Faculty of Mathematics and Natural Sciences

Bachelor Thesis

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Modified Newtonian Dynamics (MOND) and a Possible Microscopic Description

Author:
L.M. Mooiweer

Supervisor:
prof. dr. E. Pallante

Abstract

Nowadays, the mass discrepancy in the universe is often interpreted within the paradigm of Cold Dark Matter (CDM) while other possibilities are not excluded. The main idea of this thesis is to develop a better theoretical understanding of the hidden mass problem within the paradigm of Modified Newtonian Dynamics (MOND). Several phenomenological aspects of MOND will be discussed and we will consider a possible microscopic description based on quantum statistics on the holographic screen which can reproduce the MOND phenomenology.

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Contents

1 Introduction

Today, many physicist are trying to explain the hidden mass problem within the framework of Cold Dark Matter (CDM). Where theorist are developing new frameworks and extending their models to postulate dark matter candidates, experimentalist are actively searching for new particles. In the first part of this thesis we will introduce you to the paradigm of Modified Newtonian Dynamics (MOND) as an alternative for the hidden mass problem and discuss whether it can be a suitable framework for interpreting the hidden mass problem without postulating dark matter. Since MOND is often criticized of being an ad hoc theory its perspective would increase if it had a foundation based on physical laws. Therefore we will, in the second part of this thesis, consider recent ideas that can produce MOND and re-derive MOND via a holographic scenario. Before we start we will shortly review the beginning of hidden mass problem and try to see why CDM became the paradigm.

1.1 The Problem of the Hidden Mass

Fritz Zwicky is considered to be the first person who came up with the idea of dark matter. In his paper of 1937 he calculated, by using the virial theorem of classical mechanics, the average dynamical mass of the nebulae of the Coma cluster. He found that the lowest estimate of the mass was $4.5 \times 10^{10}\,\mathrm{M}_{\odot}$ [1], where $\mathrm{M}_{\odot}$ is the solar mass. The average luminosity of a nebula was obtained to be $8.5 \times 10^{7}\,\mathrm{L}_{\odot}$, where $\mathrm{L}_{\odot}$ is the solar luminosity. He calculated a mass-to-light ratio $\gamma = 500$ and compared this with local value obtained for the solar neighborhood ($\gamma_{\mathrm{local}} = 3$). This dynamical mass was much larger than one would expect from the observed light. Therefore he concluded that on an intergalactic scale there must be a much denser dark matter component. Later it became clear that Zwicky overestimated the mass-to-light ratio because he did not take into account the relative abundant amount of inter-cluster hot gas. Nevertheless, present day calculations still show that additional mass is needed in order to explain the observed dynamics.

A couple of years later, Horace Babcock measured the rotation curve (i.e. the radial velocity plotted against the radius) of the Andromeda galaxy (M31) with the noticeable absence of a Keplerian decline [2]. The Keplerian decline is the characteristic velocity decline of $1/\sqrt{r}$ which follows from Newtonian dynamics. In order to explain this, he proposed that the mass-to-light ratio must increase with the radius or a modification of Newtonian dynamics was required at the galactic scales. In 1954, Martin Schwarzschild reanalyzed the rotation curves of the Andromeda galaxy by using more recent data and another perspective. He assumed that there was a disk like mass distribution which completely follows the light distribution [3]. In other words he, contrary to Babcock, assumed that the mass-to-light ratio is constant with radius. By using Newtonian dynamics he obtained the shape of the velocity curve which he could nicely fit within the observational errors by using the mass-to-light ratio as a free parameter [2].

Since the development of radio astronomy it became possible to measure distances and velocities by using the 21-cm line of the hyperfine transition of hydrogen. This method was used to measure the rotation curves of, among others, the Andromeda galaxy with more precision and beyond the optical image [4]. They found that the rotation curve did not show the Keplerian $1/r^{1/2}$ decline but a slower $1/r^{1/5}$ decline.

All told, at that time there were observation based mass anomalies which did not have a clear interpretation and explanation. Aside the fact that people were trying to explain these anomalies, they were not considered to be a major problem. People thought that by improvement of the measurements and a development of the theoretical side these anomalies could be resolved in the upcoming years.

With the advent of supercomputer, theoretical astronomers were able to test their theories by running N-body simulations. In early simulations they simulated the evolution of a disk distributed set of particles in gravitational equilibrium. The initial conditions were that the centrifugal force equals the gravitational attraction (i.e. a rotationally supported system). Surprisingly, these systems where not stable. The galaxies evolved rapidly into barred shaped unstable configurations which, at longer time scales, where evolving in pressure supported systems. In such systems the gravitational force is not balanced by the centrifugal force but by the random motion of the particles. This was a very surprising result because the spiral galaxies which we observe seem to be stable. In order to explain this, Jerry Ostriker used classical theory of fluid spheroids to argue that the simulated systems where indeed unstable. By splitting the kinetic term of the virial theorem into an orbiting velocity term and a random velocity term and comparing this with stability conditions of classical fluid spheroids, one could indeed argue that the N-body simulated galaxies are instable [5]. This must be a problem because we know that for instance our own galaxy, the Milky Way, is a stable system. In order to obtain stability, a spheroidal mass component must be added to the system that makes a sufficient contribution to the gravitational potential energy. Therefore a dark matter spheroidal component with a much larger mass-to-light ratio was postulated. This was the beginning of the dark matter halo.

Besides this theoretical argument for adding of a dark component to the mass distribution, the observational need was also growing. Since the development of radio interferometers, large distance galaxies could be observed with more precision. It was found that, besides the absence of the Keplarian decline, the rotation velocity remained constant beyond the optical mass distribution. Surprisingly, this problem could be fixed in the same way the stability problems of spiral galaxies was fixed, namely by postulating a dark matter halo.

Nowadays it is widely accepted that the Navarro-Frenk-White (NFW) potential gives a universal shape (i.e. there is no need of free parameters to obtain the correct shape) of the dark matter halo [6]. The profile is obtained by running N-body simulations and compare this to cosmological structure. The NFW potential can explain the flat rotation curves when the dark matter potential dominates the gravity due to luminous matter.

In 1965, Arno Penzias and Robert Wilson accidentally discovered the existence of the cosmic microwave background (CMB) [7], a radiation emerging from the photon decoupling (i.e. the recombinations of baryons and electrons) in an early universe. Before decoupling the early universe was radiation dominated, meaning that the gravitational force was not sufficient to overcome the radiational pressure. Therefore, the formation of large scale structure due to gravitational collapse of baryonic density fluctuations could only start after decoupling. In order to explain the observed structure, the fractional density variation must be of the order of $\delta\rho/\rho \approx 10^{-4}$ while the CMB observations show a variation of $\delta\rho/\rho \approx 10^{-5}$ [2]. So in order to explain the large scale structure that we observe one needs higher density fluctuations or more time. Besides photons as a remnant of the early universe it has been stretched out that non-baryonic particles that have a weaker coupling with photons (i.e. dark matter candidates) decouple in an earlier epoch. This can explain the large scale structure that we see today because these particles can start with structure formation in an earlier epoch.

2 Modified Newtonian Dynamics

In order to explain the hidden mass problem, Milgrom proposed a modification of the dynamics. In his original paper [8] he proposed an acceleration based modification by introducing the constant a_0. This constant has the dimensions of an acceleration and sets a boundary between the old and the new dynamics. He postulated that in the low acceleration deep MOND limit (i.e. when $a \ll a_0$) one obtains Modified Newtonian Dynamics (MOND). He proposed that in this regime the dynamics become space-time scale invariant [9] meaning that the equations of motion are invariant under the transformation $(t, r) \to (\lambda t, \lambda r)$. On the other hand, at high accelerations (i.e. when $a \gg a_0$) the old dynamics must be restored. The introduction of this new constant is very similar to, for example, the introduction of the Planck constant $\hbar$ in quantum mechanics. Like in the case of quantum mechanics, where one can take the limit $\hbar \to 0$ to restore classical dynamics, one can take the limit $a_0 \to 0$ to restore Newtonian dynamics. The transition between Newtonian dynamics and MOND is established by an interpolating function $\tilde{\mu}(x)$. The exact form of this function is not determined by the theory, but is an arbitrary function with the required asymptotic behaviour. One can select the most preferable interpolating function based on observations. The most commonly used functions are the so called standard interpolation function $\tilde{\mu}(x) = x/\sqrt{x^2+1}$ and the simple interpolating function $\tilde{\mu}(x) = x/(x+1)$. Because of a modification of acceleration the second law of Newton gets

$$\frac{F}{m} = a\tilde{\mu}(a/a_0). \tag{2.1}$$

The question arises how we should interpret the MOND theory. Equation (2.1) can be seen as a modification of Newton's second law or as a modification of the gravitational potential. In the first case, MOND is a modification of inertia where all forces, not only the gravitational force, that cause an acceleration $a \ll a_0$ show non-linear behaviour. The non-linear behaviour is showed in figure 1. On the other hand MOND theory could be seen as a modification of the gravitational potential leaving the inertia law intact. In this case equation (2.1) becomes the less rigorous proposition

$$\frac{F}{m} = g\tilde{\mu}(g/a_0), \tag{2.2}$$

where g is the conventional Newtonian gravitational acceleration. When we only look at gravitational effects it is not necessary to make a distinction between the two interpretations. This only plays a role at a more fundamental level.

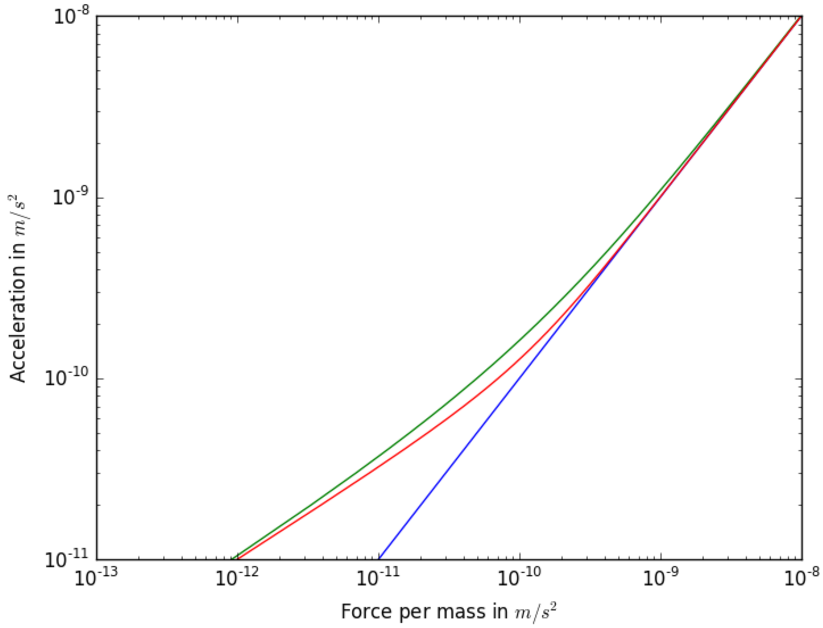

Figure 1: The difference between MOND and Newtonian dynamics. The blue line demonstrates (linear) Newtonian dynamics while the other lines make a transition around $a_0 = 10^{-10}\mathrm{m\,s^{-2}}$. For the red line the simple transition function $\tilde{\mu}(x) = x/(x+1)$ is used while for the green line the standard interpolation function $\tilde{\mu}(x) = x/\sqrt{x^2+1}$ is used.

2.1 The Acceleration Constant a_0

One of the axioms of MOND theory is that there must be a universal constant a_0 which sets the boundary between Newtonian dynamics and MOND. The value of the constant $a_0 \approx (1.2 \pm 0.2) \times 10^{-8}\mathrm{cm\,s^{-2}}$ [9]. Besides the fact that the constant sets a boundary it will show up in disparate phenomena. It has been found that there is a connection between a_0 and cosmological significant quantities

$$a_0 \approx \frac{cH_0}{2\pi} \approx \frac{c^2}{2\pi}(\Lambda/3)^{1/2}, \tag{2.3}$$

where H_0 is the Hubble constant, Λ is the cosmological constant and c the speed of light. This could be an indication that the MOND theory could be derived from underlying principles.

The deep MOND limit must be space-time scale invariant meaning that the dynamics must be invariant under the scale transformation. We can investigate how the physical constants of a gravitational theory behave in the deep MOND limit by applying the transformation $(t, r) \to (\lambda t, \lambda r)$ and let

$\lambda \to \infty$. By dimensional reasons we see that the constants transform in following way: $a_0 \to \lambda^{-1} a_0$, $c \to c$ and $G \to \lambda G$. Therefore, within the deep MOND limit only c and the combination $Ga_0 \equiv \mathcal{A}_0$ can exist.

2.2 MOND Phenomenology

Starting with equation (2.1), it is relatively easy to make certain predictions. If we look at the deep MOND limit, the region where $a \ll a_0$ and the dynamics is modified, we can calculate the new gravitational acceleration. If we take the modified law of Newton with $\tilde{\mu}(a/a_0) = a/a_0$ and set it equal to gravitational force we obtain

$$a^2 = Ga_0 \frac{M}{r^2}. \tag{2.4}$$

By using the centrifugal acceleration $a = v^2/r$, we obtain a velocity

$$v^4 = MGa_0, \tag{2.5}$$

that is fully determined by the enclosed mass M, the gravitational constant G and the acceleration constant a_0. Thus, at low accelerations MOND predicts an asymptotically flat rotation curve that converges to the value $(MGa_0)^{1/4}$. We clearly see the combination of G and a_0 indicating that this value is scale invariant, as it should be. This is a major success of MOND if we compare it with the CDM paradigm where the flat rotation curves can only be reproduced after constructing the dark matter halo. Although the parameters of the NFW halo are deduced from the current cosmological model (ΛCDM), it cannot successfully reproduce all observed rotation curves [10]. Even though the asymptotic flat rotation curves appear naturally within MOND, the theory is also not able to reproduce all observations [11].

Because we used the centrifugal acceleration, equation (2.5) is only valid for rotational supported systems. Such a relationship can also be derived for pressure supported systems where the gravitational force is balanced by the pressure due to the random motion of the particles [12]. The relation is given by

$$\sigma^2 = \frac{2}{3}(MGa_0)^{1/2}, \tag{2.6}$$

where σ is the mean velocity dispersion.

2.2.1 The Tully-Fischer and Jackson-Faber relation

The original Tully-Fischer law describes an observational based relationship between the luminosity (L) of spiral galaxies and the band width (ΔV) of a particular emission line [13]. The form of the relation is given by $L \propto \Delta V^{\gamma}$, where the power γ is determined by observations. By using red shift, the spectral emission lines can be used to observe rotation velocities of galaxies. Therefore, the Tully-Fischer law relates the luminosity to the velocity by $L \propto v^{\alpha}$. Because in MOND the velocity for rotational supported systems is given by $M \propto v^4$ and we assume a constant mass-to-light ratio we conclude that $L \propto v^4$. The question now is, which α do we observe? If we measure the near infrared (H-band) luminosity of spiral galaxies we see indeed that this is related to the velocity by $L \propto v^4$ [14]. Observations in other regimes leads to α values between 3 and 4 [15]. Yet, the Tully-Fischer relationship breaks down for low mass systems which might be expected because in the lower mass regime the number of gas-rich galaxies is higher. If one adds the mass of the gas to the stellar mass, one obtains the Baryonic Tully-Fischer Relation (BTFR) shown in figure 2 (left) where we see that the observed BTFR is predicted by MOND.

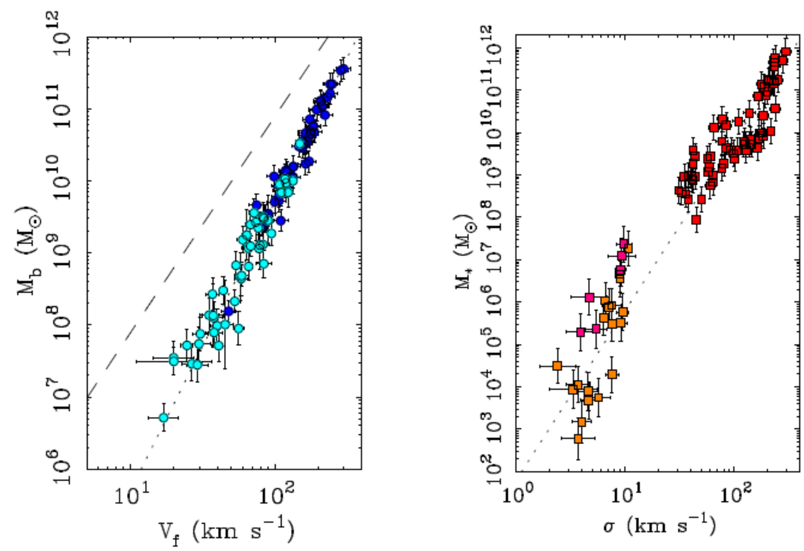

Figure 2: The left picture [16] shows the BTFR of galaxies where the mass is given by total baryonic mass (i.e. the sum of the stellar and gas mass). The dotted line indicates the MOND predicted with slope $\alpha = 4$ while the dashed line of slope $\alpha = 3$ is what one would expect from CDM paradigm. The right picture [16] shows the Jackson-Faber relations where the dotted line of slope $\alpha = 4$ is the MOND predicted relation between the mass and the velocity dispersion.

While the Tully-Fischer relationship gives an observational relation for rotationally supported systems there is also a relation for pressure supported systems, the Jackson-Faber relationship [16]. The Jackson-Faber relation observes a luminosity velocity relation that is predicted by MOND, see figure 2 (right).

2.2.2 The external field effect

Assuming Newtonian dynamics as valid, the internal dynamics of a subsystem is completely determined by its internal gravitational fields and is independent of the external fields assuming that the external fields are uniform across the system. Because MOND is a non-linear in the acceleration, one has to measure the absolute acceleration due to internal and external fields and compare the value with the acceleration constant a_0 to determine the dynamics. This is in contradiction with Einstein's equivalence principle [17] in which he stated that an uniform accelerating frame of reference can be considered to be at rest, if the presence of a gravitational field that cause the same (opposite) acceleration is assumed. In the framework of MOND this is not the case because one has to determine the dynamics by considering the interpolating function

$$\tilde{\mu}\left(\frac{|\vec{a_i}+\vec{a_e}|}{a_0}\right) = \begin{cases} \text{Newtonian limit} & \text{if } |\vec{a_i}+\vec{a_e}| \gg a_0 \\ \text{Deep MOND limit} & \text{if } |\vec{a_i}+\vec{a_e}| \ll a_0 \end{cases}$$

where a_i is the acceleration due to internal fields and a_e the acceleration due to external fields. The different limits of $\tilde{\mu}$ are determined in the same way as before but now by comparing the absolute acceleration with the acceleration constant a_0. Another aspect of the equivalence principle is that the inertial mass and gravitational mass are equivalent. In the case where MOND is a modification of gravity this is no longer the case.

Another question that arises is how one should defines and measures absolute accelerations. This is a fundamental problem that cannot be answers by General Relativity. The answer to this question has to come from a underlying theory of MOND.

2.3 The Non-Relativistic Field Formulation

2.3.1 Conservation of energy

The original formulation of MOND as a modification of gravity has problems with the general conservation laws. Because the inertia of a body is not linear in the acceleration (figure 1), the modification of Milgrom does not conserve momentum and energy. We can show this by considering the amount of work required to accelerate a particle of mass m to a certain velocity v. In order to have conservation of energy we must obtain that the work required to accelerate a particle to a velocity v equals the kinetic energy associated with that velocity

$$E_k = \int \vec{F}\,\mathrm{d}\vec{s} = \frac{1}{2}mv^2. \tag{2.7}$$

In order to check if this is the case, let us consider two forces F_1 and F_2 acting both on a particle of mass m. Let us say that both forces act in the same direction, F_1 is a force that causes an acceleration $a_1/a_0 \ll 1$ and F_2 causes an acceleration $a_2/a_0 \gg 1$. In order to obtain the same kinetic energy we state that

$$F_1 \Delta x_1 = F_2 \Delta x_2, \tag{2.8}$$

where Δx_1 and Δx_2 are chosen such that the particles end up with the same velocity. Because $a_1 \ll a_0$, the particle of the left hand side is traveling through the deep MOND regime while the particle of right hand side is though the Newtonian regime. If we use equation (2.1) we obtain that

$$m\frac{a_1^2}{a_0}\Delta x_1 = m a_2 \Delta x_2. \tag{2.9}$$

Because the particles have the same velocity we can identify that $a_1 \Delta x_1 = a_2 \Delta x_2$ and obtain

$$\frac{a_1}{a_0} = 1. \tag{2.10}$$

Since this is in contradiction with the fact that $a_1/a_0 \ll 1$ we conclude that (2.1), the MOND equation of Milgrom, in general does not conserve energy.

2.3.2 A quadratic Lagrangian formalism (AQUAL)

In order to overcome this problem, MOND can be reformulated in terms of a (non-relativistic) Lagrangian theory [18]. Before we show the field formulation of MOND, let us review the Lagrangian formulation of Newtonian dynamics. The usual Newtonian gravitational potential can be derived from the Lagrangian density

$$\mathcal{L} = -\rho\phi_N - \frac{(\vec{\nabla}\phi_N)^2}{8\pi G}, \tag{2.11}$$

where ϕ_N is the Newtonian gravitational field and ρ the mass density. In order to obtain the equations of motion for ϕ_N we solve the Euler-Lagrange equation for classical fields

$$\partial_i \left(\frac{\partial \mathcal{L}}{\partial(\partial_i \phi)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0. \tag{2.12}$$

Applying this equation to the Lagrangian density of equation (2.11) we immediately obtain the Poisson equation for gravity

$$\vec{\nabla}^2 \phi_N = 4\pi G\rho. \tag{2.13}$$

If we consider the fact that gravity is a conservative force (i.e. $-\vec{\nabla}\phi_N = \vec{g}$) one obtains the Newtonian acceleration. In order to obtain a Lagrangian formulation of MOND something completely analog to the modification of Milgrom to the second law of Newton is done, but now the Lagrangian density of Newtonian gravity is modified as follows.

$$\mathcal{L} = -\rho\phi_N - \frac{a_0^2}{(8\pi G)} \mathcal{F} \left(\frac{|\vec{\nabla}\phi_N|^2}{a_0^2} \right) \tag{2.14}$$

Because the Lagrangian is quadratic in $|\nabla\phi_N|$, this formulation of MOND is known as 'a quadratic Lagrangian' (AQUAL). In this formulation the function $\mathcal{F}$ is just an arbitrary function which defines the MOND interpolating function. In order to obtain an interpolating function that produces the same transition Milgrom proposed, the function has to obey $\mathrm{d}\mathcal{F}(x^2)/\,\mathrm{d}x^2 = \tilde{\mu}(x)$. In order to obtain the right asymptotic behaviour another constraint is that the function must have the limits $\mathcal{F}(x^2) \to x^2$ for $x \gg 1$ and $\mathcal{F}(x^2) \to \frac{2}{3}x^3$ for $x \ll 1$.

Using the least action principle (i.e. applying equation (2.12)) one obtains the equation of motion

$$\vec{\nabla} \cdot \left[\tilde{\mu} \left(\frac{|\vec{\nabla}\phi_N|}{a_0} \right) \vec{\nabla}\phi_N \right] = 4\pi G\rho. \tag{2.15}$$

Since the Lagrangian is invariant under translations and rotations the usual conservation laws, the conservation of energy, linear and angular momentum are preserved. For a full derivation of the conserved charges we refer to [18]. In order to show that this formulation is consistent with the MOND formulation of Milgorm we combine the equations of motion (2.15) and (2.13) to obtain that

$$\vec{\nabla} \cdot \left[\tilde{\mu} \left(\frac{|\vec{\nabla}\phi_N|}{a_0} \right) \vec{\nabla}\phi_N - \vec{\nabla}\phi_N \right] = 0. \tag{2.16}$$

Because the divergence of a curl field has to be zero it is sufficient to identify the expression between the brackets as the curl of a vector field $\vec{h}$.

$$\tilde{\mu}(a/a_0)\vec{a} = \vec{g} + \vec{\nabla} \times \vec{h} \tag{2.17}$$

It is theoretically shown [18] that in the limit where the radius around an enclosed mass goes to infinity, one can neglect the curl term and obtain equation (2.1).

2.4 The Relativistic Field Formulation

In order to make connection with cosmology and make cosmological predictions, a relativistic formulation of MOND is needed. In 2004 Bekenstein published a paper [19] where he, building on the ideas of Sanders, provided a relativistic formulation of MOND. The dynamics of this theory is, instead of a single tensor field in the case of General Relativity, governed by a Tensor, Vector and a Scalar field. For this reason the theory is better known as TeVeS.

One of the main successes of the CDM paradigm is that it can very well explain the large scale structure formation of the universe. If MOND wants to be a successfully theory, there must be a relativistic formulation that can explain the large scale structure. In [20] it was pointed out that by running simulations it is possible to produce large scale structure comparable to observations.

2.5 MOND Difficulties

One of the observational problems that MOND has encountered is that there is still a mass discrepancy between predicted dynamical mass (M_{dyn}) and observed mass (M_{obs}) in the core of rich X-ray clusters [15]. In [21] Sanders calculated the virial discrepancy of a sample of 93 X-ray emitting clusters within the framework of Newtonian dynamics and MOND. His result is shown figure 3. Where within the paradigm of Newtonian dynamics (left plot) the average mass discrepancy $< M_{dyn}/M_{obs} >= 4.4 \pm 1.6$, MOND (right plot) can only reduce this to a discrepancy of $< M_{dyn}/M_{obs} >= 2.1 \pm 1.2$. This is one of the major problems of MOND. It can reduce the mass discrepancy but cannot remove it.

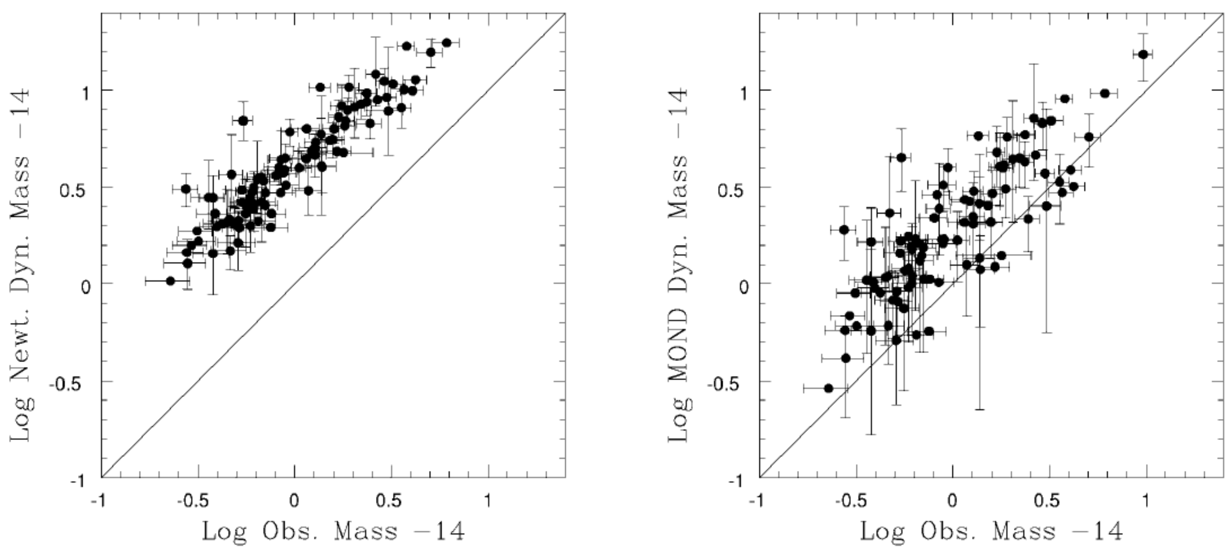

Figure 3: On the left the dynamical mass of the 93 X-ray clusters is plotted against the observed mass within the paradigm of Newtonian dynamics while on the right the clusters are plotted within the framework of MOND. The solid line indicates no mass discrepancy.

Furthermore, a well know argument in favour of explaining the mass problem in terms of a non-baryonic particle is the Bullet cluster, where two galaxy clusters merges. The mass distribution is probed by combination of observation in the optical regime to detect the stellar mass, X-ray observation to determine the mass distribution of the hot (luminous) inter-cluster gas and gravitational lensing to determine the total mass distribution [2]. The remarkable fact is that the mass distribution determined by gravitational lensing does not coincide with the X-ray distribution (figure 4). This observation is consistent with the idea of non-barionic cold dark matter because the luminous gas is interacting during collision and feels a drag force while the dark matter particle of the halo can fly through because it is non (or weakly) interacting. To investigate the Bullet cluster within the paradigm of MOND one has to use gravitational lensing of a relativistic MOND formulation instead of General Relativity. By use of a toy model that simulates the merging Bullet cluster it turned out that TeVeS is not capable of reproducing the Bullet observations without additional mass [22] [23].

Another problem has been pointed out in [24] where the rotation curves of a sample of 12 galaxies is investigated. By letting a_0 and the mass-to-light ratio vary as free parameters the individual rotation curves could be reproduced. However, this results in a wide range of predicted values of a_0. In the attempt to fit the data with a universal value of a_0 it turns out that not all the rotation curves can be reproduced.

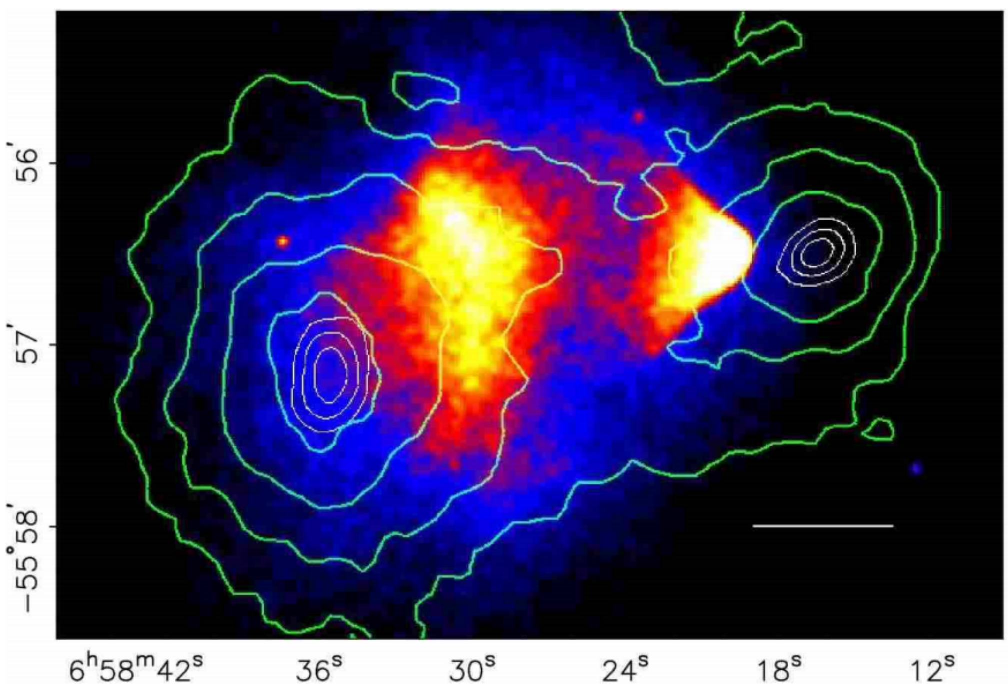

Figure 4: The merging Bullet Cluster (1E0657558) [25]: The red/yellow color indicates the X-ray observed mass distribution of the hot-gas where the blue color indicates the mass distribution due to gravitational lensing. The green contour lines indicate the weak lensing reconstructions where the outer indicates $\kappa = 0.16$ and increases with steps of 0.07.

3 A Possible Microscopic Description of MOND

3.1 The Holographic Principle

In 1993, 't Hooft came up with the holographic principle [26] arguing that at the Planck scale the degrees of freedom of our 3+1 dimensional world must be described on a 2 dimensional dynamical lattice. In general one speaks of a holographic theory if the physics of a $(n+1)$-dimensional object can be described or is related to the physics in n-dimensions.

If we consider a black hole, its entropy is given by the Hawking-Bekenstein entropy relation $S = k_B A c^3/4G\hbar$, where A is the area of the black hole horizon. Because the entropy is a measure of the number of microstates of a certain configuration, the degrees of freedom of a black hole are described by the area. Given its entropy one can derive other thermodynamical quantities such as temperature. However, the question still remains how one should count the degrees of freedom.

Another example of a holographic theory is the idea of the Anti-de-Sitter/Conformal Field Theory (AdS/CFT) correspondence [27,28], originating from super sting theory, that points out the equivalence between a 5-dimensional gravitational theory and a 4-dimensional strongly coupled gauge theory.

3.2 Emergent Gravity as an Entropic Force

Nowadays there are several theories of emergent gravity. In his original paper from 2010 [29], Verlinde derived Newton's law through a holographic scenario. He assumed the validity of the holographic principle and stated that the information associated with a part of space that has not yet been emerged is contained on a holographic screen. Here we will shortly review three of the assumption he made in order to derive Newtonian gravity.

- **There is a change in entropy in the emergent direction.**
  Bekenstein came up with the idea that the area of a black hole increases when a particle of mass m is one Compton wavelength away from the horizon. Verlinde generalized and extended this idea by postulating that there is a change in entropy that is proportional to the change in position. He stated that, when a particle is approaching the holographic screen, the entropy of the screen

is increasing by $2\pi k_B$ for every travelled Compton wavelength $\hbar/mc$

$$\Delta S = 2\pi k_B \frac{mc}{\hbar} \Delta x. \tag{3.1}$$

Because now the change in entropy (ΔS) is related to a change in position (Δx) we can argue that gravity is an entropic force. Since he postulated that a particle moving to the screen is causing an increase in entropy we can make the analogy with osmosis or elasticity. Because nature has the tendency to increase its entropy, there will be an effective force that cause a displacement toward the screen. Like in the case of osmosis across a semi-permeable membrane the effective force is given by

$$F = \frac{T\Delta S}{\Delta x}. \tag{3.2}$$

- **The number of degrees of freedom is proportional to the area of the screen.**

  The number of degrees of freedom on the screen is determined by the area (A) of the screen divided by the Planck area.

$$N_s = \frac{A}{l_p^2} = \frac{Ac^3}{G\hbar} \tag{3.3}$$

- **The energy is evenly distributed over the degrees of freedom.**

  By assuming that the energy is equally distributed over the degrees of freedom, and treating the system as a classical ideal gas, each degree of freedom carries an energy $k_B T/2$. This is known as the law of equipartition

$$E = \frac{1}{2} N_{\mathrm{s}} k_B T. \tag{3.4}$$

Starting with these assumptions we are able to derive the Newton's law of gravitation. If we take Einstein equation

$$E = Mc^2, \tag{3.5}$$

which relates the total energy to the total mass we can write the equality

$$Mc^2 = \frac{1}{2} N_{\mathrm{s}} k_B T. \tag{3.6}$$

This looks like an trivial step but in fact this is really non-trivial. This equation essentially states that the energy of the screen is the same as the energy of the bulk. In section 3.2.1 we will investigate this in more detail but for now let us first see what comes out. Using equation (3.3) for the number of degrees of freedom and the fact that the area of the holographic screen is given by $A = 4\pi R^2$ we get

$$Mc^2 = \frac{2\pi R^2 c^3}{G\hbar} k_B T. \tag{3.7}$$

If we consider Unruh's relation [29], the relation between the acceleration with respect to the inertial frame of the vacuum and a temperature

$$k_B T = \frac{1}{2\pi}\frac{\hbar a}{c}, \tag{3.8}$$

and apply this to equation (3.7) we obtain the Newtonian acceleration

$$a = \frac{M}{R^2} G. \tag{3.9}$$

It should be noted that if we, instead of using Unruh's relation, combine equation (3.1) and (3.2) to express the temperature and substitute this in equation (3.7) we obtain the Newtonian expression for the gravitational force

$$F = \frac{mM}{R^2} G. \tag{3.10}$$

3.2.1 The connection between the bulk and the surface

The connection between the bulk and the surface is essentially made in equation (3.6) by stating that the energy on the screen is equal to the energy of the bulk. To get a better view on why this is a reasonable statement lets reverse our line of reasoning and start with the Poisson equation of gravity

$$\vec{\nabla}^2 \phi = 4\pi G \rho, \tag{3.11}$$

where again, $\vec{\nabla}\phi = -\vec{g}$. If we integrate the mass density ρ over a volume V and use Einstein's mass-energy relation (3.5) we obtain

$$E_{\text{bulk}} = \frac{c^2}{4\pi G} \iiint_V -\vec{\nabla} \cdot \vec{g} \, \mathrm{d}V. \tag{3.12}$$

If we now apply Gauss' law, we can replace the volume integral by a closed surface integral, where we take the surface S to be an equipotential surface (i.e. $\vec{g}$ is perpendicular to the surface) in order to replace $\vec{g}$ with its magnitude and get

$$E_{\text{bulk}} = \frac{c^2}{4\pi G} \oiint_S g \, \mathrm{d}S. \tag{3.13}$$

Without any assumptions we see that it is a convenient step to describe the bulk energy of a system by the surface enclosing the bulk. If we now use equation (3.8), (3.3) and integrate over the surface we get

$$E_{\text{bulk}} = \frac{1}{2} N_s k_B T. \tag{3.14}$$

So the energy of the bulk is given by the equipartition of energy over the number of degrees of freedom on the surface, like Verlinde assumed. If we define the degrees of freedom of the bulk $N_{\text{bulk}} \equiv E_{\text{bulk}}/(1/2)k_B T$ we can say that

$$N_{\text{bulk}} = N_s. \tag{3.15}$$

This equality is known as the holographic equipartition [30]. So in the context Verlinde's emergent gravity the assumptions of equipartition on the holographic screen can be seen as the assumption of holographic equipartition.

3.3 Quantum Statistical Description on the Holographic Screen

Holographic equipartition follows from the assumption of equipartition of energy on the holographic screen. One could argue that this is in general not a valid assumption because the equipartition of energy only applies for high temperature systems. Considering low temperatures, one has to take into account the quantum behaviour of the particles. In the following section we will obtain expressions for various thermodynamical quantities in the case of a two dimensional quantum gas. In section 3.3.2 we will use their low temperature limit to make the connection with MOND. Finally, in section 3.3.3 we will obtain the MOND interpolating function from the microscopic theory.

3.3.1 Two dimensional quantum gases

If one considers gases at low temperature the quantum behaviour of particles becomes very important. Because of the Pauli exclusion principle the behaviour of fermions is in general not the same as the behaviour of bosons. We first derive some properties of a two dimensional fermion gas.

Let us consider an ideal fermion gas, where we neglect the interaction between particles, and look at the mean occupation number

$$\bar{n}(\epsilon) = \frac{1}{e^{\beta(\epsilon_i - \mu)} + 1}. \tag{3.16}$$

This function, better known as the Fermi-Dirac distribution, gives the statistical fraction of the total number of particles with a certain energy ϵ. Furthermore μ is the chemical potential and $\beta = 1/k_B T$ where k_B is the Boltzmann constant and T is the temperature of the gas. It is important to realize that the chemical potential is not a constant but a function of the temperature. Because we want to describe excitations on the holographic screen, the density of states is given by a surface density

$$f(p)\,\mathrm{d}p = \frac{A}{h^2} 2\pi p\,\mathrm{d}p, \tag{3.17}$$

where $f(p)$ is the density of states, A the area of the gas, h the Planck constant and p the momentum. Because we are considering an ideal quantum gas, the Hamiltonian has only a kinetic component (i.e. we neglect interactions). In the classical limit the discrete energy levels become continuous. The non-relativistic kinetic energy becomes $\epsilon = p^2/2m$. Substituting this in equation (3.17) and using the dact that can write the reduced Plack constant as $\hbar = h/(2\pi)$ we obtain the density of states as a function of energy

$$f(\epsilon)\,\mathrm{d}\epsilon = \frac{Am}{2\pi\hbar^2}\,\mathrm{d}\epsilon. \tag{3.18}$$

Now we can calculate the total number of particles because we know the number of states as a function of energy and the mean occupation number as a function of energy. The only thing that we have to take into account is the degeneracy factor g. Because, in the case of fermions, the energy states have a spin degeneracy of $g = 2s + 1$ we have to multiply our mean occupation number with that factor. The total number of particles can then be calculated by integrating $\mathrm{d}N(\epsilon) = g\bar{n}(\epsilon)f(\epsilon)\,\mathrm{d}\epsilon$ over all possible energies

$$N = \int_0^\infty \frac{gAm}{2\pi\hbar^2} \frac{\mathrm{d}\epsilon}{e^{\beta(\epsilon-\mu)} + 1}. \tag{3.19}$$

If we look at the zero temperature limit this integral becomes relative easy. Because we are dealing with fermions, the total energy E at zero temperature is non zero because not all particles can occupy the ground state. The energy of the highest occupied energy level is called the Fermi energy. At zero temperature, the Fermi energy is defined to be the chemical potential $\mu(T = 0) \equiv \epsilon_F$ [31]. If we now take te limit $T \to 0$ the mean occupation number becomes the following:

$$\lim_{T\to 0} \frac{1}{e^{\beta(\epsilon-\mu)} + 1} = \begin{cases} 1 & \text{for } \epsilon < \epsilon_F \\ 0 & \text{for } \epsilon > \epsilon_F \end{cases}$$

This allows us to evaluate the integral in equation (3.19) and obtain the following expression for the total number of particles at zero temperature

$$N_0 = \frac{gAm}{2\pi\hbar^2}\epsilon_F. \tag{3.20}$$

The total energy will be obtained by integrating over $\mathrm{d}E(\epsilon) = \epsilon\,\mathrm{d}N(\epsilon)$

$$E = \frac{gAm}{2\pi\hbar^2}\int_0^\infty \frac{\epsilon\,\mathrm{d}\epsilon}{e^{\beta(\epsilon-\mu)}+1}. \tag{3.21}$$

For a very low but non zero temperature we can approximate this integral by using the Sommerfeld expansion derived in appendix A.1. The total energy at low temperatures is given by

$$E \approx \frac{N_0}{2}\epsilon_F + \frac{N_0\pi^2 k_B^2 T^2}{6\epsilon_F}. \tag{3.22}$$

The first term is the zero temperature energy and the second term is the contribution due to thermal excitations. Because in Verlinde's formulation of gravity he considered thermal excitations on the screen, we have to subtract the zero point term from the total energy. Let us therefore define the low temperature thermal energy as

$$E_{th} = \frac{N_0\pi^2 k_B^2 T^2}{6\epsilon_F}. \tag{3.23}$$

We could also wish to obtain an expression valid for all temperatures. To this purpose, rewrite the integral of equation (3.21) in terms of a polylogarithm, defined in appendix A.2, which is in fact a power series. If we do so, we obtain that the thermal energy is given by

$$E_{th} = \frac{-N_0}{\epsilon_F\beta^2}L_{i2}(-e^{\beta\mu}) - \frac{N_0\epsilon_F}{2}. \tag{3.24}$$

The derivation of this result is given in appendix A.3.

A remarkable property of two dimensional quantum gases is that the specific heat C_v of bosons is identical to that of fermions. A full derivation of this result is given in appendix A.4.

3.3.2 The connection with the deep MOND limit

In this section we will use the ideas of Pazy and Argaman [32] to show that a quantum particle description on the holographic screen leads to modified Newtonian dynamics. Using the thermodynamical quantities of the two dimensional Fermi gas we are able to make the connection with the deep MOND limit. If we are following Verlinde's derivation of Newtonian dynamics, while replacing the equipartition of energy with the thermal energy obtained for a Fermi gas we end up with a modified version of Newton's law. We start our derivation by saying that the low temperature thermal energy equals Einstein's formula

$$Mc^2 = \frac{N_0 \pi^2 k_B^2 T^2}{6\epsilon_F}. \tag{3.25}$$

If we take the square of Unruh's relation (3.8), we obtain an expression for temperature squared

$$T^2 = \frac{a^2 \hbar^2}{4\pi^2 c^2 k_B^2}. \tag{3.26}$$

In general, the relation between the number of degrees of freedom and the number of particles in two dimensions is given by $N_p = N/2$, where N_p is the number of particles and N the number of degrees of freedom. Using this, we can relate the number of particles at zero temperature to the number of degrees of freedom on the holographic screen (3.3) and obtain

$$N_0 = \frac{Ac^3}{2\hbar G}. \tag{3.27}$$

If we now use equation (3.26) to transform temperature to acceleration, (3.27) for the number of particles and use that $A = 4\pi R^2$ we can rewrite expression (3.25) and obtain

$$a^2 = \frac{12c\epsilon_F}{\hbar\pi}\frac{MG}{R^2}. \tag{3.28}$$

By identifying the first fraction of (3.28) as the MOND acceleration constant the equation becomes the expression for the acceleration in the deep MOND limit with

$$a_0 \equiv \frac{12c\epsilon_F}{\hbar\pi}. \tag{3.29}$$

In Verlinde's framework of gravity, an accelerating particle is causing a temperature on the holographic screen. The mondian dynamics can be understood as a result of the quantum effect that occur at temperatures that correspond to accelerations of the order of a_0. One could expect that the quantum description could lead to correction of MOND and produces a MOND-like theory instead of exactly the one of Milgrom. Nevertheless, this is not the case because in the low temperature Sommerfeld expansion of the thermal energy the higher order terms are vanishing. Since the thermal energy becomes quadratic in the temperature we obtained the familiar MOND limit.

3.3.3 The MOND interpolating function as a function of temperature

Considering the low temperature expansion for the thermal energy we can obtain the deep MOND limit. We will now show that, by starting with the MOND equation, we can obtain an expression for the interpolating function [33]. Let us start with Milgrom's equation of modified gravity

$$\frac{GM}{R^2} = a\tilde{\mu}(a/a_0). \tag{3.30}$$

Using equations (3.5), (3.8) and (3.27) we can rewrite equation (3.30) in terms of thermodynamical quantities and obtain

$$\frac{E_{\text{bulk}}}{N_0 k_B T} = \tilde{\mu}(a/a_0). \tag{3.31}$$

Because we are using Einstein's formula to replace the total mass of the system, the resulting energy is the energy of the bulk. If we now assume that the bulk energy equals the two dimensional thermal energy as we did before but with the thermal energy now given by equation (3.24), we obtain the interpolating function

$$\tilde{\mu}(T) = -\frac{1}{\epsilon_F \beta} Li_2(-e^{\beta\mu}) - \frac{\beta\epsilon_F}{2}. \tag{3.32}$$

Since the Fermi energy is related to the acceleration constant a_0 and an acceleration can be transformed into a temperature we obtain $\tilde{\mu}$ as a function of (T/T_0), where T_0 is the temperature associated with the acceleration a_0. Using Unruh's relation we obtain that

$$T_0 = \frac{\hbar a_0}{2\pi k_B c}. \tag{3.33}$$

If we use equation (3.33) and the definition of the acceleration constant (3.29) we obtain

$$\tilde{\mu}\left(\frac{T}{T_0}\right) = -\frac{6}{\pi^2}\frac{T}{T_0} Li_2(-e^{\beta\mu}) - \frac{\pi^2}{12}\frac{T_0}{T}. \tag{3.34}$$

The numerical solution [32] of the interpolating function is shown in figure 5. We observed that the curve lies between the standard and simple interpolating functions (section 2), those favored by observations.

So far we only considered the fermionic case, however, we can extent the result to bosons as well. Since we can express the thermal energy as an integral over the specific heat $C_v \equiv (\partial E/\partial T)_v$ and we showed (in appendix A.4) that the two dimensional specific heat is independent of the nature of the particles, we will use equation (3.31) to derive an interpolating that is independent of the nature of the

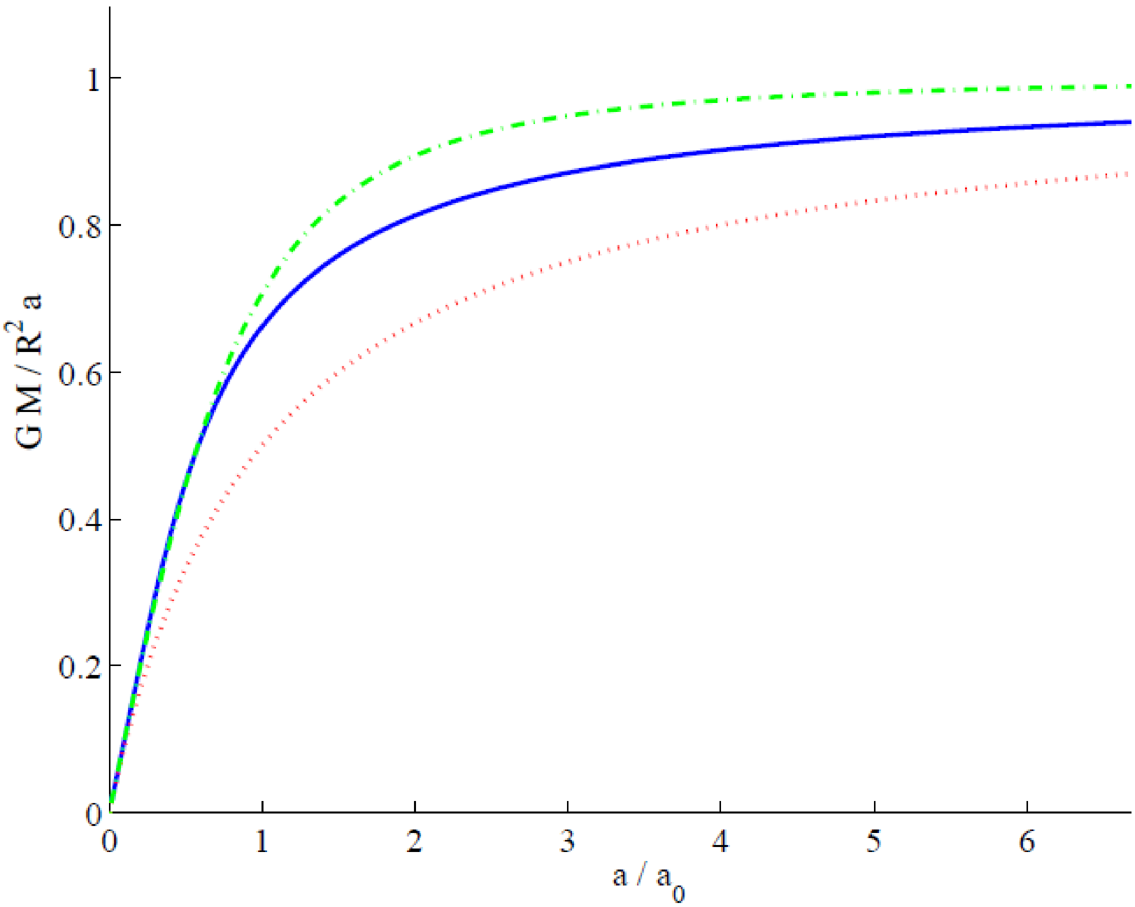

Figure 5: The blue line represents the $\tilde{\mu}$ function obtained by considering quantum statistical mechanics on the holographic screen, where the green (dashed-line) gives the standard interpolating function $\tilde{\mu} = x/\sqrt{x^2+1}$ and the red (dotted) line the simple interpolating function $\tilde{\mu} = x/(x+1)$.

particles. Furthermore, we will explicitly derive the specific heat in the low temperature limit and use this to show that the obtained interpolating function has the correct deep MOND limit.

We start with the energy expression for bosons (given in appendix A.1)

$$E^B = \frac{N_F k_B T^2}{T_F} Li_2(1 - e^{-T_F/T}), \tag{3.35}$$

where we recall that N_F is the total number of particles associated with a particular temperature T_F. To obtain a small temperature approximation we can Taylor expand the dilog function around

$T = 0$. If we use the the polylogarithm properties (A.13) and (A.14) we obtain

$$Li_2(1 - e^{-T_F/T}) \approx \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \zeta(2-n)(1 - e^{-T_F/T})^n e^{-nT_F/T}. \tag{3.36}$$

If we use the leading order term, which is a reasonable step because the higher order terms are exponentially small in the temperature, we obtain

$$E^B \approx \frac{N_F k_B T^2 \pi^2}{6T_F}. \tag{3.37}$$

By defining the temperature scale to be the Fermi temperature $T_F \equiv \epsilon_F/k_B$, which has to be the case in order to obtain the correct zero point energy, we see that the low temperature thermal energy for bosons is identical to the fermionic of equation (3.23). By using the definition we obtain the low temperature specific heat

$$C_v^{\text{Bosons}} = C_v^{\text{Fermions}} = \frac{N_0 k_B^2 \pi^2 T}{3\epsilon_F}. \tag{3.38}$$

Since we know the low temperature specific heat we can obtain the low temperature limit of a general valid interpolating function. By integrating the specific heat up to a temperature T to obtain the thermal energy and substitute this in equation (3.31), the definition of the interpolating function, we obtain

$$\tilde{\mu}(a/a_0) = \frac{k_B \pi^2 T}{6\epsilon_F}. \tag{3.39}$$

We can now easily check whether the interpolating function has the right limit. If we use the definition of the acceleration constant (3.29) and Unruh's relation (3.8) to transform the temperature into an acceleration we obtain

$$\tilde{\mu}(a/a_0) = \frac{a}{a_0}, \tag{3.40}$$

i.e. the deep MOND limit of the interpolating function. This does not come as a surprise since in the low temperature limit we already made the connection with the deep MOND limit. Although we can make the extension to bosons, one has to realize that MOND arises due to fermionic properties. Since there is no low temperature energy freeze-out in the case of bosons the acceleration constant a_0 becomes zero in the low temperature limit and hence, one will obtain Newtonian dynamics.

3.3.4 AQUAL as a consequence of modified holographic equipartition

From equation (3.31) it follows that

$$E_{\text{bulk}} = \frac{1}{2}\tilde{\mu}(a/a_0)N_{\text{s}}k_BT. \tag{3.41}$$

We saw that the equipartition of energy leads to holographic equipartition. By the same arguments given section 3.2.1 it follows that equation (3.41) is essentially a modification of the holographic equipartition [33]. It follows that the bulk degrees of freedom are given by

$$N_{\text{bulk}} = \tilde{\mu}(T/T_0)N_{\text{s}}. \tag{3.42}$$

If we write (3.3) as a surface integral we obtain

$$E_{\text{bulk}} = \oiint_A \frac{c^2 a}{4\pi G}\tilde{\mu}(a/a_0)\,\mathrm{d}A. \tag{3.43}$$

If we apply Gauss's theorem and use equation (3.5) we obtain,

$$\vec{\nabla}\cdot\left[\tilde{\mu}\left(\frac{|\vec{\nabla}\phi|}{a_0}\right)\vec{\nabla}\phi\right] = 4\pi G\rho. \tag{3.44}$$

This is exactly the equation of motion obtained from the AQUAL Lagrangian formulation of MOND. But in order to make the real connection with the Lagrangian formalism the interpolating function has to obey the restrictions mentioned in section 2.3.

4 Discussion

If we review the paradigm of CDM we see that it has a very different origin compared to Modified Newtonian Dynamics (MOND). While in the early days of the hidden mass problem a modification of dynamics was considered, the idea of dark matter became more favoured by stability arguments. Later it turned out that dark matter was also needed in order to explain the observed asymptotically flat rotation curves and large scale structure. In itself, it is a remarkable fact that CDM can explain these disparate phenomena.

An advantage of particle dark matter is that we have a good theoretical understanding of particles and their interactions. Despite the fact that the behaviour of a dark matter particle could be very different as compared with ordinary matter, postulating a dark particle fits the paradigm. In some way this is the simplest and most straightforward solution. Contrary to the paradigm of emergent gravity it does not require a revision of our interpretation of space-time and does not change the role and paradigm of particle physics. Until a dark matter candidate is observed, the dark matter halo is analogous to the phenomenological MOND equation of Milgrom because it provides an effective way of understanding and explaining certain phenomena. Nevertheless, I think we have to conclude that at the moment the CDM paradigm can explain a larger range of phenomena than MOND. Since there is still a mass discrepancy in galaxy clusters and there is not yet a fully suitable relativistic theory that provides a good understanding of the Bullet cluster, the framework of CDM is a richer paradigm.

In Verlinde's framework of emergent gravity one can, by using a quantum statistical description for the thermal excitations, obtain MOND by calculating the energy of the holographic screen. It was shown that the MOND acceleration constant a_0 is related to the Fermi energy of the screen. Although we can make the analogy for bosons, one has to realize that the connection with MOND is based on a fermionic property. In Verlinde's emergent gravity, an accelerating particle is causing, by the Unruh effect, a temperature on the holographic screen. Since small accelerations correspond to small temperatures one has to take into account quantum effects for small accelerations. In the high temperature (Maxwell-Boltzmann) limit one obtains holographic equipartition, which is the equality between the bulk and surface degrees of freedom. Nevertheless, holographic equipartition is modified in the low temperature limit where quantum effects become important. It turned out that the temperature transition function that governs the modification of the holographic equipartition can be interpreted as the MOND interpolating function.

In order to predict the dynamics of a system, the external field effect requires us to have an understanding about absolute acceleration. Within the framework of emergent gravity we can understand this as the temperature associated with the holographic screen. Although this framework can solve some problems it also raises questions. A question one could ask is to what extend the holographic principle is valid when it is combined with the framework of MOND. Considering a black hole, the equivalence principle learns us that there is a frame of reference in which an object, accelerating toward the centre, never crosses the horizon but rather joins the area of the horizon. This seems to me as a very important argument for the validity of the holographic principle. Since MOND violates the equivalence principle and the holographic principle is essential for the framework of Verline, one could ask the question if these frameworks are in conflict.

To end I would like to say that even though the deep insight and the implication of the paradigm of emergent gravity are beyond my scope, I found it very interesting to think about gravity from this perspective. And although today the paradigm of CDM is a richer framework, it might turn out that due to the framework of emergent gravity we obtain a better understanding of gravity, the hidden mass problem and Modified Newtonian Dynamics.

Acknowledgment

I would like to thank my supervisor, Elisabetta Pallante, for introducing me to the wonderful topic of modified dynamics and for her guidance throughout this project.

A Appendices

A.1 Sommerfeld Expansion

In order to calculate the integral of equation (3.21) we can do a small temperature approximation [34] which is sometimes referred to as a Sommerfeld expansion. As a starting point, lets define the integral

$$I = \int_0^\infty \frac{f(\epsilon)\,\mathrm{d}\epsilon}{e^{\beta(\epsilon-\mu)}+1}, \tag{A.1}$$

where $f(\epsilon)$ is an arbitrary function such that the integral converges. If we define $x \equiv \beta(\epsilon-\mu)$ and substitute this in equation (A.1) we get

$$I = \frac{1}{\beta}\int_{-\beta\mu}^\infty \frac{f(\mu + x/\beta)\,\mathrm{d}x}{e^x+1}. \tag{A.2}$$

If we now split the integral in two integrals and swap the integration limits of the first one we get

$$I = \frac{1}{\beta}\left[\int_0^{\beta\mu} \frac{f(\mu - x/\beta)\,\mathrm{d}x}{e^{-x}+1} + \int_0^\infty \frac{f(\mu + x/\beta)\,\mathrm{d}x}{e^x+1}\right] \tag{A.3}$$

We can now use the fact that $1/(e^{-x}+1) = 1 - 1/(e^x+1)$ we get

$$I = \frac{1}{\beta}\left[\int_{-\beta\mu}^{0} f(\mu + x/\beta)\,\mathrm{d}x + \int_0^\infty \frac{f(\mu + x/\beta) - f(\mu - x/\beta)\,\mathrm{d}x}{e^x+1}\right]. \tag{A.4}$$

$$= \int_0^\mu f(\epsilon)\,\mathrm{d}\epsilon + \frac{1}{\beta}\int_0^\infty \frac{f(\mu + x/\beta) - f(\mu - x/\beta)\,\mathrm{d}x}{e^x+1} \tag{A.5}$$

Because we want to do a small temperature expansion we can do a Taylor approximation of the numerator around the point $x/\beta = 0$.

$$f(\mu + x/\beta) - f(\mu - x/\beta) \approx 2\left[\frac{1}{\beta}f'(\mu)x + \frac{1}{6\beta^3}f'''(\mu)x^3 + ...\right] \tag{A.6}$$

$$I \approx \int_0^\mu f(\epsilon)\,\mathrm{d}\epsilon + \frac{2f'(\mu)}{\beta^2}\int_0^\infty \frac{x\,\mathrm{d}x}{e^x+1} + \frac{2f'''(\mu)}{6\beta^4}\int_0^\infty \frac{x^3\,\mathrm{d}x}{e^x+1} + ... \tag{A.7}$$

If we use that

$$\int_0^\infty \frac{x^n\,\mathrm{d}x}{e^x+1} = (1-2^{-n})\zeta(n+1)\Gamma(n+1), \tag{A.8}$$

where ζ is the Riemann Zeta function and Γ is the Euler Gamma function we can write that

$$I \approx \int_0^\infty f(\epsilon)\,\mathrm{d}\epsilon + \frac{\pi^2}{6\beta^2}f'(\mu) + \frac{7\pi^4}{360\beta^4}f'''(\mu) + ... \tag{A.9}$$

Now we can use this result to do an expansion of equation (3.21) where we use that $f(\epsilon) = \epsilon$. Because $f(\epsilon)$ vanishes after the first derivative the expression for the thermal energy becomes quadratic in the temperature.

$$E = \frac{gAm}{2\pi\hbar^2}\int_0^\infty \frac{\epsilon\,\mathrm{d}\epsilon}{e^{\beta(\epsilon-\mu)}+1} \approx \frac{N_0\mu^2}{2\epsilon_F} + \frac{N_0\pi^2k_B^2T^2}{6\epsilon_F} \tag{A.10}$$

Because the first term is independent of temperature we can identify this as the zero point energy and replace μ by ϵ_F.

A.2 The Polylogarithm

In statistical physics it is often possible to rewrite the integrals that appear in a lot of thermodynamic observables in terms of a polylogarithm. It turns out to be very useful because these mathematical expressions can be formulated in terms of an integral but also as a power series. A general expression for the polylogarithm as a power series and in integral form is given by

$$Li_n \equiv \sum_{k=1}^{\infty}\frac{z^k}{k^n}, \tag{A.11}$$

$$\pm\, Li_n(\pm z) = \frac{1}{\Gamma(n)}\int_0^\infty \frac{t^{n-1}}{e^t/z \mp 1}. \tag{A.12}$$

where Γ is the Euler Gamma function. Note that $\Gamma(1) = \Gamma(2) = 1$. Useful properties of the polylogaritm are:

$$\frac{\partial Li_n(z)}{\partial z} = zLi_{n-1} \tag{A.13}$$

$$Li_s(1) = \zeta(s) \tag{A.14}$$

where ζ is the Riemann zeta function. Note that $\zeta(1) = \infty$ and $\zeta(2) = \pi^2/6$.

$$Li_2(1-\frac{1}{z}) + Li_2(1-z) = -\frac{1}{2}\ln^2(z). \tag{A.15}$$

$$\pm\, Li_1(z) = \mp\ln(1\mp z) \tag{A.16}$$

A.3 Thermal Energy

If we start with equation (3.21) and do a substitution with $x = \beta\epsilon$ we obtain that the thermal energy can be written as

$$E = \frac{N_0}{\epsilon_F \beta^2} \int_0^\infty \frac{x\,\mathrm{d}x}{e^{x-\beta\mu}+1}\,\mathrm{d}x \tag{A.17}$$

By using the definition of (A.12) and subtracting the zero point energy we obtain an expression for the thermal energy

$$E_{th} = \frac{-N_0}{\epsilon_F \beta^2} L_{i2}(-e^{\beta\mu}) - \frac{N_0 \epsilon_F}{2}. \tag{A.18}$$

A.4 Specific Heat in two Dimensions

Here we will show that the specific heat of a two dimensional quantum gas is independent of the nature of the particle (i.e. bosons or fermions) making up the gas [35]. We start our derivation by deriving thermodynamical quantities, wherever the upper signs corresponds to a fermion gas while the lower signs correspond to boson gas. We start by expressions the particle number and the energy

$$N_B^F = \frac{gAm}{2\pi\hbar^2} \int_0^\infty \frac{\mathrm{d}\epsilon}{e^{\beta(\epsilon-\mu)} \pm 1}, \tag{A.19}$$

$$E_B^F = \frac{gAm}{2\pi\hbar^2} \int_0^\infty \frac{\epsilon\,\mathrm{d}\epsilon}{e^{\beta(\epsilon-\mu)} \pm 1}. \tag{A.20}$$

Here the degeneracy factor g is equal to 1 for bosons while in the case of fermions it is the usual expression $g = 2s + 1$. If we do a substitution $x = \beta\epsilon$ we obtain

$$N_B^F = \frac{gA}{\lambda^2} \int_0^\infty \frac{\mathrm{d}x}{e^{x-\beta\mu} \pm 1}, \tag{A.21}$$

$$E_B^F = \frac{gA}{\lambda^2\beta} \int_0^\infty \frac{x\,\mathrm{d}x}{e^{x-\beta\mu} \pm 1}. \tag{A.22}$$

Here λ is defined to be the thermal de Broglie wavelength given by

$$\lambda^2 = \frac{4\pi\hbar^2}{2mk_BT}. \tag{A.23}$$

If we use the integral representation of the polylogarithm (A.12) and identity (A.16) we obtain

$$N_B^F = \pm\frac{gA}{\lambda^2} \ln(1 \pm e^{\beta\mu}), \tag{A.24}$$

$$E_B^F = \mp\frac{gA}{\lambda^2\beta} Li_2(\mp e^{\beta\mu}). \tag{A.25}$$

The total number of particles N_F associated with a particular temperature T_F can also be obtained by dividing the area of the quantum gas by the square of the thermal de Broglie wavelength

$$N_F = \frac{gA}{\lambda_F^2}, \tag{A.26}$$

where λ_F is the thermal wavelength associated with the temperature T_F. In the case of fermions we have to include our g factor. If we use equation (A.24) and (A.26) we can write the following expression for the fugacity

$$e^{\beta\mu} = \begin{cases} e^{T_F/T} - 1 & \text{for Fermions} \\ 1 - e^{-T_F/T} & \text{for Bosons} \end{cases}$$

If we now rewrite equation (A.25) by use of (A.26) and the above identity we get

$$E^B = \frac{N_F k_B T^2}{T_F} Li_2(1 - e^{-T_F/T}) \tag{A.27}$$

$$E_F = -\frac{N_F k_B T^2}{T_F} Li_2(1 - e^{T_F/T}) \tag{A.28}$$

If we use the property of equation (A.15) we obtain that

$$E_{\text{Fermions}} = E_{\text{Bosons}} + \frac{1}{2} N_F k_B T_F \tag{A.29}$$

The energy of a fermion gas is equal to a boson gas up to a constant factor, the zero point energy. Because the specific heat is the partial derivative of the energy with respect to the temperature, we find that the zero point energy varnishes. It should be noted that the two dimensional gas is a special case and that this conclusion can not be generalized in n-dimensions. If we look at the specific heat in n-dimensions [35] for $T > T_F$

$$C_v = \frac{n}{2} N k_B \left[1 \pm \left(\frac{n}{2} - 1 \right) \frac{1}{2^{(n/2)+1}} \left(\frac{T_F}{T} \right)^{n/2} - (n-1) \left(\frac{1}{2^n} - \frac{2}{3^{(n/2)+1}} \right) \left(\frac{T_F}{T} \right)^n + ... \right],$$

we see that in the the case where $n = 2$ the particle depending $\pm$ factor disappears. Another fact that should be mentioned is that if one considers extreme relativistic particles, lets say a photon gas, the two dimensional bosonic specific heat is different from the fermionic one. In the case of relativistic particles the specific heat becomes independent of the particle nature in the case of an one dimensional gas.

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