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AdS/CFT Correspondence and Holographic Entanglement Entropy

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1 Introduction

Nowadays, Entanglement Entropy (EE) has become a key concept in many branches of physics such as quantum information or condensed matter physics. The EE is a non-local quantity defined for a subsystem A. We can think about the EE as the entropy that one observer measures when the only accessible part of the system is A , which means that we cannot access to the *information* of the rest of the system B. Precisely these situations, where the observer cannot access to a part of the system, are the most common cases in many branches of physics.

Basically, the entanglement entropy of the subsystem A is the Von Neumann's entropy S_A when we trace out the subsystem B. This *"lost information"* and the inaccessibility to B, at first sight, looks very similar to the black hole entropy. As is pointed out in many works (e.g. [19]), this similarity was the historical motivation of introducing the EE in QFT.

In Quantum Field Theories (QFT) or Quantum Many–body Systems it is usual to study the correlation functions of local operators in the theory. However, when we want to characterize some phases and phase transitions that occur in these systems, the correlation functions of local operators become featureless. The EE is expected to be able to characterize such behaviours.

The entanglement entropy is as useful in QFTs as difficult to compute. There are just a few cases where the EE can be computed analytically. One of the most successful approaches to compute the EE come from holographic arguments in the framework of AdS/CFT correspondence, which state a duality between a Conformal Field Theory (CFT) in the boundary and a gravity theory in the bulk. In this framework, it has been proposed by Ryu and Takayanagi [7] a simple formula to compute the EE in the boundary theory through simple calculations in the bulk. This proposal has been able to reproduce a lot of known results in CFTs. Also it has been used as an approach of open problems like, for example, the black hole information problem.

The aim of this work is to introduce the basic concepts needed for the understanding of the Ryu-Takanayagi proposal. We will use it to reproduce some well known results in 2–dimensional CFTs. The organization of this work is as follows: in Section 2 we review the historic context in which the AdS/CFT was developed.

In Section 3 we explain the Entanglement Entropy from the Von Neumann's definition. We explain their implications in quantum field theories, highlighting how difficult it is to compute the entanglement entropy in this framework. Finally, we focus on the case of quantum field theories with conformal symmetries and we write down a few properties of these systems. We give some explicit expressions of the entanglement entropy computed in 2–dimensional conformal field theories which we will obtain in the next sections by holographic arguments.

In Section 4 we review some features of anti de–Sitter spacetimes and we write down the different forms of the metric that we will use during the next sections.

In Section 5 we derive the AdS/CFT correspondence from the geometrization of the renormalization group. At the end of this section, we state the Ryu-Takayanagi proposal for the holographic entanglement entropy and we comment its implications.

In Section 6 we show the proof of the strong subadditivity, one of the most important properties of the entanglement entropy, through simple holographic arguments, pointing out that the proof in the framework of quantum mechanics is highly non–trivial.

In Sections 7 and 8 we perform some explicit computations in the framework AdS_3/CFT_2. In the former we proceed in a static environment and in the latter we discuss the dynamic thermalization of a CFT_2 system.

We draw conclusions in Section 9, summarizing the results obtained and some interesting features about the proposal not mentioned in this work. At the end, we have added an appendix with the main philosophical interpretations on the idea of holography; even though we are moving in a really speculative area we think it could be useful for a better understanding of the conjecture.

2 A brief history of the AdS/CFT correspondence

In the 1960s, more than one hundred of *hadrons* were appearing every day in the resonances at the laboratories of the particle physicists. The situation can be summarized in the famous Fermi's advise to his student:

> *Young man, if I could remember the names of these particles, I would have been a botanist.*

During this period was coined as well the term *particle zoo.* In order to explain the strong nuclear force a lot of models were created and, in this way, the huge amount of recently discovered particles. The race for the *holy grail* was reached by the Quark Model in 1964 and completed and fully "understood" by the establishment of Quantum Chromodynamics decades later, a race full of Nobel prizes. But before the phrase *"Three quarks for Muster Mark"* became part of the Particle Physics History and give them physical reality, there was plenty of theories were proposed, and one of these theories was the origin of the String theory.

During the 1960s, some physicists tried to describe the strong interactions through the S-matrix theory, initially proposed by Heisenberg in 1943. In 1968, using the S-matrix theory and its properties such as unitarity maximal analyticity and a recently-discovered duality (which states that the scattering amplitudes can be decomposed into a set of s-channel or t-channel), Veneziano discovered a simple formula given by a sum of ratios of some specific Euler gamma functions. However, even though the equation worked well (he wanted to describe mesons interactions due to strong force), the physical interpretation of the formula wasn't clear. Working on this idea, later in 1970, Nambu, Nielsen and Susskind came to one conclusion—the physical reality under this formula is a formalism where a relativistic string is propagated. A really important concept was proposed inside this early string theory—the supersymmetry, which was introduced later in some QFTs and nowadays plays a big role and creates enormous expectations. It was in 1973 when quantum chromodynamics struck and people abandoned the theory.

It wasn't until one decade after that the theory was taken into account once more. During the 1980s came the so-called *"First Superstring Revolution"* by the hand of Green and Schwarz, they had found out how to cancel an anomaly in type I superstring theory, obtaining in this way a quantum theory where the coupling with supergravity is included. The superstring theory became then a really active field.

By the earlies 1990s there was five different superstring theories depending of the symmetry of each one. In the mid of this decade, Witten proposed the famous M-theory which states that all these superstring theores plus supergravity were in fact part of a single framework.

The idea of holography started in a place that, since it was discovered, it has been the witness of the birth of many others: the black holes. A hot issue was (and still is) whether the information is destroyed or not in these scenarios. Hawking claimed that it was while 't Hooft opposed to the idea and defended the information conservation. In 1993 t' Hooft published a paper that contained what would be called *the holographic principle* of quantum gravity. Taking the example of the entropy of a black hole, which is proportional to its area, the principle states: "we can represent all that happens inside [the volume] by degrees of freedom on the surface. This suggests that quantum gravity should be described entirely by a topological quantum field theory, in which all physical degrees of freedom can be projected onto the boundary"[26].

The AdS/CFT correspondence was proposed by Maldacena in 1997 [1]. He was studying the states of a stack of N D3-branes, a particle type of solution spatially extended in 3 dimensions such that the Dirichlet boundary condition is imposed on the string, when he discovered there was two equivalent ways of describing it. He discovered that the branes excitations (surface) can be described by a Yang-Mills theory with a gauge group $SU(N)$ while the bulk surrounding the branes, it can be described by a IIB supergravity in $\mathrm{AdS}_5 \times S^5$. Since then, the AdS/CFT framework has become in a really intense area of theoretical physics. This opened the doors of the study of superstring theory non-perturbatively.

Recently, in 2008 Ryu and Takayanagi proposed a way to compute the entanglement entropy in CFT in the AdS/CFT framework. The idea came from the black holes entropy in a similar way that the holographic principle was inspired by 't Hooft. It has become in an important area of study thanks specially to the huge information available in $\mathrm{AdS}_3/\mathrm{CFT}_2$ due to it reproduces a lot of hard-of-computing results in CFT_2 through simply calculations in classical AdS_3 gravity.

3 Entanglement entropy

In this section we will define and review the basic properties of Entanglement Entropy (EE). Roughly, the EE measures *how much* are quantum systems entangled.

Depending on the context or the frame we are using the EE, we can find different interpretations (not selective), e.g. in Quantum Information Theory it can be interpreted as how much quantum information is stored in a quantum state. Through the next sections we will define and see how we can interpret the EE.

3.1 Von Neumann Entropy

Suppose a quantum system defined by one of a number of states $|\psi_i\rangle$ with probability p_i in a Hilbert space $\mathcal{H}$, we define the density matrix as

$$\rho \equiv \sum_i p_i |\psi_i\rangle\langle\psi_i| \tag{3.1}$$

We can use ρ to describe completely the properties of the quantum system defining the postulates of QM in terms of it (see for example [2]). This formulation is useful for many purposes and simplify a lot of calculations in situations where we are dealing with mixed states. The Von Neumann Entropy is defined by the formula:

$$S(\rho) \equiv -\mathrm{tr}(\rho \log \rho) \tag{3.2}$$

Where ρ is the density matrix of the quantum system as defined above and the logarithm is taken in base 2. If λ_x are the eigenvalues of ρ we can obtain the completely equivalent relation:

$$S(\rho) \equiv -\sum_x \lambda_x \log \lambda_x \tag{3.3}$$

Which is more useful for calculations involving finite dimensional systems. We have to define $0 \log 0 \equiv 0$ for consistency. We can understand this intuitively comparing it with the *Shannon Entropy* which has exactly the same expression than (3.3) but λ_x now represent probabilities (for a deep discussion see [2]). In this frame, we precisely argue that an event with zero probabilities cannot contribute to the total entropy.

Properties of ρ and EE

The Von Neumann entropy and the density matrix have plenty of well studied properties. Here we are going to point out some of them which are going to be useful for our purposes.

- Equality between the trace and the inner product $\mathrm{tr}(|b_1\rangle\langle b_2|) = \langle b_2|b_1\rangle$
- **Trace condition**. ρ has a trace equal to one.

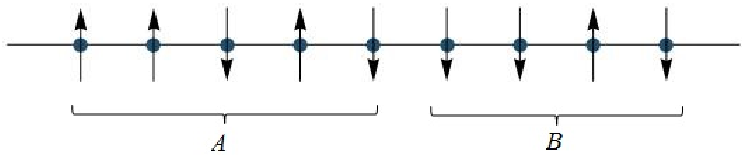

Figure 1: 1–dimensional spin chain divided into two subsystems A and B. The "cut" does not change the system and the choice of the length of A and B is completely arbitrary for our description.

- The entropy is **zero** if and only if the system is in a pure state and **non-zero** otherwise.
- In a d-dimensional Hilbert space the entropy is at most $\log d$
- Entanglement entropy satisfies the inequalities called *Subadditivity* and *Strong subadditivity* which we will review in the next sections.

For further information and proofs [2] can be consulted.

Subsystems

We can decompose our system defined by ρ into two subsystems A and B, so we can write the Hilbert space of the total system as the direct product of the Hilbert spaces of the two subsystems $\mathcal{H} \equiv \mathcal{H}_A \otimes \mathcal{H}_B$. We have illustrated this in Figure 1. The system is formed by a spin chain, which we cut at some point into two different systems. Notice that this procedure doesn't change the system, it's just an *artificial* cut. This is useful, for example, if at some laboratory the experimentalist has only access to one of the subsystems. We can define the density matrix for each one of the subsystems by:

$$\rho_A \equiv \mathrm{tr}_B \rho \tag{3.4}$$

where tr_B is the partial trace of ρ over the subsystem B. Perhaps this is the most useful concept in the density operator formalism of QM and, in a lot of cases, indispensable in the analysis of huge range of quantum systems.

Maybe is not intuitive how trace out the subsystem B to the total system can give us the density matrix which describe the subsystem A. In order to illustrate this, we suppose a quantum system in the product state $\rho = \rho_A \otimes \rho_B$. So we get

$$\rho_A \equiv \mathrm{tr}_B(\rho_A \otimes \rho_B) = \rho_A \mathrm{tr}(\rho_B) = \rho_A$$

where we have used that the trace of a density operators is equal to one. We now have all the ingredients that we need for define the **entanglement entropy** which is *simply* the Von Neumann entropy of the subsystem A.

$$S_A(\rho_A) = -\mathrm{tr}(\rho_A \log \rho_A) \tag{3.5}$$

Bell state

We are going to compute the entanglement entropy of the *Bell state* in order to illustrate better the previous section. Let $|\psi\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$ where $|00\rangle$ actually means $|0\rangle_1 \otimes |0\rangle_2$, omitted for simplicity, and the subindices of the vectors denoting the Hilbert space they are define in. Clearly, the dimension of the Hilbert space is four due to the system is composed by two entangled qubits. Computing the equation (3.1) we get:

$$\rho = |\psi\rangle\langle\psi| = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

We find that the eigenvalues are degenerate and the possible values are $\lambda = 0, 1$. We see immediately from the definition of Von Neumann entropy (3.2) that $S(\rho) = 0$. This confirms that the system is in a pure state. The state of a two entangled qubits it is known *exactly* and it is determined by $|\psi\rangle$. Historically, this result was quite remarkable in the development of QM because it means that two entangled qubits are completely determined by $|\psi\rangle$, even when the qubits are spatially separated.

If we want to describe the subsystem A formed by one of the qubits we have to compute the reduced density matrix ρ_A. Notice that the Hilbert space describing one qubit has dimension 2, so we expect that the reduced density matrix will be a 2×2 matrix. Following the definition (3.4) we get:

$$\rho_A \equiv \mathrm{tr}_B \rho = \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Computing the entanglement entropy as defined in (3.5) we get

$$S(\rho_A) = \log 2$$

We obtain important conclusions from this result. First of all, notice that while ρ describes a pure state, the subsystem A is a mixed state. We can interpret this in two different but equivalent ways. Firstly, we can say that it measures *how* closely entangled the qubit A respect to the qubit B and secondly, *how much* information about the system we miss if we only have access to A.

3.2 Entanglement Entropy in QFT

To define the *entanglement entropy* in Quantum Field Theories is comparatively harder than in Quantum Mechanics. We can think of QFT as a infinitely direct product over all points in space where each point have a finite number of degrees of freedom. All that we are meaning is that in QFT we are dealing with *fields*

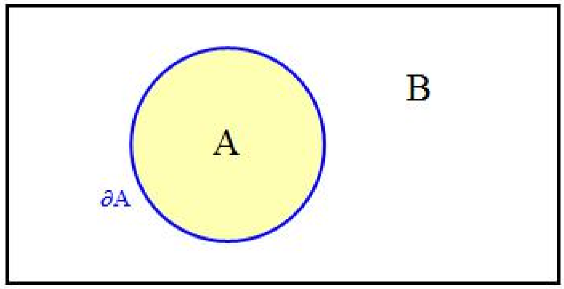

Figure 2: QFT divided into two subsystems A and B. The shape of A has been arbitrarily chosen. ∂A is the boundary of the subsystem A as well as the boundary of the subsystem B as can be appreciated. Thus, since B is just the complementary of A $\partial A = \partial B$.

rather than single particles. We describe a given d-dimensional QFT by a $\mathbb{R} \times N$ manifold where N represent the $(d-1)$ dimensions of the space-like manifold. It is convenient to point out this distinction between the space and time dimensions due to the fact we will define the entanglement entropy at a fixed t. We can understand the EE in QFT as the correlation between field operators at spatially separated points.

Following the same strategy as in the previous section, the Hilbert space of a given QFT is split up into two subsystems A and B at a fixed time t_0 as it's shown in Figure 2. Notice that both subsystems A and B are contained in the manifold N. We see clearly from the Figure 2 that it's the boundary ∂A which divide the system, that's why sometimes this is called the *geometric entropy*. After this, all we have to do is apply the equation (3.5) and we'll get the EE for the subsystem A.

Two-point function at a ground state in QFT

Let's take one of the simplest QFT possibles, a scalar massive field $\phi(t, x)$, and compute the two-point function (or how correlated are two points) at vacuum. For small distances $|x - y| \ll m^{-1}$, the correlation function scales like

$$\langle 0|\phi(0, x)\phi(0, y)|0\rangle \propto \frac{1}{|x - y|^2}$$

Even though this result comes from the massive scalar field the basic structure at short distances is expected in any QFT. For further information and a more detailed development for the above result see [3]. This result shows that at short distances the two-point correlation function is divergent. If we apply this to our description of EE, we obtain that the entanglement entropy of A is certainly divergent. In order to fix this we have to introduce an UV cut-off a, which means we are defining a lattice spacing in the theory.

Area law and BH entropy

Intuitively, we can see how the boundary ∂A is going to be the leading term. Since in the EE of A and B, as we has seen, the closest points are the most correlated they are going to contribute the most to S_A and these points are the ones that forms the boundary ∂A. This has been called the *Area law*

$$S_A = \gamma \frac{Area(\partial A)}{a^{d-1}} + \mathcal{O}(a^{-(d-2)}) \tag{3.6}$$

where γ depends on the system. Even though this law has been directly proven for free field theories there's no direct proof about interacting field theories.

However, holographic calculations implies that this law has to be true for any QFT with UV cut-off [7]. It is an important exception the case of a 2-dimensional QFT, which follows a logarithmic law, as we will see.

As a remark, in the context of this work this is a result of great importance. If we look at the Bekenstein-Hawking entropy of the black holes it reads

$$S_{BH} = \frac{\text{Area of horizon}}{4G_N} \tag{3.7}$$

where G_N is the Newton's gravitational constant. The similarities with the equation (3.6) are obvious and, indeed, this was the original motivation for the proposal of holographic entanglement entropy. For further information and references, see [5].

3.3 Entanglement Entropy in CFT

The Conformal Field Theories are QFTs whose symmetry group contains *local conformal transformations*. Even though the requirement of this symmetry is very restrictive (*e.g.* the particles have to be massless), the CFTs are widely used beyond the topic of this work. The reasons behind the study of this theories (concretely the 2-dimensional CFT) at the beginning were two: it was possible to solve non-perturbatively some of this theories and, at sufficient high energies, any (2-dimensional) QFT is approximately massless. See for example [6] or the discussion in *Peskin and Schroeder* [4].

We can state that a conformal transformation is a change of coordinates $x'^{\mu} \to x^{\mu}$ such that the metric changes in this way

$$g'_{\mu\nu}(x) \to \Omega^2(x^{\mu}) g_{\mu\nu}(x^{\mu}) \tag{3.8}$$

One of the symmetries that conformal invariance implies is the scale invariance. The scale invariance states that the systems stay invariant under the rescaling of coordinates such that

$$x'^{\mu} \to \lambda x^{\mu} \tag{3.9}$$

We can understand this symmetry as describing a theory which does not care about the lengths; all that matters in the CFTs are the angles. The bigger symmetry that CFTs possess is translated (in 2-dimensional CFTs) as an infinitely number of conserved currents.

The entanglement entropy of a CFT divided into two subsystems has been done analytically as we can see, for example, in [5]. It is not the main purpose of this work to compute the entanglement entropy through QFT methods, so we are just going to show some results and point out some useful properties.

1. The entanglement entropy of a 2D CFT, assuming that the total system is infinitely long at $T = 0$, is given by

$$S_A = \frac{c}{3} \log \frac{l}{a} \tag{3.10}$$

where l is the length of the subsystem A and c is the central charge.

2. For a 2D CFT in the same conditions than above, but with a different temperature, the total entropy is given by

$$S_A = \frac{c}{3} \log \left(\frac{\beta}{\pi a} \sinh \left(\frac{\pi l}{\beta} \right) \right) \tag{3.11}$$

where $\beta = T^{-1}$

4 AdS spacetime

In this section we are going to describe the Anti de–Sitter spacetime in the different coordinates that we are going to use through the next sections. Given the Einstein's field equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \tag{4.1}$$

where R is the Ricci scalar curvature, $R_{\mu\nu}$ the Ricci curvature tensor and $T_{\mu\nu}$ the stress–energy tensor. The left–hand side of the equation roughly can be understood as the geometry of the space time: all the terms are described in function of the metric and/or its derivatives. The right–hand side take account on the energy on the spacetime. The AdS spacetime is a maximally symmetric solution of these equations such that the cosmological constant is define negative $\Lambda < 0$ and there is no matter $T_{\mu\nu} = 0$. For an space to be maximally symmetric means that it has the same number of symmetries than a Euclidean space; this can be proven through an "if and only if" relation that the Riemman tensor has to obey. If we solve the remain equation we get the AdS metric.

4.1 The global coordinates

The AdS$_{d+1}$ line element in the so-called global coordinates reads [8]:

$$ds^2 = \frac{1}{\cos^2\left(\frac{\rho}{R}\right)} \left(-dt^2 + d\rho^2 + \sin^2 \left(\frac{\rho}{R} \right) d\Omega^2_{d-1} \right) \tag{4.2}$$

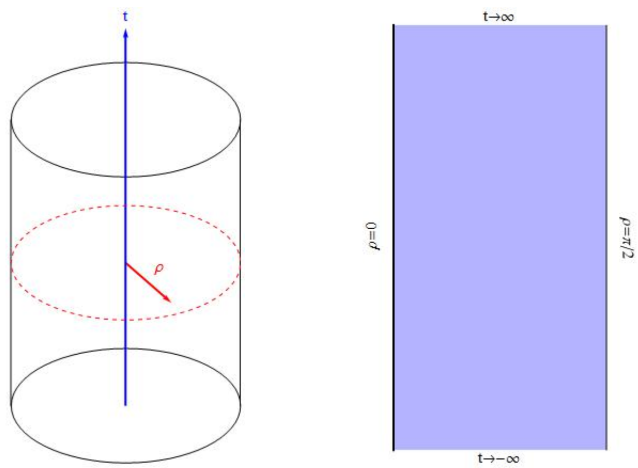

Figure 3: (a) AdS spacetime in global coordinates. We see that the vertical direction is equivalent to the time t and ρ is the radial coordinate from 0 in the centre to $\pi/2$ in the boundary. (b) Penrose diagram for AdS spacetime. The right side is the boundary of the space. The null geodesic form 45 degrees with the vertical lines, as usual. We appreciate that there are no horizons. Thus, every geodesic can access to all points in a finite proper time.

Where R is the curvature of the AdS spacetime and the angular coordinates Ω cover the S^{d-1} sphere. The notation $d+1$ on the dimension of the AdS spacetime comes from the AdS/CFT framework as we will see. We can easily chek in (4.2) that the radial coordinate ρ take the values in the range $\big[0, \frac{\pi}{2}\big)$ and the time coordinate t take the values $\big(-\infty, \infty\big)$. It will be important the fact that the metric diverges at $\rho \to \frac{\pi}{2}$. In this coordinates is easy to make an sketch of the AdS spacetime embedded in a infinite cylinder (Figure 3a).

In order to study the causal structure of the AdS spacetime we use Penrose diagrams. Penrose diagram holds the notion of timelike, spacelike and null geodesics of a given spacetime while we are distorting vastly the geometry to make finite infinite distances. In order to distort the geometry without affecting this causal structure we multiply the whole metric by the Weyl factors, that is $g_{\mu\nu}(x) \to f(x) g_{\mu\nu}(x)$. In our metric (4.2) we can set $R = 1$ and drop the term $\cos^2\big(\rho\big)$ to get

$$ds^2 = -dt^2 + d\rho^2 + \sin^2\big(\rho\big) d\Omega^2_{d-1} \tag{4.3}$$

We can then draw the Penrose diagram shown in Figure 3b, where $\rho \to \frac{\pi}{2}$ is the AdS boundary while t runs from $-\infty$ to ∞. We see then that a masless particle (which follows a null geodesic, that means, a 45^{o} trajectory in Figure 3b) sent from $r = 0$ can reach the AdS boundary and back in a finite proper time. We formalize this property saying that the AdS boundary is *timelike*, with the topology $\mathbb{R} \times \mathbb{S}^{d-1}$ with $\mathbb{R}$ being the temporal dimension. Another property that

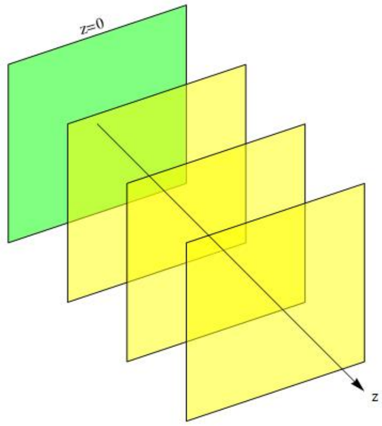

Figure 4: AdS spacetime in Poincaré coordinates. The green square represents the boundary at $z = 0$ which represents just an ordinary Minkowski space in one dimension less than the total spacetime.

we can point out is that AdS has no horizons, which it will be important in the next sections.

4.2 The Poincaré Patch

For our purposes, we can perform a change of coordinates and obtain the so-called Poincaré metric of the AdS spacetime. We define de new variables in terms of (4.2) as

$$\begin{aligned} t &= R\frac{\sin t_g}{\cos t - \Omega_d \sin\rho} \\ z &= R\frac{\cos\rho}{\cos t - \Omega_d \sin\rho} \\ \vec{x}_i &= R\frac{\Omega_i}{\cos t - \Omega_d \sin\rho} \end{aligned} \tag{4.4}$$

where we denoted t_g as the time coordinate in the global coordinates and simply t in this new coordinates. First, we notice that, even if t runs from $-\infty$ to ∞, it just cover a finite range in t_g, which means that the Poincaré coordinates don't cover the entire AdS space (although it can be extended to do it). The z coordinate cover the range $(0, \infty)$ where $z \to 0$ now plays the role of the AdS boundary which, we recall, in the global coordinates was $\rho \to \frac{\pi}{2}$. We rewrite the metric and we obtain the Poincaré coordinates

$$ds^2 = \frac{R^2}{z^2}\Big(- dt^2 + dz^2 + \sum_{i=1}^{d-1} dx_i^2\Big) \tag{4.5}$$

where is clear that $z \to 0$ is the boundary of AdS spacetime as we pointed out above. This coordinates are interesting by the next property: we can view the AdS spacetime as a multiple copies of a Minkowski spacetime along the z direction as is shown in Figure 4 where the green slice is the AdS boundary in the Poincaré coordinates. Hence, each slice represented in Figure 4 is spanned by the coordinates $\left(t, \vec{x}_i\right)$ for a fixed z.

4.3 Asymptotically AdS spaces times and the BTZ metric

Another sort of metrics that are going to be useful for our purposes are those which asymptotically are equivalent to AdS spacetime but different in the interior. One of the definitions is given by [3]. We say that a spacetime is asymptotically AdS if its only boundary is timelike and near of this boundary the metric approaches to the AdS metric. We are interested in this kind of metrics for reasons that we will explain in the next sections. One of the characteristics of the AdS space, as we can have guessed is that there is no horizons. So we are going to looking for metrics which contains horizon and they will be asymptotically AdS. One of this metrics is the Bañados–Teitelboim–Zanelli (BTZ) metric, which was discovered in 1992. This metric is a black hole solution for (4.1) with a negative cosmological constant value. For a non-charged non-rotating 2+1–dimensional black hole, its line element is given by

$$ds^2 = -\left(r^2 - r_+^2\right)d\tau^2 + \frac{R^2}{r^2 - r_+^2}dr^2 + r^2 d\phi^2 \tag{4.6}$$

where r_+ is the black hole horizon. If we make the substitution $r = \frac{1}{z}$ we see that in the limit $r \to \infty$ we recover (4.5). Hence, the boundary of the AdS metric is now $r \to \infty$.

This metric is given, as usual, in its Lorentzian signature, as we can see for the minus sign in the time coordinate. For our purposes, it is convenient to give the metric with an Euclidean signature, which will be $(+,+,+)$ in 3–dimensional spacetime. This can be done through an analytical continuation of the Lorentzian metric [9]. In order to get the Euclidean version of the BTZ in Poincaré coordinates, we make a Wick rotation in the time coordinate such that $\tau \to -it$, and then we get the metric with an Euclidean signature

$$ds^2 = \left(r^2 - r_+^2\right)dt^2 + \frac{R^2}{r^2 - r_+^2}dr^2 + r^2 d\phi^2 \tag{4.7}$$

where t denotes the Euclidean time. The reasons of using this metric is that we can deal with the black hole's thermodynamics easier than in Lorentzian signature. We observe that there is a singularity at r_+, which is called *conical singularity*, but we want the geometry to be smooth here. There is not a *real* singularity in the black hole horizon. To fix this, we are going to follow [10]. Making the transformations

$$t' = r_+ t \qquad r'^2 = \frac{r^2}{r_+^2} - 1 \qquad \phi' = r_+\phi \tag{4.8}$$

the metric reads

$$ds^2 = r'^2 dt'^2 + \frac{R^2}{r'^2+1} dr'^2 + (r'^2+1) d\phi'^2 \tag{4.9}$$

It is clear that the black hole horizon r_+ in the new coordinates is at $r' = 0$. Studying the metric near the horizon and ignoring the angular part, it reads

$$ds^2 = r'^2 dt'^2 + dr'^2 \tag{4.10}$$

where we can identify the new coordinate t' precisely as an angle with period of 2π. Hence, going to our original coordinates, the Euclidean time has to have a period of

$$t \sim t + \frac{2\pi R}{r_+} \tag{4.11}$$

The angle ϕ has a period of 2π as usual. Hence, imposing this conditions, we get a smooth geometry and we have been able to get rid of the conical singularity. A extensive discussion on BTZ black hole solution can be found in the original paper [11].

5 AdS/CFT correspondence and Holographic EE

In this section we will review some characteristics about AdS/CFT correspondence and we will present the Ryu-Takayanagi proposal.The Ryu-Takayanagi proposal was made in 2006 in the paper *Holographic Derivation of Entanglement Entropy from AdS/CFT"* [7] where they proposed how to compute the EE in the AdS/CFT framework.

5.1 The AdS/CFT correspondence

As we said in Section 2, the AdS/CFT is a conjecture that was made by Maldacena in 1997 [1] in the context of string theory. Sometimes the AdS/CFT correspondence is also called *gauge/gravity* duality. Even though the AdS/CFT fundamental mechanism remains unknown, the theoretical achievements make the AdS/CFT a strong candidate toward a quantum gravity theory. We are just going to present the AdS/CFT as a framework where we will work and we won't derive it from string theory.

The correspondence relates the quantum physics of a strongly correlated many-body systems with the physics of a weakly coupled gravity.

This correspondence is used these days in both directions. E.g. conformal field theories are widely used in condensed matter physics where a lot of computations are rather difficult or not possible analytically due to the strong coupling. The correspondence provides a scheme where these computations can be related with a weakly coupled theory where it can be done. On the other way, some conformal field theories have been studied and understood quite well. Those results can help us to understand better some aspects of gravitational theories as can be the black holes paradoxes or the very nature of those objects. In order to sketch the AdS/CFT duality we are going to follow the example of [12] through the normalization group on a QFT.

The renormalization group

The renormalization group is the formalism that allows us to relate the physics on different length scales or, equivalently, physics on different energy scales [13]. The modern approach to renormalization group came out by Kadanoff, Fisher and Wilson. The renormalization group is one of the most important concepts in QFT. As we know, the coupling *constant* (with its confusing name) inform us about the strength of a interaction. We have known for years that this coupling constants depend of the energy scale that we are dealing with. The historically name of *constant*, today just means the value of the coupling at some specific energy levels. The evolution with the energy scale μ of a coupling constant g is given by the renormalization flow equation

$$\mu \frac{dg}{d\mu} = \beta(g) \tag{5.1}$$

We consider a QFT in a lattice with a lattice spacing a and a Hamiltonian given by

$$H = \sum_{x,i} g_i(x,a)\mathcal{O}^i(x) \tag{5.2}$$

where x denotes de position and $\mathcal{O}^i$ the different operators with a determined coupling constant g_i. According to the renormalization group, we can *zoom* in or out in our system, that means, change the energy scale and therefore the lattice spacing a. Hence, our operators $\mathcal{O}^i$ will have a coupling constant with a different value due to (5.1). The key point of the AdS/CFT is to consider μ as an extra dimension, so the lattices at different μ are layers of a new high-dimensional space. The fact that μ represents a new local direction, that means, the theory which describes the bulk is local (as gravity is), is not explained yet. Therefore, its mechanism remains unknown. In addition, we identify g_i as fields with one extra-dimension

$$g_i(x,\mu) = \phi_i(x,\mu) \tag{5.3}$$

This new field ϕ_i is going to be described by some action and, according to AdS/CFT correspondence, this action is going to be described in a gravitational theory, *ergo* will be described by some metric. *Therefore, one can consider the holographic duality as a geometrization of the quantum dynamics encoded by the renormalization group* [12]. So that we can relate the UV cutoff in our QFT with the values of the boundary of the new higher-dimensional space, which later will turn out to be the IR cutoff of the gravitational theory; so we have just related the UV cutoff in a QFT with the IR cutoff in a gravity theory, which remind us the connection between a strong coupled and a weakly coupled theory. Hence, we can state that the QFT lives on the boundary of the higher

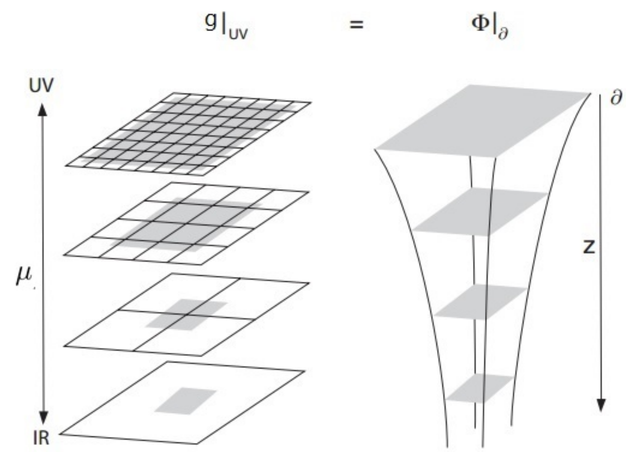

Figure 5: Taken from [12]. Through AdS/CFT correspondence the lattice at different scales are the layers of a high dimensional space

dimensional space as we can see in Figure 5, that's why we usually refer the gravitational theory as *the bulk* and the QFT as the boundary.

Degrees of freedom

For consistency, the fields ϕ_i must have the same tensor structure as the dual operator $\mathcal{O}$. The degrees of freedom have to match in both sides as well. We know that one way to measure the degrees of freedom is through the entropy. The entropy in a QFT is a extensive quantity which depends of the spatial dimensions. For a 4-dimensional QFT, the entropy will be given by the volume of a spatial region at some fixed time t

$$S_{QFT_d} \propto Volume_{d-1} \tag{5.4}$$

Hence, if we accept that the degrees of freedom have to match and, therefore, the *information* in both sides has to be the same, the entropy measured on our high dimensional gravitational theory cannot be proportional to its volume like the given QFT. The answer arrives from the known black hole entropy, given by the Bekenstein-Hawking formula

$$S_{BH} \propto \text{Area of the horizon} \tag{5.5}$$

Which, as we see, in a d-dimensional gravity theory, the entropy is proportional to a $d-2$ surface. Hence, both entropies behave in the same way.

The AdS spacetime and CFTs

However, to find a geometry which fits with a given QFT is rather difficult. Here is where the symmetries that CFTs possess became really important. A CFT is a QFT with a fixed point of (5.1) which provides the theory with the conformal symmetry. So we are going to find a metric with Poincaré invariance and the conformal symmetry. We write down the general expression for a $d+1$-dimensional metric with Poincaré invariance in d dimensions

$$ds^2 = \Omega^2\big(z\big)\big(-dt^2 + dz^2 + d\vec{x}^2\big) \tag{5.6}$$

where the extra dimension is denoted by z. Recalling the scale invariance from conformal symmetry (3.9) and imposing it on the metric (5.6) we got the condition that the function $\Omega\big(z\big)$ must transform as

$$\Omega\big(z\big) \to \lambda^{-1}\Omega\big(z\big) \tag{5.7}$$

which fixes $\Omega\big(z\big) = \frac{R}{z}$ and we get the metric, which we write in the form

$$ds^2 = \frac{R^2}{z^2}\Big(-dt^2 + dz^2 + \sum_{i=1}^{d-1} dx_i^2\Big) \tag{5.8}$$

We get exactly the metric (4.5) and then we have obtained the metric which governs the physics in the bulk. This metric is nothing but the AdS spacetime which, as we have derived, is dual to a CFT in the boundary. However, one difficult question that is still addressed in many works is "which part of the AdS spacetime corresponds to a particular information of the CFT?".

5.2 Ryu-Takayanagi proposal

Let us start by reminding the Bekenstein-Hawking entropy for a black hole

$$S_{BH} = \frac{\text{Area of horizon}}{4G_N} \tag{5.9}$$

This result can be calculated using QFT tools in a curved spacetime in the vicinity of a black hole. It was a remarkable result and surprising since it relates the entropy to the area instead of the volume. This formula was celebrated due to the connection between two concepts: on one side, the entropy in the context of quantum mechanical quantity and on the other side the area, which is purely geometrical data. This equation seems to already encode the idea that we presented in the above section, it relates a microscopical quantity with a geometric (gravitational) one.

Recalling the area law (3.6) and interpreting the entanglement entropy of a subsystem A in a QFT as the entropy of an observer in this subsystem who has not access to the rest of the system, we have all the pieces to state the Ryu-Takayanagi proposal. The similarities between the black hole entropy and the

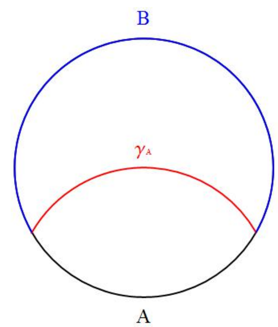

Figure 6: Constant time slice in global coordinates. The QFT formed by A and B lives in the boundary of the cylinder and γ_A is the minimal surface in the bulk.

area law for the QFT are significant and we want to establish a correspondence between this two quantities through the AdS/CFT. But then, as we pointed out in the last section, “which part or quantity of the AdS corresponds to the entanglement entropy of a given subsystem A (with boundary ∂A) in a CFT$_{d+1}$?”. The holographic formula proposed by Ryu and Takayanagi [7] reads

$$S_A = \frac{\text{Area}(\gamma_A)}{4G_N^{d+1}} \tag{5.10}$$

where G_N^{d+1} is the d+1-dimensional Newton's constant and γ_A is the d-dimensional static minimal surface in AdS$_{d+1}$ whose boundary is given by ∂A. In Figure 6 we illustrate this in the global coordinates.

The next physical interpretation is made in [5]. We can consider the entanglement entropy of the subsystem A as the one measured by an observer in A who is not accessible to B, where B is the rest of the system. In the higher dimensional AdS spacetime this would be equivalent to ”hiding” a part of the bulk. If we have some region in the AdS from where we are not accessible to the rest of the space is like we are separated from that region by an horizon γ_A in a similar sense that a black hole horizon doesn't allow to communicate the black hole interior with the exterior. Hence, γ will be covering the subsystem A and then, we expect that $\partial A = \partial\gamma$.

Even though the formula (5.10) has not been directly derived from AdS/CFT, it reproduces a large amount of known results in CFT$_2$ as we will show in the next sections. In addition, it help us to understand phenomena such as phase transitions and properties highly non-trivial of the entanglement entropy such as strong subadditivity.

6 Strong Subadditivity

As we said in the section 3.1, the entanglement entropy has many properties, among them there is one called *Strong Subadditivity*. The strong subadditivity is the most powerful inequality obeyed for EE that has been derived until nowadays for any quantum system so we have to be able to obtained it in the AdS/CFT frame.

Suppose that we have a system A which is divided into two subsystems A_1 and A_2, we get the relation

$$\mid S_{A_1} - S_{A_2} \mid \leq S_A \leq S_{A_1} + S_{A_2} \tag{6.1}$$

It is straightforward to obtain the last inequality from the Von Neumann entropy. The first inequality is called the *Araki-Lieb inequality* which was obtained in the 1970 through a non-trivial derivation using the properties of the density matrix and Von Neumann entropy [14]. If the system A is in a pure state, its total entropy is $S_A = 0$ and $S_{A_1} = S_{A_2}$. It is easy to see that inequalities (6.1) hold.

If we now divide the system into three subsystems A_1, A_2 and A_3 and we would get the *strong subadditivity* inequalities

$$\begin{gathered} S_{A_1+A_2} + S_{A_2+A_3} \geq S_{A_1+A_2+A_3} + S_{A_2} \\ S_{A_1+A_2} + S_{A_2+A_3} \geq S_{A_1} + S_{A_3} \end{gathered} \tag{6.2}$$

This inequalities can be obtained from (6.1) adding one subsystem. If we set the subsystem C to be empty we recover, as we expect, the subadditivity inequalities (6.1). Even though (6.2) seems simple, the proof is highly non-trivial and, despite of we can encounter many simple proofs nowadays [15, 16], it took seven years since it was conjectured to prove it [14]. It can be proof using elementary properties of the Hilbert space and its represents basically the concavity of Von Neumann entropy. This property precisely can be used to characterize the Von Neumann entropy and that's why is so important for us to derive from AdS/CFT.

Holographic proof of SSA

Originally, Takayanagi and Hirata [17], aware that they have to prove that (6.2) holds in the holographic description of the proposal, computed by *brute force* the entanglement entropy through (5.10) and checked that the Strong Subadditivity holds in all the examples. But definitely proof didn't come after one year when they developed a simple geometrical proof [18]. After that, simple proofs, following the same idea, have been developed in the form that we can find in [5, 19]. We are going to present the proof for SSA from (5.10) following the original paper where it was proposed [18] but we will state at the end of the section the proofs that we find in the modern reviews of the topic.

First of all, we rewrite (6.2) using just two different subsystems A and B that overlaps in some region. In this notation the three subsystems are given

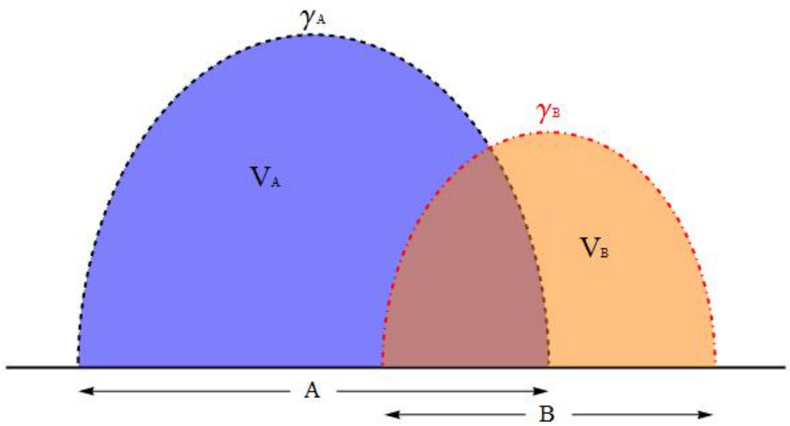

Figure 7: Holographic entanglement entropy in a constant time slice for two subsystems A and B. Following (6.2): A_1 is the boundary of the blue area, A_2 is the boundary of the red area and A_3 is the boundary of the orange area.γ_A and γ_B represents the minimal surface areas that end in ∂A and ∂B respectively.

by the combinations of the new regions A and B such that

$$\begin{aligned} A_1 &= A \cap B^c \\ A_2 &= A \cap B \\ A_3 &= A^c \cap B \end{aligned} \tag{6.3}$$

where the exponent c means that we are taking the complementary of that group. Therefore, we rewrite (6.2) in terms of A and B and we get

$$\begin{aligned} S_A + S_B &\geq S_{A\cap B} + S_{A\cup B} \\ S_A + S_B &\geq S_{A\cap B^c} + S_{B\cap A^c} \end{aligned} \tag{6.4}$$

We illustrate this in Figure 7. In this Figure, the CFT in the boundary is represented by the straight line, divided in the two subsystems A and B.

According to (5.10) this entanglement can be computed through the minimal surface area in the theory in the bulk, which are denoted in Figure 7 by γ_A and γ_B for each of the two subsystems. Of course, γ_A and γ_B end on ∂_A and ∂_B respectively. It is clear that these surfaces and the CFT in the boundary encloses, respectively, the regions in the bulk V_A and V_B so we write the boundary of this regions ∂V_A and ∂V_B as

$$\begin{aligned} \partial V_A &= \gamma_A \cup A \\ \partial V_B &= \gamma_B \cup B \end{aligned} \tag{6.5}$$

By simply examining the Figure 7, we can point out the next equality:

$$\text{Area}(\gamma_A) + \text{Area}(\gamma_B) = \text{Area}(\gamma_{A\cap B}) + \text{Area}(\gamma_{A\cup B}) \tag{6.6}$$

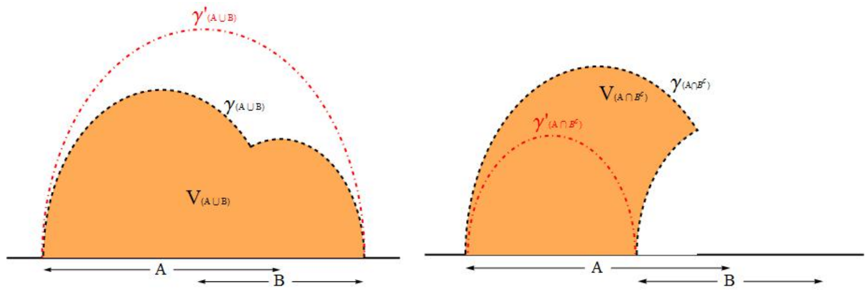

Figure 8: (a) Comparison of $\gamma'_{A\cup B}$ and $\gamma_{A\cup B}$. Following (6.2), the geodesics ends both in $\partial(A_1 \cup A_2 \cup A_3)$. The red geodesic represent is the minimal surface area. (b) Comparison of $\gamma'_{A\cap B^c}$ and $\gamma_{A\cap B^c}$. Both geodesics ends in ∂A_1; the red one is the minimal surface area.

This is obvious simply looking at the regions in the bulk. We see that, if we "sum" both regions V_A and V_B we are counting twice the red region, the region in which intersect. We see that this is equivalent to the first inequality in (6.4) when its saturated. Now, the key point is to realize that, while γ_A and γ_B are the minimal area surfaces that end in ∂A and ∂_B, we don't know if $\gamma_{A\cup B}$ and $\gamma_{A\cap B}$ are minimal surface areas as well. So we define the minimal surface area that ends in $\partial A \cup B$ as $\gamma'_{A\cup B}$ and the the one that ends in $\partial A \cap B$ as $\gamma'_{A\cap B}$. We have schematically represented this in Figure 8.

Hence, if $\gamma_{A\cap B}$ is not the minimal surface, it only can be greater than the minimal surface area by definition. The same argument for $\gamma_{A\cup B}$. So we can write the next inequalities

$$\begin{aligned} \gamma_{A\cap B} &\geq \gamma'_{A\cap B} \\ \gamma_{A\cup B} &\geq \gamma'_{A\cup B} \end{aligned} \tag{6.7}$$

We plug these results into (6.6) and we obtain:

$$\text{Area}\big(\gamma_A\big) + \text{Area}\big(\gamma_B\big) \geq \text{Area}\big(\gamma'_{A\cap B}\big) + \text{Area}\big(\gamma'_{A\cup B}\big) \tag{6.8}$$

Due to the fact we defined all the γ present in (6.8) as the minimal surfaces, we can compute the entanglement entropy through (5.10) and finally we obtain

$$S_A + S_B \geq S_{A\cap B} + S_{A\cup B}$$

Which is precisely the first inequality in (6.4). For the demonstration of the second equation we will follow exactly the same procedure. Notice that we are going to use now the boundary of the blue and the orange region in Figure 7. In the same way than above, we write the next equality

$$\text{Area}\big(\gamma_A\big) + \text{Area}\big(\gamma_B\big) = \text{Area}\big(\gamma_{A\cap B^c}\big) + \text{Area}\big(\gamma_{B\cap A^c}\big) \tag{6.9}$$

In the same way, we don't know if $\gamma_{A\cap B^c}$ and $\gamma_{B\cap A^c}$ are the minimal surfaces that end in $\partial\big(A \cap B^c\big)$ and $\partial\big(B \cap A^c\big)$ respectively. Nevertheless we know that they represent an upper bound for the real ones. Hence, we define $\gamma'_{A\cap B^c}$ and $\gamma'_{B\cap A^c}$ as the minimal surfaces and we write in the similar way than (6.7)

$$\begin{aligned}\gamma_{A\cap B^c} &\geq \gamma'_{A\cap B^c}\\ \gamma_{B\cap A^c} &\geq \gamma'_{B\cap A^c}\end{aligned} \tag{6.10}$$

And plugin these inequalities in (6.9) and using (5.10) we obtain

$$S_A + S_B \geq S_{A\cap B^c} + S_{B\cap A^c}$$

Hence, we have proved (6.2) through simply geometrical arguments *Q.E.D.*.

7 Holographic EE from $\mathrm{AdS}_3/\mathrm{CFT}_2$

In this section we will compute explicitly the EE from $\mathrm{AdS}_3/\mathrm{CFT}_2$ using the Ryu-Takayanagi proposal (5.10). As we said in Section 3, to compute the entanglement entropy in QFT systems is not an easy task, even if we are working with conformal field theories. There are just a few cases where the EE has been obtained analytically; some of those cases has been computed precisely in 2–dimensional CFTs. Thus, the framework of $\mathrm{AdS}_3/\mathrm{CFT}_2$ becomes in perfect lab to test the holographic entanglement entropy. We find that we can reproduce the known results computed in CFT_2 which gives theoretical support to the Ryu-Takayanagi proposal.

7.1 CFT_2 in a pure state

For simplicity, we will use the Poincaré coordinates to make this computation. Recalling the Poincaré coordinates (4.5), in the 3–dimensional AdS:

$$ds^2 = \frac{R^2}{z^2}\Big(- dt^2 + dz^2 + dx^2\Big) \tag{7.1}$$

Where we can think in z as the additional dimension. We represented this in Figure 9, where the "x axis" is the boundary, where the CFT_2 lives. It is clear that the total entropy of the system is zero since it is in a pure state. We select the segment $\left(\frac{-l}{2}, \frac{l}{2}\right)$ in the CFT and we compute the entanglement entropy of the subsystem. According to the holographic formula, this entropy will be simply the length of the geodesic which ends in those two points in time–constant slice in AdS_3. So all we have to do is to compute the geodesics in (7.1) for a constant time.

However, as we pointed out, the Poincaré metric diverges at $z = 0$, so the length of the geodesic that connect two points in the boundary will diverge. To fix this we introduce the cutoff a as we can see in Figure 9. We can point out some properties that the geodesic will have. We see that the geodesic is symmetric in some point in the middle of the subsystem which will depend on our choice of origin, even if we don't know *a priori* which shape the geodesic will have.

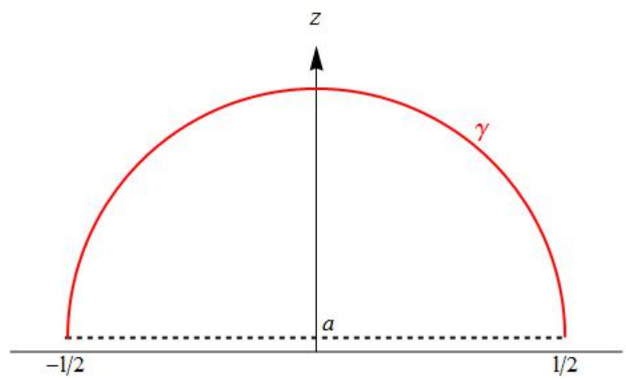

Figure 9: Constant time slice in AdS_3/CFT_2. We are taking the subsystem A of the CFT_2 to be the interval between $-l/2$ and $l/2$

We will use elementary methods to compute the geodesic. The Lagrangian of the AdS_3 parametrized by its proper length can be written as

$$\mathcal{L} = \frac{1}{2}\Big(\frac{R^2}{z^2}\Big(-\dot{t}^2 + \dot{z}^2 + \dot{x}^2\Big)\Big) \tag{7.2}$$

A geodesic is a trajectory that extremizes the action. We know since classical mechanics, that this kind of trajectories are given by the Euler–Lagrange equations. We parametrize this curve with a parameter. Usually, it is convenient to take an affine parameter to parametrize the geodesics. For more information and definitions about this, one can check [20]. For example, the trajectory that follows a particle in a free falling is a timelike geodesic. But since we have defined the entanglement entropy for a system at a given time t, which is fixed, we are going to compute the geodesics for a constant time, which means $\dot{t} = 0$.

Just to avoid ambiguities, we can think in the spacelike geodesic as the extremal trajectory at a fixed t in a given metric, therefore is the extremal length connecting two points at that t. We cannot speak about the spacelike geodesic as the length without fixing a t in order to give it a physical meaning. For example, a metric which changes with the time will have different spacelike geodesic connecting the same two points at different times. However, luckily that doesn't happen in the AdS metric and we can define the minimal length between two points unambiguously.

An affinely parametrized geodesic is a curve in a N–dimensional Riemannian or pseudo–Riemannian manifold which satisfies:

$$\frac{d}{ds}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}\right) - \frac{\partial \mathcal{L}}{\partial x^\mu} = 0 \tag{7.3}$$

$$g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = 1 \tag{7.4}$$

In what follows we set $R = 1$ since it is just a constant and we will add later. From (7.3) we obtain $\dot{x}/z^2 = C_1$ if we solve it for the coordinate x. We can plug this in (7.4) and we obtain the following differential equation

$$\dot{z} = \pm\sqrt{z^2 - C_1 z^4} \tag{7.5}$$

Notice that we have two signs, each of them corresponding to a different region. We choose the $-$ sign, which means that we are taking the half right part in Figure 9 where the slope of the z is negative, that is $0 < x < l/2$. Observe that solving (7.5) we obtain an extra integration constant C_2. Obtaining the equation for x from $\dot{x}/z^2 = C_1$ we get another integration constant C_3. At this point, we have the functions $z(s)$ and $x(s)$. In order to get rid of these three constants we have to use three different conditions:

- $x(0) = 0$ We choose the origin. Note that this choice will not affect the length of the geodesic, which is the value that we want to compute
- $\dot{z}(0) = 0$ as we see in the Figure 9, which indicates that the geodesic is symmetric respect to $s = 0$
- $z(s) = a \implies x(s) = l/2$ which is the boundary condition as we can appreciate in Figure 9.

Using these conditions we obtain the functions $x(s)$ and $z(s)$ parametrized by the proper length s.

$$\begin{aligned} x(s) &= C\,\tanh(s) \\ z(s) &= C\,\operatorname{sech}(s) \end{aligned} \qquad C = \frac{1}{2}\sqrt{4a^2 + l^2} \tag{7.6}$$

Notice that these solutions are valid only in the right side of the Figure 9. The length of the curve is given by

$$\text{Length}(\gamma) = \int ds \tag{7.7}$$

Between the two points that we want to compute it. One advantage of use the proper length as our affine parameter is that we don't have to compute the last integral plugging the metric (7.1) inside. The total length of the geodesic will be twice (since we are computing everything for just a half side) the s where $z(s) = a$ or, equivalently where $x(s) = l/2$. s is given by

$$s = \frac{1}{2}\log\left(4a^2 + l^2\right) + \log\left(\frac{1 + \sqrt{\frac{l^2}{4a^2 + l^2}}}{2a}\right) \tag{7.8}$$

Since a is the cutoff that we imposed, we have to *normalize* the length now. In order to do that, we expand (7.8) in powers of a and we obtain the leading term

$$s = \log\frac{l}{a} + \mathcal{O}(a^2) \tag{7.9}$$

Therefore, reintroducing the constant R and summing twice s we obtain the total length of γ

$$\text{Length}(\gamma) = 2R \log \frac{l}{a} \tag{7.10}$$

Once that we have the length we just have to compute the entropy through (5.10). At this point, in order to express the entanglement entropy in terms of the CFT_2 parameters, we have to use the relation [5]

$$c = \frac{3R}{2G_N^{(3)}} \tag{7.11}$$

Thus, finally we obtain the entanglement entropy of the subsystem A

$$S_A = \frac{\text{Length}(\gamma)}{4G_N^3} = \frac{c}{3} \log \frac{l}{a} \tag{7.12}$$

As we can see, we have reproduced the known result (3.10). We have already pointed out that the entanglement entropy obeys the so–called *Area law* (see section 3.2) except for 1–dimensional quantum systems. In this systems, the entropy follows a logarithm scale, which we can reproduce as well from holographic formulas.

Notice that the total CFT system is in a pure state. This can be seen perfectly in Figure 6 if we take the whole CFT system. Given two points that divide the CFT system into two subsystem we appreciate that there is a uniquely geodesic which ends in these points. This fact is telling us that the entanglement entropy of both subsystems A and B is exactly the same. It is not a surprising property and is just indicating the well-known property of the entanglement entropy, that is a intensive property. We have already seen this, for example, in the *Bell state* example that we gave in Section 3.

7.2 CFT_2 at finite temperature T

QFTs at finite temperatures

Usually, QFTs are studied at "zero temperature". From statistical physics, we can characterize the temperature of a classical system and generalize it to quantum physics. The partition function of a quantum field theory at finite temperature T described by H is given by

$$Z = \text{tr}\left(e^{-\beta H}\right) = \int D\varphi e^{-\int_0^\beta d\tau \int d^D x \mathcal{L}(\varphi)} \tag{7.13}$$

where $\beta = 1/T$. In this path integral formalism, we are integrating over all field configurations $\varphi(\vec{x}, \tau)$ such that $\varphi(\vec{x}, 0) = \varphi(\vec{x}, \beta)$. This can be seen as a simply rotation to Euclidean space, where we have replaced the time T in the path integral formalism

$$Z = \int D\varphi e^{i \int d^d x \mathcal{L}(\varphi)} \tag{7.14}$$

and imposing the periodic boundary conditions mentioned above. This formalism is called Thermal Field Theory. This theory has a central role in areas like Cosmology and processes like heavy–ion collisions.

In the limit $\beta \to \infty$ we simply recover the QFT at $T = 0$. We won't enter in more details about this. We just wanted to remark that the temperature, mathematically in QFT, is equivalent to a periodic imaginary time. As Zee point out in [13] Section V.2, this connection comes from the central objects in quantum physics e^{-iHt} and thermal physics $e^{-\beta H}$. However, there may be some kind of connection between this two concepts beyond their mathematical behaviour. For further information, one check [13] Section V.

We write the density matrix of a quantum system at finite temperature T given by

$$\rho = \frac{e^{-\beta H}}{Z} \tag{7.15}$$

We appreciate that if we compute the entanglement entropy (3.5), this quantity will take account of the thermal entropy as well. Therefore, the entanglement entropy in thermal quantum systems is not a good quantity for describing *how much* the systems are entangled. To obtain the measure of the entanglement, we should give rid of the thermal contribution.

The entanglement entropy in finite temperature systems becomes of greater importance when we talk about phase transitions. It is expected to be an order parameter describing these transitions [19]. Its non–locality property, as opposed to correlation functions, can capture the information of the ground state at low energies in strongly coupled systems.

Another important property of a thermal state, is that if we divide the system into A and B the relation $S_A = S_B$ is no longer held. Of course, the total entropy of the system is not zero since it has some given temperature T. Having a quick look to the pure AdS metric (4.5) we cannot derive these properties, so pure AdS space is no longer dual to a QFT at finite temperature.

Geodesics in 3–dimensional BTZ metric

Since the main idea of the Holographic Entanglement Entropy is that the boundary of AdS describe a dual CFT, we can extend our attention to those metrics that are asymptotically AdS spacetimes. For a two dimensional CFT, if we assume that the total length L of system is infinite, which holds $\beta/L \ll 1$, we find that the dual gravity theory to the boundary is the Euclidean BTZ metric (4.7). We observe that this metric is a black hole solution. Precisely, the presence of this black hole in the "middle" of the bulk is what will deform the geodesics making them to wrap the horizon when the length of the subsystem will become bigger. We can appreciate this in Figure 10a.

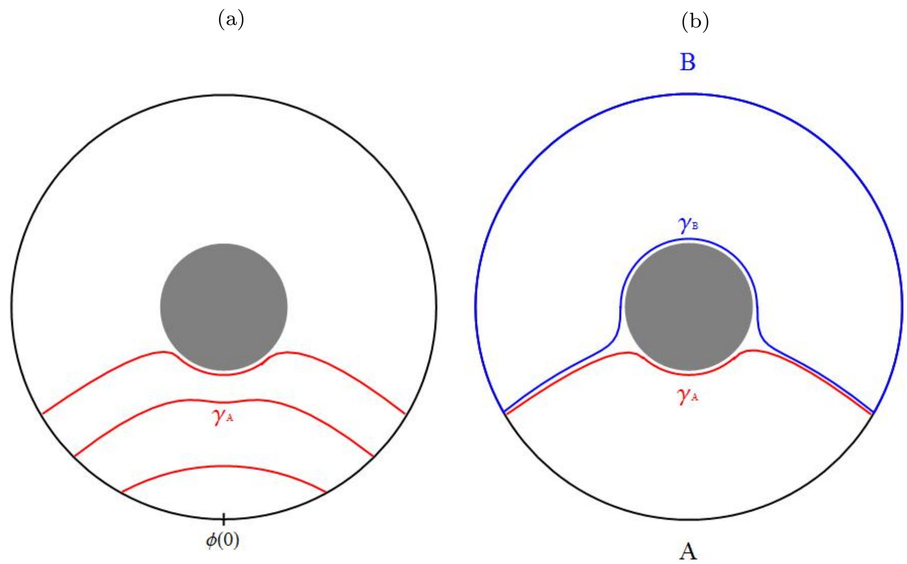

Figure 10: (a)Geodesics in the BTZ metric for different chosen length of the subsystem. When the length of the subsystem is greater (which means a greater $\Delta\phi$), the geodesic starts to wrap the black hole horizon. At sufficient small $\Delta\phi$ the geodesic is described approximately with the pure AdS space (4.2). (b) Two different geodesics that connects the same pair of points in the boundary, each of them covering a different side of the black hole horizon. Each one describe the region in the boundary which they are homologous with. The red line is the geodesic for the subsystem A and the blue line the geodesic for the subsystem B

The process that we will follow to get the geodesic will be exactly the same that we have used for the pure AdS except one important remark. As we can appreciate in Figure 10b, the presence of the black hole leads to the fact that each pair of points defining the length of the CFT_2 will have two geodesics. In these cases we have to take the geodesic which is homologous to the subsystem for which we are computing the entanglement entropy. This already fix the condition that in thermal states $S_A \neq S_B$. This statement is extended to higher dimensional cases as well. From (4.7), we write down the Lagrangian of the metric with Euclidean signature

$$\mathcal{L} = \left(r^2 - r_+^2\right)\dot{t}^2 + \frac{R^2}{r^2 - r_+^2}\dot{r}^2 + r^2\dot{\phi}^2 \tag{7.16}$$

where r_+ is the black hole horizon. Once again, since we are computing the geodesic for a fixed time t, we have $\dot{t} = 0$. The geodesics will satisfy the equations (7.3) and (7.4) like in the $T = 0$ case. Solving the Euler–Lagrange equations for ϕ we get the condition $r^2\dot{\phi} = C_1$. Plugging this into (7.4) we got

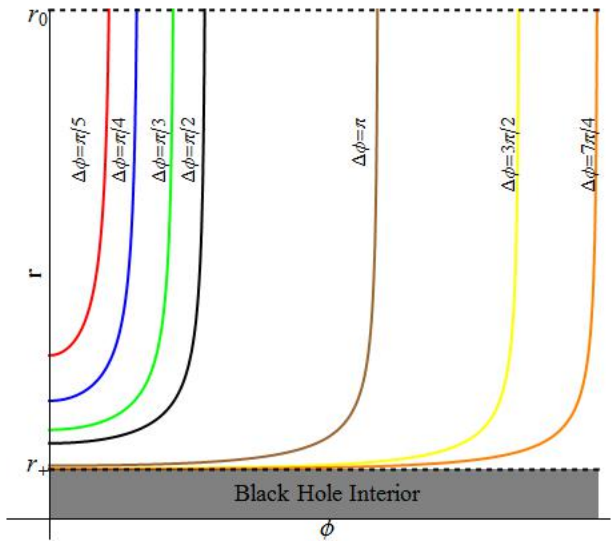

Figure 11: Geodesics in BTZ. The graphic just represent the half of the geodesic, it is clear that the other part is symmetric. The different coloured lines indicate the length of the subsystem A where the red one is the smaller and the orange is almost the whole system. It can be appreciated like in Figure 10a how the geodesics wrap to the black hole horizon when we take greater $\Delta\phi$.

the next differential equation

$$\dot{r} = \pm\sqrt{\left(1 - \frac{C_1^2}{r^2}\right)\frac{r^2 - r_+^2}{R^2}} \tag{7.17}$$

As in the zero temperature case, we have to choose on of the signs, which represent two different areas. We can represent the behaviour of the geodesics plotting r vs. ϕ as well to make this choice more clear than just looking at Figure 10a. We expect that our solutions behave as in Figure 11. We see that the geodesics cannot have a smaller value than r_+ since they cannot penetrate inside the black hole. We impose the cutoff r_0 when $r \to \infty$, that means, near the boundary, since the length of the geodesic which ends in the boundary diverge; this process is analogous to impose the cutoff a in the previous section. We select the + sign, which corresponds with the represented part of the geodesics in Figure 11.

Integrating (7.17) we obtain $r(s)$ and we get an extra constant C_2 from the integration. From $r^2\dot{\phi} = C_1$ we immediately obtain $\phi(s)$ and we get another constant C_3 from the integration.

$$r(s) = \frac{1}{2}\sqrt{e^{-\frac{2(C_2-s)}{R}} + e^{\frac{2(C_2-s)}{R}}\left(C_1^2 - r_+^2\right)^2 + 2\left(C_1^2 + r_+^2\right)} \tag{7.18}$$

$$\phi(s) = \frac{R}{r_+}\text{arctanh}\Big(\frac{C_1^2 + e^{-\frac{2(C_2-s)}{R}} + r_+^2}{2C_1 r_+}\Big) + C_3$$

In order to determine the three constants we have to impose three conditions

- $\phi(0) = 0$ We choose the origin. The origin is placed as it is shown in Figure 10a. So, if we define $\Delta\phi$ as the total length of the subsystem, each of the extremes will be $\Delta\phi/2$.
- $\dot{r}(0) = 0$ as we see in the Figure 11, which indicates that the geodesic is symmetric respect to $s = 0$. Notice that in the papers [5, 7] the authors choose other origin, thus this condition is displaced.
- $r(s) = r_0 \implies \phi(s) = \Delta\phi/2$ which is the boundary condition as we can appreciate in Figure 10a and in Figure 11.

With these conditions we determine the three constants which are given in terms of the constants that we have R, r_+ and $\Delta\phi$. The final expressions have the form

$$r(s) = \sqrt{A + B\cosh\left(\frac{2s}{R}\right)} \tag{7.19}$$

$$\phi(s) = C - D\text{arctanh}\Big(E + Fe^{2s/R}\Big)$$

where A, B, C, D, E and F are some combinations of the constants R, r_0 and r_+. These solutions are only valid for the half of the geodesic, but using its symmetry properties, we see that the total length will be simply twice the total length of or geodesic from $\phi = 0$ to $\phi = \Delta\phi/2$. Once again, due to we have used the proper length as the affine parameter, the length of the geodesic is simply s such that $\phi(s) = \Delta\phi/2$ or $r(s) = r_0$. We have to normalize it. Hence, we expand the length s in powers of $1/r_0$ and we get

$$s = R\log\left[\frac{2r_0}{r_+}\sinh\left(\frac{\Delta\phi r_+}{2R}\right)\right] + \mathcal{O}\Big(\frac{1}{r_0^2}\Big) \tag{7.20}$$

Now we have to relate the parameters in the bulk with the parameters in the boundary. There exist the so-called AdS/CFT dictionary which relates these pairs of parameters. In some way, the temperature in the boundary is related holographically to the Hawking temperature in the bulk. We are not going to enter in details here and we are going to follow the procedure of [21]. As we showed in Section 4.3, the Euclidean time has a period of $t \sim t + 2\pi R/r_+$. This period of the Euclidean time is related precisely with the inverse of the temperature $1/T = 2\pi R/r_+$ and, as usual, we have the definition $\beta \equiv 1/T$. Taking the boundary limit $r \to \infty$ we can find the relation $\beta/L = R/r_+$ [22]. Note that the total length of the boundary L is given by $L = 2\pi r_0$, hence the total length of the CFT tends to infinity. The relation between the cut-off in the CFT and the one in the bulk is given by $\beta = 2\pi a r_0/r_+$ [19]. Rearranging this, we got

$$\frac{r_+}{R} = \frac{2\pi r_0}{\beta} \tag{7.21}$$

Plugging these into (7.20) and adding a factor of 2 we got the total length. Simply using the holographic entanglement entropy equation (5.10) we get finally the entanglement entropy of the subsystem A

$$S_A = \frac{c}{3}\log\left(\frac{\beta}{\pi a}\sinh\left(\frac{\pi r_0 \Delta\phi}{\beta}\right)\right) \tag{7.22}$$

where we have used once again the relation (7.11).

Finally, if we choose $\Delta\phi$ to take the value l/r_0 where l will be the length of the subsystem A we get exactly the equation (3.11) as we expected. We can check if the entanglement entropy is equivalent to (7.12) when the temperature gets smaller. That means that $\beta \to \infty$. We can do the following approximation

$$\sinh\left(\frac{\pi l}{\beta}\right) \approx \frac{\pi l}{\beta} \tag{7.23}$$

So plugging this result into (7.22) we get

$$S_A = \frac{c}{3}\log\left(\frac{\beta}{\pi a}\frac{\pi l}{\beta}\right) = \frac{c}{3}\log\left(\frac{l}{a}\right) \tag{7.24}$$

Thus, we recover the solution for the pure state as we expected.

Phase transition

From Figure 10b we can see that there are two different geodesics connecting the pair of points. As we have said, each one of them corresponds to a different subsystem, A or B respectively. We can immediately write down the entanglement entropy for the subsystem B which is, of course, the complementary of A by simply setting the different ϕ that we are looking at $\Delta\phi' = 2\pi - \Delta\phi$. Thus, we get

$$S_B = \frac{c}{3}\log\left(\frac{\beta}{\pi a}\sinh\left(\frac{\pi r_0(2\pi - \Delta\phi)}{\beta}\right)\right) \tag{7.25}$$

Both entropies are equal at $\Delta\phi = \pi$. For $\Delta\phi < \pi$, S_A is smaller than S_B and for $\Delta\phi > \pi$ happens the opposite. As we have said, each of the geodesics represent different subsystems since they have to be homologous to their boundary regions. However, there is another way to hold the homology using the geodesic which is supposed to describe the complementary region. This construction can be made using the "complementary" geodesic (that means, for the region A, using S_B) plus a spatially closed geodesic which wraps the black hole horizon. This is shown schematically in Figure 12a.

Since this way to construct the minimal surface has to be taken into account, we cannot longer claim that the entanglement entropy of the subsystem A is

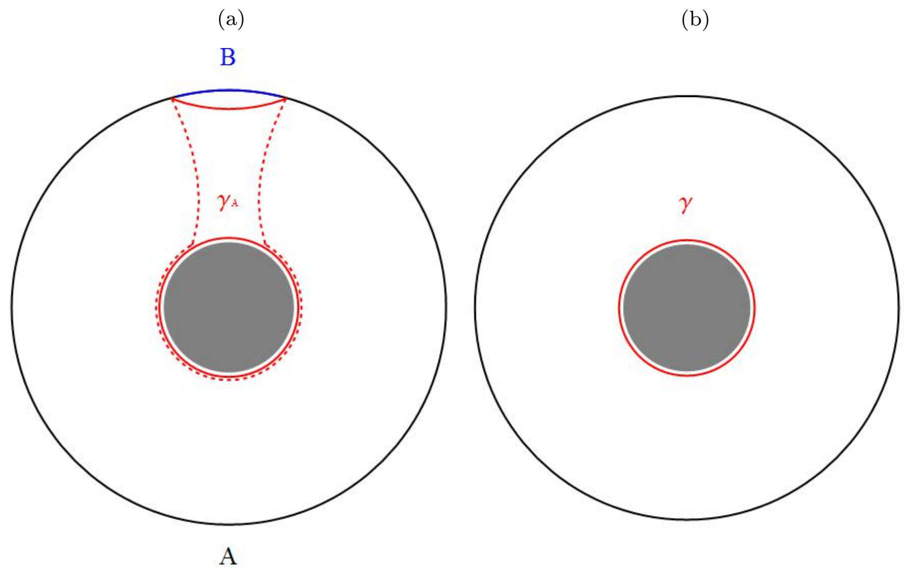

Figure 12: (a) Two different ways of construct the minimal surface area in the BTZ metric. The dotted line corresponds to the length which finally gives the entropy S_A given by the equation (7.22). The straight line corresponds to the two disconnected components of the minimal surface, one wraps the black hole horizon and the other is just the geodesic computed for the subsystem complementary to A. (b) Geodesic which wraps the black hole horizon when we select the entire CFT system. In contrast to the pure CFT, the total entropy is not equal to zero.

given entirely by (7.22). However, we can predict that (7.22) will be the good choice at not really high values of $\Delta\phi$.

The length of the complementary geodesic will be given by (7.25) as we said. Thus, we just have to compute the closed geodesic around the black hole horizon. This geodesic is exactly the area of the black hole horizon, so its entropy will be given by the familiar Bekenstein-Hawking entropy (3.7). Since we are in a time-constant slice, the area will be just the length. We can rewrite the black hole entropy as [21]

$$S_{BH} = \frac{\text{Length}(\gamma)}{4G_N} \tag{7.26}$$

where the length is given by $2\pi r_+$. Using once again the relation (7.11) we get

$$S_{BH} = \frac{c}{3}\frac{2\pi^2 r_0}{\beta} \tag{7.27}$$

Notice that we could rewrite this expression just in terms of the boundary parameters using $L = 2\pi r_0$. Therefore, the entanglement entropy for the subsystem A given with this two disconnected geodesics is given by

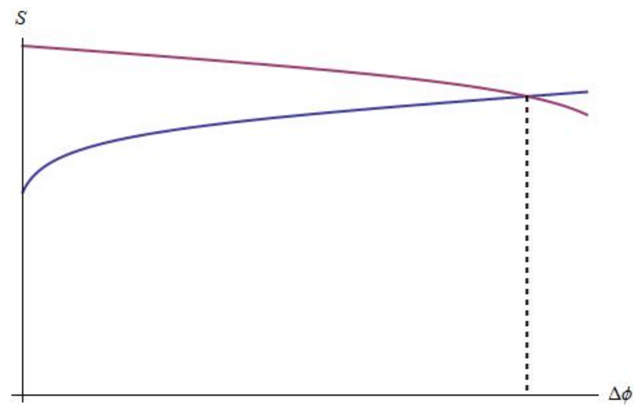

Figure 13: Phase transition at sufficient high temperature with constants arbitrarily chosen keeping $R \gg a$. The dashed black line denote the point where the phase transition occurs. The blue line represents the entropy S_A given by the equation (7.22). The purple line represents the entropy S'_A given by the equation (7.28).

$$S'_A = \frac{c}{3}\left(\log\left(\frac{\beta}{\pi a}\sinh\left(\frac{\pi r_0(2\pi - \Delta\phi)}{\beta}\right)\right) + \frac{2\pi^2 r_0}{\beta}\right) \tag{7.28}$$

Since we have to select the geodesics with the minimal length, the actual entropy for the subsystem A will be precisely the minimum between S_A and S'_A. It is clear that for small $\Delta\phi$ the correct entropy will be S_A. However, when we increase its value, can occur than the S'_A get a smaller value than S_A. Then, at this point, the entanglement entropy will experience a phase transition from S_A to S'_A as we have represented in Figure 13. However, this phase transition only will occur above some temperature T_c. Since at low temperatures we have seen that the system is just approximately the pure AdS_3 spacetime, that means that we can use the equation (7.12), so the phase transition will not occur. At higher temperatures such that $r_0/\beta \gg 1$ we can find that the length in which the phase transition occur goes as [21]:

$$\Delta\phi \sim 2\pi - \log 2\frac{\beta}{2\pi r_0} \tag{7.29}$$

As a remark, we can point out that, if we choose the whole system, that is, we compute the entanglement entropy of the entire CFT_2 system we get exactly the Bekenstein-Hawking entropy (7.27). We can appreciate this in Figure 12b. In this case, we can be sure that the entanglement entropy computed represents exactly the thermal entropy of the CFT_2 system.

7.3 Comments on CFT in higher dimensional cases

In this section we will just present the results obtained by Ryu, Takayanagi et al. in [5, 7, 19, 22] on higher dimensional CFT cases. These cases have not been studied as deeply as CFT_2 cases due, mainly, to their computational complexity, being impossible in a lot of cases to get an analytical solution. Nevertheless, since the Ryu-Takayanagi proposal (5.10) has to hold for every chosen dimension, we are going to summarize the main obtained results for any d–dimensional system.

Since we are looking in higher dimensions, the subsystem in the CFT is no longer described by its length. Rather, we have to choose a shape in the CFT given by as many coordinates as its dimension. Two are the mainly shapes computed in higher dimensions:

- An straight belt A_S such that one of the spatial dimension is chosen as a segment between $(-l/2, l/2)$ and the rest of the spatial dimensions goes from $(-\infty, \infty)$. L denotes the length of the other dimensions which goes to infinity.
- A circular disk A_D defined by the radius l as $r = \sqrt{\sum_{i=1}^{d-1} x_i^2} \leq l$.

Computing the minimal area surfaces that ends in the boundary of A_S and A_D and using (5.10) the authors get in the papers the expressions:

$$S_{A_S} = \frac{1}{4G_N^{d+1}} \left[\frac{2R^d}{d-2} \left(\frac{L}{a} \right)^{d-2} - \frac{2^{d-1}\pi^{d-1/2} R^{d-1}}{d-2} \left(\frac{\Gamma\left(\frac{d}{2(d-1)}\right)}{\Gamma\left(\frac{1}{2(d-1)}\right)} \right)^2 \left(\frac{L}{l} \right)^{d-2} \right] \tag{7.30}$$

$$S_{A_D} = p_1 \left(\frac{l}{a} \right)^{d-2} + p_3 \left(\frac{l}{a} \right)^{d-4} + \begin{cases} p_{d-2}\left(\frac{l}{a}\right) + p_{d-1} + \mathcal{O}\left(\frac{a}{l}\right) \\ p_{d-3}\left(\frac{l}{a}\right)^2 + q \log\left(\frac{l}{a}\right) + \mathcal{O}(1) \end{cases} \tag{7.31}$$

where Γ is the usual Gamma function and the coefficients p_i and q are some combinations of Gamma function and constants depending on which dimension we are working with. Without entering in technical details, we can see that the leading terms in both results are proportional to the area of the shape chosen. That is, in the straight belt case the term L^{d-2} and in the circular disk the term l^{d-2}. Hence, the Ryu-Takayanagi proposal (5.10) reproduces the known result of the entanglement entropy in CFTs, the so–called *Area law* which we stated in Section 3.2.

8 EE thermalization from AdS_3/CFT_2

In the previous sections we have been computing the holographic entanglement entropy for a given time t. Hence, we have explored the dynamic of the system, that is, how it evolves. In this section we are going to study a system evolving towards an equilibrium following the paper of Liu and Suh [23].

8.1 Quantum quenches and the Vaidya metric

In the CFT side, the evolution is always governed by an unitary transformation, as quantum mechanics requires. We suppose a CFT in a pure state when suddenly it gets excited and evolves toward a thermal equilibrium. This excitation can be made by changing the parameters of the Hamiltonian which describes the CFT. This way is one of the simplest cases where we can study the dynamical evolution of the entanglement entropy. The process of parameters change instantaneously is known by the name of *quantum quenches*. The state that reaches the CFT after evolving is characterized by some equilibrium temperature T_c.

Intuitively, the dual theory should be represent precisely a black hole formation. There are various metrics which can describe a black hole formation. However, it seems to be an agreement supported by a good theoretical evidence that the good geometry on the bulk would be described by *Vaidya metric*. This metric can describe holographically the properties observed in the EE in the CFT systems.

The Vaidya metric was developed as a model for a spherically symmetric and radiating star by the Indian physicist *Prahalad Chunnilal Vaidya*. The Vaidya is a full family of solutions to Einstein's Field Equations. It can describe the gravitational collapse of null dust, feature which we are going to use. For further information about the Vaidya metric concerning hollographic purposes, Section 3 of [24] can be checked.

We consider the Poincaré version of the 3–dimensional Vaidya metric which describes the quantum quenche, given by

$$ds^2 = \frac{R^2}{z^2}\big(-(1-m(v)z^2)dv^2 - 2dzdv + dx^2\big) \tag{8.1}$$

where the function $m(v)$ is called the mass function and it goes from 0 to a finite value m. This function represent the black hole formation and it can be shown that, if it is constant, the metric would be the Schwarzschild metric. We set that $m(v)$ goes from 0 to 1, so we can define it as

$$m(v) = \frac{1+\tanh v/v_0}{2} \tag{8.2}$$

The coordinate v represents an infalling null and v_0 is the thickness of a shell falling along $v=0$. Since v_0 is a constant and it has been checked that physical results does not depends significantly when it is small, we set $v_0 = 0$. It is clear that the metric diverges at $z=0$, where the metrics reduces to AdS spacetime when $v<0$. The increment of the geodesic length to respect at its vacuum value (in the AdS pure space) which ends in the points $(\tau, z_0, \mp l/2)$ has been obtained in [25] and it is given by

$$\Delta\text{Length}(\gamma) = 2\log\left(\frac{\sinh\tau}{l s(l,\tau)}\right) \tag{8.3}$$

where the function $s(l,\tau)$ is defined as

$$l = \frac{c}{s\rho} + \frac{1}{2}\log\left(\frac{2(1+c)\rho^2 + 2s\rho - c}{2(1+c)\rho^2 - 2s\rho - c}\right) \tag{8.4}$$

$$\rho = \frac{1}{2}\coth(\tau) + \frac{1}{2}\sqrt{\frac{1}{\sinh^2(\tau)} - \frac{1-c}{1+c}} \tag{8.5}$$

$$c = \sqrt{1-s^2}$$

and it is defined in the interval $[0, 1]$. Basically, they have obtained the length of the geodesic for a given time τ and a given length (size) of the CFT l. It is obvious that this length has to remain constant for any given l independently of τ for a sufficient large times, that is, after reach the equilibrium. In the equilibrium, as we have seen in the previous section, the metric is just given by the BTZ black hole solution. Hence, if we define τ_c as the time such that the system reach the equilibrium, the length and therefore the entanglement entropy, is given by the computed values (7.22) and (7.28).

8.2 Evolution of EE and scaling regimes

The entanglement entropy will we given by (5.10). As a remark, despite of the proposal was originally made for spacelike geodesics, that means, for a time constant slice in the given metric, it has been extended to dynamical systems like the one we are looking at. Using the result (8.3) obtained by [25] we can easily find the increment of EE for a subsystem A respect to its vacuum value

$$\Delta S_A = \frac{c}{3}\log\left(\frac{\sinh\tau}{ls(l,\tau)}\right) \tag{8.6}$$

As we pointed out, the solution in the last section is valid in the dynamical thermalization regimen, when the system evolves toward a thermal equilibrium. We can easily find the time in which the system will reach the equilibrium by direct comparison of (7.22), which we can rewrite as

$$S_A = \frac{c}{3}\left[\log\left(\frac{l'}{a}\right) + \log\left(\frac{\beta}{\pi l'}\sinh\left(\frac{\pi l'}{\beta}\right)\right)\right] \tag{8.7}$$

where notice that we have replaced l in (7.22) by l' in order to avoid to use the same variable that in (8.6). From the last expression, it is clear that the first term is associated by the vacuum value and the second term with the increment of entropy respect to it, which we can denote as ΔS_A^{eq}. We state then the next equality which will allow us to determine τ_c, the time in which the system will reach the equilibrium.

$$\Delta S_A(\tau_c, l) = \Delta S_A^{eq} \tag{8.8}$$

we can identify immediately that $l = \pi l'/\beta$ and therefore the time in which the system reach the equilibrium is simply

$$\tau_c = \frac{\pi l'}{\beta} = l \tag{8.9}$$

Hence, we are going to study the system in times $\tau < \tau_c$. The variables l and τ are defined as dimensionless as we can observe. Comparing with the equations for the equilibrium that we got in the previous section (7.22) and (7.25), notice that we have taken the first expression as the EE in the equilibrium. This means that we are looking at the dynamics in the region where $\Delta\phi$ and β are sufficiently small to be described simply by (7.22).

Following [21] we can distinguish four different regimes.

Early times

If we want to check how the system evolves right after apply the quantum quenche we have to look at small times when $\tau \ll \tau_c$ and $s \to 0$. We first expand ρ in series of τ letting $s \to 0$ and we get

$$\rho = \frac{1}{\tau} + \mathcal{O}(\tau) \tag{8.10}$$

Plugging this approximation into (8.4) and expanding l in series of s we get

$$s = \frac{\tau}{l} + \mathcal{O}(t^2) \tag{8.11}$$

Finally, introducing this value in the equation (8.6) and expanding in terms of τ, we obtain how the entanglement entropy evolves at early times

$$\Delta S_A = \frac{c}{3}\frac{\tau^2}{6} + \mathcal{O}(\tau^4) \tag{8.12}$$

Hence, we observe that the EE grows quadratically at early times for all l values which entanglement entropy in equilibrium is given by (7.22).

Linear regimen

To study the behaviour at later times we approximate $\tau_c \gg \tau \gg 1$. It is convenient to introduce the angle ϕ defined as

$$\begin{aligned} \sin\phi &= s \\ \cos\phi &= c \end{aligned} \tag{8.13}$$

where the equilibrium is reached at $\phi = \pi/2$ and its minimum value its $\phi = 0$ as we can see. In terms of ϕ the approximation is [21] $e^{-\tau} \ll \phi \ll e^{-2\tau/5}$ with ρ and l given by

$$\rho = \frac{1}{2} + \frac{\phi}{4} + \mathcal{O}\big(e^{-2\tau}/\phi\big), \qquad l = \frac{2}{\phi} + \tau + \log\phi + \mathcal{O}(1) \tag{8.14}$$

where we have used the explicit definitions of hyperbolic sine and hyperbolic cotangent in terms of exponentials. Plugging this in into (8.6) we get [21]

$$\Delta S = \frac{c}{3}\tau - \frac{c}{3}\log 4 + \mathcal{O}\big(\tau/l, \log l/l, e^{-2\tau}\big) \tag{8.15}$$

which, as we can see, is linear with τ.

Near equilibrium regime

To study the dynamics of the EE near to the equilibrium τ_c, we set $\phi = \pi/2 + \epsilon$, where $\epsilon \to 0$. Expanding (8.5) in powers of ϵ we get

$$\rho = \coth\tau - \frac{\epsilon}{2}\tanh\tau + \mathcal{O}(\epsilon^2) \tag{8.16}$$

Taking the first term, plugging it into (8.4) and expanding once again in powers of ϵ we get

$$l = \tau + \frac{\epsilon^2}{2}\tanh\tau + \mathcal{O}(\epsilon^3) \tag{8.17}$$

Finally, using this expression for l and writing s in terms of ϕ in the way we showed it, we get the entanglement entropy as

$$\Delta S_A = \frac{c}{3}\log\frac{\sinh\tau}{\tau} + \frac{1}{2}\left(1 - \frac{\tanh\tau}{\tau}\right)\epsilon^2 + \mathcal{O}(\epsilon^3) \tag{8.18}$$

In the last step, we set $\tau = \tau_c + \delta$ where $\delta \to 0$ and we expand the above expression in powers of δ

$$\delta S = \Delta S_{eq} - \frac{\sqrt{2}}{3}\sqrt{\tanh\tau_c}\,\delta^{3/2} + \mathcal{O}(\delta^2) \tag{8.19}$$

where we see that the growth of the EE is proportional to $(\tau_c - \tau)^{3/2}$. In the paper [21], Liu and Suh point out that, for large systems $l \gg 1$ we can approximate $\tanh\tau_c \approx \tau_c$

Memory loss regime

The transition between the linear regimen and the equilibrium regime is not direct. There exists a forth last regime which in [21] is called as *memory loss regime*. They show that the evolution of entanglement entropy is proportional to a single combination of τ and l. This is interpreted as the memory loss of the size l.

This region is characterized by $\tau, l \gg 1$ and $\tau < \tau_c$. There exists a minimum value of $\rho_{min} = 1/2(1+\tan(\phi/2))$ as we can see from (8.5). This minimum value of ρ correspond to the values $\tau, l \to \infty$. Hence, the approach in this regime will be similar to the near–equilibrium regime. We will study the system in the limit $\rho = \rho_{min} + \epsilon$ such that $\epsilon \to 0$. Following the same procedure than above, we get the expression

$$\Delta S = \Delta S_{eq} + \frac{c}{3}\big(\tau - l - \log(\sin\phi)\big) + \mathcal{O}\big(e^{-2\tau}, e^{-2l}\big) \tag{8.20}$$

This can be rewritten just in terms of the interval $l-\tau$ as

$$\Delta S = \Delta S_{eq} + \frac{c}{3}\lambda(l-\tau) + \mathcal{O}(e^{-2\tau}) \tag{8.21}$$

where the function $\lambda(\tau - l)$ is given by

$$\lambda(l-\tau) = -(l-\tau) - \log\left(\sinh^{-1}(l-\tau)\right) \tag{8.22}$$

In [21] the authors shows how for the conditions $l \gg \tau \gg 1$ and $\delta = \tau_c - \tau \ll 1$ the above entropy gives the linear regime and the nearly–equilibrium regime respectively.

9 Conclusions

In this work we have introduced the Ryu-Takayanagi proposal for holographic entanglement entropy. We have reviewed the basic properties of the EE in different frameworks and we gave an explicit example of EE in QM.

In Section 5 we have derived the AdS/CFT correspondence from the geometrization of the renormalization group. Even though the duality was discovered in a different way, this derivation gives an useful way to understand the AdS/CFT correspondence as well as make to more clear the mathematical beauty of the conjecture and the still unknown mechanism that generate it. The AdS/CFT was discovered in string theory framework, more concretely the type IIB compactified on $\mathrm{AdS}_5 \times \mathrm{S}^5$. Despite this fact, the AdS/CFT correspondence is, in some way, independent of string theory, since one of their *sides* is described by a QFT without "mentioning" neither gravity nor strings and can be derived as we have shown from QFTs.

In the last sections we have just used the proposal and we have computed the geodesics using simple mathematical tools from $\mathrm{AdS}_3/\mathrm{CFT}_2$. All the results computed until now in this framework are in perfect agreement with the ones obtained from CFT_2. However, to test the proposal in higher dimensions is a difficult task since we have no direct results computed in the CFTs. Nevertheless, we can find in papers like [5] approximations for these computations, which agree with the results computed in general $\mathrm{AdS}_{d+1}/\mathrm{CFT}_d$. One important result that still has to be achieved, is the derivation of the proposal from the AdS/CFT correspondence.

The Ryu-Takayanagi proposal, for its very own nature, has been applied to the understanding of black holes. For example in [19], they identify the black hole entropy as entanglement entropy from $\mathrm{AdS}_2/\mathrm{CFT}_1$. In our Section 7, for example, we have computed the entanglement entropy for a CFT in a finite T we have used the black hole entropy. There exists a huge amount of papers relating these two concepts, so everything seems to indicate that there is a connection between the black hole entropy and the entanglement entropy.

The holographic entanglement entropy has been applied successfully in the branch of condensed matter physics and is currently an active research area. As we have pointed out, the entanglement entropy could play a crucial role as an order parameter in phase transitions [19] leading this field towards a better understanding of, for example, the quark–gluon plasma phase transition.

A Holography and philosophy

In this appendix we will review some of the main ideas and interpretations that the concept of holography has. The holography idea was first formulated, as we mentioned, by 't Hooft in 1993 as a principle which a quantum gravity theory should follow. For reminder, the holography states a relation between the physics on the surface and the physics on the bulk. 't Hooft stated that at the Planck scale, our world is not 3+1 dimensional, that the theory on the surface is more fundamental that the theory on the bulk. Even though 't Hooft is ambiguous at this point in his original paper where he writes "the observables in our world can best be described as *if*" (where one can think 't Hooft is suggesting just a *representation*), he assumed that the fundamental ontology lies on the spacetime boundary and its degrees of freedom. With the arise of AdS/CFT correspondence (a clear example of holography) and the huge amount of papers on this area, this question became more and more important.

One can argue that, due to the AdS/CFT theory is still a conjecture, why should we take care about its deeply interpretation and its ontology? The amount of theoretical "evidence" and known solutions reproduced in holographic scenarios is an inarguable fact, which makes the theory the first candidate for the so-called quantum gravity. With no doubt, a better understanding of the basic principles and physical interpretations of the theory will be translated into new ideas and different approaches by the hand of the string theory researchers. This argument become unquestionable when we point out the lack of empirical evidence (although the important labour of many groups on phenomenology at low energies that could be detectable).

On the interpretation of AdS/CFT correspondence we can find two different and opposed views. The first of them claims that the AdS/CFT is, as a matter of fact, a duality between two theories. This implies that the two frameworks, the physics on the boundary and the physics on the bulk, are exactly duals and they represent exactly the same physics; that means, there is a one-to-one map between the theories. The second point of view claims that gravity has to be interpreted as an emergent phenomenon which does not exists at microscopic level, so the duality of the theory breaks down at some order in perturbation theory; so we are treating with thermodynamic ideas. On this idea is remarkable the Verlinde's scheme, where he obtains the Newton's law based on holography and thermodynamic principles.

Stretching the last viewpoint and taking the CFT as the fundamental theory, some authors claim that it implies the exciting consequence that the spacetime would become non-fundamental, which means that at very high energies the notion of distance loses its meaning. However, other authors prefer a moderate approach in which we interpret the correspondence just in an approximate way, leaving the question of which theory is more fundamental to an empirical

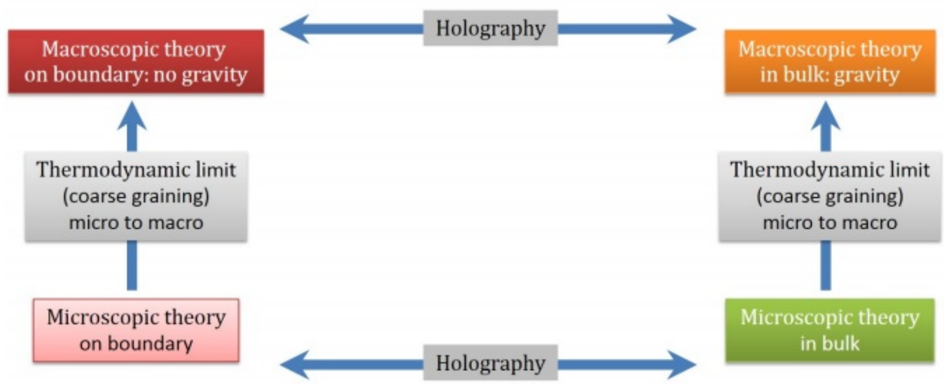

Figure 14: Scheme of holography and emergent gravity taken from [27]

adequacy.

Of course, one can think on the bulk theory as the fundamental one. Nevertheless, through the literature we find out that this possibility has not been explored as much as those ones with gravity as an emergent phenomenon. This can be explained through the way we interpret the gravity in contrast with the other forces and their quantization. Even Albert Einstein expressed that if someone wants a theory of quantum gravity, one should get rid of the *continuum* spacetime.

In the recent years a lot of papers on the concept of emergent gravity or spacetime (as Verlinde's paper mentioned above) has been published. At a first sight on can think that this idea is inside the point of view where the duality breaks down at some point. But exists the possibility where the gravity is an emergent phenomenon and the duality is still exact. In this case, the bulk theory has a microscopic theory with no gravity and this one emerges from its microscopical description through thermodynamic limit and the duality still holds. We can see this schematically in Figure 14. In the framework where the duality breaks down and we still have a gravity emergence, there would be no microscopic theory in the bulk.

As de Haro point out in his paper [27] we can make a useful analogy with the position-momentum duality in QM. The description of the systems in terms of momentum p and position x have had different physical interpretations. With the advent of QM we learnt that the description in p or x terms are different representations of the same theory (with Fourier transforms as the map between both concepts). This taught us an exciting new reality of the description of the nature: particles are neither waves nor particles. In the same way, we can expect that AdS/CFT will teach us something new about the nature of gravity and spacetime.

References

[1] J. Maldacena, *The Large N Limit of Superconformal field theories and supergravity*,arXiv:hep-th/9711200v3

[2] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information*, 10th Anniversary Edition, Cambridge University Press. *Sections 2.4 & Section 11*

[3] D. Harlow, *Jerusalem Lectures on Black Holes and Quantum Information*, arXiv:1409.1231v2 [hep-th]

[4] M. E. Peskin and D. V. Schroeder, *An Introduction To Quantum Field Theory*, First Edition, Perseus Books

[5] S. Ryu and T. Takayanagi, *Aspects of Holographic Entanglement Entropy*, arXiv:hep-th/0605073v3

[6] M. R. Gaberdiel, *An Introduction to Conformal Field Theory*, arXiv:hep-th/9910156v2

[7] S. Ryu and T. Takayanagi, *Holographic Derivation of Entanglement Entropy from AdS/CFT*, arXiv:hep-th/0603001v2

[8] J. Kaplan, *Lectures on AdS/CFT from the Bottom Up*, `www.pha.jhu.edu/ jaredk/AdSCFTCourseNotesPublic.pdf`

[9] I. Bengtsson, *Anti de–Sitter space*, Lecture notes `http://www.fysik.su.se/ ingemar/`

[10] E. van der Mast, *The BTZ Black Hole*, Master Thesis, University of Amsterdam

[11] M. Bañados, A. Gomberoff and C. Martínez, *Anti-de Sitter space and black holes*, arXiv:hep-th/9805087v1

[12] A. V. Ramallo, *Introduction to the AdS/CFT correspondence*, arXiv:1310.4319v3 [hep-th]

[13] A. Zee, *Quantum Field Theory in a Nutshell*, 2nd Edition, Princeton University Press

[14] H. Araki and E. H. Lieb, *Entropy inequalities*, Commun. Math. Phys. 18 (1970) 160170

[15] M. B. Ruskai, *Another Short and Elementary Proof of Strong Subadditivity of Quantum Entropy*, arXiv:quant-ph/0604206v1

[16] M. A. Nielsen and D. Petz, *A simple proof of the strong subadditivity inequality*, arXiv:quant-ph/0408130v3

[17] T. Hirata and T. Takayanagi, *AdS/CFT and Strong Subadditivity of Entanglement Entropy*, arXiv:hep-th/0608213v2

[18] M. Headrick and T. Takayanagi, *A holographic proof of the strong subadditivity of entanglement entropy*, arXiv:0704.3719v1 [hep-th]

[19] T. Nishioka, S. Ryu and T. Takayanagi, *Holographic Entanglement Entropy: An Overview*, arXiv:0905.0932v2 [hep-th]

[20] R. d'Inverno, *Introducing Einstein's Relativity*, Clarendon Press

[21] D. D. Blanco, H. Casini, L. Hung and R. C. Myers, *Relative Entropy and Holography*, arXiv:1305.3182v2 [hep-th]

[22] T. Takayanagi, *Entanglement entropy from a holographic viewpoint*, arXiv:1204.2450v2 [gr-qc]

[23] H. Liu and S. J. Suh, *Entanglement growth during thermalization in holographic systems*, arXiv:1311.1200v1 [hep-th]

[24] J. Abajo-Arrastia, J. Aparício and E. López, *Holographic Evolution of Entanglement Entropy*, arXiv:1006.4090v1 [hep-th]

[25] V. Balasubramanian, A. Bernamonti, J. de Boer, N. Copland, B. Craps, E. Keski-Vakkuri, B. Müler, A. Schäfer, M. Shigemori and W. Staessens, *Thermalization of Strongly Coupled Field Theories*, arXiv:1012.4753v3 [hep-th]

[26] D. Dieks, J. van Dongen and S. de Haro, *Emergence in Holographic Scenarios for Gravity* , arXiv:1501.04278v1 [hep-th]

[27] S. de Haro, *Dualities and Emergent Gravity: AdS/CFT and Verlindes Scheme*, arXiv:1501.06162v1 [physics.hist-ph]