The maximum number of ordinary double points on a surface

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Abstract

In this thesis the maximum number of ordinary double points on an algebraic surface in complex three-space is investigated. For lower degree surfaces a proof of this maximum number can be given; for higher degree surfaces various examples of surfaces with many ordinary double points are discussed.

Contents

1 Introduction

Singular points on algebraic surfaces have always been a major subject of interest in the field of algebraic geometry. These points are different from the other points of the surface, the regular points, in a certain way and they are often easily recognized from the plots of the surface. Initially, singular points were observed in the study of algebraic plane curves, the zeros of a polynomial in two variables. In a singular point the tangent space has a dimension exceeding the dimension of the variety, which makes the point qualitatively different from the regular points. Such singular points were quickly generalized to algebraic varieties of higher dimensions. We will give an exact explanation of the way in which these singularities differ from regular points later on.

In this bachelor's thesis we will specifically be concerned with a certain type of singularity: ordinary double points. In two variables there is a complete answer to the question of how many of these singularities are possible depending on the degree d of the curve: at most $\frac{(d-1)(d-2)}{2}$ (see for example chapter IV, remark 3.11.1 in *Algebraic Geometry* [6]). The first nontrivial case is for three variables and this is the case we will be interested in. It was proved already in the 19th century that an algebraic surface in complex three-space admits at most finitely many ordinary double points and we will investigate the global question of the maximum possible number depending on the degree of the surface. For lower degree surfaces this number is known; they are given in the following table where d is the degree of the surface and $\mu(d)$ stands for the maximum possible number of ordinary double points.

d	$\mu(d)$
1	0
2	1
3	4
4	16
5	31
6	65

For higher degree surfaces the maximum number of ordinary double points is not yet known, but lower and upper bounds have been found and are still improved nowadays.

d	$\mu(d) \geq$	$\mu(d) \leq$
7	99	104
8	168	174
9	216	246
10	345	360
11	425	480
12	600	645

In this thesis we will thus investigate the maximum number of ordinary double points on surfaces in complex three-space. The following chapter will go deeper into what it means for a point to be singular. We will introduce some basic definitions that we will be using throughout this thesis and we will try to establish some lower and upper bounds ourselves. In the chapters that follow we will study surfaces of a specific degree and the maximum possible number of ordinary double points on them. In the case of lower degree surfaces we will be able to give full proofs of this maximum number. For higher degree we will only give examples of surfaces allowing the maximum number of ordinary double points and study them. This way we will gain some fundamental understanding of ordinary double points on algebraic surfaces and see various examples of them.

2 Singularities

2.1 Properties

As was already mentioned, we will be mainly interested in ordinary double points on algebraic surfaces in complex three-space. Let us first define properly what is meant by this.

Definition 1. *An algebraic surface of degree d in $\mathbb{C}^3$ is the set of solutions of an equation $f(x, y, z) = 0$ in which $f \in \mathbb{C}[x, y, z]$ is an irreducible polynomial of degree d.*

These surfaces can have sharp peaks, known as singularities.

Definition 2. *A singularity of a surface in $\mathbb{C}^3$ is a point $z_0 \in \mathbb{C}^3$ such that $f(z_0) = f_x(z_0) = f_y(z_0) = f_z(z_0) = 0$.*

Ordinary double points (also called nodes) form a specific type of singularity, defined as follows. [3]

Definition 3. *An ordinary double point of a surface in $\mathbb{C}^3$ is a point $z_0 \in \mathbb{C}^3$ that is a singularity with the additional property that $det(Hf(z_0)) \neq 0$, i.e. $Hf(z_0)$ is of rank 3. Here Hf represents the Hessian of f, given by:*

$$Hf(x, y, z) = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial x \partial z} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial y \partial z} \\ \frac{\partial^2 f}{\partial z \partial x} & \frac{\partial^2 f}{\partial z \partial y} & \frac{\partial^2 f}{\partial z^2} \end{bmatrix}$$

Now what can we say about singular points? Suppose we take an arbitrary surface in complex three-space: $S = \{f(x, y, z) = 0\}$ and a singular point $p = (a, b, c)$ on it. Since p is singular, f and all its first partial derivatives vanish in (a, b, c). Now let us define local coordinates $(\xi, \eta, \zeta) = (x - a, y - b, z - c)$; it follows that $(x, y, z) = (\xi + a, \eta + b, \zeta + c)$. In these coordinates $(0, 0, 0)$ is singular and the Taylor expansion of f in the point $(\xi, \eta, \zeta) = (0, 0, 0)$ will be of the form:

$$\begin{aligned} f(x, y, z) = f(\xi + a, \eta + b, \zeta + c) &\approx f(p) + \alpha\xi + \beta\eta + \gamma\zeta + \text{higher order terms} \\ &= \alpha\xi + \beta\eta + \gamma\zeta + \text{higher order terms} \end{aligned}$$

where

$$\begin{aligned} \alpha &= \frac{\partial f}{\partial x}(p) \\ \beta &= \frac{\partial f}{\partial y}(p) \\ \gamma &= \frac{\partial f}{\partial z}(p). \end{aligned}$$

But since p is a singular point, we see that all partial derivatives of f vanish and thus that the Taylor expansion in p starts quadratically. We will be seeing this many times during the rest of this thesis.

The last concept I would like to introduce is that of the complex *projective* three-space $\mathbb{CP}^3$, which can be thought of as all lines in $\mathbb{C}^4$ through $(x, y, z, w) = (0, 0, 0, 0)$. A point of $\mathbb{CP}^3$ is defined by four homogeneous complex coordinates that are not all equal to zero; these coordinates are thus considered up to a scalar multiple.[15] A point (x, y, z) in $\mathbb{C}^3$ corresponds to a point $(x : y : z : 1)$ in $\mathbb{CP}^3$. We will often consider surfaces in complex projective space instead of just complex space because of simplicity and clarity.

2.2 Lower and upper bounds

We will now use the definition of singularities to create singular points on surfaces of a given degree. If we take an arbitrary surface of degree d in complex projective three-space, then it will be given by a polynomial

$$f(x, y, z, w) = a_0 + a_1x + a_2y + a_3z + a_4w + a_5x^2 + a_6y^2 + a_7z^2 + a_8w^2 + ... = 0.$$

The number of monomials $x^k y^l z^m w^n$ with degree $d = k + l + m + n$ is equal to the number of multicombinations of d elements chosen among $n = 4$ variables, given by the binomial coefficient $\binom{d+n-1}{n-1} = \binom{d+(4-1)}{4-1} = \binom{d+3}{3}$. [2] Therefore the total amount of possible combinations of coefficients, and thus the total amount of possible surfaces, is given by:

$$\binom{d+3}{3} = \frac{(d+1)(d+2)(d+3)}{6} = \frac{1}{6}d^3 + d^2 + \frac{11}{6}d + 1.$$

Thus a surface of degree d has

$$N := (\frac{1}{6}d^3 + d^2 + \frac{11}{6}d + 1) - 1 = \frac{1}{6}d^3 + d^2 + \frac{11}{6}d$$

degrees of freedom, which is in fact the dimension of the vector space spanned by the coefficients. We distract one from the total amount of possible surfaces, because the surface only depends on the ratio between the coefficients; multiplying the given equation by a constant does not change the surface.

If we want an arbitrary point p to be a singularity on the surface, we get four linear conditions on the coefficients, namely:

$$\begin{aligned} f(p) &= 0 \\ \frac{\partial f}{\partial x}(p) &= 0 \\ \frac{\partial f}{\partial y}(p) &= 0 \\ \frac{\partial f}{\partial z}(p) &= 0. \end{aligned}$$

It follows that if we want to have n singularities on a surface, we get $4n$ linear conditions on the coefficients of the surface. If we take our n singularities to be (p_i, q_i, r_i, s_i), we get the following system that we want to solve:

$$\begin{pmatrix} 1 & p_1 & q_1 & r_1 & s_1 & p_1^2 & q_1^2 & r_1^2 & s_1^2 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 2p_1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & 0 & 2q_1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2r_1 & 0 & \cdots \\ 1 & p_2 & q_2 & r_2 & s_2 & p_2^2 & q_2^2 & r_2^2 & s_2^2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2r_n & 0 & \cdots \end{pmatrix} \cdot \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ \vdots \\ a_N \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

where we do not want the trivial solution $(a_0, \ldots, a_N)^T = (0, \ldots, 0)^T$. Now we can use a formula from linear algebra called the rank-nullity theorem which states that the rank of a matrix and the nullity (dimension of the kernel) of a matrix add up to the number of columns of the matrix (see for example *Matrix Analysis and Applied Linear Algebra* [12]). In our case the number of columns of the matrix (let us call it A) is equal to $N + 1$ and we know that the rank of A is at most $4n$, so we can deduce the following formula for the dimension of the kernel of A:

$$dim(ker(A)) = N + 1 - rank(A) \geq N + 1 - 4n.$$

Now we want a nontrivial solution, thus we want the dimension of the kernel to be at least one. Combining this with the previous result, we get:

$$n \leq \frac{N}{4}$$

where n is the number of singularities we have imposed on the surface and $N = \frac{1}{6}d^3 + d^2 + \frac{11}{6}d$. We have to be careful before drawing any conclusions from this calculation. We want the $a_i' s$ to be such that they define an equation that is irreducible and it is not so easy to see when this is the case. What we *can* conclude from this argument, is that if $a_0, \ldots, a_N$ define an irreducible f, then we can certainly find n singularities where $n \leq \frac{N}{4}$. We may expect that the maximum number of ordinary double points on a surface is $\geq \frac{N}{4}$, depending on whether the equation is irreducible. The following table lists the values of $\frac{N}{4}$ for the lower degree surfaces.

d	N	$\frac{N}{4}$
1	3	$\frac{3}{4}$
2	9	$2\frac{1}{4}$
3	19	$4\frac{3}{4}$
4	34	$8\frac{1}{2}$
5	55	$13\frac{3}{4}$

We see that our expectation is not valid for surfaces of degree two, but for other degrees the maximum number is indeed $\geq \frac{N}{4}$. Still this is not the best bound we can find, so we will now try a different approach to find more accurate ones.

Let us take $2n$ lines in the xy-plane such that every triple l, m, n satisfies $l \cap m \cap n = \emptyset$ and every pair l, m intersects. Suppose two lines intersect in the point $(x, y) = (a, b)$, or $(\xi, \eta) = (0, 0)$ in local coordinates. These lines are of the form $0 = \eta - \lambda_i \xi$, where $i = 1, 2...$ and $\lambda_i \neq \lambda_j$ for all i and j. Then the product of these equations is a polynomial of degree $2n$ with no constant and linear part in ξ and η. Now if we set the product of these equations equal to z^2 (which is quadratic in z), we get the generally irreducible polynomial $\prod_{i=1}^{2n}(a_i x + b_i - y) = z^2$ of which the point $(\xi, \eta, 0) = (0, 0, 0)$ is singular. We can prove that this point is an ordinary double point as well. We can write our equation, call it f, as follows:

$$f = \prod_{i=1}^{2n}(a_i x + b_i - y) - z^2 = x \cdot y \cdot (\ldots) - z^2 = 0$$

where $(\ldots)$ is some term in x and y. For the Hessian matrices we need to calculate all second partial derivatives. So let us begin with the first partial derivatives:

$$\begin{aligned}
\frac{\partial f}{\partial x} &= x \cdot (y \cdot (\ldots))_x + y \cdot (\ldots) = x \cdot y \cdot (\ldots)_x + y \cdot (\ldots) \\
\frac{\partial f}{\partial y} &= y \cdot (x \cdot (\ldots))_y + x \cdot (\ldots) = y \cdot x \cdot (\ldots)_y + x \cdot (\ldots) \\
\frac{\partial f}{\partial z} &= -2z.
\end{aligned}$$

For the second partial derivatives we get:

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} &= x \cdot (y \cdot (\ldots)_x)_x + 2y \cdot (\ldots)_x = x \cdot y \cdot (\ldots)_{xx} + 2y \cdot (\ldots)_x \\
\frac{\partial^2 f}{\partial y^2} &= y \cdot (x \cdot (\ldots)_y)_y + 2x \cdot (\ldots)_y = y \cdot x \cdot (\ldots)_{yy} + 2x \cdot (\ldots)_y \\
\frac{\partial^2 f}{\partial z^2} &= -2.
\end{aligned}$$

Filling in our singular point $(0, 0, 0)$, we see that the first two partial derivatives vanish. It is also obvious that the mixed second partial derivatives of x and z and of y and z vanish as well. Now we only need to calculate the mixed second partial derivative $\frac{\partial^2 f}{\partial x \partial y}$, which is of course equal to $\frac{\partial^2 f}{\partial y \partial x}$.

$$\begin{aligned}
\frac{\partial^2 f}{\partial x \partial y} &= x \cdot (\ldots)_x + y \cdot (x \cdot (\ldots)_x)_y + (\ldots) + y \cdot (\ldots)_y \\
&= x \cdot (\ldots)_x + y \cdot (\ldots)_{xy} + (\ldots) + y \cdot (\ldots)_y
\end{aligned}$$

Filling in our singular point $(0,0,0)$ again, we see that $\frac{\partial^2 f}{\partial x \partial y} = (\ldots)$, which is a term in x and y that is not zero in $(0,0,0)$. The resulting Hessian matrix is of the following form:

$$Hf(0,0,0) = \begin{bmatrix} 0 & \neq 0 & 0 \\ \neq 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

which is obviously of rank three. Therefore the singular point we found in this way is an ordinary double point. Since we can always make the coordinate change so that $(0,0,0)$ is our singular point, we know that all singularities we find in this manner are ordinary double points.

In general if we take $2n$ lines of the form $y = a_i x + b_i$ such that they have $\binom{2n}{2}$ different intersections, then a surface of degree $2n$ can be given by:

$$z^2(z-1)^2 \cdots (z-n+1)^2 = \prod_{i=1}^{2n}(a_i x + b_i - y).$$

The number of singularities we can find is equal to the number of intersections of the lines on the right side of the equation times the number of terms on the left side. Reasoning as before, we know that all singularities we find in this manner are ordinary double points. This surface therefore has $\binom{2n}{2} \cdot n = \frac{1}{2} \cdot 2n(2n-1) \cdot n = 2n^3 - n^2$ ordinary double points, which results in the following table.

$d = 2n$	n	$2n^3 - n^2$
2	1	1
4	2	12
6	3	45
8	4	112
10	5	225

Repeating the same argument for $2n+1$ lines with $\binom{2n+1}{2}$ different intersections, we get surfaces of degree $2n+1$ with $\binom{2n+1}{2} \cdot n = \frac{1}{2} \cdot (2n+1)(2n) \cdot n = 2n^3 + n^2$ ordinary double points, which results in the following table.

$d = 2n+1$	n	$2n^3 + n^2$
1	0	0
3	1	3
5	2	20
7	3	63
9	4	144

What we actually just did was create surfaces of degree $2n$ with $2n^3 - n^2$ ordinary double points and of degree $2n+1$ with $2n^3 + n^2$ ordinary double points, which gives a lower bound for the possible number of those singularities on the given surfaces. We know that the actual lower bound is higher, but still this is a lower bound that goes up very fast and is pretty accurate for a second attempt.

Now let us have a look at upper bounds. There have been found many upper bounds that come very close to the actual maximum number (at least for lower degree surfaces) of ordinary double points. We will discuss one that was found in the 19th century and it has to do with the so-called dual of an algebraic surface.

Given the projective space $\mathbb{P}^n = \{(a_0 : \cdots : a_n)\}$, we can consider any linear hypersurface in it: $X_{b_0:\cdots:b_n} = \{(x_0 : \cdots : x_n) \in \mathbb{P}^n \mid b_0x_0 + \ldots + b_nx_n = 0\}$, which is defined uniquely by the values of $b_0, \ldots, b_n$. The dual of the projective space is defined to be all of these linear hypersurfaces together, which defines itself a projective space: $\mathbb{P}^n = \{(b_0 : \cdots : b_n)\}$. We denote this dual of the projective space as $\check{\mathbb{P}}^n$. Now consider any hypersurface $X_f = \{f = 0\} \subset \mathbb{P}^n$. Its dual is given by $\check{X}_f = \{X_b \in \check{\mathbb{P}}^n | X_b \subset \mathbb{P}^n$ is a tangent space of $X_f\}$. Generally for a given hypersurface $X_f \subset \mathbb{P}^n$ with f a homogeneous equation and $p = (a_0 : \cdots : a_n) \in X_f$ a smooth point, the tangent space of X_f in p is given by $\sum_{i=0}^{n} \frac{\partial f}{\partial x_i}(p) \cdot x_i = 0$. This explains why the dual is conventionally defined as follows.

Definition 4. *The dual of a hypersurface $X_f = \{f = 0\}$ is the closure of the image of the nonsingular points of X_f under the map $x = (x_0, \ldots, x_n) \mapsto (\frac{\partial f}{\partial x_0}(x), \ldots, \frac{\partial f}{\partial x_n}(x))$.*

In case X_f is not a cone (it does not consist of lines passing through a fixed point), it is known that the dual is also a hypersurface. Therefore it has a certain degree, which is generally called the class of the original surface.

We are going to use the class of a surface to create an upper bound for the maximum possible number of ordinary double points on it. This is based on the reasoning in *Classical Algebraic Geometry* [4]; we will sketch the idea here, but the full proofs of some of the statements can be found there. The degree d of the dual of a *nonsingular* hypersurface X is found in the following way. For a point $b = (b_0, \ldots, b_n) \in \mathbb{P}^n$, its polar is given by $P_b(X) = \sum b_i \frac{\partial f}{\partial x_i}$. We fix $n-1$ general points $a_1, \ldots, a_{n-1}$ in $\mathbb{P}^n$ and the polars $P_{a_i}(X)$ through them. The set of hyperplanes through a general set of $n-1$ points is a line in the dual space. [4] Since geometrically, the degree of X is equal to the number of its intersections with a general line in the projective space, the degree of its *dual* is likewise equal to the number of intersections of $\check{X}$ with a general line in the *dual* projective space. Therefore we get that

$$deg(\check{X}) = \#X \cap P_{a_1}(X) \cap \cdots \cap P_{a_{n-1}}(X) = d(d-1)^{n-1},$$

since the polars are made up of first partial derivatives and are therefore of degree one less than the original surface. This formula holds for nonsingular hypersurfaces X, but we are interested in *singular* hypersurfaces, or more specifically, in hypersurfaces with ordinary double points, and to calculate their degree we need to adapt the formula slightly. This is because a singular point lowers the degree of the dual. [8] Geometrically, an ordinary double point on

X corresponds to a tangent space that is tangent to two points on $\check{X}$ simultaneously. [14] Since the degree of $\check{X}$ is found by considering the number of tangent spaces to X passing through a generic point in the projective space and each ordinary double point is passed through with multiplicity two, we get the following formula for the degree of the dual d^* of singular hypersurfaces:

$$d^* = deg(\check{X}) = d(d-1)^{n-1} - 2\delta$$

where δ stands for the number of ordinary double points on X. Now we will use the following relationship between the class $d*$ and the degree d of X:

$$d^* \geq d - 1. [10]$$

Combining this with the formula for d^* and taking into account that we are working in complex three-space, so that $n = 3$, we get that:

$$\delta \leq \frac{1}{2}[d(d-1)^2 - (d-1)]$$

which is an upper bound for the number of ordinary double points on a surface X in complex three-space, given in the following table.

d	$\frac{1}{2}[d(d-1)^2 - (d-1)]$
3	5
4	$16\frac{1}{2}$
5	38
6	72
7	123

This upper bound holds for surfaces of degree ≥ 3, since the general definition of the dual surface does not hold for quadrics. We see that the upper bound is accurate for low degree surfaces; the actual upper bound for surfaces of degree four is indeed sixteen. Now that we have found both a lower and an upper bound for the maximum possible number of ordinary double points on surfaces of a given degree, we will turn to the specific surfaces containing that many nodes.

3 Lower degree surfaces

The maximum number of ordinary double points on surfaces of degree ≤ 2 is easy to compute. We will start with the trivial case, that is, where the surface is of degree one. The case where the degree of the surface is two is already not trivial anymore, but the maximum number of ordinary double points on these surfaces is still easily calculated.

3.1 Planes

A surface of degree one is called a plane in $\mathbb{C}^3$. It can be easily seen that planes cannot have any singularities and thus in particular cannot have any ordinary double points.

A plane in $\mathbb{C}^3$ can be given by the following equation:

$$f(x, y, z) = ax + by + cz + d = 0$$

where a,b,c,$d \in \mathbb{C}$. A singularity of such a plane is a point for which $f = f_x = f_y = f_z = 0$. In this case we find that:

$$\begin{aligned} f_x &= a \\ f_y &= b \\ f_z &= c. \end{aligned}$$

For all these partial derivatives to be zero, we get that $a = b = c = 0$. But for f to vanish as well, we get that d must equal zero too. The resulting equation of the plane is $0 = 0$, but this does not restrict x, y, z in any way and thus does not describe a plane. Therefore a plane cannot have any singularities and in particular cannot have any ordinary double points.

3.2 Quadrics

A surface of degree two is called a quadric and it is generally of the following form:

$$f(x, y, z) = ax^2 + by^2 + cz^2 + dxy + fxz + gyz + hx + jy + kz + l = 0$$

where a,b,c,d,f,g,h,j,k,$l \in \mathbb{C}$. To find the maximum number of ordinary double points on these surfaces we need a different method than in the case of a plane. Let us first consider the question geometrically. The following claim will then be of great help:

Theorem 1. *If a point p is a singularity of a quadric S in $\mathbb{C}^3$, then every line through p intersects S at least doubly in p.*

Proof. A line through p can be given by the equation:

$$y = p + t \cdot \vec{r}$$

where t is a parameter and $\vec{r}$ a direction vector. Then $f(p + t \cdot \vec{r}) = 0$ gives us all intersections of S and the line y. The first-order Taylor expansion of this equation in the point p will be:

$$f(p + t \cdot \vec{r}) \approx f(p) + t \cdot \langle grad(f(p)), \vec{r} \rangle$$

But since $f(p) = 0$ and $grad(f(p)) = 0$, we see that the full Taylor expansion in p starts quadratically. Therefore every line through a singularity of S intersects the quadric at least doubly in that point. □

Now take another point of the surface q, then l_{pq} (the line through p and q) is given by $y = p + t \cdot (q - p)$. Then $f(p + t \cdot (q - p))$ is an element of $\mathbb{C}[t]$ with degree ≤ 2 and $t = 0$ a double zero because of Theorem 1. Then $f(p+t\cdot(q-p))$ must be of the form $\alpha \cdot t^2$ with $\alpha \in \mathbb{C}$. But $t = 1$ must be a (single) zero as well, since q is a point of S and therefore $\alpha = 0$ and $f(p + t \cdot (q - p))$ is identically zero. Therefore $l_{pq} \subset S$. So we see that $p \in S \subset \mathbb{C}^3$ (with S a quadric and p a singular point) implies that $l_{pq} \subset S$ for a point $q \in S$ with $p \neq q$. The quadric thus consists of lines through p which results in a cone. We will see later on that p is the only ordinary double point that this surface can have and thus that any quadric can have at most one ordinary double point.

The fact that a quadric can only have one ordinary double point does not imply that it cannot have more singularities. A quadric can in fact have infinitely many singularities. Take for example the quadric S given by $f = xy = 0$. Points on the quadric are of the form $\{(x, y, z)|xy = 0\} = \{(0, \cdot, \cdot)\} \cup \{(\cdot, 0, \cdot)\}$. Then the singularities are exactly $\{(0, 0, \cdot)\}$, because in $(0, 0, \cdot)$ we have that $f = f_x = f_y = f_z = 0$. But the thing is that these singularities form the line of intersection of two planes and are therefore not isolated. Therefore they cannot be ordinary double points.

We can prove that a quadric can have at most one ordinary double point analytically by using the general classification of quadrics (see for example [11]).

Theorem 2. *Every quadric can be moved into the form of a cone, an ellipsoid or a hyperbolic paraboloid and their projective equivalents through a rotation and/or a translation. These forms are given by the following equations respectively:*

1. $ax^2 + by^2 + cz^2 = 0$
2. $ax^2 + by^2 + cz^2 - 1 = 0$
3. $ax^2 + by^2 - z = 0$

where a, b, c are constants.

We can now consider all possible singularities in these three cases.

Case 1.

$$f_x = 2ax$$
$$f_y = 2by$$
$$f_z = 2cz.$$

For $a, b, c \neq 0$, we get that $x = y = z = 0$, which is indeed a point of the quadric. If $det(Hf(0,0,0)) = 2a \cdot 2b \cdot 2c \neq 0$, this is indeed an ordinary double point. Otherwise we may assume (after permuting x,y,z if necessary) that $a = 0$, and then $by^2 + cz^2 = (\sqrt{b} \cdot y + \sqrt{-c} \cdot z) \cdot (\sqrt{b} \cdot y - \sqrt{-c} \cdot z)$ is reducible.

Case 2.

$$f_x = 2ax$$
$$f_y = 2by$$
$$f_z = 2cz.$$

For $a, b, c \neq 0$, we get again that $x = y = z = 0$, but in this case this point is not on the quadric, since $0{+}0{+}0{-}1 \neq 0$. Thus no singularities are found in this case.

Case 3.

$$f_x = 2ax$$
$$f_y = 2by$$
$$f_z = -1$$

Since $-1 \neq 0$, no singularities can be found in this case either.

We can conclude that the only singularity that a quadric can have is the point $(0,0,0)$ in the case where the quadric looks like $ax^2 + by^2 + cz^2 = 0$. As we have seen before, the resulting quadric is a cone, the classic form of which is for $a, b > 0$ and $c < 0$. Now we have also shown that the one singularity of this cone is really an ordinary double point. The cone is plotted in the following figure, where we thus have taken $a, b > 0$ and $c < 0$.

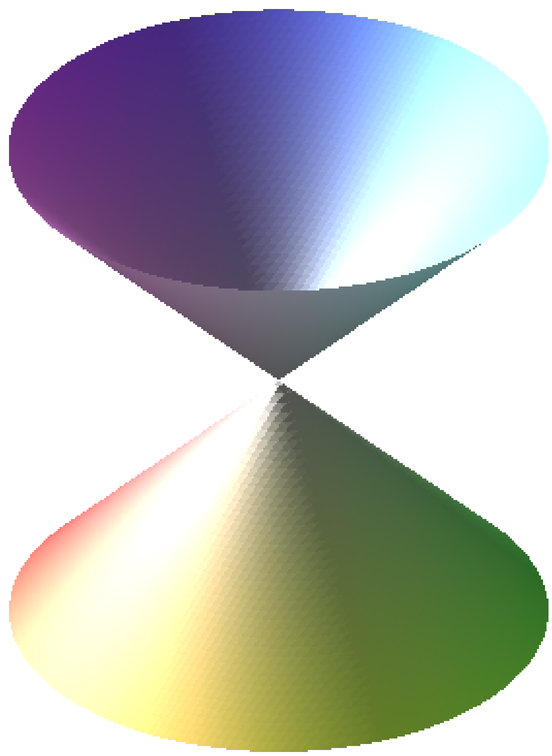

Figure 1: Cone with one ordinary double point.

4 Cubics and quartics

If the surface is of degree higher than two, finding the maximum number of ordinary double points is not so easy anymore. For degree three and four this number is known though and we will investigate two of the most famous surfaces in history: Cayley's cubic and Kummer's quartic.

4.1 Cubics

In the 19th century a lot of research was done on cubic surfaces, and specifically on singular cubic surfaces. Ludwig Schläfli (1814-1895) made a classification of singular cubics by their singularities in 1863 and in 1869 Arthur Cayley (1821-1895) elaborated on this. It is appraised as one of the most important contributions to the study of cubic surfaces to date. The unique surface of degree three having four ordinary double points is named after Cayley.

First I would like to prove that a cubic cannot have any more singular points than four. The proof is based on the proof given in *Theory and History of Geometric Models* [13] and it is actually a construction of the unique cubic surface with four ordinary double points.

Let $S = \{f(x,y,z,w) = 0\}$ be a cubic in $\mathbb{CP}^3$ defined by an irreducible equation f with finitely many singularities. Let us suppose that the number of singularities on S is at least four. If we take four of the singularities and suppose they are all contained in one plane, then we know that $V \cap S$ is a curve of degree three in $\mathbb{P}^2$, and p_1,p_2,p_3,p_4 are singular points on this curve. Reasoning as in subsection 3.2, we know that the line through any combination of two singular points is contained in the curve. This implies that all four singular points must be on the same line, since otherwise our cubic would contain more than three lines. So what we have is a line contained in a cubic surface with four singular points on it. By possibly applying a linear transformation we may assume that the line is such that $x = y = 0$ with singular points $(0,0,0)$, $(0,0,1)$ and two more of them. The cubic surface can be given by the following equation in $\mathbb{CP}^3$:

$$f = \alpha w^3 + w^2 \cdot F_1(x,y,z) + w \cdot F_2(x,y,z) + F_3(x,y,z) = 0$$

which reduces to

$$f = \alpha + F_1(x,y,z) + F_2(x,y,z) + F_3(x,y,z) = 0$$

for $w = 1$. Here $F_i(x,y,z)$ is an expression in x,y,z of degree i. If we fill in $x = y = 0$, we get that

$$\alpha + F_1(z) + F_2(z) + F_3(z) = 0$$

which implies that all terms of this equation individually must be zero. What remains for the equation of our cubic surface is of the form:

$$f = \beta x + \gamma y + \delta x^2 + \epsilon xy + \kappa y^2 + \lambda xz + \mu yz + G_3(x,y) + z \cdot G_2(x,y) + z^2 \cdot G_1(x,y)$$

where again $G_i(x, y)$ is an expression in x, y of degree i. One condition for this equation is that $(0, 0, 0)$ is singular, so we get that $\beta = 0$ and $\gamma = 0$. Likewise, for the three other singular points we get restrictions on the values of our coefficients. Since all three partial derivatives must equal zero we get many restrictions and we actually get so many that the value of z will not matter anymore, so that we get a whole line of singularities. Since we assumed that our cubic has finitely many of them, we conclude that four singularities cannot be contained in one plane.

An alternative way to see why we cannot have three singularities on one line ℓ runs as follows. Consider any plane containing ℓ. The intersection of the cubic surface with the plane contains the line ℓ and a curve given by a quadratic equation. Since there are more than two singularities contained in ℓ, the only possibility is that the equation of ℓ divides the quadratic one. As a result, every point of ℓ is singular on the cubic surface.

Since this is the case, we can apply a linear transformation such that the singular points are $(1 : 0 : 0 : 0)$, $(0 : 1 : 0 : 0)$, $(0 : 0 : 1 : 0)$ and $(0 : 0 : 0 : 1)$. Let us have a look at what it means for the equation of our surface that $(1 : 0 : 0 : 0)$ is singular. We can write it in the following manner:

$$f = \upsilon x^3 + x^2 \cdot F_1(y, z, w) + x \cdot F_2(y, z, w) + F_3(y, z, w) = 0.$$

Filling in $(1 : 0 : 0 : 0)$, we see that $\upsilon = 0$. But since also all partial derivatives must vanish in $(1 : 0 : 0 : 0)$, we see that the linear part in y, z, w vanishes as well. We are left with the equation $x \cdot F_2(y, z, w) + F_3(y, z, w) = 0$ and therefore we conclude that if $(1 : 0 : 0 : 0)$ is singular, then f has degree ≤ 1 in x. But the same holds for y, z, w for the singular points $(0 : 1 : 0 : 0)$, $(0 : 0 : 1 : 0)$ and $(0 : 0 : 0 : 1)$. The equation we are left with is of the form:

$$f = c_1 yzw + c_2 xzw + c_3 xyw + c_4 xyz$$

where $c_1, ..., c_4$ are nonzero constants: if any of them were zero, then f would be reducible. Now if we apply the transformation where

$$\begin{aligned} x &\mapsto c_1 x \\ y &\mapsto c_2 y \\ z &\mapsto c_3 z \\ w &\mapsto c_4 w \end{aligned}$$

and we divide by $c_1 ... c_4$, we get the standard equation of Cayley's cubic:

$$f = wxy + xyz + yzw + zwx = 0$$

where $w = 1 - x - y - z$. We can verify by hand that this surface has indeed four ordinary double points and this shows that a cubic cannot have more singularities, and thus cannot have more ordinary double points than these.

The partial derivatives of f are given by:

$$\begin{aligned} f_x &= wy + w_x xy + yz + w_x yz + zw + w_x zx \\ &= wy - xy + yz - yz + zw - zx \\ &= wy - xy + zw - zx = 0 \end{aligned}$$

$$
\begin{aligned}
f_y &= wx + w_y xy + xz + zw + w_y yz + w_y zx \\
&= wx - xy + xz + zw - yz - zx \\
&= wx - xy + zw - yz = 0
\end{aligned}
$$

$$
\begin{aligned}
f_z &= w_z xy + xy + yw + w_z yz + wx + w_z zx \\
&= -xy + xy + yw - yz + wx - zx \\
&= yw - yz + wx - zx = 0.
\end{aligned}
$$

We see that both f and all partial derivatives vanish in the points where three of the four coordinates equal zero and these are indeed the four singularities of Cayley's cubic surface that we have seen before: $(1 : 0 : 0 : 0)$, $(0 : 1 : 0 : 0)$, $(0 : 0 : 1 : 0)$ and $(0 : 0 : 0 : 1)$. Now to verify if these singularities are ordinary double points we need to compute the Hessian of f, where we use that $w = 1 - x - y - z$:

$$
Hf(x, y, z) = \begin{bmatrix} -2y - 2z & 1 - 2x - 2y - 2z & 1 - 2x - 2y - 2z \\ 1 - 2x - 2y - 2z & -2x - 2z & 1 - 2x - 2y - 2z \\ 1 - 2x - 2y - 2z & 1 - 2x - 2y - 2z & -2x - 2y \end{bmatrix}
$$

The singular points we found give us the following Hessian matrices:

$$
Hf(0,0,0) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad Hf(1,0,0) = \begin{bmatrix} 0 & -1 & -1 \\ -1 & -2 & -1 \\ -1 & -1 & -2 \end{bmatrix}
$$

$$
Hf(0,1,0) = \begin{bmatrix} -2 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & -2 \end{bmatrix} \quad Hf(0,0,1) = \begin{bmatrix} -2 & -1 & -1 \\ -1 & -2 & -1 \\ -1 & -1 & 0 \end{bmatrix}
$$

We see that all these matrices are of rank three. Since singular points with Hessian matrices of full rank are ordinary double points, we see that Cayley's cubic indeed contains four of these and they are all real points of the surface. We have thus shown the existence of cubic surfaces with the maximum number of ordinary double points. Cayley's cubic is plotted in the figure below.

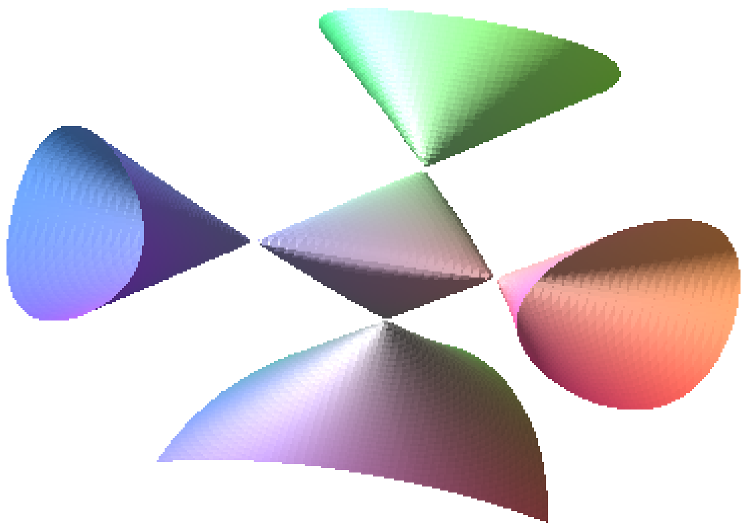

Figure 2: Cayley's cubic with four ordinary double points.

4.2 Quartics

One year after Schläfli wrote his classification of singular cubic surfaces, Ernst Kummer (1810-1893) found and published the most important results around quartic surfaces to date. Though a lot of research into the maximum number of ordinary double points on quartic surfaces had already been done, Kummer was the first one to note that there actually existed a surface with this many nodes: Fresnel's wave surface, that was found by Augustin-Jean Fresnel in 1821. It is given by the following equation in projective space:

$$\frac{a^2x^2}{x^2+y^2+z^2-a^2w^2}+\frac{b^2y^2}{x^2+y^2+z^2-b^2w^2}+\frac{c^2z^2}{x^2+y^2+z^2-c^2w^2}=0$$

for suitable $a, b, c \in \mathbb{C}$. [8] Of course we can set w equal to 1 for simplicity. It is not that clear from the equation that this surface is of degree four, but in the appendix a Maple worksheet is given that show that the numerator of the full equation is equal to the product of $(x^2+y^2+z^2)$ and another term. This other term is the quartic surface we consider here. It is given by the equation

$$\begin{aligned}a^2b^2c^2 - a^2b^2x^2 - a^2b^2y^2 - a^2c^2x^2 - a^2c^2z^2 + a^2x^4 + a^2x^2y^2 + a^2x^2z^2 \\ - b^2c^2y^2 - b^2c^2z^2 + b^2x^2y^2 + b^2y^4 + b^2y^2z^2 + c^2x^2z^2 + c^2y^2z^2 + c^2z^4 = 0.\end{aligned}$$

Now it *is* clear that the surface is indeed a quartic. In the following figure Fresnel's wave surface is plotted with values $a = 1$, $b = 0,3$ and $c = 0,5$.

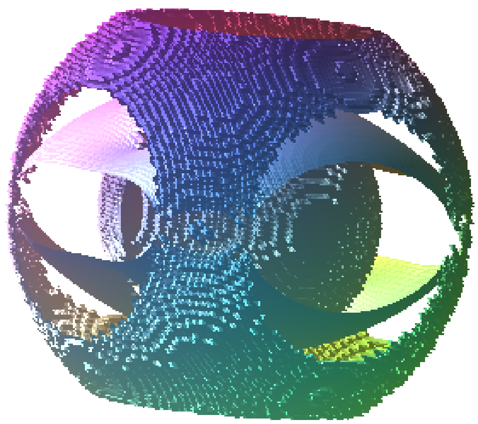

Figure 3: Fresnel's wave surface with sixteen ordinary double points.

After this discovery, Kummer wanted to find all quartic surfaces with the maximum number of ordinary double points and the result was a family of quartics, all containing sixteen of these points.

Kummer's quartic surfaces can be given by the following equation. An extended discussion of this surface is given in *Kummer's Quartic Surface* [7].

$$f = (x^2 + y^2 + z^2 - \mu^2 w^2)^2 - \lambda pqrs = 0$$

where $\mu \in \mathbb{R}$ and

$$\begin{aligned}
\lambda &= \frac{3\mu^2 - 1}{3 - \mu^2} \\
p &= w - z - \sqrt{2}x \\
q &= w - z + \sqrt{2}x \\
r &= w + z + \sqrt{2}y \\
s &= w + z - \sqrt{2}y.
\end{aligned}$$

We see that we need $\mu^2 \neq 3$ and typically also the cases $\mu^2 = \frac{1}{3}$, which gives $\lambda = 0$, and $\mu^2 = 1$, which gives $\lambda = 1$, are excluded from the general analysis of Kummer's quartic. We often let $w = 1$ for simplicity. It is known that the Kummer surfaces have sixteen ordinary double points and these can be real or complex, depending on the value of μ^2.

μ^2	real singularities	complex singularities
$0 \leq \mu^2 < \frac{1}{3}$	4	12
$\frac{1}{3} < \mu^2 < 1$	4	12
$1 < \mu^2 < 3$	16	0
$\mu^2 > 3$	16	0

Let us take a closer look at the equation to see what happens at a singular point. In a point $(x, y, z) = (\alpha, \beta, \gamma)$ where two of the equations p, q, r and s equal zero, say for example $q = 0$ and $s = 0$, these equations must be of the following form:

$$\begin{aligned}
q &= (-z + \gamma) + \sqrt{2} \cdot (x - \alpha) \\
s &= (z - \gamma) - \sqrt{2} \cdot (y - \beta).
\end{aligned}$$

Then we see that the second term of the surface equation $\lambda pqrs$ consists of quadratic terms of the form ξ^2, $\xi\eta$, $\xi\zeta$ et cetera, in terms of the local coordinates $(\xi, \eta, \zeta) = (x - a, y - b, z - c)$. For the point to be singular we need that also the first term of the surface equation consists of terms that are at least quadratic and we see that we manage that by setting $\alpha^2 + \beta^2 + \gamma^2 - \mu^2$ equal to zero. We then get the sphere $\alpha^2 + \beta^2 + \gamma^2 = \mu^2$. This implies that we can find singular points by finding the intersections of the sphere $x^2 + y^2 + z^2 = \mu^2$ and the lines that we get by letting q and s (or any other two of the equations) equal zero. In this manner we may find twelve singular points of the surface already.

Let us try this method by letting $w = 1$ and choose $\mu^2 = 2$. We know that there should be sixteen real singular points on this surface. If we take q and s to be zero, we see that $z = 1 + \sqrt{2}x$ and $z = -1 + \sqrt{2}y$, which gives us the line $y = \sqrt{2} + x$. Now filling in $z = 1 + \sqrt{2}x$ and $y = \sqrt{2} + x$ in $x^2 + y^2 + z^2 = 2$ and solving for x, we get $x = -\frac{1}{2}\sqrt{2} \pm \frac{1}{2}$ and it follows that $y = -\frac{1}{2}\sqrt{2} \mp \frac{1}{2}$ and $z = \pm\frac{1}{2}\sqrt{2}$. Thus we have found two singularities. We can find two singularities for every line we intersect with the sphere, but the singularities found by letting $q = r = 0$ are equal to the ones found by letting $q = s = 0$ and the same goes for $p = r = 0$ and $p = s = 0$. Thus we find eight singularities in this way and they are indeed eight of the sixteen we can find in total:

$$
\begin{aligned}
&(0,1,1) && (0,-1,1)\\
&(1,0,-1) && (-1,0,-1)\\
&(0,\frac{1}{\sqrt{2}},\frac{1}{2}) && (0,-\frac{1}{\sqrt{2}},\frac{1}{2})\\
&(\frac{1}{\sqrt{2}},0,-\frac{1}{2}) && (-\frac{1}{\sqrt{2}},0,-\frac{1}{2})\\
&(\frac{1}{2}-\frac{1}{2}\sqrt{2},\frac{1}{2}+\frac{1}{2}\sqrt{2},\frac{1}{2}\sqrt{2}) && (\frac{1}{2}-\frac{1}{2}\sqrt{2},-\frac{1}{2}-\frac{1}{2}\sqrt{2},\frac{1}{2}\sqrt{2})\\
&(\frac{1}{2}+\frac{1}{2}\sqrt{2},\frac{1}{2}-\frac{1}{2}\sqrt{2},-\frac{1}{2}\sqrt{2}) && (\frac{1}{2}+\frac{1}{2}\sqrt{2},-\frac{1}{2}+\frac{1}{2}\sqrt{2},-\frac{1}{2}\sqrt{2})\\
&(-\frac{1}{2}-\frac{1}{2}\sqrt{2},\frac{1}{2}-\frac{1}{2}\sqrt{2},-\frac{1}{2}\sqrt{2}) && (-\frac{1}{2}-\frac{1}{2}\sqrt{2},-\frac{1}{2}+\frac{1}{2}\sqrt{2},-\frac{1}{2}\sqrt{2})\\
&(-\frac{1}{2}+\frac{1}{2}\sqrt{2},\frac{1}{2}+\frac{1}{2}\sqrt{2},\frac{1}{2}\sqrt{2}) && (-\frac{1}{2}+\frac{1}{2}\sqrt{2},-\frac{1}{2}-\frac{1}{2}\sqrt{2},\frac{1}{2}\sqrt{2}).
\end{aligned}
$$

I have listed the specific points like this, because they make it clear that there is a lot of symmetry going on in Kummer's quartic. This is because of the way in which p,q,r and s are defined. Later on we will see that higher degree surfaces with many ordinary double points also have a certain symmetry, so that if you find one ordinary double point, you immediately find many of them. Of course this makes it easier to find them and as in Kummer's case it makes that we can find many of the ordinary double points by hand.

Let us verify that the singularities we find are indeed ordinary double points. In order to do this, we need to calculate the Hessian of the equation in the singular points that we found. We will verify that the rank of the Hessian in the singular point $(0,1,1)$ has indeed full rank; the Hessians in the other points will have the same properties.

$$Hf(0,1,1) = \begin{bmatrix} 40 & 0 & 0 \\ 0 & 8 & 8 \\ 0 & 8 & -12 \end{bmatrix}$$

Since this matrix is of rank three, the point $(0,1,1)$ is indeed an ordinary double point. The other singularities will turn out to be ordinary double points as well, which proves that there exists indeed a surface of degree four with sixteen ordinary double points. Kummer's quartic is plotted in the figure below.

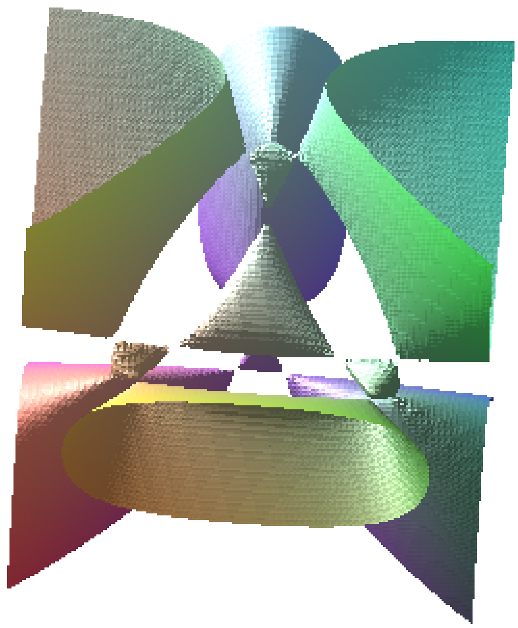

Figure 4: Kummer's quartic with sixteen ordinary double points.

5 Higher degree surfaces

For surfaces of degree higher than four it becomes very difficult to determine the maximum possible number of ordinary double points on them. For degree five and six this number is known though, they are 31 and 65 respectively. But for higher degree there is only a lower and an upper bound known yet. In this chapter I will therefore not try to prove why these surfaces cannot have more ordinary double points, but for degree five, six and seven I will show the surfaces with this maximum (known) number and see what we can say about them.

5.1 Quintics

In 1937 Eugenio Giuseppe Togliatti (1890-1977) found a surface of degree five with 31 ordinary double points, many years after Kummer constructed his quartic surfaces. At that time the known upper bound for surfaces of degree five was 34 ordinary double points, so nobody knew that Togliatti's surfaces contained the maximum possible number. Finally in 1980 the French mathematician Arnaud Beauville (1947) was able to show that a quintic cannot have more than 31 singularities. Wolf Barth (1942) has made a construction of a quintic with this many ordinary double points, that I would like to discuss here. For an extended discussion, see *World Record Surfaces* [9].

We first define the following parameters and polynomials:

$$\begin{aligned}
a &= -\frac{5}{32} \\
b &= -\frac{(5-\sqrt{5})}{20} \\
d &= -1-\sqrt{5} \\
P &= \frac{1}{16}\cdot(x^5-10x^3y^2+5xy^4-5x^4-10x^2y^2-5y^4+20x^2+20y^2-16) \\
Q &= bz^2+x^2+y^2+d+z.
\end{aligned}$$

Then Barth's quintic is given by:

$$\begin{aligned}
f &= P - Q^2\cdot az \\
&= \frac{5}{32}((-\frac{1}{4}+\frac{1}{20}\sqrt{5})z^2+x^2+y^2-1-\sqrt{5}+z))^2z+\frac{1}{16}x^5 \\
&\qquad -\frac{5}{8}x^3y^2+\frac{5}{16}xy^4-\frac{5}{16}x^4-\frac{5}{8}x^2y^2-\frac{5}{16}y^4+\frac{5}{4}x^2+\frac{5}{4}y^2-1.
\end{aligned}$$

Barth's construction uses pentagonal symmetry and a value of d that is closely related to the golden ratio $\frac{1}{2}+\frac{1}{2}\sqrt{5}$. I am not going into this deeper, but the use of (pentagonal) symmetry is clear from just looking at the following figure of Barth's quintic.

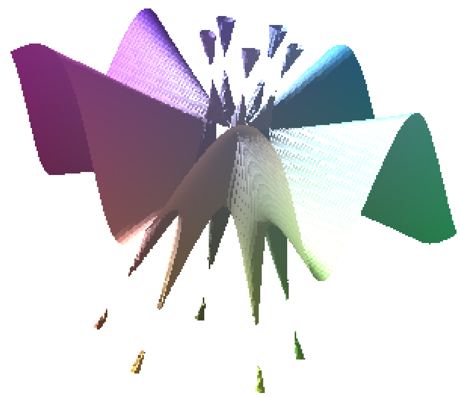

Figure 5: Barth's quintic with 31 ordinary double points.

We can let Maple find the singular points of this surface for us; in the appendix the worksheet is given that confirms that there are indeed 31 singularities on this surface. Also the ranks of the Hessian matrices in the singular points are calculated: they are all equal to three, so all singularities are ordinary double points. A first look at the singular points gives the impression that all of them are in $\mathbb{Q}$, the extension $\mathbb{Q}(\sqrt{5})$ and a fourth-degree extension of the form $\mathbb{Q}(\sqrt{\alpha + \beta\sqrt{5}})$ for some integers α and β, which is again related to the golden ratio.

5.2 Sextics

Barth is also the one after whom a very famous sextic is called, one with the maximum possible number of ordinary points for surfaces of degree six, namely 65. He found this in 1996 and shortly after his construction a couple of mathematicians were able to prove that there were no more ordinary double points possible on a sextic. [9]

We first define the following parameters:

$$
\begin{aligned}
K &= x^2 + y^2 + z^2 - 1 \\
t &= \frac{1}{2}(1+\sqrt{5}) \\
a &= \frac{1}{4}(2t+1) \\
P &= (t^2 \cdot x^2 - y^2) \cdot (t^2 \cdot y^2 - z^2) \cdot (t^2 \cdot z^2 - x^2).
\end{aligned}
$$

Then Barth's sextic is given by:

$$
\begin{aligned}
f &= -K^2 \cdot a + P \\
&= -(x^2+y^2+z^2-1)^2 \cdot (\frac{1}{2} + \frac{1}{4}\sqrt{5}) + ((\frac{1}{2} + \frac{1}{2}\sqrt{5})^2 x^2 - y^2) \\
&\qquad \cdot ((\frac{1}{2} + \frac{1}{2}\sqrt{(5)})^2 y^2 - z^2) \cdot ((\frac{1}{2} + \frac{1}{2}\sqrt{5})^2 z^2 - x^2).
\end{aligned}
$$

We see that again the golden ratio plays an important part in the equation: t is exactly equal to it. Where Barth's quintic surface was constructed by using the symmetry of a pentagon, the construction of this sextic is based on the symmetry of an icosahedron, a polyhedron with 20 faces. [9] The appendix also contains the Maple worksheet that finds the singular points of Barth's sextic, but it can only find 50 of them, since the other 15 are singular points at infinity. We see again that all Hessian matrices in the singular points are of rank three. The surface is plotted in the following figure.

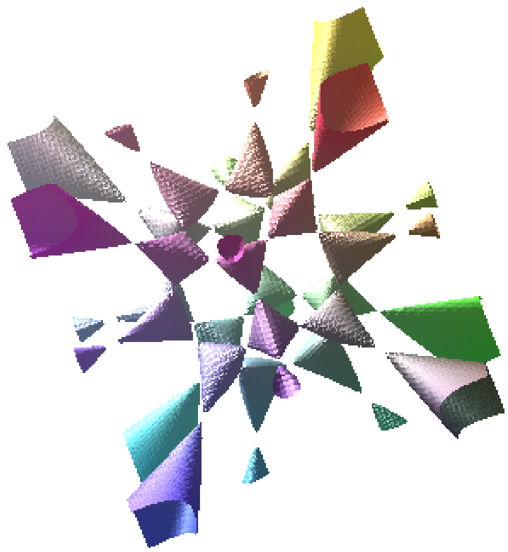

Figure 6: Barth's sextic with 65 ordinary double points.

5.3 One last example

Since the maximum possible number of ordinary double points on a surface is not yet known for degree higher than six, many mathematicians are still searching for new constructions of these surfaces with many of these points and thus new examples may still be found nowadays. The last example that I would like to mention in this thesis is one found by Oliver Labs in 2004. It is a septic (a surface of degree seven) with 99 real ordinary double points and up to now no one has been able to find a septic with more of them, while the known upper bound is 104. Labs constructed this septic while writing his PhD dissertation *Hypersurfaces with Many Singularities* [8]; a thorough description of his construction can be found there.

We first define a to be the only real solution of $7x^3 + 7x + 1 = 0$. Furthermore, we define:

$$
\begin{aligned}
w &= 1\\
a_1 &= -\frac{12}{7}a^2 - \frac{384}{49}a - \frac{8}{7}\\
a_2 &= -\frac{32}{7}a^2 + \frac{24}{49}a - 4\\
a_3 &= -4a^2 + \frac{24}{49}a - 4\\
a_4 &= -\frac{8}{7}a^2 + \frac{8}{49}a - \frac{8}{7}\\
a_5 &= 49a^2 - 7a + 50\\
U &= (a_5 w + z) \cdot ((z+w)(x^2+y^2) + a_1 z^3 + a_2 z^2 w + a_3 z w^2 + a_4 w^3)^2\\
P &= x(x^6 - (3\cdot 7)x^4y^2 + (5\cdot 7)x^2y^4 - 7y^6) + 7z((x^2+y^2)^3\\
&\qquad\qquad - 2^3 z^2 (x^2+y^2)\cdot(x^2+y^2) + 2^4 z^4 (x^2+y^2)) - 2^6 z^7.
\end{aligned}
$$

The actual surface is given by:

$$f = P - U$$

and it is plotted in the following figure.

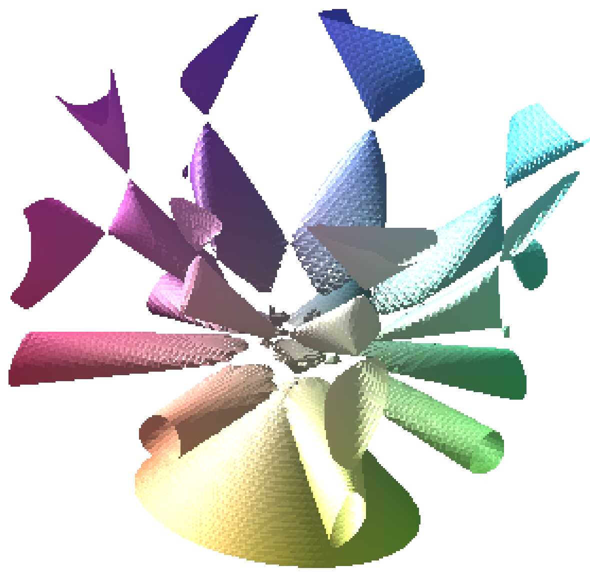

Figure 7: Labs's sextic with 99 real ordinary double points.

What we see in many surfaces is that the construction makes use of the symmetry of a certain polyhedron. In general, this makes it easier to find many of the singularities: if you find one, you immediately find many of them. This symmetric aspect is also apparent from the solutions, that often only differ from one another in sign.

Of course there are many other constructions of higher degree surfaces with many ordinary double points that I am not going to discuss in this thesis. Many optimal examples come from the 1990s and it seems to be very difficult to construct surfaces with even more ordinary double points. But Labs's construction shows that this is not impossible and that examples are still found in present times.

6 Conclusion

At the beginning of this thesis, I wrote that I wanted to elaborate on the fundamentals of ordinary double points on algebraic surfaces, on the maximum possible number of this type of singularity on surfaces of a given degree and discuss various examples of surfaces containing many ordinary double points. In the first chapter I have examined the basic properties of ordinary double points and how we find simple upper and lower bounds for their maximum possible number. After that we saw the surfaces of degree two through six with the maximum possible number of ordinary double points. For the lowest degree surfaces we were able to prove why this is the maximum number, but from degree five on this becomes very difficult, and too involved for this bachelor's thesis. We did verify that the examples presented in the literature indeed have the asserted number of ordinary double points.

We saw that the equations of the quintic, sextic and septic we considered are very long. Since for surfaces of degree higher than six the maximum possible number of ordinary double points is not yet known, it will be very interesting to see how these numbers *are* established. The example by Labs dates from 2004, so that is not that long ago. It proves that nowadays people are still involved in this topic and are still trying to find optimal examples of surfaces with many ordinary double points. Labs himself is one of the people who is mainly involved in this subject, but he is not the only one. All in all, it is a very interesting topic to keep following: who knows what surfaces with many ordinary double points people will come up with in the future.

References

[1] Arnold, V.I. et al. *Singularity Theory I*. Springer Science and Business Media, New York, 1998.

[2] Bertram, A. *Polynomials*. University of Utah, Salt Lake City, 2014. Available at: *http://www.math.utah.edu/ bertram/4800/PolyIntroduction.pdf*.

[3] Brugallé, E. and O. Labs. *Surfaces With Many Solitary Points*. Geometriae Dedicata, 141 (2009): 123-135.

[4] Dolgachev, I.V. *Classical Algebraic Geometry: a Modern View*. Cambridge University Press, Cambridge, 2011.

[5] Fulton, W. *Intersection Theory*. Springer-Verlag, Berlin-Heidelberg, 1984.

[6] Hartshorne, R. *Algebraic Geometry*. Springer-Verlag, New York, 1977.

[7] Hudson, R.W.H.T. *Kummer's Quartic Surface*. Cambridge University Press, Cambridge, 1990.

[8] Labs, O. *Hypersurfaces with Many Singularities*. PhD dissertation, Johannes Gutenberg University, Mainz, 2005.

[9] Labs, O. *World Record Surfaces*. IMAGINARY platform, license CC BY-NC-SA, 2014. Available at: *http://imaginary.org/background-material/world-record-surfaces*.

[10] Lanteri, A. and C. Turrini. *Projective Surfaces with Class Less Than or Equal to Twice the Degree*. Mathematische Nachrichten, 175 (1995): 199-207.

[11] Laval, P.B. *Vectors and the Geometry of Space*. Department of Mathematics, Kennesaw State University, 2009. Available at: *http://math.kennesaw.edu/ plaval/math2203/func3d.pdf*.

[12] Meyer, C.D. *Matrix Analysis and Applied Linear Algebra*. Society for Industrial and Applied Mathematics, Philadelphia, 2001.

[13] Polo-Blanco, I. *Theory and History of Geometric Models*. PhD dissertation, Rijksuniversiteit Groningen, Groningen, 2007.

[14] Tevelev, E. *Projective Duality and Homogeneous Spaces*. Springer-Verlag, Berlin-Heidelberg, 2005.

[15] Tevelev, E. *Projectively Dual Varieties*. Journal of Mathematical Sciences, 117 (6) (2003): 4585-4732.

Appendices

A Maple worksheet Fresnel's wave surface

$>$ $w := 1:$
$>$ $f := a^2 * x^2/(-a^2 * w^2 + x^2 + y^2 + z^2) + b^2 * y^2/(-b^2 * w^2 + x^2 + y^2 + z^2)$
$>$ $+c^2 * z^2/(-c^2 * w^2 + x^2 + y^2 + z^2):$
$>$ $numer(f):$
$>$ $factor(\%);$
$>$ $-(x^2+y^2+z^2)(a^2b^2c^2 - a^2b^2x^2 - a^2b^2y^2 - a^2c^2x^2 - a^2c^2z^2 + a^2x^4 + a^2x^2y^2$
$>$ $+a^2x^2z^2 - b^2c^2y^2 - b^2c^2z^2 + b^2x^2y^2 + b^2y^4 + b^2y^2z^2 + c^2x^2z^2 + c^2y^2z^2 + c^2z^4)$

B Maple worksheet Barth's quintic surface

$>$ $a := -5/32:$
$>$ $b := -(5 - sqrt(5)) * (1/20):$
$>$ $d := -1 - sqrt(5):$
$>$ $P := 1/16 * (x^5 - 10 * x^3 * y^2 + 5 * x * y^4 - 5 * x^4 - 10 * x^2 * y^2 - 5 * y^4 +$
$>$ $20 * x^2 + 20 * y^2 - 16):$
$>$ $Q := b * z^2 + x^2 + y^2 + d + z:$
$>$ $f := -Q^2 * a * z + P:$

$>$ $sols := solve(\{f, diff(f, x), diff(f, y), diff(f, z)\}, \{x, y, z\}, explicit):$
$>$ $list := [seq([evalf(E[1]), evalf(E[2]), evalf(E[3])], E \text{ in } sols)]:$
$>$ $S := \{\}:$
$>$ $\text{for } E \text{ in } list \text{ do } S := union(S, \{E\}) \text{ end } do:$
$>$ $nops(S);$
$>$ $\qquad 31$

$>$ $with(VectorCalculus):$
$>$ $with(LinearAlgebra):$
$>$ $A := \{\}:$
$>$ $\text{for } E \text{ in } S \text{ do } A := union(A, \{Rank(Hessian(f, [x, y, z], E))\}) \text{ end } do:$
$>$ $A;$
$>$ $\qquad \{3\}$

C Maple worksheet Barth's sextic surface

$> \quad K := x^2 + y^2 + z^2 - 1 :$
$> \quad t := 1/2 * (1 + sqrt(5)) :$
$> \quad a := 1/4 * (2 * t + 1) :$
$> \quad P := (t^2 * x^2 - y^2) * (t^2 * y^2 - z^2) * (t^2 * z^2 - x^2) :$
$> \quad f := -K^2 * a + P :$

$> \quad sols := solve(\{f, diff(f, x), diff(f, y), diff(f, z)\}, \{x, y, z\}, explicit) :$
$> \quad list := [seq([evalf(E[1]), evalf(E[2]), evalf(E[3])], E \,\mathrm{in}\, sols)] :$
$> \quad S := \{\} :$
$> \quad \mathrm{for}\, E \,\mathrm{in}\, list \,\mathrm{do}\, S := union(S, \{E\}) \,\mathrm{end}\, do :$
$> \quad nops(S);$
$> \qquad\qquad 50$

$> \quad with(VectorCalculus) :$
$> \quad with(LinearAlgebra) :$
$> \quad A := \{\} :$
$> \quad \mathrm{for}\, E \,\mathrm{in}\, S \,\mathrm{do}\, A := union(A, \{Rank(Hessian(f, [x, y, z], E))\}) \,\mathrm{end}\, do :$
$> \quad A;$
$> \qquad\qquad \{3\}$