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# Tautologies and Logical Equivalence in <br> Intuitionistic 

# Propositional Logic without Implication 

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#### Abstract

Intuitionistic Propositional Logic, or IPL, is based on the idea that a formula $A$ is valid when there is a proof for it and invalid when there is a proof for $\neg A$. As long as a formula has not been proven or disproven the truth value remains unknown. This means that the law of excluded middle, $A \vee \neg A$, is not a tautology.

In this thesis we will look a fragment of IPL, namely IPL without implication. The questions we would like to answer are: - When is a formula a tautology in IPL without implication? - How many formulas exist that are not logically equivalent in IPL without implication?

We will study IPL without implication in possible world semantics with the use of Kripke models. This will prove very useful, because we can identify a class of models with the property that every model in IPL without implication is similar to a model in that class. Two models are similar when exactly the same formulas are true. The models in this class will help answer our questions.

In chapter 1 we will start by investigating IPL in Kripke models. We will restrict ourselves to finite models that are of a special form, namely rooted. In a rooted model all the nodes are accessible from one special node, which is called the root.

Next, in chapter 2, we shall exclude implication. This leads to a method of simplifying finite rooted models without changing the truth values of formulas. The resulting models are called simple. Each model in IPL without implication is similar to a simple model.

Finally, in chapter 3 we will study some properties of simple models, which will lead to a formula to calculate the number of simple models. One model is of particular interest: the largest simple model. This model is the simple model with the maximal number of nodes. It is necessary and sufficient to check a formula in the largest model to see whether it is a tautology. We will prove that none of the simple models of depth 1 are similar, so in order to check whether two formulas are logically equivalent, we need all of them. We can use this result to calculate an upper bound for the number of formulas that are not logically equivalent.


## Contents

1 Intuitionistic Propositional Logic ..... 2
1.1 IPL in Kripke models ..... 2
1.2 Logical principles in IPL ..... 4
1.3 Rooted models ..... 8
1.4 Finite models ..... 9
1.5 Bisimulation ..... 10
1.6 Conclusion ..... 13
2 IPL without Implication ..... 14
2.1 Simplification of Kripke models in IPL without implication ..... 14
2.1.1 Omitting in-between nodes ..... 14
2.1.2 Omitting duplicate maximal nodes ..... 15
2.1.3 The simplified model $\tau(M)$ ..... 16
2.2 Conclusion ..... 19
3 Tautologies and Logical Equivalence in IPL without Implication ..... 20
3.1 Properties of simple models ..... 20
3.2 Number of simple models ..... 23
3.3 The largest simple model ..... 24
3.4 Logically equivalent formulas ..... 27
3.5 Conclusion ..... 31
4 Conclusion ..... 32

## Chapter 1

## Intuitionistic Propositional Logic

Intuitionistic Logic or IPL is based on the idea that a formula $A$ is valid when there is a proof for it. And as long as there is no proof of $A$ and there is no proof of $\neg A$, the truth value of $A$ remains unknown. This immediately shows an important difference between IPL and classical logic, the law of the excluded middle $A \vee \neg A$ is not a tautology in IPL [4].

The language of IPL has four connectives, which are $\wedge, \vee, \neg$ and $\rightarrow$. In terms of proofs, their meaning is characterized as follows [3]:

- A proof of $A \wedge B$ means there is a proof of $A$ and a proof of $B$;
- A proof of $A \vee B$ means there is a proof of $A$ or a proof of $B$;
- A proof of $\neg A$ is a construction that can rewrite any proof of $A$ to a contradiction;
- A proof of $A \rightarrow B$ is a construction that, given any proof of $A$, can be applied to give a proof of $B$

We sometimes write $[P, \wedge, \vee, \neg, \rightarrow]$ instead of IPL where $P$ is the set of all possible atoms and $\# P<\infty$.

### 1.1 IPL in Kripke models

A way to look at IPL is in possible-world semantics with the use of Kripke models. Each node $k$ in a Kripke model $M$ is a possible world where several atoms are true. A Kripke models is represented by a triple

$$
M=\langle W, \leqslant, V\rangle, \text { where }
$$

- $W$ is the collection of worlds and $W$ is not empty;
- $\leqslant$ is a binary relation on $W$ also known as the accessibility relation, writing $k \leqslant k^{\prime}$ means $k^{\prime}$ is accessible from $k$;
- $V: W \rightarrow \mathcal{P}(P)$ is a valuation that maps a node in $W$ to an element of the power set of $P$, such that $p \in V(k)$ if $p$ is true at node $k$.

Remark 1.1. We use a few variations on $\leqslant$, they are as follows:

- $k<k^{\prime}$, which is defined as $k \leqslant k^{\prime}$ and $k \neq k$
- $k \geqslant k^{\prime}$, which is defined as $k^{\prime} \leqslant k$


Figure 1.1: The dotted lines in (a) and (b) indicate relations that must hold in IPL, but are generally omitted in figures to keep them clear. The dotted lines in (c) are relations that can never hold. In (d), because the underlined atom $p$ is true in the bottom node, it is true in all the above nodes. The overlined atom $q$ becomes true in the middle left node, and therefore in the two nodes above. Note that even though $q$ is true in the top right node, it is not because of heredity.

In IPL the following conditions must hold (see also figure 1.1):

- Reflexivity, which means

$$
\forall k \in W(k \leqslant k)
$$

- Transitivity, which means

$$
\forall k_{1}, k_{2}, k_{3} \in W\left(k_{1} \leqslant k_{2} \text { and } k_{2} \leqslant k_{3} \Rightarrow k_{1} \leqslant k_{3}\right)
$$

- Antisymmetry, which means

$$
\forall k_{1}, k_{2} \in W\left(k_{1} \leqslant k_{2} \text { and } k_{2} \leqslant k_{1} \Rightarrow k_{1}=k_{2}\right)
$$

- Heredity, which means

$$
\forall k_{1}, k_{2} \in W\left(k_{1} \leqslant k_{2} \Rightarrow V\left(k_{1}\right) \subseteq V\left(k_{2}\right)\right)
$$

Heredity is a consequence of the intended meaning of the models. Each node $k$ represents a knowledge state. Once a formula is known to be true, it remains true in all the nodes accessible from $k$.
Together transitivity and antisymmetry ensure that there can be no loops in our models. So when you go from node $k$ to node $k^{\prime}$ with $k<k^{\prime}$, there is no way to get back to $k$.

The notation $M, k \vDash A$ means that $A$ is true in model $M$ at node $k$. The interpretation of formulas is defined inductively by [5]:

$$
\begin{array}{ll}
M, k \vDash p & \Longleftrightarrow p \in V(k) \\
M, k \vDash A \wedge B & \Longleftrightarrow M, k \vDash A \text { and } M, k \vDash B \\
M, k \vDash A \vee B & \Longleftrightarrow M, k \vDash A \text { or } M, k \vDash B \\
M, k \vDash \neg A & \Longleftrightarrow \forall k^{\prime} \geqslant k\left(M, k^{\prime} \not \models A\right) \\
M, k \vDash A \rightarrow B & \Longleftrightarrow \forall k^{\prime} \geqslant k\left(M, k^{\prime} \vDash A \Rightarrow M, k^{\prime} \vDash B\right)
\end{array}
$$

As is clear from these definitions, in order to check the validity of formulas whose only connectives are disjunction and conjunction, we can restrict ourselves to node $k$. But when we are faced with negation or implication we also have to check all the nodes above $k$, which can be infinitely many. We would like to develop a procedure to simplify models in such a way that the same formulas are true, but it is a lot easier to check the validity of a formula. Then we only have to check the validity of a formula in the simplified model to know its truth value in the original model.

In the next section we are going to investigate a few logical principles that hold in classical logic. What happens to these principles in IPL, do they still hold?

### 1.2 Logical principles in IPL

We know that the law of excluded middle $A \vee \neg A$ is not a tautology in IPL. We are going to look at four more formulas that are tautologies in classical logic and see how they behave in IPL. The first two formulas are De Morgan's laws for disjunction and conjunction and the last two formulas contain multiple negations.

## De Morgan's law for disjunction

We will prove that De Morgan's law for disjunction holds in IPL, which is the following proposition.

Proposition 1.2. In IPL it holds that

$$
\begin{equation*}
\neg(A \vee B) \Longleftrightarrow \neg A \wedge \neg B \tag{1}
\end{equation*}
$$

Proof. To see this take an arbitrary node $k$ in a model $M \in[P, \wedge, \vee, \neg, \rightarrow]$ and apply
the interpretations from $\S 1.1$ to the left-hand side of (1).

```
    \(M, k \vDash \neg(A \vee B)\)
\(\Longleftrightarrow \quad\) \{definition of negation\}
    \(\forall k^{\prime} \geqslant k\left(M, k^{\prime} \not \models A \vee B\right)\)
\(\Longleftrightarrow \quad\) \{negation of a disjunction\}
    \(\forall k^{\prime} \geqslant k\left(M, k^{\prime} \not \models A\right.\) and \(\left.M, k^{\prime} \not \models B\right)\)
\(\Longleftrightarrow\)
    \(\forall k^{\prime} \geqslant k\left(M, k^{\prime} \not \models A\right)\) and \(\forall k^{\prime} \geqslant k\left(M, k^{\prime} \not \models B\right)\)
\(\Longleftrightarrow \quad\) \{definition of negation\}
    \(M, k \vDash \neg A\) and \(M, k \vDash \neg B\)
\(\Longleftrightarrow\)
    \(M, k \vDash \neg A \wedge \neg B\)
```

We can see that De Morgan's law holds for disjunction.

## De Morgan's law for conjunction

De Morgan's law for conjunction does not hold in IPL.
Proposition 1.3. In IPL the following does not hold

$$
\begin{equation*}
\neg(A \wedge B) \Longleftrightarrow \neg A \vee \neg B . \tag{2}
\end{equation*}
$$

Proof. This proof is by counterexample, it suffices to give one model where (2) does not hold.

Let $M=\langle W, \leqslant, V\rangle$ be a model in IPL with the following properties:

- $W=\left\{k, k_{1}, k_{2}\right\}$
- $k \leqslant k_{1}$ and $k \leqslant k_{2}$ (and the reflexive relations)
- $V(k)=\varnothing, V\left(k_{1}\right)=\{p\}$ and $V\left(k_{2}\right)=\{q\}$


Figure 1.2: A counterexample for $\neg(A \wedge B) \Longleftrightarrow \neg A \vee \neg B$.

Take $A=p$ and $B=q$ and apply the interpretations in $\S 1.1$ to the left-hand side of (2), then

$$
\begin{array}{ll}
\quad M, k \vDash \neg(p \wedge q) & \\
\Longleftrightarrow & \text { \{definition of negation\} } \\
\Longleftrightarrow & \text { \{negation of a conjunction\} } \\
\quad \forall k^{\prime} \geqslant k\left(M, k^{\prime} \not \models p \wedge q\right) &
\end{array}
$$

We can see that this holds in model $M$, because in every node either $p$ is untrue or $q$ is untrue.

The right-hand side of (2) is interpreted as follows

$$
\begin{array}{ll}
\quad M, k \vDash \neg p \vee \neg q & \\
\Longleftrightarrow & \text { \{definition of disjunction\} } \\
\Longleftrightarrow & \text { \{definition of negation } \\
& \forall k^{\prime} \geqslant k\left(M, k^{\prime} \not \models p\right) \text { or } \forall k^{\prime} \geqslant k\left(M, k^{\prime} \not \models q\right)
\end{array} \quad .
$$

For this to hold in a node, all accessible nodes must make $p$ untrue or all accessible nodes must make $q$ untrue. This is not the case in model $M$.
The model $M$ is a counterexample for our assumption, this leads to the conclusion

$$
\neg(A \wedge B) \nLeftarrow \neg A \vee \neg B .
$$

De Morgan's law for conjunction does not hold in IPL.

## Double negation

When we encounter a double negation in classical logic they cancel each other out. This is not the case in IPL. We will prove the following proposition.

Proposition 1.4. In IPL the following does not hold

$$
\begin{equation*}
\neg \neg A \Longrightarrow A \tag{3}
\end{equation*}
$$

In the proof of this proposition, we will use the negation of the definition of negation. We will give it for clarity, although the reader is probably familiar with negating a statement.

Definition 1.5 (Negation of the definition of negation in IPL).

$$
M, k \not \models \neg A \Longleftrightarrow \exists k^{\prime} \geqslant k\left(M, k^{\prime} \vDash A\right)
$$

Proof of proposition 1.4. To disprove (3) we only need to find a counterexample.
Let $A=p$ and define model $M=\langle W, \leqslant, V\rangle$ as follows

- $W=\left\{k, k^{\prime}\right\}$
- $k \leqslant k^{\prime}$ (and both reflexive relations $k \leqslant k$ and $k^{\prime} \leqslant k^{\prime}$ )
- $V(k)=\varnothing$ and $V\left(k^{\prime}\right)=p$


Figure 1.3: A counterexample for $\neg \neg A \Rightarrow A$.

Clearly $p$ is not true at $k$, so

$$
\begin{equation*}
M, k \not \models p . \tag{4}
\end{equation*}
$$

To see that $M, k \vDash \neg \neg p$ holds, we apply definition 2.1

$$
\forall k_{1} \geqslant k\left(M, k_{1} \not \models \neg p\right)
$$

Since all nodes are accessible from $k$ the expression $\forall k_{1} \geqslant k$ simply means $\forall k_{1} \in W$. Then we apply definition 1.5 and we get

$$
\forall k_{1} \in W\left(\exists k_{2} \geqslant k_{1}\left(M, k_{2} \vDash p\right)\right) .
$$

So for all nodes in our model, there has to be a node accessible where $p$ is true. For both nodes $k$ and $k^{\prime}$ this is the case, namely node $k^{\prime}$. We can conclude that in model $M$

$$
M, k \vDash \neg \neg p .
$$

Together with (4) this leads to

$$
M, k \vDash \neg \neg p \nRightarrow M, k \vDash p .
$$

So model $M$ is a counterexample for (3), we conclude

$$
\neg \neg A \nRightarrow A .
$$

So in IPL a double negation does not cancel out.

## Triple negation

However, when we have a triple negation, two negations do cancel out. Therefore a triple negation is the same a single negation.

Proposition 1.6. In IPL it holds that

$$
\neg \neg \neg A \Longrightarrow \neg A
$$

Proof. Let $M=\langle W, \leqslant, V\rangle$ be a model and let $k \in W$ be a node. Suppose

$$
M, k \vDash \neg \neg \neg A
$$

If we apply definition of negation from $\S 1.1$, we get

$$
\forall k_{1} \geqslant k\left(M, k_{1} \not \models \neg \neg A\right) .
$$

Then we apply definition 1.5 and we get

$$
\forall k_{1} \geqslant k\left(\exists k_{2} \geqslant k_{1}\left(M, k_{2} \vDash \neg A\right)\right) .
$$

We apply the definition of negation again and get

$$
\forall k_{1} \geqslant k\left(\exists k_{2} \geqslant k_{1}\left(\forall k_{3} \geqslant k_{2}\left(M, k_{2} \not \models A\right)\right) .\right.
$$

Because of heredity, this is the same as saying

$$
\forall k_{1} \geqslant k\left(M, k_{1} \not \models A\right),
$$

which is exactly the definition of

$$
M, k \vDash \neg A .
$$

So when we encounter a triple negation we can treat this as a single negation. And since we picked our model and node arbitrarily we can conclude

$$
\neg \neg \neg A \Longrightarrow \neg A
$$

Remark 1.7. So you can always cancel out two negations as long as there are more than three negations in a row.

We have seen in this section that IPL differs from classical logic. De Morgan's law for disjunction still holds, but not for conjunction. And a double negation does not always cancel out.

Next we will define a specific type of model, namely a rooted model.

### 1.3 Rooted models

Let $M=\langle W, \leqslant, V\rangle$ be a model in IPL. When we check the validity of a formula in a node $k \in W$, we only have to consider the nodes that are accessible from $k$. This gives rise to a new model, which we will call a rooted model,

$$
M_{k}=\left\langle k, W_{k}, \leqslant_{k}, V_{k}\right\rangle, \text { with } W_{k}=\left\{k^{\prime} \in W \mid k \leqslant k^{\prime}\right\} .
$$

In this model $\leqslant_{k}$ is defined as $\leqslant$ restricted to the nodes accessible from $k$. The same holds for $V_{k}$, which is defined as $V$ restricted to the nodes accessible from $k$. In this model we call $k$ the root.
When a formula is valid in the root of a rooted model, it is valid in the whole model because of heredity.

Note that the class of rooted models is a subclass of all Kripke models.
Next we will impose a condition on the models we consider, namely that they are finite.

### 1.4 Finite models

A Kripke model $M=\langle W, \leqslant, V\rangle$ is finite if the number of nodes in $W$ is finite.
According to [6] for every node in an infinite model for $[P, \wedge, \vee, \rightarrow, \neg]$ with $\# P<\infty$, there is a finite rooted model, where the same formulas hold. In other words, the subclass of finite rooted models is sound and complete.

In order look at some of the properties of finite Kripke models we define the set of maximal nodes as in [5], these are the end nodes of a model.

Definition 1.8. A node $k \in W$ is maximal in model $M$ if there is no larger node:

$$
\neg \exists k^{\prime} \in W\left(k<k^{\prime}\right) .
$$

The set of all maximal nodes in $W$ is defined by

$$
\max (W)=\{k \in W \mid k \text { maximal in } M\} .
$$

The only node that is accessible from a maximal node, is the node itself, because of reflexivity. We can use this fact to prove the following claim.

Claim 1.9. In a maximal node classical logic holds.
Proof. It is clear from the interpretations of disjunction and conjunction in $\S 1.1$ that they behave the same in IPL as in classical logic. This holds for any node, so also a maximal node. It only remains to show the cases for negation and implication.

To prove the case for negation we have to show that for any maximal node $k$

$$
M, k \vDash \neg A \Longleftrightarrow M, k \not \models A .
$$

Let $k \in W$ be a maximal node, then

$$
\begin{aligned}
& M, k \vDash \neg A \\
& \forall k^{\prime} \geqslant k\left(M, k^{\prime} \not \models A\right)
\end{aligned} \quad \text { \{interpretation of } \neg \text { in IPL\} }
$$

It holds for negation.
For implication the following has to hold for any maximal node $k$

$$
M, k \vDash A \rightarrow B \Longleftrightarrow(M, k \vDash A \Rightarrow M, k \vDash B)
$$

Again, let $k \in W$ be a maximal node, then

$$
\begin{array}{ll} 
& M, k \vDash A \rightarrow B \\
& \forall k^{\prime} \geqslant k\left(M, k^{\prime} \vDash A \Rightarrow M, k^{\prime} \vDash B\right) \\
\Longleftrightarrow & \text { \{interpretation of } \rightarrow \text { in IPL \} } \\
& \text { \{the only node accessible from } k \text { is } k \text { itself }\}
\end{array}
$$

It also holds for implication.
So in a maximal node all the connectives behave the same as in classical logic, therefore classical logic holds.

When classical logic holds at node $k$, only the atoms at $k$ make a formula $A$ true (or false). When this is the case we sometimes write

$$
V(k) \vDash A(\text { or } V(k) \not \models A) .
$$

So in a maximal node the following holds

$$
\begin{aligned}
& M, k \vDash A \Leftrightarrow V(k) \vDash A, \\
& M, k \not \models A \Leftrightarrow V(k) \not \models A .
\end{aligned}
$$

Next we will define a property of a node, namely the depth. This definition is the same as in [5].
Definition 1.10. The depth $d(k)$ of a node $k \in W$ is defined inductively by

$$
d(k)=\sup \left\{d\left(k^{\prime}\right)+1 \mid k<k^{\prime}\right\}, \text { where } \sup (\emptyset)=0
$$

So what the depth of a node tells you, is the maximum number of steps it takes to reach a maximal node. Given that each step must take you from one node to another, so the reflexive relation does not count as a step. Therefore, it is clear that the depth of a maximal node is 0 .
Sometimes we use the depth as a property of a model.
Definition 1.11. The depth $d(M)$ of a model $M=\langle W, \leqslant, V\rangle$ is defined as

$$
d(M)=\sup \{d(k) \mid k \in W\} .
$$

We make the following observation.
Proposition 1.12. For a finite Kripke model $M=\langle W, \leqslant, V\rangle$ the following holds

$$
d(M)<\# W-1<\infty
$$

What this means is that when you keep going upwards in a finite model, you will always reach a maximal node at some point. This is clear from the fact that there is only a finite amount of nodes in a finite model.

As we stated at the beginning of this section we will restrict ourselves to the models that are finite.

In the next section we will define a method to check whether the same formulas are true in two nodes.

### 1.5 Bisimulation

Checking the validity of a formula is easier when there are less nodes to check. If we can simplify a model in such a way that the same formulas hold with the minimum amount of nodes, this can save effort. To determine whether the same formulas hold in two nodes in different models we use the concept of bisimulation.
Definition 1.13. Let $M=\langle W, \leqslant, V\rangle$ and $M^{\prime}=\left\langle W^{\prime}, \leqslant^{\prime}, V^{\prime}\right\rangle$ be two models with $k_{0} \in W$ and $k_{0}^{\prime} \in W^{\prime}$. Then $k_{0}$ and $k_{0}^{\prime}$ are bisimilar if there exists a relation $R \subseteq W \times W^{\prime}$ such that the following properties hold:
(i) $\left(k_{0}, k_{0}^{\prime}\right) \in R$
(ii) $\left(k, k^{\prime}\right) \in R \Rightarrow V(k)=V^{\prime}\left(k^{\prime}\right)$
(iii) $\left(k, k^{\prime}\right) \in R$ and $k \leqslant k_{1} \Rightarrow \exists k_{1}^{\prime} \in W^{\prime}\left(\left(k_{1}, k_{1}^{\prime}\right) \in R\right.$ and $\left.k^{\prime} \leqslant k_{1}^{\prime}\right)$
(iv) $\left(k, k^{\prime}\right) \in R$ and $k^{\prime} \leqslant k_{1}^{\prime} \Rightarrow \exists k_{1} \in W\left(\left(k_{1}, k_{1}^{\prime}\right) \in R\right.$ and $\left.k \leqslant k_{1}\right)$

In this case $R$ is called a bisimulation. We write $k_{0} \overleftrightarrow{\leftrightarrow} k_{0}^{\prime}$ to indicate that a bisimulation exists.

Remark 1.14. Notice that a bisimulation is a symmetric relation. If $k \leftrightarrows k^{\prime}$ than also $k^{\prime} \leftrightarrow k$ with bisimulation $R^{\prime}=\left\{\left(k_{1}^{\prime}, k_{1}\right) \mid\left(k_{1}, k_{1}^{\prime}\right) \in R\right\}$.


Figure 1.4: The dotted lines indicate the relations that must exist if there is a bisimulation between $k$ and $k^{\prime}$. Note that in (a) $k^{\prime}$ and $k_{1}^{\prime}$ may be equal, and in that case, the dotted line between them is the reflexive relation. The same holds for $k$ and $k_{1}$ in (b).

From (ii) it is clear that the same atoms are true in two nodes when there exists a bisimulation between them. In fact the same formulas are true. We are going to prove the following theorem, which states that in bisimilar nodes the same formulas hold.

Theorem 1.15. Given two rooted models $M=\left\langle k_{0}, W, \leqslant, V\right\rangle$ and $M^{\prime}=\left\langle k_{0}^{\prime}, W^{\prime}, \leqslant^{\prime}, V^{\prime}\right\rangle$ such that $k_{0} \leftrightarrow k_{0}^{\prime}$ with bisimulation $R$. Then

$$
\forall k \in W \forall k^{\prime} \in W^{\prime}\left(\left(k, k^{\prime}\right) \in R \Longrightarrow\left(M, k \vDash A \Leftrightarrow M^{\prime}, k^{\prime} \vDash A\right)\right)
$$

Proof. The proof of this theorem is by induction over the complexity of $A$.
First we consider the case $A=p$. Pick arbitrary $k \in W$ and $k^{\prime} \in W^{\prime}$, if $\left(k, k^{\prime}\right) \in R$, then

| $M, k \vDash p$ |  |
| :---: | :---: |
| $\Longleftrightarrow$ | \{definition of double turnstile\} |
| $p \in V(k)$ |  |
| $\Longleftrightarrow$ | \{by definition 1.13(ii) \} |
| $p \in V^{\prime}\left(k^{\prime}\right)$ |  |
| $\Longleftrightarrow$ | \{definition of double turnstile\} |
| $M^{\prime}, k^{\prime} \vDash p$ |  |

So it holds for any atom.
The cases for conjunction and disjunction are straightforward.
To prove that the theorem holds for negation and implication assume

$$
\begin{aligned}
& \forall k \in W \forall k^{\prime} \in W^{\prime}\left(\left(k, k^{\prime}\right) \in R \Longrightarrow\left(M, k \vDash B \Leftrightarrow M^{\prime}, k^{\prime} \vDash B\right)\right), \\
& \forall k \in W \forall k^{\prime} \in W^{\prime}\left(\left(k, k^{\prime}\right) \in R \Longrightarrow\left(M, k \vDash C \Leftrightarrow M^{\prime}, k^{\prime} \vDash C\right)\right),
\end{aligned}
$$

this is the induction hypothesis.
Let us prove the case for implication. Pick $k \in W$ and $k^{\prime} \in W^{\prime}$ arbitrarily and suppose $M, k \vDash B \rightarrow C$, then by the definition of implication in $\S 1.1$ we know

$$
\begin{equation*}
\forall k_{1} \geqslant k\left(M, k_{1} \vDash B \Rightarrow M, k_{1} \vDash C\right) \tag{5}
\end{equation*}
$$

If $\left(k, k^{\prime}\right) \in R$, then we know by definition 1.13 (iv)

$$
\forall k_{1}^{\prime} \geqslant k^{\prime}\left(\exists k_{1} \geqslant k\left(\left(k_{1}, k_{1}^{\prime}\right) \in R\right)\right)
$$

So for all nodes $k_{1}^{\prime}$ with $k^{\prime} \leqslant^{\prime} k_{1}^{\prime}$ there is a node $k_{1} \in W$ such that $\left(k_{1}, k_{1}^{\prime}\right) \in R$. Then we know by the induction hypothesis that

$$
\begin{aligned}
& M, k_{1} \vDash B \Leftrightarrow M^{\prime}, k_{1}^{\prime} \vDash B, \\
& M, k_{1} \vDash C \Leftrightarrow M^{\prime}, k_{1}^{\prime} \vDash C .
\end{aligned}
$$

If we combine this with the result of (5) we can ony conclude that the following must hold

$$
\forall k_{1}^{\prime} \geqslant k^{\prime}\left(M^{\prime}, k_{1}^{\prime} \vDash B \Rightarrow M^{\prime}, k_{1}^{\prime} \vDash C\right),
$$

which is the definition of implication, so

$$
M^{\prime}, k^{\prime} \vDash B \rightarrow C .
$$

The proof in the other direction is analogous because of symmetry of bisimulation. So we have proven the case for implication.

Now the case of negation. Pick $k \in W$ and $k^{\prime} \in W^{\prime}$ arbitrarily and suppose $M, k \vDash \neg B$, then by the definition of negation in $\S 1.1$ we know

$$
\begin{equation*}
\forall k_{1} \geqslant k\left(M, k_{1} \not \models B\right) \tag{6}
\end{equation*}
$$

If $\left(k, k^{\prime}\right) \in R$, then from definition 1.13 (iv) we know

$$
\forall k_{1}^{\prime} \geqslant^{\prime} k^{\prime}\left(\exists k_{1} \geqslant k\left(\left(k_{1}, k_{1}^{\prime}\right) \in R\right)\right)
$$

So for all nodes $k_{1}^{\prime}$ with $k^{\prime} \leqslant^{\prime} k_{1}^{\prime}$ there is a node $k_{1} \in W$ such that $\left(k_{1}, k_{1}^{\prime}\right) \in R$. Then we know by the induction hypothesis that

$$
M, k_{1} \vDash B \Leftrightarrow M^{\prime}, k_{1}^{\prime} \vDash B
$$

but we already concluded in (6) that the opposite of the left-hand side is the case, therefore the opposite of the right-hand side must hold

$$
\forall k_{1}^{\prime} \geqslant^{\prime} k^{\prime}\left(M^{\prime}, k_{1}^{\prime} \not \models B\right) .
$$

This is the definition of negation, so we conclude

$$
M^{\prime}, k^{\prime} \vDash \neg B,
$$

Which is what we wanted to prove. The proof in the other direction is analogous because of symmetry of bisimulation.

Now we know that theorem 1.15 holds for all four connectives, we can conclude by induction that it must hold for all formulas.

This means that when we simplify a model and we can show there exists a bisimulation between a node of the original model and a node of the simplified model, we know the same formulas are true in that node.

### 1.6 Conclusion

In this chapter we have started by introducing IPL and determining the conditions for in IPL in possible-world semantics with the use of Kripke models.

We have seen that IPL differs from classical logic by looking at De Morgan's laws and multiple negations.

Next we showed that we can impose two conditions on the Kripke models in IPL, namely that they are finite and that they are rooted. If we can find a bisimulation between two nodes in different models, we know that the same formulas hold. We can use this to simplify models.

In the next chapter we are going to investigate a fragment of IPL, namely IPL without implication. When we rule out implication, we can further simplify models, and we will see that with a limited number of models we can distinguish between all the different formulas.

## Chapter 2

## IPL without Implication

In this chapter we are going to look at a fragment of IPL, namely IPL without implication, also denoted by $[P, \wedge, \vee, \neg]$.
When we only consider formulas containing disjunctions, conjunctions and negations, we can further simplify the Kripke models and we can distinguish between these formulas with a limited number of different models.

### 2.1 Simplification of Kripke models in IPL without implication

So far we have restricted ourselves to models that are rooted and finite. Now that we have excluded formulas with implication, we can simplify our models in two ways without changing the truth values of formulas, they will be discussed in the following two sections.

### 2.1.1 Omitting in-between nodes

We have seen in $\S 1.1$ that for conjunction and disjunction we only have to consider the node in which we are checking. When we want to check a formula with negation we had the following definition

Definition 2.1 (Definition of negation).

$$
M, k \vDash \neg A \Longleftrightarrow \forall k^{\prime} \geqslant k\left(M, k^{\prime} \not \models A\right) .
$$

So none of the nodes accessible from $k$ must make $A$ true, in particular the maximal nodes above $k$. This leads to the following theorem.

Theorem 2.2. In a rooted finite model $M=\langle k, W, \leqslant, V\rangle$ the following holds

$$
M, k \vDash \neg A \Longleftrightarrow \forall k^{\prime} \in \max (W)\left(M, k^{\prime} \not \models A\right)
$$

Proof. $(\Rightarrow)$ Since all maximal nodes are accessible from $k$ it is clear that $M, k^{\prime} \not \models A$ must hold.
$(\Leftarrow)$ This part is a proof by contradiction. Suppose
(i) $\forall k^{\prime} \in \max (W)\left(M, k^{\prime} \not \models A\right)$
(ii) $M, k \not \models \neg A$.

For assumption (ii) to be true, the following must hold

$$
\exists k_{1} \geqslant k\left(M, k_{1} \vDash A\right)
$$

by definition 1.5. This $k_{1}$ can not be a maximal node because of assumption (i). But by the heredity condition we know that

$$
\forall k_{2} \geqslant k_{1}\left(V\left(k_{1}\right) \subseteq V\left(k_{2}\right)\right) .
$$

What this means is that once a formula is true, it will remain true. So if $A$ is true at node $k_{1}$, it is true at all the nodes above $k_{1}$, including one or more maximal nodes. This contradicts assumption (i) and therefore we can conclude that

$$
\forall k^{\prime} \in \max (W)\left(M, k^{\prime} \not \models A\right) \Longrightarrow M, k \vDash \neg A
$$

So we have proven theorem 2.2 in both directions.
What this means is that in order to check the validity of a formula in IPL without implication we only have to look at two kinds of nodes: The root and, when the formula contains a negation, the maximal nodes. So if we want to simplify a model, but we do not want to change the validity of formulas, it makes sense to drop all the in-between nodes. Then we are left with models of depth 1 (or depth 0 if the original model contained only one node).


Figure 2.1: This figure shows a simplification of a model by omitting the inbetween nodes, where $\operatorname{In}$-between $(W):=W \backslash\left(\left\{k_{0}\right\} \cup \max (W)\right)$.

### 2.1.2 Omitting duplicate maximal nodes

So far we have restricted ourselves to rooted finite models of depth 1. But even these models can be very large since they can have any amount of maximal nodes. We can do one more thing to simplify models without changing the values of formulas. For this we will need to have a closer look at $P$, the set of atoms.

If we look at a fragment of IPL, we also define a set $P$ of all the possible atoms. A valuation $V$ of a node in a model is a subset of $P$. So there is a finite amount of possible valuations of a node, namely the power set $\mathcal{P}(P)$.
When we have to check a formula in the root, we only have to consider the nodes above if the formula contains a negation. So let's look again at theorem 2.2.

$$
M, k \vDash \neg A \Longleftrightarrow \forall k^{\prime} \in \max (W)\left(M, k^{\prime} \not \models A\right)
$$

Since $M, k^{\prime} \not \models A$ must hold for all maximal nodes $k^{\prime}$, having two maximal nodes with the same valuation, does not change the truth value of $A$. It makes sense to simplify the models by removing maximal nodes that have the same valuation as another maximal node, we will call them duplicate nodes.

To see that this simplification does indeed lead to a model where the same formulas still hold, we can use bisimulation from §1.5.

Let $M=\left\langle k_{0}, W, \leqslant, V\right\rangle$ be a finite rooted model of depth 1 and let $M^{\prime}=\left\langle k_{0}^{\prime}, W^{\prime}, \leqslant^{\prime}, V^{\prime}\right\rangle$ be the simplified model where all the duplicate nodes are removed. If we define $R$ as follows

- $\left(k_{0}, k_{0}^{\prime}\right) \in R ;$
- $\forall k \in \max (W) \forall k^{\prime} \in \max \left(W^{\prime}\right)\left(V(k)=V^{\prime}\left(k^{\prime}\right) \Longrightarrow\left(k, k^{\prime}\right) \in R\right)$,
then $R$ is a bisimulation and the same formulas hold in $k$ and $k^{\prime}$.


Figure 2.2: This figure shows a simplification of a model by omitting the duplicate nodes. There are two duplicate nodes in the left picture. Note that a root and a maximal node can not be duplicates.

The two methods of simplifying a model $M$ leads to the simplified model $\tau(M)$.

### 2.1.3 The simplified model $\tau(M)$

We use the methods of omitting in-between nodes and then omitting duplicate nodes from the previous two sections to define the model $\tau(M)$.

Definition 2.3. Let $M=\left\langle k_{0}, W, \leqslant, V\right\rangle$ be a rooted finite model, then the simplified model $\tau(M)$ is defined as follows

$$
\tau(M)=\left\langle k_{0}, W^{\prime}, \leqslant^{\prime}, V^{\prime}\right\rangle
$$

where

- $k_{0}$ is the same as in model $M$
- $W^{\prime}=\left\{k_{0}\right\} \cup\{V(k) \mid k$ maximal in $W\}$
- $\leqslant^{\prime}$ is such that $\forall k \in W^{\prime}\left(k_{0} \leqslant^{\prime} k\right.$ and $\left.k \leqslant^{\prime} k\right)$
- $V^{\prime}$ is defined as follows
$-V^{\prime}\left(k_{0}\right)=V\left(k_{0}\right)$
$-V^{\prime}(Q)=Q$ with $Q \in \max \left(W^{\prime}\right) \subseteq \mathcal{P}(P)$
Now that we have the definition of a simplified model, we would like to prove that the same formulas hold in the root. Unfortunately, we can not use bisimulation, because there may be in-between nodes in the model $M$ whose valuation is not present in the set $W^{\prime}$ of model $\tau(M)$. These nodes can not be paired with a node in $\tau(M)$, so bisimulation fails. We will use induction to prove the following

Theorem 2.4. Given a rooted finite model $M=\left\langle k_{0}, W, \leqslant, V\right\rangle$, then

$$
M, k_{0} \vDash A \Longleftrightarrow \tau(M), k_{0} \vDash A
$$

Proof. The proof of this theorem is by induction over the complexity of $A$.
First we consider the case where $A$ is a single atom, let $A=p$.

$$
\begin{array}{ll}
\quad M, k_{0} \vDash p & \\
\Longleftrightarrow p \in V\left(k_{0}\right) & \text { \{definition of double turnstile }\} \\
\Longleftrightarrow & \text { \{definition of } \tau\}
\end{array}
$$

It holds for a single atom.
To prove that the theorem holds for all the connectives assume that $\forall M, k_{0}$ we have

$$
\begin{aligned}
& M, k_{0} \vDash B \Leftrightarrow \tau(M), k_{0} \vDash B, \\
& M, k_{0} \vDash C \Leftrightarrow \tau(M), k_{0} \vDash C,
\end{aligned}
$$

this is the inductive hypothesis.
To prove it holds for conjunction let $A=B \wedge C$.

$$
\left.\right)
$$

It holds for conjunction.

Next is the case of disjunction, let $A=B \vee C$.

$$
\begin{aligned}
& M, k_{0} \vDash B \vee C \\
& \Longleftrightarrow \quad \text { \{definition of disjunction } \\
& M, k_{0} \vDash B \text { or } M, k_{0} \vDash C \\
& \Longleftrightarrow \quad \text { \{inductive hypothesis\} } \\
& \tau(M), k_{0} \vDash B \text { or } \tau(M), k_{0} \vDash C \\
& \Longleftrightarrow \quad \text { \{definition of disjuction } \\
& \tau(M), k_{0} \vDash B \vee C \\
& \text { \{definition of disjunction\} } \\
& \text { \{inductive hypothesis\} } \\
& \text { \{definition of disjuction }\}
\end{aligned}
$$

It also holds for disjunction.
Before we prove the theorem for negation, let's recall claim 1.9 from $\S 1.4$

$$
\text { If } k \in \max (W) \text { then classical logic holds in } k .
$$

In the case of negation this meant

$$
M, k \vDash \neg A \Leftrightarrow M, k \not \models A .
$$

If the atoms of a node $k$ make a formula true, we wrote

$$
\begin{equation*}
M, k \vDash A \Leftrightarrow V(k) \vDash A, \tag{7}
\end{equation*}
$$

this is always the case if $k \in \max (W)$.
Now let $A=\neg B$

$$
\begin{array}{ll} 
& M, k_{0} \vDash \neg B \\
& \forall k \in \max (W)(M, k \not \models B) \\
\Longleftrightarrow & \text { \{by definition } 2.2\} \\
& \forall k \in \max (W)(V(k) \not \models B) \\
\Longleftrightarrow & \text { \{by use of }(7)\} \\
& \forall Q \in\{V(k) \mid k \in \max (W)\}(Q \not \models B) \\
& \forall Q \in \max \left(W^{\prime}\right)\left(V^{\prime}(Q) \not \models B\right) \\
\Longleftrightarrow & \text { \{rewrite this }\} \\
& \\
& \tau(M), k_{0} \vDash \neg B
\end{array}
$$

It also holds for negation.
By induction we can conclude that

$$
M, k_{0} \vDash A \Leftrightarrow \tau(M), k_{0} \vDash A .
$$

With this proof we have found our method of simplification for models in IPL without implication. We will refer to models of this form as simple models.

### 2.2 Conclusion

In this chapter we started with looking at IPL without implication. We have seen that when we do this, we can further simplify finite rooted models without changing the truth of formulas by omitting the in-between nodes and then omitting the duplicate nodes. This led to the definition of $\tau(M)$. Models of the form $\tau(M)$ are called simple models.

In the next chapter we will calculate how many of these simple models exist given a set of atoms $P$. Then we will look at a particular simple model, namely the largest one. This model is useful when determining whether a formula is a tautology or not

## Chapter 3

## Tautologies and Logical Equivalence in IPL without Implication

Now that we have defined simple models in IPL without implication, another question is raised. How many different simple models exist? And how many models do we need to check in order to determine whether a formula is a tautology.

We are also going to look at formulas that are logically equivalent. How many simple models do we need to check before we may conclude whether two formulas are logically equivalent? And how many different formulas exist that are not logically equivalent?

First we are going to look at the properties of simple models in order to determine the number of simple models there are.

### 3.1 Properties of simple models

When calculating the number of simple models it is easier if all the models are of the same depth. We know from $\S 2.1 .1$ that the depth of a simple model $M$ is $d(M) \leqslant 1$. So there are simple models of depth 0 . These models contain only a root. We will prove that we can do not need these models, because there is a model of depth 1 where the same formulas hold.

Claim 3.1. When we have a model $M=\langle W, \leqslant, V\rangle$ with only one node, so $W=\{k\}$, and the model $M^{\prime}=\left\langle W^{\prime}, \leqslant^{\prime}, V^{\prime}\right\rangle$ with the following properties

- $W^{\prime}=\left\{k^{\prime}, k_{1}^{\prime}\right\}$
- $k^{\prime} \leqslant k_{1}^{\prime}$ (and the reflexive relations $k^{\prime} \leqslant^{\prime} k^{\prime}$ and $k_{1}^{\prime} \leqslant k_{1}^{\prime}$ )
- $V^{\prime}\left(k^{\prime}\right)=V^{\prime}\left(k_{1}^{\prime}\right)=V(k)$
then $k$ and $k^{\prime}$ are bisimilar.
Proof. If we define $R$ as follows $R=\left\{\left(k, k^{\prime}\right),\left(k, k_{1}\right)\right\}$ (see also figure 3.1), then $R$ is a bisimulation and all the properties from definition 1.13 hold. So claim 3.1 holds.


Figure 3.1: If it holds that $V(k)=V^{\prime}\left(k^{\prime}\right)=V\left(k_{1}^{\prime}\right)$, then $k$ and $k^{\prime}$ are bisimilar with bisimulation $R$.

Even though the model of depth 0 is a "simpler" model, we are going to consider the model of depth 1 instead, because it is easier when all the models are uniformly.

From $\S 2.1 .2$ we know that the valuation of each maximal node in a simple model has to be unique. So the number of different simple models we can have is depending on the number of possible atoms $\# P$. Let's look at the set $P$ containing two atoms, so

$$
P=\{p, q\} .
$$

The valuation of a node is an element in the power set of $P$. In the case of $\# P=2$ this is

$$
\mathcal{P}(P)=\{\varnothing,\{p\},\{q\},\{p, q\}\} .
$$

The number of elements in the power set $\# \mathcal{P}(P)=2^{\# P}=4$. So for the roots there are 4 possible valuations. But the valuation of the root influences the possible valuations of the maximal nodes, because of heredity.

Let $k_{0}$ be the root. If $V\left(k_{0}\right)=\{p, q\}$, then there is only one possible simple model $M=\left\langle k_{0}, W, \leqslant, V\right\rangle$ of depth 1 . This is model where

$$
W=\left\{k_{0},\{p, q\}\right\} .
$$



Figure 3.2: The only possible simple model when $P=\{p, q\}$ and $V\left(k_{0}\right)=\{p, q\}$.

However, when $V\left(k_{0}\right)=\{p\}$ there are two possible valuations for a maximal node, they are $\{p\}$ and $\{p, q\}$. This leads to three possible models, namely the models with the following $W$ 's

- $W=\left\{k_{0},\{p\}\right\}$
- $W=\left\{k_{0},\{p, q\}\right\}$
- $W=\left\{k_{0},\{p\},\{p, q\}\right\}$

| $\begin{gathered} p \\ \bullet \\ k_{0} \bullet p \end{gathered}$ |  |  |
| :---: | :---: | :---: |

Figure 3.3: The three possible simple models when $P=\{p, q\}$ and $V\left(k_{0}\right)=\{p\}$.

Clearly, the same holds for $V\left(k_{0}\right)=\{q\}$.
When $V\left(k_{0}\right)=\varnothing$, the maximal nodes inherit no atoms from the root, so a maximal node can be any element of the power set $\mathcal{P}(P)$. This means that the set $W \backslash k_{0}$ is an element of the power set of $\mathcal{P}(P)$, which is

$$
\mathcal{P}(\mathcal{P}(P))=\{\varnothing,
$$

4 sets with 1 element

$$
\{\varnothing\},\{\{p\}\},\{\{q\}\},\{\{p, q\}\},
$$

6 sets with 2 elements $\quad\{\varnothing,\{p\}\},\{\varnothing,\{q\}\},\{\varnothing,\{p, q\}\}$,

$$
\{\{p\},\{q\}\},\{\{p\},\{p, q\}\},\{\{q\},\{p, q\}\}
$$

4 sets with 3 elements

$$
\begin{aligned}
& \{\varnothing,\{p\},\{q\}\},\{\varnothing,\{p\},\{p, q\}\} \\
& \{\varnothing,\{q\},\{p, q\}\},\{\{p\},\{q\},\{p, q\}\}
\end{aligned}
$$

1 set with 4 elements $\quad\{\varnothing,\{p\},\{q\},\{p, q\}\}$
$\}$
The number of elements in this set is

$$
\# \mathcal{P}(\mathcal{P}(P))=2^{\# \mathcal{P}(P)}=2^{2^{\# P}}=2^{4}=16
$$

Notice that the first element, the empty set $\varnothing$ (not the set $\{\varnothing\}$, which contains the empty set ), corresponds with the model that only contains a root. We already concluded that we can exclude this model, because it is bisimilar with a simple model of depth 1 . So when $V\left(k_{0}\right)=\varnothing$ there are $2^{2^{\# P}}-1=15$ possible models if $\# P=2$.

In the case where $\# P=2$ there are 22 simple models of depth 1 . We would like to generalize this for any $\# P$ in a formula. This is the subject of the next section.

### 3.2 Number of simple models

We will develop a formula that tells us how many simple models there are, given a set of atoms $P$. To do this we first need to find a way to calculate the number of simple models given the valuation in the root.

Proposition 3.2. Given a set of atoms $P=\left\{p_{1}, p_{2}, \ldots, p_{n}\right\}$, then the number of simple models in $[P, \wedge, \vee, \neg]$ where the valuation of the root $k_{0}$ is $V\left(k_{0}\right)=\left\{p_{1}, p_{2}, \ldots, p_{m}\right\}$ with $m<n$, is given by

$$
\begin{equation*}
M(n, m)=M(n-m, 0)=2^{2^{n-m}}-1 . \tag{8}
\end{equation*}
$$

Proof. When an atom is true in $k_{0}$, it is true in all the nodes and therefore it reduces the possibilities of valuations in the maximal nodes. Observe that the number of possibilities for the maximal nodes is equal to

$$
\# \mathcal{P}\left(\mathcal{P}\left(P \backslash V\left(k_{0}\right)\right)\right)=2^{2^{n-m}}
$$

The minus 1 is the result of the fact that we excluded the model of depth 0 . This leads to the formula

$$
M(n, m)=M(n-m, 0)=2^{2^{n-m}}-1
$$

Now we want to extend this formula to calculate the number of the simple models given a $P$ without specifying the valuation of the root.

Theorem 3.3. Given a set of atoms $P=\left\{p_{1}, p_{2}, \ldots, p_{n}\right\}$, then the number of simple models in $[P, \wedge, \vee, \neg]$, is given by

$$
\begin{equation*}
M(n)=\sum_{m=0}^{n}\binom{n}{m}\left(2^{2^{n-m}}-1\right) \tag{9}
\end{equation*}
$$

Proof. It is clear that (8) also holds when $V\left(k_{0}\right)$ is not the first $m$ atoms of $P$, but any other subset of $P$, as long as $\# V\left(k_{0}\right)$ is still equal to $m$.

There are $\binom{n}{m}$ ways to pick $m$ atoms out of $n$ possibilities. So in order to calculate the number of simple models where there are precisely $m$ atom true, without stating which atoms, we just multiply the formula from proposition 3.2 by $\binom{n}{m}$.

Then we sum over the number of atoms that are true in the root and the formula follows.

Remark 3.4. Theorem 3.3 has been proven by Hendriks in [2] for the fragment $[P, \wedge, \neg]$ except he adds 1 to the number given by (9), because he allows for a model of depth 0 . Hendriks also stated that the fragment $[P, \wedge, \vee, \neg]$ is isomorphic to $[P, \wedge, \neg]$, so it clear that these fragments result in the same number of simple models.

Here are some results for different values of $n$.

| $n=\# P$ | $M(n)$ |
| :---: | ---: |
| 0 | 1 |
| 1 | 4 |
| 2 | 22 |
| 3 | 310 |
| 4 | 66658 |
| 5 | 4295297686 |

We see that the number of models increases rapidly when $n$ gets bigger. In fact, it increases superexponentially with $A(n) \geqslant 2^{2^{n}}-1$.

In the next section we are going to look at the largest model, that is the model with $2^{\# P}$ maximal nodes.

### 3.3 The largest simple model

The largest simple model has some interesting properties that we are going to look at in this section. First we will give the definition.

Definition 3.5. The largest simple model $M=\left\langle k_{0}, W, \leqslant, V\right\rangle$ is the model with the following properties

- $V\left(k_{0}\right)=\varnothing$
- $W=\left\{k_{0}\right\} \cup \mathcal{P}(P)$.

The largest simple model is the only simple model in $[P, \wedge, \vee, \neg]$ with $2^{\# P}$ maximal nodes.

If we want to check whether a formula is a tautology it suffices to check the truth of the formula in the largest simple model. We will prove the following claim.

Claim 3.6. Let $M=\left\langle k_{0}, W, \leqslant, V\right\rangle$ be the largest simple model in $[P, \wedge, \vee, \neg]$, then for all models $M^{\prime}=\left\langle W^{\prime}, \leqslant^{\prime}, V^{\prime}\right\rangle$ it holds $\forall k \in W^{\prime}$

$$
M, k_{0} \vDash A \Longrightarrow M^{\prime}, k \vDash A .
$$

Proof. We begin by observing that every model is equivalent to a simple model, because of 2.4. So we only have to prove this claim for the simple models.

The proof is by induction over the complexity of $A$. Let $M=\left\langle k_{0}, W, \leqslant, V\right\rangle$ be the largest simple model in $[P, \wedge, \vee, \neg]$ and let $M^{\prime}=\left\langle k_{0}^{\prime}, W^{\prime}, \leqslant^{\prime}, V^{\prime}\right\rangle$ be a simple model.

First we will look at the case where $A$ is a single atom, let $A=p$. We know that $p \notin V\left(k_{0}\right)=\varnothing$, therefore $M, k_{0} \not \models p$. So our claim holds for any atom.

Assume the following inductive hypothesis:

$$
\begin{aligned}
& M, k_{0} \vDash B \Rightarrow M^{\prime}, k_{0}^{\prime} \vDash B, \\
& M, k_{0} \vDash C \Rightarrow M^{\prime}, k_{0}^{\prime} \vDash C .
\end{aligned}
$$

For conjunction, let $A=B \wedge C$

$$
\begin{array}{cl} 
& M, k_{0} \vDash B \wedge C \\
& \text { \{definition of conjunction\} } \\
\Longrightarrow & \text { \{inductive hypothesis\} } \\
& M^{\prime}, k_{0}^{\prime} \vDash B \text { and } M, k_{0} \vDash C \\
& M^{\prime}, k_{0}^{\prime} \vDash B \wedge C
\end{array}
$$

It holds for conjunction.
To prove it for disjunction, let $A=B \vee C$

$$
\begin{aligned}
& M, k_{0} \vDash B \vee C \\
& \text { \{definition of conjunction\} } \\
\Longrightarrow & \text { \{inductive hypothesis \}} \\
& M^{\prime}, k_{0}^{\prime} \vDash B \text { or } M, k_{0} \vDash C \\
& M^{\prime}, k_{0}^{\prime} \vDash B \vee M^{\prime}, k_{0}^{\prime} \vDash C \\
& \text { \{definition of conjuction\} }
\end{aligned}
$$

It holds for disjunction.
In order to prove our claim for negation, let $A=\neg B$ and assume $M, k_{0} \vDash \neg B$. When we apply the definition of negation, we get

$$
\forall k \geqslant k_{0}(M, k \not \models B) .
$$

In particular, it holds for all the maximal nodes

$$
\forall k \in \max (W)(M, k \not \models B) .
$$

If $k \in \max (W)$, then classical logic holds in $k$. So

$$
\forall k \in \max (W)(V(k) \nvdash B) .
$$

Since $k$ can be any maximal node in our model, $V(k)$ can be any element $Q \in \mathcal{P}(P)$, we can write

$$
\begin{equation*}
\forall Q \in \mathcal{P}(P)(Q \not \models B) . \tag{10}
\end{equation*}
$$

This formula no longer contains any reference to the largest model $M$. So it holds for any maximal node in any model in $[P, \wedge, \vee, \neg]$.

Now we use a contradiction to prove that it holds in $M^{\prime}$.
Suppose that

$$
M^{\prime}, k_{0}^{\prime} \vDash B
$$

Because of heredity, once a formula is true, it remains true in all the accessible nodes, including the maximal nodes. This contradicts (10), so we can conclude

$$
M^{\prime}, k_{0}^{\prime} \not \models B
$$

And from (10) we know that

$$
\forall Q \in \max \left(W^{\prime}\right)(Q \not \models B) .
$$

So we can write

$$
\forall k \geqslant k_{0}^{\prime}(k \not \models B),
$$

or

$$
M^{\prime}, k_{0}^{\prime} \vDash \neg B .
$$

Which proves our claim in the case of negation.
Since claim 3.6 holds for all connectives, we can conclude that it holds for all formulas.

Since we have proven that when a formula is true in the largest model it is true in all the simple models and therefore it is true in all the models in $[P, \wedge, \vee, \neg]$, we can conclude that when a formula is true in the largest model, it is a tautology.

In fact, it not only suffices to check the largest model, it is also necessary, because it is possible that a formula holds in every simple model except in the largest one. We will prove the following claim.
Claim 3.7. There is an $A \in[P, \wedge, \vee, \neg]$ such that
(i) For all simple $M=\left\langle k_{0}, W, \leqslant, V\right\rangle$, with $\max (W)<2^{\# P}$ we have $M, k_{0} \vDash A$;
(ii) $\not \models A$.

Proof. The formula $A$ is given by

$$
A=\bigvee_{Q \subseteq P} \neg\left(\bigwedge_{p \in Q} p \wedge \bigwedge_{q \in P \backslash Q} \neg q\right)
$$

We will prove that both (i) and (ii) hold for this $A$ for any set of atoms $P$, with $\# P>0$.
The formula $A$ is a disjunction of several formulas $A_{1}, \ldots, A_{m}$, each of which will turn out to be false in the largest model.

Each $A_{i}$ is be of the form

$$
A_{i}=\neg\left(p_{1} \wedge \ldots \wedge p_{n} \wedge \neg q_{1} \wedge \ldots \wedge \neg q_{m}\right)
$$

with

$$
\begin{aligned}
& \left\{p_{1}, \ldots, p_{n}\right\} \cup\left\{q_{1}, \ldots, q_{m}\right\}=P \\
& \left\{p_{1}, \ldots, p_{n}\right\} \cap\left\{q_{1}, \ldots, q_{m}\right\}=\varnothing
\end{aligned}
$$

In order for $A_{i}$ to be true in the root $k_{0}$ of a simple model $M$, we get

$$
\begin{aligned}
& M, k_{0} \vDash A_{i} \\
& M, k_{0} \vDash \neg\left(p_{1} \wedge \ldots \wedge p_{n} \wedge \neg q_{1} \wedge \ldots \wedge \neg q_{m}\right) .
\end{aligned}
$$

Apply the interpretation of negation

$$
\forall k_{1} \geqslant k_{0}\left(M, k_{1} \not \models p_{1} \wedge \ldots \wedge p_{n} \wedge \neg q_{1} \wedge \ldots \wedge \neg q_{m}\right) .
$$

The negation of conjunctions can be written as

$$
\forall k_{1} \geqslant k_{0}\left(M, k_{1} \not \models p_{1} \wedge \ldots \wedge p_{n} \text { or } M, k_{1} \not \models \neg q_{1} \text { or } \ldots \text { or } M, k_{1} \not \models \neg q_{m},\right) .
$$

Applying the negation of the interpretation of a negation gives

$$
\begin{aligned}
& \forall k_{1} \geqslant k_{0}\left(M, k_{1} \not \models p_{1} \wedge \ldots \wedge p_{n}\right. \text { or } \\
& \left.\quad \exists k_{2} \geqslant k_{1}\left(M, k_{2} \vDash q_{1}\right) \text { or } \ldots \text { or } \exists k_{2} \geqslant k_{1}\left(M, k_{2} \vDash q_{m}\right)\right) .
\end{aligned}
$$

We can distribute the existential quantifier over disjunctions

$$
\forall k_{1} \geqslant k_{0}\left(M, k_{1} \not \models p_{1} \wedge \ldots \wedge p_{n} \text { or } \exists k_{2} \geqslant k_{1}\left(M, k_{2} \vDash q_{1} \vee \ldots \vee q_{m}\right)\right)
$$

This means that $A_{i}$ is true in all the simple models where the maximal node $k$ with $V(k)=\left\{p_{1}, p_{2}, \ldots, p_{n}\right\}$ is missing and false in all the models where this maximal node is present, including the largest model.

So when we can show that there is an $A_{i}$ for each possible subset of $P$, we are done. Because every model $M$ with $\max (W)<2^{\# P}$ misses at least one of the possible maximal nodes.

The power set of $P$ gives all the possible subsets $\mathcal{P}(P)=\left\{Q_{1}, Q_{2}, \ldots, Q_{2 \# P}\right\}$, we can write each $A_{i}$ as

$$
\begin{equation*}
A_{i}=\neg\left(\bigwedge_{p \in Q_{i}} p \wedge \bigwedge_{q \in P \backslash Q_{i}} \neg q\right) \tag{11}
\end{equation*}
$$

The formula $A$ is the disjunction of all these $A_{i}$ 's

$$
A=\bigvee_{i=1, \ldots, 2 \neq P} A_{i}
$$

or

$$
\begin{equation*}
A=\bigvee_{Q \subseteq P} \neg\left(\bigwedge_{p \in Q} p \wedge \bigwedge_{q 1 P \backslash Q} \neg q\right) \tag{12}
\end{equation*}
$$

The formula $A$ is false in the largest simple model and nowhere else.
We have proven that (12) is true in every simple model except in the largest model. This proves that it is necessary to check the largest model in order to establish whether a formula is a tautology.

We will use this proof to reach another conclusion; in order to check whether two formulas are logically equivalent, you need to check all the simple models of depth 1.

### 3.4 Logically equivalent formulas

We have seen in the previous section that we can construct a formula that is true everywhere except in the largest model. Now we are going to show that for every two different simple models of depth 1 we can construct a formula that has a different truth value in both models.

Theorem 3.8. Given two simple models of depth $1, M=\left\langle k_{0}, W, \leqslant, V\right\rangle$ and $M^{\prime}=$ $\left\langle k_{0}^{\prime}, W^{\prime}, \leqslant^{\prime}, V^{\prime}\right\rangle$ in $[P, \wedge, \vee, \neg]$ and $M \neq M^{\prime}$, then there exists a formula $A$ such that

$$
\begin{array}{ll}
\text { either } & M, k_{0} \vDash A \text { and } M^{\prime}, k_{0}^{\prime} \not \models A, \\
\text { or } & M, k_{0} \not \models A \text { and } M^{\prime}, k_{0}^{\prime} \vDash A .
\end{array}
$$

Proof. We can distinguish two cases, the case were $V\left(k_{0}\right) \neq V^{\prime}\left(k_{0}^{\prime}\right)$ and the case where $V\left(k_{0}\right)=V^{\prime}\left(k_{0}^{\prime}\right)$.

If $V\left(k_{0}\right) \neq V^{\prime}\left(k_{0}^{\prime}\right)$ it is simple to construct the formula $A$. Pick an atom $p$ that is true in one of the models, but not both. Set $A=p$ and this formula has different truth values in $k_{0}$ and $k_{0}^{\prime}$.

If $V\left(k_{0}\right)=V^{\prime}\left(k_{0}^{\prime}\right)$ we know that $\max (W) \neq \max \left(W^{\prime}\right)$, or else it would be the same model. So there is a valuation of a maximal node $Q \in \mathcal{P}(P)$ present in one of the models that is not present in the other. We can use (11) to construct the formula $A$ as follows

$$
A=\neg\left(\bigwedge_{p \in Q} p \wedge \bigwedge_{q \in P \backslash Q} \neg q\right)
$$

This $A$ is true at the model where $Q$ is absent and false in the model where $Q$ is present.
What theorem 3.8 tells us is that when we want to check whether to formulas are logically equivalent, we need to check every simple model of depth 1.

Now we would like to know how many formulas exist that are not logically equivalent given a $\# P=n$, we will call this number $F(n)$. For $n \leqslant 2$ the number of formulas $F(n)$ is already known and for $n=3$ a lower bound is known. These numbers can be found in [1].

We know how many simple models there are for each $P$, so we can find a upper bound for the number of formulas that are not logically equivalent. Since they are either true or false in each model

$$
F(n) \leqslant 2^{M(n)}
$$

Since $M$ is already increasing superexponentially, this upper bound for $F$ is increasing at an immense rate as you can see in the table below. The rightmost column are the values found in [1].

| $n=\# P$ | $F(n) \leqslant$ | $F(n)$ |
| :---: | ---: | ---: |
| 0 | 2 | 2 |
| 1 | 16 | 7 |
| 2 | 4194304 | 626 |
| 3 | $2^{310}$ | $\geqslant 2^{70}$ |

We can improve upon this upper bound, because we know that when a formula is true in the largest model, it is true everywhere. This leads to the following upper bound

$$
F(n) \leqslant 2^{M(n)-1}+1
$$

This reduces the upper bound by almost a factor 2 as can be seen in the table below. The rightmost column are once again the values found in [1].

| $n=\# P$ | $F(n) \leqslant$ | $F(n)$ |
| :---: | ---: | ---: |
| 0 | 2 | 2 |
| 1 | 9 | 7 |
| 2 | 2097153 | 626 |
| 3 | $2^{309}+1$ | $\geqslant 2^{70}$ |

It might seem impossible to create 2 formulas in the case of $\# P=0$ since there are no atoms. But we can use $\top$ and $\perp$, since $M, k \vDash \top$ and $M, k \not \models \perp$ for every model $M$.

In the case of $\# P=1$, we know that the number of formulas $F(1)=7$. In order to determine where the difference of 2 with our upper bound, $F(1) \leqslant 9$, comes from we have listed the formulas for each of the possible truth values in each simple model and the result is in table 3.1.

|  | Largest $M$ | $M_{1}$ | $M_{2}$ | $M_{3}$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\max (W)=$ | $\{\varnothing,\{p\}\}$ | $\{\varnothing\}$ | $\{\{p\}\}$ | $\{\{p\}\}$ |  |
| $V\left(k_{0}\right)=$ | $\varnothing$ | $\varnothing$ | $\varnothing$ | $\{p\}$ | $M, k_{0} \vDash$ |
|  | F | F | F | F | $p \wedge \neg p$ |
|  | F | F | F | T | $p$ |
|  | F | F | T | F | $\times$ |
|  | F | F | T | T | $\neg \neg p$ |
|  | F | T | F | F | $\neg p$ |
|  | F | T | F | T | $p \vee \neg p$ |
|  | F | T | T | F | $\times$ |
|  | F | T | T | T | $\neg p \vee \neg \neg p$ |
|  | T | T | T | T | $\neg(p \wedge \neg p)$ |

Table 3.1: Formulas for each of the possible truth values in the root of each of the simple models $M_{1}, M_{2}, M_{3}$ and the largest $M$ in $[P, \wedge, \vee, \neg]$ with $P=\{p\}$.

We can see that in line 3 and in line 7 we can not find a formula. The reason seems to be the fact that a formula can not be false in $M_{3}$ and at the same time be true in $M_{2}$. We will prove the following claim.

Claim 3.9. Let $M=\left\langle k_{0}, W, \leqslant, V\right\rangle$ be a simple model of depth 1 and let $M^{\prime}=\left\langle k_{0}^{\prime}, W^{\prime}, \leqslant^{\prime}, V^{\prime}\right\rangle$ be a simple model with the following properties

- $\max (W)=\max \left(W^{\prime}\right)$
- $V\left(k_{0}\right) \subset V^{\prime}\left(k_{0}^{\prime}\right)$
then for any formula $A$, it holds

$$
M, k_{0} \vDash A \Longrightarrow M^{\prime}, k_{0}^{\prime} \vDash A
$$

Proof. This proof is by induction over the complexity of $A$.
First the case when $A$ is an atom, let $A=p$. If

$$
M, k_{0} \vDash p
$$

then $p \in V\left(k_{0}\right)$. Since $V\left(k_{0}\right)$ is a subset of $V^{\prime}\left(k_{0}^{\prime}\right)$, we know $p \in V^{\prime}\left(k_{0}^{\prime}\right)$ and therefore

$$
M^{\prime}, k_{0}^{\prime} \vDash p
$$

The cases for disjunction and conjunction are straightforward by stating an induction hypothesis, which we do not need for negation.

Let $A=\neg B$.

$$
\begin{array}{ll}
\quad M, k_{0} \vDash \neg B & \text { \{by theorem } 2.2\} \\
\Longleftrightarrow \forall k \in \max (W)(M, k \not \models B) & \left\{\max (W)=\max \left(W^{\prime}\right)\right\} \\
\Longleftrightarrow & \\
\Longleftrightarrow \forall k \in \max \left(W^{\prime}\right)\left(M^{\prime}, k \not \models B\right) & \text { \{because of heredity \}} \\
\Longleftrightarrow \forall k \geqslant k_{0}^{\prime}\left(M^{\prime}, k_{0}^{\prime} \not \models B\right) & \\
\Longleftrightarrow & \text { \{definition of negation\} }
\end{array}
$$

This proves the case of negation.
Since the claim holds for all connectives, we can conclude by induction that it holds for all formulas.

Now that we have proven this claim, we have another restriction for possible truth values of a formula in simple models. So it restricts the number of possible formulas that are not logically equivalent. In the case of $\# P=1$ this restriction leads to all the different possibilities for formulas that are not logically equivalent. This means that every formula we can construct in $[P, \wedge, \vee, \neg]$ with $\# P=1$ is logically equivalent to one of the 7 formulas in table 3.1.

When we look at the case of $\# P=2$ there are 5 models that are influencing the possible truth values of formulas in 7 other models. For example, the simple model $M=\left\langle k_{0}, W, \leqslant, V\right\rangle$ with

- $V\left(k_{0}\right)=\varnothing$,
- $\max (W)=\{\{p, q\}\}$,
is influencing the 3 models with the same maximal node but with valuations $\{p\},\{q\}$ and $\{p, q\}$ in the root.

So even when we apply the restriction that follows from claim 3.9 we can still choose the truth value of 14 models freely ( 22 minus the largest model minus the 7 that are influenced by other models). Unfortunately, $2^{14}$ is still a lot more that the number $F(2)=626$, which is found in [1]. So it is clear that there are more restrictions to the possible truth value of formulas in the simple models, but it has proven quite challenging to find those restrictions.

### 3.5 Conclusion

In this chapter we have focused on the simple models. We have excluded the simple model of depth 0 , which we could do because there is a bisimilar model of depth 1 .

Then we created a formula to calculate the number of simple models given any set of atoms $P$.

We looked at the largest simple model and found some interesting properties. In order to check whether a formula is tautology it suffices and it is necessary to check the largest simple model.

Finally we have proven that we need all the simple models of depth 1 in order to distinguish between different formulas. And we have attempted to improve upon the upper bound for the number of formulas that are not logically equivalent.

## Chapter 4

## Conclusion

When we look at truth values in Kripke models in IPL, we have found that we can impose to conditions on our models, namely that they are finite and that they are rooted.

If we omit implication as a connective, we can further simplify the models by omitting the in-between nodes and the duplicate maximal nodes. This has led to the definition of simple models.

There are only a finite number of simple models, but the number increases superexponentially as the set of atoms get bigger. There is one model, the largest model, with the special property that it is necessary and sufficient to check a formula in that largest model to determine if a formula is a tautology.

We need all the simple models of depth 1 to distinguish between formulas that are not logically equivalent. In the cases $\# P=0$ and $\# P=1$ we have found all the possible formulas that are not logically equivalent and we have found upper bounds for the number of formulas when $\# P>1$.

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