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Conformal window for QCD-like theories

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Abstract

In this thesis we will give a necessary and sufficient condition such that scale invariance implies invariance under the full conformal group for local continuous, Lorentz invariant and renormalizable lagrangian. First we will go through the construction of jet spaces, from which we deduce the first Noether theorem. Also we will discuss what the conformal transfomations are and what it means for a lagrangian to be renormalizable. We will prove that if a certain restriction for the energy momentum tensor holds. A scale invariant lagrangian will also be conformally invariant.

Acknowledgements

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Contents

1 Introduction and motivation

In this thesis we are going to research when scale invariance implies invariance under the full conformal group. This research is motivated by the fact that Quantum Chromo Dynamics (QCD) develops an infrared fixed point (IRFP) for a sufficiently large fixed point. Where the flavours are those observed in nature until now: the up, down, charm, strange, bottom, top quarks. For a review on this topic see article [1]. At this fixed point scale invariance can be guaranteed but not invariance under the full conformal group. Therefore we are going to investigate the following question: what are the sufficient and/or necessary conditions for scale invariance to imply invariance under the full conformal group.

In order to be able to give a good mathematical description and answer to this question we are first going to construct the so called jet spaces. Then we derive the first Noether theorem, which allows us to define what it means for a lagrangian to be invariant under an infinitesimal transformation. After this we will have a look at the infinitesimal conformal transformations. Here we will see that scale transformations are a subset of the conformal transformations.

Next we will arrive at the main question of this thesis. Here we will find a restriction for the energy momentum tensor, which if satisfied will guarantee that scale invariance implies invariance under the full conformal group. We will prove this for a renormalizable lagrangian. In the last section we will have an intuitive and only slightly technical look at renormalization and what it means for a lagrangian to be renormalizable.

2 Non-Abelian gauge theories and fixed points

In this section we are going to look at the symmetries of the prototype Quantum Chromo Dynamics (QCD) lagrangian, which governs the strong force. The prototype of the lagrangian for QCD has the following form:

$$\mathcal{L} = -\frac{1}{4}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) + \overline{\psi}(i\gamma^\mu D_\mu - m_f)\psi \tag{2.1}$$

Here $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ for the gauge field A^μ, D_μ is the covariant derivative defined as $D_\mu\psi(x) = (\partial_\mu + gA_\mu(x))\psi(x)$. This Lagrangian is invariant under the non-Abelian $SU(N)$ gauge transformations. Here N is the number of colors, which in real world QCD equals three, i.e., $N = 3$. Now we are going to have a look at the symmetries of this lagrangian. A symmetry of a lagragian is a coördinate and/or field transformation that changes the Lagrangian at most by a total derivative.

We are going to show that the QCD lagragian is chirally symmetric if and only if the mass of the fermions m_f equals zero. The chiral transformation is given by:

$$\psi'(x) = e^{i\alpha\gamma_5}\psi(x) \tag{2.2}$$

With α constant. Then for $\overline{\psi}(x)$ we have the following transformation:

$$\begin{aligned}(\overline{\psi})'(x) &= (\psi^\dagger)'(x)\gamma^0 \\ &= \psi^\dagger(x)e^{-i\alpha\gamma^5}\gamma^0 \\ &= \overline{\psi}(x)e^{i\alpha\gamma^5}\end{aligned} \tag{2.3}$$

where we have used $\{\gamma^5, \gamma^\mu\} = 0$. Under a chiral transformation the QCD lagrangian transforms as:

$$\begin{aligned}\mathcal{L}' &= -\frac{1}{4}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) + (\overline{\psi})'(i\gamma^\mu D_\mu - m_f)\psi' \\ &= -\frac{1}{4}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) + \overline{\psi}(x)e^{i\alpha\gamma_5}(i\gamma^\mu D_\mu - m_f)e^{i\alpha\gamma_5}\psi(x) \\ &= -\frac{1}{4}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) + i\overline{\psi}(x)\gamma^\mu D_\mu e^{-i\alpha\gamma_5}e^{i\alpha\gamma_5}\psi(x) - \overline{\psi}(x)e^{i\alpha\gamma_5}m_f e^{i\alpha\gamma_5}\psi(x) \\ &= -\frac{1}{4}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) + i\overline{\psi}(x)\gamma^\mu D_\mu\psi(x) - m_f\overline{\psi}(x)e^{2i\alpha\gamma_5}\psi(x)\end{aligned} \tag{2.4}$$

We see that the QCD lagrangian is chirally symmetric if and only if $m_f = 0$. We say that $m_f \neq 0$ breaks the chiral symmetry explicitly. Chiral symmetry is spontaneously broken if we have that the vacuüm expectation value of $\psi\overline{\psi}$ is non-zero, i.e.,

$$< 0|\psi\overline{\psi}|0 > \neq 0 \tag{2.5}$$

2.1 Pertubation theory

In perturbation theory we are breaking the exact theory up into a series of solvable problems. It is a mathematical trick to expand the problem into a power series depending on a constant which we call the renormalization scale μ. A more detailed description will be given in section 9. In perturbation theory we can define a so called beta-function in terms of a coupling g. In the case of the QCD lagrangian (2.1) the coupling g is the coupling of the fermions with the gauge bosons. The beta-function or β-function is defined as

$$\beta(g) = \frac{\partial g}{\partial \log(\mu)} \tag{2.6}$$

This tells us that if the β-function is positive, increasing the renormalization scale, results into an increase in the coupling g. In article [2], the beta function for an $SU(N)$ gauge theory with N_f massless Dirac fermions in the representation R of the gauge group was calculated to be:

$$\beta(g) = -b_0 \frac{g^3}{16\pi^2} + b_1 \frac{g^5}{(16\pi^2)^2} + \mathcal{O}(g^7) \tag{2.7}$$

where

$$\begin{aligned} b_0 &= \frac{11}{31} C_2(G) - \frac{4}{3} T(R) N_f \\ b_1 &= \frac{34}{3} C_2(G)^2 - \frac{20}{3} C_2(G) T(R) N_f - 4 C_2(R) T(R) N_f \end{aligned} \tag{2.8}$$

For N_f massless Dirac fermions in the representation R of the compact Lie Gauge group G. $C_2(G)$ and $C_2(R)$ are the Casimir operators of the adjoint and the fundamental representations respectively and $T(R)$ is the trace of R. In order to make it more explicit, we will see what this means for N_f Dirac fermions in the fundamental representation. Also we take $G = SU(3)$ (thus $N = 3$), we then have:

$$\begin{aligned} b_0 &= 11 - \frac{2}{3} N_f \\ b_1 &= 102 - \frac{38}{3} N_f \end{aligned} \tag{2.9}$$

We plot the beta-function as a function of the coupling g in figure 1. When $N_f < 16.5$ the origin is reached when the energy to infinity since the beta function is defined as $\beta(g) = \frac{\partial g}{\partial \log(\mu)}$; increasing the energy for $\beta(g)$ negative results into a decrease in the coupling. This is known as asymptotic freedom and the zero of the beta-function at the origin is corresponds to an ultraviolet fixed point (UVFP) of the theory. By the same reasoning the zero at g^* corresponds to a fixed point at zero energy, which is known as an infrared fixed point (IRFP).

At a fixed point we have, since the beta-function is zero $\frac{\partial g}{\partial \log(\mu)} = 0$, that $g(\mu) = g^* = \text{const}(\mu)$. Which implies that the coupling is independent of the energy scale. In other words at a fixed point of the beta-function we have scale invariance. However we often want to have invariance under the full conformal group. This motivates us to investigate the following question: when does scale invariance imply invariance under the full conformal group?

Before we can answer this question we first need to construct some mathematical concepts called jet spaces. In jet spaces we can formally define what it means for a lagrangian to be invariant under a specific type of transformations.

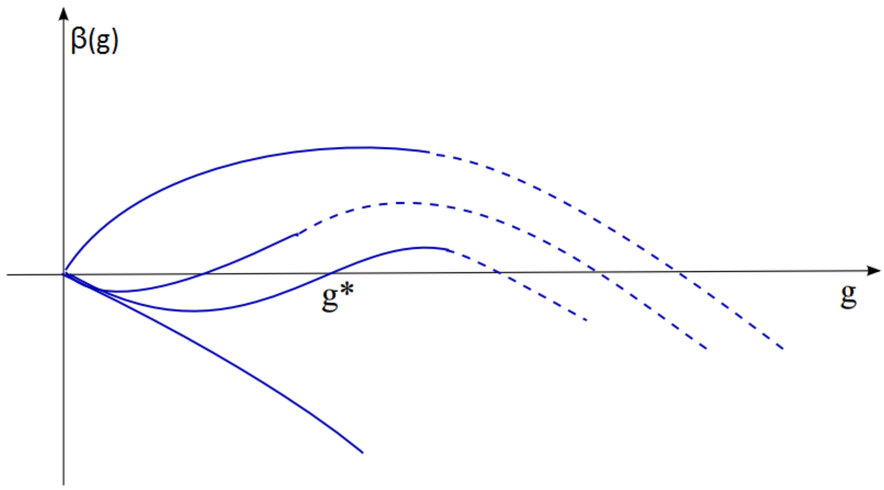

Figure 1: The qualitative behavior of the two-loop beta-function $\beta(g)$ as a function of the coupling g for $SU(3)$. The number of massless flavours N_f in the fundamental representation increases form bottom to top. For $N_f < 8.05$ the beta function stays negative. For $8.05 < N_f < 16.5$ the beta function develops a zero (IRFP) at a non zero coupling g. For $N_f > 16.5$, the beta function becomes positive, implying that $g* = 0$ is no longer an UV stable fixed point and asymptotic freedom is lost. The beta function might develop an additional zero (UVFP) at strong coupling (dashed lines), a property to be investigated in future studies. Figure from article [1]

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3 Jet Spaces

We are going to define the jet spaces according to article [3]. First we are going to create k-th jet spaces. The sections in these jet spaces consist of a postion x in the base and the corresponding partial derivatives. We define that two sections in the k-th jet space are equivalent to each other if their corresponding Taylor series till order k are the same.

Let $\pi : E^{m+n} \to M^n$ be a vector bundle. Let $x \in M^n$ be a point and $u \in \pi^{-1}(x)$ be a point in the fibre of x. First note that two sections $s_1, s_2 \in \Gamma_{\text{loc}}(\pi)$ being equivalent at a point $x_0 \in M^n$ is equivalent to the statement that their difference: $(s_1 - s_2) \in \Gamma(\pi)$ has a zero at x_0. Which we can write as

$$\begin{aligned}(s_1 - s_2) \in \mu_{x_0}\Gamma(\pi) := \{s \in \Gamma_{\text{loc}}(\pi) | \exists r \in \Gamma_{\text{loc}}(\pi), \exists \mu \in C^{\infty}_{\text{loc}}(M^n) \\ \text{such that} \quad \mu(x_0) = 0, s = \mu \cdot r\}\end{aligned} \tag{3.1}$$

Now we define that two sections $s_1, s_2 \in \Gamma_{\text{loc}}$ are tangent at $x_0 \in M^n$ with tangency order $k \geq 0$ in terms of it's Taylor expansion. Therefore we define two sections to be tangent to each other of order k if their Taylor expansion till order k is the same. Which we formally state in the definition beneath.

Definition 3.1. two sections $s_1, s_2 \in \Gamma_{\text{loc}}$ are tangent at $x_0 \in M^n$ with tangency order $k \geq 0$ if $(s_1 - s_2)(x) = \mathcal{O}(|x - x_0|^k)$ for all x near $x_0 \in M^n$. By convention two sections are tangent of order 0 if their values at x_0 only coincide.

Now we have a look at the following equivalence classes of sections at a point

$$[s]^k_{x_0} := \{\Gamma(\pi)/(\mu^{k+1}_{x_0}\Gamma(\pi)\} \tag{3.2}$$

This class marks the set of sections $s^* \in \Gamma_{\text{loc}}$ such that $s - s^* = \mathcal{O}(|x - x_0|^k)$. In other words if two sections are tangent with order k to each other then we call them equivalent in the equivalence class $[s]^k_{x_0}$. With these equivalence classes we can define the jet spaces.

Definition 3.2. The space $J^k(\pi)$ of k-th jets of sections for the vector bundle π is defined as

$$J^k(\pi) := \bigcup_{\substack{x \in M^n, \\ s \in \Gamma_{\text{loc}}(\pi)}} [s]^k_x \tag{3.3}$$

This definition states that in the k-th jet spaces two section are equivalent if their Taylor series till order k is the same. When we now take the limit of k to infinity we obtain the so called infinite jet spaces.

Definition 3.3. The infinite jet space $J^\infty(\pi)$ is defined as

$$J^\infty := \varprojlim_{k\to+\infty} J^k(\pi) \tag{3.4}$$

That is, the minimal object such that there exists an infinite chain of epimorphisms $\pi_{\infty,k} : J^\infty(\pi) \to J^k(\pi)$ for every $k \geq 0$. Also there is the vector bundle structure $\pi_{\infty,-\infty} : J^\infty(\pi) \to M^n$ such that the diagram with all admissible compositions o $\pi_{k+1,k}$, $\pi_{\infty,k}$ and $\pi_{\infty,-\infty}$ is commutative

With these definitions we prove the following lemma, which basically states that we can find a "smooth" point θ^∞ in the infinite jet space, which contains all the information about a point x and all it's partial derivatives.

Lemma 3.4 (Borel). *Every point $\theta^\infty \in J^\infty(\pi)$, where $\theta^\infty = (x, u, u_x, \cdots, u_\sigma, \cdots)$ is the infinite sequence of (collections of) real numbers; here $|\sigma| \geq 0$ and $u_\emptyset := u$, does encode a genuine infintely smooth local section with the values of its partial derivatives at x prescribed by θ^∞.*

The proof of this lemma for the case $m = 1$, $n = 1$ can be found in article [3]. Now let us take a step back and look at what we have constructed so far

Remark 3.5. Take a point $\theta^\infty \in J^\infty(\pi)$ then this point determines the class of sections $[s]^\infty_{x_0}$ of local sections $s \in \Gamma_{\text{loc}}(\pi)$ of the initial bundle π. Such that for $\theta^\infty = (x, u, u_x, \cdots, u_\sigma, \cdots)$, the partial derivatives are given by $\frac{\partial^{|\sigma|}}{\partial x^\sigma}|_{x_0}(s) = u_\sigma$, at the given point $x_0 \in M^n$ and remain continuous in a finite neighborhood $U_{x_0} \subset M^n$ of x_0. Also we have a map $\pi_{\infty,-\infty}$ which maps $\theta^\infty \mapsto x_0$. This gives us the local section $j_\infty(s); U_{x_0} \to J^\infty(\pi)$ of the bundle $\pi_{\infty,-\infty} : J^\infty \to M^n$ for all $x \in U_{x_0}$:

$$j_\infty(s)(U_{x_0}) = \{u = s(x), u_x = (\frac{\partial}{\partial x}s)(x), \cdots, u_\sigma = (\frac{\partial^{|\sigma|}}{\partial x^\sigma}s)(x), \cdots\}. \tag{3.5}$$

Note that $\pi_{\infty,-\infty} \circ (j_\infty(s))(x) := x \ \forall x \in U_{x_0}$. The lift $j_\infty : s \in \Gamma(\pi) \mapsto j_\infty(s) \in \Gamma(\pi_\infty)$ is called the infite jet of s. Since in terms of the Borel lemma indeed the infite jet space $J^\infty(\pi)$ is the space of infite jets $[s]^\infty_x = j_\infty(s)(x)$ for sections s of π. This space can be visualized in figure 2. In this figure we have also included the time. We will introduce the time component later in this section.

3.1 Functions on the infinite jet space

Now that we have constructed the infinite jet spaces, we are going to define some functions on this space. Including the total differential and the evolutionary derivation which in term allow us to construct infinitesimal symmetries. We can define the total derivative in an intuitive way by the chain rule since we know all the partial derivatives.

Let $\mathcal{F}(\pi)$ be the ring of smooth functions on $J^\infty(\pi)$, then for $f, g \in \mathcal{F}(\pi)$ and $r \in J^\infty(\pi)$ the following axioms are satisfied:

1. (addition) $(f + g)(x) = f(x) + g(x)$
2. (multiplication) $(fg)(x) = f(x)g(x)$
3. (identities) There exists a function $r(x) \in \mathcal{F}(\pi)$, such that $r(x) = r$.
4. (additive inverse) $(-f)(x) = -(f(x))$
5. (multiplication inverse) If $f(x) \neq 0$ for all x in some subset of $j^\infty(\pi)$, then we can define the multiplicative inverse of f, written f^{-1} as

$$f^{-1} = \frac{1}{f(x)} \tag{3.6}$$

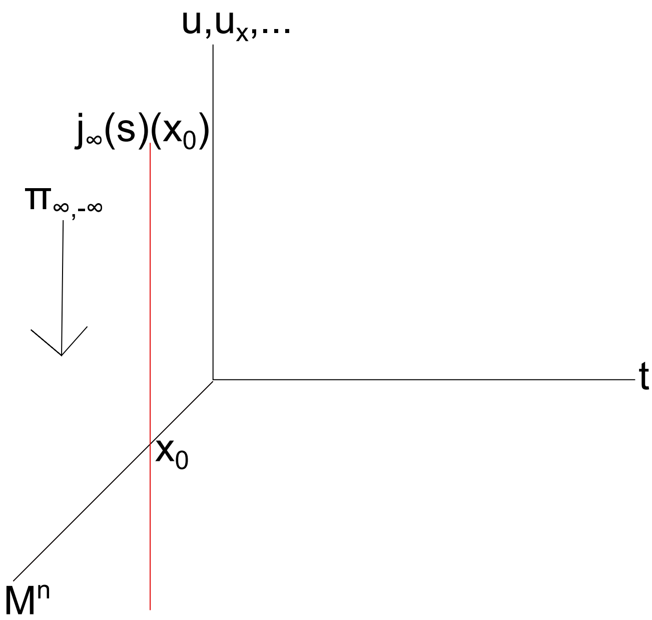

Figure 2: sketch of the space, where on the vertical axis the partial derivatives $[u]$ are and on the horizontal axis we have the time. On the axis towards us we have the postition, note that x denotes all position coördnates, i.e., x is a vector. Therefore also all $[u]$ are vector-like. Note also that we have a mapping $\pi_{\infty,-\infty}$ which take the entire jet and produces its position.

Let us define the total derivative in terms of the chain rule as stated before.

Definition 3.6. The total derivative $\frac{\mathrm{d}}{\mathrm{d}x} : \mathcal{F}(\pi) \to \mathcal{F}(\pi)$ is defined by the following formula

$$(\frac{\mathrm{d}}{\mathrm{d}x^i} f)(j_\infty(s))(x) = \frac{\partial}{\partial x^i}(j_\infty(s) * (f))(x), \quad f \in \mathcal{F}, \quad s \in \Gamma(\pi) \tag{3.7}$$

For $1 \leq i \leq n = \dim(M^n)$. In other words, the derivative $\frac{\mathrm{d}}{\mathrm{d}x}$ is determined by its application to the infinite jets $j_\infty(s)(\mathcal{U} - \alpha)$ of local sections on $\mathcal{U}_\alpha \subset M^n$.

To see what this definition has to do with the chain rule we have the following theorem

Theorem 3.7. *When we restrict our self to a section we have*

$$\frac{\mathrm{d}}{\mathrm{d}x^i} = \frac{\partial}{\partial x^i} + \sum_{j=1}^{m} \sum_{|\sigma| \geq 0} u^j_\sigma \cdot \frac{\partial}{\partial u^j_\sigma} \tag{3.8}$$

Proof. Because we talked about the chain rule so often this prove will most definitely involve the chain rule. Let us restrict our self to the section s and take f as a trial function.

$$\begin{aligned}(\frac{\mathrm{d}}{\mathrm{d}x^i} f)(j_\infty(s))(x) &= \frac{\mathrm{d}}{\mathrm{d}x^i}(f(x, u_x, u_{xx}, \cdots, u_\sigma, \cdots) \\ &= \frac{\partial}{\partial x^i} f + \frac{\partial}{\partial x^i}(u^j_x \frac{\partial}{\partial u^j_x} f + u^j_{xx} \frac{\partial}{\partial u^j_{xx}} f + \cdots) \\ &= \frac{\partial}{\partial x^i} + \sum_{j=1}^{m} \sum_{|\sigma| \geq 0} u^j_\sigma \cdot \frac{\partial}{\partial u^j_\sigma}\end{aligned} \tag{3.9}$$

Note that here in the second step we have used the "chain rule" by which we defined our total derivative. □

Now that we have defined the total derivative let us take a look at the k-forms. As we know a k-form has the following expression

$$\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \cdots \wedge \mathrm{d}x_k \tag{3.10}$$

where the 1-forms have the following properties when applied to the partial differential.

$$\begin{aligned} \mathrm{d}x^n(\frac{\partial}{\partial x^i}) &= 0 \quad \forall n = i \\ \mathrm{d}x^n(\frac{\partial}{\partial x^i}) &= 1 \quad \forall n \neq i \end{aligned} \tag{3.11}$$

Also we have the usual k-form multiplication rules, that is

$$\begin{aligned} \mathrm{d}x^i \wedge \mathrm{d}x^n &= -\mathrm{d}x^n \wedge \mathrm{d}x^i \qquad && \forall i \neq n \\ \mathrm{d}x^i \wedge \mathrm{d}x^n &= 0 && \forall i = n \end{aligned} \tag{3.12}$$

We will denote the space of all k-forms by $\Lambda^k(\pi)$. Now that we have defined k-forms and the total derivative. Let us define the so called horizontal derivative $\overline{\mathrm{d}}$.

Definition 3.8. Let $1 \leq i \leq n = \dim(M^n)$. The horizontal differential $\overline{\mathrm{d}}$ is given by the following formula:

$$\overline{\mathrm{d}} = \sum_{i=1}^{n} \mathrm{d}x^i \frac{\mathrm{d}}{\mathrm{d}x^i} \tag{3.13}$$

Now let us define the evolutionary derivation along the infinite jet bundle. We are going to define the evolutionary derivation in a similar from as we have for the total derivative. Before we define the evolutionary derivative we define the generating section ϕ such that

$$\dot{u} = \frac{\partial u}{\partial t} = \phi(x, t, u, u_x, \cdots) \tag{3.14}$$

Definition 3.9. The evolutionary derivation $\partial_\phi^{(u)}$, for a generating section ϕ and the corresponding fibre variable u is given by:

$$\partial_\phi^{(u)} = \sum_{|\sigma| \geq 0} D_\sigma(\phi) \frac{\partial}{\partial u_\sigma} \tag{3.15}$$

Note that here we have used the time component for the first time. We already included it in figure 2, now we have completed that figure. When we restrict our self onto a section s, we have

$$\dot{u}_\sigma = D_\sigma(\dot{u}) = D_\sigma(\phi) \tag{3.16}$$

Now let us define the so called Cartan forms ω_σ^j by the following formula

$$\omega_\sigma^j = \mathrm{d}u_\sigma^j - \sum_{i=1}^{n} u_{\sigma+i}^j \mathrm{d}x^i \tag{3.17}$$

From this Cartan forms we define the de Rham differential d_{dR}, such that

$$\begin{aligned} \mathrm{d}_{dR} &= \sum_{|\sigma| \geq 0} \omega_\sigma \frac{\partial}{\partial u_\sigma} \\ &= \sum_{i=1}^{n} \mathrm{d}x^i \frac{\mathrm{d}}{\mathrm{d}x^i} + \mathrm{d}_c \\ &= \overline{\mathrm{d}} + \mathrm{d}_c \end{aligned} \tag{3.18}$$

We call d_c the vertical derivative. From the equation above we see that they need to be given by the following definition

Definition 3.10. The vertical derivative d_c is given by the following formula

$$\mathrm{d}_c = \sum_{|\sigma| \geq 0} \omega_\sigma \frac{\partial}{\partial u_\sigma} \mathrm{d} - \overline{\mathrm{d}} \tag{3.19}$$

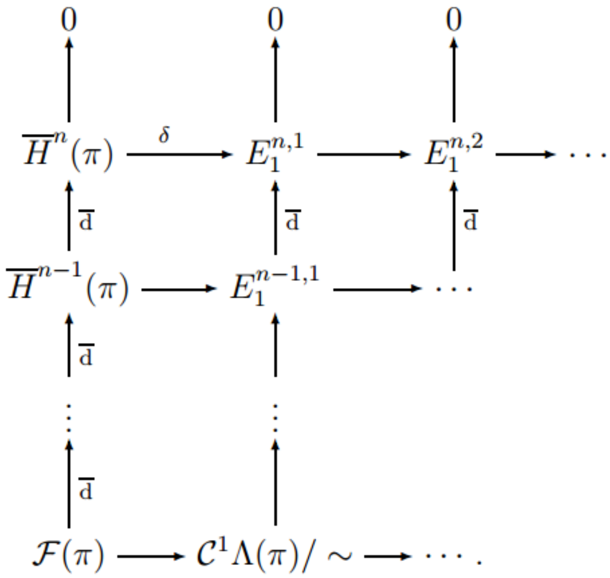

Figure 3: From [3], note that in the figure we have $\delta =: \mathrm{d}_c$:, whihc is the restriction of the Cartan differential onto the horizontal cohomology classes, such that the normalization throws all the derivatives off the Cartan forms by the multiple integration by parts.

This allows us to construct figure 3. Where an element $\eta \in \overline{H}^{n-1}(\pi)$ is given by:

$$\eta = \sum_{i=1}^{n} (-1)^{i+1} \eta_i \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^n \tag{3.20}$$

Here we use the notation such $\widehat{\mathrm{d}x^{i-1}}$ to denote that we are omitting the i-th differential form. To get an understanding of this new space in figure 3. Let us consider the following example:

Example 3.11. Take $n = 3$ then we have that $\eta = \eta_x \mathrm{d}y \wedge \mathrm{d}z - \eta_y \mathrm{d}x \wedge \mathrm{d}z + \eta_z \mathrm{d}x \wedge \mathrm{d}y \in \overline{H}^{3-1}$. We apply a horizontal differential $\overline{\mathrm{d}}$ to η

$$\begin{aligned}\overline{\mathrm{d}}\eta &= \frac{\mathrm{d}}{\mathrm{d}x}\eta_x \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z - \frac{\mathrm{d}}{\mathrm{d}y}\eta_y \mathrm{d}y \wedge \mathrm{d}x \wedge \mathrm{d}z + \frac{\mathrm{d}}{\mathrm{d}z}\eta_z \mathrm{d}z \wedge \mathrm{d}x \wedge \mathrm{d}y \\ &= (\frac{\mathrm{d}}{\mathrm{d}x}\eta_x + \frac{\mathrm{d}}{\mathrm{d}y}\eta_y + \frac{\mathrm{d}}{\mathrm{d}z}\eta_z)\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z\end{aligned} \tag{3.21}$$

Note that in the first step we have omitted the terms with the same 1-forms, since $\mathrm{d}x^i \wedge \mathrm{d}x^i$ and in the last step we used $\mathrm{d}x^i \wedge \mathrm{d}x^j = -\mathrm{d}x^j \wedge \mathrm{d}x^i$. Also we can now try to apply a horizontal differential once more. We then gain:

$$\overline{\mathrm{d}}(\overline{\mathrm{d}}\eta) = \overline{\mathrm{d}}(\frac{\mathrm{d}}{\mathrm{d}x}\eta_x + \frac{\mathrm{d}}{\mathrm{d}y}\eta_y + \frac{\mathrm{d}}{\mathrm{d}z}\eta_z)\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \quad = 0 \tag{3.22}$$

where again we have used $\mathrm{d}x^i \wedge \mathrm{d}x^i = 0$

This example can be generalized for arbitrary but finite n. First let's take an other look at figure 3. Note that in the figure applying a horizontal differential correspond to a vertical arrow up and a "vertical" differential corresponds to a horizontal arrow from left to right. Also we see by the example above that applying a horizontal differential on an element from $\overline{H}^n$ always gives 0.

In the example above we saw that for $n = 3$ all terms have a positive sign. This feature can be generalized to a more general form in the following lemma

Lemma 3.12. *For* $\eta \in \overline{H}^{n-1}$

$$\overline{\mathrm{d}}\eta = \sum_{i=1}^{n} D_{x^i}(\eta_i)\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n \tag{3.23}$$

Proof.

$$\begin{aligned}\overline{d}\eta &= D_x \sum_{i=1}^{n} \{(-1)^{i-1}\eta_i \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^{i-1} \wedge \mathrm{d}x^{i+1} \wedge \cdots \wedge \mathrm{d}x^n \\ &= D_{x^1}(\eta_1)\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n - D_{x^2}(\eta_2)\mathrm{d}x^2 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^3 \wedge \cdots \wedge \mathrm{d}x^n + \cdots + \\ &+ D_{x^n}(\eta_n)\mathrm{d}x^n \wedge \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^{n-1} \\ &= \sum_{i=1}^{n} D_{x^i}(\eta_i)\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n\end{aligned} \tag{3.24}$$

Where as in the example we have in the second step omitted the terms with double $\mathrm{d}x^i$. In the third step we switched the i-th term $i-1$ times resulting in an $(-1)^{i-1}$ term and since $(-1)^{i-1}\cdot(-1)^{1+i} = 1$ each term has a positive sign. □

Let us consider the following example to get a little bit more familiar with the vertical differential d_c. The lagrangians of a physical system live in the space $\overline{H}^n$. The (conserved) currents and the conservation laws live in the space $\overline{H}^{n-1}$. The equations of motion live in the space $E_1^{n,1}$. We will see later in the thesis why the conservation laws live the space $\overline{H}^{n-1}$. But in order to see why the equations of motion live in the space $E_1^{n,1}$. Take $\mathcal{L} \in \overline{H}^n$, $\mathcal{L} = \int L(x,t,[u])\mathrm{d}^n x$ where $\mathrm{d}^n x = \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$. Then apply a vertical differential d_c

$$\begin{aligned}\mathrm{d}_c\mathcal{L} &= \{\frac{\partial L}{\partial u}\omega_\emptyset + \frac{\partial L}{\partial u_x}\omega_x + \frac{\partial L}{\partial u_{xx}}\omega_{xx} + \cdots\}\mathrm{d}^n x \\ &\cong \{\frac{\partial L}{\partial u} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial L}{\partial u_x} + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\frac{\partial L}{\partial u_{xx}} - \cdots\}\omega_\emptyset \wedge \mathrm{d}^n x\end{aligned} \tag{3.25}$$

Here we used that by integration by parts we have

$$\frac{\partial L}{\partial u_x}\omega_x \wedge \mathrm{d}^n x \cong \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial L}{\partial u_x}\omega_\emptyset \wedge \mathrm{d}^n x \tag{3.26}$$

Where we assume that there are no boundary terms. Thus also

$$\frac{\partial L}{\partial u_{x^i}}\omega_{x^i} \wedge \mathrm{d}^n x \cong \frac{\mathrm{d}^i}{\mathrm{d}x^i}\frac{\partial L}{\partial u_{x^i}}\omega_\emptyset \wedge \mathrm{d}^n x \tag{3.27}$$

Also note that $\overline{\mathrm{d}}\mathcal{L} = 0$, thus the vertical derivative of $\mathcal{L}$ equal the de Rham derivative of $\mathcal{L}$. The term in equation (3.25) between $\{\quad\}$ is know as the famous Euler Lagrange equation. It looks a little bit different as you might be used to. But when we restrict our self to a section s we have as usual $u_{xx} = s''(x)$ and thus we need to rewrite the $\frac{\mathrm{d}}{\mathrm{d}x}$ terms into $\frac{\mathrm{d}}{\mathrm{d}t}$ and also x needs to be differentiated with respect to the time. Take u for example as the position x_i. Notation may be a bit confusing here but this results, omitting $\frac{\mathrm{d}^2}{\mathrm{d}t^2}$ and higher order terms into:

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \frac{\partial x}{\partial t}} \tag{3.28}$$

The reason that you are allowed to drop terms of $\frac{\mathrm{d}^2}{\mathrm{d}t^2}$ and higher is because often the lagrangian depends at most on a second derivative. Which means that $\frac{\partial \mathcal{L}}{\partial \frac{\partial^i x}{\partial t^i}} = 0$ for $i \geq 2$. Also you might be used that equation (3.28) equals zero, therefore we are going to look at the following theorem

Lemma 3.13. *If $\mathcal{L} = \overline{\mathrm{d}}\eta$, for $\eta \in \overline{H}^{n-1}$ and $\mathcal{L} \in \overline{H}^n$. Then $\mathrm{d}_c\mathcal{L} = 0$, i.e., the Euler Lagrange equation* (3.25) *equals zero. We write this as $\mathcal{E}_{EL}(\overline{\mathrm{d}}\eta) = 0$.*

Proof. If $\mathcal{L} = \overline{\mathrm{d}}\eta$ for an $\eta \in \overline{H}^{n-1}$ then we have from lemma 3.12 that

$$\mathcal{L} = \sum_{i=1}^{n} D_{x^i}(\eta_i)\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n \tag{3.29}$$

Let us now look at the specific case where $n = 1$ then we have

$$\mathcal{L} = \frac{\mathrm{d}}{\mathrm{d}x}\eta \tag{3.30}$$

The Euler Lagrange equation of $\mathcal{L} = \overline{\mathrm{d}}\eta$ then becomes

$$\begin{aligned}
\mathcal{E}_{\mathrm{EL}}(\overline{\mathrm{d}}\eta) &= \sum_{i=0}^{\infty}(-\frac{\mathrm{d}}{\mathrm{d}x})^i \circ \frac{\partial}{\partial u_i} \circ \frac{\mathrm{d}}{\mathrm{d}x}\eta \\
&= \sum_{i=0}^{\infty}(-\frac{\mathrm{d}}{\mathrm{d}x})^i \circ \frac{\partial}{\partial u_i} \circ \{\frac{\partial}{\partial x}\eta + \sum_{j=1}^{\infty} u_j \frac{\partial}{\partial u_{j-1}}\eta\} \\
&= \sum_{i=0}^{\infty}(-\frac{\mathrm{d}}{\mathrm{d}x})^i \circ \{\frac{\partial}{\partial u_i}\frac{\partial}{\partial x}\eta + \frac{\partial}{\partial u_{i+1}}\eta + \sum_{j=1}^{\infty} u_j \frac{\partial}{\partial u_i}\frac{\partial}{\partial u_{j-1}}\eta\} \\
&= 0
\end{aligned} \tag{3.31}$$

□

This last lemma allows us to proof the following theorem.

Theorem 3.14. *For $\eta \in \overline{H}^{n-1}$ and $\mathcal{L} \in \overline{H}^n$ we have that:*

$$\mathcal{E}_{EL}(\mathcal{L}) = \mathcal{E}_{EL}(\mathcal{L} + \overline{\mathrm{d}}\eta) \tag{3.32}$$

Meaning that the Euler Lagrange equation for $\mathcal{L}$ equals the Euler-Lagrange equation of $(\mathcal{L} + \overline{\mathrm{d}}\eta)$.

Proof. By lemma 3.13 we are done, if we can prove $\mathcal{E}_{\mathrm{EL}}(\mathcal{A} + \mathcal{B}) = \mathcal{E}_{\mathrm{EL}}(\mathcal{A}) + \mathcal{E}_{\mathrm{EL}}(\mathcal{B})$ for $\mathcal{A}, \mathcal{B} \in \overline{H}^n$, since $\mathcal{E}_{\mathrm{EL}}(D_x\eta) = 0$. This is true since

$$\begin{aligned}
\mathcal{E}_{\mathrm{EL}}(\mathcal{A} + \mathcal{B}) &\cong \{\frac{\partial(\mathcal{A} + \mathcal{B})}{\partial u} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial \mathcal{L}}{\partial u_x} + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\frac{\partial(\mathcal{A} + \mathcal{B})}{\partial u} - \cdots\}\omega_{\emptyset} \wedge \mathrm{d}^n x \\
&= \{\frac{\partial \mathcal{A}}{\partial u} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial \mathcal{A}}{\partial u_x} + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\frac{\partial \mathcal{A}}{\partial u} - \cdots\}\omega_{\emptyset} \wedge \mathrm{d}^n x \\
&+ \{\frac{\partial \mathcal{B}}{\partial u} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial \mathcal{B}}{\partial u_x} + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\frac{\partial \mathcal{B}}{\partial u} - \cdots\}\omega_{\emptyset} \wedge \mathrm{d}^n x \\
&= \mathcal{E}_{\mathrm{EL}}(\mathcal{A}) + \mathcal{E}_{\mathrm{EL}}(\mathcal{B})
\end{aligned} \tag{3.33}$$

□

Now we assume that we can describe a physical system $\mathcal{E}$ by finite linear differential equations $F = 0$, i.e., $\mathcal{E} = \{F = 0\}$. Then we can define the infinite prolongation of $\mathcal{E}$: $\mathcal{E}^{\infty}$ in the following way:

Definition 3.15. For a system of finite linear differential equations $\mathcal{E} = \{F = 0\}$ we define it's prolongation $\mathcal{E}^n$ as:

$$\mathcal{E}^n = \{F = 0, D_x(F) = 0, \cdots, D_n(F) = 0\} \tag{3.34}$$

And we define

$$\mathcal{E}^{\infty} = \lim_{n\to\infty} \mathcal{E}^n \tag{3.35}$$

4 Continuous Symmetries

Let us consider a continuous (classical) system, since these systems are continuous they admit an infinitesimal form. In this section we are going to have a look at these symmetries. As we have discussed before in this article a symmetry is a certain transformation. We want these symmetry transformations of $\mathcal{E}^{\infty} \subset J^{\infty}(\pi)$ to respect the correspondence $\frac{\mathrm{d}^{\tau}}{\mathrm{d}x^{\tau}}(u_{\sigma}) = u_{\sigma+\tau}$. That is, the fields we are interested are sections $j_{\infty}(s)$ of $\pi_{-\infty,\infty}$ to sections again. Therefore we want the field to preserve the distribution of the space spanned by the total derivative. These fields X are the field that satisfy the following relationship:

$$X = \sum_{i=1}^{n} a^i \cdot \frac{\mathrm{d}}{\mathrm{d}x^i} + \partial_\psi^{(u)} \tag{4.1}$$

This can be proven in the following way. Let us denote the space spanned by the total derivatives by $\mathcal{C}$, where

$$\mathcal{C} := \mathrm{span}_{\mathcal{F}(\pi)} \langle \frac{\mathrm{d}}{\mathrm{d}x^i} \rangle \tag{4.2}$$

Note that the total derivative maps from $\mathcal{F}(\pi)$ to $\mathcal{F}(\pi)$. Now we are interested in symmetries of this distribution $[X, \mathcal{C}] \subset \mathcal{C}$ but which do not belong to it: $x \notin \mathcal{C}$, we call this distribution the Cartan distribution. For this reason it is always possible to remove the horizontal part of the field. Consequently the field X we seek in a vertical field with respect to $\pi_{-\infty,\infty}$. This can be reduced to the following condition for X in order to be a symmetry of $\mathcal{C}$:

$$[X, \frac{\mathrm{d}}{\mathrm{d}x^i}] = 0, \quad 1 \leq i \leq n \tag{4.3}$$

Now we show that it is always possible to remove the horizontal part, i.e., the horizontal derivatives induce a trivial transformation. We claim that the fields $X = a^i \cdot \frac{\mathrm{d}}{\mathrm{d}x^i} \in \mathcal{C}$ induce the trivial transformations. This can be easily seen with the relationship (4.3) above. Therefore we mod these equations out. This motivates us to consider the space of evolutionary vector fields which preserve the Cartan distribution {equation (4.3)} and preserve the infinite prolongation $\mathcal{E}^\infty$ of a given system $\mathcal{E}$. Now let us define a popper infinitesimal symmetry by the following theorem. This theorem is proven in article [3], but let us now except is as a definition.

Theorem 4.1. *The restriction of the $\partial_\phi^{(u)}$ onto $\mathcal{E}^\infty$ is a* proper infinitesimal symmetry *of the equation $\mathcal{E} = \{F = 0\}$ if and only if*

- *the determining equation*

$$\partial_\phi^{(u)}|_{\mathcal{E}^\infty}(F) \doteq 0 \tag{4.4}$$

  We denote the symbol $\doteq$ for the equality which is valid on-shell, i.e., on $\mathcal{E}^\infty$.

- *Besides, there is a linear total differential operator $\nabla = \nabla_\phi$ (moreover, linear also in ϕ) such that*

$$\partial_\phi^{(u)}(F) = \nabla_\phi(F) \quad on \quad J^\infty(\pi) \tag{4.5}$$

- *The space of proper infinitesimal symmetries $\partial_\phi^{(u)}$ retain the Lie algebra structure of vector fields. We denote by sym $(\mathcal{E})$ the Lie algebra of proper infinitesimal symmetries for $\mathcal{E}$.*

This gives us that the symmetries are given by

$$X = \sum_{i=1}^{n} a^i \cdot \frac{\mathrm{d}}{\mathrm{d}x^i} + \partial_\psi^{(u)} \tag{4.6}$$

4.1 Conservation Laws

In view of Noether's laws let us also look at conserved charges which we will define in the following way.

Definition 4.2. A conserved current η for a system $\mathcal{E}$ is the continuity equation

$$\sum_{i=1}^{n} \frac{\mathrm{d}}{\mathrm{d}x^i}|_{\mathcal{E}^\infty}(\eta_i) \doteq 0 \quad \text{on} \quad \mathcal{E}^\infty \tag{4.7}$$

Where $\eta_i(x, [u])$ are the coefficients of the horizontal $(n-1)$-form

$$\eta = \sum_{i=1}^{n} (-1)^{i+1} \eta_i \mathrm{d}x^1 \wedge \cdots \wedge \widehat{\mathrm{d}x^i} \wedge \cdots \wedge \mathrm{d}x^n \in \overline{\Lambda}^{n-1}(\pi) \tag{4.8}$$

The conservation of η is the equality

$$\overline{\mathrm{d}}|_{\mathcal{E}^\infty} \eta \doteq 0 \tag{4.9}$$

i.e., the form η is $\overline{\mathrm{d}}$-closed on-shell

This definition makes sense because if the total derivative in all direction for a continuous function, the current, equals zero. Then we naturally define that current is conserved, since it doesn't change. Now let us consider the following example to motivate the following definition for a conservation law.

Example 4.3. For simplicity let us take $n = 1$. Consider a system of ordinary differential equations $\mathcal{E}$ and a function $\mathcal{C}(t, [u]) \in \mathcal{F}_k(\pi) \to \overline{\Lambda}^{n-1}(\pi)$ such that $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{C}(t, [u]) \doteq 0$ by virtue of $\mathcal{E}$. This means that $\mathcal{C}(t, [s]) = \mathrm{const}(s)$ for every solution s of $\mathcal{E}$. In other words the conserved quantity $\mathcal{C}$ is the first integral of the equation $\mathcal{E}$.

From this example we see that the conservation laws live in $\overline{H}^{n-1}(\mathcal{E})$, this motivates us for the following defintition. This definition is quite similar to definition 4.2 for a conserved current.

Definition 4.4. A conservation law $\int \eta \in \overline{H}^{n-1}(\mathcal{E})$ for an equation $\mathcal{E}$ is the equivalence class of conserved currents

$$\overline{\mathrm{d}}|_{\mathcal{E}^\infty}(\eta) \doteq 0 \quad \text{on} \quad \mathcal{E}^\infty \tag{4.10}$$

modulo the globally defined exact currents $\overline{\mathrm{d}}\xi \in \int 0$. That is we mod out the trivial conserved currents.

5 Euler-Lagrange equations

In this section we are going to bring together our concepts of symmetry and conservation. This fundamental relation of nature is expressed in the first and second Noether theorem. The first Noether theorem states that for each symmetry of the lagrangian there is a conserved current and thus a conserved charge. Where the second Noether theorem tells you how to find all conserved currents. In this thesis we will only present the first Noether theorem. The second can be found in article [3]. Also we will not prove the general form of the first Noether theorem in this article, this also can be found in article [3]. However in the next section, section 6, we will state and prove the first Noether theorem in terms of the four vector notation.

Consider diagram 3, here we see that it remains in our power to pick an element from $\overline{H}n(\pi)$ and apply δ, which is the function which is restricted to the Cartan differential. Note that we have $\mathrm{d}_c u_\sigma = \mathrm{d}_c(\frac{\mathrm{d}^\sigma}{\mathrm{d}x^\sigma}(u)) = \frac{\mathrm{d}^\sigma}{\mathrm{d}x^\sigma}(\mathrm{d}_c u_\emptyset)$. WE have the following convention $\delta =: \mathrm{d}_c$:,

$$\overrightarrow{\delta}\mathcal{L} = \frac{\overrightarrow{\delta\mathcal{L}}}{\delta u} \cdot \delta u \tag{5.1}$$

Then by equation (3.25) we have

$$\frac{\overrightarrow{\delta\mathcal{L}}}{\delta u} = \sum_{|\sigma|\geq 0} (-1)^\sigma \frac{\mathrm{d}^\sigma}{\mathrm{d}x^\sigma}\left(\frac{\overrightarrow{\partial L}}{\partial u_\sigma}\right), \quad \text{where} \quad \mathcal{L} = \int L(x, [u]) \mathrm{d}^n x \tag{5.2}$$

The system of Euler-Lagrange equations then becomes

$$\mathcal{E}_{EL} = \{F = \frac{\delta\mathcal{L}}{\delta u} = 0 | \mathcal{L} \in \overline{H}^n(\pi)\} \subset J^k(\pi) \tag{5.3}$$

Let us now define what is means for an evolutionary derivative to be a Noether symmetry. As it shall turn out to every Noether symmetry we have a conserved current by the Euler Lagrange equation since every Noether symmetry is also a symmetry of the system Euler Lagrange equations.

Definition 5.1. The evolutionary vector field $\partial_\phi^{(u)}$ is a *Noether symmetry* of the Lagrangian $\mathcal{L} \in \overline{H}^n(\pi)$ if the following equation holds

$$\partial_\phi^{(u)} \mathcal{L} = \int \bar{\mathrm{d}}\xi \quad \text{on} \quad J^\infty(\pi), \quad \xi \in \overline{\Lambda}^{n-1}(\pi) \tag{5.4}$$

We denote the space of all symmetries of the lagrangian $\mathcal{L}$ by $\mathrm{sym}(\mathcal{L})$

Theorem 5.2 (First Noether Theorem)**.** *Let $\mathcal{E}_{EL} = \{F = \frac{\delta \mathcal{L}}{\delta u} = 0 \in P \simeq \widehat{\varkappa(\pi)} | \mathcal{L} \in \overline{H}^n(\pi)\}$ be a system of Euler-Lagrange equations. A section $\phi \in \varkappa(\pi)$ is a Noether symmetry of the Lagrangian $\mathcal{L}$ if and only if $\phi \in \varkappa(\pi) \simeq \hat{P}$ is the generating section of a conserved current $\eta \in \overline{\Lambda}^{n-1}(\pi)$ for the Euler-Lagrange equation $\mathcal{E}_{EL}$:*

$$\phi \in sym(\mathcal{L}) \quad \Longleftrightarrow \quad \exists \eta \in \overline{\Lambda}^{n-1}(\pi) : \bar{\mathrm{d}}\eta = \langle 1, \nabla(F) \rangle, \quad \phi = \nabla^\dagger(1) \tag{5.5}$$

where $\nabla \in \mathcal{C}Diff(P, \overline{\Lambda}^{n-1}(\pi))$

6 4-Vector Notation

Now we are going to introduce the usual notation which we use in Quantum Field Theory. Let us consider a Miskowski space with p time dimensions and q space dimension, where $p + q = n$. When we write A^μ, we use a short notation for writing $A^{t_1}, \cdots, A^{t_p}, A^{x_1}, \cdots, A^{x_q}$, where t_i and x_i are respectively the i-th time component en the i-th space component. You can visualize this a vector. Let us now consider $A^{\mu\nu}$, where μ and ν run again from 1 to n, here we mean all combinations of μ and ν, which you can visualize as square matrix of size n. In principle we could have $A^{\mu_1 \cdots \mu_m}$, where for $m \geq 3$ you cannot easily visualize this. More information on this notation can be found in article [4] and [5].

In Miskowski space we have the following metric $\eta_{\mu\nu} = \eta^{\mu\nu} = \mathrm{diag}(+1, \cdots, +1, -1, \cdots, -1)$. Here all the time coördinates have a positive sign and all the space coördinates have a negative sign. This metric can be used to lower and raise indices, i.e.,

$$\eta_{\mu\nu} A^\mu = A_\nu \tag{6.1}$$

As the equation above shows, when we multiply a vector by the metric it's indices is raised or lowered and changed by the other index of the metric. Now when we multiply two tensors with the same index, where one has an index up and the other has an index down we imply a summation of the index. In terms of formula this can be written as:

$$A_\mu B^\mu = \sum_{i=1}^{n} A_i \cdot B^i \tag{6.2}$$

This notation is known as Einstein's implicit summation convention. Also let us introduce the notation $\partial_\mu \psi(x) = \frac{\partial}{\partial x^\mu} \psi(x)$. In most physical systems we don't work with an arbitrary number of dimension, instead we work in a Miskowski $3+1$ space, denoted as $M^{3,1}$. The components are then usually labelled (t, x, y, z), where obviously t is time component and x, y, z are the space components. Using this notation we can rewrite the first Noether theorem.

Theorem 6.1. *For a Lagrangian $\mathcal{L}(x, \psi, \partial_\mu \psi)$ and for an infinitesimal transformation $\phi(x) \to \phi(x) + \delta\phi(x)$, where $\delta\phi = X(\phi)$. We say that this transformation is a symmetry, as in definition 5.1, if the Lagrangian changes at most by a total derivative, i.e.,*

$$\delta \mathcal{L} = \partial_\mu F^\mu \tag{6.3}$$

Here again we use that $\mathcal{L} \to \mathcal{L} + \delta\mathcal{L}$. Then there exists a current j^μ such that $\partial_\mu j^\mu = 0$ with

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} X(\phi) - F^\mu(\phi) \tag{6.4}$$

Proof. Let us consider the change of the Lagrangian.

$$\begin{aligned}\delta\mathcal{L} &= \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu(\delta\phi) \\ &= \{\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\}\delta\phi + \partial_\mu(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi)\end{aligned} \tag{6.5}$$

In the first step we have used the chain rule and in the second step we used the product rule. When the Euler-Lagrange equations are satisfied, that is, we are on shell. Then the term between { } is zero, thus implying

$$\delta\mathcal{L} = \partial_\mu(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi) \tag{6.6}$$

When we now define

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}X(\phi) - F^\mu(\phi) \tag{6.7}$$

Then we see that

$$\begin{aligned}\partial_\mu j^\mu &= \partial_\mu\{\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}X(\phi)\} - \partial_\mu F^\mu(\phi) \\ &= \partial_\mu\{\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\} - \partial_\mu F^\mu(\phi) \\ &= \partial_\mu\{\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu\delta\phi - \partial_\mu F^\mu(\phi) \\ &= \partial\mathcal{L} - \partial_\mu F^\mu(\phi) \\ &= 0\end{aligned} \tag{6.8}$$

Here in the third step we have used the product rule and in the last step we used the hypothesis that the lagrangian changes at most by a total derivative, i.e., $\delta\mathcal{L} = \partial_\mu F^\mu$. Which gives our desired result, on shell. □

We are going to use this form of Noether's theorem to proof our main question for the thesis. Since now we know that system is scale invariant if the infinitesimal scale transformation is a symmetry of the lagrangian. The same holds for all other infinitesimal transformations. But before we go to the main question of this thesis let us first take a look at the following lemma

Lemma 6.2. *The divergence of an anti symmetric tensor $A^{\mu\nu} = -A^{\nu\mu}$ equals zero.*

$$\partial_\mu A^{\mu\nu} = 0 \tag{6.9}$$

Proof.

$$\begin{aligned}\partial_\mu A^{\mu\nu} &= \frac{1}{2}\partial_\mu(A^{\mu\nu} - A^{\nu\mu}) \\ &= \frac{1}{2}(\partial_\mu A^{\mu\nu} - \partial_\mu A^{\nu\mu}) \\ &= 0\end{aligned} \tag{6.10}$$

□

With this lemma we can prove the following theorem

Theorem 6.3. *Suppose $J^{\mu\nu}$ is a conserved tensor, i.e., $\partial_\mu J^{\mu\nu} = 0$ and given by*

$$J^{\mu\nu} = J_+^{\mu\nu} + J_-^{\mu\nu} \tag{6.11}$$

where $J_-^{\mu\nu}$ is anti symmetric. Then $J_+^{\mu\nu}$ is also a conserved tensor.

Proof. Since $J^{\mu\nu}$ is conserved we have $\partial_\mu J^{\mu\nu} = 0$, and form lemma 6.2 we have that $\partial_\mu J_-^{\mu\nu} = 0$. And since

$$\partial_\mu J^{\mu\nu} = \partial_\mu J_+^{\mu\nu} + \partial_\mu J_-^{\mu\nu} \tag{6.12}$$

We also have that $\partial_\mu J_+^{\mu\nu} = 0$. And thus also J_+^μ is also a conserved tensor. □

This theorem states that out of a conserved current we can always define a new conserved current by dropping out the anti-symmetric part. The following theorem will also be useful in redefining the current:

Theorem 6.4. *For a second order tensor $T^{\mu\nu}$ it is always possible to split it into a symmetric $T_+^{\mu\nu}$ and an anti-symmetric part $T_-^{\mu\nu}$, such that $T^{\mu\nu} = T_+^{\mu\nu} + T_-^{\mu\nu}$.*

Proof. This is always possible due to the following equality

$$T^{\mu\nu} = \frac{1}{2}[T^{\mu\nu} + T^{\nu\mu}] + \frac{1}{2}[T^{\mu\nu} - T^{\nu\mu}] \tag{6.13}$$

Thus we define $T_+^{\mu\nu}$

$$T_+^{\mu\nu} := \frac{1}{2}[T^{\mu\nu} + T^{\nu\mu}] = T_+^{\nu\mu} \tag{6.14}$$

And $T_-^{\mu\nu}$

$$T_-^{\mu\nu} := \frac{1}{2}[T^{\mu\nu} - T^{\nu\mu}] = -T_-^{\nu\mu} \tag{6.15}$$

□

7 Conformal transformations

This section is based upon the notes from article [6]. Let us consider the space $\mathbb{R}^n$ with the metric $\eta_{\mu\nu}$ with p time dimensions and q space dimensions ($p + q = n$). Also we write an infinitesimal line element $ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu$. We define the conformal group as the subgroup which leaves the metric invariant up to a scale change, i.e.,

$$\eta_{\mu\nu}(x) \to \eta'_{\mu\nu}(x') = \Omega(x)\eta_{\mu\nu}(x) \tag{7.1}$$

These are consequently the transformations that conserve the angle between two vectors. Since if you take two vectors v and w then the angle between the two vectors is defined as $\frac{v \cdot w}{\sqrt{v^2 w^2}}$, where $v \cdot w = \eta_{\mu\nu}(x)v^\mu w^\nu$. And the angle between two vectors is independent of the metric, since

$$\begin{aligned} \frac{\mathrm{d}v \cdot \mathrm{d}w}{\sqrt{\mathrm{d}v^2 \mathrm{d}w^2}} &= \frac{\eta_{\mu\nu}(x)\mathrm{d}v^\mu \mathrm{d}w^\nu}{\sqrt{\eta_{\mu\nu}(x)\eta_{\rho\sigma}(x)\mathrm{d}x^\mu \mathrm{d}x^\nu \mathrm{d}x^\rho \mathrm{d}x^\sigma}} \\ &= \frac{\mathrm{d}v^\mu \mathrm{d}w^\nu}{\sqrt{(\mathrm{d}x^\mu)^2(\mathrm{d}x^\nu)^2}} \end{aligned} \tag{7.2}$$

Thus the angle between two vectors is conserved under a conformal transformation. Next we look at the infinitesimal generators of the conformal group. The infinitesimal generators can be determined by considering the infinitesimal coördinate transformation $x^\mu \to x'^\mu = x^\mu + \epsilon^\mu$. For a general change of coordinates $x \to x'$, we have:

$$\eta_{\mu\nu}(x) \to \eta'_{\mu\nu}(x') = \frac{\partial x'^\alpha}{\partial x^\mu}\frac{\partial x'^\beta}{\partial x^\nu}\eta_{\alpha\beta}(x) \tag{7.3}$$

Then for the coördinate transformation $x^\mu \to x'^\mu = x^\mu + \epsilon^\mu$ we have:

$$\frac{\partial x'^\alpha}{\partial x^\mu}\frac{\partial x'^\beta}{\partial x^\nu}\eta_{\alpha\beta}(x) = (\delta^\alpha_\mu + \frac{\partial \epsilon^\alpha}{\partial x^\mu})(\delta^\beta_\nu + \frac{\partial \epsilon^\beta}{\partial x^\nu})\eta_{\alpha\beta}(x) \tag{7.4}$$

which gives us:

$$\mathrm{d}s^2 \to \mathrm{d}s^2 + (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu)\mathrm{d}x^\mu \mathrm{d}x^\nu \tag{7.5}$$

Where we can drop the $\mathcal{O}(\epsilon^2)$, since we are considering infinitesimal transformations. In order to have a conformal transformation, thus to satisfy equation (7.1), we must require $\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu$ to be proportional to $\eta_{\mu\nu}$, i.e.,

$$\begin{aligned} \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu &= A\eta_{\mu\nu} \\ \Rightarrow 2\partial \cdot \epsilon &= An \\ \Rightarrow A &= \frac{2}{n}\partial \cdot \epsilon \end{aligned} \tag{7.6}$$

Where we have multiplied both sides with $\eta^{\mu\nu}$, for notation we have used $(\eta_{\mu\nu})^2 = \eta_{\mu\nu}\eta^{\mu\nu} = n$ and $\partial^\mu \epsilon_\mu = \partial \cdot \epsilon$.

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{n}(\partial \cdot \epsilon)\eta_{\mu\nu} \tag{7.7}$$

Note that we can rewrite equation (7.5), using $\mathrm{d}s^2 = \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$, in the following way

$$\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu \rightarrow (\eta_{\mu\nu} + \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu)\mathrm{d}x^\mu \mathrm{d}x^\nu \tag{7.8}$$

From this we conclude that

$$\begin{aligned} \eta'_{\mu\nu} &= \eta_{\mu\nu} + \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu \\ &= \eta_{\mu\nu} + \frac{2}{n}(\partial \cdot \epsilon)\eta_{\mu\nu} \\ &= \Omega(x)\eta_{\mu\nu}(x) \end{aligned} \tag{7.9}$$

Where $\Omega(x) = 1 + \frac{2}{n}(\partial \cdot \epsilon)$. Now we multiply equation (7.7) by ∂^μ

$$\begin{aligned} \Box \epsilon_\nu + \partial_\nu \partial \cdot \epsilon &= \frac{2}{n}\partial_\nu \partial \cdot \epsilon \\ \Rightarrow \Box \epsilon_\nu &= (\frac{2}{n} - 1)\partial_\nu \partial \cdot \epsilon \\ \Rightarrow n\Box \epsilon_\nu &= (2-n)\partial_\nu \partial \cdot \epsilon \end{aligned} \tag{7.10}$$

Now we multiply by ∂_μ

$$\begin{aligned} n\Box \partial_\mu \epsilon_\nu &= (2-n)\partial_\mu \partial_\nu \partial \cdot \epsilon \\ \Rightarrow \frac{2}{n}\Box(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) &= (2-n)\partial_\mu \partial_\nu \partial \cdot \epsilon \\ \Rightarrow \Box \partial \cdot \epsilon &= (2-n)\partial_\mu \partial_\nu \partial \cdot \epsilon \end{aligned} \tag{7.11}$$

Where as usual we define $\Box = \partial^\mu \partial_\mu$. In the last step we have used equation (7.7). Rewriting this equation gives us:

$$(\eta_{\mu\nu}\Box + (n-2)\partial_\mu \partial_\nu)\partial \cdot \epsilon = 0 \tag{7.12}$$

If $n > 2$ we require that the third derivatives of ϵ must vanish, in order to satisfy equations (7.7) and (7.12). Such that ϵ is at most quadratic in x. This results in the following possibilities for ϵ

1. $\epsilon^\nu = a^\nu$, ordinary translations independent of x.
2. $\epsilon^\nu = \omega^\mu{}_\nu x^\nu$ (where ω is antisymmetric), the rotations.
3. $\epsilon^\nu = \lambda x^\nu$, the scale transformation.
4. $\epsilon^\nu = b^\nu x^2 - 2x^\nu b \cdot x$, the special conformal transformations.

Let us now turn these infinitesimal transformation into finite transformations. First of all we find the Poincaré group

$$\begin{aligned} x &\rightarrow x' = x + a \\ x &\rightarrow x' = \Lambda x \quad (\Lambda^\mu{}_\nu \in SO(p,q)) \end{aligned} \tag{7.13}$$

Since the infinitesimal form of the translations simply is: a^μ and for the Lorentz transformation the infinitesimal form is: $\delta^\mu{}_\nu + \omega^\mu{}_\nu$. Note that here we have $\Omega = 1$. Adjoined to it, we have dilatations

$$x \rightarrow x' = \lambda x \tag{7.14}$$

The infinitesimal transformations here are, due to Taylor expansion, $x^\mu \rightarrow x^\mu + \lambda x^\mu$. Note also that here we have $\Omega = \lambda^{-2}$. For the same reasoning as before we have that the special conformal transformations are:

$$x \rightarrow x' = \frac{x + bx^2}{1 + 2b \cdot x + b^2 x^2} \tag{7.15}$$

Here $\Omega = (1 + 2b \cdot x + b^2 x^2)^2$. Now we see that in order for scale invariance to imply invariance under the full conformal group. There must be an extra condition such that the lagrangian is also invariant under the special conformal transformations.

8 From scale invariance to conformal invariance

In this section we are going to find necessary and sufficient conditions such that scale invariance implies invariance under the full conformal group. We are going to do this based on the article [7] and assuming that the lagrangian is renormalizable. As we have seen before the infinitesimal scale transformations are:

$$x \to x' = x + \lambda x \tag{8.1}$$

We are in particular interested in the case when the field infinitesimal transforms linear, i.e.,

$$\phi(x) \to \phi'(x) = e^{\lambda d}\phi(e^{\lambda x} \tag{8.2}$$

for an arbitrary matrix d. Now we have

$$\begin{aligned} \phi'(x') &= (1+\lambda d)(\phi(x) + \partial_\mu \phi(x)\lambda x^\mu) \\ &= \phi(x) + \lambda(d + x_\mu \partial^\mu)\phi(x) \\ \Rightarrow \delta\phi(x) &= (d + x_\mu \partial^\mu)\phi(x) \end{aligned} \tag{8.3}$$

Suppose we have a Lagrangian that is invariant under the scale transformation: $\phi \to \phi + \delta\phi$, with $\delta\phi = D \cdot \phi + x^\mu \partial_\mu \phi$. Here we have D as the dimension matrix, i.e., a generalization of d above and $\phi \sim \begin{pmatrix} \text{spin } 0 \\ \text{spin } 1/2 \\ \text{spin } 1 \end{pmatrix}$. Then the associated scale current is given by

$$\begin{aligned} J^\mu &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi + x^\mu \mathcal{L} \\ &= \pi^\mu \cdot \delta\phi + x^\mu \mathcal{L} \\ &= \pi^\mu \cdot D \cdot \phi + \pi^\mu x^\nu \partial_\nu \phi - x^\mu \mathcal{L} \\ &= \pi^\mu \cdot D \cdot \phi + x_\nu T_c^{\mu\nu} \end{aligned} \tag{8.4}$$

Here we have $\pi^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}$. Also we have defined the canonical energy-momentum tensor $T_c^{\mu\nu} = \pi^\mu \partial^\nu \phi - \eta^{\mu\nu}\mathcal{L}$ which differs from the conventional symmetric energy-momentum tensor [7].

$$\begin{aligned} T^{\mu\nu} &= \pi^\mu \partial^\nu \phi - \eta^{\mu\nu}\mathcal{L} + \frac{1}{2}\partial_\lambda(\pi^\lambda \cdot \Sigma^{\mu\nu}\phi - \pi^\mu \cdot \Sigma^{\lambda\nu}\phi - \pi^\nu \cdot \Sigma^{\lambda\mu}\phi) \\ &= T_c^{\mu\nu} + \frac{1}{2}\partial_\lambda(\pi^\lambda \cdot \Sigma^{\mu\nu}\phi - \pi^\mu \cdot \Sigma^{\lambda\nu}\phi - \pi^\nu \cdot \Sigma^{\lambda\mu}\phi) \end{aligned} \tag{8.5}$$

Where the spin matrix $\Sigma^{\mu\nu}$ is defined by the transformation properties of the fields under an infinitesimal Lorentz transformation,

$$\delta^{\mu\nu}\phi = [x^\mu \partial^\nu - x^\nu \partial^\mu + \Sigma^{\mu\nu}]\phi \tag{8.6}$$

Now we are going to constuct an energy momentum tensor $\Theta^{\mu\nu}$ such that:

$$J^\mu = x_\nu \Theta^{\mu\nu} \tag{8.7}$$

which obeys the following relation:

$$\partial_\mu J^\mu = \Delta \tag{8.8}$$

We shall show that a sufficient and necessary condition for (8.7) to exist for some $\Theta^{\mu\nu}$ is

$$\pi^\mu \cdot D\phi + \pi_\lambda \cdot \Sigma^{\mu\nu}\phi = \partial_\lambda \sigma^{\mu\lambda} \tag{8.9}$$

where $\sigma^{\mu\lambda}$ is some tensor. Later we shall show that this condition is also a necessary and sufficient condition for a scale-invariant theory to be conformally invariant. Let us first rewrite J^μ from scale-invariance,

$$\begin{aligned} J^\mu &= \pi^\mu \cdot D \cdot \phi + x_\nu T_c^{\mu\nu} \\ &= \pi^\mu \cdot D \cdot \phi + x_\nu T^{\mu\nu} - \frac{1}{2}x_\nu \partial_\lambda(\pi^\lambda \cdot \Sigma^{\mu\nu}\phi - \pi^\mu \cdot \Sigma^{\lambda\nu}\phi - \pi^\nu \cdot \Sigma^{\lambda\mu}\phi) \\ &= \pi^\mu \cdot D \cdot \phi + x_\nu T^{\nu\mu} + \pi_\lambda \Sigma^{\mu\lambda}\phi - \frac{1}{2}\partial_\lambda\{x_\nu(\pi^\lambda \cdot \Sigma^{\mu\nu}\phi - \pi^\mu \cdot \Sigma^{\lambda\nu}\phi - \pi^\nu \cdot \Sigma^{\lambda\mu}\phi)\} \end{aligned} \tag{8.10}$$

We may discard the last term, since it is the divergence of an antisymmetric tensor. And thus by theorem 6.3 we can define a new conserved current without it. Therefore we redefine the current to be:

$$\begin{aligned} J^\mu &= \pi^\mu \cdot D \cdot \phi + x_\nu T^{\mu\nu} + \pi_\lambda \Sigma^{\mu\lambda}\phi \\ &= x_\nu T^{\mu\nu} + \partial_\nu \sigma^{\mu\nu} \end{aligned} \tag{8.11}$$

where $\partial_\nu \sigma^{\mu\nu} = \pi^\mu \cdot D \cdot \phi + \pi_\lambda \Sigma^{\mu\lambda}\phi$, without specifying the exact form of $\sigma^{\mu\nu}$ for now. Let us write $\sigma^{\mu\nu}$ in a symmetric $\sigma_+^{\mu\nu}$ and an antisymmetric part $\sigma_-^{\mu\nu}$ this is always possible by theorem 6.4.

$$\sigma^{\mu\nu} = \sigma_+^{\mu\nu} + \sigma_-^{\mu\nu} \tag{8.12}$$

By the same reasoning we define a new current without the anti-symmetric part

$$J^\mu = x_\nu T^{\mu\nu} + \partial_\nu \sigma_+^{\mu\nu} \tag{8.13}$$

Now we define

$$X^{\lambda\rho\mu\nu} = \eta^{\lambda\rho}\sigma_+^{\mu\nu} - \eta^{\lambda\mu}\sigma_+^{\rho\nu} - \eta^{\lambda\nu}\sigma_+^{\mu\rho} + \eta^{\mu\nu}\sigma_+^{\lambda\rho} - \frac{1}{3}\eta^{\lambda\rho}\eta^{\mu\nu}\sigma_{+\alpha}^{\alpha} + \frac{1}{3}\eta^{\lambda\mu}\eta^{\rho\nu}\sigma_{+\alpha}^{\alpha} \tag{8.14}$$

Then $\frac{1}{2}\partial_\lambda\partial_\rho X^{\lambda\rho\mu\nu}$ is symmetric and divergence-less. This follows because $\frac{1}{2}\partial_\lambda\partial_\rho X^{\lambda\rho\mu\nu} = \eta^{\mu\nu}\partial_\lambda\partial_\rho\sigma_+^{\lambda\rho}$. And as we know $\eta^{\mu\nu}$ is symmetric and divergence-less. This allows us to add this term to the energy momentum tensor without changing the four-momentum or the total angular momentum in the following way

$$\Theta^{\mu\nu} = T^{\mu\nu} + \frac{1}{2}\partial_\lambda\partial_\rho X^{\lambda\rho\mu\nu} \tag{8.15}$$

The reason that $\Theta^{\mu\nu}$ is also a conserved current is that we have $\partial_\mu(\frac{1}{2}\partial_\lambda\partial_\rho X^{\lambda\rho\mu\nu}) = 0$. It follows that

$$\begin{aligned} J^\mu &= x_\nu \Theta^{\mu\nu} + \partial_\nu \sigma_+^{\mu\nu} \\ &= x_\nu \Theta^{\mu\nu} - x_\nu \frac{1}{2}\partial_\lambda\partial_\rho X^{\lambda\rho\mu\nu} + \partial_\nu \sigma_+^{\mu\nu} \\ &= x_\nu \Theta^{\mu\nu} - \frac{1}{2}\partial_\lambda\partial_\rho(x_\nu X^{\lambda\rho\mu\nu}) + \frac{1}{2}(\eta_{\lambda\nu}\partial_\rho + \eta_{\rho\nu}\partial_\lambda)X^{\lambda\rho\mu\nu} + \partial_\nu \sigma_+^{\mu\nu} \end{aligned} \tag{8.16}$$

We note that $\frac{1}{2}\partial_\lambda\partial_\rho(x_\nu X^{\lambda\rho\mu\nu})$ is the divergence of an anti-symmetric tensor and therefore we redefine a new current without it. Also note that

$$\begin{aligned} \eta_{\rho\nu}\partial_\lambda X^{\lambda\rho\mu\nu} &= -\eta_{\rho\nu}\partial_\lambda(\eta^{\mu\nu}\sigma_+^{\lambda\rho}) \\ &= -\partial_\lambda \sigma_+^{\mu\lambda} \end{aligned} \tag{8.17}$$

Which follows straight from equation (8.14). Thus also $\eta_{\lambda\nu}\partial_\rho X^{\lambda\rho\mu\nu} = -\partial_\rho\sigma^{\mu\rho}$. Note here that after applying ∂_ρ we gain three terms two of which cancel against each other. Now we redefine the current to be

$$\begin{aligned} J^\mu &= x_\nu \Theta^{\mu\nu} + \frac{1}{2}(\eta_{\lambda\nu}\partial_\rho + \eta_{\rho\nu}\partial_\lambda)X^{\lambda\rho\mu\nu} + \partial_\nu \sigma_+^{\mu\nu} \\ &= x_\nu \Theta^{\mu\nu} \end{aligned} \tag{8.18}$$

Since the second and third term cancel. We have now achieved that we can write the scale current as $x_\nu\Theta^{\mu\nu}$ if equation (8.9) is satisfied. This is useful since when we now define

$$K^{\lambda\mu} = x^2\Theta^{\lambda\mu} - 2x^\lambda x_\rho \Theta^{\rho\mu} \tag{8.19}$$

then we see that

$$\partial_\mu K^{\lambda\mu} = 2x_\mu \Theta^{\lambda\mu} - 2x_\rho\Theta^{\rho\lambda} - 2x^\lambda \Theta_\rho^\rho = 0 \tag{8.20}$$

Where the first two terms cancel and the last terms equals zero because $\partial_\mu J^\mu = \Theta_\mu^\mu = 0$. Thus it seem that when equation (8.9) is satisfied scale invariance implis invariance under four other infinitesimal transformations. We are going to show that these four other infinitesimal transformations are the special conformal transformations.

The most general form of a renormalizable lagrangian is the sum of the free part and an interaction part,

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 \tag{8.21}$$

The free part is given by

$$\begin{aligned}\mathcal{L}_0 = &\frac{1}{2}\partial_\mu\phi^a\partial^\mu\phi^a - \frac{1}{2}(\mu_0^2)^a\phi^a\phi^a + \overline{\psi}^a(i\partial^\mu\gamma_\mu - m_0^a)\psi^a \\ &- \frac{1}{4}(\partial_\mu {A_\nu}^a - \partial_\nu {A_\mu}^a)^2 + \frac{1}{2}(M_0^2)^a {A_\mu}^a A^{\mu a}\end{aligned} \tag{8.22}$$

The interaction part is given by

$$\begin{aligned}\mathcal{L}_1 = &\alpha^a\phi^a + \beta^{abc}\phi^a\phi^b\phi^c + \lambda^{abcd}\phi^a\phi^b\phi^c\phi^d \\ &+ g^{abc}\overline{\psi}^a\psi^b\psi^c + ih^{abc}\overline{\psi}^a\gamma_5\psi^b\psi^c + e^{abc}\overline{\psi}^a\gamma^\mu\psi^b {A_\mu}^c \\ &+ 2f^{abc}(\partial^\mu\phi^a)\phi^b {A_\mu}^c + f^{abc}f^{ade}\phi^b\phi^d {A_\mu}^c A^{\mu e}\end{aligned} \tag{8.23}$$

We know that the special conformal transformations are given by $x^\mu \to x^\mu + \epsilon^\mu$, with $\epsilon^\mu = b^\mu x^2 - 2x^\mu b\cdot x$. We can denote this with two free indices as ${\delta_c}^\mu x^\mu = -2x^\mu x^\nu + \eta^{\mu\nu}x^2$. The scale transformation properties of field at arbitrary points, as we have seen before is given by

$$\delta\phi = D\phi + x^\mu\partial_\mu\phi \tag{8.24}$$

And for the conformal transformation of the fields we have

$${\delta_c}^\mu\phi = (2x^\mu x^\nu - \eta^{\mu\nu}x^2)\partial_\nu\phi + 2x^\mu D\phi + 2x_\nu\Sigma^{\mu\nu}\phi \tag{8.25}$$

Now for the conformal transformation of the lagrangian, dependent on the field ϕ and it's first derivative where the field ϕ could have spin $0, 1/2$ or 1, we have

$$\begin{aligned}{\delta_c}^\mu\mathcal{L}(x,\phi,\partial_\lambda\phi) &= {\delta_c}^\mu x^\nu\cdot\frac{\partial\mathcal{L}}{\partial x^\nu} + {\delta_c}^\mu\phi\cdot\frac{\partial\mathcal{L}}{\partial\phi} + {\delta_c}^\mu\partial_\lambda\phi\cdot\frac{\partial\mathcal{L}}{\partial(\partial_\lambda\phi)} \\ &= (2x^\mu x^\nu - \eta^{\mu\nu}x^2)\cdot\partial_\nu\mathcal{L} + \{(2x^\mu x^\nu - \eta^{\mu\nu}x^2)\partial_\nu\phi + 2x^\mu D\phi + 2x_\nu\Sigma^{\mu\nu}\phi\}\cdot\frac{\partial\mathcal{L}}{\partial\phi} \\ &+ [\partial_\lambda{\delta_c}^\mu\phi]\cdot\pi^\lambda \\ &= (2x^\mu x^\nu - \eta^{\mu\nu}x^2)\cdot\partial_\nu\mathcal{L} + \{(2x^\mu x^\nu - \eta^{\mu\nu}x^2)\partial_\nu\phi + 2x^\mu D\phi + 2x_\nu\Sigma^{\mu\nu}\phi\}\cdot\frac{\partial\mathcal{L}}{\partial\phi} \\ &+ [2(\eta^{\mu\lambda}x^\nu + \eta^{\nu\lambda}x^\mu - \eta^{\mu\nu}x^\lambda)\partial_\nu\phi + 2{\eta_\lambda}^\nu x^\mu D\partial_\nu\phi + 2D\phi \\ &+ 2x_\nu\Sigma^{\mu\nu}\partial_\lambda\phi + 2\Sigma^{\mu\nu}\phi]\pi^\lambda\end{aligned} \tag{8.26}$$

In the first step we have used the chain rule. The term in round brackets corresponds to ${\delta_c}^\mu x^\nu$, the term in curly brackets corresponds to ${\delta_c}^\mu\phi$ and the term between square brackets corresponds to $\partial_\lambda{\delta_c}^\mu\phi$. Also we have defined $\pi^\lambda = \frac{\partial\mathcal{L}}{\partial(\partial_\lambda\phi)}$.

Thus for a renormalizable lagrangian $\mathcal{L}$ we have that the change in the lagrangian due to a conformal transformation ($\delta_c^\mu\mathcal{L}$) is given by

$$\begin{aligned}\delta_c^\mu\mathcal{L} = &(2x^\mu x^\nu - \eta^{\mu\nu}x^2)\partial_\nu\mathcal{L} + 2\pi_\lambda\cdot(\eta^{\mu\lambda}x^\nu + \eta^{\nu\lambda}x^\mu - \eta^{\mu\nu}x^\lambda)\partial_\nu\phi \\ &+ 2x^\mu\frac{\partial\mathcal{L}}{\partial\phi}\cdot D\phi + 2x^\mu\pi^\nu\cdot D\partial_\nu\phi \\ &+ 2\pi^\mu\cdot D\phi + 2x_\nu\frac{\partial\mathcal{L}}{\partial\phi}\cdot\Sigma^{\mu\nu}\phi \\ &+ 2x_\nu\pi^\lambda\cdot\Sigma^{\mu\nu}\partial_\lambda\phi + 2\pi_\nu\cdot\Sigma^{\mu\nu}\phi\end{aligned} \tag{8.27}$$

For a Lorentz invariant lagrangian $\mathcal{L}$ we have

$$\frac{\partial\mathcal{L}}{\partial\phi}\cdot\Sigma^{\mu\nu}\phi + \pi^\lambda\cdot\Sigma^{\mu\nu}\partial_\lambda\phi - \pi^\nu\cdot\partial^\mu\phi + \pi^\mu\cdot\partial^\nu\phi = 0 \tag{8.28}$$

For a scale invariant lagrangian $\mathcal{L}$ we have

$$\frac{\partial \mathcal{L}}{\partial \phi} \cdot D\phi + \pi^\lambda \cdot D\partial_\lambda \phi + \pi^\lambda \cdot \partial_\lambda \phi = 4\mathcal{L} \tag{8.29}$$

Thus the conformal transformations of a Lorentz and scale invariant lagrangian $\mathcal{L}$ are given by

$$\begin{aligned} \delta_c^\mu \mathcal{L} &= (2x^\mu x^\nu - \eta^{\mu\nu} x^2)\partial_\nu \mathcal{L} + 4x^\mu \mathcal{L} + 2\pi^\mu \cdot D\phi + 2\pi_\nu \cdot \Sigma^{\mu\nu}\phi \\ &= \partial_\nu\{(2x^\mu x^\nu - \eta^{\mu\nu} x^2)\mathcal{L}\} + 2\pi^\mu \cdot D\phi + 2\pi_\nu \cdot \Sigma^{\mu\nu}\phi \end{aligned} \tag{8.30}$$

Thus in order for $\mathcal{L}$ to be conformally invariant, two independent conditions must hold: scale invariance, and

$$\pi^\mu \cdot D\phi + 2\pi_\nu \cdot \Sigma^{\mu\nu}\phi = \partial_\mu \sigma^{\mu\nu} \tag{8.31}$$

Since then we can write the change of the lagrangian due to a special conformal transformation as the total derivative of a tensor.

9 Renormalization

Quantum Field theory is where renormalization becomes truly interesting. Let us therefore consider a simple example from quantum electrodynamics (QED). Namely the process of an electron and a positron annihilating to create a muon-antimuon pair. We write this as $e^-e^+ \to \mu^-\mu^+$. We can visualise this process in terms of so called Feynman diagram, which is shown in figure 4.

The order of these expressions can be inferred by the number of vertices, and so in this process the order is e^2, where e is the charge of the particles. Now when we consider the second order in perturbation theory, which can be visualised by the Feynman diagram in figure 5. This diagram has two more vertices. Note that in principle we could add an arbitrary amount of loops such that there are infinite of such Feynman diagrams. When we create a loop, we create virtual particles, for example in the figure 5 we create a fermion anti-fermion pair. This pair has energy and momentum and due to the superposition principle we must integrate over all these momenta. Each loop involves an integral over four-momenta and each particle contributes a factor of inverse four-momentum to the integrand. Hence the "'blod" in figure 5 has an integral which goes as momentum squared, which diverges when we take the ultraviolet cutoff of the integral over momenta to infinity.

This method we used is called *power counting*; we count the powers of the internal momentum to deduce hoe the integral behaves when the momentum is large. With this method one derives the so called *superficial degree of divergence* of the Feynman diagrams. The latter gives the way the integral diverges at large momenta, unless a symmetry of the lagrangian lowers the divergence.

9.1 Renormalizability

Checking whether a lagrangian is renormalizable can be done by so called dimension counting. When we denote the smallest dimension of a coupling factor by $[g]$, then we can label the lagrangian in the

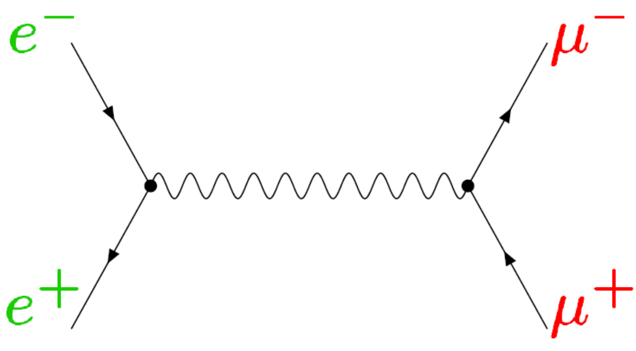

Figure 4: $e^-e^+ \to \mu^-\mu^+$ at leading order

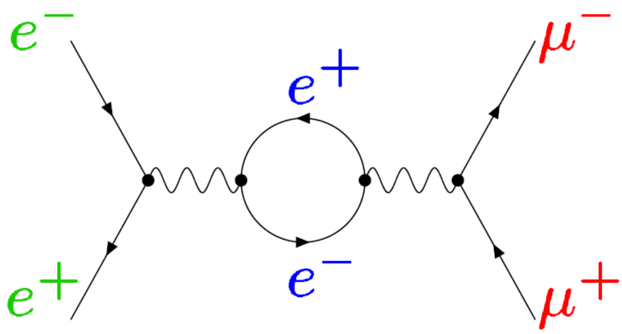

Figure 5: $e^-e^+ \to \mu^-\mu^+$ with one loop

following way

$$\begin{array}{ll} \text{Super renormalizable} & [g] > 0 \\ \text{Renormalizable} & [g] = 0 \\ \text{Non-renormalizable} & [g] < 0 \end{array} \tag{9.1}$$

Note that we measure the dimension in terms of energy. Let us illustrate how this process works by the following example.

Example 9.1. Consider a field whose lagrangian has the following form

$$\mathcal{L} = \overline{\psi}(\not{\partial} - m)\psi + c(\overline{\psi}\psi)^2 + d\phi^4 + e\phi^2 \tag{9.2}$$

Where c, d and e are the coupling factors. Here we have that ψ represents a fermion field and ϕ represents a scalar field. The first part $\overline{\psi}(\not{\partial} + m)\psi + e\phi^2$, is called the *free part of the lagrangian.* The other two terms are called *the interaction terms.* The action of a lagrangian must always be dimensionless. In $d = 4$ space-time dimensions we have that the action S for a free fermion field is given by

$$S = \int \overline{\psi} \not{\partial}\psi \mathrm{d}^4x \tag{9.3}$$

And the action of a free scalar field is given by

$$S = \frac{1}{2}\int \eta^{\mu\nu}\partial^\mu\phi\partial_\mu\phi \mathrm{d}^4x \tag{9.4}$$

By free we mean that there are no interaction terms present in the lagrangian. We know that the action always must be dimensionless. Also we see that the dimension of d^4x, equals -4, since length $\sim 1/\text{energy}$. Thus we see that the dimension of ψ equals $\frac{3}{2}$, which we denote by $[\psi] = \frac{3}{2}$, also we see that $[\phi] = 1$. Here we have used that the dimension of $[\not{\partial}] = 1$ and $[\partial_\mu] = 1$ since they both contain a first derivative.

Therefore also the dimension of the first interaction term $[c(\overline{\psi}\psi)^2] = 4$. From this we conclude that $[c] = -2$. Thus by (9.1) this term and as a consequence the lagrangian is non-renormalizable. Since one of the coupling factors has a negative dimension.

We could also try to apply this mechanism to check whether the prototype lagrangian of QCD is renormalizable.

Example 9.2. The lagrangian is given by the formula

$$\mathcal{L} = -\frac{1}{4}\mathrm{Tr}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}(i\gamma^\mu[\partial_\mu - gA_\mu]\psi - m_f)\psi \tag{9.5}$$

Which we rewrite as

$$\mathcal{L} = -\frac{1}{4}\mathrm{Tr}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}i\gamma^\mu\partial_\mu\psi - g\overline{\psi}i\gamma^\mu A_\mu\psi - m_f\overline{\psi}\psi \tag{9.6}$$

From this equation it follows that $[g] = 0$, thus $g\overline{\psi}i\gamma^\mu A_\mu\psi$ is a renormalizable term. Also we note that $[m_f] = 1$, thus $m_f\overline{\psi}\psi$ is a super renormalizable term.

Now that we have seen how to classify whether a lagrangian is renormalizable or not, it would only be unsatisfying to not know why this works. The basic reason why dimension counting works is the following. As I have told you before we want to absorbed the divergence of the integral in the physical parameters of the system. We can have infinitely many non-renormalizable terms by adding derivatives and fields. They are increasingly suppressed however by powers of the UV cut-off. Thus they go to zero if the cut-off goes to infinity. The problem is to have control over these terms when the cut-off remains finite because it corresponds to some physical scale. Imposing renormalizability avoids dealing with these infinitely many terms.

Renormalizability is thus a restrictive condition that reduces the most general renormalizable lagrangian in $d = 4$ to the one in (8.21). Renormalizability was used to answer the main question of this thesis, following article [7]. In fact, we have used the property that the form of the lagrangian (8.21) is not modified by quantum corrections, i.e., no new terms will appear.

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