The ϵ-window in three-dimensional bootstrap percolation

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April 2015

Abstract

In this thesis an introduction to bootstrap percolation and some applications will be given and an attempt will be made to determine the magnitude of the so-called ϵ-window, which is length of the transition from being unlikely to percolate to being the likely to percolate.

Contents

0.1 Introduction

0.1.1 What is bootstrap percolation

In a few words one can describe the process of bootstrap percolation as the following deterministic process on a grid. In the initial state sites(or points) are either infected or healthy. When a site has sufficient(for example 3) direct neighbours being infected it becomes infected as well. Afterwards sites which gets infected can infect other sites in the same way. One could for example wonder whether the final state consists of all sites being infected or not [1]. In the rest of the thesis the use of the notion occupied will be preferred to infected. In the rest of this section a more mathematical notation will be used to get the reader acquainted with the notation, which is similar to the one used in [2].

This bachelor thesis will consider bootstrap percolation on a cubic grid contained in $\mathbb{Z}^3$. Initially a one or zero is put at each site of the grid. To the sites with ones will be referred as *occupied* (also called *infected* in the literature [3]) and to the sites with zeros as *unoccupied*. On this grid an iterative process is defined. This process keeps every one unchanged and changes a zero into a one if at least half of its neighbouring sites contain a one. In some cases all sites will be occupied after some time, in such a case the system is said to *percolate*.

The bootstrap percolation process can be regarded as a discrete iterative process. The process is discrete in both time and space. In the notation of this thesis x_i will be used for discrete coordinates at a grid in 3-dimensions with length l and t for the discrete time starting at zero. The following function can be defined for the bootstrap process.

$$X_t(x_1, x_2, x_3) : \mathbb{N} \cup \{0\} \times \{1, 2, ..., l\}^3 \to \{0, 1\}$$

In this notation time is represented by $\mathbb{N} \cup \{0\}$ and the discrete coordinates are represented by $\{1, 2, ..., l\}^3$.

The time evolution rule is given by:

$$X_{t+1}(x_1, x_2, x_3) = \begin{cases} 1 \text{ if } X_t(x_1, x_2, x_3) = 1 \text{ or } \Sigma_{i=1}^{6} X_t(y_i) \geq 3 \\ 0 \text{ else} \end{cases}$$

Where y_i are the next neighbours of $x = (x_1, x_2, x_3)$, given by: $y_{i+\frac{3\pm3}{2}} = x \pm e_i$, where e_i are the basis vectors on the cubic grid.

So for a given initial state at $t = 0$ all later states are fixed. A random initial state will be considered and statements are made about $lim_{t\to\infty} X_t$. One says that if $\forall(x_1, x_2, x_3) :\ \lim_{t\to\infty} X_t(x_1, x_2, x_3) = 1$ the system *percolates*. With X_t the set of all occupied sites will be denoted.

As in [3] this bachelor thesis will be concerned with the chance to percolate for a $q - random$ initial state. This q-random initial state is obtained in the following way: Each site on the grid has a chance of q to be initially occupied.

The chance that a q-random initial state at a cubic grid $\mathbb{G}_n$ with sides of length l percolates is denoted by $P_{\mathbb{G}_n}(q)$. It will in some case be preferred to use $n = l^3$. Note that when $P_{\mathbb{G}_n}(q)$ is considered as a function of q, it is a function with the following properties. $P_{\mathbb{G}_n}(q)$ is increasing. The map $P_{\mathbb{G}_n}(q)$ maps zero to zero and maps the value one to the value one. $P_{\mathbb{G}_n}(q)$ is continuous. $P_{\mathbb{G}_n}(q)$ maps the interval $[0, 1]$ to $[0, 1]$ in a one-to-one manner.

In the notation of this thesis there is a chance denoted by $q_{\mathbb{G}_n}(\alpha)$ such that $P_{\mathbb{G}_n}(q_{\mathbb{G}_n}(\alpha)) = \alpha$. An illustration of a sketch of such a graph is given below.

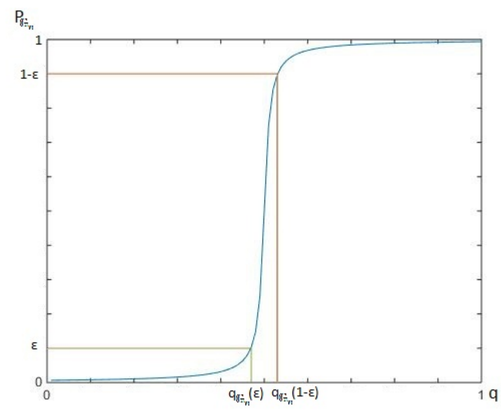

Figure 1: Possible graph of $P(\mathbb{G}_n)$ vs. q

By virtue of the notation for any $x \in [0, 1]$ holds that $x = q_{\mathbb{G}_n}(P_{\mathbb{G}_n}(x))$.

0.1.2 Relevance to physics

The bootstrap percolation model as explained has a few physical properties. The model should be considered as a nearest neighbour interaction. In physics nearest neighbour approximations are not uncommon. However in reality the influence of non-nearest neighbours is most often not zero, but in some cases negligible. Another property of the bootstrap percolation model is being a two-states model. The model assumes a grid and therefore discretization of space. As noted in [1] this feature seems to be present in physical systems such as a crystals. The bootstrap percolation model should be regarded as a simplification for real world problems.

The notion of bootstrap percolation is introduced in 1978 by J.Chalupa, P.L. Leath, P.L. and G.R Reich [1]. In their article bootstrap percolation is suggested as a model for state transition between a magnetic and anti-magnetic state. The bootstrap percolation model is a simplification of the Ising model and they therefore share several characteristics. Instead of bootstrap percolation on a grid, as done in this thesis, they considered bootstrap percolation on the Bethe lattice, a graph without loops where any site has a fixed number of neighbours. In 1988 M. Aizenman and J.L. Lebowitz [4] gave the first result for a finite grid. Their motivation was given by the study of metastability. Both articles were published in the Journal of Physics. As mentioned before, the bootstrap

percolation model is a simplification of the Ising model for magnetic spins [5]. The Ising model is a well-known model in Physics and has applications as well in other fields of science. The Ising model is as the bootstrap percolation model a model for sites at a grid. The sites in the Ising model represent particles with non-zero magnetic moment. As in the bootstrap percolation process, each site can be in two states. These states are often referred to as spin up(+1) or spin down(-1). Particles interact with their nearest neighbour by their spin. The spin of a particle interact with an external magnetic field as well. The Hamiltonian of the system is given by:

$$H = \Sigma_j \Sigma_{y_{i,j}} E^{i,j} \sigma_{y_{i,j}} \sigma_j - \Sigma_j B \sigma_j$$

j is the index that runs over the number of particles and $y_{i,j}$ are the nearest nearest neighbours of the particle corresponding to index j. $E^{i,j}$ is the interaction constant between particle i and its neighbour. σ_i represents the spin of particle i. For $B = 0$ and $E^{i,j} = E^{k,l}$ for all i, j, k and l the system can analysed. As outlined in [5] perturbation theory can be used if $B \neq 0$. The bootstrap percolation model can be thought of a situation where $B \neq 0$ and has such a magnitude such that the spins up don't flip to spin down.

In [6] a relation in biology about the saturation of hemoglobin is confirmed by use of the Ising model. In this article the authors use a lattice of four sites, which represent places where oxygen can bind to hemogloblin. As the model is quite small it seems to be solvable, which gives the same result as obtained in an earlier article.

Even some scientist have tried to apply Ising-like models to sociology as described in [7]. The most noticeable example given in the article is a model about business confidence. The sites represent managers who share their confidence and the external field is replaced by the economical facts.

Another social application is given in [8]. In this article the authors consider a modified bootstrap percolation model. In this model a site has far-away-neighbours, the interaction decreases with the distance. For example: a number of n occupied far-away-neighbours at distance d are needed to have the same effect as an occupied nearest neighbour. Unoccupied sites become occupied if an equivalent of k its nearest neighbours are occupied. However in contrast with this thesis, the authors are computing results for finite simulations, where the interest of this article concerns asymptotic behaviour.

In [9] the bootstrap percolation model is used to describe jamming transitions. These are transitions between non-equilibrium states such as glasses and gels. In this model a site becomes jammed if a certain amount of it neighbours are jammed. Comparable to the bootstrap percolation model it is assumed that jammed sites stay jammed over all time.

In 2009 Tim Hulshof [10] used the bootstrap percolation model to describe state transitions in a physical systems for his master thesis. He considered phase transitions between phases(solid, liquid and gas) and transitions between a paramagnetic and ferromagnetic state.

Theoretical results about asymptotic behaviour for the bootstrap percolation does not give accurate results [9], if the number of lattice points for asymptotic behaviour is not reached. Therefore relying at Theoretical results for asymptotic behaviour.

0.1.3 Aims of this thesis

When the function $P_{\mathbb{G}_n}(q)$ is considered as a function of q, there is a sharp threshold, i.e. there is a sudden increase of the chance of percolation at some point. The quantity of $q_{\mathbb{G}_n}(1-\epsilon) - q_{\mathbb{G}_n}(\epsilon)$ is referred to as the ϵ-window. Balogh and Bollobas[3] have proven that for any dimension larger than one the threshold is sharp, in case an unoccupied site gets occupied if it has at least two occupied neighbours. Friedgut and Kalai[11] have proven a more general statement for the presence of a sharp threshold. This theorem will be used to show that the bootstrap percolation process has a sharp threshold, for the case an unoccupied site gets occupied if at least half of it neighbours is occupied. In this thesis the considered grid will have dimension 3.

0.2 Theorem of Friedgut and Kalai

As mentioned earlier a theorem of Friedgut and Kalai [11] will be used. To state this theorem a few concepts need to be mentioned first. This theorem deals with properties of subgraphs of graphs. Such properties can be for example: set A has a diameter of at least l or set A will lead to percolation using the bootstrap percolation process. In the bootstrap percolation the graph will be the grid and the subgraph will be the random initially occupied sites. A graph is a collection of sites and edges, where an edge connects two sites.

Definition 0.2.1. *A property $\mathbb{P}$ is increasing if for every set B containing set A and set A satisfies property $\mathbb{P}$, then set B also satisfies property $\mathbb{P}$.*

Note the property of a set leading to percolation is increasing. To state the next theorem the notion of symmetric is needed. In this thesis the same definition as in Balogh and Bollobas[11] will be used.

Definition 0.2.2. *A property $\mathbb{P}$ in a space S is symmetric if for every x, y in S, there is a permutation σ of S, such that $\sigma(x) = y$ and the set of all sets satisfying property $\mathbb{P}$ is invariant.*

Using this definition of symmetric it won't be easy to find whether the bootstrap percolation on a large grid is not symmetric.

To state the next theorem the following notation of Friedgut and Kalai[11] is used. Let $\mu_q(\mathbb{P})$ be the chance that a random set satisfies property $\mathbb{P}$ when the edge probability is q. Edge probability is the chance that for a given lattice an edge is active. A lattice consist of both edges and sites. For the edge probability case, all sites are present, instead there is a probability of q for an edge to be active. In the notation introduced earlier for bootstrap percolation $P_{\mathbb{G}_n}(q)$ will be used, which is the probability of percolation when each initial site has chance q to be occupied.

The following theorem is by Friedgut and Kalai [11] and its proof will not be given in this thesis.

Theorem 0.2.1. *There is a constant c such that for every symmetric increasing property $\mathbb{P}$ on a graph with n sites with $\mu_q(\mathbb{P}) > \epsilon$ then*

$$\mu_{q+c\cdot q\frac{\log(1/\epsilon)\log(1/q)}{\log(n)}}(\mathbb{P}) \geq 1-\epsilon$$

This theorem is trivial if $\epsilon \geq \frac{1}{2}$, so it is only useful if $\epsilon < \frac{1}{2}$. This theorem will be applied to the torus. The torus will be introduced in the next section. After the desired relation for the torus is obtained it will be shown that the result can be converted to the grid.

0.3 Introducing the torus

An important object in this thesis will be the higher-dimensional torus. As an illustration it is shown below how to turn a two dimensional surface in a torus. This procedure includes connecting the top side to the bottom side and connecting the left and right side.

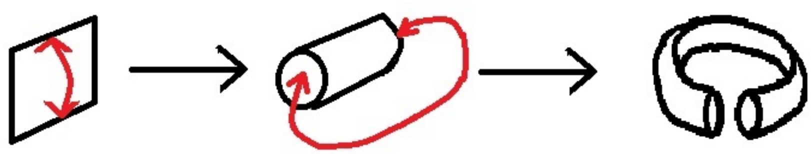

Figure 2: Turning a two dimensional plane into a torus (the intersections still need to be connected)

In a similar way if one wants to turn a normal grid into a torus, one should connect the edges of the object. In three dimension this means that to turn a box into a torus, one should connect the top side with the bottom, the front side with the back side and the left side with the right side. If coordinates are assigned to all sites on the grid. sites which are on the edges on a grid have coordinates $(..., 1, ...)$ or $(..., k, ...)$, where k is the length of the grid in that certain direction. At the torus $(..., k, ...)$ is considered to be a neighbour of $(..., 1, ...)$ and the other way around. This means that every site which was on an edge, gets at least one extra neighbour. On the torus all sites have equal amount of neighbours.

The torus is denoted by $\mathbb{T}_n$. For the torus we will use comparable notation as for grid. The chance of percolation is given by $P_{\mathbb{T}}(q)$ and the required value of q to have a certain chance of percolation P is given by $q_{\mathbb{T}_n}(P)$.

It is important to note that any initial state leading to percolation on the grid also leads to percolation on the torus. Therefore the following relation holds:

$$q_{\mathbb{T}_n}(P) < q_{\mathbb{G}_n}(P)$$

Similary the following relation holds.

$$P_{\mathbb{T}}(q) > P_{\mathbb{G}}(q)$$

Another important property of the torus is related to symmetric properties.

Proposition 0.3.1. *The property for initial states to lead to percolation on the torus is symmetric.*

Proof Take sites x,y in the torus with sides of length l and dimension d. Assign the spatial coordinates to every site. A site a has coordinates given by $(a_1, a_2, ..., a_d)$, where $1 \leq a_i \leq l$. Define the permutation σ by

$$\sigma(a) = (a_1 + (y_1 - x_1)(modl), a_2 + (y_2 - x_2)(modl), ..., a_d + (y_d - x_d)(modl))$$

Clearly $\sigma(x) = y$ and the set of sets satisfying the property of percolation is invariant, as sites which are neighbours stay neighbours after the permutation. $\square$

0.4 Applying the theorem of Friedgut and Kalai at the torus

In this section it will be shown that theorem 1 can be used for the bootstrap percolation process on the torus. Recall that the 3-dimensional torus can be considered to have the same structure as the grid, except for the difference that the sites on the top are considered to be neighbours of the sites on the bottom. The same relation holds for sites on the front side and back side and sites on the left side and right side of the box. The following corollary holds.

Corollary 0.4.1. *There is a constant c such that for sufficiently large n if $\frac{1}{2} \geq P_{\mathbb{T}_n}(q) > \epsilon$ then*
$P_{\mathbb{T}}(q + cq_{\mathbb{T}_n}(1/2) \cdot \log(\frac{1}{\epsilon})\frac{\log(1/q_{\mathbb{T}_n}(1/2))}{\log(n)}) \geq 1 - \epsilon$

Proof As shown earlier: The property of an initial state leading to percolation on the torus is symmetric. Obviously the property is monotone. In theorem 1 edge probability is used. For $l \geq 2$ there are more edges than sites. So to every site an edge can be assigned. To get a q-random site probability one should check whether the edge assigned to the site is active. So an active edge can be related to a 1 in the initial state of the bootstrap percolation process. So theorem 1 can be used. This theorem states that there is a constant c such that $q + c \cdot q\frac{\log(1/\epsilon)\log(1/q)}{\log(n)}$ suffices.

First it will be shown that

$$q_{\mathbb{T}_n}\log(1/q_{\mathbb{T}_n}(1/2))) > q\log(1/q) \tag{1}$$

Note that $P_{\mathbb{T}_n}(q) \leq P_{\mathbb{T}_n}(q_{\mathbb{T}_n}(1/2)) = 1/2$. As the function $P_T(x)$ is increasing in x, it can be found that $q < q_{\mathbb{T}_n}(1/2)$. Moreover we will use that $q_{\mathbb{T}_n}(1/2)$ is typically smaller than $\frac{1}{10}$ for large n. This is confirmed by [12]. For $0 < q < \frac{1}{10}$ we find that $q \cdot \log(1/q)$ is increasing. This leads to the following $q \cdot \log(1/q) < q_{\mathbb{T}_n}(1/2) \cdot \log(1/q_{\mathbb{T}_n}(1/2))$. For the given range of q, all terms are positive. So $\cdot q \cdot \log(1/q) < \cdot q_{\mathbb{T}_n}(1/2) \cdot \log(1/q_{\mathbb{T}_n}(1/2))$. this leads to equation (1).

Recall that theorem 1 stated that $q + c \cdot q\frac{\log(1/\epsilon)\log(1/q)}{\log(n)}$ suffices. Because of the inequality given by (1) it is clear that $c\log(\frac{1}{\epsilon} \cdot q\frac{\log(1/q)}{\log(n)}$ also suffices.

□

0.5 Deriving the relation for the grid

The interest of this article is showing that there is a number k such that

$$q_{\mathbb{G}_n}(1-\epsilon) - q_{\mathbb{G}_n}(\epsilon) \leq k \tag{2}$$

To achieve such a result we start with the following:

$$q_{\mathbb{G}_n}(1-\epsilon) = q_{\mathbb{T}_n}\Big(P_{\mathbb{T}_n}(q_{\mathbb{G}_n}(1-\epsilon))\Big) = q_{\mathbb{T}_n}\Big(1 - \Big(1 - P_{\mathbb{T}_n}(q_{\mathbb{G}_n}(1-\epsilon))\Big)\Big)$$

The corollary of the previous section leads to the existence of a constant c:

$$q_{\mathbb{T}_n}\Big(1 - \Big(1 - P_{\mathbb{T}_n}(q_{\mathbb{G}_n}(1-\epsilon))\Big)\Big)$$

$$\leq q_{\mathbb{T}_n}\Big(1 - P_{\mathbb{T}_n}(q_{\mathbb{G}_n}(1-\epsilon))\Big) + c\log(\frac{1}{1 - P_{\mathbb{T}_n}(q_{\mathbb{G}_n}(1-\epsilon))}) \cdot q_{\mathbb{T}_n}(\frac{1}{2})\frac{-\log(q_{\mathbb{T}_n}(\frac{1}{2}))}{\log(n)}$$

Note that:

$$P_{\mathbb{T}_n}(q_{\mathbb{G}_n}(1-\epsilon)) \geq P_{\mathbb{G}_n}(q_{\mathbb{G}_n}(1-\epsilon)) = 1-\epsilon$$

So

$$1 - P_{\mathbb{T}_n}(q_{\mathbb{G}_n}(1-\epsilon)) \leq 1 - P_{\mathbb{T}_n}(q_{\mathbb{G}_n}(1-\epsilon)) = \epsilon$$

Using the previous line and the fact that $q_{\mathbb{T}_n}(x)$ is increasing we get:

$$q_{\mathbb{T}_n}\Big(1 - P_{\mathbb{T}_n}(q_{\mathbb{G}_n}(1-\epsilon))\Big) \leq q_{\mathbb{T}_n}(\epsilon)$$

Adding an extra term to the previous line gives

$$
\begin{aligned}
&q_{\mathbb{T}_n}\Big(1-P_{\mathbb{T}_n}(q_{\mathbb{G}_n}(1-\epsilon))\Big)+c\log(\frac{1}{1-P_{\mathbb{T}_n}(q_{\mathbb{G}_n}(1-\epsilon))})\cdot q_{\mathbb{T}_n}(\frac{1}{2})\frac{-\log(q_{\mathbb{T}_n}(\frac{1}{2}))}{\log(n)}\\
&\qquad\le q_{\mathbb{T}_n}(\epsilon)+c\log(\frac{1}{1-P_{\mathbb{T}_n}(q_{\mathbb{G}_n}(1-\epsilon))})\cdot q_{\mathbb{T}_n}(\frac{1}{2})\frac{-\log(q_{\mathbb{T}_n}(\frac{1}{2}))}{\log(n)}\\
&\qquad\le q_{\mathbb{G}_n}(\epsilon)+c\log(\frac{1}{1-P_{\mathbb{T}_n}(q_{\mathbb{G}_n}(1-\epsilon))})\cdot q_{\mathbb{T}_n}(\frac{1}{2})\frac{-\log(q_{\mathbb{T}_n}(\frac{1}{2}))}{\log(n)}
\end{aligned}
$$

After connecting the top line with the previous line the following is obtained:

$$
q_{\mathbb{G}_n}(1-\epsilon)\le q_{\mathbb{G}_n}(\epsilon)+c\log(\frac{1}{1-P_{\mathbb{T}_n}(q_{\mathbb{G}_n}(1-\epsilon))})\cdot q_{\mathbb{T}_n}(\frac{1}{2})\frac{-\log(q_{\mathbb{T}_n}(\frac{1}{2}))}{\log(n)}
$$

This is the form which resembles the desired form given in (2). The next goal is to determine the magnitude of

$$
q_{\mathbb{G}_n}(1-\epsilon)-q_{\mathbb{G}_n}(\epsilon)
$$

From the previous line it is clear that this is smaller than

$$
c\log(\frac{1}{1-P_{\mathbb{T}_n}(q_{\mathbb{G}_n}(1-\epsilon))})\cdot q_{\mathbb{T}_n}(\frac{1}{2})\frac{-\log(q_{\mathbb{T}_n}(\frac{1}{2}))}{\log(n)} \tag{3}
$$

In [13] the magnitude $q_{\mathbb{T}_n}(\frac{1}{2})$ is determined.

To determine the magnitude of $q_{\mathbb{T}_n}(\frac{1}{2})$ the following notation will be used. The statement $x=O(y)$ means that there is a bounded constant C such that $x<Cy$.

Let d be the dimension of the grid and r be the required number of occupied neighbours needed to turn a zero in one. Then according to [13] $q_{\mathbb{T}_n}(\frac{1}{2})=O\Big(\frac{1}{\log_{r-1}(n)}\Big)^{d-r+1}$, where $\log_{r-1}$ is an $r-1$ times iterated logarithm. Using this (3) can be rewritten as :

$$
q_{\mathbb{T}_n}(\frac{1}{2})\frac{-\log(q_{\mathbb{T}_n}(\frac{1}{2}))}{\Big(1-P_{\mathbb{T}_n}(q_{\mathbb{G}_n}(1-\epsilon))\Big)\log(n)}=O\Big(\frac{\log_r(n)}{\log_{r-1}(n)\log(n)}\Big)\log(\frac{1}{1-P_{\mathbb{T}_n}(q_{\mathbb{G}_n}(1-\epsilon))}) \tag{4}
$$

To find the magnitude of $\frac{1}{1-P_{\mathbb{T}_n}(q_{\mathbb{G}_n}(1-\epsilon))}$ is not trivial. In the next section this magnitude will be discussed.

0.6 Final step to the magnitude of the ϵ-window

This section will continue on the previous one, with the aim to determine the magnitude of $\frac{1}{1-P_{\mathbb{T}_n}(q_{\mathbb{G}_n}(1-\epsilon))}$.

First an extra definition will be given.

Definition 0.6.1. *Let x be an occupied site, then the connected occupied neighbourhood of x at time t, denoted with $Y_t(x)$, is the set of sites such that it contains x and all occupied next neighbours of any site in the set.*

Note that by definition, if x is in $Y_t(x)$, there are no occupied next neighbours who are not in $Y_t(x)$. The statement in the previous sentence will be used a few times. If X_t denotes the set of all occupied sites, there is a one-to-one correspondence with the set of all distinct sets $Y_t(x)$, where x is an occupied site, denoted by Ξ_t. To update Ξ_t to Ξ_{t+1} the following updates need to occur [4]:

- If there is are unoccupied sites x_j, such that x_j has at least 3 neighbours in k distinct connected occupied neighbourhoods, $Y_t(x_i)$ where $i = 1, ..., k$ and $j = 1, ..., \gamma$. Then the new connected neighbourhood is given by the union of x_j and $Y_t(x_i)$ with $i = 1, ..., k$. Mathematically this can be defined as $\Xi_{t+1,j} = \big(\Xi_{t,j} \backslash \{Y_t(x_1), ..., Y_t(x_k)\}\big) \cup \{Y_t(x_1), ..., Y_t(x_k) \cup x_j\}$. One should remember that $\Xi_{t,j}$ is a set of sets and $Y_t(x_i)$ and $\{Y_t(x_1), ..., Y_t(x_k) \cup x_j\}$ are possible elements of such a set of sets.
- If there is after the previous process occupied sites y_p in $Y_t(y_r)$ and it has a occupied neighbour z_p which is not in $Y_t(x_r)$, where $p = 1, .., \rho$. Then those occupied connected neighbourhoods should merge to a new occupied connected neighbourhood. In mathematical notation one could define it in this way: $\Xi_{t+1,\gamma+p} = \big(\Xi_{t,\gamma+p-1} \backslash \{Y_t(y_p), Y_t(z_p)\}\big) \cup \{Y_t(y_p) \cup Y_t(z_p)\}$
- If no such site exists, the algorithm stops.

Note that $\Xi_{t,\gamma+\rho} = \Xi_{t+1}$ and that all sets in $\Xi_{t,\tau}$ are connected occupied neighbourhoods.

Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ be two sites in a set A. Then define $D_i(A) = max\{|x_i - y_i|, \forall x, y \in A\}$, where $1 \leq i \leq 3$ Using this notation the following lemma can be proved, which is similar to a lemma from [4].

Lemma 0.6.1. *If an initial state percolates, then $\forall k$ there is a t, τ such that in $\Xi_{t,tau}$ is a set such that $k \leq max(D_1, D_2, D_3) < 3k$*

Proof At any stage during the algorithm a set in $\Xi_{t,tau}$ can at most magnify its length by a factor 3. However for the largest set at the end of the algorithm it will hold that $max(D_1, D_2, D_3) = l$. □

In [4] it is shown that there is a value λ_c such that if $p\log(L) > \lambda$ then there is a constant c such that the chance that a two dimensional rectangle with length L percolates is larger than $1 - e^{-cL}$ and in [14] it is shown that $\lambda = \frac{\pi^2}{18}$. From [2] is follows that $q_{\mathbb{T}_n}(1 - \frac{\epsilon}{6}) = \Theta\Big(\frac{1}{\log(\log(L))}\Big)$, where $y = \Theta(x)$ means that there are two constants $c_1 \leq c_2$ such that $c_1 x \leq y \leq c_2 x$.

An object used in the next proof will be the square plate with side length d. This is a Cuboid with one side of length 1 and two others side of length d. These observations will lead to the proof of the theorem. The proof following theorem will be the final step to determining the length of the ϵ-window.

Theorem 0.6.1. *For $\frac{1}{2} > epsilon > 0$ and sufficiently large l and if it can be assumed that that the largest plate that has a chance of $1 - \frac{\epsilon}{6}$ to be present in a set, generated by the bootstrap percolation process, with length $\frac{\epsilon}{18}l$ has length larger than $\frac{1}{log(l)}$, then the following equation holds $q_{\mathbb{G}_n}(1-\epsilon) < q_{\mathbb{T}_n}(1-\frac{\epsilon}{6})$*

Before proving this theorem lets take a look at the relevance of this result. If the theorem holds, clearly $\frac{1}{1-P_{\mathbb{T}_n}(q_{\mathbb{G}_n}(1-\epsilon))} < \frac{1}{1-P_{\mathbb{T}_n}(q_{\mathbb{T}_n}(1-\frac{\epsilon}{a}))} = \frac{a}{\epsilon}$. Inserting this in (4) completes the goal of the thesis. However the theorem needs first to be proven.

Proof of theorem 6.1 To show that $q_{\mathbb{G}_n}(1-\epsilon) < q_{\mathbb{T}_n}(1-\frac{\epsilon}{6})$ it is sufficient to show that $P_{\mathbb{G}_n}(q_{\mathbb{G}_n}(1-\epsilon)) < P_{\mathbb{G}_n}(q_{\mathbb{T}_n}(1-\frac{\epsilon}{6})) = 1-\epsilon$.

By the previous lemma it can be found that there is a set A with $\frac{\epsilon}{18} \leq L_2 = max\{D_1(A), D_2(A), D_3(A)\} \leq \frac{\epsilon}{6}$, if the initial state percolates.

By assumption it holds that it can be assumed that that the largest plate that has a chance of $1-\frac{\epsilon}{6}$ to be present in a set, generated by the bootstrap percolation process, with length $\frac{\epsilon}{18}l$ has length larger than $\frac{1}{log(l)}$.

Recall that $q_{\mathbb{T}_n}(1-\frac{\epsilon}{6}) = \Theta\left(\frac{1}{\log(\log(l))}\right)$. Using this it can be shown that $q_{\mathbb{T}_n}(1-\frac{\epsilon}{6})\log(L_1) > \lambda$. So the chance that a rectangle with side length L_1 percolates is $1-e^{-cL_1}$.

On the edge of the plate the bootstrap percolation process is the following: All points have 1 occupied neighbour in the plate 4 on the edge of the plate and one more. If the neighbour that is not on the edge is neglected, the three-dimensional bootstrap percolation process is similar to the two-dimensional bootstrap percolation. This process can be repeated until $l-1$ slabs are added. Afterwards there is an occupied area of thickness L_1 and length l. Comparable to the plate this occupied area can grow in the other directions. Let $\lfloor x \rfloor$ denote the integer part of x. The chance at a sufficiently large plate filling the area will be at least $(1-e^{-cL_1})^{l+\lfloor \frac{l}{L_1} \rfloor l+l} \geq (1-(l+\lfloor \frac{l}{L_1} \rfloor l + l)e^{-cL_1})$. For sufficiently large L this is larger than $1-\frac{\epsilon}{6}$.

If all the previous is combined it can be found that: There is a chance of at least $1-\frac{\epsilon}{6}$ the system percolates at the torus. There is a chance of at least $(1-\frac{\epsilon}{6})^3$ that the occupied connected neighbourhood does not cross the edges of the grid, as its position is random in the torus. There is a chance of at least $1-\frac{\epsilon}{6}$ that there is a plate of length L_1 in the largest connected occupied neighbourhood. For sufficient large l the chance that the presence of such a plate leads to percolation is larger than $1-\frac{\epsilon}{6}$. So the total chance is larger than $(1-\frac{\epsilon}{6})^6 > 1-\epsilon$. □

0.7 Relevance of the ϵ-window to computer simulations

Since it is shown that the ϵ-window approaches zero and logarithmically depends on ϵ, it should be discouraged to think that any ϵ-window is a good measure for the error in a computer simulations, unless it is know that the error goes faster to zero than the size of the ϵ-window. Before an analytical expression was given in [12] some computer simulations were done to determine the value of $q_{\mathbb{G}_n}(\frac{1}{2})$. In [15] and [16] computer simulations were done to determine the asymptotic value of the bootstrap percolation model in two and three dimensions.

0.8 Conclusion and outlook

It is a pity that to obtain the result I had to assume that the largest plate that has a chance of $1-\frac{\epsilon}{6}$ to be present in a set, generated by the bootstrap percolation process, with length $\frac{\epsilon}{18}l$ has length larger than $\frac{1}{\log(l)}$, instead of proving it.

It can be understood that if the longest side of the set A given in the proof is in the x_3 direction then $D_1(A) > \frac{1}{3}\log(\log(l))$ and $D_2(A) > \frac{1}{3}\log(\log(l))$. The surface of an occupied connected neighbourhood does not increase during the bootstrap percolation process. So to end with a surface of $2D_3(A)D_1(A)+2D_3(A)D_2(A)+2D_2(A)D_1(A)$ one needs to start with an equal surface. Therefore one needs $\frac{2D_3(A)D_1(A)+2D_3(A)D_2(A)+2D_1(A)D_2(A)}{6}$ initially occupied sites. The expected number of initially occupied sites is $V\cdot p \leq D_1(A)D_2(A)D_3(A)\frac{1}{\log(\log(l)}$. This number should be as large as $\frac{D_3(A)D_1(A)+D_3(A)D_2(A)+D_1(A)D_2(A)}{3}$. Therefore

$$D_1(A) > \frac{1}{3}\log(\log(l)) \text{ and } D_2(A) > \frac{1}{3}\log(\log(l))$$

However this result wasn't sufficient to show the validation of the assumption.

From the derivation of section 5 it is clear that

$$q_{\mathbb{G}_n}(1-\epsilon) \leq q_{\mathbb{T}_n}\Big(1-P_{\mathbb{T}_n}(q_{\mathbb{G}_n}(1-\epsilon))\Big)+c\log(\frac{1}{1-P_{\mathbb{T}_n}(q_{\mathbb{G}_n}(1-\epsilon))})\cdot q_{\mathbb{T}_n}(\frac{1}{2})\frac{-\log(q_{\mathbb{T}_n}(\frac{1}{2}))}{\log(n)}$$

So a sharper approximation of the ϵ-window would be

$q_{\mathbb{G}_n}(1-\epsilon)-q_{\mathbb{G}_n}(\epsilon) \leq \Big(q_{\mathbb{T}_n}\Big(1-P_{\mathbb{T}_n}(q_{\mathbb{G}_n}(1-\epsilon))\Big)-q_{\mathbb{G}_n}(\epsilon)\Big)+c\log(\frac{1}{1-P_{\mathbb{T}_n}(q_{\mathbb{G}_n}(1-\epsilon))})\cdot q_{\mathbb{T}_n}(\frac{1}{2})\frac{-\log(q_{\mathbb{T}_n}(\frac{1}{2}))}{\log(n)}$

This is just to say that it is unknown to me whether the ϵ-window is the largest part of $c\log(\frac{1}{1-P_{\mathbb{T}_n}(q_{\mathbb{G}_n}(1-\epsilon))})\cdot q_{\mathbb{T}_n}(\frac{1}{2})\frac{-\log(q_{\mathbb{T}_n}(\frac{1}{2}))}{\log(n)}$ or $-\Big(q_{\mathbb{T}_n}\Big(1-P_{\mathbb{T}_n}(q_{\mathbb{G}_n}(1-\epsilon))\Big)-q_{\mathbb{G}_n}(\epsilon)\Big)$

Further research could include the validation of the assumption for the final theorem or the ratio between the ϵ-window and $-\Big(q_{\mathbb{T}_n}\Big(1-P_{\mathbb{T}_n}(q_{\mathbb{G}_n}(1-\epsilon))\Big)-q_{\mathbb{G}_n}(\epsilon)\Big)$.

If a validation is given for the assumption in the final theorem and that assumption could be generalised to higher dimension, then it is likely that the result obtained in this thesis also might be generalised to higher dimensions.

0.9 Acknowledgements

I would like Dr. Van Enter for introducing me to the problem and the useful conversations we had.

0.10 Appendix, validation of an assumption

In this appendix a validation of the assumption in theorem 6.1 will be given. The assumption which needs to be validated is that if there is a set with one side such that $D_3(A) > \epsilon l$, then for an arbitrary large chance $(1-\epsilon)$, there is an l such that it contains a plate with sides at least $\log(l)$ with chance $1-\epsilon$. It is useful to realise that it doesn't matter whether you initially skip some updates to the total state of the system during the bootstrap percolation process. Therefore the bootstrap percolation process on a set can be considered, and the outer sites can be considered a q-random. To estimate some quantities, Wolfram Mathematica has been used. The following definition will be given to state some concepts.

Definition 0.10.1. *Let A be a set, which is a connected occupied neighbourhood, then for $i = 1, 2, 3$ a slice of A, denoted by $S_i(y)$, is the set $S_i(y) = \{x \in A \text{ such that } x_i = y\}$.*

Lemma 0.10.1. *Let l be the length of the grid and A a set generated by the bootstrap percolation process with $D_3(A) > \epsilon l$ for some $\epsilon > 0$. $\forall a > 0$ $\exists l$ such that there are less than $\frac{l}{a(\log(l))^2}$ sites z_i such that both $D_1(S_3(z_i)) < \sqrt[4]{\log l}$ and $D_2(S_3(z_i)) < \sqrt[4]{\log l}$*

sketch to proof The idea behind this proof is that if the result does not hold, then it is unlikely that a set with $D_3(A) > \epsilon l$ exists anywhere on the grid.

Let $a > 0$. Assume that there are more than $\frac{l}{a\log(l)}$ of sites z_i such that both $D_1(S_3(z_i)) < \sqrt[4]{\log l}$ and $D_2(S_3(z_i)) < \sqrt[4]{\log l}$. For the proof of this lemma it is useful to consider slices of A. To determine whether a slice $S_3(x_3)$ is filled the following construction is used. Impose that if for some $x \in A$ if $x = (x_1, x_2, x_3) \in S_3(x_3)$ then $(x_1, x_2.x_3 + 3) \in S_3(x_1 + 1)$ and if $y = (y_1, y_2, y_3) \in S_3(y_1 + 1)$ then $(y_1, y_2, y_3 - 1) \in S_3(y_1)$. A site $v = (v_1, v_2, v_3)$ with $v_3 < x_3$ is at most occupied, so every empty site in $S_3(x_3)$ needs at least two neighbours in $S_3(x_3)$ to become occupied(by construction a site in $S_3(x_3)$ is only empty if it neighbour in $S_3(x_3+1)$ is empty). For the sites in $S_3(x_3+1)$ holds a similar relation, with the remark that a sites v with $v_3 > y_1 + 1$ is at most occupied. Note that the modification is only a useful mathematical tool to get a 'bound' on the influence of $S_3(x_3 + 1)$ on $S_3(x_3)$.

Some points have neighbours with the same coordinate in the third direction, but not in the slice of A. These points are by construction empty, or do not affect the bootstrap percolation process(in case the modification is not applied). If these points affected the bootstrap percolation process, there would be a connection between the occupied point and the slice, contradicting the fact that

A is occupied connected neighbourhood and the assumption that this site is not in the slice of A.

Note that this resembles a two-dimensional bootstrap percolation problem, with a sites initially being occupied with a chance of $2q-q^2 < 2q$. Suppose there is a largest occupied connected neighbourhood in $S_3(y_1)$ with at most $D_1(A) = \sqrt[4]{\log(l)}$ and $D_2(A) = \sqrt[4]{\log(l)}$. If in two dimensions an occupied connected neighbourhood is present, then the smallest rectangle surrounding it percolates. So the chance, that the smallest rectangle surrounding it percolates, is smaller than the chance of on occupied connected neighbourhood to be present. If $(2q-q^2)\log(l) < \frac{\pi^2}{18}$, then the chance for a rectangle to percolate is given by [4][14]

$$exp\Big(-(\frac{\pi^2}{18} - (2q-q^2)\log(l) + o(1))\frac{1}{p}\Big)$$

$$\leq exp\Big(-(\frac{\pi^2}{18} - (2q-q^2)\log(l) + o(1))\Theta\big(\log(\log(l))\big)\Big)$$

$$= \Theta\big(\log(l)^{-(\frac{\pi^2}{18}-(2q-q^2)\log(l)+o(1))}\big)$$

Where $o(1)$ approaches zero in the limit if $\log(l) \to \infty$. Since $q_{\mathbb{T}_n}(1-\frac{\epsilon}{6}) = \Theta\Big(\frac{1}{\log(\log(l))}\Big)$ [2] this condition is met. For shorter notation in the following the following will be introduced $b = -\big(\frac{\pi^2}{18} - (2q-q^2)\log(l) + o(1)\big)$. For $l \to \infty$ it holds that $b \to -\frac{\pi^2}{18}$

If in $S_3(y_1)$ there is a connected occupied neighbourhood, then there is a smallest rectangle containing that neighbourhood. The rectangle has at least a chance of $1-\log(l)^b$ at not percolating for sufficiently large l. If we characterise a rectangle by its side lengths and the coordinates of the lower right corner, then there are in a square area of $\sqrt[4]{\log l}$ at most $\log l$ rectangles. The chance that at least one set in $S_3(y_1)$ is an occupied connected neighbourhood is therefore less than $1-(1-\log(l)^b)^{\log l}$.

Now it can be shown that it is unlikely that there are more than $\frac{l}{a(\log(l))^2}$ of sites z_i such that either $D_1(S_3(z_i)) < \sqrt[4]{\log l}$ or $D_2(S_3(z_i)) < \sqrt[4]{\log l}$. There needs to be a set, that percolates in two dimensions, in every of these slices. Furthermore it needs to be so unlikely that it is not only unlikely to happen for a single case, but even so unlikely that there isn't likely to happen anywhere on the grid. A good measure to determine whether it doesn't happen anywhere on the grid is considering the chance that it doesn't happen and raise it to the power l^3. In the end it is likely that there is no such a set which violates the result, as for $l \to \infty$:

$$\Big(1 - \big(1-(1-\log(l)^b)^{\log l}\big)^{\frac{\epsilon l}{a(\log(l))^2}}\Big)^{l^3} \to 1$$

$\square$

As a larger value of $D_1(S_3(z_i))$ does not affect $D_2(S_3(z_i))$ in a negative way (but rather are positively correlated), it is reasonable to say that the chance for $D_i(S_3(z_i))$ (i=1,2) to be smaller than $\sqrt[4]{\log l}$ is at most the square root the chance of both being smaller than $\sqrt[4]{\log l}$.

This result shows that the largest part of the sides of a face-to-face connected set B with $D_3(B) > \epsilon l$ are larger than $\sqrt[4]{\log l}$. To continue on this result, squares on the sides of B will be considered. These squares have an occupied neighbour inside set B. So they need only two neighbours to percolate. Therefore it can again be considered as a two dimensional problem. The chance that a square will percolate on the edge is not very large $\Theta\big(\log(l)^b\big)$. As a result of the length of B, there is still a reasonable chance that there is at least one. For the next step there should 'grow'(percolate) $\log(l)$ squares with sides $\sqrt[4]{\log l}$ on each other. The probability that this happens at some position is:

$$1 - \Big(1 - \Theta\big(\log(l)^{b\log(l)}\big)\Big)^{\frac{\epsilon l}{\sqrt[4]{\log(l)}}}$$

Up to now it is shown that if the set B has $D_3(B) > \epsilon l$, then at the majority of places $D_1(S_3(z_i)) \geq \sqrt[4]{\log l}$ and $D_2(S_3(z_i)) \geq \sqrt[4]{\log l}$, and it is likely to have a pile of occupied squares on it. This pile has 'height' $log(l)$ and has a width of $D_3(B) > \epsilon l$. On the corner of the pile and the rest of set B, sites have at least two occupied neighbours(at least one in the pile and one in set B). As the corner has length $log(l)$, there is a chance at $1 - (1 - \frac{1}{\log(\log(l))})^{\sqrt[4]{\log(l)}}$ that at least one site is initially occupied. The sites next to an initially occupied site(at the corner) have at least 3 neighbours (at least one in the pile and one in set B and the one which is initially occupied) and become occupied as well. This continues until that entire row in the corner is occupied. Once the entire row is occupied, any site above or next to percolated row has 2 occupied neighbours. So these rows each need only one initially occupied point to percolate. The idea is that in this way a cuboid with sides $\log(l) \times \log(l) \times \sqrt[4]{\log l}$ will be formed. The chance that this happen is $\Big(1-(1-\frac{1}{\log(\log(l))})^{\sqrt[4]{\log(l)}}\Big)^{(\log(l))^2} \to 1$ if $l \to \infty$. This block contains the desired plate.

in the last two paragraphs, it might have seemed whether a set B with flat surfaces was considered. It doesn't have to be the case that B has a flat surface. However if this is not the case, there are points outside B on its edge with two or more neighbours. Comparable arguments can be used if this the surface is not flat, but the adjustments should be quite case-specific. For example: if there is a flat surface with a few occupied sites on top, the ones on top can be neglected and it can be treated in the proof as a flat surface, with random occupation on the top. Making a few occupied sites randomly occupied, lowers the chance at the presence of a plate and therefore can be done, without drastically changing to the proof. In case there are a sufficient occupied sites on top, then the occupied sites are likely to make a square on top percolate. If there isn't anything at all looking at a flat surface (for example: diagonal plane), it means that most sites have at least 2 neighbours. If most sites have at least 2 neighbours, only a few extra sites are needed to make the surface grow. For the argument of piling squares on set B it is sufficient if B contains $\frac{l}{alog(l)}$ plates with side length $\sqrt[4]{\log l}$, which haven't any sites above a site in another one. The sets containing only a few plates with sufficient side length are probably unlikely to exist.

To estimate the limiting behaviour of some quantities given earlier, the following codes in Wolfram Mathematica are used.

- $Plot[(1-(1-(1-1/Sqrt[x])\hat{}x)\hat{}(E\hat{}x/(x\hat{}2)))\hat{}(E\hat{}x), \{x, 0, 20\}]$
- $Plot[(1-1/(x\hat{}(x/2)))\hat{}(E\hat{}x/(x\hat{}0.25)), \{x, 0, 1000\}]$
- $Plot[(1-(1-1/(Log[x]))\hat{}(x\hat{}0.25))\hat{}(x\hat{}2), \{x, 0, 2*10\hat{}12\}]$

  These equations are written in a slightly different form. In these equation $x = \log(l)$ and it is used that for large l it holds that $b < -1/2$.

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