensions we have the chirality projection, in which we generalize the notion of chiralty developed in chapter 2. The generalization is done by substituting $\gamma^5 \to \Gamma_*$. Using this projection we can define right- and left-handed parts of spinors in the following way

$$\lambda_L = \frac{1}{2}\left(1+\Gamma_*\right)\lambda, \quad \lambda_R = \frac{1}{2}\left(1-\Gamma_*\right)\lambda. \tag{3.10}$$

Another projection is the *reality condition*. To construct this condition, we combine (3.6) and (3.9) to get

$$\begin{aligned}
{\Gamma_a}^* &= \left({\Gamma_a}^\dagger\right)^\top \\
&= (-1)^t \left(A^{-1}\right)^\top {\Gamma_a}^\top A^\top \\
&= (-1)^t(-\eta)\underbrace{\left(A^{-1}\right)^\top C}_{B}\,\Gamma_a\,\underbrace{C^{-1}A^\top}_{B^{-1}} \\
&= (-1)^t(-\eta)B\Gamma_a B^{-1}
\end{aligned}$$

Concluding,

$$B^\top = C^\top A^{-1}$$

Since A and C are unitary, B is also unitary. Now take a spinor λ and start with the following ansatz for a reality condition

$$\lambda^* = \tilde{B}\lambda.$$

Here $\tilde{B}$ is a yet undetermined matrix. Of course we want to guarantee consistency with Lorentz transformation. So taken the Lorentz transformation on both sides of (3.4) gives us

$$\begin{aligned}
\left(-\frac{1}{4}\Gamma_{ab}\lambda\right)^* &= -\frac{1}{4}\tilde{B}\Gamma_{ab}\lambda \\
(-1)^t(-\eta)B\Gamma_{ab}B^{-1}\tilde{B} &= \tilde{B}\Gamma_{ab}
\end{aligned}$$

From which we see that for example

$$\tilde{B} = \alpha B,$$

satisfies this relation. Furthermore we know from $\lambda^{**} = \lambda$, that $\tilde{B}^* \tilde{B} = 1$. Combining this with (3.4) gives us $|\alpha| = 1$. It turns out that this reality condition is satisfied for certain values of $s - t$ mod 8, see tabel 3.2. For these spacetime dimensions, one can impose the reality condition of (3.4). Spinors that satisfy these conditions are called Majorana spinors. A less known spinor is the *symplectic Majorana spinor*. These spinors satisfy a slightly different reality condition, namely

$$\lambda_i{}^* := \left(\lambda^i\right)^* = B\Omega_{ij}\lambda^j.$$

Here Ω is an antisymmetric matrix that satisfies $\Omega\Omega^* = -1$. Finally, one could also study the possibility of imposing a reality condition that respects chirality. In other words can there be *(symplectic) Majorana Weyl spinors*? It turns out that in certain spacetime dimensions this is also possible. For an overview of the possible irreducible spinors in different spacetime dimensions, see the table below.

$d\downarrow/t\rightarrow$	0	1	2	3
1	M	M		
2	M^-	MW	M^+	
3		M	M	
4	SMW	M^+	MW	M^-
5			MW	M^-
6	M^+	SMW	M^-	MW
7	M			M
8	MW	M^-	SMW	M^+
9	M	M		
10	M^-	MW	M^+	SMW
11		M	M	
12	SMW	M^+	MW	M^-

Table 3.2: Possible spinors in different spacetime dimensions. M indicates a Majorana spinor. The superscript indicates which sign of η should be used ± 1. MW stands for a Majorana-Weyl spinor. For even dimensions ($d = 2k$) one can always have Weyl spinors. Also one always can have symplectic Majorana spinors, when there are no Majorana spinors. So both those spinors have been omitted from the table.

Chapter 4

Clifford algebras and "condensed matter Majorana's"

In this chapter we will finally focus on the possible realization of "Majorana fermions" or a generalization of them as quasiparticles in condensed matter physics. There are many proposed realizations of Majorana quasiparticles in condensed matter physics [10]. Only experiments which use one-dimensional semiconductor nanowires as an environment for their quasiparticles have provided real evidence for realizing the Majorana quasiparticle. The first experiment succesfully using this approach was executed by the research group of Leo Kouwenhoven at the Kavli Institute in Delft [26]. This is also the reason why we will primarily focus on this research. In their experiment the nanowire approach was combined with the presence of superconductivity to show the existence of this quasiparticle.

Before we can understand what the research group from Delft did, let us first familiarize ourselves with the concept of a "Majorana fermion" in condensed matter physics. Of course the most important and actually the only fermionic particles in solid state physics are the electrons. Since electrons are electrically charged, they cannot be Majorana fermions. Instead, they are Dirac fermions with ${\Psi_{Dir}}^C \neq \Psi_{Dir}$. The story could end here with the conclusion that no Majorana fermions exist in solids. But actually it is very interesting to study whether certain *Majorana-like* objects, quasiparticles or modes, can be realized. Here *quasiparticles* stands for a collective excitation of the quantum many-body system made of interacting electrons in the solid. In other words a specific collection of electrons and nuclei of the solid that move in a certain organized way is a quasiparticle. Hence quasiparticles only exist in solids, contrary to ordinary particles.[1] The "Majorana" adjective here indicates that we are talking about a quasiparticle which is identical to its antiquasiparticle. A, in

[1]For a more elaborate description on quasiparticles, see Kittel.

this case, relevant example of a quasiparticle is the *exciton*, which is a combination of electronic bound states. This electronic bound states are made of states of electrons and "holes". These holes can be thought of as the absence of an electron in a mode, which is occupied in the ground state. These holes act like they are electrons with opposite charge in the solid, i.e. they are the electron's antiparticles.
Let us now construct a set of creation and annihilation operators describing such a quantum many-body state in a solid. The creation operator for an electron state with quantum number j, where j collectively indicates the spatial degree of freedom as well as the spin or other quantum numbers, is $c_j^\dagger$. At the same time $c_j^\dagger$ is the annihilation operator for a hole with quantum number j. Since we are dealing with spin-$\frac{1}{2}$ fermions, we should use Fermi-Dirac statistics

$$\begin{aligned}\{c_j, c_k\} = \left\{c_j^\dagger, c_k^\dagger\right\} = 0,\\ \left\{c_j^\dagger, c_k\right\} = \delta_{jk}.\end{aligned}$$

In this system, charge conjugation can be realized by swapping $c_j \leftrightarrow c_j^\dagger$. So did we cross a "Majorana fermion" yet? After the exclusion of the electron, also the exciton is not a Majorana particle. Since it is a superposition of electron and hole states, they are created by operators of the general form $c_j^\dagger c_k + c_j c_k^\dagger$. Indeed under charge conjugation $c_j^\dagger \leftrightarrow c_j^\dagger$ this operator stays the same. However conventional excitons are bosons, meaning that they have an integer spin quantum number , thus they are not "Majorana fermions".

4.1 Superconductivity's solution

If we, after the exclusion of the electron and the exciton as a Majorana (quasi)fermion, stubbornly continue studying our system, we see that we can manipulate our system in the following way. Let us transform our basis to the "Majorana basis" without any loss of generality

$$c_j = \frac{1}{2}(\gamma_{j1} + i\gamma_{j2}), \qquad c_j^\dagger = \frac{1}{2}(\gamma_{j1} - i\gamma_{j2}).$$

Inverting this relation gives

$$\gamma_{j1} = c_j^\dagger + c_j, \qquad \gamma_{j2} = i(c_j^\dagger - c_j). \tag{4.1}$$

One can check that the γ-operators satisfy:

$$\{\gamma_{i\alpha}, \gamma_{j\beta}\} = 2\delta_{ij}\delta_{\alpha\beta}, \qquad \gamma_{i\alpha}^\dagger = \gamma_{i\alpha}. \tag{4.2}$$

Here the first index of γ-indicates the associated fermion and the second index indicates the left or right Majorana operator in figure 4.1.

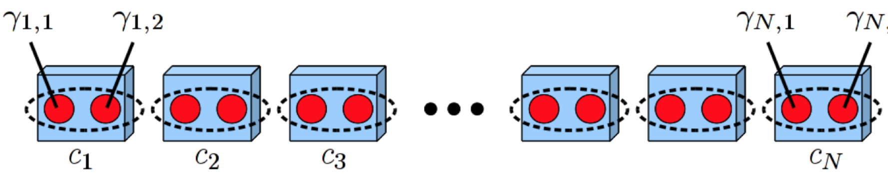

Figure 4.1: Visualization of the Majorana basis. Adapted from [18]

If we look at the last expression on the right side, we see why we call this a Majorana basis, the γ-operator creates a particle as well as its antiparticle, hence the particle and antiparticle must be the same. We see from above that any electronic system can be described in terms of the γ-operators. However, in most situations this transformation brings no benefit. The physical reason for this is that the Majorana-operators which can be used to describe an electron are related, so a separate description of the system in the two Majorana terms very often does not make sense. There is a situation in which this transformation does have physical relevance, namely in the situation of superconductivity. This can be seen directly from (4.1), since the theory of superconductivity predicts the realization of a coherent superposition of electrons and holes. Furthermore, the γ-operators are known to act only non-trivially on states with an unknown number of particles. Hence we need superconductivity to guarantee these states of unknown particle number. According to Schrieffer, superconductivity occurs when

"In a superconducting material, a finite fraction of electrons are in a real sense condensed into a "macromolecule" (or "superfluid") which extends over the entire volume of the system and is capable of motion as a whole. At zero temperature the condensation is complete and all the electrons participate in forming this superfluid, although only those electrons near the Fermi surface have their motion appreciably affected by the condensation."[29]

Here the Fermi surface is a surface of constant energy in three dimensional momentum space for the electrons. The Fermi surface is the surface that separates the occupied states of electrons in a solid from unoccupied states in which there are no electrons at $T = 0K$.[2] A more etymological description of superconductivity is the sudden drop

[2]See Kittel and Schrieffer.

in electrical resistivity of a metal or an alloy when the temperature is decreased far enough. The theory of superconductivity was developed in the fifties by Bardeen, Cooper and Schrieffer from which the term BCS theory originates, the theory that describes superconductivity.
The positive effect of superconductivity in our context, lies in the fact that superconductivity supersedes the initially very clear difference between holes and electrons [37]. This happens due to the so-called *Cooper pairs*. At low energies near the Fermi surface, there is an attractive force between electrons. This attractive force leads to a decrease in energy when two electrons form a pair, hence we have a bound pair state. The two fermions behave in some sense as a composite particle[3] Due to their bosonic nature, Cooper pairs can approach each other more closely, because of the absence of Pauli's exclusion principle and form a condensate. This condensate accounts for the superconductivity of the solid. A normal (=ground) state electron mode can lower its energy even further by forming a superposition with a normal state hole mode combined with a Cooper pair. This can again be seen mathematically by a mixing of the creation and annihilation operators, since we have a superposition of electrons and holes. The formalism describing this situation is the *Bogoliubov-Valatin formalism* and its corresponding *Bogoliubov-Valatin modes* have the form

$$\cos\theta c_j + \sin\theta c_k^\dagger.$$

Obviously, except for the specific case $j = k$ and $\theta = \frac{\pi}{4}$, they are not Majorana quasiparticles. The Cooper pairing has two more important properties. Firstly, conservation of the electron number is violated in a "Cooper condensate", i.e. a Cooper pair can be taken from or added to the condensate without really altering properties of the condensate. Secondly, the Cooper pairs shield electric fields and constrain magnetic fields. As a consequence of this phenomenon, charge is no longer observable, so that we have got rid of our original "charge problem" of creating Majorana quasiparticles. In figure 4.2.a the coupling of a hole and a Cooper pair is shown. Figure 4.2.b shows the "charge invisibility", i.e. due to the violation of the electron number, one cannot determine if one sees a hole with 6 Cooper pairs or an electron with 5 Cooper pairs.

[3] To be more precise on this, Cooper pairs have a length of approximately $100nm$, three orders of magnitude larger then the average spacing of a lattice of a solid. So the justifiability of equating this fermion pair to a single particle can be questioned.

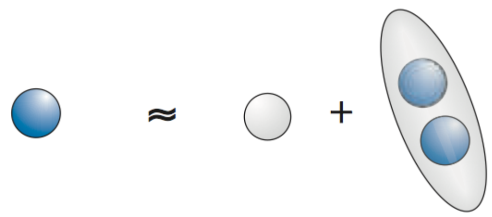

Figure 4.2.a: Hole-Cooper pair coupling

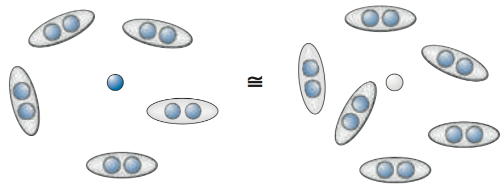

Figure 4.2.b: Charge invisiblity in a superconductor

Figure 4.2: Figures are adapted from [37]

So how do we find those Majorana quasiparticles?

1. For two- and three-dimensional systems the solution lies in the realm of *Abrikosov vortices* or *magnetic flux tubes*. A vortex is a concept from fluid dynamics and can also be applied for currents in superconductivity. The *Vorticity* ω is defined as $\omega = \nabla \times \mathbf{u}$. It can be seen as an indicator for the way particles move relatively to each other as they follow the flow $\mathbf{u}$ in a fluid. Here $\mathbf{u}(\mathbf{x}, t)$ is defined as the velocity of an *element of fluid* at position $\mathbf{x}$ and time t. The vortex is a region in the fluid in which the particles are mostly spinning around a line; the vortex line. The vortex lines form a vortex tube as the family of all vortex lines passing through a closed and reducible[4] curve in the fluid. A magnetic flux tube is a topologically cylindrical volume whose sides are defined by magnetic field-lines[3]. The magnetic flux tube is sometimes used as a simplification of magnetic behaviour. The magnetic flux through the surface of the defined space of the magnetic flux tube is constant. An Abrikosov vortex (or fluxon) is a magnetic flux tube in a *type-II* superconductor around which a superconducting current runs. Basically there are two types of superconductors, *type-I* and *type-II*. Type-I superconductors have *one* critical value of an applied magnetic field H_c, above which superconductivity is completely destroyed. Type-II superconductors have *two* critical values, H_{c1} and H_{c2}, of applied magnetic field at which their conductive behaviour changes drastically. Below H_{c1} superconductivity occurs. Between H_{c1} and H_{c2} magnetic flux tubes are formed and their density increases with an increase of the applied magnetic field. One could see this as the leaking of the magnetic field into the superconductor. Above the H_{c2}-limit superconductivity is destroyed.

[4]Reducible means here that the curve can be reduced to a point by continuously deformating the curve, without passing the surface of the fluid

The Abrikosov vortices can trap *zero modes*. Zero modes are with very low, formally zero, energy. Each vortex has a finite number of associated zero modes.[5]. Now we get to the solution provided by the Abrikosov vortices for creating Majorana fermions. Since the zero modes are low-energy *excitons*, it are quasiparticles. Furthermore, the zero modes are superpositions of particle- and hole-states in equal measure. Thus, let us call these quasiparticles *partiholes*. We have already seen the operators that create the partiholes, see (4.1). The invariance under charge conjugation tells us that we have found operators for localized quasiparticles, that are equal to their own antiquasiparticles. It is therefore justified to call the zero modes associated with the partihole *Majorana quasiparticles* or *Majorana zero modes*.
Now that we have been dreaming theoretically about Majorana modes, let us make it more practical; Where do we find these Majorana modes? Majorana modes are only predicted to occur in very special instances of superconductors. If the electrons have orbital angular momentum 0 (s-wave) and the electrons behave non-relavistically, then the Majorana modes cannot be predicted theoretically. Basically one can be distinguish two sorts of experiments creating Majorana modes in the context of superconductivity.

- p-wave electrons, this case can occur in a special fractional quantum Hall state, namely the *Pfaffian* or *Moore-Read* state. In this state the filling factor is $\nu = \frac{5}{2}$.[6]
- relativistic s-wave electrons, in this case the Majorana modes could be found at the surface of topological insulators or in graphene. As Joel Moore explains [24], a topological insulator is "an insulator that always has a metallic boundary when placed next to a vacuum or on 'ordinary' insulator. These metallic boundaries originate from topological invariants, which cannot change as long as a material remains insulating." A topologically invariant property of an object is a property that is invariant under smooth deformations. For example, a coffee cup and a donut can be turned

[5]A fundamental mathematical theorem, the Atiyah-Singer index theorem, relates the number of zero modes with the shape of the Abrikosov vortex. More specifically with the genus of the vortex

[6]In the quantum Hall effect, which is the quantum-mechanical brother of the classical Hall effect, the conductance of a two-dimensional electron system in strong magnetic fields at low temperatures takes quantized values. The conductance is defined as $\sigma = \frac{I_{channel}}{V_{Hall}}$ and in the quantum Hall effect the conductance takes on the values $\sigma = \nu\frac{e^2}{h}$. Here e is the elementary charge quantum, h is Planck's constant and finally ν is the filling factor, which accounts for the quantization. ν can attain fractional and integer values. $\nu = \frac{5}{2}$ is a fractional Hall state.

into each other by continuous deformation, they have both genus 1. For more information on topological insulators, see Moore [24].

2. For one-dimensional systems there are also several possibilities for creating Majorana modes. For example [10],

   - at the edges of a two-dimensional topological insulator.
   - in nanowires from a 3D-topological insulator.
   - in helical spin chains
   - in semiconductor quantum wires

   As already said, one-dimensional experiments with nanowires have provided real evidence for realizing the Majorana zero modes. So that is why this article will mainly focus on the last bullet. In order to get a good idea of Kouwenhoven's experiment, we first study a simple theoretical model of Majorana zero modes in one-dimensional quantum wires.

4.2 The Kitaev Chain model

This is a very simple 'toy model' introduced by Kitaev [15].[7][8] We already know from (4.1), that we can write every fermion as a combination of Majorana operators. Often the Majorana modes are localized close to each other, meaning that they overlap undeniably and so the separate description does not lead to anything useful. However sometimes the two Majorana modes are prevented from overlapping by for example a large spatial separation. Such a state, consisting out of two spatially separated Majorana modes, is safe for most types of decoherence (information loss) since local energy perturbations can only effect one of its Majorana parts. Furthermore, this state can be manipulated due to its non-Abelian statistics. So if we could store information in these modes, it is rather safe for pertubations and manipulatable[9]. This is why there is so much interest in Majorana modes from the quantum computation sector.

To see how above the statements can be realized, we introduce the following Hamiltonian. It describes a spinless p-wave superconductor, having eigenstates that are

[7]Actually this is even a more simplified and pedagogical version than Kitaev's original model, adapted from [18], but it is very similar.

[8]It is important to notice that Kouwenhoven's experiment even as this model is one-dimensional.

[9]The term "manipulatable"" refers here to the *braiding*, that will be discussed in section 4.3

spatially isolated Majorana modes. The 1D p-wave binding chain has N sites for operators to be localized on.

$$\mathcal{H}_{chain} = \mu \sum_{i=1}^{N} n_i - \sum_{i=1}^{N-1} (tc_i^\dagger c_{i+1} + \Delta c_i c_{i+1} + h.c.)$$

Here h.c stands for hermitian conjugate, μ stands for the chemical potential, c_i is the annihilation operator for an electron on site i, and consequently $n = c_i^\dagger c_i$ is then the electron number operator. t is the hopping amplitude and Δ is the superconducting gap. Δ and t are assumed to be equal for all sites. We now recall the transformations (4.1).

$$\gamma_{j,1} = c_j^\dagger + c_j, \qquad \gamma_{j,2} = i(c_j^\dagger - c_j).$$

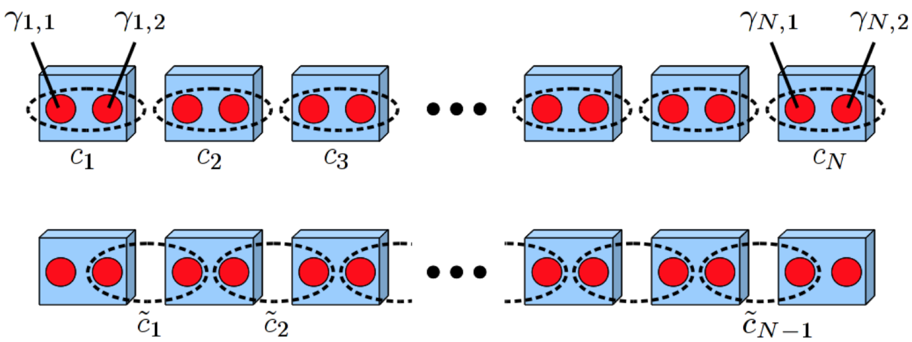

Figure 4.3: The Kitaev Chain Model. Figure adapted from [18]

In the upper subfigure of figure 4.3, one can see the pairing of the Majorana operator forming a fermionic operator. If we now set $\mu = 0$, $t = \Delta$[10], and insert the γ-operators in our Hamiltonian, we acquire;

$$\mathcal{H}_{chain} = -it \sum_{i=1}^{N-1} \gamma_{i,2}, \gamma_{i+1,1}.$$

The pairing of Majorana modes can now be seen in a different way. The pairing can be concretized by constructing the following fermionic operators.

$$\tilde{c}_i = \frac{\gamma_{(i+1)1} + i\gamma_{i2}}{2}.$$

We now have

$$2\tilde{c}_i^\dagger \tilde{c}_i = 2\tilde{n}_i = 2\left(\frac{\gamma_{i+1,1}^\dagger - i\gamma_{i,2}^\dagger}{2}\right)\left(\frac{\gamma_{i+1,1} + i\gamma_{i,2}}{2}\right) = -i\gamma_{i,2}\gamma_{i+1,1}.$$

[10]This is known as the ideal quantum wire limit

So our Hamiltonian in terms of our new fermionic operators, constructed out of a different choice of Majorana operators, becomes

$$\mathcal{H} = 2t \sum_{i=1}^{N-1} \tilde{c}_i^\dagger \tilde{c}_i. \tag{4.3}$$

The creation of $\tilde{c}_i$ fermionic mode has energy $2t$. A closer look at our new Hamiltonian fin (4.3) reveals something very interesting. The Majorana mode terms γ_{N2} and γ_{11}, which are localized at the two ends of the wire, do not appear in this Hamiltonian anymore! We can combine these two Majorana modes in one fermionic operator as

$$\tilde{c}_M = \frac{\gamma_{N2} + i\gamma_{11}}{2}. \tag{4.4}$$

Obviously these Majorana modes are delocalized since they appear on opposite ends of the wire. The mere fact that this modes do not appear in the Hamiltonian leaves us no other option than to conclude that this mode has zero energy.

The analysis used above, was carried out in the 'ideal quantumwire'-limit, meaning $\Delta = t$ and $\mu = 0$, but it turns out that the Majorana end-states can exist as long as

$$|\mu| < 2t. \tag{4.5}$$

Furthermore, the Majorana modes remain at zero energy as long as the wire is long enough so that the modes do not overlap. Kitaev's chain model is not a very realistic model, however it is a good way to give lay readers intuition for the way Majorana modes can be created in more complex systems, for example continuous 1D, p-wave superconductors or in honeycomb 2D structures.

4.3 Statistics of Majorana modes in two dimensions

We know that *Fermi-Dirac statistics* apply to fermions. From the so-called *spin-statistics theorem*, we know that for a system of fermions the wavefunction changes sign under the exchange of two particles. More mathematically,

$$\Psi(\mathbf{r}_1, \mathbf{r}_2) = -\Psi(\mathbf{r}_2, \mathbf{r}_1).$$

As a consequence of this, the wavefunction vanishes if you try to put two identical fermions in the same place. This is a manifestation of the Pauli exclusion principle for fermions. This is also why the square of a fermionic creation operator must be 0. Applying methods from statistical mechanics, we can derive a distribution function for a system of identical fermions. This theoretical statistical distribution is known as the Fermi-Dirac statistics. For a boson, which follows the Bose-Einstein statistics, the square of its creation operator should create identical particles. However, we know from (4.2) ${\gamma_{i\alpha}}^2 = 1$, thus the adding of a partihole to a state already occupied by a partihole results in

$$\gamma_{i\alpha}\gamma_{i\alpha} |0\rangle = \gamma_{i\alpha}^2 |0\rangle = |0\rangle ,$$

i.e. in the vacuum state again.[11] Hence, Majorana modes cannot be fermions nor bosons.[12] So if a Majorana mode is not a fermion nor a boson, what is it then, what statistical behaviour does it have? Well, a Majorana quasiparticle is correctly described as a non-Abelian anyon or an *Ising anyon*. For convenience the treatment of anyons and Majorana modes statistics is confined here to the two-dimensional case, i.e. we are considering a plane in which the vortex lines have reduced to vortex points, around which the supercurrents flow. The non-Abelian anyonic exchange behaviour is the main reason for the scientific interest in Majorana zero modes.

For the simplest Abelian anyons the effect on the phase of an anticlockwise exchange between two anyons can vary continuously between the fermionic or bosonic anticlockwise exchange effect. More mathematically, the exchange effect for anyons can be summarized as

$$|\Psi_1\Psi_2\rangle = e^{i\theta} |\Psi_2\Psi_1\rangle . \tag{4.6}$$

One can see that for $\theta = \pi$, we obtain Fermi-Dirac statistics and for $\theta = 2\pi$ we obtain Bose-Einstein statistics. However, anyons are way more general as (4.6) shows. For a Majorana mode it turns out that we find a more complex factor instead of the exponent, which results in non-Abelian statistics.

4.3.1 Non-Abelian statistics

To have non-Abelian statistics, a degenerate ground state, separated from other (higher energy) states by an energy gap , is essential. In the case of such a degenerate ground state, adiabatic exchanges of quasiparticles can bring the system from one

[11] In some way this reminds us of the $\mathbb{Z}_2$-grading of the Clifford algebra. However this is something completely different.

[12] So note that we can never speak of Majorana fermions in solid state physics, since they simply are not fermions.

groundstate to another. This process is called *braiding*[13]. To give a concrete example of braiding and its associated non-Abelian statistics, a transparent simplified analysis of Ivanov [14] is used for the p-wave semiconductor, where the electrons have orbital angular momentum one.

Recalling (4.1), but now for simplicity writing the relations down with only one index, gives

$$\gamma_{2i-1} = c_i^\dagger + c_i, \qquad\qquad \gamma_{2i} = i(c_i^\dagger - c_i).$$

Let us now have two Abrikosov vortices in a two-dimensional topological superconductor, trapping Majorana modes with corresponding operators γ_1 and γ_2. Associated with each vortex, there is a superconducting phase ϕ with a winding of 2π. The phase of each vortex has a branch cut and we choose the branch lines to be parallel. A clockwise exchange of the vortices one and two results in a crossing of vortex 1 through the branch cut of vortex 2, as shown in figure 4.4. Consequently vortex 1 acquires a phase shift of 2π. The Majorana quasiparticle in vortex 1, made up from single fermion operators (rather than of products of two), takes a phase shift of π by crossing the branch cut. This results in

$$\begin{aligned}\gamma_1 &\to -\gamma_2,\\ \gamma_2 &\to +\gamma_1.\end{aligned}$$

An anticlockwise exchange would have resulted in

$$\begin{aligned}\gamma_1 &\to \gamma_2,\\ \gamma_2 &\to -\gamma_1.\end{aligned}$$

These exchange transformations can be represented by so-called *braid-operators.* The braid operators act as

$$\gamma_i \to B_{12}\gamma_i B_{12}^\dagger$$

The clockwise braid operator is $B_{12} = \frac{1}{\sqrt{2}}(1+\gamma_1\gamma_2)$, or more generally $B_{ij} = \frac{1}{\sqrt{2}}(1+\gamma_i\gamma_j)$, whereas the anticlockwise exchange has the braid operator $\tilde{B}_{12} = \frac{1}{\sqrt{2}}(1-\gamma_1\gamma_2)$. One can easily check that these operators have the required effect on the γ-operators. In the right part of figure 4.4, we have to bring vortex 1 around vortex 2 and bring

[13]The analysis will stay focussed on two-dimensional systems. Reference [1] shows how to develop non-Abelian statistics in a one-dimensional wire using T-junctions.

it back to its original position. This is topologically equivalent to two subsequent exchanges. Hence, we must calculate

$$(B_{12})^2 = \frac{1}{2}(1+\gamma_1\gamma_2)(1+\gamma_1\gamma_2) = \frac{1}{2}(1+2\gamma_1\gamma_2+\gamma_1\gamma_2\gamma_1\gamma_2) = \frac{1}{2}(1-1+2\gamma_1\gamma_2) = \gamma_1\gamma_2$$

So, we have

$$\gamma_1 \to (\gamma_1\gamma_2)\gamma_1(\gamma_1\gamma_2)^\dagger = -\gamma_1$$
$$\gamma_2 \to (\gamma_1\gamma_2)\gamma_2(\gamma_1\gamma_2)^\dagger = -\gamma_2$$

So a loop of one vortex around the other vortex gives a minus sign to each Majorana operator. This can also be seen from figure 4.4: due to the loop, the γ_1-operator crosses the branch cut of vortex 2, however vortex 2 gets also swiped through the branch cut of vortex 1.

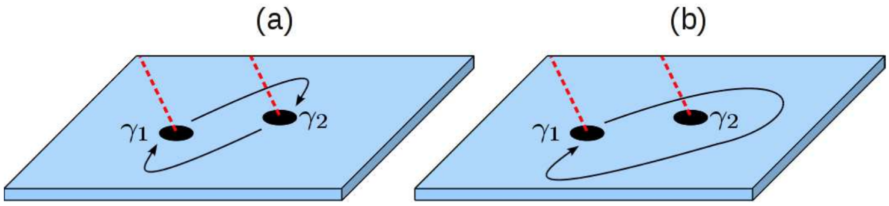

Figure 4.4: A visualization of the exchange process. The red dashed lines represent the branch lines. Adapted from [18]

It is useful to evaluate the braid operator B_{12} on the vacuum state $|0\rangle$ at the fermionic site c_1, which can be split in Majorana states γ_1 and γ_2. We thus have

$$B_{12}|0\rangle = \frac{1}{\sqrt{2}}(1+\gamma_1\gamma_2)|0\rangle,$$

where

$$\gamma_1\gamma_2|0\rangle = (c_1^\dagger + c_1)i(c_1^\dagger - c_1)|0\rangle$$

Since $c_1|0\rangle = 0$

$$= (c_1^\dagger + c_1)i(c_1^\dagger)|0\rangle$$
$$= i((c_1^\dagger)^2 + c_1c_1^\dagger)|0\rangle$$

Since c is a fermionic operator, $(c_1^\dagger)^2 = 0$.

$$= i|0\rangle$$

So we conclude

$$B_{12}\,|0\rangle = \frac{1}{\sqrt{2}}(1+i)\,|0\rangle\,.$$

To see the non-Abelian nature of the Majorana modes, we need a system with at least 4 Majorana modes.[14] These 4 Majorana modes can be described by 2 fermionic number states $|n_1 n_2\rangle$. Choose the branch cuts of all Majorana modes to be in the same direction. Number the Majorana modes based on their position orthogonal to the branch cut. This way, we have only one crossing of Majorana mode $(i+1)$ by Majorana mode (i), when exchanging Majorana modes i and $(i+1)$ in a clockwise manner. In this system it turns out that the braid operators of adjacent Majorana modes do not commute, so for example

$$[B_{12}, B_{23}] = \gamma_2\gamma_3 \neq 0$$

For B_{21} one has,

$$B_{12}\,|00\rangle = \frac{1}{\sqrt{2}}(1+\gamma_1\gamma_2)\,|00\rangle$$

with

$$\begin{aligned}\gamma_1\gamma_2\,|00\rangle &= i(c_1^\dagger + c_1)(c_1^\dagger - c_1)\,|00\rangle\,^{15}\\ &= i((c_1^\dagger)^2 + c_1 c_1^\dagger)\,|00\rangle = i\,|00\rangle\,.\end{aligned}$$

So, this gives

$$B_{12}\,|00\rangle = \frac{1}{\sqrt{2}}(1+i)\,|00\rangle\,.$$

Analogously,

$$B_{23}\,|00\rangle = \frac{1}{\sqrt{2}}(1+\gamma_2\gamma_3)\,|00\rangle\,,$$

with

$$\begin{aligned}\gamma_2\gamma_3\,|00\rangle &= i(c_1^\dagger - c_1)(c_2^\dagger + c_2)\,|00\rangle\,,\\ &= i((c_1^\dagger c_2^\dagger + c_1^\dagger c_2 - c_1 c_2^\dagger - c_1 c_2)\,|00\rangle\,.\end{aligned}$$

Here the annihilation operators on the vacuum let certain terms vanish, this gives

$$\gamma_2\gamma_3\,|00\rangle = i c_1^\dagger c_2^\dagger\,|00\rangle = i\,|11\rangle\,.$$

Leading to,

$$B_{23}\,|00\rangle = \frac{1}{\sqrt{2}}(|00\rangle + i\,|11\rangle)$$

[14]This is a simplified procedure. However it does give a very good idea why the Majorana obey non-Abelian statistics.

If we now combine these two expressions in the commutator, this gives

$$
\begin{aligned}
B_{12}B_{23}\left|00\right\rangle &= B_{12}\left(\frac{1}{\sqrt{2}}(\left|00\right\rangle + i\left|11\right\rangle)\right), \\
&= \frac{1}{\sqrt{2}}(B_{12}\left|00\right\rangle + iB_{12}\left|11\right\rangle), \\
&= \frac{1}{2}(1+i)\left|00\right\rangle + \frac{1}{2}i(1+\gamma_1\gamma_2\left|11\right\rangle), \\
&= \frac{1}{2}(1+i)\left|00\right\rangle + \frac{1}{2}i(1+i(c_1^\dagger + c_1)(c_1^\dagger - c_1)\left|11\right\rangle.
\end{aligned}
$$

Here, the creation operators on the $\left|11\right\rangle$ let certain terms vanish, since c is a fermionic operator

$$
\begin{aligned}
&= \frac{1}{2}(1+i)\left|00\right\rangle + \frac{1}{2}i(1+i(c_1^\dagger + c_1)(0-\left|01\right\rangle), \\
&= \frac{1}{2}(1+i)\left|00\right\rangle + \frac{1}{2}i(1-i\left|11\right\rangle), \\
&= \frac{1}{2}(1+i)(\left|00\right\rangle + \left|11\right\rangle).
\end{aligned}
$$

We can also evaluate the same expression the other way around.

$$
\begin{aligned}
B_{23}B_{12}\left|00\right\rangle &= B_{23}\frac{1}{\sqrt{2}}(1+i)\left|00\right\rangle, \\
&= \frac{1}{\sqrt{2}}(1+i)B_{23}\left|00\right\rangle.
\end{aligned}
$$

Using the expression we find above for $B_{23}\left|00\right\rangle$.

$$
\begin{aligned}
&= \frac{1}{\sqrt{2}}(1+i)(\frac{1}{\sqrt{2}}(\left|00\right\rangle) + i\left|11\right\rangle) \\
&= \frac{1}{2}(1+i)\left|00\right\rangle) + \frac{1}{2}(-1+i)\left|11\right\rangle)
\end{aligned}
$$

Evidently, these two expressions are not the same. Moreover one has

$$[B_{12}, B_{23}]\left|00\right\rangle = \left|11\right\rangle.$$

But it turns out that

$$\gamma_1\gamma_3\left|00\right\rangle = (c_1^\dagger + c_1)(c_2^\dagger + c_2)\left|00\right\rangle = \left|11\right\rangle$$

This whole procedure can be summarized as

$$[B_{i-1,i}, B_{i,i+1}] = \gamma_{i-1}\gamma_{i+1} \neq 0,$$

from which we can conclude that the Majorana modes obey a non-Abelian statistics. Figure 4.5 shows the typical way to visualize the braiding of the γ-operators.

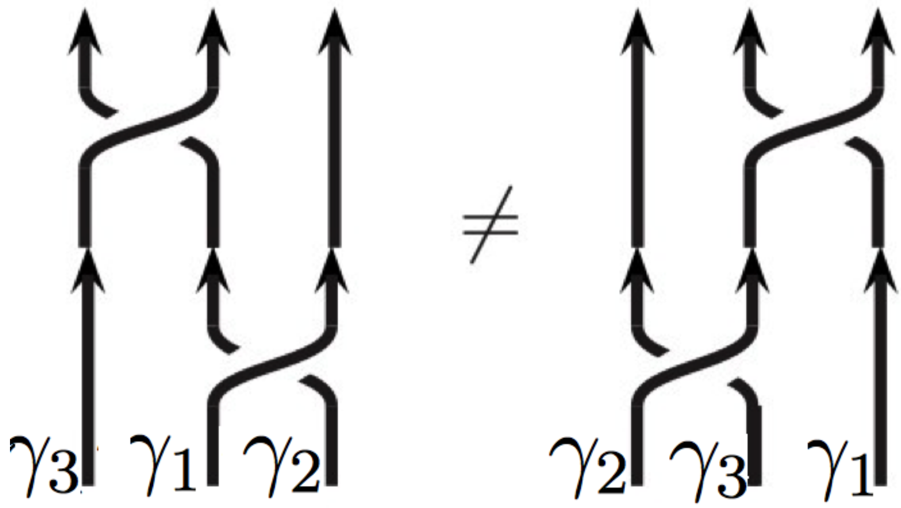

Figure 4.5: Typical visualization of the braiding operations from the example above.

4.4 Kouwenhoven's experiment

Now that we have an idea how Majorana modes are predicted by theorists, let us give a briefly discuss the first real evidence for a Majorana mode[16] [26].

4.4.1 Setup of the experiment

The experiment was done in a chip at the nanoscale. One can distinguish the following parts of the experimental setup. A silicon substrate on which golden electrodes are fabricated with widths ranging from 50 to 300 nm. These golden electrodes work as gates in the experiment. By applying voltages to these gates you can change the potential electrostatic fields that the electrons feel. On this substrate, an indium antimonide quantum wire is placed (silver), contacted with a normal conductor on one side (gold coloured) and a superconductor (blue) on the other side. The used materials are gold and niobium titanium nitride, respectively. To prevent electrical conduction between the gates and the quantum wire, a dielectric film of silicon nitride is applied to the silicon substrate. This system is cooled down to several mK. This is done to make sure no thermal excitations are present anymore. Furthermore, a strong magnetic field is applied, which combined with the large Landé-factors of indium antimonide nanowires, gives rise to a strong spin orbit coupling. This is done to guarantee the existence of Majorana modes at the endpoints of the part of the nanowire that is attached to the superconductor (from now on called endpoints).

[16]For what we have explained above, we do not refer to these entities as Majorana fermions.

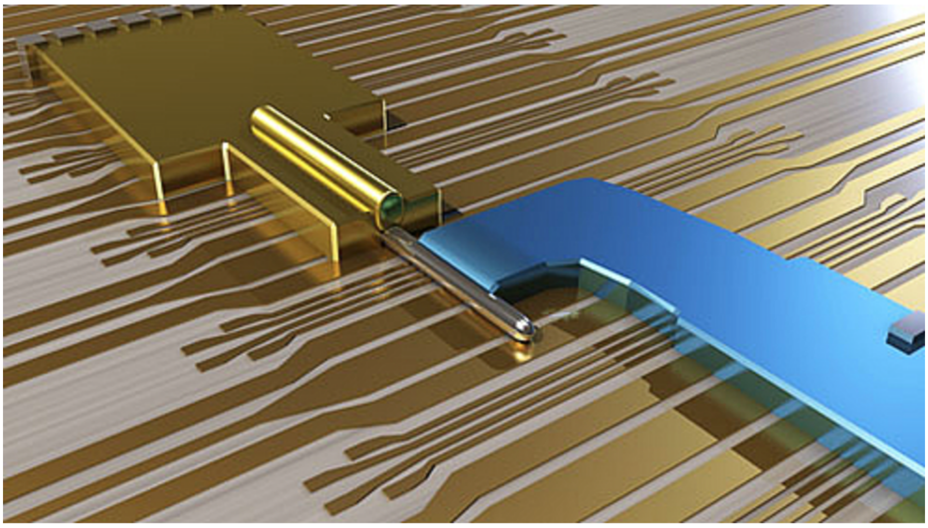

Figure 4.6: Sketch of the setup of the experiment. Adapted from [8]

4.4.2 Detection

To detect the Majorana modes, the following procedure is followed. In the case of no magnetic field $B = 0$, no Majorana modes are present; so when electrons from the gold electrode are shot towards the nanowire, they simply cannot get any further and return back to the gold. As a consequence, no tunneling current is measured. If one now does apply a magnetic field, Majorana modes at the endpoints are formed. By again shooting electrons from the gold towards the superconductor, certain electrons with the right energy can interact with the Majorana modes. In this interaction, due to a process called Andreev reflection, new Cooper pairs are formed in the superconductor. These Cooper pairs cause a small tunneling current, which can be measured.

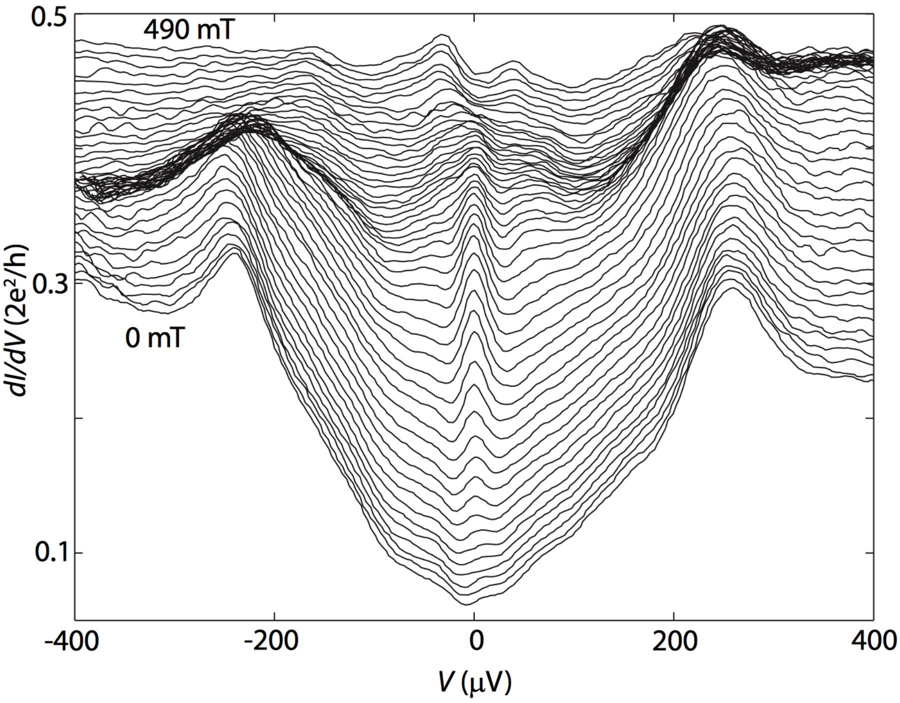

Figure 4.7: Graph of the conductivity of the nanowire versus the energy represented in voltage. The zero energy peak for increasing magnetic fields tells us that the mode does not have a magnetic dipole moment. Since otherwise we would have seen an increase in energy of the conductance peak. Adapted from [26]

In the above graph of the conductivity, which depends on the creation of the Cooper pairs, is plotted against the voltage which corresponds to energy. The lower line, which corresponds to $B = 0$, is important. At $B = 0$ and at $V = 0$, no current is expected to be measured. When the magnetic field is increased the Majorana modes appear and even for the "zero energy" case a small peak is starting to appear. Furthermore, for the increasing magnetic field the peak stays at zero energy. This immediately implies that the tested quasiparticle does not have a magnetic dipole moment. We can also change the electric field, and again the peak occurs and stays at zero energy for increasing values of the electric field. Summarizing the "three times nothing"-property of the Majorana mode is proved; no charge, no magnetic dipole moment, no energy.

Chapter 5

Final comparison and conclusion

Having understood what we have discussed in the preceding chapters, the differences between the Majorana fermion as first predicted by Ettore Majorana in the early twentieth century and the Majorana mode (or quasiparticle) as probably detected by the research group at Delft and other experimental groups, cannot be missed. A final summary of the main differences is shown in the table below

	Majorana fermion	Majorana zero mode or quasiparticle
Statistics	1	2
Clifford algebra used in description	3	4
Space-time dimension	5	6

Table 5.1: Summary of the differences the Majorana fermion and the Majorana zero mode

1. Fermi-Dirac statistics.
2. Non-abelian anyonic/Ising anyonic statistics.
3. Yes, the Clifford algebra used in the description is a pseudo-Euclidan Clifford algebra *in spacetime* formed by Γ-matrices. These Γ-matrices can be constructed in arbitrary spacetime dimension following the method from section 3.4.
4. Yes, the creation and annihilation operators of the Majorana modes satisfy a Euclidean Clifford algebra. However, this is a totally different Clifford algebra then the Clifford algebra from 3. This Clifford algebra namely lives in the abstract space of zero modes, i.e. the space of modes that have zero energy.
5. Majorana spinors can be found in a selection of space-time dimensions, see table 3.2. They can also live, for example, in $d = t + s = 1 + 1$ spacetime dimensions.

6. Majorana zero modes are conjectured to at least exist in $d = t + s = 1 + 1$ or $d = t + s = 1 + 2$ space-time dimensions.

After this elaborate description of the difference of these two physical phenomena, one could wonder why the term “Majorana fermion” is so often used for the Majorana zero mode. Probably, this has happened since the will to detect the true Majorana particle was so large, that when experimentalists found a quasiparticle which showed some resemblance to the Majorana fermion, they could not resist to call this the “Majorana particle”, or worse, the article of Kouwenhoven is “Signature of Majorana *fermions* ...”. As explained in chapter 4.3.1, the Majorana quasiparticles are Ising anyons so this title cannot be categorized other than wrong.
It is not that experimentalists do not know of this silly particle taxonomy, Kouwenhoven admits openly that “his” quasiparticle is not a fermion and that it is very different from the real Majorana fermion. In his Kronig lecture in June 2012 he says: “I think it is obvious, right, that our Majorana fermions are different from the ones in the cosmos? They obey the same definition, but they appear in a completely different way.”[16]. One could argue about the “same definition”- statement, but if we ignore that dangerous statement we see that the misnaming is rather born out of a theoretical carelessness then out of a real different view of physics.
My proposal (and hopefully the reader agrees after this article) would be to, from now on, make a clear distinction between the proper[1] Majorana fermion and the Majorana quasiparticle or Majorana (zero) mode.

[1] Instead of the adiective “proper”, one could use the word“real”. However this might again lead to confusion, since the Majorana fermion has never been observed in nature. This problem would be solved if the Majorana nature of neutrino’s would finally be confirmed by particle and astroparticle physics experiments. Then Majorana fermions would then truely be real.

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