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Multiperson strategic interactions in non-cooperative game theory

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> "*Like all of mathematics, game theory is a tautology whose conclusions are true because they are contained in the premises.*"
>
> Thomas Flanagan, *Game Theory and Canadian Politics Chapter 10, What Have We Learned?, p. 164.*, 1998.

Abstract

Using non-cooperative game theory we will study the strategic interdependence of multiperson interaction. This new mathematical area is a very useful tool to study the strategic interaction of players in certain games. Nowadays, it is extensively applied to economic and business applications, but also many other fields such as sports. In this thesis we will consider static and dynamic games of complete and incomplete information. All these type of games are based on the solution concept of Nash equilibrium and its refinements. Since the situation in the oligopolistic market appropriately reflects the general strategic interaction of players in all kind of games, we will mainly employ examples in this setting, such as car sales. At last we will apply the Nash equilibrium and its refinements to extended games that are encountered in the real world.

Keywords: strategic interaction, Nash equilibrium, static games, dynamic games, applications.

Contents

Chapter 1

Terminology

Due to a large number of notions the reader could consult this list as a guidance.

$\{1, 2, \ldots, I\}$	: set of all I players;
H	: information set;
$\mathcal{H}_i$	: collection of information sets for player i;
s_i	: strategy of player i;
$s = (s_1, \ldots, s_I)$	: strategy profile: vector of strategies of each player;
$s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_I)$	: strategy profile excluded s_i;
S_i	: strategy set of player i;
$\{S_i\}$	: set of strategy sets of all players;
$S = S_1 \times \ldots \times S_I$	: set of strategy profiles of all players;
$S_{-i} = S_1 \times S_{i-1} \times S_{i+1} \times \ldots \times S_I$	: set of strategy profiles of all players but player i;
$u_i : S \to \mathbb{R}$	: payoff function of player i;
$\{u_i(\cdot)\}$	: set of payoff functions of all players;
$\mathcal{X}$	: set of nodes;
$\mathcal{A}$	: set of possible actions;
$p : \mathcal{X} \to \{\mathcal{X} \cup \emptyset\}$	: assign a decision node $x \in \mathcal{X}$ to its predecessor node;
$s : \{\mathcal{X} \cup \emptyset\} \to \mathcal{X}$	: assign a decision node $x \in \{\mathcal{X} \cup \emptyset\}$ to a successor node;
$Z = \{x \in \mathcal{X} : s(x) = \emptyset\}$	: set of terminal nodes;
$T = \mathcal{X} \backslash Z$	: set of decision nodes;
$\alpha : \mathcal{X} \backslash \{x_0\} \to \mathcal{A}$	: assign an action to each decision node;
$c(x) = \{a \in \mathcal{A} : a = \alpha(x'), x' \in s(x)\}$	: set of possible actions at decision node x;
$H : \mathcal{X} \to \mathcal{H}$	: assign a decision node to an information set;
$C(H) = \{a \in \mathcal{A} : a \in c(x), x \in H\}$	: set of possible actions at information set H;
$\mathcal{H}$	: collection of information sets;
$\iota : \mathcal{H} \to \{0, 1, \ldots, I\}$	: assign information set to each player;
$\mathcal{H}_i = \{H \in \mathcal{H} : i = \iota(H)\}$	: collection of information sets of player i;
$\rho : \mathcal{H}_0 \times \mathcal{A} \to [0, 1]$	: assign probability to actions at the information sets of Nature;
$u_i : T \to \mathbb{R}$	: assign the utility at each terminal node for player i; players;
$\sigma_i(\cdot)$	: mixed strategy for player i;
$\sigma_i(s_{ki}) = \sigma_{ki}$	: probability that player i will use strategy s_{ki} for some $k \in \mathbb{N}$;

$\sigma = (\sigma_1, \ldots, \sigma_I)$	: profile of mixed strategy of each player;
$\sigma_{-i} = (\sigma_1, \ldots, \sigma_{i-1}, \sigma_{i+1}, \ldots, \sigma_I)$	: profile of mixed strategies excluded σ_i;
$\Delta(S_i) = \{(\sigma_{1i}, \sigma_{2i}, \ldots)\}$	: set of all mixed strategies for player i;
$S_i^+ \subset S_i$	: set of pure strategies with positive probability in the profile of mixed strategies σ for player i;
θ_i	: type of player i;
Θ_i	: set of all types for player i;
$\Theta = \Theta_1 \times \ldots \times \Theta_I$	: set of all of types for each player;
$F(\theta_1, \ldots, \theta_I)$	: joint probability distribution of the type of all player;
$s_i(\theta_i)$	: strategy choice given type θ_i for player i (i.e. decision rule);
s^B	: profile of decision rules;
$\bar{u}_i(\cdot)$	: expected utility function for player i;
$p(\cdot)$	: price function (or inverse demand function);
$x(\cdot)$	: demand function;
c	: cost per unit;
q	: quantity of a good produced.

Furthermore we would like to stress out the common orthography.

$\blacksquare$	: the end of an example;
subscript i	: referring to player i;
subscript $-i$	: referring to all players except player i.

Chapter 2

Motivation

Game theory distinguishes itself from the common statistical methods thanks to its plain approach. In the real world it turns out that game theory is very suitable to apply. That is due to the fact that many different scenarios can be described in a game form, as we will see throughout this thesis.

Constructing a game can be as tedious as one wants. Often, we should restrict ourselves to some assumptions, even though this limits the practical usefulness of the research. Obviously, more (relevant) information will provide us to study more complicated situations. Nowadays, however, there are just a handful of experimental studies done using game theory, partly due to the lack of practical sufficient game theoretical information.

Big companies such as Microsoft agreed that game theory is extremely advantageous in assisting (risky) decisions [1]. But we have to stress that game theory should not be used solely on making a big decision, but merely as an additional, helpful tool to get a deeper insight in the possible consequences of certain decision making. We urge for that this 'tool' strengthens the validity of statistical evidence as to whether decisions should be fortified or not. In spite of that, the topics we will discuss are indubitably interesting. One may argue that the results are straightforward, but this is not always the case. Some of the results are counter-intuitive, for example adding a road can increase the total travel time (Hagstrom and Abrams, 2001), Nigerian scammers should mention that they are from Nigeria even though most people are familiar with the scam (Herley, 2012), and competitors on a market will mostly settle near each other instead of spread out (Hotelling, 1929).

We start our study by understanding the definition of a game. Then, we will consider the basic (and most important) class of games, namely static games of complete information. Without this knowledge we would not be able to study non-cooperative game theory at all. To be more explicit, the solution concept of Nash equilibrium (that will be discussed in the section of static games of complete information) will be reformulated to obtain solution concepts for the other type of games, such as static games of incomplete information, but also dynamic games. The last bit of theory we will encounter is about the oligopoly market models, which has a more practical environment. This thesis will be finished by exploiting the theory to some real world situations.

[1] Microsoft even designed two games, *Project Waterloo* and *Doubloon Dash*, that can be played on Facebook that are based on the strategic interactions between real people in social networks.

Chapter 3

History

This section is partly thanks to “Introduction To Game Theory” (2007), retreived from http://www.rijpm.com/pre_reading_files/Open_Options_Introduction_To_Game_Theory.pdf.

Game Theory is a relatively new branch in mathematics. It studies strategic interactions among individuals (also called players, or agents) in games. Mathematical statisticians have found game theory extremely useful, not just for their discipline, but also for economics, business, philosophy, biology, computing science etc.

John Von Neumann and Oskar Morgenstern published in 1944 the book *Theory of Games and Economic Behavior*. This book is considered to be the start of game theory, and so it is the seminal work in areas of game theory. The work was about obtaining the optimal set of strategy equilibria for every possible strategy of their own by considering the possible payoffs of other players. However it is mainly focused on cooperative games - games where the player are allowed to form a coalition (but may compete against other coalitions).

Shortly thereafter, between 1950 and 1953, John (Forbes) Nash made seminal contribution to game theory, but he focused on non-cooperative games - games in which the players make their decisions independently. He developed the very famous solution concept of *Nash equilibrium*. Even the Nobel Prize in Economics (1994) has been awarded to him for his work to game theory. The latter concept of game theory will be the main focus of this thesis [1].

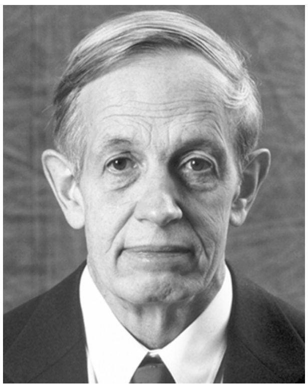


Figure 3.1: John Forbes Nash (Frängsmyr, 1995)

[1] On May 23, 2015, John Nash and his wife Alicia died in a car crash in New Jersey during their trip back home after John received the Abel prize in Norway (McCormack, 2015).

More than a decade later, in 1965, Reinhard Selten introduced the concepts of subgame perfect equilibrium, which is a refinement of the Nash equilibrium. Namely, it excludes the Nash equilibria that consist of unreasonable strategies using backward induction. Also R. Selten has been awarded the Nobel Prize in Economics (1994) for his work on game theory. Two years later, John Harsanyi refined the work of J. Nash for games of incomplete information games by introducing an 'external player'. And indeed, also J. Harsanyi won the Nobel Prize for Economics in 1994. We will pay some attention to both works as well.

Chapter 4

Introduction to non-cooperative game theory

Non-cooperative game theory is based on games where the players choose their own strategy independently, as its name already suggests. Depending on the rules of the game, the players may be able to observe the strategy chosen by the other players (Dragasevic et al., 2011). For example this is the case in a (non-cooperative) game where the players chose their strategy one after the other. But still, a player could hide her strategy so that the next player(s) are still uninformed.

For the study of non-cooperative games we are interested in the possible solution concepts in a game. A solution point where the players cannot improve their payoff is the so called a *Nash equilibrium* (or non-cooperative equilibrium) (Osborne and Rubinstein, 1994). This is currently the most important solution concept in non-cooperative game theory. Of course the usefulness of the solution concept and corresponding mathematical theorems depend heavily on the rules of the game. Therefore, we will consider several refinements of the Nash equilibrium. The refinements are based on the following type of games: static games of incomplete information and dynamic games of complete or incomplete information. The existence of the (mixed strategy) Nash equilibrium has been proven by John Nash.

Throughout this thesis we make the assumption that all players are *rational* (unless specified differently). In other words, by applying a certain strategy the players try their best to win the game, or at least try to maximize their own utility, or minimize the loss. In the context of oligopolistic market, the oligopolistic firms are the players. Also, we will only consider (finite) games with a finite number of players where each player has a finite set of strategies. Sometimes we will provide definitions that are formulated in a infinite sense (i.e. for games that are not finite), but these are just extended versions of the finite case and thus can easily be considered for finite games (Ferguson).

4.1 The game

The analysis in this section follows from Mas-Colell et al. (1995).

A game consists of the next four basic elements:

(i) The players: strategic decision makers participating in the game;

(ii) The rules: available actions/moves;

(iii) The outcomes: possible results after the performed action(s);

(iv) The rewards: payoff expressed as profit, happiness, quantity, utility etc.

Additionally, there are two ways of representing a game:

(i) The *normal* (or *strategic*) *form representation*, in which information is implicitly described using a cross table.

(ii) The *extensive form representation*, in which the information is explicitly described using game trees and information sets.

So we can regard the normal form representation as a brief version of the extensive form representation. Any game can be represented in normal and extensive form representation. For the analysis of games, however, we will just use the representation that is most convenient (and sometimes both). Often, for simultaneous-move games, the normal form representation is enough for the analysis, whereas the extensive form representation is recommended for sequential-move games. Examples of these two form representation will be given in sections 4.1.1 and 4.1.2.

Before we explain these form representations, we should understand two concepts of importance in non-cooperative game theory. Namely, the *information set* and the *strategy* of a player. For the extensive form representation, we require in addition a third concept, namely the *game tree*. We will give the definition of this concept in section 4.1.2. Now we introduce the two concepts that are used in both game form representations.

Definition 4.1.1. An *information set* for a player is a set of all possible actions that could be performed at a certain stage in a game for that particular player, given that the player does not know what the other players did before. We denote the information set that contains the decision node x by $H(x)$. Since an information set could contain several decision nodes, it follows that if $x_j \in H(x)$ for $j = 1, \dots, J$, then $x \in H(x_j)$ for all $j = 1, \dots, J$.

A strategy is a series of actions a player could perform at each information sets. Mathematically we define strategy as follows:

Definition 4.1.2. Denote the collection of information sets of player i by $\mathcal{H}_i$, the set of possible actions by $\mathcal{A}$ and the set of possible actions at information set H by $C(H) \subset \mathcal{A}$. Then a *strategy* for player i is the function $s_i : \mathcal{H}_i \to \mathcal{A}$ with the property that $s_i(H) \in C(H)$ for all $H \in \mathcal{H}_i$.

Remark 4.1. For convenience we will usually say that a player 'chooses (or performs) a strategy'. Yet, in some textbooks it is reported as a player that 1) does a move, 2) makes an action or 3) plays a strategy. Sometimes we will use these descriptions if it is more suitable.

4.1.1 The normal form representation

Definition 4.1.3. Consider a game with I players. Then a *normal form representation* Γ_N itemizes for each player i a set of strategies S_i and a utility function $u_1(s_1, \dots, s_I) : S_1 \times \ldots \times S_I \to \mathbb{R}$, where the Cartesian product of the set of strategies $S_1 \times \ldots \times S_I$ is the set of all strategy profiles. We write $\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}]$.

From definition 4.1.3 we inspect that a utility function $u_i(\cdot)$ assigns a payoff (indicated by a real number) for player i to a certain strategy profile.

Remark 4.2. The set of strategies for player i in the normal form representation will preferably be denoted as $S_i := \{s_{1i}, s_{2i}, \ldots\}$, where s_{ji} (for $j \in \mathbb{N}$) is the jth strategy for player i.

Regularly the normal form representation Γ_N provides the most relevant information in quite a simple fashion. Therefore, when possible, we wish to use this representation to report a game.

Example 4.1.1. *Matching Pennies, sequential-move.*
Consider the following matching pennies game:

(i) The players: player 1 and player 2;

(ii) The rules: player 1 starts off by putting a coin on the table, either heads up or tails up. Then player 2 starts to play by putting another coin on the table, either heads up or tails up;

(iii) The outcomes: if both coins show heads up or tails up, player 1 receives 1 euro from player 2. Otherwise, if one coin shows heads up, and the other one shows tails up, then player 2 receives 1 euro from player 1 (so it is very favorable to be player 2) [1];

(iv) The rewards: the payoff function for player 1 is:

$$u_1(s_1, s_2) = \begin{cases} +1 & \text{if} \quad (s_1, s_2) = (H, H) \text{ or } (T, T) \\ -1 & \text{if} \quad (s_1, s_2) = (H, T) \text{ or } (T, H) \end{cases}$$

where s_1, s_2, H and T denote the strategy for player 1, the strategy for player 2, heads up and tails up respectively. The payoff function for player 2 is simply $u_2(s_1, s_2) = -u_1(s_1, s_2)$.

Since this is a sequential-move game, player 1 has two possible strategies ($S_1 = \{s_{11}, s_{21}\}$), whereas player 2 has four possible strategies ($S_2 = \{s_{12}, \ldots, s_{42}\}$):

1. s_{11}: Play H;
2. s_{21}: Play T;
3. s_{12}: Play H if player 1 plays H or T;
4. s_{22}: Play T if player 1 plays H or T;
5. s_{32}: Play H if player 1 plays T, or play T if player 1 plays H;
6. s_{42}: Play T if player 1 plays T, or play H if player 1 plays H.

Using the information of the game above, we represent the normal form as below:

		Player 2			
		s_{12}	s_{22}	s_{32}	s_{42}
Player 1	s_{11}	$1, -1$	$-1, 1$	$-1, 1$	$1, -1$
	s_{21}	$-1, 1$	$1, -1$	$-1, 1$	$1, -1$

where in each entry the left number denotes the payoff of player 1, while at the same time the right number denotes the payoff of player 2. For example, in the left top entry we have $u_1(s_{11}, s_{12}) = 1$ and $u_2(s_{11}, s_{12}) = -1$. ■

[1] Recall from the first paragraph of section 4 that the players may observe the strategy choice of other players.

4.1.2 The extensive form representation

As mentioned earlier, the third concept of importance to describe an extensive form representation is the game tree. In fact, an extensive form representation completely describes a game by using just a game tree.

Definition 4.1.4. A *game tree* is tree-structured graph consisting of the following elements [2]:

1. *Initial decision node* (x_0): the very first node where the first action in the game is done.
2. *Nodes* (x): represents the stage of the game where a certain player could perform an action (given the action of the players before).
3. *Branches* (a): represents the possible action for a player at a stage in the game. A branch connects two nodes.
4. *Terminal nodes* (z): the nodes that do not have successor nodes.

Since we now know how a game and a game tree are constructed, it is not difficult to study an extensive form representation.

From the extensive form representation we can extract the following:

$\mathcal{X}$	: set of nodes;
$\mathcal{A}$	: set of possible actions;
$\{0, 1, \ldots, I\}$	: set of players;
$p : \mathcal{X} \to \{\mathcal{X} \cup \emptyset\}$	: assign a decision node $x \in \mathcal{X}$ to its predecessor node;
$s : \{\mathcal{X} \cup \emptyset\} \to \mathcal{X}$	: assign a decision node $x \in \{\mathcal{X} \cup \emptyset\}$ to a successor node [3];
$Z = \{x \in \mathcal{X} : s(x) = \emptyset\}$	: set of terminal nodes;
$T = \mathcal{X} \backslash Z$	: set of decision nodes;
$\alpha : \mathcal{X} \backslash \{x_0\} \to \mathcal{A}$	: assign an action to each decision node [4];
$c(x) = \{a \in \mathcal{A} : a = \alpha(x'), x' \in s(x)\}$	: set of possible actions at decision node x;
$H : \mathcal{X} \to \mathcal{H}$	: assign a decision node to an information set;
$C(H) = \{a \in \mathcal{A} : a \in c(x), x \in H\}$	: set of possible actions at information set H;
$\mathcal{H}$	: collection of information sets;
$\iota : \mathcal{H} \to \{0, 1, \ldots, I\}$	: assign information set to each player;
$\mathcal{H}_i = \{H \in \mathcal{H} : i = \iota(H)\}$	: collection of information sets of player i;
$\rho : \mathcal{H}_0 \times \mathcal{A} \to [0, 1]$	: assign probability to actions at the information sets of Nature;
$u_i : T \to \mathbb{R}$	: assign the utility at each terminal node for player i;
$u = \{u_1(\cdot), \ldots, u_I(\cdot)\}$	: set of payoff functions for all players.

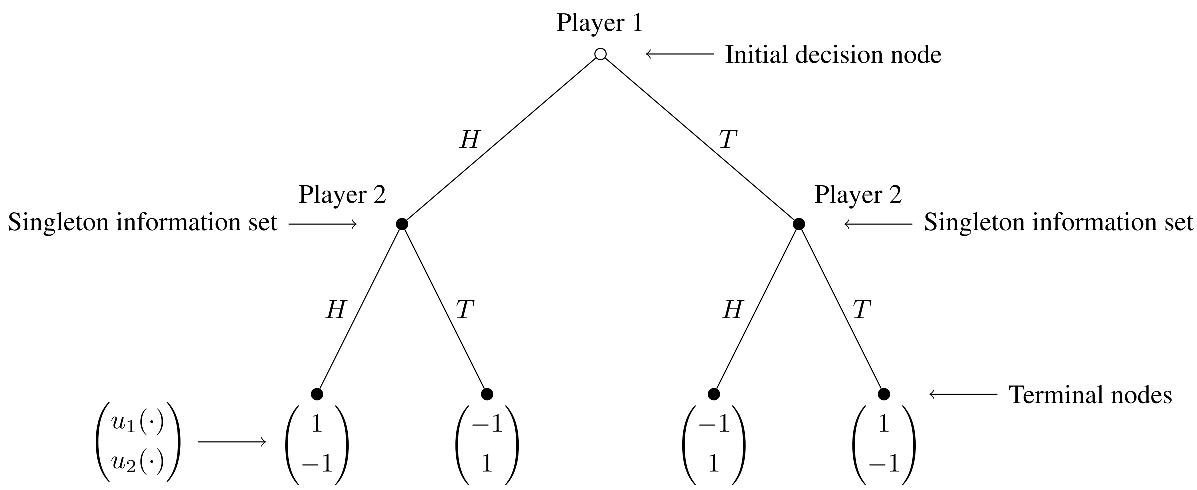
Formally we write $\Gamma_E = \{\mathcal{X}, \mathcal{A}, I, p(\cdot), \alpha(\cdot), \mathcal{H}, H(\cdot), \iota(\cdot), \rho(\cdot), u\}$. Even though the extensive form representation provides us comprehensive information about a game, we will mainly

[2] Any game tree constructed in this thesis are thanks to Chen (2013).
[3] We must have $s(x) = p^{-1}(x)$ and $\{s(x)\} \cup \{p(x)\} = \emptyset$, otherwise we would not have a game tree structure.
[4] The action from decision node x to node $x' \in s(x)$ and to node $x' \in s(x)$ are different if $x' \neq x''$.

use the normal form representation (if it is clearly not necessary to use an extensive form game), since it presents the relevant information in a neat way.

Remark 4.3. To avoid any confusions we should mention that player 0 represents *Nature*, which we will introduce in section 6. However it would not affect the interpretation of extensive form representation. The reader may find it convenient to suppose $\{1, 2, \ldots, I\}$ as the set of players.

Example 4.1.2. Recall example 4.1.1. The extensive form representation is depicted in the game tree below.

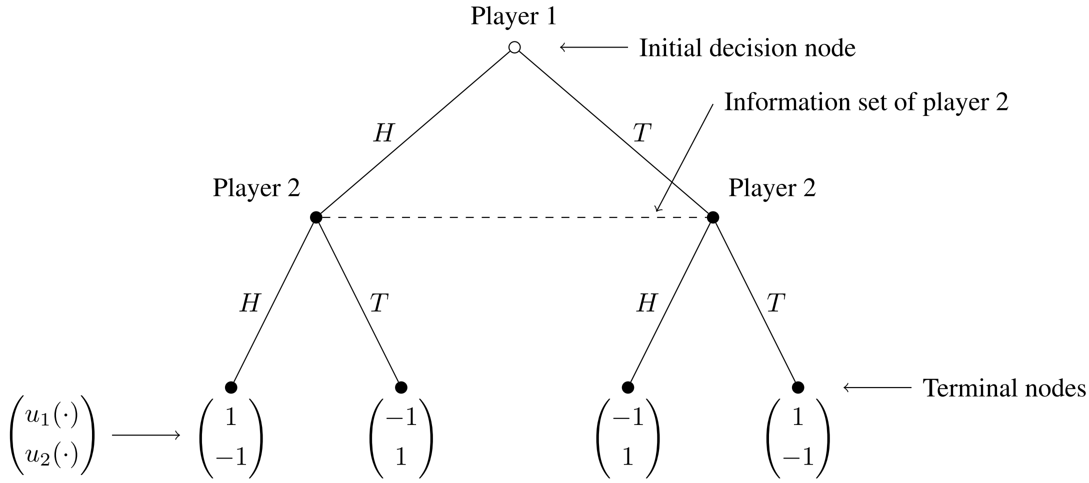


An information set that contains only a single node is called a *singleton* information set. ■

Example 4.1.3. *Matching Pennies, simultaneous-move.*
Now assume that player 1 and player 2 have to put down their coin simultaneously. This implies that player 2 does not know about the action of player 1. In other words, player 2 has two strategies, instead of four as in example 4.1.1: play H or play T. The normal form representation is therefore:

		Player 2 H	Player 2 T
Player 1	H	$1, -1$	$-1, 1$
	T	$-1, 1$	$1, -1$

The structure of this simultaneous-move game is almost similar to the sequential-move version. The difference is that player 2 (here) does not observe any information played prior to her choice. Stated differently, the information set of player 2 contains two decision nodes. Commonly we indicate the information set by drawing a dashed line that connects the decision nodes, as shown below.



■

Remark 4.4. We could also use the above extensive form representation in a sequential game with the rule that player 1 puts her coin down but she keeps her hand on top of the coin, or at least so that player 2 cannot see the coin and thus does not know at which decision node she is.

In example 4.1.2 we see that we do not have such an information set as in example 4.1.3, since player 2 could observe the action of player 1 and so each of the two possible actions of player 2 are predetermined by the action of player 1. Therefore both of the nodes corresponding to player 2 are singleton information sets in that case.

4.2 Some classifications of games

It is sufficient to use examples 4.1.1 and 4.1.3 to outline a triplet of different classification of games that are common in most game theory literature (Gibbons, 1992; Leyton-Brown, 2008).

1. An *one-shot/stage game* is a game in which each player only can perform one strategy, but they do not observe the strategy choice of other players. The game in example 4.1.3 is a *stage game*. However, the game in example 4.1.1 consists of two stages. Namely, player 1 has to do a move in the first stage, and player 2 then continues in the second stage. Such games are called *multi-stage games* [5]. Games that repeat an one-shot game more than once are called *repeated games*. These kind of games will be explained in section 7;

2. The game in either examples could be classified as *symmetric games*. That are games where $u_i(s_1, \ldots, s_I) = u_{\pi(i)}(s_{\pi(1)}, \ldots, s_{\pi(I)})$ for any permutation π. Notice that in such kind of games all players share the same set of available strategies. The general 2×2 normal form representation of a symmetric 2-players game is:

		Player 2 A	Player 2 B
Player 1	A	α, α	β, γ
	B	γ, β	δ, δ

[5]There is no direct link between the term ‘stage’ in stage game and ‘stage’ mentioned in definition 4.1.4.

An *asymmetric game* is a game that is not symmetric;

3. The last classification of games is the so called *zero-sum* games. That are games for which the sum of the utilities equals zero for every strategy profile:

$$\sum_{i=1}^{I} u_i(s) = 0 \quad \text{for all } s \in S$$

Accordingly the matching pennies game (both sequential- and simultaneous-move) is an example of a zero-sum game. A *nonzero-sum* game is a game for which there exists a strategy profile $s' \in S$ such that:

$$\sum_{i=1}^{I} u_i(s') \neq 0$$

4.3 Further classification of games

We just presented three different classification of games, namely one-shot vs. repeated games, symmetric vs. asymmetric games and zero-sum vs. nonzero-sum games. We only introduced these three classifications briefly, since these are non-essential for the purpose of our further analysis, but yet common classifications in game theory. A classification of greater importance, which we actually encountered earlier (see the examples in section 4.1), is the following:

1. *Perfect information:* All information sets for all players are singleton information sets. In other words, at any stage in the game all players are perfectly informed about the actions done before (by all players) (Sandholm, 2014).

2. *Imperfect information:* If the game is not of perfect information. That is, if there is a stage in the game where a player (that has to do an action) does not know exactly which action is performed just before. This is equivalent to say that there exists an information set containing at least two decision nodes (Sandholm, 2014).

Remark 4.5. All players in example 4.1.1 are informed about the action done before (except for player 1, but no action is performed prior to the action of player 1). So that game is an example of a perfect information game. However, all players in example 4.1.3 are not informed by the action done before. This implies that the game in example 4.1.3 is an example of an imperfect information game.

4.4 Mixed strategies

In section 4.1 we presented the notion of strategy. So far, we supposed that each player deterministically choose their strategy. Formally, we say that a deterministically chosen strategy is a *pure strategy*. For convenience we make no distinction between strategy and pure strategy. So we also say that $S_i = \{s_{1i}, s_{2i}, \ldots\}$ is the set of pure strategies for player i (cf. remark 4.2). When we assign to each pure strategy a certain probability, we allow the players to choose between their strategies randomly (Alizon and Cownden, 2009).

Definition 4.4.1. A *mixed strategy* for player i is the function $\sigma_i : S_i \to [0,1]$, that assigns to each pure strategy $s_{ji} \in S_i$ a probability $\sigma_i(s_{ji}) \in [0,1]$ such that $\sum_{s_{ji}\in S_i} \sigma_i(s_{ji}) = 1$. Intuitively, $\sigma_i(s_{ji}) := \sigma_{ji}$ is the probability that player i will use the pure strategy s_{ji}, so $\sigma_i = (\sigma_{1i}, \sigma_{2i}, \ldots)$ is the probability distribution over the pure strategies $s_{1i}, s_{2i}, \ldots$ for player i.

We denote the normal form representation of a game with mixed strategies in a similar manner as in a game with pure strategies, namely by $\Gamma_N = [I, \{\triangle(S_i)\}, \{u_i(\cdot)\}]$, where $\triangle(S_i) = \{(\sigma_{1i}, \sigma_{2i}, \ldots)\}$ is the set of all mixed strategies for player i.

We have to note that the concept of mixed strategies can sometimes lead to confusions regarding to the interpretation (or rather notation) of strategy profiles, but with a probabilistic mindset it should at least be more bearable. For convenience, we will illustrate the interpretation of mixed strategies (compared to pure strategies) in the next example by mainly focusing on the notational aspect.

Example 4.4.1. *Pure vs. mixed strategies.*
Consider a simultaneous-move game with 2 players. Player 1 can choose between strategies A_1 and A_2, and player 2 between strategies B_1 and B_2. Then the four possible strategy profiles (consisting of pure strategies) are (A_1, B_1), (A_1, B_2), (A_2, B_1) and (A_2, B_2).

Now, let us assume that a mixed strategy for player 1 is for which A_1 is chosen with probability $\alpha \in [0,1]$ and A_2 with probability $1-\alpha$, i.e. $\sigma_1 = (\alpha, 1-\alpha)$. Similarly, assume that a mixed strategy for player 2 is for which B_1 is chosen with probability $\beta \in [0,1]$ and B_2 with probability $1-\beta$, i.e. $\sigma_2 = (\beta, 1-\beta)$. Then, a strategy profile of mixed strategy is $\sigma = (\sigma_1, \sigma_2) = ((\alpha, 1-\alpha), (\beta, 1-\beta))$. Observe the difference in notation (e.g. here we have numbers instead of letters) and we could have infinitely many strategy profiles of mixed strategies! ■

Remark 4.6. We consider mixed strategies when we allow players to randomize over their pure strategies. This implies that the outcome is random as well. Therefore in games with mixed strategies we should consider the expected utility function:

$$u_i(\sigma) := E_\sigma[u_i(s)] = \sum_{s\in S} u_i(s)\,[\sigma_1(s_1)\sigma_2(s_2)\cdots\sigma_I(s_I)]$$

Notice the slight abuse of notation: we use $u_i(\cdot)$ for the utility function as well as for the expected utility function. The reason for this is that in none of the cases this notation will have a high impact on the result of the analysis.

Remark 4.7. A game with pure strategies is a special case of a game with mixed strategies. Namely, a game with pure strategy is for which each player plays one and only one mixed strategy with probability 1 and the remaining mixed strategies with probability 0. So therefore we could consider pure strategies as well when talking about a game with mixed strategies.

4.5 Additional notations

We will end this section with some notations that will be useful in the remaining of this thesis.

$s = (s_1, \ldots, s_I)$	: strategy profile: a vector of the strategy for all players;
$s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_I)$	: strategy profile excluded s_i;
$S_{-i} = S_1 \times S_{i-1} \times S_{i+1} \times \ldots \times S_I$	: set of strategy profiles of all players but player i; of all players;
$\sigma = (\sigma_1, \ldots, \sigma_I)$	: profile of mixed strategies for all players;
$\sigma_{-i} = (\sigma_1, \ldots, \sigma_{i-1}, \sigma_{i+1}, \ldots, \sigma_I)$	: profile of mixed strategies excluded σ_i;

Notice that the strategy profile (as well as the profile of mixed strategies) consists of only one strategy for each player. Pay attention to the difference resulting from the subscripts: e.g. s is a I-vector, s_i is an element of the vector s, and s_{-i} is a $(I-1)$-vector. Similar arguments applies for the remaining terms in the list above.

Chapter 5

Static games of complete information

The theory considered in this chapter is thanks to *Microeconomic Theory* (Mas-Colell et al., 1995).

Static games of complete information (not to confuse with (im)perfect information, see section 4.3) are one-shot games where the players simultaneously choose one strategy, so that each player does not have any knowledge about the strategy choice of the other players. But, each player does know the complete information of the game: available strategies and payoff functions for every player. Actually, one-shot sequential games can also be regarded as static games, as long as each player have no knowledge of the strategy choice of the other players.

This section covers the basic idea of non-cooperative game theory, such as dominant and dominated strategies, and also the Nash equilibrium. These are fundamental for understanding the solution concepts that we will discuss in the upcoming sections. In the next section we will turn our attention to static games of incomplete information, which is slightly more complicated. The complement of a static game is a dynamic game, which we will discuss in section 7. It turns out that we 'only' have to refine the solution concept of the original Nash equilibrium (that we will introduce in this section) to obtain solution concepts in dynamic games. For this reason, the structure of this thesis gradually encourage the understanding of the solution concepts.

5.1 Dominant and dominated strategies

We start this section with a definition that we will frequently engage for solving game.

Definition 5.1.1. Consider a game with pure strategies, so $\Gamma_N = \{I, \{S_i\}, \{u(\cdot)\}\}$. Then a strategy $s_i \in S_i$ is *strictly dominant* for player i if $u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$ for all $s_i' \in S_i \backslash \{s_i\}$ and all $s_{-i} \in S_{-i}$.

Example 5.1.1. *Prisoner's Dilemma.*
This is a very famous example in game theory with a very memorable outcome. Two players are arrested because of committing a horrible crime. The court of justice proposed the following: if one of the two criminal confesses, then the one who confesses has to stay in jail for just 1 year, whereas the criminal that does not confess has to stay in jail for 10 years. However, if both criminals confess or both do not confess, then each of them have to stay in jail for 5 or 2 years respectively. The two criminals have been separated from each other, so that they cannot communicate. The game can be summarized as follows:

(i) The players: criminal 1 and criminal 2;

(ii) The rules: each of the criminals have to simultaneously make a choice to confess or not.

(iii) The outcomes: if both criminals confess, then each of them has to stay in jail for 5 years, whereas if they do not confess they have to stay in jail for 2 years each. Otherwise the one who confesses has to stay in jail for just 1 and the other one for 10 years.

(iv) The rewards: the payoff function for player 1 is:

$$u_1(s_1, s_2) = \begin{cases} -1 & \text{if} \quad (s_1, s_2) = (C, NC) \\ -2 & \text{if} \quad (s_1, s_2) = (NC, NC) \\ -5 & \text{if} \quad (s_1, s_2) = (C, C) \\ -10 & \text{if} \quad (s_1, s_2) = (NC, C) \end{cases}$$

where s_1, s_2, C and NC denote the strategy for criminal 1, the strategy for criminal 2, confessing and not confessing respectively. This is a symmetric game (see section 4.2), so the payoff function for player 2 is simply $u_2(s_1, s_2) = u_{\pi(2)}(s_{\pi(1)}, s_{\pi(2)}) = u_1(s_{-1}, s_{-2}) = u_1(s_2, s_1)$.

The normal form representation is as follows:

		Criminal 2	
		C	NC
Criminal 1	C	$-5, -5$	$-1, -10$
	NC	$-10, -1$	$-2, -2$

Let us take a look at the situation for criminal 1:

▷ If criminal 2 does confess, then criminal 1 should also confess, because then she should stay in jail for 1 year instead of 2 years;

▷ If criminal 2 does not confess, then criminal 1 should confess as well, because then she has to stay in jail for 5 years instead of 10 years.

Mathematically we have $u_1(C, s_2) > u_1(C, s_2)$ for all $s_2 = s_{-1} \in S_{-1} = S_2$. Therefore the strategy of confessing is the strictly dominant strategy for player 1. The same reasoning applies for criminal 2 due to symmetry. ■

Due to the rationality of both criminals, each of them should stay in jail for 5 years (since they will confess). However, from the normal form representation above we see that if both criminals do not confess, then they should stay in jail for just 2 years each. So although both players applied their strictly dominant strategy, the outcome is on the contrary jointly undesirable.

Many situations in the competitive oligopolistic markets follow the same principle as the Prisoner's Dilemma. Non-cooperative oligopolies will choose the price of their goods that eventually results in a lower profit. Instead, if the firms were allowed to cooperate, then this situation would be favorable for both firms, but since a cartel is illegal the firms would be guided by the Nash equilibrium (which is beneficial for the consumers).

A second definition that we will provide is the complement of the strictly dominant strategy.

Definition 5.1.2. Consider a game with pure strategies, so $\Gamma_N = \{I, \{S_i\}, \{u(\cdot)\}\}$. Then a strategy $s_i \in S_i$ is *strictly dominated* for player i if there exists $s_i' \in S_i$ such that

$$u_i(s_i, s_{-i}) < u_i(s_i', s_{-i})$$

for all $s_{-i} \in S_{-i}$. The strategy s_i' is said to strictly dominate strategy s_i.

Example 5.1.2. *Car sales 1.*
Consider 2 car salesmen that have to determine what type of car they will sell in the city. Salesman Alef can choose to either sell car type A or B, whereas salesman Ernst can choose to sell either car type X, Y or Z. Let $u_i(\cdot)$ be the utility function representing the yearly profit in million euros for player i. The normal form representation is shown below:

		Ernst		
		X	Y	Z
Alef	A	2, 1	2, 3	1, 2
	B	1, 4	1, 2	3, 1

By definition neither A nor B are dominated strategies for Alef. Besides, by comparing strategies Y and Z for Ernst, we see that choosing Y is always better than choosing Z, regardless of the strategy of Alef:

- ▷ if Alef chooses strategy A, then Ernst would choose strategy Y since the profit for Ernst will be 3 million euros per year, whereas the profit will be 2 million euros for strategy Z;
- ▷ if Alef chooses strategy B, then again strategy Y (2 million profit) would be a better than strategy Z (1 million profit) for Ernst.

Therefore in this case we have that strategy Z is strictly dominated by strategy Y for Ernst. We could also say that strategy Y is a strictly dominant strategy. ■

Definition 5.1.3. Consider a game with pure strategies, so $\Gamma_N = \{I, \{S_i\}, \{u(\cdot)\}\}$. Then a strategy $s_i \in S_i$ is *weakly dominant* for player i if there exists another strategy $s_i' \in S_i$ such that

$$u_i(s_i, s_{-i}) = u_i(s_i', s_{-i}) \quad \text{and} \quad u_i(s_i, s_{-i}) > u_i(s_i'', s_{-i})$$

for some $s_i'' \neq s_i$ and $s_i'' \neq s_i'$ for all $s_{-i} \in S_{-i}$.

Definition 5.1.4. Consider a game with pure strategies, so $\Gamma_N = \{I, \{S_i\}, \{u(\cdot)\}\}$. Then a strategy $s_i \in S_i$ for player i is *weakly dominated* if there exists a strategy $s_i' \in S_i$ that weakly dominates strategy s_i.

Example 5.1.3. *Car sales 2.*
Reconsider example 5.1.2, but now Alef has the choice to sell either car type C or D. The corresponding normal form representation is:

		Ernst		
		X	Y	Z
Alef	C	0, 3	3, 3	1, 2
	D	1, 4	3, 5	0, 5

In this new setting, we observe:

- ▷ if Alef chooses strategy C, then Ernst should either choose strategy X or Y;
- ▷ if Alef chooses strategy D, then Ernst should choose strategy Y or Z.

So Y is always good choice for Ernst. But now he could also choose strategy X or Z, depending on the strategy of Alef. Moreover, for Ernst, the strategy X and Z are weakly dominated by strategy Y. Or, strategy Y weakly dominates strategy X and Z. ■

5.2 Iterated deletion of strictly dominated strategies

Consider example 5.1.2 once again, because we used the solution concept of the so called *iterated deletion of strictly dominated strategies* (IDOSDS) partly there. In the IDOSDS we assume that any player is rational and they know about the rationality of the other players. By this we mean that every player knows that every player is rational, and that every player knows that every player knows that every player is rational, and that every player knows that every player knows that every player knows that every player is rational, and so on. This assumption is related to *rationalizability*, which we will describe in more detail in section 5.4.

Recall that strategy Z is strictly dominated by strategy Y. So here, Alef is sure about the fact that Ernst would not choose strategy Z. The first iteration of IDOSDS suggests us to delete the strictly dominated strategies:

		Ernst		
		X	Y	Z
Alef	A	2, 1	2, 3	1, 2
	B	1, 4	1, 2	3, 1

$\xrightarrow{\text{delete strategy} Z}$

		Ernst	
		X	Y
Alef	A	2, 1	2, 3
	B	1, 4	1, 2

By considering the new normal form representation, we see that strategy B is strictly dominated by strategy A for Alef. Therefore, by the IDOSDS, we are suggested to delete strategy B as well:

		Ernst	
		X	Y
Alef	A	2, 1	2, 3
	B	1, 4	1, 2

$\xrightarrow{\text{delete strategy } B}$

		Ernst	
		X	Y
Alef	A	2, 1	2, 3

By the assumption of IDOSDS, Ernst knows about the rationality of Alef (and that Alef knows about the rationality of Ernst). Hence we find that the optimal strategy profile for this game is $s = (s_1, s_2) = (A, Y)$ so that the optimal utility for Alef and Ernst is $u_1(s_1, s_2) = u_1(A, Y) = 2$ and $u_2(s_1, s_2) = u_2(A, Y) = 3$ respectively.

5.3 Generalization to mixed strategies

So far we considered the definition of dominances for pure strategies. We can simply generalize this to mixed strategies.

Definition 5.3.1. Consider a game with mixed strategies, so $\Gamma_N = [I, \{\Delta(S_i)\}, \{u(\cdot)\}]$. Then a strategy $\sigma_i \in \Delta(S_i)$ is *strictly dominant* for player i if

$$u_i(\sigma_i, \sigma_{-i}) > u_i(\sigma_i', \sigma_{-i})$$

for all $\sigma_i' \neq \sigma_i$ and all $\sigma_{-i} \in \prod_{j \neq i} \Delta(S_j)$.

Definition 5.3.2. Consider a game with mixed strategies, so $\Gamma_N = [I, \{\Delta(S_i)\}, \{u(\cdot)\}]$. Then a strategy $\sigma_i \in \Delta(S_i)$ is *strictly dominated* for player i if there exists $\sigma_i' \in \Delta(S_i)$ such that

$$u_i(\sigma_i, \sigma_{-i}) < u_i(\sigma_i', \sigma_{-i})$$

for all $\sigma_{-i} \in \prod_{j \neq i} \Delta(S_j)$. The strategy σ_i' is said to strictly dominate strategy σ_i.

5.4 Rationalizable strategies

As mentioned in section 5.2, *rationalizability* yields strategies that are based on common knowledge of rationality among players. It will turn out that the solution concept of rationalizable strategies is closely related to the solution concept of Nash equilibrium (and thus also many other solution concepts). Therefore, the importance of this solution concept should not be underrated. Relating to these solution concepts, the notion of *best-responses* will turn out to play a crucial role in the further analysis of this thesis.

Definition 5.4.1. Consider a game with mixed strategies, so $\Gamma_N = [I, \{\Delta(S_i)\}, \{u(\cdot)\}]$. Then a strategy $\sigma_i \in \Delta(S_i)$ is a *best-response* for player i to the opponent's strategy $\sigma_{-i} \in \prod_{j \neq i} \Delta(S_j)$ if

$$u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i})$$

for all $\sigma_i' \in \Delta(S_i)$.

Note the subtle difference between definition 5.3.1 and 5.4.1. Namely, definition 5.3.1 regards all $\sigma_{-i} \in \prod_{j \neq i} \Delta(S_j)$. Whereas definition 5.4.1 says that a strategy σ_i for player i is a best-response to an opponent's strategy σ_{-i} when player i expects the opponent to choose strategy σ_{-i} and at the same time σ_i is the optimal strategy for σ_{-i}. Now knowing what a best-response is, we will introduce the concept of rationalizable strategies:

Definition 5.4.2. Consider a game with mixed strategies, so $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$. The strategies that remain left after IDOSDS are the *rationalizable strategies*. The set of rationalizable strategies is the set of strategies from a player that are optimal in a game where all players are aware of each other's rationality. A strategy profile consisting of only rationalizable strategies is also called a *rationalizable strategy profile*.

Recall from remark 4.7 that a pure strategy is a special case of a mixed strategy, so definition 5.4.2 can easily be generalized for games with pure strategies.

Remark 5.1. A strictly dominated strategy is never a best-response. That is, the set of rationalizable strategies is a subset of the set of strategies that are left after IDOSDS.

Example 5.4.1. Consider an example with the following normal form representation:

		Player 2			
		b_1	b_2	b_3	b_4
	a_1	$7, -1$	$1, 1$	$1, 1$	$1, -2$
Player 1	a_2	$1, 2$	$6, 1$	$1, 6$	$3, 5$
	a_3	$1, 2$	$1, 6$	$6, 1$	$3, 5$
	a_4	$1, 2$	$5, 3$	$5, 3$	$4, 4$

Strategy b_1 is strictly dominated by strategies b_2 and b_3 whenever $\sigma_2(b_2) = \frac{1}{2}$ and $\sigma_2(b_3) = \frac{1}{2}$. Therefore using IDOSDS we could delete strategy b_1. This leads to the fact that strategy a_1 is then strictly dominated by strategy a_4. So instead we could use the following normal form representation:

		Player 2		
		b_2	b_3	b_4
	a_2	$6, 1$	$1, 6$	$3, 5$
Player 1	a_3	$1, 6$	$6, 1$	$3, 5$
	a_4	$5, 3$	$5, 3$	$4, 4$

Now there are no dominated strategies and thus we cannot apply IDOSDS to delete some of the remaining strategies. Let us look for the rationalizable strategies for both players. First, we consider the situation for player 1:

- ▷ if player 2 chooses strategy b_2, then strategy a_2 is the best-response for player 1;
- ▷ if player 2 chooses strategy b_3, then strategy a_3 is the best-response for player 1;
- ▷ if player 2 chooses strategy b_4, then strategy a_4 is the best-response for player 1;

Similarly for player 2 we observe:

- ▷ if player 1 chooses strategy a_2, then strategy b_3 is the best-response for player 2;
- ▷ if player 1 chooses strategy a_3, then strategy b_2 is the best-response for player 2;
- ▷ if player 1 chooses strategy a_4, then strategy b_4 is the best-response for player 2;

Hence the sets $\{a_2, a_3, a_4\}$ and $\{b_2, b_3, b_4\}$ are sets of rationalizable strategies for player 1 and 2 respectively. ■

5.5 The Nash equilibrium

In section 5.2 we have seen that we could use the solution concept IDOSDS to obtain the optimal strategy profile in a game with dominated strategies. Unfortunately in many games there are no dominated strategies. Therefore we should come up with alternative solution concepts. In this section we will introduce the solution concept of *Nash equilibrium*, which is currently the most common in non-cooperative game theory.

Definition 5.5.1. Consider a game with mixed strategies, so $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$. Then the strategy profile of mixed strategies σ is a *Nash equilibrium* if

$$u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i})$$

for all $\sigma_i' \in \Delta(S_i)$, $i = 1, \ldots, I$.

In words, a Nash equilibrium is a strategy profile for which each player has no incentive to change their strategy given the strategy choice of the other players.

Remark 5.2. Keeping in mind that a game with pure strategies is a special case of a game with mixed strategies, we can simply reformulate definition 5.5.1 in terms of pure strategies by simply replacing $\Delta(S_i) \to S_i$, $\sigma_i \to s_i$, $\sigma_i' \to s_i'$ and $\sigma_{-i} \to s_{-i}$.

One could ask themselves what the difference between rationalizable strategies (for a certain strategy of the opponent) and Nash equilibrium is. So let us explain in some detail the striking difference. Consider the point of view of player i. For rationalizability we assume that all players know about each other's rationality and payoff functions. Then the rationalizable strategies consists of the strategies that is the best-response for a strategy that player i expects the opponent chooses (i.e. when player i make a conjecture about the opponent's strategy). However, a Nash equilibrium is a strategy profile (either pure or mixed) for which the strategy for each player is a best-response to the other strategies in the strategy profile. Or else, in a more mathematical fashion, the strategy profile $\sigma = (\sigma_1, \ldots, \sigma_I)$ is a Nash equilibrium if σ_i is a best-response to σ_{-i} for all $i \in \{1, \ldots, I\}$. As a result, a game could have more than one Nash equilibrium.

To relate the Nash equilibrium to rationalizability, we could say the following: the Nash equilibrium is obtained when we take for granted (so we confirm the conjecture from the rationalizability) that a certain strategy is chosen by the opponents. In game theoretical sense, this implies that any Nash equilibrium consists of rationalizable strategies only, but not any rationalizable strategy is necessarily contained in a Nash equilibrium. This is the noteworthy difference between the rationalizability and the Nash equilibrium that the reader should not be confused about.

For illustration, let us consider the normal form representation in example 5.4.1. From the perspective of player 1, the optimal strategy profiles are (a_2, b_2), (a_3, b_3) and (a_4, b_4). However, from the perspective of player 2, the optimal strategy profiles are (a_2, b_3), (a_3, b_2) and (a_4, b_4). The only optimal strategy profile that fits both players is (a_4, b_4), so (a_4, b_4) is the unique Nash equilibrium. Indeed, the Nash equilibrium consists of rationalizable strategies only, while at the other hand not all rationalizable strategies are contained in a Nash equilibrium.

We can reformulate the definition of the Nash equilibrium in terms of *best-response correspondence*.

Definition 5.5.2. Consider a game with mixed strategies, so $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$. Then the *best-response correspondence* $b_i : \Delta(S_{-i}) \to \Delta(S_i)$ for player i to each $\sigma_{-i} \in \Delta(S_{-i})$ is the set of $\sigma_i \in \Delta(S_i)$ such that $u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i})$ for all $\sigma_i' \in \Delta(S_i)$. In other words, a best-response correspondence for player i is the set of best-responses (for player i) to each mixed strategy $\sigma_{-i} \in \Delta(S_{-i})$:

$$b_i(\sigma_{-i}) = \{\sigma_i \in \Delta(S_i) : u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i}) \text{ for all } \sigma_i' \in \Delta(S_i)\}$$

Remark 5.3. A best-reponse correspondence is a set of best-responses for one particular strategy. A best-response correspondence that consists of just one element is identical to a best-response.

PROPOSITION 5.1. Consider a game with mixed strategies, so $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$. Then the strategy profile of mixed strategies σ is a Nash equilibrium if and only if $\sigma_i \in b_i(\sigma_{-i})$ for $i = 1, \ldots, I$ [1].

Proof. By definition of the Nash equilibrium. □

Let us illustrate the use of the proposition in the following example.

Example 5.5.1. *Penalty shootout.*
After some busy days of selling cars, the two salesmen, Alef and Ernst, from example 5.1.2 decided to go the local football field and play a penalty shootout. Alef claims to be a great goalkeeper in his younger days and Ernst claims to be a top-notch striker. Both salesmen want to substantiate their statement, so Alef will be the goalkeeper, while Ernst will be the penalty taker. For each penalty shootout series Alef should decide either to dive to the left (L) or the right (R) corner. In a like manner, Ernst should decide to aim either for the left or right corner. For simplicity we say that the left (and right) corner for Alef is the same left (and right) corner for Ernst.

We may argue that a penalty shootout is a sequential-move game, but in practice it is most likely that the goalkeeper decides the corner to dive to beforehand. Otherwise he will not even get close to the shot ball. So we make the assumption that Alef chooses his strategy before Ernst kicks the ball. As a consequence we could regard the penalty shootout as a simultaneous-move game.

The regulation is that Alef wins the penalty shootout if he manages to stop the ball or if Ernst simply misses the target (e.g. by hitting the outside of the post, crossbar or shooting wide or high), whereas Ernst only wins if he succeed to hit the back of the net. So there is always one winner and one loser. The corresponding normal form representation is shown below.

		Ernst	
		L	R
Alef	L	90, 10	30, 70
	R	40, 60	70, 30

where the payoff is the fraction that the corresponding player wins a particular combination multiplied by 100%.

Suppose the mixed strategy for Alef and Ernst is $\sigma_1 = (\sigma_1(L), \sigma_1(R)) = (\alpha, 1-\alpha)$ and $\sigma_2 = (\sigma_2(L), \sigma_2(R)) = (\beta, 1-\beta)$ respectively. The Nash equilibrium is the profile of mixed strategy $\sigma = (\sigma_1, \sigma_2)$ that maximizes the expected utility of both players:

[1] In *Microeconomic Theory* (Mas-Colell et al., 1995) this proposition is regarded as an alternative definition of the Nash equilibrium.

▷ Alef:

$$\begin{aligned}
&\max_{\alpha}\,[\alpha u_1(L,\sigma_2)+(1-\alpha)u_1(R,\sigma_2)]\\
=&\max_{\alpha}\,[90\alpha\beta+30\alpha(1-\beta)+40(1-\alpha)\beta+70(1-\alpha)(1-\beta)]\\
=&\max_{\alpha}\,[90\alpha\beta+30\alpha-30\alpha\beta+40\beta-40\alpha\beta+70-70\alpha-70\beta+70\alpha\beta]\\
=&\max_{\alpha}\,[90\alpha\beta-40\alpha-30\beta+70]\\
=&\max_{\alpha}\,[(90\beta-40)\alpha-30\beta+70]
\end{aligned}$$

Now we consider different values for β to find the best-response correspondence. It is easily observed that:

- if $\beta > \frac{4}{9}$, then $\alpha = 1$;
- if $\beta = \frac{4}{9}$, then $\alpha \in [0,1]$;
- if $\beta < \frac{4}{9}$, then $\alpha = 0$.

▷ Ernst:

$$\begin{aligned}
&\max_{\alpha}\,[\beta u_2(L,\sigma_1)+(1-\beta)u_2(R,\sigma_1)]\\
=&\max_{\beta}\,[10\alpha\beta+60(1-\alpha)\beta+70\alpha(1-\beta)+30(1-\alpha)(1-\beta)]\\
=&\max_{\beta}\,[10\alpha\beta+60\beta-60\alpha\beta+70\alpha-70\alpha\beta+30-30\alpha-30\beta+30\alpha\beta]\\
=&\max_{\beta}\,[-90\alpha\beta+40\alpha+30\beta+30]\\
=&\max_{\beta}\,[(-90\alpha+30)\beta+40\alpha+30]
\end{aligned}$$

- if $\alpha > \frac{1}{3}$, then $\beta = 0$;
- if $\alpha = \frac{1}{3}$, then $\beta \in [0,1]$;
- if $\alpha < \frac{1}{3}$, then $\beta = 1$.

Finally, we compare the conditions for α and β to find the Nash equilibrium:

▷ if $\beta > \frac{4}{9}$, then $\alpha = 1 > \frac{1}{3}$. But for $\alpha > \frac{1}{3}$ we should have $\beta = 0 < \frac{4}{9}$. Therefore this combination of α and β is not valid;

▷ if $\beta < \frac{4}{9}$, then $\alpha = 0 < \frac{1}{3}$. But for $\alpha < \frac{1}{3}$ we should have $\beta = 1 > \frac{4}{9}$. Therefore this combination of α and β is also not valid;

▷ if $\beta = \frac{4}{9}$, then $\alpha \in [0,1]$. And when $\alpha = \frac{1}{3} \in [0,1]$, then also $\beta = \frac{4}{9} \in [0,1]$. Therefore this combination of α and β is valid.

The strategy profile is $\sigma = (\sigma_1,\sigma_2)$ a Nash equilibrium for $\sigma_1 = (\alpha, 1-\alpha) = (\frac{1}{3},\frac{2}{3})$ and $\sigma_2 = (\beta, 1-\beta) = (\frac{4}{9},\frac{5}{9})$. In other words, Alef should dive twice as much to the right corner as to the left corner, and Ernst should aim for the left (respectively right) corner 4 (respectively 5) times out of 9 to maximize their own payoff. ■

5.5.1 Alternative method to find a Nash equilibrium

Obviously, there are three possible types of strategy profiles of mixed strategies for player i:

(i) play all strategies with positive probability, i.e. $\sigma_i(s_{ji}) = \sigma_{ji} > 0$ for all $j \in \mathbb{N}$. This is known as a *completely mixed strategy*;

(ii) play at least two strategies with positive probability, and the other strategies with zero probability, i.e. there exists a $k, l \in \mathbb{N}$ such that $\sigma_i(s_{ki}) = \sigma_{ki} > 0$ and $\sigma_i(s_{li}) = \sigma_{li} > 0$. This is known as a *semi mixed strategy*;

(iii) play one and only one strategy with probability 1, and the other strategies with zero probability. This is known as a pure strategy.

Due to these distinct types of strategy profiles, it is convenient to denote the set of strategies played with positive probability in the profile of mixed strategies σ for player i by $S_i^+ \subset S_i$.

In general we would not encounter many problems when we try to find a Nash equilibrium of a simple game where we allow mixed strategies. Still it could be very challenging to find a Nash equilibrium in a game (where we allow mixed strategies) with large pure strategy sets. Fortunately, the following proposition provides us a very useful way to tackle this issue.

PROPOSITION 5.2. Consider a game with mixed strategies, so $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$. Then the strategy profile of mixed strategies $\sigma = (\sigma_1, \ldots, \sigma_I)$ is a Nash equilibrium if and only if the following two conditions are satisfied:

(i) $u_i(s_i, \sigma_{-i}) = u_i(s_i', \sigma_{-i}) \quad$ for all $s_i, s_i' \in S_i^+$;

(ii) $u_i(s_i, \sigma_{-i}) \geq u_i(s_i'', \sigma_{-i}) \quad$ for all $s_i \in S_i^+$ and all $s_i'' \notin S_i^+$.

Proof. ($\Rightarrow$): Firstly, we will show that condition (i) holds, and secondly that condition (ii) also holds. Assume that $\sigma = (\sigma_1, \ldots, \sigma_I)$ is a Nash equilibrium. Suppose for contradiction that (i) does not hold, i.e. $u_i(s_i, \sigma_{-i}) \neq u_i(s_i', \sigma_{-i})$ for some $s_i, s_i' \in S_i^+$. Without loss of generality, assume that $u_i(s_i, \sigma_{-i}) > u_i(s_i', \sigma_{-i})$. This implies that player i would choose s_i' with probability 0, i.e. $s_i' \notin S_i^+$, which contradicts our supposition.

Likewise, to prove that condition (ii) has to hold, we suppose for contradiction that condition (ii) does not hold, i.e. $u_i(s_i, \sigma_{-i}) < u_i(s_i'', \sigma_{-i})$ for some $s_i \in S_i^+$ and some $s_i'' \notin S_i^+$. This implies for player i that strategy $s_i'' \notin S_i^+$ (for which player i plays with probability 0) results in a higher expected payoff than for strategy $s_i \in S_i^+$. This cannot be true, since otherwise player i prefers s_i'' to s_i, which contradicts our supposition.

($\Leftarrow$): Assume that conditions (i) and (ii) hold. From condition (i) we notice that changing the probability of choosing the pure strategies $s_i, s_i' \in S_i^+$ would not have effect on the expected payoff of player i. And from condition (ii) we notice the strategy chosen with probability zero $s_i'' \notin S_i^+$ will regrettably decrease the expected payoff of player i. So then player i would have no intention to change her strategy. In other words, the mixed strategy σ_i is the best-response to σ_{-i} for $i = 1, \ldots, I$. With that, every player chooses a best-response and hence σ is a Nash equilibrium. This completes the proof. $\square$

We will apply this proposition to the *Penalty shootout 1* game to illustrate its finesse.

Example 5.5.2. *Penalty shootout 2.*
Reconsider the same setting as in example 5.5.1. From the perspective of Alef, Ernst must be indifferent between strategies L and R. By that we mean that the payoff for Ernst is equal for both strategies, so that Ernst does not bother about choosing L or R. Let $\sigma_1 = (\alpha, 1-\alpha)$ and $\sigma_2 = (\beta, 1-\beta)$. Then the first condition of proposition 5.2 assures the corresponding mixed strategies for Alef as follows (recall that $u_2(\cdot)$ is the payoff function of Ernst):

$$\begin{aligned} u_2(L, \sigma_1) &= u_2(R, \sigma_1) \\ 10\alpha + 60(1-\alpha) &= 70\alpha + 30(1-\alpha) \\ 10\alpha + 60 - 60\alpha &= 70\alpha + 30 - 30\alpha \\ 90\alpha &= 30 \\ \alpha &= \frac{1}{3} \end{aligned} \tag{5.1}$$

Similarly Alef must be indifferent between strategies L and R from the perspective of Ernst:

$$\begin{aligned} u_1(L, \sigma_2) &= u_1(R, \sigma_2) \\ 90\beta + 30(1-\beta) &= 40\beta + 70(1-\beta) \\ 90\beta + 30 - 30\beta &= 40\beta + 70 - 70\beta \\ 90\beta &= 40 \\ \beta &= \frac{4}{9} \end{aligned}$$

We find that the profile of mixed strategies $\sigma = (\sigma_1, \sigma_2)$ is a Nash equilibrium for $\sigma_1 = (\frac{1}{3}, \frac{2}{3})$ and $\sigma_2 = (\frac{4}{9}, \frac{5}{9})$, which agrees with the Nash equilibrium found in example 5.5.1. ■

Remark 5.4. In the previous example, all pure strategies are considered in the profile of mixed strategies, so we do not have to check the second condition of proposition 5.2. If Ernst would have, for example, $(n-2)$ additional pure strategies, say $s_{32}, s_{42}, \ldots, s_{n2}$, then the obtained n-vector profile of mixed strategies $\sigma_2 = (\frac{4}{9}, \frac{5}{9}, 0, \ldots, 0)$ is a Nash equilibrium if additional (cf. condition (ii)):

$$u_2(L, \sigma_1) \geq u_2(s_{i2}, \sigma_1) \quad \text{for all } i = 3, 4, \ldots, n \tag{5.2}$$

In equation 5.2 we could replace $u_2(L, \sigma_1)$ by $u_2(R, \sigma_1)$, since $u_2(L, \sigma_1) = u_2(R, \sigma_1)$ (see equation 5.1).

5.5.2 Existence of the Nash equilibrium

We will come straight to the point: the Nash equilibrium does not always exist. For example, there is no Nash equilibrium in the simultaneous-move Matching Pennies game, because one of the players (actually the one who loses) can always improve her payoff by switching to a different strategy. Under certain conditions we can guarantee the existence of a Nash equilibrium.

PROPOSITION 5.3. A Nash equilibrium exists in every game with mixed strategies, $\Gamma_N = [I, \{\triangle(S_i)\}, \{u_i(\cdot)\}]$, if for every $i = 1, \ldots I$ there exists a $k \in \mathbb{N}$ such that $S_i = \{s_{1i}, s_{2i}, \ldots, s_{ki}\}$, i.e. $S_1, \ldots, S_I$ consist of a finite number of strategies.

Proof. This proposition follows indirectly from a generalization of the Brouwer's fixed point theorem, namely the Kakutani fixed point theorem (Kakutani, 1941). Unfortunately, this theorem is beyond the scope of our study. See Appendix A of chapter 8 in *Microeconomic Theory* (Mas-Colell et al., 1995) for the proof. □

Proposition 5.3 tells us that if we allow for mixed strategies in a game with finitely many players and strategies, then the Nash equilibrium exists. Yet in many games there are even more than one Nash equilibrium.

Example 5.5.3. *Two Nash equilibria.*
Consider the following normal form representation:

		Player 2	
		C	D
Player 1	A	1, 1	0, 0
	B	0, 0	2, 2

where A, B, C and D are pure strategies. Like we did before, we will consider the situation of player 1 first:

- ▷ when player 2 chooses strategy C, then strategy A is the best-response for player 1;
- ▷ when player 2 chooses strategy D, then strategy B is the best-response for player 1.

Now, consider the situation of player 2:

- ▷ when player 1 chooses strategy A, then strategy C is the best-response for player 2;
- ▷ when player 1 chooses strategy B, then strategy D is the best-response for player 2.

We conclude that there are two pure Nash equilibria in this game, namely $s = (s_1, s_2) = (A, C)$ and $s = (s_1, s_2) = (B, D)$. ■

PROPOSITION 5.4. A Nash equilibrium exists in game with pure strategies, $\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}]$, if the following two conditions hold for every $i = 1, \ldots, I$:

(i) $S_i \subset \mathbb{R}^k$ is nonempty, convex and compact for some $k \in \mathbb{N}$;

(ii) $u_i(s_1, \ldots, s_I)$ is continuous in $(s_1, \ldots, s_I)$ and quasiconcave in s_i.

Proof. See Appendix A of chapter 8 in *Microeconomic Theory* (Mas-Colell et al., 1995) for the proof. □

Due to the fact that a finite strategy set S_i can never be a convex , we should realize that proposition 5.3 is restricted to infinite strategy sets only. But looking on the bright side, this proposition could be used for the existence of pure strategies in many economic applications where the pure strategies are considered as continuous variables.

Remark 5.5. Propositions 5.3 and 5.4 are sufficient statements, but not necessary statements. This means that there may be a Nash equilibrium without the validity of the conditions.

Chapter 6

Static games of incomplete information

In the previous section we considered static (one-shot simultaneous-move) games of complete information (all relevant information are common knowledge). However, in practice, players in a game usually do not have complete information. For example, in an oligopolistic market, not all firms are aware of the competitor's payoff function (Gibbons, 1992). In this section we will inspect these kind of games: *static games with incomplete information*.

6.1 The Bayesian game

Consider an incomplete game with two players. Suppose that player 1 is not informed of the payoff function of player 2. Then, how could we even describe the normal form representation of such a game? The economist J.C. Harsanyi (1967) proposed a model that answers this question. This model allows an incomplete information game for randomness of a certain type for each player, and with the use of an extensive form representation we could reinterpret the incomplete information game as of imperfect information. A Bayesian game is constructed by the four basic elements (i)-(iv) of a game (see section 4.1) plus an additional element (v): the *types*. Somes examples of types are: strong, weak, skilled, smart, heroic, humorous and betrayer.

In a Bayesian game we have one additional player, named *Nature*. Nature assigns to each player a preference *type* with a certain probability (Harsanyi, 2004). For player i this is denoted by the random variable $\theta_i \in \Theta_i$. Additionally, player i is supposed to be the only player in the game that knows about her actual type. Let us denote $\mu_i(\theta_{-i}|\theta_i)$ the uncertainty of player i about the type of the other players given its own type, which we call the *belief* of player i (Mas-Colell et al., 1995). In section 7.2 we will provide a formal definition of an extended version of beliefs, but for now it is not necessary.

The joint probability distribution of the type of all players is denoted by $F(\theta_1, \ldots, \theta_I)$. The *decision rule* $s_i(\theta_i)$ in a Bayesian game indicates the strategy choice given type θ_i for player i. Consistently we also denote the set of the profile of types by $\Theta = \Theta_1 \times \ldots \times \Theta_I$, which is common knowledge of the player (i.e. all players know the possible types for each player) and the set of all decision rules for player i by $S_i^B = \{s_i(\theta_i) : \theta_i \in \Theta_i\}$. Then a Bayesian game can be represented as $\Gamma^B = [I, \{S_i\}, \{u_i(\cdot)\}, \Theta, F(\cdot)]$ [1].

Furthermore, given the profile of decision rules $s^B = (s_1(\cdot), \ldots, s_I(\cdot))$, the expected utility

[1] Due to the setting of different types of the players, the game is typically described by multiple normal form representation.

function $\bar{u}_i : S_i^B \times \Theta \to \mathbb{R}$ for player i modifies to:

$$\begin{aligned}\bar{u}_i(s_1(\cdot), \ldots, s_I(\cdot)) &:= E_\theta[u_i(s_1(\theta_1), \ldots, s_I(\theta_I), \theta_i)] \\ &= \sum_{\theta_{-i} \in \Theta_{-i}} u_i(s_1(\cdot), \ldots, s_{i-1}(\cdot), s_i(\theta_i), s_{i+1}(\cdot), \ldots, s_I(\cdot)) \cdot \mu_i(\theta_{-i}|\theta_i)\end{aligned}$$

Remark 6.1. In the definition of the Bayesian game, I is the number of players excluded player Nature.

This bring us to the definition of the *Bayesian (Nash) equilibrium*:

Definition 6.1.1. Consider a Bayesian game $\Gamma^B = [I, \{S_i\}, \{u_i(\cdot)\}, \Theta, F(\cdot)]$. Then the profile of decision rules $s^B = (s_1(\cdot), \ldots, s_I(\cdot))$ is a Bayesian equilibrium if for every $i = 1, \ldots, I$ we have that

$$\bar{u}_i(s_i(\cdot), s_{-i}(\cdot)) \geq \bar{u}_i(s_i'(\cdot), s_{-i}(\cdot))$$

for all $s_i'(\cdot) \in S_i^B$. In other words, s^B is a Bayesian equilibrium if s^B characterizes a Nash equilibrium in a Bayesian game.

The following proposition facilitates the search of a Bayesian equilibrium.

PROPOSITION 6.1. Consider a Bayesian game $\Gamma^B = [I, \{S_i\}, \{u_i(\cdot)\}, \Theta, F(\cdot)]$. Then the profile of decision rules $s^B = (s_1(\cdot), \ldots, s_I(\cdot))$ is a Bayesian equilibrium if and only if

$$E_{\theta_{-i}}[u_i(s_i(\theta_i'), s_{-i}(\theta_{-i}), \theta_i')|\theta_i'] \geq E_{\theta_{-i}}[u_i(s_i', s_{-i}(\theta_{-i}), \theta_i')|\theta_i'] \tag{6.1}$$

with positive probability for all $\theta_i' \in \Theta_i$ and all $s_i' \in S_i$, $i = 1, \ldots, I$. In other words, the Bayesian equilibrium consists of the best-response to the conditional distribution of the opponent's strategy for each type of a player.

Proof. ($\Rightarrow$): Assume that $s^B = (s_1(\cdot), \ldots, s_I(\cdot))$ is a Bayesian equilibrium. Suppose for contradiction that equation (6.1) does not hold. That is, there is a player i for whom there exist a $\theta_i' \in \Theta_i$ such that equation (6.1) does not hold with positive probability. But this implies that there is a strategy choice $s_i''(\theta_i')$ ($s_i'' \in S_i$) that is a better strategy than $s_i(\theta_i')$ for player i. However, this contradicts the fact that $s^B = (s_1(\cdot), \ldots, s_I(\cdot))$ is a Bayesian equilibrium.

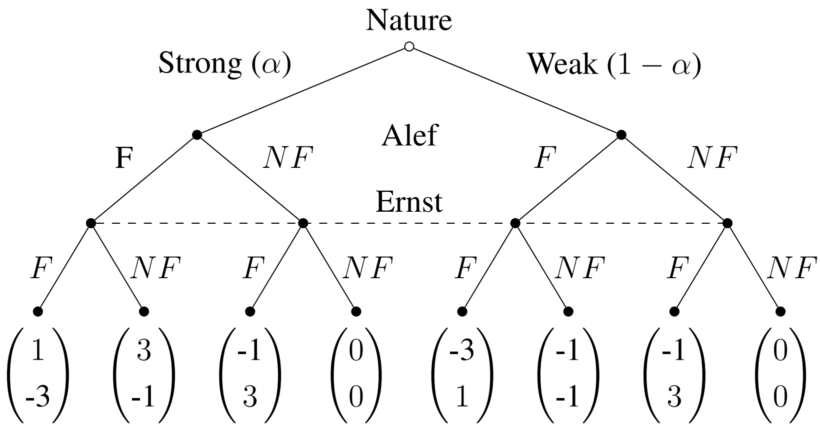
($\Leftarrow$): Assume that for every $i = 1, \ldots, I$ equation (6.1) holds with positive probability for all $\theta_i' \in \Theta_i$ and all $s_i' \in S_i$. Then for every $i = 1, \ldots, I$ the strategy choice $s_i(\theta_i')$ results in an expected utility that is maximal, since $s_i(\theta_i')$ is the best-response to $s_{-i}(\theta_{-i})$ for every $i = 1, \ldots, I$. Hence $s^B = (s_i(\cdot), \ldots, s_I(\cdot))$ is a Bayesian equilibrium. □

Remark 6.2. In the case that $|\Theta_i| = \infty$, we could slightly modify propositon 6.1 by requiring that equation (6.1) holds with probability 1 for *almost every* $\theta_i' \in \Theta_i$.

Example 6.1.1. *Fighting.*
Salesman Ernst from example 5.1.2 suspect the other salesman Alef to cheat in the car sale business. Therefore Ernst decided to confront Alef, which is almost about to turn into an awkward fight. Despite, Ernst does not know whether Alef is either strong or weak. Contrary, Alef obviously knows whether he is either strong or weak. In this game we are wondering

whether each of the two salesmen should fight (F) or not fight (NF). Since Ernst is unaware of the type of Alef, he does not exactly know the payoff for each strategy. Therefore this is a game of incomplete information.

In the Bayesian game setting, Nature attaches probability $\alpha \in [0, 1]$ to Alef of strong type, and probability $1 - \alpha \in [0, 1]$ to Alef of weak type. Thus the extensive form representation is:



By adding Nature to this game, we indeed transformed the game of incomplete information to a game of imperfect information. From this extensive form representation we infer for each type a normal form representation:

Alef (strong), Prob.: α	Ernst: F	Ernst: NF
F	$1, -3$	$3, -1$
NF	$-1, 3$	$0, 0$

and

Alef (weak), Prob.: $1 - \alpha$	Ernst: F	Ernst: NF
F	$-3, 1$	$-1, -1$
NF	$-1, 3$	$0, 0$

Let us find the expected payoff for Ernst. To do this, we should reduce the normal form representations with the help of IDOSDS if possible.

Suppose that we are in the case that Alef is strong. Then Alef is always willing to fight, since F is the strictly dominant strategy for him.
Now suppose that we are in the case that Alef is weak. Then, NF is the strictly dominated strategy for Alef. So instead of using the two normal form representations above, we use:

Alef (strong), Prob.: α	Ernst: F	Ernst: NF
F	$1, -3$	$3, -1$

and

Alef (weak), Prob.: $1 - \alpha$	Ernst: F	Ernst: NF
F		
NF	$-1, 3$	$0, 0$

▷ Now, if Ernst chooses F, the expected payoff for Ernst is:

$$u_{F,2} := u_2(\sigma_{-2}, \sigma_2) = u_2(\sigma_1, F) = -3\alpha + 3(1 - \alpha) = 3 - 6\alpha$$

▷ Instead, if Ernst chooses NF, then the expected payoff for Ernst would be:

$$u_{NF,2} := u_2(\sigma_{-2}, \sigma_2) = u_2(\sigma_1, NF) = -1\alpha + 0(1 - \alpha) = -\alpha$$

Equating the two payoffs we find $5\alpha = 3$, or $\alpha = \frac{3}{5}$. If $\alpha < \frac{3}{5}$, then $u_{F,2} > u_{NF,2}$ and thus Ernst should choose to fight. Intuitively, as α decreases, Alef is less likely to be of strong type, which is in the advantage of Ernst (and thus Ernst rathers to fight). If $\alpha > \frac{3}{5}$, then $u_{F2,2} < u_{NF,2}$ and so Ernst should choose not to fight. If $\alpha = \frac{3}{5}$, then $u_{F,2} = u_{NF,2}$ and Ernst is indifferent and would have no strict preference. ■

Chapter 7

Dynamic games

So far we have only studied one-shot simultaneous-move games, but often we have situations where the players should choose their actions (may be more than once) sequentially. For example, the firms in a oligopolistic market are not required to stick to a certain price forever, and the choice of a price could depend on the price of the competitor. Essentially, some players in a *dynamic game* have knowledge about the strategies chosen in the past. In this section, we will briefly introduce the two classifications of dynamic games: *one-shot dynamic games* and *repeated games* (Romp, a). With the knowledge of dynamic games we are interested in the possible solution concepts. But before we study these solution concepts, we should have some sense about the two classifications of dynamic games:

▷ *One-shot dynamic games*: Any static games can be regarded as an one-shot simultaneous-move game, where the players choose their strategy independently. Despite, sometimes the next players' strategy choice may be influenced by the strategy choice of the current player (cf. sequential-move games). With regard to the oligopolistic market, it is often the case that a new entering firm have knowledge about the price chosen by the already settled competitors. Therefore the strategy choice of the new firm would definitely be affected by this knowledge. If we regard this as a game, then this game is called an one-shot dynamic game (Gibbons, 1992).

▷ *Repeated games*: As we all know, the activity of the oligopolistic market, and many other economic situations, repeats and takes place over a long period of time. In some extent we are interested in repeated interaction between players in a game. Repeated games are simply, as its name almost suggests, repeated stage games over time (Ferguson) [1]. Therefore the final payoff is the sum of payoffs from each stage game. A Nash equilibrium in finitely repeated game consists of a sequence of strategies that is a Nash equilibrium for each player in every game.

  For infinitely repeated games, in contrary, the players may impose *trigger strategies* which could lead to a more ideal outcome than when the players choose a Nash equilibrium in every stage game (e.g. in the prisoner's dilemma game, example 5.1.1, the players would choose NC instead of C in every stage game). Due to this, sometimes the behaviour of the players seem to be cooperatively. We would not go further in detail, since we only study finite games. For the interested reader, we refer to Appendix A of chapter 12 in *Microeconomic Theory* (Mas-Colell et al., 1995).

[1] Recall that in a stage game each player does not observes the move of the other players.

A property of the repeated game is that the players observe all the information from the previously played stage games, also known as *perfect recall*. Notice that a stage game could either be a sequential-move (but of imperfect information) or simultaneous-move game, as long as each player in the stage cannot observe any strategy choice of the other players (and therefore at least be regarded as simultaneous-move game).

Like for static games, we also distinguish between dynamic games of complete and incomplete information. In this section we will only cover dynamic games of complete information. However, with the help of dynamic games of imperfect (and complete) information it is just a matter of small modifications to understand dynamic games of incomplete information. But no worries, we will give an example of a dynamic game of incomplete information in section 9.

7.1 Subgame perfect Nash equilibrium

Sometimes a Nash equilibrium is not reasonable when we propose additional requirements. Therefore we should come up with a refinement of the Nash equilibrium that are satisfactory under these circumstances. In this section the concept of *subgame perfect Nash equilibrium* will be discussed to deal with the so called *credible treats* (Wilson, 2014). The notion of *credibility* and *subgames* are essential to understand this topic:

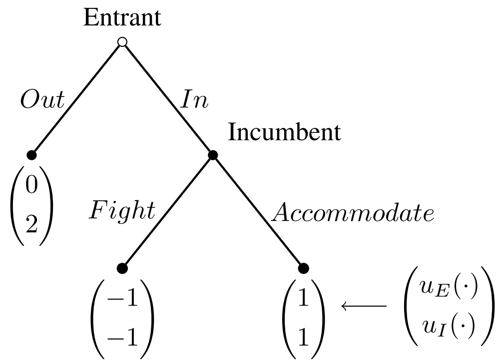
Credibility: this term is used in sequential-move games to identify unreasonable strategies. Let us illustrate this with the following simple example.

Example 7.1.1. Consider two firms: an entrant and an incumbent. The entrant can either choose to enter or stay out of the market. The two firms are competitors of each other. The strategic interaction is based on deciding how many homogeneous/identical good they should bring on the market.

If the entrant chooses not to enter the market, then the incumbent would produce the monopoly quantity. On the other hand, if the entrant does choose to enter the market, then one of the strategies for the incumbent is to produce a significant large quantity so that the entrant will always suffer a loss.

Is the treat of the incumbent reasonable? The answer is no. Namely, if we assume that the cost per unit/good is equal for both firms and the quantity of produced good is negatively correlated with the price, then the incumbent should produce the amount for which the cost and the price are equal (because then the entrant would always suffer a loss for any produced quantity). This means that the incumbent would not make any profit at all. So this strategy is also not advantageous for the incumbent. We say that this treat is not *credible* and so the incumbent should employ a different strategy.

In a game theoretical fashion, consider the corresponding game representations:



		Incumbent	
		Fight if In	*Accommodate if In*
Entrant	*Out*	$0, 2$	$0, 2$
	In	$-1, -1$	$1, 1$

Let us quickly find the Nash equilibria:

- ▷ *If entrant chooses Out*: the incumbent is indifferent between fighting and accommodate (both yields the incumbent a profit of 2);
- ▷ *If entrant chooses In*: the incumbent chooses Accommodate, since $1 > -1$;
- ▷ *If incumbent chooses Fight if In*: Entrant chooses Out, since $0 > -1$;
- ▷ *If incumbent chooses Accommodate if In*: Entrant chooses In, since $1 > 0$.

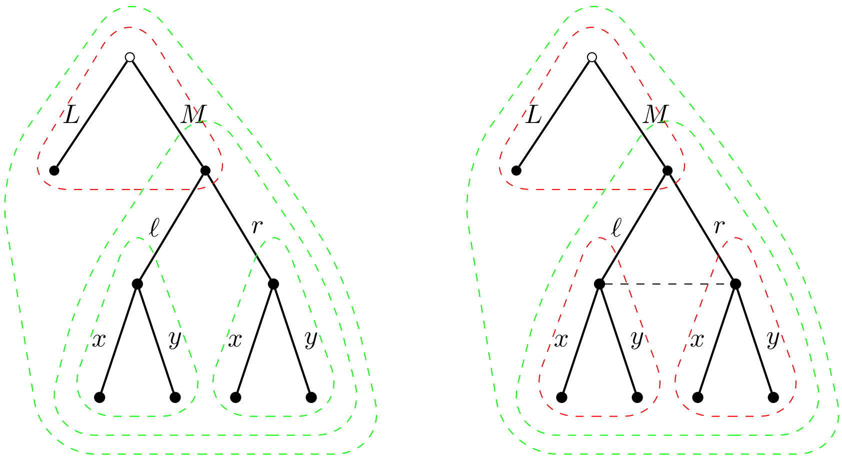
The Nash equilibria are therefore $s^{(1)} := (s_{1E}, s_{1I}) = (\text{Out}, \text{Fight if In})$ and $s^{(2)} := (s_{2E}, s_{2I}) = (\text{In}, \text{Accommodate if In})$. But $s^{(1)}$ is Nash equilibrium that consists an unreasonable treat. Namely, no matter what the entrant chooses, the incumbent should always choose to accommodate. So fighting is against the incumbents interest and therefore not credible. ■

A recent example is the sponsorship of the Fifa. Due to the speculation about corruption scandal some of the key sponsors, such as Adidas, Coca-Cola and Visa, threatens to stop the collaboration with the Fifa if the Fifa does not reform its current practise. In the game theoretic perspective, this is very likely a noncredible treat, because continuation of the sponsorship most probable results in a higher utility for these companies. This is partly due to the fact that the competitors of these companies, for example Nike, Pepsi and Mastercard, are likely to cooperate with Fifa then (Withnall, 2015).

Subgames: A subset of an extensive form game Γ_E is a *subgame* if the following two properties hold:

(i) The starting decision node is a singleton information set that constains all its successors;

(ii) If a subset contains one decision node x from an information set, then it must also contain the other decision nodes in that information set (Mas-Colell et al., 1995). It is convenient to illustrate the definition of subgames with some examples:



where the subgame within the green dashed region is in its entirely a well-defined subgame, and the subgame within the red dashed region is not a subgame.

Remark 7.1. A subgame is a well-defined game, and the whole game is also a subgame. Commonly a subgame that is not the whole game is called a *proper subgame*. It is obvious that if a game has no proper subgame, then the subgame perfect Nash equilibrium is equivalent to the Nash equilibrium.

By the first observation in remark 7.1, we are able to apply the concept of Nash equilibrium to subgames. Doing so, we define a refinement of the Nash equilibrium for dynamic games. The following definition is proposed by the economist R. Selten (1965).

Definition 7.1.1. Consider an extensive form game Γ_E with I number of players. Then a profile of mixed strategies $\sigma = (\sigma_1, \ldots, \sigma_I)$ is a *subgame perfect Nash equilibrium* if it constitutes a Nash equilibrium in every subgame of Γ_E (Mas-Colell et al., 1995).

This solution concept is a refinement of the Nash equilibrium since the strategy profile should first be a Nash equilibrium, but at the same time also survive a certain test: to be a Nash equilibrium in every subgame.

The idea behind subgame perfection is to deal with credible strategies: a Nash equilibrium may contain strategies that are not credible, and so we would like to dismiss these strategies. Because of this, recent results of subgame perfection is commonly taken in practice of the oligopolistic market study.

Remark 7.2. If a profile of mixed strategies is a subgame perfect Nash equilibrium, then it is also a Nash equilibrium. The converse does not necessarily hold.

In example 7.1.1 we argued that $s^{(1)} = (\text{Out, Fight if In})$ is not a reasonable Nash equilibrium due to the noncredible treat 'Fight'. If we use definition 7.1.1 and the fact that the strategy profile (In, Fight) is not a Nash equilibrium in the final subgame, then we conclude that $s^{(1)}$ is not a subgame perfect Nash equilibrium, as desired.

7.1.1 Backward induction

Identifying a subgame perfect Nash equilibrium in a dynamic game Γ_E could be done by using the definitions (5.5.1 and 7.1.1 principally). This is not as easy as it sounds, because it may be tedious to find all Nash equilibria in each subgame. Yet there is a simple procedure that is based on backward reasoning in time, called *backward induction* (Penta, 2011). This procedure works as follows (notice that it is based on games of complete information and the rationality of the players):

(i) Consider the *final* subgames (i.e. subgames that do not contain any other subgames);

(ii) Select one of the Nash equilibria in each of these final subgames;

(iii) Reduce the extensive form game: replace the predecessor node of the terminal nodes by a new terminal node with payoff of the Nash equilibrium;

(iv) Save the strategy that ascertains the game to run from the replaced predecessor node to the original terminal node (i.e. the rational strategy), and then dismiss any branch and node after this node with payoff;

(v) Repeat the above steps until the initial decision node is reached;

(vi) The strategy profile consisting of all strategies saved during the backward induction procedure form a subgame perfect Nash equilibria;

(vii) Repeat steps (iii) - (vi) for the remaining Nash equilibria to obtain all subgame perfect Nash equilibria in the original game.

Since we are working backwards, and considering a Nash equilibrium in every final subgame, we end up with a strategy profile that is a Nash equilibrium in every subgame. Then, by definition 7.1.1 this strategy profile is a subgame perfect Nash equilibrium.

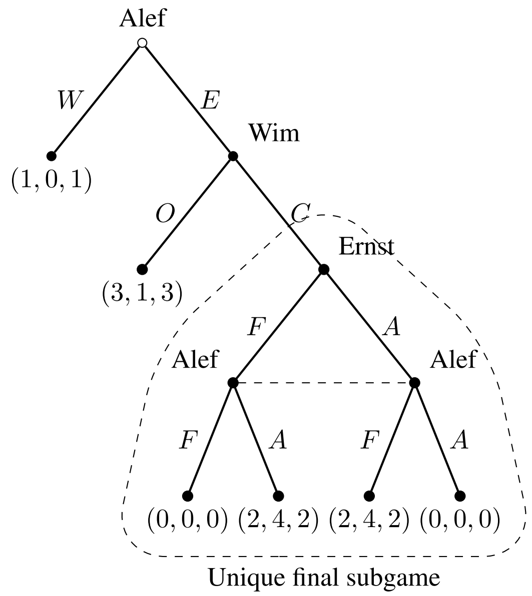
Backward induction would work fairly well in games of perfect information and it will always yield a subgame perfect Nash equilibrium [2]. In a game with no indifferences, backward induction would yield a unique subgame perfect Nash equilibrium. Otherwise there are more than one subgame perfect Nash equilibrium.

In general, backward induction causes some problems in games of imperfect information (due to the lack of information), unfortunately [3]. However we are still able to find the set of subgame perfect Nash equilibria such type of games, but this solution would practically not be sufficient. Let us illustrate the backward induction and its limitation (in games of imperfect information) with an example using game trees.

Example 7.1.2. *Three salesmen in Groningen*
Consider a 3-players dynamic game with three car salesmen: Alef, Wim and Ernst. Suppose that Ernst is already settled in the center of the city Groningen (C), and Wim in the east of the city

[2] In games of perfect information, the set of subgame perfect Nash equilibria is equal to the set of Nash equilibria.
[3] Only if a player has a dominant strategy in each nonsingleton information set, backward induction would be satisfactory.

(E). Now, Alef is confident to start a car sales in Groningen, but he could only make a choice either to settle in the west (W) or take over the car sales of Wim in the east of Groningen. If Alef takes over Wim's car sales, then, according to the government, Wim could only start a new car sales outside Groningen (O), or in the center. In the last case, the salesmen Alef and Ernst can each choose to fight (F) or accommodate (A) with Wim in the hope to increase their own payoff. Obviously Alef and Ernst will not tell each other whether they fight or accommodate. This is a market study with the following extensive form game:

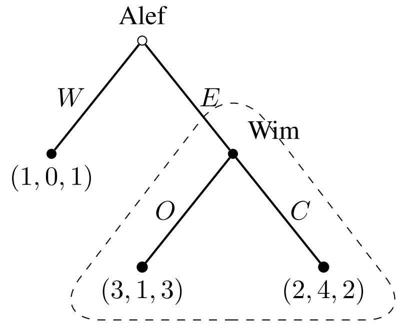


where the payoff are profits (from this whole year) expressed in millions for the salesmen in the order of Alef, Wim, Ernst, and the (unique) final subgame is indicated within the dashed region. This subgame is a game between two players: Alef and Ernst. And for this subgame we could derive the normal form representation[4]:

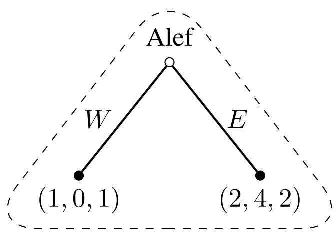
		Ernst F	Ernst A
Alef	F	$0, 0, 0$	$2, 4, 2$
	A	$2, 4, 2$	$0, 0, 0$

It is obvious that $s^{(1)} := (s_{1A}, s_{1E}) = (A, F)$ and $s^{(2)} := (s_{2A}, s_{2E}) = (F, A)$ are only two pure strategy Nash equilibria in this subgame. First, let us continue with $s^{(1)} = (A, F)$ (so Alef chooses to accommodate, and Ernst chooses to fight). The third step of the backward induction procedure tells us to reduce the extensive form game:

[4]Formally we indicate Ernst on the left (i.e. first player to move) of the normal form representation, and Alef on top (i.e. second player to move), but the order in a game of imperfect information (like this subgame) does not matter. Since Alef is the first player to move in the whole game, we indicate him on the left.



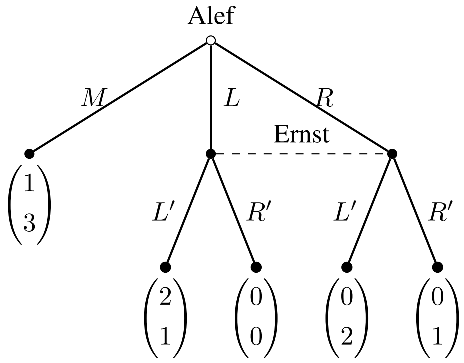
Now, instead of using a normal form representation, we could find the optimal strategy for Wim by reasoning. Namely, Wim argues that settling in the center (with a profit of \$4 million) is more beneficial than settling outside the city (with a profit of \$1 million). Therefore Wim would settle in the center. For the third and last time we should reduce the extensive form game further:



Using the same arguments as before, we could infer that Alef would take over the car sales of Wim in the east (since that will result in a profit of \$2 million) instead of settling in the west (with a profit of \$1 million). Therefore the strategy profile $s := (s_A, s_W, s_E) = ((E, A), C, F)$ is a subgame perfect Nash equilibrium with payoff $(u_A, u_W, u_E) = (2, 4, 2)$. We could repeat the same steps for the other pure strategy Nash equilibrium $s^{(2)} = (F, A)$ and obtain the subgame perfect Nash equilibrium $s = ((E, F), C, A)$ with payoff $(2, 4, 2)$. ■

In the last example we managed to find two subgame perfect Nash equilibria in a game of imperfect information. But keep in mind that in the first subgame we considered (where the information set of Alef contains two decision nodes), it may as well happen that both salesmen, Alef and Ernst, choose F or both choose A. This information is unknown (or rather imperfect). In other words, the two subgame perfect Nash equilibria found are practically insufficient for our study since the player at the nonsingleton information set does not know at which decision node she is. Because of this, a subgame perfect Nash equilibrium may contain a noncredible treat in games of imperfect information. See the next example.

Example 7.1.3. Consider a game with the following extensive and normal form representation:



		Ernst	
		L'	R'
	M	$1,3$	$1,3$
Alef	L	$2,1$	$0,0$
	R	$0,2$	$0,1$

The two (pure) Nash equilibria are $s^{(1)} := (M, R')$ and $s^{(2)} := (L, L')$. Since this game has no proper subgame, both Nash equilibria are also subgame perfect Nash equilibria.

But R' is the dominated strategy for Ernst, which implies that R' is a noncredible treat and consequently the subgame perfect Nash equilibrium $s^{(1)} = (M, R')$ should be neglected. Up to this point we cannot prove this statement. So this example will be continued. ■

An approach, that deal with nonsingleton information set (as in the previous example), relies on the *belief* of the player at this information set. This is done by specifying a probability for the player of being at each node in this information set. This topic is related to the solution concept of *perfect Bayesian equilibrium* that we will discuss in the next section.

7.2 Perfect Bayesian equilibrium

The theory covered in the section are found from Osborne and Rubinstein (1994); Gibbons (1992); Bonanno (a).

Another refinement of the Nash equilibrium is provided by the solution concept of *perfect Bayesian equilibrium*. As the subgame perfect Nash equilibrium rules out the noncredible treats in games of complete information, the concept of perfect Bayesian equilibrium is used to deal with noncredible treats in games of incompleteinformation. This solution concept relies on the notion of *system of beliefs* and *sequential rational* strategy profiles.

Definition 7.2.1. Consider an extensive form game Γ_E. Then a *system of beliefs* μ is a function that assigns to each information set a probability for each decision node in the information set such that

$$\sum_{x \in H} \mu(x) = 1$$

We can think of that the system of beliefs specifies a probabilty of being at a certain decision node in the information set for a player. If a player has a belief $\mu = 1$ of being at a decision node in a information set, then the player is sure she is at that decision node and by definition this corresponds to a singleton information set. Related to the system of beliefs, is the principle of *sequentially rational*:

Definition 7.2.2. Consider an extensive form game Γ_E. Then a profile of mixed strategies $\sigma = (\sigma_1, \ldots, \sigma_I)$ is *sequentially rational at information set H given a system of beliefs μ* if for every player i, $i = 1, \ldots, I$, at information set $H \in \mathcal{H}_i$ we have that

$$E[u_i(\sigma_i, \sigma_{-i})|H] \geq E[u_i(\sigma_i', \sigma_{-i})|H]$$

for all $\sigma' \in \Delta(S_i)$. If the condition holds for every $H \in \mathcal{H}_i$, then σ is said *sequentially rational given system of belief μ.*

Intuitively, a profile of mixed strategies $\sigma = (\sigma_1, \ldots, \sigma_I)$ is sequentially rational if every player expects that the strategy imposed at each of her information set is optimal given her beliefs what happened before and her opponents' strategies.

We introduce one more terminology (actually two) before we provide the definition of the perfect Bayesian equilibrium.

Definition 7.2.3. Consider an extensive form game Γ_E and the Nash equilibrium σ. Then an information set H is *on the equilibrium path* if $P(H|\sigma) > 0$, i.e. when H will be reached with positive probability when played according to the equilibrium strategies. On the contrary, an information set H is *off the equilibrium path* if it is not on the equilibrium path.

Finally, we are arrived at defining the perfect Nash equilibrium:

Definition 7.2.4. Consider an extensive form game Γ_E. Then a strategy profile σ and system of beliefs μ is a *perfect Bayesian equilibrium* under the following two conditions [5]:

(i) The strategy profile σ is sequentially rational given μ;

(ii) At information sets H on the equilibrium path, the beliefs are derived by the Bayes' rule and σ. That is:

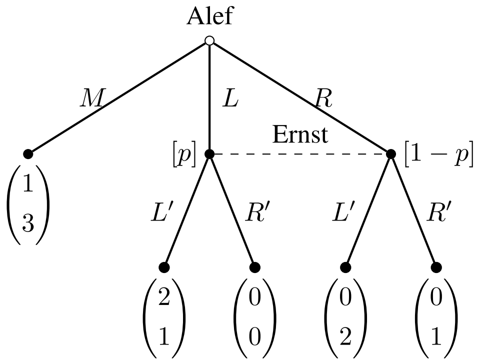
$$\mu(x) = \frac{P(x|\sigma)}{P(H|\sigma)}$$

The first condition provides the players to have beliefs and react optimally given these beliefs. The second condition adds that beliefs are reasonable (or *consistent*), by the Bayes' rule.

Example 7.2.1. Now we know what a perfect Bayesian equilibrium is, we can provide a formal 'proof' that the subgame perfect Nash equilibrium $s^{(1)} = (M, R')$ in example 7.1.3 is unreasonable.

The system of beliefs implies that Ernst has a belief that Alef choses L and R with probability p and $1 - p$, where $p \in (0, 1)$, respectively (see the extensive form representation below).

[5] Many other authors use different definitions for the perfect Bayesian equilibrium (see e.g. Gibbons (1992)), but this definition provides the core of the perfect Bayesian equilibrium.



Given this belief, the expected payoff of Ernst for playing R' is:

$$u_2(R') = p \cdot 0 + (1-p) \cdot 1 = 1 - p$$

while for playing L' it is:

$$u_2(L') = p \cdot 1 + (1-p) \cdot 2 = 2 - p$$

This implies that the expected payoff is higher for playing L' than for playing R' (because $u_2(L') > u_2(R')$ for all $p \in (0, 1)$). Then, sequential rationality tells us that Ernst would not choose R': R' is a noncredible treat. Therefore $s^{(1)} = (M, R')$ is implausible and hence dismissed.

The other (subgame perfect) Nash equilibrium $\sigma = (L, L')$ is a perfect Bayesian equilibrium since it satisfies the two conditions (cf. definition 7.2.4):

(i) We have just argued that playing L results in a higher expected payoff than by playing R';

(ii) The information set of Ernst will be reached with probability $P(H|\sigma) = 1$ given that Alef chooses to play L with probability $q_2 = 1$. With the result:

$$p = \mu([p]) = \frac{P([p]|\sigma)}{P(H|\sigma)} = \frac{q_2}{1} = 1$$

which is in accordance with the equilibrium (L, L').

■

Remark 7.3. It may feel like that $\mu = 1$ is derived a by circular reasoning, but this is not true. To 'illustrate' this, suppose that Alef has a mixed strategy equilibrium, i.e. $\sigma_1 = (q_1, q_2, 1 - q_1 - q_2) := (\sigma_M, \sigma_L, \sigma_R)$ with $q_1, q_2 \in [0, 1]$. Then, at condition (ii), we would have obtained:

$$p = \frac{q_2}{q_2 + (1 - q_1 - q_2)} = \frac{q_2}{1 - q_1}$$

Indeed, in the example above we would have replaced $q_1 = \sigma_M = 0$.

So far we have considered dynamic games of complete information. A more complex kind of game is the dynamic game of incomplete information. In section 6 we have seen that static games of incomplete information can be solved by introducing the Nature so that the game transforms into a dynamic game of imperfect information. Likewise, introducing Nature in dynamic games of incomplete information works similarly, and together with the knowledge from section 7 we can readily solve these kind of dynamic games. In section 9 we will entertain ourselves with an example.

Chapter 8

Oligopolistic market

An oligopoly is a market situation where a relatively small number of firms have most influence on the market and entry of new firms is limited. Examples of oligopolies are computer system operators, telecom, oil companies, airlines and the auto industry.

The three oligopoly models that we will discuss in this section are originally developed before the modern game theory. Exclusively, it is ideally to use game theory for the analysis since the competition among firms are strongly linked with strategic interaction. It turns out that the models produce different result of the Nash equilibrium. We will omit the phenomenon of *collision* or *cartel* (i.e. agreement of cooperation to maximize the joint profit) among oligopolies, because we are focusing on non-cooperative situations [1].

8.1 Cournot oligopoly model

The first oligopoly model that we will study is the Cournot (oligopoly) model, which is originally been developed by the philosopher and mathematician A.A. Cournot (1838). This model is designed for the analysis of one period with N leading firms where the N firms simultaneously determine the optimal quantity of good to produce, and common knowledge is assumed (Keskinocak, 2004; Erhun and Keskinocak, 2003). So this is a static model of complete information. Originally, the Cournot model assumes that firm produce homogeneous goods, but we will formulate an extended form of the Cournot model where we allow for differentiated goods. This extended Cournot model is characterized by the following [2]:

- ▷ N firms selling a substituted good;
- ▷ The total quantity produced is $Q := q_1 + \ldots + q_N \geq 0$, where q_i is the quantity produced by firm i;
- ▷ Cost per unit for firm i is c_i;
- ▷ Price function $p_i(Q)$ for firm i assigns a price of a good given the quantity Q of the goods on the market. The price function $p_i(\cdot)$ satisfies the next conditions:
  - - $p_i(\cdot)$ is differentiable;
  - - $p_i(Q) \geq 0$ for all Q;

[1] In this section we primarily refer players to firms.

[2] This model is inspired by Tremblay and Tremblay (2012).

- $p_i'(Q) < 0$ at all $Q \geq 0$, i.e. the price decreases as the quantity increases;
- $p_i(0) > c_i$, i.e. the price is originally higher than the required cost;
- there exists a quantity $\bar{Q} \in (0, \infty)$ such that $p_i(\bar{Q}) = c_i$.

▷ The total profit of firm i for $i = 1, \dots, N$ given the quantity produced by firms $-i$ (or: firms j for $j = 1, \dots, N$ and $j \neq i$), is equal to the profit per unit, $(p_i(Q) - c_i)$, multiplied by the quantity produced, q_i:

$$\begin{aligned} \text{Profit} &= \text{Revenue} - \text{Cost} \\ u_i(Q) &= (p_i(Q) - c_i)q_i = p_i(Q)q_i - c_i q_i \end{aligned} \tag{8.1}$$

The strategy profile $(q_1^*, \dots, q_N^*)$ is a Nash equilibrium where:

$$\begin{aligned} q_1^* &= \underset{q_1}{\operatorname{argmax}}\, u_1(q_1, q_2^*, \dots, q_N^*) \\ &\vdots \\ q_N^* &= \underset{q_N}{\operatorname{argmax}}\, u_1(q_1^*, \dots, q_{N-1}^*, q_N^*) \end{aligned}$$

Remark 8.1. In the original design of the Cournot model (i.e. where firms produce homogeneous goods), we make the assumption that:

$$\begin{aligned} c_i &= c \quad \text{ for all } i = 1, \dots, N \\ p_i(\cdot) &= p(\cdot) \text{ for all } i = 1, \dots, N \end{aligned}$$

We could easily derive the first order condition for firm i, that should hold for maximizing the utility function, by taking the first derivative of $u_i(Q)$ with respect to q_i and subsequently setting this first derivative equal to zero:

$$\begin{aligned} u_i'(Q) = p'(Q)q_i + p_i(Q) - c_i &= 0 \\ p_i'(Q)q_i + p_i(Q) &= c_i \end{aligned} \tag{8.2}$$

If we denote $b_i(q_j)$ the best-response correspondence to quantity q_j for firm i, then by proposition 5.1 we deduce that the quantity profile $(q_1^*, \dots, q_N^*)$ is a Nash equilibrium if and only if $q_i^* \in b_i(q_j^*)$ for all $i, j = 1, \dots, N$, $i \neq j$. Or, in terms of the first condition in equation 8.2, the quantity profile $(q_1^*, \dots, q_N^*)$ is a Nash equilibrium if and only if:

$$p_1'(Q)q_1^* + p_1(Q) = c_1 \tag{8.3}$$

$$\vdots$$

$$p_N'(Q)q_N^* + p_N(Q) = c_N \tag{8.4}$$

Example 8.1.1. *Car sales 3, simultaneous-move.*
Consider the car salesmen Alef and Ernst (again...), and suppose that the two salesmen sell the same type of car. As a result they both cope with equal cost $c > 0$ per car and price function $p(q) = a - bq$ with $0 < c < a$ and $b > 0$. Then, by equations 8.3 and 8.4, the quantity profile (q_1^*, q_2^*) is a Nash equilibrium if and only if the following two equations hold:

$$-bq_1^* + a - b(q_1^* + q_2^*) = c \tag{8.5}$$

$$-bq_2^* + a - b(q_1^* + q_2^*) = c \tag{8.6}$$

We have two equations, 8.5 and 8.6, and two unknowns, q_1^* and q_2^*. Simple algebra is favorable to solve for q_1^* and q_2^* here. First, we find an expression for q_1^* in terms of the remaining variables. Rewritting and ordering the variables in equation 8.5 yields:

$$\begin{aligned} -2bq_1^* &= c - a + bq_2^* \\ q_1^* &= \frac{a - c - bq_2^*}{2b} \end{aligned} \tag{8.7}$$

Similarly:

$$q_2^* = \frac{a - c - bq_1^*}{2b}$$

Therefore the best-response functions are given by:

$$b_i(q_j) = \max\left\{0, \frac{a - c - bq_j}{2b}\right\} \quad \text{for } i, j = 1, 2, \ i \neq j$$

Plugging this expression for q_1^* into equation 8.6 we find:

$$\begin{aligned} -bq_2^* + a - \frac{a - c - bq_2^*}{2} - bq_2^* &= c \\ -2bq_2^* + \frac{b}{2}q_2^* &= c - a + \frac{-(c-a)}{2} \\ -\frac{3}{2}bq_2^* &= \frac{1}{2}(c - a) \\ q_2^* &= \frac{a - c}{3b} \end{aligned} \tag{8.8}$$

Finally, observe from the first order conditions 8.5 and 8.6 that $q_1^* = q_2^*$, so that the quantity profile (q_1^*, q_2^*) is a Nash equilibrium if and only if $q_1^* = q_2^* = \frac{a-c}{3b}$. Rather, we could also fill in the expression for q_2 into expression 8.7 to derive $q_1^* = \frac{a-c}{3b} = q_2^*$. Hence the total quantity produced by the firms is:

$$Q_c := q_1^* + q_2^* = \frac{2(a - c)}{3b}$$

and the profit for each salesman

$$\begin{aligned} u_i(q_i^* + q_j^*) &= (p(q_i^* + q_j^*) - c)q_i^* \\ &= (a - b(q_i^* + q_j^*) - c)q_i^* \\ &= (a - b\frac{2(a-c)}{3b} - c)\frac{a - c}{3b} \\ &= (a - \frac{2}{3}(a - c) - c)\frac{a - c}{3b} \\ &= (\frac{1}{3}a - \frac{1}{3}c)\frac{a - c}{3b} \\ &= \frac{(a - c)^2}{9b} \end{aligned}$$

The best-responses and the Nash equilibrium are shown in figure 8.1 below.

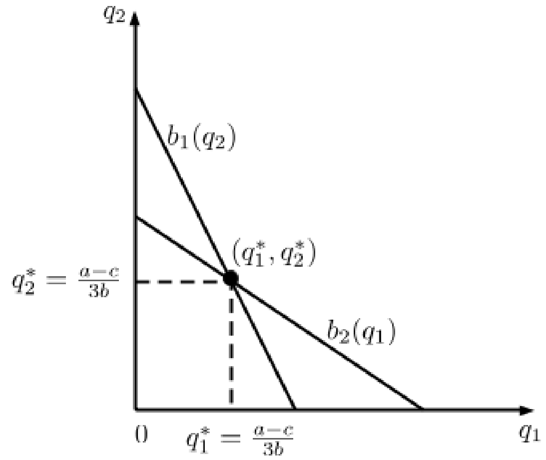


Figure 8.1: Cournot duopoly model.

■

8.2 Bertrand oligopoly model

The second oligopoly model that we will study is the Bertrand (oligopoly) model, which is originally been developed by mathematician J.L.F. Bertrand (1883). This model is, like the Cournot model, applicable to simultaneous-move games of the oligopoly market. But here, the strategic interaction is based on determining the optimal price of a firm's good instead of the quantity of a firm's good (Dixon, 2001). Also, common knowledge is assumed to hold, like in the Cournot model. Therefore this is also a variety of a static model of complete information. Similarly, we will provide a formulation of the extended Bertrand model to allow for differentiable goods:

- ▷ N firms selling a substituted good;
- ▷ The price profile is denoted by $P := (p_1, \ldots, p_N)$, where p_i is the price imposed by firm i;
- ▷ Demand function $x_i(P)$ for firm i assigns the demand given the prices imposed by the firms. This demand function is $x_i(P) > 0$ for all possible P for which $x_i(P)$ is continuous and strictly decreasing. Further, there exists a $\bar{P} < \infty$ such that $x_i(P) = 0$ for all $P \geq \bar{P}$ (i.e. $p_1 \geq \bar{p}_1, \ldots, p_N \geq \bar{p}_N$). Intuitively, the demand decreases as the price increases, but at certain prices $\bar{P}$ there are no consumer willing to pay any price larger (or equal) to this price $\bar{P}$. Relating to the price function $p_i(\cdot)$ for the Cournot model, we could write $x_i(\cdot) = p_i^{-1}(\cdot)$;
- ▷ Cost $c_i > 0$ per unit/good and for which $x_i(c_i) \in (0, \infty)$ (or $c_i < p_i$ cost is strictly lower than upper limit price) [3];

[3] $\bar{Q}_i \in (0, \infty)$ for which $p_i(\bar{Q}_i) = c_i$ from the Cournot model can be expressed as $\bar{Q}_i = p_i^{-1}(p_i(\bar{Q}_i)) = p_i^{-1}(c_i) = x_i(c_i)$. On this account, the price function is also called the inverse demand function.

- ▷ The firm with the strictly lower price will take all the consumers. Then the other firms, with the strictly higher price, will not sell any. If the price of $M \leq N$ firms are equal (and are lower than the price of the remaining $M - N$ firms), then the demand is fairly shared between the M firms. Mathematically, denote $x(\cdot) = \sum_i x_i(\cdot)$, let p_i be the price charged by firm i and assume without loss of generality that $p_1 \leq \ldots \leq p_N$:

$$x_i(P) = \begin{cases} x(Pe_i) & \text{if } \ p_i < p_j \text{ for } j = 1, \ldots, N, i \neq j \\ \frac{1}{M} x(Pe_i) & \text{if } \ p_i = p_m \text{ and } p_i < p_j \text{ for } i, m = 1, \ldots, M, \ j = M+1, \ldots, N \\ 0 & \text{if } \ p_i > p_j \text{ for } j = 1, \ldots, N, i \neq j \end{cases} \tag{8.9}$$

  where $Pe_i = (0_1, \ldots, 0_{i-1}, p_i, 0_{i+1}, \ldots, 0_N)$. It is important to keep in mind that the firms choose their prices simultaneously;

- ▷ The total profit (cf. utility function) of firm i is the product of the profit per unit, $(p_i - c)$, and the demand $x_i(P)$ given the prices, i.e. $u_i(P) = (p_i - c)x_i(P)$ [4].

Remark 8.2. In the original design of the Bertrand model (i.e. where the firms sell a homogeneous good), we make the assumption that $c_i = c$ for all $i = 1, \ldots, N$ (Dixon, 2001).

In the Cournot model, the utility function should be maximized using differential optimization techniques, which differs from the approach in the Bertrand model. This is because the price function in the Cournot model is continuous for every Q. The firms can face infinitely many scenarios: for every combination of $q_1, \ldots, q_N$ a certain price corresponds. In the Bertrand model, however, the firms could face just three different scenarios (see equation 8.9). Instead, using direct game theoretic approach (cf. Nash equilibrium) we will be able to determine the optimal utility.

Often, the study of Bertrand model is stricted to the duopoly case, so where there are just two leading firms competing against each other. The next proposition is formulated for a duopoly market situation, but the result can be extended to an oligopoly situation of N firms.

PROPOSITION 8.1. The price profile $p = (p_1^*, p_2^*) = (c, c)$ is a unique Nash equilibrium in the Bertrand duopoly model.

Proof. (*Existence*): Let $p = (p_1^*, p_2^*) = (c, c)$ so that $x_1(p_1^*, p_2^*) = x_2(p_1^*, p_2^*) = 0$. Let us examine the situation for firm 1.

- ▷ If firm 1 chooses a higher price $p_1 := p_1^* + \varepsilon > p_2^*$ for $\varepsilon > 0$, then firm 1 would not sell any of its goods, i.e. $x_1(p_1, p_2^*) = 0$.

- ▷ Alternatively, if firm 1 chooses a lower price $p_1 := p_1^* - \varepsilon < p_2^*$ for $\varepsilon > 0$, then the profit per unit of firm 1 would be $p_1 - c = (p_1^* - \varepsilon) - c = (c - \varepsilon) - c = -\varepsilon$, i.e. firm 1 would incur loss.

[4] We will say that a firm sells a 'unit' or a 'good' instead of a 'product'. This is to avoid any confusions with the 'product' known as the multiplicative operator.

An analogous analysis could be done for firm 2. Taking the analysis of both firms into consideration, we pinpoint that no other price profile than $p = (c, c)$ yields a higher payoff for either firms. Hence indeed the price profile $p = (c, c)$ is a Nash equilibrium.

(*Uniqueness*): Likewise, let $p = (p_1^*, p_2^*) = (c, c)$ so that $x_1(p_1^*, p_2^*) = x_2(p_1^*, p_2^*) = 0$ and examine the situation for firm 1. Here, as well, a completely analogous analysis could be done for firm 2. Consider different possible cases to prove that there are no other Nash equilibra:

(i) Firm 1 chooses a lower price $p_1 := p_1^* - \varepsilon < p_2^*$ for $\varepsilon > 0$. As before, this results in the profit per unit $p_1 - c = (p_1^* - \varepsilon) - c = (c - \varepsilon) - c = -\varepsilon$ for firm 1 and therefore $p = (p_1, p_2^*) = (c - \varepsilon, c)$ is not a Nash equilibrium;

(ii) Firm 1 chooses a higher price $p_1 := p_1^* + \varepsilon > p_2^*$ for $\varepsilon > 0$. But on that occassion firm 2 will be better off by choosing a higher price p_2 for which $c < p_2 < p_1$, because then firm 2 will make a strictly positive profit. Thus, an increase in the price of firm 1 does not establish a Nash equilibrium;

(iii) Both firms choose a strictly higher price, i.e. $c < p_1 \leq p_2$. The profit for firm 1 is at most $u_1(p_1, p_2) = \frac{1}{2}(p_1 - c)x_1(p_1, p_2)$. However, firm 1 could have a higher profit by undercutting the price of firm 2 (and thus lure all the consumers) by choosing $p_1^{**} := p_1 - \varepsilon$ such that $u_1(p_1^{**}, p_2) = (p_1^{**} - c)x_1(p_1^{**}, p_2) > \frac{1}{2}(p_1 - c)x_1(p_1, p_2)$ for sufficiently small $\varepsilon > 0$. Because of this, the price profile $p = (p_1, p_2)$ does not constitute a Nash equilibrium.

Together with the analysis of firm 2 (that we have left out), we conclude that any price profile other than $p = (c, c)$ cannot be a Nash equilibrium. This completes the proof. □

We just behold a mind-blowing feature: there is a unique proper competitive outcome in a duopoly where the price is equal to the cost of the good. But, is this result realistic? The answer is ‘no’. Because, in reality, profit-seeking firms tend to set the price of their good higher than the corresponding cost. There is an alternative model that shows why firms set the price higher than the cost, namely the Hotelling linear city model (Hotelling, 1929). See Appendix A for the detail.

Remark 8.3. As mentioned before, proposition 8.1 tells us that the price of a homogeneous good is at the same level as the cost, which implies that the firms make no profit at all. Since this is generally not the case, this is referred to as the *Bertrand Paradox*. Notice that as the number of firms in the oligopoly increases, the proposition is more sensible (since the price converges to the cost). All in all, the results of the Cournot model are more in line with the real world competition, where the firms earn positive profit [5].

8.3 Stackelberg duopoly model

grand models are concerning static (one-shot simultaneous-move) games. The Stackelberg (duopoly) model is originally been developed by the economist H. von Stackelberg (1934) that,

[5] This is an emperical result, but many would agree to say that the Cournot equilibrium is more realistic than the Bertrand's.

in contrast, concerns one-shot dynamic (sequential-move) games instead. Likewise the Cournot model, the strategic interaction in the Stackelberg model is based on producing an optimal quantity of a firm's good. But in this model the so called *dominant player* or *leader* is the first firm to choose a quantity to produce the good, followed by a quantity choice of the other firm, the *follower* (Keskinocak, 2004). Generally speaking, the leader has *commitment* power: the leader moves first and cannot undo the strategy before until the follower has moved (Hubert, 2006). The consequence of this is that the second firm has the 'benefit' to observe the leader's strategy choice. However, we will see that the follower is actually worse off than the leader.

We make the assumption that the leader knows *ex ante* that the follower observes the strategy of the leader. Additional, the follower should not commit a non-Stackelberg follower action, which is also known by the leader [6].

The goal is to find an equilibrium of the game. The games in the Stackelberg models are dynamic games of complete and perfect information. So it is sufficient to apply the solution concept of subgame perfect Nash equilibrium. To do so, we should find the best-response functions using backward induction. We start by analyzing the final subgame, which is the subgame after the leader's strategy choice, and gradually get to the initial decision node of the leader.

Assume, from now on, that firm 1 is the leader, and firm 2 has to choose a strategy secondly. As firm 2 can observe the quantity q_1 produced by firm 1, we could define a quantity function for firm 2 that depends on q_1. Here, the profit function for firm i with $i = 1, 2$ is the same as we have seen in section 8.1:

$$u_i(q_i + q_j) = (p(q_i + q_j) - c)q_i = p(q_i + q_j)q_i - cq_i \tag{8.10}$$

Recall that we apply backward induction. Therefore we first should find an expression for q_2^* that depends on q_1, secondly we can derive the optimal quantity q_1^* for the leader and finally obtain the optimal quantity q_2^* for the follower.
Given the quantity q_1 produced by firm 1, the first order condition (for firm 2) that should be fulfilled (see equation 8.4) is:

$$p'(q_1 + q_2^*)q_2^* + p(q_1 + q_2^*) = c$$

Now this equation could be solved to obtain the expression of the optimal quantity $q_2^* = f(q_1)$ for firm 2, where $f : \mathbb{R} \to \mathbb{R}$. Now, q_1^* is found by maximizing $u_1(q_1 + q_2^*) = u_1(q_1 + f(q_1))$ with respect to q_1. Once we have done that, it is just a matter of substituting q_1^* into the earlier obtained expression of q_2^*:

$$q_2^* = f(q_1^*)$$

Example 8.3.1. *Car sales 4, sequential-move.*
Consider the same situation as in example 8.1.1, but now where Alef is the leader and Ernst the follower. Recall that the price function is $p(q) = a - bq$ with $0 < c < a$ and $b > 0$.

[6] Ex ante means '*before the event*' and is regularly used in economics. In this context it means that the leader keeps in mind the behaviour of the follower beforehand.

Straightforward substitution yields the first order condition:

$$\begin{aligned} p'(q_1 + q_2^*)q_2^* + p(q_1 + q_2^*) &= c \\ -bq_2^* + a - b(q_1 + q_2^*) &= c \end{aligned}$$

The expression for q_2^* is easily obtained by rearranging the last equation:

$$-2bq_2^* + a - bq_1 = c \tag{8.11}$$

$$-2bq_2^* = c - a + bq_1 \tag{8.12}$$

$$q_2^* = \frac{a - c - bq_1}{2b} \tag{8.13}$$

Notice that the quantity q_2^* for Ernst is equivalent to as in the Cournot model (see equation 8.7). The only difference is that q_2^* here is the true quantity choice, whereas in the Cournot model the quantity q_2^* was a hypothesized quantity. But, Alef is a smart salesman who would choose his quantity q_1 (or rather q_1^*) wisely. Recall the assumption that the leader, Alef, knows ex ante the behaviour of the follower, Ernst. So then it is fortunate to apply backward inductive reasoning to derive q_1^*. That is, we want to maximize the utility function u_1 of Alef given $q_2^* = \frac{a-c-bq_1}{2b}$. In other words:

$$\begin{aligned} \max_{q_1}\ u_1(q_1, q_2^*) &= \max_{q_1}\ [p(q_1 + q_2^*)q_1 - cq_1] \\ &= \max_{q_1}\ [aq_1 - b(q_1 + q_2^*)q_1 - cq_1] \\ &= \max_{q_1}\ \left[aq_1 - b\left(q_1 + \frac{a - c - bq_1}{2b}\right)q_1 - cq_1\right] \\ &= \max_{q_1}\ \left[aq_1 - bq_1^2 - \frac{aq_1 - cq_1 - bq_1^2}{2} - cq_1\right] \\ &= \max_{q_1}\ \left[aq_1 - bq_1^2 - \frac{a}{2}q_1 + \frac{c}{2}q_1 + \frac{b}{2}q_1^2 - cq_1\right] \\ &= \max_{q_1}\ \left[-\frac{b}{2}q_1^2 + \left(\frac{a}{2} - \frac{c}{2}\right)q_1\right] \end{aligned}$$

Taking the first derivative of the last function, equate it to zero and isolate the term of q_1 to obtain the optimal quantity:

$$\begin{aligned} -bq_1^* + \frac{a - c}{2} &= 0 \\ q_1^* &= \frac{a - c}{2b} \end{aligned}$$

Finally, the optimal quantity q_2^* of Ernst is simply obtained by replacing q_1 by $\frac{a-c}{2b}$ in equation 8.13:

$$\begin{aligned} q_2^* &= \frac{a - c - bq_1}{2b} \\ &= \frac{a - c - b\frac{a-c}{2b}}{2b} \\ &= \frac{a - c}{4b} \\ &= \frac{1}{2}q_1^* \end{aligned}$$

The total produced quantity in the Stackelberg game is

$$Q_s := q_1^* + q_2^* = \frac{a-c}{2b} + \frac{a-c}{4b} = \frac{3(a-c)}{4b}$$

Notice that $Q_s = \frac{3(a-c)}{4b} > \frac{2(a-c)}{4b} = Q_c$, since we make the assumption that $a > c$ and $b > 0$. In particular $p(Q_s) < p(Q_c)$, since $p(q)$ is a strictly decreasing function. The difference in quantity production between the two salesmen results in different payoffs:

$$\begin{aligned} u_1(q_1^* + q_2^*) &= (p(q_1^* + q_2^*) - c)q_1^* \\ &= (a - b(q_1^* + q_2^*) - c)q_1^* \\ &= (a - b(\frac{3(a-c)}{4b}) - c)\frac{a-c}{2b} \\ &= (a - \frac{3}{4}(a-c) - c)\frac{a-c}{2b} \\ &= (\frac{1}{4}a - \frac{1}{4}c)\frac{a-c}{2b} \\ &= \frac{(a-c)^2}{8b} \end{aligned}$$

and

$$u_2(q_1^* + q_2^*) = \frac{(a-c)^2}{16b}$$

A remarkable result is that the first player to move, Alef, is better off than the follower, Ernst: Alef makes more profit than Ernst. This seems a bit counterintuitive, since the second mover has more information and therefore always seem to be beneficial. The clue for this counterintuitive result is that Alef knows that Ernst knows about q_1. ■

Chapter 9

Applications

So far we have considered the theoretical aspect of non-cooperative game theory. The examples in the previous sections illustrate the use of the many definitions. As mentioned before, game theory is very useful tool to study interactions between players in many fields. Therefore, in this section we will apply the theory that we have discussed so far regarding real world data from different fields. The first application is about entry-deterrence of a company, which is not difficult to analyze, but it is very practical. Thereafter, we will extend this model to a more complicated (but less reliable) model. Then, since sports is worldwide respected, we will also provide an analysis regarding to penalty shootout in football. Next to that, roughly every week a serious crime is the topic that is discussed on the news. Thus it may useful to have an understanding of the strategic interaction during a certain crime. And at last, we will analyze the competition of two internet providers in the Netherlands.

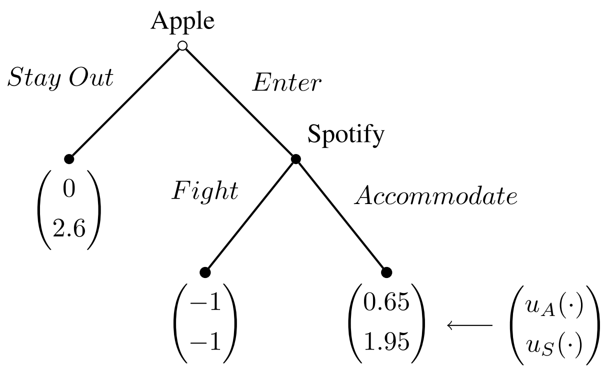
9.1 Entry-deterrence of the on-demand music streaming service in the UK

Spotify is one of the biggest, if not the biggest, on-demand music streaming service in the world. Monday, June 8, 2015, Apple officially announced during the Worldwide Developers Conference (WWDC) to compete against Spotify with the launch of Apple Music (Kelion, 2015). This is a typical example of a dynamic game of complete and perfect information studied in non-cooperative game theory, namely the entry-deterrence game.

Should or should not Apple compete against Spotify on
the on-demand music streaming service?

The game is as follows: there is an incumbent company in a certain market (here: Spotify), but a ‘new’ company (here: Apple with Apple Music) may want to enter this market. However, Apple makes this decision based on whether enterring is profitable or not. We have to mention that Spotify has not been very profitable from it existence till 2013 (in particular, it has incurred loss). Unfortunately, Spotify has not yet released their financial report of 2014. But during the years, the net loss of the company was decreasing dramatically. It could even be that Spotify did enjoy a net profit in 2014.

Some good news for the company is that it did make profit in 2013 considering Spotify in the United Kingdom (UK). Since we only consider games with rational players, it is of importance that Spotify takes on a positive profit for operating in the market. So let us take under consideration the financial numbers in the Spotify's UK (Dredge, 2014). The corresponding game representations are:



		Spotify	
		$Fight$	$Accommodate$
Apple	$Stay\ Out$	$0, 2.6$	$0, 2.6$
	$Enter$	$-1, -1$	$0.65, 1.95$

In example 7.1.1 we considered similar representations, but in that example it was used for the illustration of credible strategies. With the knowledge of subgame perfect Nash equilibrium, we are able to draw a conclusion about the optimal strategy.

Like in example 7.1.1, the only two Nash equilibria are $s^{(1)} := (s_A, s_S) = (Stay\ Out, Fight)$ and $s^{(2)} := (s_A, s_S) = (Enter, Accommodate)$. Since Spotify is always better off by accommodating, it is unlikely that Spotify would fight anyway. Thus $Fight$ is a noncredible treat and so $s^{(1)}$ cannot be a subgame perfect Nash equilibrium. On the other hand, $s^{(2)}$ is a subgame perfect Nash equilibrium. Realizing that we prefer a subgame perfect Nash equilibrium to a Nash equilibrium (since in a game of complete information the subgame perfect Nash equilibrium consists of credible strategies only), the conclusion is that Apple should enter the on-demand music streaming service in the UK since in turn Spotify would accommodate. Notice that entering is in line with what is in fact the case.

9.2 Entry-deterrence of the on-demand music streaming service worldwide

In the previous application, it was fairly easy to model the entry-deterrence of Apple Music in the UK thanks to the positive profit. In this application we will model an extended entry-deterrence of Apple Music. Namely, we use the worldwide financial data (of Spotify) from 2012 and 2013 to predict the net loss and profit for the next two years [1].

Should or should not Apple compete against Spotify on
the on-demand music streaming service worldwide?

[1] Prior to 2012, Spotify's net loss was increasing over time. In the period of 2012-2013 the company managed to decrease their net loss for the very first time.

Like before, Apple could choose whether to enter the market or not, while Spotify could choose to fight (F) or accommodate $(Acc.)$. Since this game has two time instances of decision making, we add the option that Apple could also make the same decision the next year and so Spotify could choose fight or accommodate in that year as well.

In 2012 Spotify reached a net loss of €86.7 million, while the year after the net loss decreased to €57.8 million. This is a difference of €28.9 million (Ingham, 2015).

Since this is the only data we have at hand, we should make some strict assumptions. All of these assumptions are based on the payoff of the firms after one and two years. Let $j = 0, 1, 2$ denote the number of years after 2013 and $I = \{A, S\}$ so that $u_I^{(j)}$ defines the payoff in millions after j years (from 2013) of Apple and Spotify if $I = A$ and $I = S$ respectively. Then we make the following assumptions for $j > 0$:

- ▷ If Apple stays out of the market, then $u_A^{(j)} = 0 =: u_A^{(j)}(O, \cdot)$ and $u_S^{(j)} = u_S^{(j-1)} + 28.9 \cdot 1.5^j =: u_S^{(j-1)}(O, \cdot)$. In other words, the net loss (or profit) of Spotify decreases (or increases) with $+28.9 \cdot 1.5^j$;

- ▷ If Apple enters the market (either in the first or second year) and Spotify accommodates for the first time, then the payoff of the companies are:

$$
\begin{aligned}
u_A^{(j)}(E, A) &:= \frac{j}{2+2j} u_S^{(j-1)}(O, \cdot) \\
u_S^{(j)}(E, A) &:= \left(1 - \frac{j}{2+2j}\right) u_S^{(j-1)}(O, \cdot) \\
&= \frac{2+j}{2+2j} u_S^{(j-1)}(O, \cdot)
\end{aligned}
$$

  Notice that $\frac{j}{2+2j} \to \frac{1}{2}$ as $j \to \infty$;

- ▷ If Apple enters the market (either in the first or second year) and Spotify fights for the first time, then the payoff of the companies are:

$$u_I^{(j)}(E, F) := u_I^{(j)}(E, A) - 0.25 \cdot |u_I^{(j)}(E, A)|$$

  for any $I \in \{A, S\}$. That is, the net loss (or profit) increases (or decreases) with 25% for both companies if Spotify chooses to fight;

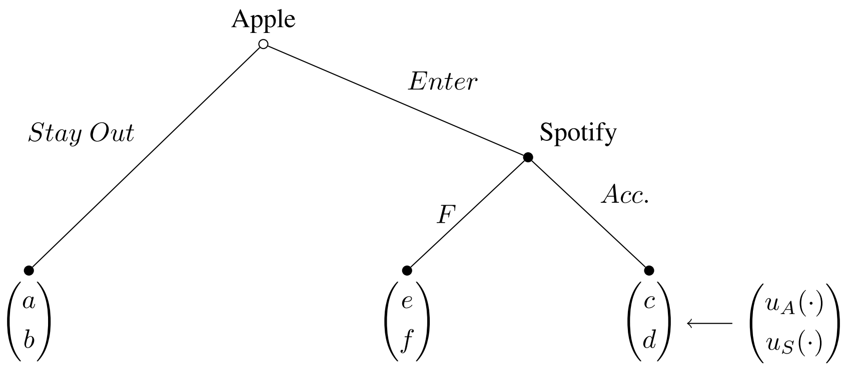
- ▷ If Apple entered the market in the first year while Spotify chose $\{S_S\} \in \{Acc., F\}$ in the first year, and $Acc.$ in the second year, then the payoffs of the companies are:

$$
\begin{aligned}
u_A^{(j)}(E, (\{S_S\}, Acc.)) &:= \frac{j}{2+2j} u_S^{(j)}(E, \{S_S\}) \\
u_S^{(j)}(E, (\{S_S\}, Acc.)) &:= \frac{2+j}{2+2j} u_S^{(j)}(E, \{S_S\})
\end{aligned}
$$

- ▷ If Apple entered the market in the first year while Spotify chose $\{S_S\} \in \{Acc., F\}$ in the first year, and F in the second year, then the payoffs of the companies are:

$$u_I^{(j)}(E, (\{S_S\}, F)) := u_I^{(j)}(E, (\{S_S\}, Acc.)) - 0.25 \cdot |u_I^{(j)}(E, (\{S_S\}, Acc.))|$$

For $j = 1$ we have the same structure for the extensive form as in the previous application:

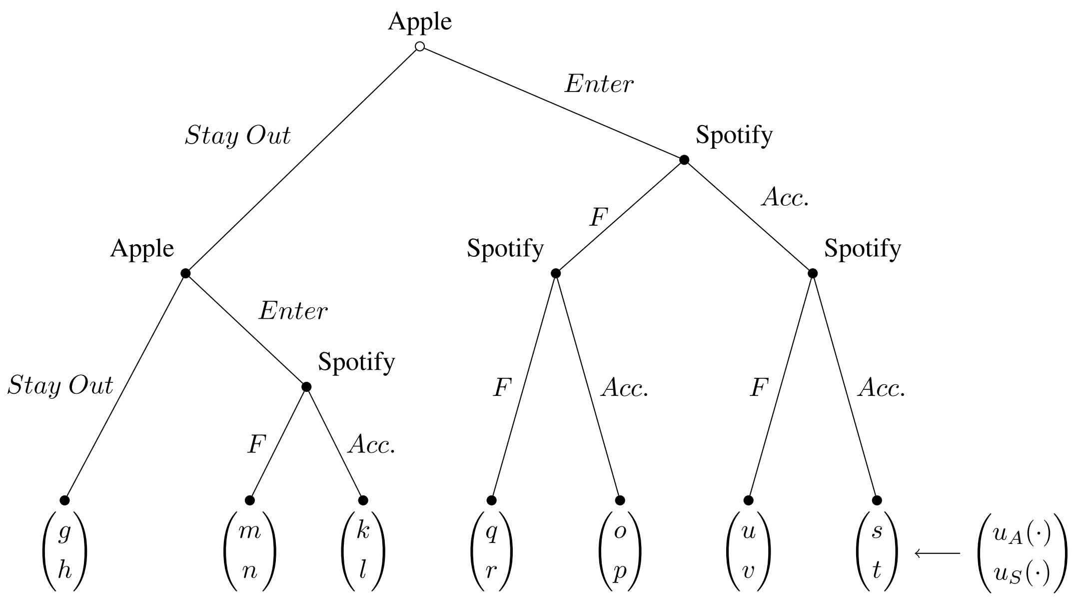


where by the first three assumptions we get:

$$
\begin{aligned}
a &= 0\\
b &= -57.8 + 28.9 \cdot 1.5^1 = -14.45\\
c &= \frac{1}{2 + 2 \cdot 1} \cdot b = -3.535\\
d &= \frac{2 + 2}{2 + 2 \cdot 1} \cdot b = -10.605\\
e &= c - 0.25 \cdot |c| = -4.41875\\
f &= d - 0.25 \cdot |d| = -13.25625
\end{aligned}
$$

The payoffs of the first year will be used to determine the payoffs after two years. This approach yields payoffs that are actually obtained cumulatively, as we will see in a moment (or by simply examining the assumptions carefully).

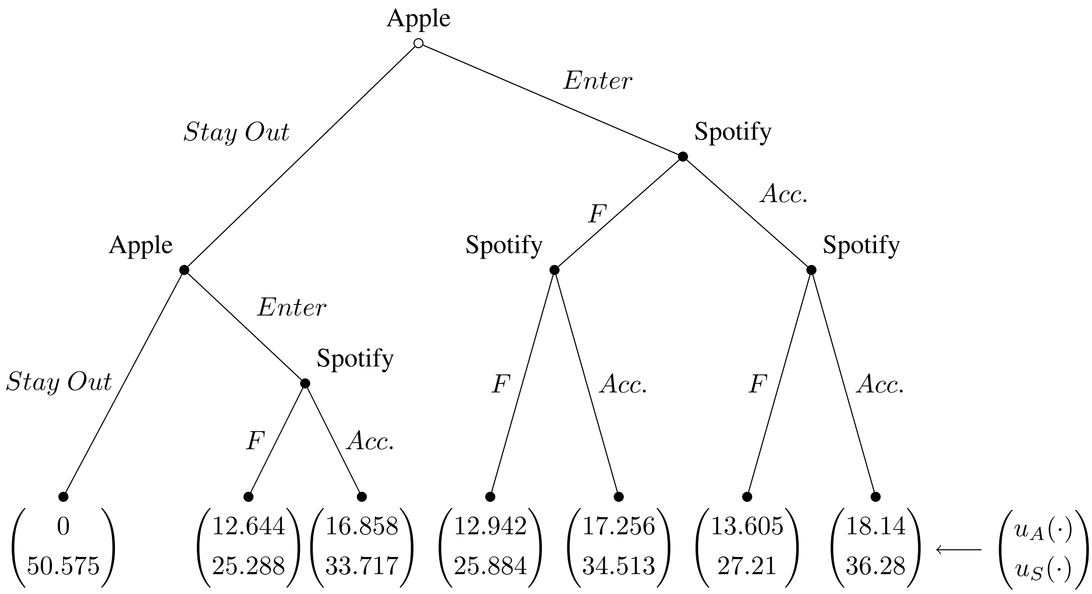
The extensive form of the whole game is as follows:



where by the last two assumptions we derive:

$$\begin{aligned}
g &= 0\\
h &= b + 28.9 \cdot 1.5^2 = 50.575\\
k &= \frac{2}{2+2\cdot 2} \cdot h = 16.858\\
l &= \frac{2+2}{2+2\cdot 2} \cdot h = 33.717\\
m &= k - 0.25 \cdot |h| = 12.64375\\
n &= l - 0.25 \cdot |l| = 25.2875\\
o &= \frac{2}{2+2\cdot 2} \cdot (f + 28.9 \cdot 1.5^2) = 17.25625\\
p &= \frac{2+2}{2+2\cdot 2} \cdot (f + 28.9 \cdot 1.5^2) = 34.5125\\
q &= o - 0.25 \cdot |o| = 12.9421875\\
r &= p - 0.25 \cdot |p| = 25.884375\\
s &= \frac{2}{2+2\cdot 2} \cdot (d + 28.9 \cdot 1.5^2) = 18.14\\
t &= \frac{2+2}{2+2\cdot 2} \cdot (d + 28.9 \cdot 1.5^2) = 36.28\\
u &= s - 0.25 \cdot |s| = 13.605\\
v &= t - 0.25 \cdot |t| = 27.21
\end{aligned}$$

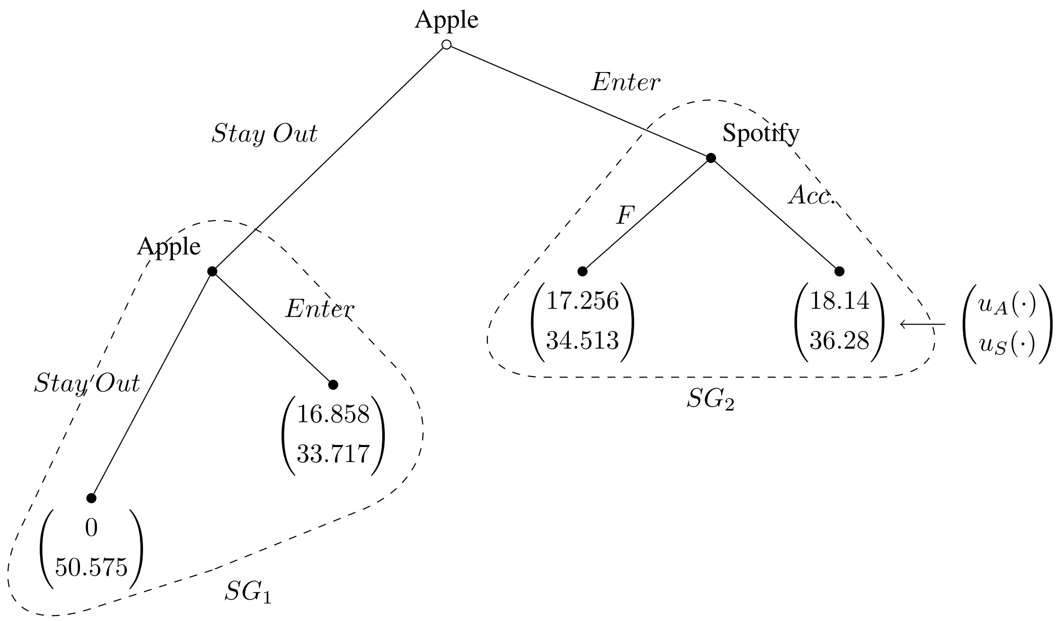
In a more favorable fashion:



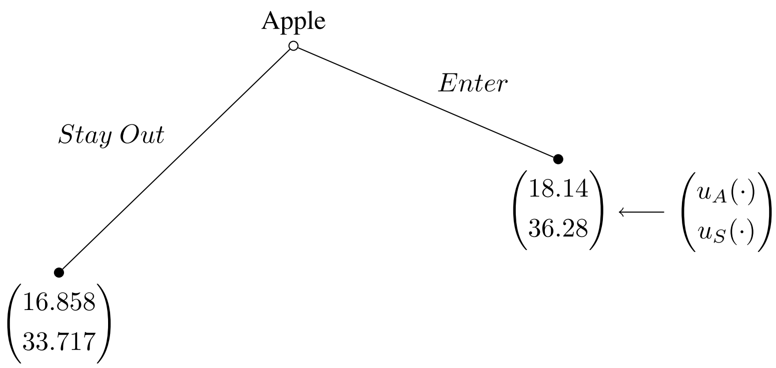
By construction, Spotify would always accommodate after entering of Apple [2]. This is in some sense reasonable, because fighting is most likely to be disadvantageous. Especially in such a

[2] Choosing F always results in 25% less profit (or 25% more losses, depending on the sign of the payoff) with respect to $Acc.$.

short time interval. But the fact that $Acc.$ is a strictly dominant strategy for Spotify is not of importance in this study. Instead, it is of importance to find out whether Apple would choose to enter or stay out of the market. So first let us reduce the extensive form representation [3]:



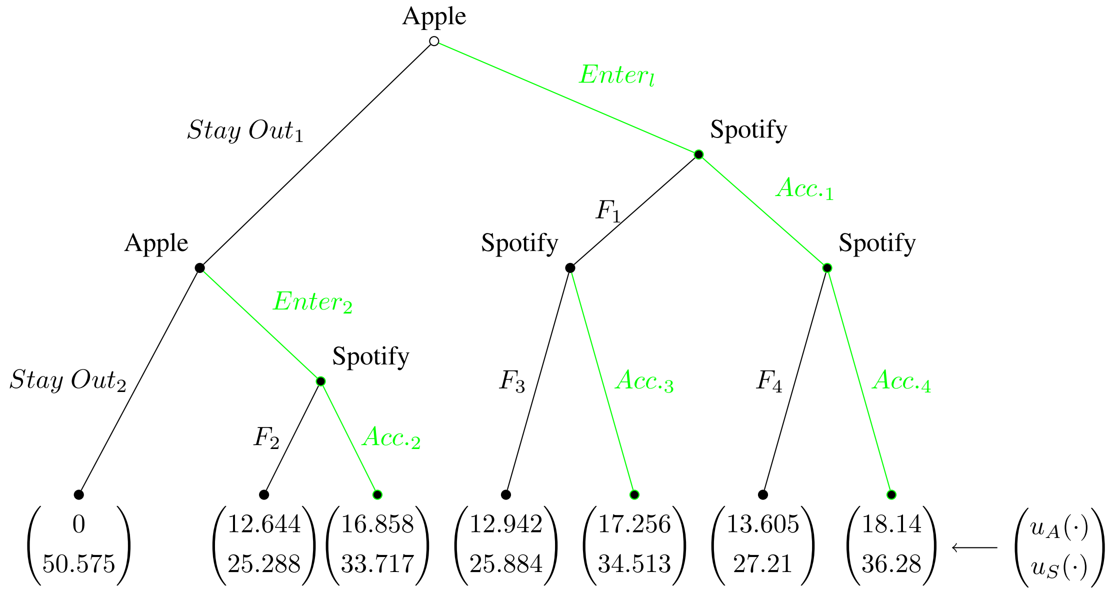
Yet there are two proper subgames as indicated, so we could apply backward induction again. In subgame 1 (SG_1) it is clear that $Enter$ is the optimal choice for Apple. In the remaining subgame, SG_2, Spotify would choose $Acc.$ over F. This brings us the next extensive form representation:



As Apple is the first player to move, we compare $u_A(Stay\ Out)$ and $u_A(Enter)$. As a result, entering is the best option.

For the very last time consider the original extensive form representation of this game. For convenience of defining the subgame perfect Nash equilibrium we add a subscript to each of the available actions.

[3] We are tempted to consider the payoffs of the first year in this extensive form representation, but this would be complete wrong. The payoffs of the first year are solely used to derive the (cumulative) payoffs of the second year. When we work backwards, like here, it is of main interest to consider the final payoffs. The reason is that this is the payoff that each company eventually will receive after two years.



where a green colored branch denotes the optimal choice in the game. From the game tree above we observe that there is only one path that connect the initial decision node to a terminal node. This implies that there is a unique subgame perfect Nash equilibrium: $s := (s_A, s_S) = ((Enter_1, Enter_2), (Acc._1, Acc._2, Acc._3, Acc._4)$ with payoffs $\binom{u_A(s)}{u_S(s)} = \binom{18.14}{36.28}$ [4]. We conclude that Apple should enter the market directly, which yields a profit of €18.14 million after two years of operation.

9.2.1 Discussion

From this application we learned that, even with a negative payoff in the first year, a strategy could also be optimal as long as the (cumulative) payoff is positive at the end. This result is quite straightforward, but games like this become more realistic if we had more data to accurately describe the future payoffs. Also, it would become more interesting if this leads to totally different payoffs. For instance if fighting becomes less worse in one of the time instances, or if entering in the second time instance is more profitable than entering directly.

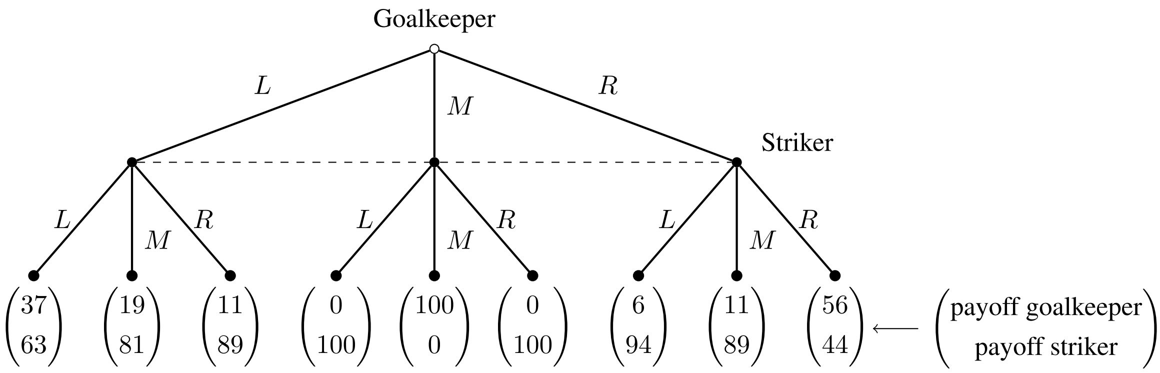
9.3 Penalty shootout

We have considered in example 5.5.1 a penalty shootout game, but now we will study a renewed version of the penalty shootout game that is more realistic. Likewise, we make the assumption that the strategy choices are made simultaneously (which is reasonable, since it takes approximately 0.3 seconds for the ball to reach the goal, which is way faster than the reaction time of most people) and that the game is of complete information. That is, both the goalkeeper and the striker know all information about the payoffs of penalty shootouts. Additional, we assume that the players are rational (and this is common knowledge) and do not participate any match fixing.

[4] Formally we include $Enter_2$ and $Acc._3$ in the subgame perfect Nash equilibrium, even though the two actions will not be reached in the second year. Recall that a subgame perfect Nash equilibrium is a Nash equilibrium that constitutes in every subgames, see definition 7.2.4.

*How should the goalkeeper and the striker distribute
their options in a penalty shootout?*

We use the data from "Testing Mixed-Strategy Equilibria When Players Are Heterogeneous: The Case of Penalty Kicks in Soccer" (Chiappori et al., 2002). In particular we use table 4 from the paper, which can be represented in the following extensive and normal form representation (where the percentages are rounded off to the nearest integer for calculation simplicity):



		Striker (*P*.2) L	M	R
Goalkeeper (*P*.1)	L	$37, 63$	$19, 81$	$11, 89$
	M	$0, 100$	$100, 0$	$0, 100$
	R	$6, 94$	$11, 89$	$56, 44$

The data consists of samples (denoted in the probabilities of winning the shootout $\times 100\%$) from all French and Italian premier league penalty kicks from 1997-1999 and 1997-2000 respectively. Like in example 5.5.1, the goalkeeper wins the penalty shootout if the goalkeeper stops the ball or if the striker misses the target. Otherwise, if the striker manages to score a goal, then the striker wins the penalty shootout.

It is easily observed that there does not exist a pure strategy Nash equilibrium:

Goalkeeper's perspective: if the striker chooses L, then my best-response is also L;
if the striker chooses M, then my best-response is also M;
if the striker chooses R, then my best-response is also R;
Striker's perspective: if the goalkeeper chooses L, then my best-response is R;
if the goalkeeper chooses M, then my best-responses are L and R;
if the goalkeeper chooses R, then my best-response is also L.

No two combinations are best-responses for each other, so indeed there is no pure strategy Nash equilibrium in this game. Also, observe that there is no strictly dominant strategy, so we should bear with this normal form representation (and thus not a smaller one). Proposition 5.3 guarantees us that there is always a mixed strategy Nash equilibrium in finite games. So let us determine the mixed strategy Nash equilibrium.

We will only analyze the extreme case where both the goalkeeper and the striker choose their strategy with positive probability (i.e. completely mixed strategy). For two reason, 1). it is reasonable that the players deviate between all three possibilities and 2). if we would also consider semi mixed strategies, then we would have considered in total of 16 combinations of mixed strategy profiles! Namely, any possible combination of $\sigma_i = (\sigma_i(L), \sigma_i(M), \sigma_i(R))$, $(\sigma_i(L), \sigma_i(M), 0)$, $(\sigma_i(L), 0, \sigma_i(R))$, $(0, \sigma_i(M), \sigma_i(R))$ where $\sigma_i(\cdot) > 0$ for $i = 1, 2$ in $\sigma = (\sigma_1, \sigma_2)$. That is, $\sigma = (\sigma_1, \sigma_2)$ where:

1. $\sigma_1 = (p_1, p_2, 1 - p_1 - p_2)$, $\sigma_2 = (q_1, q_2, 1 - q_1 - q_2)$ with $p_1, p_2, p_1 + p_2, q_1, q_2, q_1 + q_2 \in (0, 1)$;
2. $\sigma_1 = (p_1, p_2, 1 - p_1 - p_2)$, $\sigma_2 = (q, 1 - q, 0)$ with $p_1, p_2, p_1 + p_2, q \in (0, 1)$;
3. $\sigma_1 = (p_1, p_2, 1 - p_1 - p_2)$, $\sigma_2 = (q, 0, 1 - q)$ with $p_1, p_2, p_1 + p_2, q \in (0, 1)$;
4. $\sigma_1 = (p_1, p_2, 1 - p_1 - p_2)$, $\sigma_2 = (0, q, 1 - q)$ with $p_1, p_2, p_1 + p_2, q \in (0, 1)$;
5. $\sigma_1 = (p, 1 - p, 0)$, $\sigma_2 = (q_1, q_2, 1 - q_1 - q_2)$ with $p, q_1, q_2, q_1 + q_2 \in (0, 1)$;
   $\vdots$
16. $\sigma_1 = (0, p, 1 - p)$, $\sigma_2 = (0, q, 1 - q)$ with $p, q \in (0, 1)$.

We only pay attention to the first case, so let $\sigma = (\sigma_1, \sigma_2)$ where σ_1, σ_2 are defined as in the first case, and the task is to find the values of p_1, p_2, q_1 and q_2 now.

As usual, we use proposition 5.2 to find the mixed strategy Nash equilibrium. First, let us consider player 1. Then by proposition 5.2 condition (i) we should have [5]:

$$u_1(L, \sigma_2) \overset{(1)}{=} u_1(M, \sigma_2) \overset{(2)}{=} u_1(R, \sigma_2)$$

(1).

$$\begin{aligned} 37q_1 + 19q_2 + 11(1 - q_1 - q_2) &= 100q_2 \\ 37q_1 + 19q_2 + 11 - 11q_1 - 11q_2 &= 100q_2 \\ 26q_1 &= 92q_2 - 11 \\ q_1 &= \frac{92}{26}q_2 - \frac{11}{26} \end{aligned}$$

[5]As mentioned before, we make no distinction between the notation of the utility function and the expected utility function. For (completely or semi) mixed strategies we always conduct the expected utility function.

(2).

$$\begin{aligned}
6q_1 + 11q_2 + 56(1 - q_1 - q_2) &= 100q_2 \\
6\left(\frac{92}{26}q_2 - \frac{11}{26}\right) + 11q_2 + 56 - 56\left(\frac{92}{26}q_2 - \frac{11}{26}\right) - 56q_2 &= 100q_2 \\
-50\left(\frac{92}{26}q_2 - \frac{11}{26}\right) - 45q_2 + 56 &= 100q_2 \\
-\frac{4600}{26}q_2 + \frac{550}{26} - 45q_2 + 56 &= 100q_2 \\
-\frac{8370}{26}q_2 &= \frac{2006}{26} \\
q_2 &= \frac{1003}{4185}
\end{aligned}$$

Filling $q_2 = \frac{1003}{4185}$ into the expression of q_1 we have obtained earlier, we get:

$$\begin{aligned}
q_1 &= \frac{92276}{108810} - \frac{11}{26} \\
&= \frac{92276}{108810} - \frac{46035}{108810} \\
&= \frac{3557}{8370}
\end{aligned}$$

We do not have to check the second condition of proposition 5.2 since we analyze the completely mixed strategy (and thus have no strategies left that is chosen with probability 0). Therefore, the mixed strategy for the striker (player 2) is $\sigma_2 = (q_1, q_2, 1 - q_1 - q_2) = (\frac{3557}{8370}, \frac{1003}{4185}, \frac{2807}{8370})$.

Second of all, consider player 2. Then by proposition 5.2 condition (i) we should have:

$$u_2(L, \sigma_1) \overset{(1)}{=} u_2(M, \sigma_1) \overset{(2)}{=} u_2(R, \sigma_1)$$

(1).

$$\begin{aligned}
63p_1 + 100p_2 + 94(1 - p_1 - p_2) &= 81p_1 + 89(1 - p_1 - p_2) \\
63p_1 + 100p_2 + 94 - 94p_1 - 94p_2 &= 81p_1 + 89 - 89p_1 - 89p_2 \\
-23p_1 &= -95p_2 - 5 \\
p_1 &= \frac{95}{23}p_2 + \frac{5}{23}
\end{aligned}$$

(2).

$$
\begin{aligned}
89p_1 + 100p_2 + 44(1 - p_1 - p_2) &= 81p_1 + 89(1 - p_1 - p_2) \\
89p_1 + 100p_2 + 44 - 44p_1 - 44p_2 &= 81p_1 + 89 - 89p_1 - 89p_2 \\
53\left(\frac{95}{23}p_2 + \frac{5}{23}\right) + 145p_2 &= 45 \\
\frac{5035}{23}p_2 + \frac{265}{23} + \frac{3335}{23}p_2 &= 45 \\
\frac{8370}{23}p_2 &= \frac{770}{23} \\
p_2 &= \frac{77}{837}
\end{aligned}
$$

Filling $p_2 = \frac{77}{837}$ into the expression of p_1 we have obtained earlier, we get:

$$
\begin{aligned}
p_1 &= \frac{7315}{19251} + \frac{5}{23} \\
&= \frac{7315}{19251} + \frac{4185}{19251} \\
&= \frac{500}{837}
\end{aligned}
$$

For the same reason as above we do not have to check the second condition of proposition 5.2. Therefore, the mixed strategy for the goalkeeper (player 1) is $\sigma_1 = (p_1, p_2, 1 - p_1 - p_2) = (\frac{500}{837}, \frac{77}{837}, \frac{260}{837})$.

Hence, the strategy profile of mixed strategy $\sigma = (\sigma_1, \sigma_2)$ is a Nash equilibrium for $\sigma_1 = (\frac{500}{837}, \frac{77}{837}, \frac{260}{837})$ and $\sigma_2 = (\frac{3557}{8370}, \frac{1003}{4185}, \frac{2807}{8370})$.

The interpretation is that the best choice of the goalkeeper in the French (1997-1999) and Italian (1997-2000) premier league is to dive to the left 500 out of 837 times, stay in the middle 77 out of 837 times and dive left 260 out of 837 times. This yields an expected payoff of:

$$
\begin{aligned}
u_1(\sigma) &= 37q_1 + 19q_2 + 11(1 - q_1 - q_2) \\
&= 100q_2 \\
&= 6q_1 + 11q_2 + 56(1 - q_1 - q_2) \\
&= \frac{20060}{837} \approx 23.97,
\end{aligned}
$$

i.e. the goalkeeper will win approximately 23.97% of the penalty shootouts using this mixed strategy given the optimal strategy of the striker. Similarly, the striker in the two premier leagues should aim left 3557 out of 8370 times, aim middle 1003 out of 4185 times and aim right 2807 out of 8370 times. The (expected) payoff is then:

$$
\begin{aligned}
u_2(\sigma) &= 63p_1 + 100p_2 + 94(1 - p_1 - p_2) \\
&= 81p_1 + 89(1 - p_1 - p_2) \\
&= 89p_1 + 100p_2 + 44(1 - p_1 - p_2) \\
&= \frac{63640}{837} \approx 76.03
\end{aligned}
$$

i.e. the striker will score approximately 76.03% of the penalties using this mixed strategy given the optimal mixed strategy of the goalkeeper.

9.3.1 Discussion

In this application we used the data of all goalkeepers and all penalty takers in a particular region. So the result is associated to the average goalkeeper and penalty taker in that region. It may be more interesting if we just used the penalty shootout data from a certain player, like C. Ronaldo or L. Messi, because then we could find the best mixed option for a goalkeeper to increase the chance to win the penalty shootout. The downside we would be dealing with is the relatively high variance due to the low number of observations.

9.4 Robbery

Criminal activity is a daily topic in the today's news. Unfortunately, most of the people have witnessed a maltreatment at least once in their life. We will strict us to the most common crime in the Netherlands, namely mugging. Although this is not considered as a serious crime, the consequences should be taken seriously. Of course, we hope never to get involved in an assessment, but in case it is helpful to have a strategy in such situations. So our research question is:

What should we do when someone with a gun approaches us
and intrusively ask for our wallet?

To answer this question, we use the data from de Rijksoverheid about mugging in the Netherlands (2009), and also the data from the police about the declarations of the victims in the Netherlands (2010-2012) (Rovers et al., 2010; Mesu et al., 2013). From this data we can predict the type and the strategy/behaviour of the robber. Say, we walk on a public street in the Netherlands, the last thing we would (or at least hope to) expect is to face a robber with a gun. The things we immediately ask ourselves in that scenario are: what will the robber do when I do not give my wallet? Will he shoot? Is the gun real? What if I do not give him anything and just run? In this model we will carry these (sub)questions to answer the main question.

For this model, let us categorise the three main types of a robber:

1. *Bad*: he will try to kill you (which could result in either death or serious injuries) if you do not give him your money.;
2. *Violent*: he will 'only' beat you up when you do not give him your money;
3. *Good*: he will not harm (only threaten) you, in any case.

For any three types, if you give your wallet, he will not harm you and just run away. As a citizen of the Netherlands, we do not know whether the robber is of type 1, 2 or 3. But, from our data, we know that about 14% of the robberies result in death or serious injuries [6]. Approximately

[6] From the data it is not clear whether it is based on that the robber assassinates at least one innocent citizen or if the robber is the only one who get killed, but we assume the prior (i.e. there is always an innocent citizen that get killed).

56% of the muggings result in violence, and about 30% of the times the robber will not harm you [7]. From the data of the police we know that 24% of the declaration are successful, and thus 76% are not [8]. Notice that the data about the types of a robber is from 2009, while the data about the declarations is from 2010 (- 2012). But we may argue that the successfulness of declarations takes at least 1 year and so these data are well related.

To represent this game in a extensive form, we impose the following abbreviations first:

- ▷ B : robber is of bad type;
- ▷ V : robber is of violent type;
- ▷ G : robber is of good type;
- ▷ $\$$: give money;
- ▷ $\emptyset$: do not give money;
- ▷ P : punish the other;
- ▷ NP: do not punish the other;
- ▷ D : declarate, which results in imprisonment if the robber stays in the Netherlands;
- ▷ ND: declaration, which does not results in no imprisonment if the robber stays in the Netherlands;
- ▷ A : break away to a foreign country;
- ▷ NA: do not break away to a foreign country;
- ▷ k : $\times 1.000$.

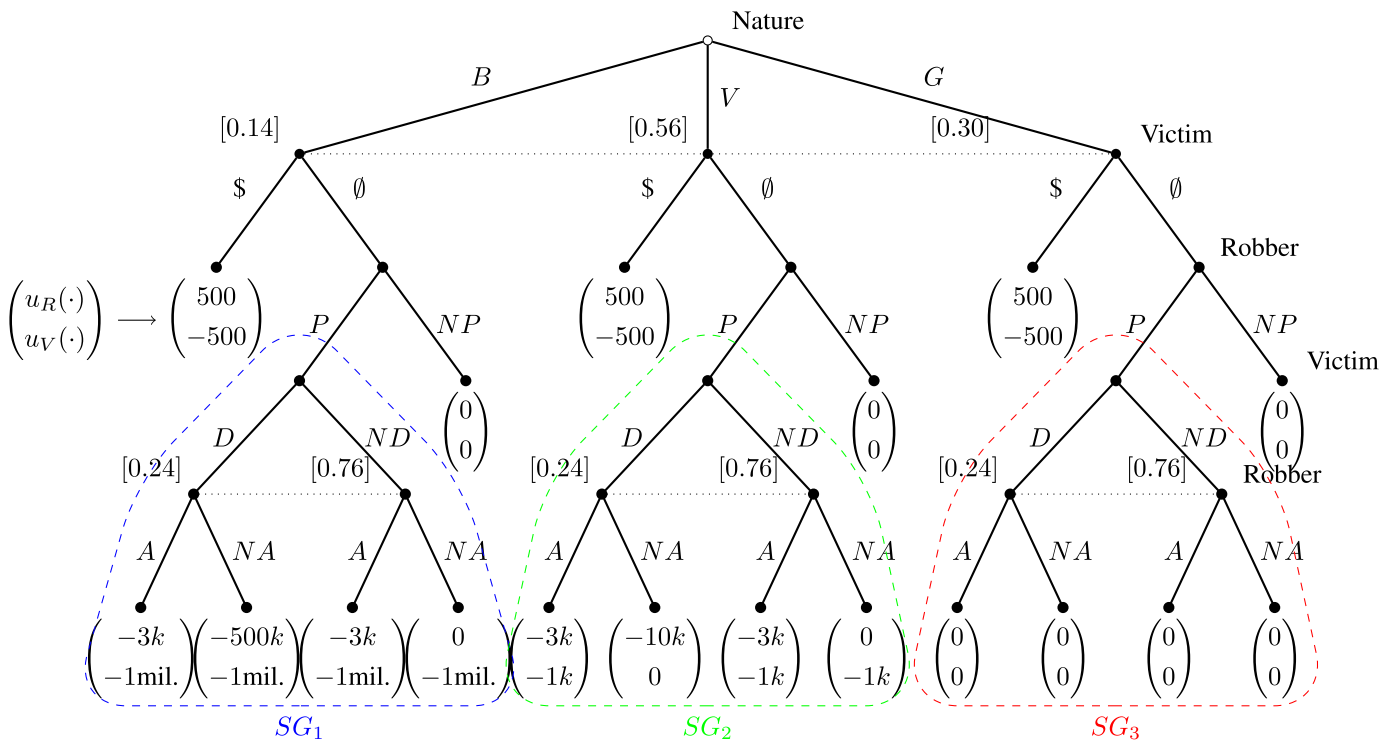
Additional, we make the following assumptions:

- ▷ The players are rational;
- ▷ The victim will only and always make a declaration when he got punished;
- ▷ If the robber moves to a foreign country, he will not get caught (since we study the situation in near future only).

Then, the game that is played after the robber treats can be express in the following extensive form representation where the corresponding payoffs are determined with common sense:

[7] We assume that the data is based on these three situations: kill or serious injuries, violence and nothing. Otherwise the percentage that nothing will happen is less than 30%.

[8] It is reasonable to assume that a declaration can either be successful or not, even though the data from the police only tells us the percentage of successful declarations.



We do not have to assign the beliefs using Bayes' rule, since the robber only has singleton information sets.

Obviously, the victim is not able to say whether her declaration will be successful or not, but as long as we assume that the victim makes a declaration after punishment this is not of importance (and so we assign a probability to reach each decision node in the information set). The main importance is that the robber does not know whether he will get caught in the near future or not.

The extensive form representation is tiring to solve, at least that is what we expect at first sight. Let us gather some courages and work step by step to find the equilibria. The first thing we are concerning of is whether this extensive form can be reduced by working backwards. Since the extensive form consists of subgames (three of them are highlighted above), it is a good chance that some reductions are possible. Let us consider:

▷ SG_1: The expected payoff of the robber by playing A and NA is respectively:

$$u_R(A) = -3.000 \times 0.24 - 3.000 \times 0.76 = -3.000$$
$$u_R(NA) = -500.000 \times 0.24 + 0 \times 0.76 = -120.000$$

So the robber would break away to a foreign country.

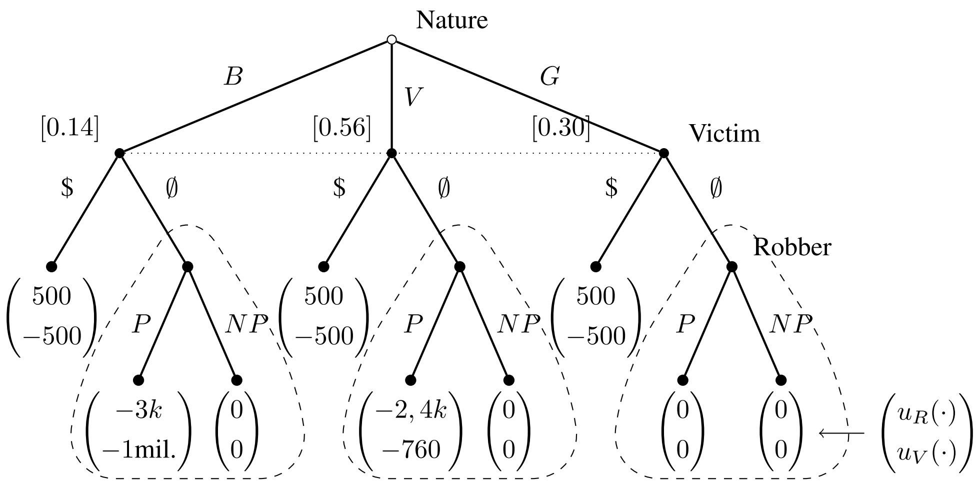
▷ SG_2: Similarly:

$$u_R(A) = -3.000 \times 0.24 - 3.000 \times 0.76 = -3.000$$
$$u_R(NA) = -10.000 \times 0.24 + 0 \times 0.76 = -2.400$$

In this case the robber's optimal choice would be to stay in the Netherlands. Then, the expected payoff for the victim is:

$$u_V(NA) = 0 \times 0.24 - 1.000 \times 0.76 = -760$$

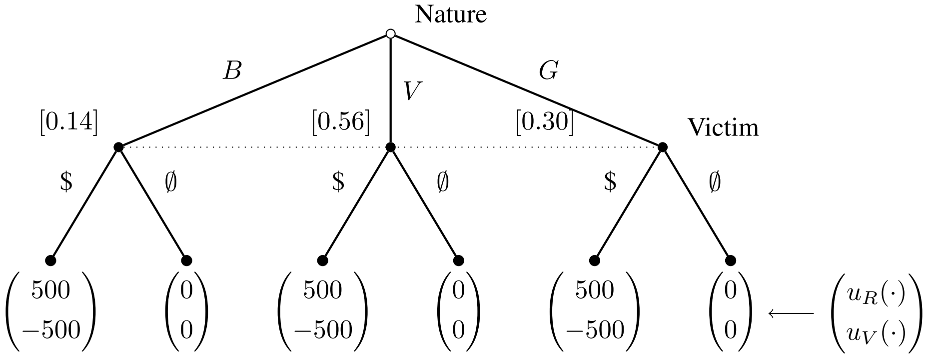
- ▷ SG_3: The robber of good type does actually not have any intention to punish the victim. This explains the corresponding payoffs, and so we could simply remove this subgame. The extensive form representation becomes:



Likewise, there are proper subgames and so we use backward induction to reduce the extensive form representation even further. As we will see soon, this extensive form representation has something interesting. Recall that we always make the assumption that all players are rational:

- ▷ If the victim does not give the wallet to the robber (in the bad and violent case), then the robber is better off by not punishing;
- ▷ If the robber is of good type, then he would not do anything to the victim and thus the payoff of punishment is equal to not punishment. In that case, the robber is indifferent between P and NP.

We infer that P is a noncredible treat. The reduced extensive form representation becomes:



Since the victim is unsure about the type of the robber (or rather unsure about the situation he is at during the treatment of the robber), we can determine the expected payoffs for the victim (or observe that $\emptyset$ is a strictly dominant strategy):

$$u_V(\$) = -500 \cdot 0.14 - 500 \cdot 0.56 - 500 \cdot 0.30 = -500$$
$$u_V(\emptyset) = 0 \cdot 0.14 + 0 \cdot 0.56 + 0 \cdot 0.30 = 0$$

We see that $u_V(\emptyset) > u_V(\$)$, which means that the victim maximizes her (expected) payoff by not giving her wallet. This strategy choice is in agreement with the previous analysis: if the robber would never punish, then it is obvious that the victim should not hand over her wallet. Hence, any sequentially rational strategy profile consists of NP and $\emptyset$, say $s := (s_R, s_V) = ((NP, \cdot), (\emptyset, \cdot))$. Therefore we conclude that the victim should never ever give money since the robber would not choose to punish.

9.4.1 Discussion

In reality, unfortunately, a robber is not rational at all. Put differently, a bad or voilent robber in general ‘lives’ in a different universe with different payoffs (e.g. he would assign a nonnegative payoff if he punishes the victim, so that punishment becomes a dominant strategy...). In future studies we may want to remove the assumption of rationality to deal with this problem.

9.5 Cable operators

Last year, 2014, there were rumours about merging of UPC and Ziggo. Yet, on April 13, 2015 both companies officially announce the merge and will continue under the name Ziggo (although Ziggo is taken over by Liberty Global). Due to this, the current Ziggo enjoys an satisfactory market position, which is in turn less pleasant for the competitors, like KPN.

According to the World Bank, about 94% of all households in the Netherlands in 2013 were connected to the internet and it is unlikely that this whopping percentage is decreasing within the last two years. Since the internet speed is an important aspect for consumers to consider, especially with the immense growth of internet usage, the companies have to strategically decide at what price they should sell a certain type of membership. So we are interested in the competition between Ziggo and KPN about the internet speed and the related price.

What is the most profitable combination of internet speed and price that could be offered by Ziggo and KPN?

Remark 9.1. Since we consider a modified version of the Stackelberg model, we will derive the mathematical expression for quantity and price first. Then use data from the two companies we are able to determine the exact value of the optimal quantity and price.

To study this competition, we will use a modified version of the Stackelberg model. The main modification of this model (with respect to the original Stackelberg model, see section 8.3) is that the companies do not compete about the number of production of packets, but the level of the internet speed for the basic membership. So, the Stackelberg model is build up as follows:

- ▷ Ziggo is the leading company (corresponds to subscript 1), and KPN is the follower (corresponds to subscript 2);
- ▷ Internet speed q_i at company i for $i = 1, 2$;
- ▷ Price function (Selim, 2010):

$$p_i = \max\{a + b_1 q_i - b_2 q_{-i}, 0\}$$

where $a, q_1, q_2, b_1, b_2 \geq 0$. Notice that p_i increases as q_i increases, but it decreases as q_{-i} increases. The intuition is that the price of company i increases as its internet speed increases, while it decreases as the internet speed of the competitor increases;

- ▷ Cost per unit (/packet) of company i is $c_i = \gamma q_i$ with $\gamma > 0$ for $i = 1, 2$, i.e. cost is positively correlated with internet speed: e.g. a higher internet speed requires more database, improved cabling, R & D, etc.;
- ▷ Profit (per packet) of company i becomes:

$$u_i(q_1, q_2) = p_i q_i - c_i q_i$$

for $i = 1, 2$;

- ▷ Additional assumptions:

$$\begin{aligned} &b_1 > \gamma \\ &b_2 > \sqrt{2}(b_1 - \gamma) \end{aligned}$$

The approach of finding the optimal internet speed, q_1^* and q_2^*, is similar to as in the Stackelberg model we have seen in section 8.3. Recall that this approach is based on backward induction.

So first we find the expression of $q_2^*(q_1)$ that maximizes the profit of KPN:

$$\begin{aligned} \max_{q_2} \ u_2(q_1, q_2) &= \max_{q_2} \ p_2 q_2 - c_2 q_2 \\ &= \max_{q_2} \ (a + b_1 q_2 - b_2 q_1) q_2 - \gamma q_2^2 \end{aligned} \tag{9.1}$$

Subsequently take the first derivative of equation 9.1 with respect to q_2:

$$\begin{aligned} \frac{\partial}{\partial q_2} u_2(q_1, q_2) &= b_1 q_2 + (a + b_1 q_2 - b_2 q_1) - 2\gamma q_2 \\ &= a - b_2 q_1 + (2b_1 - 2\gamma) q_2 \end{aligned} \tag{9.2}$$

Now by setting equation 9.2 equal to zero and solving for q_2, we obtain:

$$q_2^*(q_1) = \frac{b_2 q_1 - a}{2(b_1 - \gamma)}$$

This is the optimal strategy choice for KPN. Notice that it still depends on q_1. To get rid of this variable, we should find the optimal strategy choice of Ziggo by an almost similar approach as we did for KPN:

$$\begin{aligned}\max_{q_1}\ u_1(q_1, q_2^*(q_1)) &= \max_{q_1}\ p_1 q_1 - c_1 q_1 \\ &= \max_{q_1} (a + b_1 q_1 - b_2 q_2^*(q_1))q_1 - \gamma q_1^2 \\ &= \max_{q_1} \left(a + b_1 q_1 - b_2 \frac{b_2 q_1 - a}{2(b_1 - \gamma)}\right) q_1 - \gamma q_1^2 \end{aligned} \tag{9.3}$$

Let us find the expression for q_1 that maximizes equation 9.3:

$$\frac{\partial}{\partial q_1}\ u_1(q_1, q_2^*(q_1)) = \left(b_1 - \frac{b_2^2}{2(b_1 - \gamma)}\right) q_1 + \left(a + b_1 q_1 - \frac{b_2^2 q_1 - ab_2}{2(b_1 - \gamma)}\right) - 2\gamma q_1 = 0$$

From the last equation, by simply rearranging terms we will end up with the following:

$$\begin{aligned} b_1 q_1 - \frac{b_2^2}{2(b_1 - \gamma)} q_1 + b_1 q_1 - \frac{b_2^2}{2(b_1 - \gamma)} q_1 - 2\gamma q_1 &= -\frac{ab_2}{2(b_1 - \gamma)} - a \\ \left(2b_1 - \frac{b_2^2}{(b_1 - \gamma)} - 2\gamma\right) q_1 &= -\left(\frac{ab_2 + 2a(b_1 - \gamma)}{2(b_1 - \gamma)}\right) \end{aligned}$$

Hence, the optimal internet speed and the (corresponding) price to be set by Ziggo and KPN is respectively:

$$\begin{aligned} q_1^* &= -\left(\frac{ab_2 + 2a(b_1-\gamma)}{2(b_1-\gamma)}\right)\left(\frac{b_1-\gamma}{2(b_1-\gamma)^2 - b_2^2}\right) \\ &= \frac{a(b_2+2(b_1-\gamma))}{2(b_2^2 - 2(b_1-\gamma)^2)} \end{aligned} \tag{9.4}$$

$$\begin{aligned} q_2^* &= \frac{b_2 \frac{a(b_2+2(b_1-\gamma))}{2(b_2^2-2(b_1-\gamma)^2)} - a}{2(b_1-\gamma)} \\ &= \frac{ab_2(b_2+2(b_1-\gamma)) - 2a(b_2^2 - 2(b_1-\gamma)^2)}{4(b_2^2-2(b_1-\gamma)^2)(b_1-\gamma)} \\ &= \frac{2a(b_1-\gamma)(b_2+2(b_1-\gamma)) - ab_2^2}{4(b_2^2-2(b_1-\gamma)^2)(b_1-\gamma)} \end{aligned} \tag{9.5}$$

$$\begin{aligned} p_1^* &= a + b_1 q_1^* - b_2 q_2^* \\ &= a + b_1 \frac{2a(b_2+2(b_1-\gamma))(b_1-\gamma)}{4(b_2^2-2(b_1-\gamma)^2)(b_1-\gamma)} - b_2 \frac{2a(b_1-\gamma)(b_2+2(b_1-\gamma)) - ab_2^2}{4(b_2^2-2(b_1-\gamma)^2)(b_1-\gamma)} \\ &= a + \frac{2a(b_1-\gamma)(b_1-b_2)(b_2+2(b_1-\gamma)) + ab_2^3}{4(b_2^2-2(b_1-\gamma)^2)(b_1-\gamma)} \end{aligned} \tag{9.6}$$

$$\begin{aligned} p_2^* &= a + b_1 q_2^* - b_2 q_1^* \\ &= a + b_1 \frac{2a(b_1-\gamma)(b_2+2(b_1-\gamma)) - ab_2^2}{4(b_2^2-2(b_1-\gamma)^2)(b_1-\gamma)} - b_2 \frac{2a(b_2+2(b_1-\gamma))(b_1-\gamma)}{4(b_2^2-2(b_1-\gamma)^2)(b_1-\gamma)} \\ &= a + \frac{2a(b_1-\gamma)(b_1-b_2)(b_2+2(b_1-\gamma)) - ab_1b_2^2}{4(b_2^2-2(b_1-\gamma)^2)(b_1-\gamma)} \end{aligned} \tag{9.7}$$

Notice that, with the two additional assumptions we have that q_1^* and q_2^* are well-defined, and in particular $q_1^*, q_2^* \geq 0$ [9]. Notably, if a is strictly positive (which is likely the case), then we would have strict inequality instead.

Now, we obtained have the required expressions and yet we have to find the values for a, b_1, b_2 and γ. From the annual financial report of the companies we are not able to find a reliable and accurate value for γ unfortunately. Therefore, we assume $\gamma = 1$. Both companies offer a great amount of different memberships, but we limit this study to the All-in-1 basic packet of each company. Further, we ignore any differences in extra utilities included between the two packets, since both have their (dis)advantages and so this allows us to focus solely on the internet speed and price of both packets.

[9] It is obvious that $q_1^* \geq 0$, since the numerator is nonnegative and the denominator is strictly positive. Also, it is clear that the denominator of q_2^* is strictly positive. For the numerator to be nonnegative, we would need that $2ab_2(b_1-\gamma) - ab_2^2 + 4a(b_1-\gamma)^2 \geq 0$, or $2a(b_1-\gamma)(b_2+2(b_1-\gamma)) \geq ab_2^2$. This is the case, since $2a(b_1-\gamma)(b_2+2(b_1-\gamma)) \geq 2a(b_1-\gamma)(\sqrt{2}+2)(b_1-\gamma) = 4a(b_1-\gamma)^2(1+\frac{1}{\sqrt{2}}) \geq ab_2^2$.

Ziggo offers the All-in-1 packet for approximately €46,- at the download speed of 40 Mbit/s, while KPN offers a similar packet for approximately €59,- at the download speed of 50 Mbit/s (it is not necessary to consider the upload speed as well, since in both cases it is equal to 1/10 of the download speed) [10].

The equations of interest become:

$$46 = a + 40b_1 - 50b_2 \tag{9.8}$$
$$59 = a + 50b_1 - 40b_2 \tag{9.9}$$

Rewrite equation 9.8 to obtain the expression for b_1:

$$b_1 = \frac{46}{40} - \frac{1}{40}a + \frac{5}{4}b_2 \tag{9.10}$$

And subsequently filling this expression into equation 9.9:

$$\begin{aligned}
59 &= a + 50\left(\frac{46}{40} - \frac{1}{40}a + \frac{5}{4}b_2\right) - 40b_2 \\
&= a + \frac{5}{4}\cdot 46 - \frac{5}{4}a + \frac{250}{4}b_2 - \frac{160}{4}b_2 \\
&= \frac{230}{4} - \frac{1}{4}a + \frac{90}{4}b_2 \\
b_2 &= \frac{4}{90}\left(\frac{236}{4} - \frac{230}{4}\right) + \frac{1}{90}a \\
&= \frac{1}{15} + \frac{1}{90}a
\end{aligned} \tag{9.11}$$

From substituting b_2 into equation 9.10 we get:

$$\begin{aligned}
b_1 &= \frac{46}{40} - \frac{1}{40}a + \frac{5}{4}\left(\frac{1}{15} + \frac{1}{90}a\right) \\
&= \frac{37}{30} - \frac{1}{90}a
\end{aligned} \tag{9.12}$$

As we know, from this point on, any value of a will theoretically work for the next calculations, but if we would take $a = 1$, then we get the following optimal outputs [11]:

$$\begin{aligned}
q_1^* &= -2.8162 \\
p_1^* &= -2.2287 \\
q_2^* &= -2.7428 \\
p_2^* &= -2.1333
\end{aligned}$$

which is (practically) not reasonable (and violates the assumption that $q_1 \geq 0$ too). Instead of making further assumptions, it would be convenient if we could find the value for a that is most representative. That is, we want to minimize the following error with respect to a:

$$e(a) := \sqrt{(p_1^* - 46)^2 + (q_1^* - 40)^2 + (p_2^* - 59)^2 + (q_2^* - 50)^2} \tag{9.13}$$

[10]The data are obtained from `https://www.ziggo.nl` and `http://www.kpn.com` at July 15, 2015.

[11]The Matlab code can be found in Appendix B under *FAILURE*.

It is possible to solve this very long algebra, but we rather use a numerical approach to save some work. Actually, we will run two different algorithms with the help of Matlab R2014b.

The first algorithm is in line with equations 9.8, 9.9 and 9.13. So we should be able to find the upper- and lower bound of a using the exact expressions of b_1 and b_2. Assumption 1 (i.e. $b_1 > \gamma = 1$) and 2 (i.e. $b_2 > \sqrt{2}(b_1 - \gamma)$) imply:

$$\begin{aligned}\frac{37}{30} - \frac{1}{90}a &> 1\\ a &< 21\end{aligned}$$

$$\begin{aligned}\frac{1}{15} + \frac{1}{90}a &> \sqrt{2}\left(\frac{7}{30} - \frac{1}{90}a\right)\\ \frac{1+\sqrt{2}}{90}a &> \frac{7\sqrt{2}-2}{30}\\ a &> \frac{3(7\sqrt{2}-2)}{1+\sqrt{2}} \approx 9.8162\end{aligned}$$

With this, the description of the first algorithm is as follows:

1. Let $a = 9.8$;
2. Calculate b_1 and b_2 (see equations 9.11 and 9.12);
3. Determine q_1^*, p_1^*, q_2^* and p_2^*;
4. Calculate $e(a)$ and save this value in row $(a \cdot 10 - 97)$ of the (301×1)-vector e;
5. Repeat steps 2-4 for $a = 9.9, \ldots, 21.0$;
6. Determine the value for a, say a^*, that minimizes $e(a)$, i.e. $e(a^*) < e(a)$ for all $a \in \{9.8, 9.9, \ldots, 21.0\}$.

See Appendix B for the code of this algorithm.

The optimal value of a is $a^* = 13.8$ and correspondingly:

$$\begin{aligned}b_1 &= 1.0800\\ b_2 &= 0.2200\\ q_1^* &= 73.6517\\ p_1^* &= 90.0392\\ q_2^* &= 15.0211\\ p_2^* &= 13.8194\\ e(a^*) &= 123.9756\end{aligned}$$

In other words, Ziggo should increase the internet speed from 40 Mbit/s to 73.65 Mbit/s at the price of €90.04, while on the contrary KPN should decrease the internet speed from 50 Mbit/s to 15.02 Mbit/s at the price of €13.82.

As mentioned before, this optimal value a^* is in line with equations 9.8, 9.9 and 9.13. However, we may be more interested in only minimizing the error $e(a)$. So we are wondering if there is $a^{**} \in \mathbb{R}$ such that $e(a^{**}) < e(a^*)$. For this we use a second algorithm, which extends the first algorithm and does not take into account equations 9.11 and 9.12. So we cannot find the upper- and lower bound of a like before, but instead we guess a certain range. The first part of this algorithm is:

1. Let $a = 0.0$;
2. Let $b_1 = 0.1$ [12];
3. Let $b_2 = 0.11$;
4. Determine q_1^*, p_1^*, q_2^* and p_2^* to calculate $e(a)$ and store this value in row $(b_1 \cdot 10)$ and column $(b_2 \cdot 100 - 10)$ of the (100×100)-matrix m;
5. Repeat step 4-5 for $b_2 = 0.12, 0.13, \dots, 1.10$;
6. Repeat step 3-6 for $b_1 = 0.1, 0.2, \dots, 10.0$;
7. Save the minimum value of matrix m in row $(a \cdot 10 + 1)$ of the (51×1)-vector a_{vec};
8. Repeat step 2-8 for $a = 0.1, 0.2, \dots, 5.0$;
9. Find the minimal value of a_{vec} (which is the optimal value a^{**}).

This algorithm yields $a^{**} = 0.8$. Still, as we do not have expression for b_1 and b_2, we should determine these values backward wise in the second part of the algorithm:

10. Let $b_1 = 0.1$;
11. Let $b_2 = 0.11$;
12. Calculate $e(a_{opt})$ and store this value in row $(b_1 \cdot 10)$ and column $(b_2 \cdot 100 - 10)$ of the (100×100)-matrix b;
13. Repeat step 3 for $b_2 = 0.12, 0.13, \dots, 1.10$;
14. Repeat step 2-3 for $b_1 = 0.1, 0.2, \dots, 10.0$;
15. Find the minimum value in b and the corresponding entry in the matrix;
16. Determine b_1 and b_2 based on the entry found in step 15.

[12] This defies assumption 1 (and most likely assumption 2 as well), but as we increase b_1 this would not matter. Soon we will see that this eventually does not affect the result.

As a result we get [13]:

$$b_1 = 1.1$$
$$b_2 = 0.15$$

Finally, we find the values of interest:

$$q_1^* = 56.00$$
$$p_1^* = 56.70$$
$$q_2^* = 38.00$$
$$p_2^* = 34.20$$
$$e(a^*) = 30.7755$$

Notice that the error is dramatically reduced, while at the same time q_1^*, p_1^*, q_2^* and p_2^* are more in line with the true values. Hence, we conclude that that Ziggo should increase the internet speed to 56 Mbit/s at the price of €56.70 while on the other hand KPN should decrease the internet speed to 38 Mbit/s at the price of €34.20 to maximize their profit.

9.5.1 Discussion

The results accurately describe the situation in the real world. We may argue that this model is satisfactory, but for future studies we could try to increase the accuracy. An option is to consider different price and/or cost function by simply add a cubic or quadratic term in these function. Or we could make the analysis even more complicated by considering the effect of the extensions (e.g. the included tools for phone and television) in each packet. Besides (recall that we only take into account the All-in-1 membership of both companies), it is useful to make a similar analysis for other memberships as well to gain more insight in the competition.

[13] Notice that $b_1 > \gamma$ and $b_2 > \sqrt{2}(b_1 - \gamma)$.

Conclusion

Non-cooperative game theory relies on the solution concept of the Nash equilibrium. This solution concept is principally employed in static games of complete information. When at least one player in a static game is not sure about the payoff of the other players, then we use one of the refinements of the Nash equilibrium instead: the Bayesian Nash equilibrium.

As the Nash equilibrium could consist credible strategies in a dynamic game setting, it is prevalent to apply backward induction to get rid of these equilibria. The resulting Nash equilibrium is called a subgame perfect Nash equilibrium. However, when the dynamic game is of incomplete information, it is not possible to obtain reliable equilibria using backward induction. In that case the perfect Bayesian equilibrium provides the most desirable solution.

The four solution concepts (see above) capture the most possible game scenarios. Because of this, these are ideally for the analysis of real world problems. However, we have to keep in mind that the assumptions we make sometimes yield unrealistic results. These unrealistic results can partly be avoided by changing the rules of the game. Nevertheless it is possibly the case that the new assumptions do not perfectly describe the true situation too. So non-cooperative game theory should mainly be adopted as a tool to get a better understanding of the interactions between players in a game.

> “ *Essentially, all models are wrong, but some are useful.* ”
>
> George E.P. Box, *Empirical Model-Building and Response Surfaces, p. 424.*, 1987.

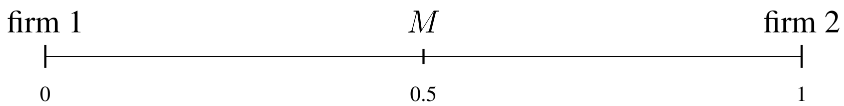
Appendix A

Hotelling linear city model

Obviously, it is very unlikely that profit-seeking firms set the price of their good equal to the corresponding cost like in the Bertrand model. An extension to the Bertrand model that deals with this problem, is the Hotelling linear city model named after H. Hotelling (1929). In this model, two firms also compete in prices, but additionally the distance between the firms (commonly regarded as an interval $[0, 1]$) are taken into account. On this unit interval, the M number of consumers are uniformly distributed. The intuition behind this is that some consumers may prefer to buy a (differentiated) good from a certain firm depending on how far she has to travel. Product differentiation provides each firm a certain market power due to the uniqueness of its good. As we will see, the result is that the price is strictly higher than the corresponding cost for each firm and thus the profits are strictly positive (Hotelling, 1929).

For example, some consumers are willing to buy coffee from Starbucks instead of the less costly coffee from a restaurant. Or, in case of a homogeneous good like Heineken beer, some consumers prefer to buy the beer from the nearest supermarket even though it is more expensive than at the other supermarket (because of the travel time and so the additional cost).

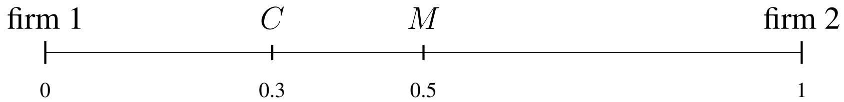
Let us consider the unit interval $[0, 1]$, where firm 1 is at the far left end and firm 2 is at the far right end of this interval. Assume that the M number of consumers are uniformly distributed on the interval. See below.



As in the Bertrand model, let $c_i > 0$ and $p_i > 0$ denote the cost and the price of the good respectively of firm i for $i = 1, 2$. (From now on, always consider i to take only value 1 and 2.) Also, assume without loss of generality that $c_1 \leq c_2$ (in case of $c_1 < c_2$, we write $c_1 + \epsilon = c_2$ for some $\epsilon > 0$.) Due to the extension of the model, we denote the total travel time by $t > 0$ and the placement of the consumers on the unit interval by $x \in [0, 1]$. Finally, we assume $t > \epsilon$ (so that the optimal price $p_2^* > c_2$ as we will see soon). The consumer is indifferent if the total cost is equal for going to firm 1 or firm 2, i.e.:

$$p_1 + tx = p_2 + t(1 - x) \tag{A.1}$$

For example, consumer C at $x = 0.3$ does not have a preferance to either buy at firm 1 or firm 2 if $p_1 + 0.3t = p_2 + 0.7t$. In this case it is obviously that $p_1 > p_2$.



Rearrange terms in equation A.1 to get the expression for x:

$$\begin{aligned} tx &= p_2 - p_1 + t - tx \\ x &= \frac{1}{2} + \frac{p_2 - p_1}{2t} \end{aligned}$$

If $p_1 = p_2$, then the consumers at $x = \frac{1}{2}$ are indifferent between buying at firm 1 or firm 2. However, if $p_1 < p_2$, then the indifferent consumers belong at $x > \frac{1}{2}$ and thus firm 1 has a greater interval of consumers that will buy at the firm. The intuition behind these results is straightforward.

Our interest is to find the optimal price of each firm. To do so, we should determine the demand function $x_i(p_1, p_2)$ for firm i ($i = 1, 2$) first:

$$\begin{aligned} x_1(p_1, p_2) &= \left(\frac{1}{2} + \frac{p_2 - p_1}{2t}\right) M \\ x_2(p_1, p_2) &= M - x_1(p_1, p_2) \\ &= \left(\frac{1}{2} + \frac{p_1 - p_2}{2t}\right) M \end{aligned}$$

The optimal price is chosen such that the profit $u_i(p_1, p_2)$ is maximized. Notice that $u_2(p_1, p_2)$ is equal to $u_1(p_1, p_2)$ when we replace p_1 by p_2, p_2 by p_1 and c_1 by c_2. Therefore it is convenient to work out the derivation for firm 1 first:

$$\begin{aligned} u_1(p_1, p_2) &= \max_{p_1} \, (p_1 - c_1) \left(\frac{1}{2} + \frac{p_2 - p_1}{2t}\right) M \\ &= \max_{p_1} \, \frac{1}{2} p_1 M + \frac{M}{2t}(p_1 p_2 - p_1^2) - \frac{1}{2} c_1 - \frac{c_1 M}{2t}(p_2 - p_1) \end{aligned}$$

The first derivative of $u_1(p_1, p_2)$ becomes:

$$\begin{aligned} \frac{\partial}{\partial p_1} u_1(p_1, p_2) &= \frac{1}{2} M + \frac{M}{2t}(p_2 - 2p_1) + \frac{c_1 M}{2t} \\ &= \frac{M}{2t} \left(t + p_2 - 2p_1 + c_1\right) \end{aligned}$$

So the first order condition can be simplified:

$$t + p_2 - 2p_1 + c_1 = 0$$

By the argument before, we have the following two equations that should be satisfied:

$$t + p_2 - 2p_1 + c_1 = 0 \tag{A.2}$$

$$t + p_1 - 2p_2 + c_2 = 0 \tag{A.3}$$

It is a matter of some simple algebra to obtain an expression for p_1 and p_2 in terms of t, c_1 and c_2. From equation A.2 we have $p_2 = 2p_1 - t - c_1$, and filling in this expression in equation A.3 we get:

$$\begin{aligned} t + p_1 - 2(2p_1 - t - c_1) + c_2 &= 0 \\ t - 3p_1 + 2t + 2c_1 + c_2 &= 0 \end{aligned}$$

Therefore the optimal price for firm 1 and (again, by the same argument before) firm 2 becomes:

$$\begin{aligned} p_1^* &= t + \frac{2c_1 + c_2}{3} \\ p_2^* &= t + \frac{2c_2 + c_1}{3} \end{aligned}$$

Indeed, with the assumptions we made, we have $p_1^* > c_1$ and $p_2^* > c_2$ [1]. Hence the profit of the firms is strictly positive:

$$\begin{aligned} u_1(p_1^*, p_2^*) &= (p_1^* - c_1)\left(\frac{1}{2} + \frac{p_2^* - p_1^*}{2t}\right) M \\ &= \left(t + \frac{2c_1 + c_2}{3} - c_1\right)\left(\frac{1}{2} + \frac{t + \frac{2c_2+c_1}{3} - t - \frac{2c_1+c_2}{3}}{2t}\right) M \\ &= \left(t + \frac{c_2 - c_1}{3}\right)\left(\frac{1}{2} + \frac{c_2 - c_1}{6t}\right) M \\ &= \frac{t^2}{2}\left(1 + \frac{c_2 - c_1}{3t}\right)^2 M \\ u_2(p_1^*, p_2^*) &= \frac{t^2}{2}\left(1 + \frac{c_1 - c_2}{3t}\right)^2 M \end{aligned}$$

Remark A.1. One may argue that the profit of firm 2 could still be strictly positive if we did not make the assumption that $t > \epsilon$ and in particular allow $0 < t \leq \frac{\epsilon}{3}$. This is true, but not realistic for two reasons. Firstly, for each good sold by firm 2, the firm would make a loss (and so, how could they eventually make positive profit?). Secondly of all, it turns out that the demand of the firm is nonpositive as well. To illustrate this, consider the utility function of firm 2:

$$u_2(p_1, p_2) = \max_{p_2} (p_2 - c_1)\left(\frac{1}{2} + \frac{p_1 - p_2}{2t}\right) M$$

If $t \leq \frac{\epsilon}{3}$, then $p_2^* < c_1$ and so $p_2 - c_1 < 0$. Furthermore, the surprise is the sign of the demand

[1] To see that $p_1^* > c_1$ (and similarly $p_2^* > c_2$), let us consider the following two cases:

- ▷ $c_1 = c_2$: $p_1^* = t + \frac{2c_1+c_1}{3} = t + c_1 > c_1$;
- ▷ $c_1 < c_2$: Take $\epsilon > 0$ such that $c_1 + \epsilon = c_2$. Then $p_1^* = t + \frac{2c_1+(c_1+\epsilon)}{3} = c_1 + \frac{3t+\epsilon}{3} > c_1$ and $p_2^* = t + \frac{2c_2+(c_2-\epsilon)}{3} = c_2 + t - \frac{\epsilon}{3} > c_2$ by the assumption that $t > \epsilon$.

function:

$$
\begin{aligned}
\left(\frac{1}{2} + \frac{p_1 - p_2}{2t}\right) M &= \left(\frac{1}{2} + \frac{t + \frac{2c_1+c_2}{3} - t - \frac{2c_2+c_1}{3}}{2t}\right) M \\
&= \left(\frac{1}{2} + \frac{c_1 - c_2}{6t}\right) M \\
&= \left(\frac{1}{2} - \frac{\epsilon}{6t}\right) M \\
&\leq \left(\frac{1}{2} - \frac{1}{2}\right) M \\
&= 0
\end{aligned}
$$

In the worst case (i.e. $t < \frac{\epsilon}{3}$) firm 2 even has a strictly negative demand.

Appendix B

Matlab code for application 9.5

```
-----------------------------------------------------------------------------
%DEFINE THE FUNCTIONS FOR QUANTITY AND PRICE FOR THE COMPANIES
q1 = @(a,b1,b2,g) ( a*(b2+2*(b1-g)) )/( 2*(b2^2-2*(b1-g)^2) )
q2 = @(a,b1,b2,g) ( 2*a*(b1-g)*(b2+2*(b1-g))-a*b2^2 )/( 4*(b2^2-2*(b1-g)^2)*
     (b1-g) )

p1 = @(a,b1,b2,g) a + ( 2*a*(b1-g)*(b1-b2)*(b2+2*(b1-g))+a*b2^3 )/( 4*(b2^2-2*
     (b1-g)^2)*(b1-g) )
p2 = @(a,b1,b2,g) a + ( 2*a*(b1-g)*(b1-b2)*(b2+2*(b1-g))-a*b1*b2^2 )/( 4*(b2^2
     -2*(b1-g)^2)*(b1-g) )


%FAILURE
a = 1;, g = 1;
b1 = 37/30 - a/90;
b2 = 1/15 + a/90;
q1(a,b1,b2,g)
p1(a,b1,b2,g)
q2(a,b1,b2,g)
p2(a,b1,b2,g)


%ALGORITHM 1
g = 1;
n = zeros(113,1);
for anew = 98:210 %STEP 1 AND 5
    a = anew/10; %RESCALE A

    b1 = 37/30 - a/90; %STEP 2
    b2 = 1/15 + a/90;

    q1(a,b1,b2,g); %STEP 3
    p1(a,b1,b2,g);
    q2(a,b1,b2,g);
    p2(a,b1,b2,g);

    n(anew-97,1) = sqrt((p1(a,b1,b2,g)-46)^2+(q1(a,b1,b2,g)-40)^2+(p2(a,b1,b2,g)
                   -55)^2+(q2(a,b1,b2,g)-50)^2); %STEP 4
end
a_opt = (find(n==min(min(n))) / 10) + 9.9 %STEP 6

b1 = 37/30 - a_opt/90
```

```
b2 = 1/15 + a_opt/90
q1(a_opt,b1,b2,g)
p1(a_opt,b1,b2,g)
q2(a_opt,b1,b2,g)
p2(a_opt,b1,b2,g)
error = sqrt((p1(a,b1,b2,g)-46)^2+(q1(a,b1,b2,g)-40)^2+(p2(a,b1,b2,g)-55)^2+
        (q2(a,b1,b2,g)-50)^2)


%FIRST PART OF ALGORITHM 2
g = 1;
m = zeros(100, 100);
a_vec = zeros(51,1);
for anew = 0:50 %STEP 1 AND 8
    a = anew/10; %RESCALE A
for b1new = 1:100 %STEP 2 AND 6
    b1 = b1new/10; %RESCALE B1
    for b2new = 11:110 %STEP 3 AND 5
        b2 = b2new/100; %RESSCALE B2
        m(b1new,b2new-10) = sqrt((p1(a,b1,b2,g)-46)^2+(q1(a,b1,b2,g)-40)^2+
                            (p2(a,b1,b2,g)-55)^2+(q2(a,b1,b2,g)-50)^2); %STEP 4
    end
end
a_vec(anew+1) = min(min(m)) %STEP 7
end
a_opt = (find(a_vec==min(min(a_vec))) / 10)-0.1 %STEP 9


%SECOND PART OF ALGORITHM 2
%FIND THE OPTIMAL VALUE OF B1 AND B2 THAT RESULTED IN A_OPT
a_opt;
b = zeros(100, 100);
for b1new = 1:100 %STEP 10 AND 14
    b1 = b1new/10; %RESCALE B1
    for b2new = 11:110 %STEP 11 AND 13
        b2 = b2new/100; %RESCALE B2
        b(b1new,b2new-10) = sqrt((p1(a_opt,b1,b2,g)-46)^2+(q1(a_opt,b1,b2,g)
                            -40)^2+(p2(a_opt,b1,b2,g)-55)^2+(q2(a_opt,b1,b2,g)
                            -50)^2); %STEP 12
    end
end
[r,c] = find(b==min(min(b))) %STEP 15
b1_opt = r/10 %STEP 16
b2_opt = 0.1+c/100


%FIND THE OPTIMAL PRICE AND QUANTITY THAT SHOULD BE SET BY THE COMPANIES
a_opt = 0.8;, b1_opt = 1.1;, b2_opt = 0.15, g = 1;
q1(a_opt,b1_opt,b2_opt,g)
p1(a_opt,b1_opt,b2_opt,g)
q2(a_opt,b1_opt,b2_opt,g)
p2(a_opt,b1_opt,b2_opt,g)
error = sqrt((p1(a_opt,b1_opt,b2_opt,g)-46)^2+(q1(a_opt,b1_opt,b2_opt,g)-40)^2
        +(p2(a_opt,b1_opt,b2_opt,g)-55)^2+(q2(a_opt,b1_opt,b2_opt,g)-50)^2)
----------------------------------------------------------------------------
```

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