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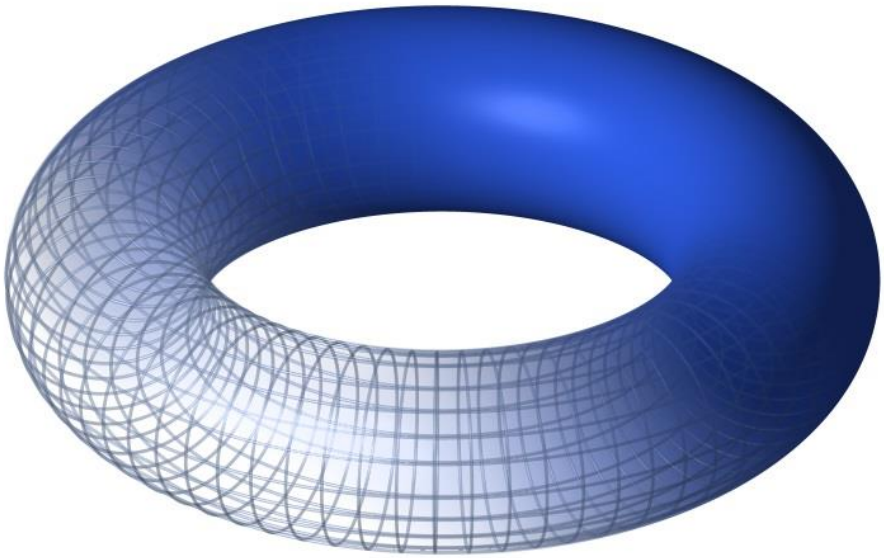
The Maslov-index



Bachelor thesis Mathematics

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Abstract

The Maslov index is defined as the degree of a closed curve in the Lagrangian Grassmanian manifold. This index can be computed by an integral. It follows however from the isomorphy between the Lagrangian Grassmanian manifold and the quotient group $U(m)/O(m)$ that it is also equal to the amount of intersections with the Maslov cycle. Finally the Maslov-index is computed for several examples.

Contents

1 Introduction

Nowadays it is widely known that classical mechanics applies to everyday life objects, like a cup of coffee or a satellite in the atmosphere. Newton's laws apply to them and we all know how to deal with them i.e. $F = m\ddot{x}$. When we are looking at very tiny objects, however, like electrons or nuclei of atoms, quantum physics describes the behaviour of the objects. We then have to use Schrödinger's equation to derive the properties of the object and the equation of motion. An interesting question arises when semi small or semi big objects are investigated, for then both quantum and classical behaviour is in play. Examples are quantum systems where some parts are describes classically and others quantum mechanically. This case is known as the semi classical limit. The Schrödinger equation is then often hard to solve or even unsolvable explicitly as one can imagine.

Many mathematicians and physicists therefore have tried to find approximations or asymptotic expressions for the solution of this equation and they have succeeded. Already in 1924 three mathematicians developed the now called Wentzel–Kramers–Brillouin method, or briefly the WKB-method. This method recasts the wave function as an exponential function and expands the equation in terms of Plancks contstant. This approximation contained a constant known as the Morse-index. The result is a series. Then in 1965 the Russian V.P. Maslov gave a rigorous treatment of multidimensional asymptotic methods in the large where certain integers appeared. These integers are know to us now as the Maslov-index and it can be assigned to a path in a set containing certain vector spaces. It turned out that this Morse-index was a special case of the Maslov-index. The Maslov-index thus has its origins in the semi classical mechanics. Since this index has something to do with symplectic geometry, it is definitely worth looking into it thoroughly and to give a precise definition of it.

Therefore the aim of this thesis is to show how the Maslov index is precisely defined mathematically. In order to do this, first some background on symplectic geometry will be given. The notion of a symplectic vector spaces and of a symplectic subspace is treated followed by the definition of the Langrangian Grassmanian manifold. Then we will see that the Maslov-index can be defined in two ways. It can either be defined quite straightforwardly or in a more algebraic way. Once the computation of the Maslov-index is well understood, the index will be computed for some applications.

2 Preliminaries

2.1 Symplectic vector spaces

In this section some preliminaries on symplectic geometry will be given in order to be able to understand the Maslov-index later on. Symplectic geometry is the branch of mathematics which deals with symplectic vector spaces and symplectic

functions. To give the definition of a symplectic vector space, we first need the notion of a symplectic form.

Let E be a finite-dimentional vector space over a field k, then a symplectic form on E is a non-degenerate antisymetric bilinear form on E. That is a mapping $\sigma : E \times E \to k$ such that for all $u \in E, v \mapsto \sigma(u, v) : E \to k$ is a linear form and for every choice of $v \in E, \sigma(u, v)$ depends linearly on u. Antisymmetric means that:

$$\sigma(u, v) = -\sigma(v, u), \qquad u, v \in E$$

By nondegenerate we mean that if $\sigma(u, v) = 0$ for every $v \in E$ implies that $u = 0$. This same σ also induces a linear mapping to the dual space of E by taking the map $\sigma^* : E \to E^*, \quad \sigma^*_{e_1}(e_2) := \sigma(e_1, e_2)$. Here we fixed e_1 in σ^* and let it operate on e_2. It follows from the nondegeneracy of σ and the fact that it is a linear function that $\sigma^* : E \to E^*$ is a bijection. Hence we can conclude that dim(E)=dim(E^*).
Eqquiped with this, we can define a symplectic vector space.

Definition 1. *Let E be a finite-dimensional vector space k equipped with a symplectic form. Then (E, ω) forms a symplectic vector space.*

A linear mapping f between two symplectic vector spaces (E, ω) and (F, ρ) is called symplectic when $f^*\rho = \omega$. The set of all symplectic mappings that map a symplectic vector space onto itself forms with a composition as group operation the group $Sp(E, \omega)$.

Proof. Since f is symplectic, f is an element of the general linear group $GL(E)$ and hence there exists an inverse f^{-1}. Since f is also a linear endomorphism, we have $(f^{-1})^* = (f^*)^{-1}$. Also $(f^{-1})^*\omega = (f^*)^{-1}\omega = (f^*)^{-1}(f^*\omega) = \omega$. Hence f^{-1} is also symplectic.
For symplectic f, g the following holds: $(f \circ g)^*(\omega) = g^* \circ f^*(\omega) = g^*\omega = \omega$.
Finally we know that this set is nonempty since the identity mapping is in it. □

From now on we will assume that E is a finite dimensional vector space over $\mathbb{R}$.

2.2 Lagrangian subspaces

As will become clear in the next chapter, it is important to have a close look at symplectic subspaces of symplectic vector spaces. Since there is an isomorphism between the dual space and the symplectic vector space, it is useful to have a look at the dual space and its subspaces as will become clear soon. To say something about them, there first follow some definitions:

Definition 2. *Let E be a vector space and E^* its dual space. Let $F \subseteq E$ and $A \subseteq E^*$ be linear subspaces. Then:*

- *The orthogonal complement or annihilator F^0 of F in E^* is the set*

$$F^0 = \{\alpha \in E^* | \alpha(v) = 0 \quad \forall v \in F\}$$

- *The orthogonal complement or annihilator A^0 of A in E is the set*

$$A^0 = \{v \in E | \alpha(v) = 0 \quad \forall \alpha \in A\}$$

Clearly dim (F^0)=dim(E^*)-dim(F)=dim(E)-dim(F)=the codimension of F in E and dim(A^0)=dim(E)-dim(A). If F is a linear subspace of E, then obviously $F \subset (F^0)^0$ and since

$$\dim((F^0)^0)=\dim(E)\text{-}\dim(F^0)=\dim(E)\text{-}(\dim(E)\text{-}\dim(F))=\dim(F)$$

we conclude that $F = (F^0)^0$. Analogously it follows that $A = (A^0)^0$. Furthermore, if F and M are linear subspaces of E, then obviously $F \subset M$ implies $M^0 \subset F^0$ and therefore in general $F^0 \subset (M \cap F)^0$ and $M^0 \subset (M \cap F)^0$. This implies that $F^0 + M^0 \subset (M \cap F)^0$ and similarly $F^0 + M^0 \supset (M \cap F)^0$.

If we now take the orthogonal complements these inclusions imply that

$$F \cap M = ((F \cap M)^0)^0 \subset (F^0 + M^0)^0 \subset (F^0)^0 \cap (M^0)^0 = F \cap M$$

Hence we can conclude that both inclusions are equalities and therefore $(F \cap M)^0 = F^0 + M^0$ and also $(F + M)^0 = F^0 \cap M^0$.

Equipped with this information we are ready to have a look at the orthogonal complement of a bilinear form and the symplectic vector space.

Definition 3. *Let (E, ω) be a symplectic vectorspace and let $F \subseteq E$ be a subspace.*

- *The ω-orthogonal complement F^ω of F in E is the set*

$$F^\omega := (\omega(F))^0 = \{u \in E | \omega(u, v) = 0 \quad \forall v \in F\}$$

- *F is called isotropic if $F \subseteq F^\omega$, Lagrangian if $F = F^\omega$*
- *F is called symplectic if $F^\omega \cap F = \{0\}$*

Note that F^ω is a subspace of E and that if ω is a symmetric or antisymmetric form that $F^\omega = F^{\omega^*}$, since we can then interchange u and v in definition 3. To make this more clear, both sets are the $u \in E$, such that $\omega(u,v) = 0$ or $\omega_u(v) = 0\ \forall v \in F$ respectively. If ω is symmetric or antisymmetric then we have $\omega_u(v) = \omega_v(u)$. Hence, these u's $\in F^\omega$ are exactly the same u's as in F^{ω^*}.

If we combine this with the fact that ω is a linear isomorphism and the rules for annihilators in the dual spaces, we have proven that if F is a Lagrangian subspace the following statement holds:

$$E = F \oplus F^\omega \tag{1}$$

From this it follows that F is a Lagrangian subspace if and only if its dimension is $\frac{1}{2}$dim(E) and F is isotropic.

Proof. If dim(F)=$\frac{1}{2}$dim(E) and $F \subset F^\omega$, then

$$\begin{aligned} dim(F) + dim(F^\omega) &= dim(E) \\ \frac{1}{2}dim(E) + dim(F^\omega) &= dim(E) \\ \rightarrow dim(F^\omega) &= \frac{1}{2}dim(E) \end{aligned}$$

We assumed that $F \subset F^\omega$. This combined with the fact that their dimensions are the same proves that F is Lagrangian.

Now if F is Lagrangian then

$$\begin{aligned} dim(F) + dim(F^\omega) &= dim(E) \\ dim(F) + dim(F) &= dim(E) \\ 2dim(F) &= dim(E) \end{aligned}$$

□

3 The Maslov index

3.1 The Lagrangian Grassmanian manifold and the Maslov index

Now that we have a deeper understanding of symplectic vector spaces and its subspaces, we are very close to our main goal: the Maslov index. Still we need a little more knowledge of vector spaces and subspaces. Therefore the following definition is introduced:

Definition 4. *Let E be a vector space over $\mathbb{R}$ with $m := dim(E) < \infty$*

- *The Grassmannian manifold of E is defined as follows:*

  $$Gr(E,n) := \{u \subseteq E | u \text{ is a } n\text{-dimensional subspace}\}$$

  Furthermore Gr(m,n):=$Gr(\mathbb{R}^m, n)$

- *Let (E, ω) be a symplectic vector space with $m := dim(E) < \infty$, then the Lagrangian-Grassmannian manifold of E is defined as follows:*

  $$\Lambda(E, \omega) := \{u \subseteq E | u \text{ is a Lagrangian subspace}\}$$

  Furthermore $\Lambda(m) := \Lambda(\mathbb{R}^{2m}, \omega)$

Since all vector spaces over $\mathbb{R}$ of fixed dimension are isomorphic to each other, it is sufficient to only investigate $Gr(m, n)$ and all $2m$-dimensional symplectic subspaces are analogously isomorphic to each other, hence we also only need to investigate $\Lambda(m)$.

Theorem 3.1. *Dim(*$\Lambda(n)) = \frac{1}{2}n(n+1)$

Proof. The Langrangian subspaces $u \subset \mathbb{R}^{2n}$ are n-dimensional subspaces with the property that $u = u^{\omega}$ and hence it is a compact submanifold of $Gr(2n, n)$. Let u_s be the orthogonal complement of u and let A be a linear map between u and u_s. Then the Graph(A) is precisely a Lagrangian subspace if

$$\omega(x + A(X), y + A(y)) = 0 \qquad (x, y \in u) \qquad (1)$$

Since both u and u_s are Lagrangian subspaces, we have

$$\omega(x, y) = 0 \quad \text{and} \quad \omega(A(x), A(y)) = 0$$

Because of (1) we now have $\omega(A(x), y) = \omega(A(y), x)$ and hence a symmetric bilinear form that maps $u \times u \to \mathbb{R} \qquad (x, y) \mapsto \omega(A(x), y)$. Since dim($u$)=$n$, we get a $\frac{1}{2}n(n+1)$-dimensional space of such an A $\in Lin(u, u_s)$.

□

We are now equipped to get to know the Maslov-index. However, we still need the next definition about the Maslov Arnol'd mapping.

Definition 5. *The Maslov Arnol'd mapping a the mapping from the Lagrangian Grassmanian manifold to the unit circle*

$$MA_m : \Lambda(m) \to \mathbb{S}^1 \qquad (m \in \mathbb{N})$$

We not yet have an explicit form of this function, but later on we will define this map explicitly.

Definition 6. *Let $c : \mathbb{S}^1 \to \Lambda(m)$ be a differentiable mapping. The the Maslov index μ is defined as the degree of a function:*

$$MA_M \circ c : \mathbb{S}^1 \to \mathbb{S}^1$$

Where the degree of a function $f \in C^1(\mathbb{S}^1, \mathbb{S}^1)$ is

$$deg(f) = \int_{\mathbb{S}^1} \frac{d}{dz} log(f(z)) \frac{dz}{2\pi i}$$

3.2 The Maslov-index in $\Lambda(m)$ as a homogeneous space

That we now defined the Maslov-index as the degree of a certain function, and the degree of a function as a certain integral, doesn't mean, however that we always have to compute the Maslov-index directly by integrating. On the contrary, this would be impossible in most cases. To see how we can compute the Maslov-index in another way, we first need to explore some properties of the Lagrangian Grassmanian manifold. Namely that it is identifiable with the quotient group $U(m)/O(m)$ and that there is an explicit formula for the MA_m mapping which is well defined.

Theorem 3.2. *$\forall m \in \mathbb{N}$ is the map*

$$U(m)/O(m) \to \Lambda(m), \quad UO(m) \mapsto \left\{ \begin{pmatrix} Re(UO)x \\ Im(UO)x \end{pmatrix} | O(m), x \in \mathbb{R}^m \right\}$$

is a homomorphism between the homogeneous space U(m)/O(m) and the Lagrangian Grassmanian manifold $\Lambda(m)$.

Proof. For a given $V \in U(m)$ the set $\left\{ \begin{pmatrix} Re(V)x \\ Im(V)x \end{pmatrix} | x \in \mathbb{R}^m \right\}$ a Lagrangian subspace, for all m-dimensional subspaces $L \subset \mathbb{R}^m \times \mathbb{R}^m$ can be interpretated as the image of a injective map

$$\mathbb{R}^m \to \mathbb{R}^m \times \mathbb{R}^m, \qquad x \mapsto \begin{pmatrix} Ax \\ Bx \end{pmatrix}$$

where A,B$\in Mat(m, \mathbb{R})$ and the rank$\begin{pmatrix} A \\ B \end{pmatrix}$=m. Now let $\mathbb{J} = \begin{pmatrix} 0 & -\mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}$. Then L is precicely Lagrangian if

$$\left\langle \begin{pmatrix} Ax \\ Bx \end{pmatrix}, \mathbb{J} \begin{pmatrix} Ax \\ Bx \end{pmatrix} \right\rangle = 0 \quad (x, y) \in \mathbb{R}^m$$

which means that $A^T B = B^T A$. The condition that the image of the basis vectors $e_1, e_2, ..., e_m \in \mathbb{R}^m$ give a orthonormal basis of L, means the same as $\begin{pmatrix} A^T & B^T \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = A^T A + B^T B = \mathbb{I}$. Both conditions are fulfilled when $V := A + iB$ is unitairy, for then

$$V^* V = (A^T - iB^T)(A + iB) = (A^T A + B^T B) + i(A^T B - B^T A) = \mathbb{I}$$

We could however, have used any basis by multiplying the base vectors with an orthogonal matrix $O(m)$. Since the bijection of $U(m)/O(m) \to \Lambda(m)$ is continous and $U(m)/O(m)$ is compact, this is a mapping is a homomorphism. $\square$

What this theorem in short says is that if $Z = \begin{pmatrix} X \\ Y \end{pmatrix}$ and $\tilde{Z} = X + iY$, with $X, Y \in Mat(m, \mathbb{R})$ is a unitary matrix. Then we have $u = Zx$ with $x \in \mathbb{R}^{2n}$ for $u \in \Lambda(m)$. With this information, we can establish a mapping from the Lagrangian Grassmanian to the unit circle.

Theorem 3.3. *The map $MA_m : \Lambda(m) \to \mathbb{S}^1$, $UO(m) \mapsto det(UO(m))^2$ is well defined.*

Proof. Independent of $O \in O(m)$, the following holds:

$$\left[det(UO)\right]^2 = \left[det(U)\right]^2 \left[det(O)\right]^2 = \left[det(U)\right]^2$$

□

If we combine the last two theorems, we can look at curves in the Lagrangian Gassmanian in another way. Every point on a curve in $\Lambda(m)$ can now be identified with an element of the quotient group $U(m)/O(m)$. This implies that if we again would have $X, Y \in Mat(m, \mathbb{R})$ and $\tilde{Z} = X+iY$ unitary, we can look at $c(z)$ as a function that maps $\mathbb{S}^1$ to $U(m)/O(m)$ by defining $c(z) := X(z)+iY(z)$. The Maslov-index is then defined as degree($MA_M \circ c$)=degree($\det(X(z) + iY(z))^2$).

With this in mind, we could also calculate the Maslov-index of a curve in the Lagrangian Grassmanian manifold by adding up the amount of intersections with a specific submanifold. If namely we single out the sets $\Lambda^k(m)$, which are the sets that consists of the Lagrangian planes that have a k-dimensional intersection with a fixed plane $v \in \Lambda(m)$, every curve on the Lagrange Grassmanian will intersect the closure of the set of subspaces with a $1-$ or higher-dimensional intersection. Let's take as fixed plane the following plane:

$$v := \mathbb{R}^m_p \times \{0\} \subset \mathbb{R}^m_p \times \mathbb{R}^m_q$$

Then we can decompose the manifold as follows:

$$\Lambda(m) = \bigcup_{k=0}^{m} \Lambda_k(m) \quad with \quad \Lambda_k(m) := \{u \in \Lambda(m) | dim(u \cap v) = k\}$$

Since all Lagrangian subspaces that are transversal to v can be represented as graph of $\{(Aq, q) \in \mathbb{R}^m_p \times \mathbb{R}^m_q\}$ where A is a symmetric $m \times m$ Matrix, $\Lambda_0(m)$ is diffeomorph to the vector space $Sym(m, \mathbb{R})$.

To make this more clear I will explain why this is true. Let's start by proving that there is a symplectic map between every two Lagrangian subspace. If $u \in \Lambda(m)$ is a Lagrangian subspace and $\Psi \in Sp(E, \omega_0)$ is a symplectomorphism, then obviously $\Psi u \in \Lambda(m)$. If we now fix a Lagrangian subspace, for example our vertical subspace then we can map it to any other Lagrangian subspace. To

do this, represent the vertical space in its unitary representation $v = 0 + iY$, then we can multiply it by a specific Ψ to get to any other namely:

$$\Psi = \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \qquad \Psi v = \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \begin{pmatrix} 0 \\ Y \end{pmatrix} = \begin{pmatrix} BY \\ AY \end{pmatrix}$$

So by this multiplication we can get to any other Lagrangian subspace by choosing the right A and B. This proves the statement.

We already now that if $u \in E$ is a (Lagrangian) subspace, that $E = u \oplus u^{\perp}$, or in our case $\mathbb{R}^{2n} = u^{\perp}$. If we consider the standard euclidian metric we see that $u^{\perp} = \mathbb{J}u$

$$\langle a, \mathbb{J}a \rangle = \left\langle \begin{pmatrix} a_1 \\ \vdots \\ a_m \\ a_{m+1} \\ \vdots \\ a_n \end{pmatrix}, \begin{pmatrix} -a_n \\ \vdots \\ -a_{m+1} \\ a_m \\ \vdots \\ a_1 \end{pmatrix} \right\rangle = \frac{1}{2} \sum_{k=1}^{k=n} a_k a_{n-k-1} - a_{n-k-1} a_k = 0 \quad \forall a \in u \tag{2}$$

And hence $\mathbb{J}u = u^{\perp}$. Two subsets of a manifold are transversal if their tangent spaces at the point of intersection seperately generate the tangent space of the manifold. Mathematically this means that $A, B \subset M$ are transversal if $T_pA + T_pB = T_pM$.

Now since the dimension of the Lagrangian subspace is $\frac{1}{2}\dim(\mathbb{R}^{2m})$ and by transversallity, the symplectic map between u and $u^{\perp}$ is bijective and hence isomorphic. We could also have seen this by the fact that $\mathbb{J}u = u^{\perp}$. Hence the Lagrangian subspace are the subspaces that are the Graph of the isomorphisms between u and $u^{\perp}$. From this we can conclude that $\Lambda_0(m) = \{(Ax, x) | x \in v^{\perp}\}$ where $A : v^{\perp} \mapsto v$. This space is only Lagrangian if A is symmetric and thus Λ_0 is diffeomorph to Sym(m,$\mathbb{R}$).

Let's rewrite now our decomposition of the Lagrangian Grassmanian manifold:

$$\Lambda(m) = \Lambda_0(m) \cup \Sigma(m) \qquad \Sigma(m) = \bigcup_{k=1}^{m} \Lambda_k(m) = \overline{\Lambda_1}(m)$$

This $\Sigma(m)$ is also known as the Maslov cycle.

If we now identify $\Lambda_0(m)$ with the symmetric matrices, we can identify every curve $c(z)$ in $\Lambda_0(m)$ with $Graph(A(z))$. This means that every closed curve c : $\mathbb{S}^1 \to \Lambda_0(m)$, where $c(z) = Graph(A(z))$ can be contracted to the constant curve with value $\{0\} \times \mathbb{R}^m_q$ by the homotopy

$$H : \mathbb{S}^1 \times [0, 1] \to \Lambda_0(m) \quad H(z, t) = Graph(tA(z))$$

Since we can contract any curve in $\Lambda_0(m)$ to a constant curve, the Maslov-index of these curves is zero. The contraction of a curve doesn't change the

Maslov-index of it, for the degree of these loops doensn't change under contraction. The numbers of rounds we make on $\mathbb{S}^1$ after transversing our initial $\mathbb{S}^1$ doesn't change if we deform the path continuously.

It can take however all values of $\mathbb{Z}$ as will be shown now. Since $\Lambda(1)$ is isomorphic to the unit circle, the map

$$c : \mathbb{S}^1 \to \Lambda(1), \quad z \mapsto span_{\mathbb{R}}(z^{I/2}) \subset \mathbb{C} \cong \mathbb{R}^2 \quad I \in \mathbb{Z}$$

shows that the Maslov-index can take all values in $\mathbb{Z}$. Here I used the notation $\Lambda(1) = \{span_{\mathbb{R}}(z) | z \in \mathbb{S}^1\}$. Hence with $span_{\mathbb{R}}(z)$ the real vectors from the origin to the points $z^{I/2}$ that are on the unit circle.

If $m > 1$ we can embed $\Lambda(1)$ in $\Lambda(m)$ by the map

$$\Lambda(1) \to \Lambda(m) \quad u \mapsto u \oplus \mathbb{R}^{m-1} \times 0.$$

Hence if a curve in the Lagrangian Grassmanian manifold doesn't have index 0, it will intersect the Maslov cycle. This cycle is a singular hypersurface of codimension one. That this is true we can deduct from the fact that the dimension of $\Lambda_k(m)$:

$$codim(\Lambda_k(m)) := dim(\Lambda(m)) - dim(\Lambda_k(m)) = \frac{k(k+1)}{2}$$

And for k=1, this means that the codemsion of $\Lambda_1(m)$ in $\Lambda(m)$ is one and hence the codemsion of $\Sigma(m)$ is also one. The hypersurface is stratified by $u \cap v$ for $u \in \Lambda(m)$ and every curve that intersects the Maslov cycle, will intersect it only in the highest stratum. These intersections are also transverse.

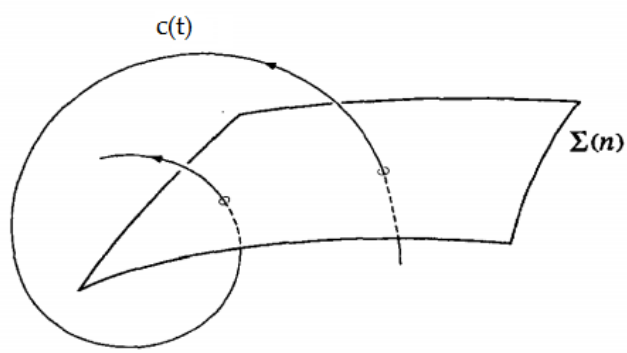

Figure 1: The intersection of a curve with the Maslov Cycle

So, if $c(z) \in \Lambda_1(m)$ and if $c'(z)$ transversal to $\Lambda_1(m)$, then the eigenvalue of the unitary representation of c imaginary and its derivative has a well defined sign, that is, the intersection are all in the mathematical right direction.

We see this by representing the curve in terms of the quotient group $U(m) \setminus O(m)$. The $MA_M \circ c(t)$ would look like $MA_A \circ c(t) := det(U(t))^2 = det(X(t) + iY(t))^2$. If $c(t)$ crosses the Maslov cycle, we see that the $\det(X(t))$ has to be zero and that the eigenvalue of the matrix representing the curve is imaginary. This indeed corresponds to a zero crossing in on the unit circle $\mathbb{S}^1$. Since the hypersurface is of codimension one, we can assign both a negative and a positive direction to a crossing depening to which side $X(t)$ is going. We can do this by the direction to which the eigenvalue of $U(t)$ is going i.e. what the sign of λ' is going. If we then normalize this derivative, we get an integer and the sign is defined as follows:

$$sign(c'(t)) = sign(\frac{\lambda'(t)}{i\lambda(t)})$$

If we sum up these signs, we get the Maslov-index. Or mathematically:

$$\mu(c) = deg(c(t)) = \sum sign(c'(t)) \tag{3}$$

4 Applications

4.1 Hamiltonian systems

Now we know what the Maslov-index is and how to compute it, it would be nice to indeed compute the Maslov-index for some applications. These applications can be found in Hamiltonian systems, which are dynamical systems governed by Hamilton's equations. Hamilton's equations gives the energy of a system and describes how it evolves over time. A Hamiltonian system is defined as follows:

Definition 7. *Let $M \subseteq \mathbb{R}^{2n}$ be an open submanifold, and let $H \in C^2(M, \mathbb{R})$. The system of differential equations:*

$$p' = \frac{\partial H}{\partial q}(p, q)$$
$$q' = -\frac{\partial H}{\partial p}(p, q)$$

or in the coordinates $x \equiv (p_1, ..., p_n, q_1, ..., q_n) \equiv (p, q)$, $\dot{x} = X_H(x)$ with the Hamiltonian vector field $X_H := \mathbb{J}\nabla H$ is what we call a Hamiltonian system.

Since H describes the energy of a given system, the energy should be conserved. This is in any case true and we can prove this simply:

$$\dot{H} = \frac{\partial H}{\partial p}\dot{p} + \frac{\partial H}{\partial q}\dot{q} = \frac{\partial H}{\partial p}\frac{\partial H}{\partial q} + \frac{\partial H}{\partial q}(-\frac{\partial H}{\partial p}) = 0$$

We can equipe this M with a symplectic form $\omega_0 = \langle \cdot, \mathbb{J} \cdot \rangle$ to make it a symplectic vector space. Furthermore, if we would take a level set of the Hamiltonian function and we would be able to parametrise the inverse image of this value by $\mathbb{S}^1$, we would be able to compute the Maslov-index. For if this inverse image could be parametrized by the unit circle, it is a closed curve in M. The tangent space at every point on this closed curve defines a Lagrangian subspace of M. This is because if we would take two elements from the tangent space at a specific point i.e. for $u, v \in T_pM$ with basis $(e_1, ..., e_n)$ and coordinates $c_i, d_i \in \mathbb{R}$:

$$u = \sum_{i=1}^{n} c_i e_i \qquad v = \sum_{i=1}^{n} d_i e_i$$

$$\langle v, \mathbb{J}u \rangle = \langle \sum_{i=1}^{n} c_i e_i, \mathbb{J} \sum_{i=1}^{n} d_i e_i \rangle = \frac{1}{2} \sum_{i=1}^{n} c_i d_{n-(i-1)} e_i \cdot e_{n-(i-1)}$$

But $e_i \cdot e_j = \delta_{ij}$ and hence $\omega_0(v, u) = 0$ for all $u, v \in T_pM$.

So if we have a closed curve in M, which is defined by $H^{-1}(E)$, where E is the energy level, we can induce a path in $\Lambda(m) = \Lambda(M, \omega_0)$. By walking along a curve in M, we cross tangent spaces, which are elements of $\Lambda(m)$ and hence, indirectly, we are walking in $\Lambda(m)$ at the same time. And thus we can compute the Maslov-index.

4.2 The one-dimensional harmonic oscillator

4.2.1 An implicit approach

Let us now focus then on the actually calculating a Maslov-index. We start with the one-dimensional harmonic oscillator, but later on we will extend the problem to the multidimensional harmonic oscillator, such that the Maslov-index could be useful for solving the Schrödinger equation of the electron in a potential well.

The Hamiltonian of the one dimensional harmonic oscillator is given by:

$$H : \mathbb{R}_p \times \mathbb{R}_q \to \mathbb{R} \qquad H(p, q) = \frac{1}{2}(p^2 + q^2)$$

If we take a level set E of the Hamiltonian, it tells us that $H(p, q) = \frac{1}{2}(p^2 + q^2) = E$. Without loss of generality we can assume that $E > 0$, since the energy of a system can't be negative. Hence, the inverse $H^{-1}(E)$ forms a topological circle in M with radius $\sqrt{2E}$.

This is a closed loop in M. If we identify M with $\mathbb{C}$ we can parametrize the solution curve by

$$\tilde{c} : \mathbb{S}^1 \to H^{-1}(E), \quad \tilde{c}(z) := \sqrt{2E} z$$

Here z are the point on the unit circle in the complex plane. We get the tangent space of $H^{-1}(E)$ at a point $\tilde{c}(z)$ by multiplying $\tilde{c}(z)$ with i. The vectors $i\tilde{c}(z)$ span the Lagrangian subspaces we cross if we traverse $\tilde{c}(z)$ and thus we can define a mapping to the Lagrangian Grassmanian manifold by:

$$c : \mathbb{S}^1 \to \Lambda(1), \quad c(z) = span(i\tilde{c}(z))$$

If we want to span the tangent space with the vectors $i\tilde{c}(z)$ the orientation doesn't matter. We can span a specific tangent space with both positive and negative base vectors and thus the tangent space at (p, q) is the same as at $(-p, -q)$.

So this means that if we traverse the unite circle once, we traverse $\tilde{c}(z)$ also once, which means that we 'hit' every tangent space twice. This means that the degree of $MA_M \circ c(z)$ is two.

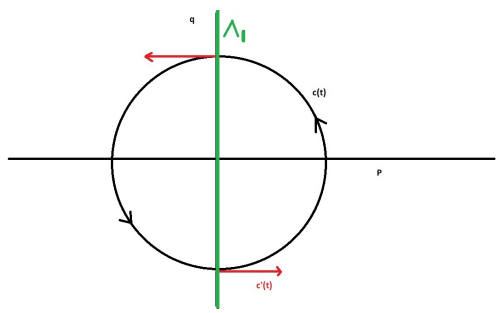

Figure 2: The intersection of a curve with the Maslov Cycle

4.2.2 A direct approach

Another way to compute the Maslov-index is by counting the amount of intersection with the Maslov cycle. For this let's take the reference frame $v := \mathbb{R}_p \times \{0\}$. Our transversal subspaces would be the horizontal Lagrangian subspace $u := \{0\} \times \mathbb{R}_q = \Lambda_0(m)$.

Without loss of generality we can take $E = \frac{1}{2}$ and an explicit function for our curve would be $c(t) = U(t) = X(t) + iY(t) = \cos(t) + i\sin(t)$, where $0 \leq t \leq 2\pi$. If we transverse $c(t)$ once, we see that we have two intersections with $v = \Lambda_1(1) \subset \overline{\Lambda_1}(1)$. The determinant of $X(t)$ is zero twice, namely at $t = \frac{\pi}{2}$ and at $t = \frac{3\pi}{2}$. The eigenvalue λ of this curves is the solution of $det(X(t) - \lambda)$ and hence $\lambda = \cos(t) + i\sin(t)$. Hence $\lambda' = -\sin(t) + i\cos(t)$ and the sign of $c'(t)$ is $\frac{\lambda'}{i\lambda} = 1$. We have two intersections, hence the Maslov-index is according to this approach $sign(c'(\frac{\pi}{2})) + sign(c'(\frac{3\pi}{2})) = 1 + 1 = 2$ as well.

4.3 The two-dimensional harmonic oscillator

If we want to compute the Maslov-index for a solution curve of the two dimensional harmonic oscillator, it is going to be a little more complicated. The formula of the Hamiltonian is now of the form:

$$H = \begin{pmatrix} H_1 \\ H_2 \end{pmatrix}, \qquad H_i = \frac{\omega_i}{2}(p_i^2 + q_i^2) \quad \text{with } \frac{\omega_1}{\omega_2} = \frac{a}{b}, \quad a, b \in \mathbb{Z}$$

We need to compute the index of a closed solution curve $\tilde{c} : \mathbb{S}^1 \to H^{-1}(E) \in \mathbb{R}^4$ with $E = (E_1, E_2)$. Since H_1 and H_2 define two topological circles, the preimage of $H(p, q)$ defines a 2-torus. An explicit form of $\tilde{c}(t)$ can be given by:

$$\tilde{c}(t) = (k_1 \sin(\omega_1(t-t_1)), k_2 \sin(\omega_2(t-t_2)), k_1 \cos(\omega_1(t-t_1)), k_2 \cos(\omega_2(t-t_2))$$

for $k_i^2 = 2\frac{E_i}{\omega_i}$.

Let's now project the two-torus on a square. If we would project the following curves in this square:

$$\{0,\mathrm{k}_2 \sin(\phi), \pm k_1, k_2 \cos(\phi)| \quad \phi \in \big[0, 2\pi\big]\}$$

$$\{\mathrm{k}_1 \sin(\phi), 0, k_1 \cos(\phi), \pm k_2| \quad \phi \in \big[0, 2\pi\big]\}$$

the projection would be vertical in the sense that the are either vertical or horizontal.The tangent space at points of the vertical projections represent elements of $\Lambda(m)$ that are in $\overline{\Lambda_1}(2)$. The tangent planes at point on these curves are spanned by:

$$\{0, k_2 \cos(\phi), 0, -k_2 \sin(\phi)\}$$
$$\{k_1 \cos(\phi), 0, -k_1 \sin(\phi), 0\}$$

These subspaces have one dimension in common with our reference space $v = \mathbb{R}^2_p \times \{0\}$ and are indeed Lagrangian. This means that if our solution curve $\tilde{c}(t)$ intersects with one of these four curves (mind indeed the plus minus signs), we intersect $\overline{\Lambda_1}(2)$ in $\Lambda(2)$. So if we know the amount of intersections with these projections of our solution curve, we know the Maslov-index.

These curves are intersected by $\tilde{c}(z)$ in a period of $T = \frac{2\pi b}{\omega_1} = \frac{2\pi a}{\omega_2}$ a and b times respectively, where $\frac{\omega_1}{\omega_2} = \frac{a}{b}, a, b \in \mathbb{R}$. Hence the total amount of intersection of the curve in $\Lambda(2)$ is $2a + 2b$ times.

In the multidimensional case, the computation works analogously and the Maslov index of an n-dimensional harmonic oscillator only depends on the different frequencies. Again we can construct curves which can be projected vertically. These curves would then represent elements of $\Lambda(m)$ and if we count the amount of intersections with these curves within a period T, we know the Maslov-index. This T is dependant on the frequencies and hence the Maslov-index would be dependant on the frequencies of the harmonic oscillator.

4.4 The simple pendulum

Last but not least we will compute the Maslov index for the simple pendulum. The simple pendulum is the pendulum with one degree of freedom and the phase portrait consists of three different regions. Before discussing these regions, it useful to first discuss the Hamiltonian of the pendulum and it's phase space.

Now lets have a look at the formula for the simple pendulum. The Hamiltonian equation of the pendulum is given by:

$$H(\dot{\theta}, \theta) = \frac{\dot{\theta} m l^2}{2} + \omega_0^2 2 \cos\theta$$

Or if we state it in terms of the momentum and we normalize the length and mass of the pendulum we get:

$$H(p, q) = \frac{p^2}{2} + \omega_0^2 2 \cos(q)$$

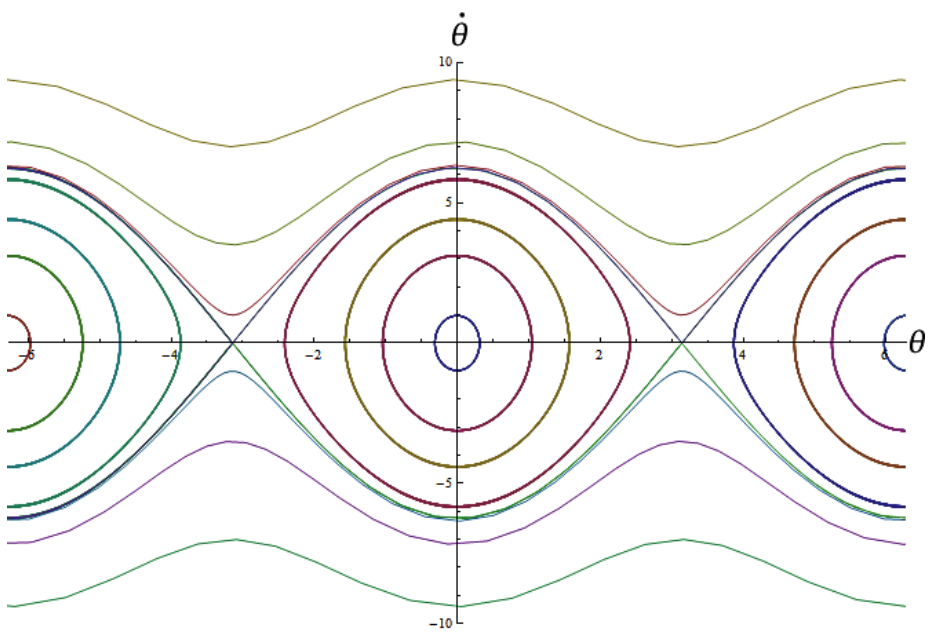

Figure 3: The phase portrait of the simple pendulum

For each energy manifold $\mathcal{H} = E$, the Hamiltonian has a geometric interpretation as a parabolic cylinder $\mathcal{P}_E = \{(x, y, z) | z^2 - 2y = h\}$ with:

$$x = \sin\theta \qquad y = \cos\theta \qquad z = \frac{p}{\omega_0}$$

and

$$h = \frac{E}{\omega_0} - 2 = \text{constant}$$

Besides the constraint $x^2 + y^2 = 1$ makes that the phase space of the simple pendulum is realized by the intersection of parabolic cylinders, given by the different energy levels of the Hamiltonian, with the surface of the cylinder $\mathcal{C} = \{(x, y, z)|x^2 + y^2 = 1\}$.

Typical trajectories on the cylinder can be displayed by means of simple contour plots of the Hamiltonian, as is shown in figure 3. Here we can see that the phase space consists of three regions. Each region represents a different state of the pendulum. The upper and lower region describes the pendulum when it is rotating in one direction. Here we can identify $\theta = -\pi$ with $\theta = \pi$ and since the coordinates are cylindrical, these curves form a topological circle. The middle region represents the pendulum oscillating, and as we can see already in figure 3, these are closed curves diffeomorphic to a circle. The stable point at $\theta = 0$ and $\theta = \pm\pi$ represent the pendulum hanging down of standing up.

Between these previous two regions, there is the separatrix. This curve represents the state of the pendulum where it has just enough energy to swing from a stable position to the other.

One way to parametrise the phase space of the simple pendulum is using the action angle variables. These can be used to get an explicit formula for a solution curve of the pendulum. However, this is not necessary to compute the Maslov-index for this case. One can see namely that the solution curves within the seperatrix are diffeomorphic to a circle. Hence we are in the same situation as we were with the one-dimensional harmonic oscillator. We would have two intersections with $\Sigma(m)$ if we would take $v := \{0\} \times \mathbb{R}_q$ as a reference frame. These intersections would have the same sign and hence the Maslov-index for these curves is hence two as well.

If we look, however to the solutions curve outside the separatrix and the separatrix itself, we can see that it would have no crossing with $\Sigma(m)$ and therefore our Maslov index would be zero.

5 Conclusion

So far we have have had a thorough introduction in the symplectic geometry and we have seen how the Maslov-index is mathematically defined. The identification of the Lagrangian Grassmanian manifold with the quotient group $U(m)/O(m)$ turned out to be a very powerful tool in helping us defining the degree of the closed curves in the manifold. It makes it possible to understand why the Maslov index equals the amount of intersections with $\overline{\Lambda_1}(m)$. We have also computed the index for some applications which could be used in the WKB-approximation. We could however by looking at $\Lambda(m)$ as a homogeneous space compute it for any closed curve.

So far we have only looked at real symplectic vector spaces. It would however be useful to see how the computation exactly is done when the vector space would be defined over an arbitrary field k and how the Lagrangian Grassmaninian manifold is immerged in $\mathbb{C}$. This would have, however, no significant mechanical applications, but mathematically this could be very interesting.

Another interesting question would be whether or not we are able to somehow assign a Maslov-index to curves that are not closed. This is however not within the scope of this thesis, but perhaps an interesting follow up research.

Word of thanks

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