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Fractional Calculus

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Abstract

This thesis introduces fractional derivatives and fractional integrals, shortly differintegrals. After a short introduction and some preliminaries the Grünwald-Letnikov and Riemann-Liouville approaches for defining a differintegral will be explored. Then some basic properties of differintegrals, such as linearity, the Leibniz rule and composition, will be proved. Thereafter the definitions of the differintegrals will be applied to a few examples. Also fractional differential equations and one method for solving them will be discussed. The thesis ends with some examples of fractional differential equations and applications of differintegrals.

Contents

1 Introduction

Fractional calculus explores integrals and derivatives of functions. However, in this branch of Mathematics we are not looking at the usual integer order but at the non-integer order integrals and derivatives. These are called fractional derivatives and fractional integrals, which can be of real or complex orders and therefore also include integer orders. In this thesis we refer to differintegrals if we are talking about the combination of these fractional derivatives and integrals.

So if we consider the function $f(t) = \frac{1}{2}x^2$, the well-known integer first-order and second-order derivatives are $f'(t) = x$ and $f''(t) = 1$, respectively. But what if we would like to take the $\frac{1}{2}$-th order derivative or maybe even the $\sqrt{\frac{1}{2}}$-th order derivative? This question was already mentioned in a letter from the mathematician Leibniz to L'Hôpital in 1695. Since then several famous mathematicians, such as Grünwald, Letnikov, Riemann, Liouville and many more, have dealt with this problem. Some of them came up with an approach on how to define such a differentiation operator. For a very interesting more detailed history of Fractional Calculus we refer to [1, p. 1-15]

First in chapter 2 we shall give some basic formulas and techniques which are necessary to better understand the rest of the thesis. Then in chapter 3 two definitions for a differintegral will be given. The Grünwald-Letnikov and the Riemann-Liouville approach will be explored. These are the two most frequently used differintegrals. Afterwards in chapter 4 some basic properties of these differintegrals will be given and proved. Then in chapter 5 we shall explore a few examples. In chapter 6 we will take a look at fractional differential equations (FDE's). Therefore we also need to explore the Laplace transforms of fractional derivatives. Chapter 6 ends with some examples of FDE's. Thereafter chapter 7 deals with a few applications of differintegrals which is followed by a conclusion in chapter 8.

2 Preliminaries

In this section we shall give some basic formulas and techniques which are necessary to better understand the rest of the thesis. We start off with the Gamma function.

2.1 The Gamma Function

The Gamma function plays an important role in Fractional Calculus and therefore it is mentioned in the Preliminaries.

Definition 2.1. *Let $z \in \mathbb{C}$, then we define the Gamma function as*

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} \, \mathrm{d}t.$$

This integral converges for $\mathrm{Re}(z) > 0$ (the right half of the complex plane).

One of the basic properties of the Gamma function is

$$\Gamma(z+1) = z\Gamma(z). \tag{1}$$

To prove this we integrate the formula for the Gamma function given in Definition 2.1 by parts

$$\Gamma(z+1) = \int_0^\infty e^{-t} t^z = [-e^{-t} t^z]_{t=0}^{t=\infty} + z \int_0^\infty e^{-t} t^{z-1} \, \mathrm{d}t,$$

where the first term drops out and the second term is equal to $z\Gamma(z)$, so identity (1) follows. We also have $\Gamma(1) = 1$ and if we use identity (1) we get

$$\begin{aligned}
\Gamma(2) &= 1 \cdot \Gamma(1) = 1 = 1! \\
\Gamma(3) &= 2 \cdot \Gamma(2) = 2 \cdot 1! = 2! \\
\Gamma(4) &= 3 \cdot \Gamma(3) = 3 \cdot 2! = 3! \\
\vdots \quad &\quad \vdots \qquad\quad \vdots \qquad \vdots \\
\Gamma(n+1) &= n \cdot \Gamma(n) = n \cdot (n-1)! = n!
\end{aligned}$$

So by induction it follows that $\Gamma(n+1) = n!$ for all $n \in \mathrm{N}$.

2.2 The Beta Function

In some cases the Beta function is more favorable than the Gamma function. Since it is convenient to use it in fractional derivatives of the Power function, we also mention the Beta function here.

Definition 2.2. *Let $z, w \in \mathbb{C}$, then we define the Beta function as*

$$B(z,w) = \int_0^1 \tau^{z-1} (1-\tau)^{w-1} \, \mathrm{d}\tau,$$

for $\mathrm{Re}(z) > 0$ and $\mathrm{Re}(w) > 0$. After we use the Laplace transform for convolutions the Beta function can be expressed in terms of the Gamma function by

$$B(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)} \tag{2}$$

and it follows from (2) that

$$B(z,w) = B(w,z). \tag{3}$$

With the Beta function it is possible to obtain two useful results for the Gamma function

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}, \tag{4}$$

$$\Gamma(z)\Gamma(z+\frac{1}{2}) = \sqrt{\pi}2^{2z-1}\Gamma(2z). \tag{5}$$

2.3 Change the Order of Integration

In section 4.4 about the composition of differintegrals we will take advantage of changing the order of an integral. If we have any function $f(t,\tau,\xi)$ which is integrable with respect to τ and ξ the change of order is given by the following formula

$$\int_a^t \int_a^\tau f(t,\tau,\xi)\,\mathrm{d}\xi\,\mathrm{d}\tau = \int_a^t \int_\xi^t f(t,\tau,\xi)\,\mathrm{d}\tau\,\mathrm{d}\xi. \tag{6}$$

2.4 The Mittag-Leffler Function

We know in integer-order differential equations the exponential function e^z plays an important role. This can also be written in its series form which is given by

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)}.$$

More generally, we can consider the expression

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \tag{7}$$

where $\alpha, \beta \in \mathbb{C}$ and $\mathrm{Re}(\alpha) > 0$. We see that in the special case of $\alpha = 1$ and $\beta = 1$ we have $E_{1,1}(z) = e^z$. This generalization is called the Mittag-Leffler function and the two-parameter function is very useful in the fractional calculus, especially in fractional differential equations, which we will discuss in section 6.

Since the series for the Mittag-Leffler function (7) is uniformly convergent on all compact subsets of $\mathbb{C}$ we can differentiate it term by term to get the following expression which is also necessary later on.

Corollary 2.1. *Let $z \in \mathbb{C}$, $\alpha, \beta \in \mathbb{C}$, $Re(\alpha) > 0$ and $m \in \mathbb{N}$, then the m-times differentiated Mittag-Leffler function is given by*

$$E_{\alpha,\beta}^{(m)}(z) = \sum_{k=0}^{\infty} \frac{(k+m)!}{k!} \frac{z^k}{\Gamma(\alpha k+\alpha m+\beta)}.$$

3 Fractional Derivatives and Integrals

In fact the term 'Fractional Calculus' is not appropriate since it does not mean the fraction of any calculus, nor the calculus of fractions. It is actually the branch of Mathematics which generalizes the integer-order differentiation and integration to derivatives and integrals of arbitrary order. If we look at the sequence of integer order integrals and derivatives

$$\ldots, \int_a^t \int_a^{\tau_2} f(\tau_1)\,\mathrm{d}\tau_1\ \mathrm{d}\tau_2, \int_a^t f(\tau_1)\,\mathrm{d}\tau_1, f(t), \frac{\mathrm{d}f(t)}{\mathrm{d}t}, \frac{\mathrm{d}^2 f(t)}{\mathrm{d}t^2}, \ldots$$

one can see the derivative of arbitrary order α as the insertion between two operators in this sequence. It is called a fractional derivative and throughout this thesis the following notation is used:

$${}_aD_t^{\alpha} f(t).$$

For a fractional integral the same notation is used, but with $\alpha < 0$. Thus an integral of order β can be denoted by:

$${}_aD_t^{-\beta} f(t).$$

In this thesis we refer to this with the term differintegral. The subscripts a and t are called the terminals of the differintegral and they are the limits of integration.

There have been different approaches to define this differintegral and this section deals with the definitions of the differintegrals from Grünwald-Letnikov and Riemann-Liouville.

3.1 The Grünwald-Letnikov construction

In this section we will derive a formula for the so-called Grünwald-Letnikov differintegral. The proof is based on the forward difference derivative given by

$$f'(t) = \lim_{h\to 0} \frac{f(t+h) - f(t)}{h}.$$

If we apply this formula again we get the well-known second-order derivative

$$f''(t) = \lim_{h\to 0} \frac{f(t+2h) - 2f(t+h) + f(t)}{h^2}.$$

We can generalize this formula for a derivative and if we use the binomial coefficient $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ we get for the n^{th}-derivative

$$f^{(n)}(t) = \lim_{h\to 0} \frac{\sum_{0\le r\le n} (-1)^r \binom{n}{r} f(t+(n-r)h)}{h^n}.$$

If we replace the integer n by $p \in \mathbb{R}$ we obtain the following definition.

Definition 3.1. *Let m be the smallest natural number such that $|p| \le m$, then we define the Grünwald-Letnikov differintegral as*

$$D^p f(t) = \lim_{h\to 0} \frac{1}{h^p} \sum_{0\le r<m} (-1)^r \binom{p}{r} f(t+(p-r)h).$$

Since we replaced the integer n by the real number p we also have to generalize the definition of the binomial coefficient. This can be done using the multiplicative formula which gives

$$\binom{p}{r} = \frac{p(p-1)(p-2)\cdots(p-r+1)}{r(r-1)(r-2)\cdots 1}, \tag{8}$$

where $r \in \mathbb{N}$. When the substition $h \to -h$ is made in Definition 3.1 we get the "direct" Grünwald-Letnikov differintegral given by

$$\begin{aligned} {}_aD_t^p f(t) &= \lim_{\substack{h\to 0 \\ mh=t-a}} h^{-p} \sum_{r=0}^{m} (-1)^r \binom{p}{r} f(t-rh) \\ &= \lim_{h\to 0} \left(\frac{t-a}{m}\right)^{-p} \sum_{r=0}^{m} (-1)^r \binom{p}{r} f\left(t - r\frac{t-a}{m}\right). \end{aligned} \tag{9}$$

When $p = n$ this can be seen as the n^{th}-order derivative and if $p = -n$ it represents the n-fold integral.

The Grünwald-Letnikov and the Riemann-Liouville fractional derivative can be related to each other. Therefore we need another expression for the Grünwald-Letnikov derivative of arbitrary order. This is given by the following formula.

Corollary 3.1.

$${}_aD_t^p f(t) = \sum_{k=0}^{m} \frac{f^{(k)}(a)(t-a)^{-p+k}}{\Gamma(-p+k+1)} + \frac{1}{\Gamma(-p+m+1)} \int_a^t (t-\tau)^{m-p} f^{(m+1)}(\tau)\,\mathrm{d}\tau.$$

In the last formula the derivatives $f^{(k)}(t)$ for $k = 1, 2, \dots, m+1$ have to be continuous in the closed interval $[a, t]$ and $m > p - 1$. The proof of Corollary 3.1 is pretty long. Therefore it won't be given in this thesis, but it can be found in [2, p. 52-55].

3.2 The Riemann-Liouville construction

Instead of beginning with the derivative as in the Grünwald-Letnikov approach, the Riemann-Liouville starts with the integral. The differintegral is given by the following expression:

$${}_aD_t^p f(t) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{m+1} \int_a^t (t-\tau)^{m-p} f(\tau)\,\mathrm{d}\tau, \tag{10}$$

where $m \in \mathbb{N}$ satisfies ($m \le p < m+1$). The expression for the Grünwald-Letnikov fractional derivative given in Corollary 3.1 can be seen as a special case of the last formula. Corollary 3.1 can be obtained from (10) by repeatedly performing integration by parts and differentiation. The requirement of $f(t)$ being integrable is a sufficient condition since then the integral given in (10) exists for $t > a$ and it is possible to differentiate it $m + 1$ times. We shall now show how to obtain the Riemann-Liouville fractional integral and thereafter how to obtain the Riemann-Liouville fractional derivative.

3.2.1 The Riemann-Liouville Fractional Integral

The Riemann-Liouville differintegral is obtained by combining integer-order derivatives and integrals. First we will generalize the definition of an integral to get the Cauchy formula. If $f(\tau)$ is integrable in every finite interval (a,t) the integral

$$f^{(-1)}(t) = \int_a^t f(\tau)\,\mathrm{d}\tau$$

exists. Next we look at the two-fold integral:

$$\begin{aligned} f^{(-2)}(t) = \int_a^t \mathrm{d}\tau_1 \int_a^{\tau_1} f(\tau)\,\mathrm{d}\tau &= \int_a^t f(\tau)\,\mathrm{d}\tau \int_\tau^t \mathrm{d}\tau_1 \\ &= \int_a^t (t-\tau) f(\tau)\,\mathrm{d}\tau. \end{aligned}$$

If the last expression is integrated we obtain the three-fold integral of $f(t)$

$$\begin{aligned} f^{(-3)}(t) &= \int_a^t \mathrm{d}\tau_1 \int_a^{\tau_1} \mathrm{d}\tau_2 \int_a^{\tau_2} f(\tau)\,\mathrm{d}\tau \\ &= \int_a^t \mathrm{d}\tau_1 \int_a^{\tau_1} (\tau_1-\tau) f(\tau)\,\mathrm{d}\tau \\ &= \frac{1}{2}\int_a^t (t-\tau)^2 f(\tau)\,\mathrm{d}\tau. \end{aligned}$$

Then, using induction, the Cauchy formula is derived

$$f^{(-n)}(t) = \frac{1}{\Gamma(n)} \int_a^t (t-\tau)^{n-1} f(\tau)\,\mathrm{d}\tau. \tag{11}$$

If we replace the integer n in the Cauchy formula (11) by the real number p we obtain an integral of arbitrary order.

Definition 3.2. *The Riemann-Liouville fractional integral of order $p \in \mathbb{R}_{>0}$ is given by*

$${}_aD_t^{-p} f(t) = \frac{1}{\Gamma(p)} \int_a^t (t-\tau)^{p-1} f(\tau)\,\mathrm{d}\tau.$$

3.2.2 The Riemann-Liouville Fractional Derivative

Now we will show how to obtain the Riemann-Liouville fractional derivative. If we fix $n \geq 1$ in formula (11) and take an integer k then it is possible to rewrite this expression as

$$f^{(k-n)}(t) = \frac{1}{\Gamma(n)} D^k \int_a^t (t-\tau)^{n-1} f(\tau)\,\mathrm{d}\tau, \tag{12}$$

where D^k represents k iterated integrations if $k \leq 0$ and k differentiations if $k > 0$. Formula (12) gives iterated integrals of $f(t)$ when $k = n-1, n-2, \ldots$, the function $f(t)$ if $k = n$ and it gives the derivatives of order $k-n = 1,2,3,\ldots$ of the function $f(t)$ when $k = n+1, n+2, n+3, \ldots$

If we replace the integer n in formula (12) by $\alpha \in \mathbb{R}$ with $k-\alpha > 0$ we obtain an expression for differentiation of non-integer order

$$ {}_aD_t^{k-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \frac{\mathrm{d}^k}{\mathrm{d}t^k} \int_a^t (t-\tau)^{\alpha-1} f(\tau)\,\mathrm{d}\tau, \tag{13} $$

where $0 < \alpha \leq 1$. If we set $p = k - \alpha$ we can rewrite the last expression and obtain a derivative of arbitrary order.

Definition 3.3. *The Riemann-Liouville fractional derivative of order $p \in \mathbb{R}_{>0}$ is given by*

$$ \begin{aligned} {}_aD_t^p f(t) &= \frac{1}{\Gamma(k-p)} \frac{\mathrm{d}^k}{\mathrm{d}t^k} \int_a^t (t-\tau)^{k-p-1} f(\tau)\,\mathrm{d}\tau \\ &= \frac{\mathrm{d}^k}{\mathrm{d}t^k} \Big({}_aD_t^{-(k-p)} f(t) \Big), \quad (k-1 \leq p < k). \end{aligned} $$

In the last equality of Definition 3.3 we used the definition of the Riemann-Liouville fractional integral given in Definition 3.2. If $\alpha = 1$ we have $p = k-1$ and we deal with the derivative of integer order with order $k-1$

$$ \begin{aligned} {}_aD_t^{k-1} f(t) &= \frac{\mathrm{d}^k}{\mathrm{d}t^k} \Big({}_aD_t^{-(k-(k-1))} f(t) \Big) \\ &= \frac{\mathrm{d}^k}{\mathrm{d}t^k} \Big({}_aD_t^{-1} f(t) \Big) = f^{(k-1)}(t). \end{aligned} $$

Obviously, if we set $p = k \geq 1$ and $t > a$ and use the zero rule given in (14), which will be proved in the next section, we obtain the usual derivative of integer order k

$$ {}_aD_t^p f(t) = \frac{\mathrm{d}^k}{\mathrm{d}t^k} \Big({}_aD_t^0 f(t) \Big) = \frac{\mathrm{d}^k f(t)}{\mathrm{d}t^k} = f^{(k)}(t). $$

4 Basic Properties of Fractional Derivatives

In this section we will discover if some basic properties, such as linearity, Leibniz's rule and composition, still apply to differintegrals.

4.1 Linearity

Linearity follows from just filling in the definitions of the fractional derivatives and integrals. If we use the expression of the Grünwald-Letnikov fractional derivative (9) we have

$$\begin{aligned}
{}_aD_t^p\Big(\lambda f(t)+\mu g(t)\Big) &= \lim_{\substack{h\to 0\\ mh=t-a}} h^{-p}\sum_{r=0}^{m}(-1)^r\binom{p}{r}\Big(\lambda f(t-rh)+\mu g(t-rh)\Big)\\
&= \lambda \lim_{\substack{h\to 0\\ mh=t-a}} h^{-p}\sum_{r=0}^{m}(-1)^r\binom{p}{r} f(t-rh)\\
&\quad + \mu \lim_{\substack{h\to 0\\ mh=t-a}} h^{-p}\sum_{r=0}^{m}(-1)^r\binom{p}{r} g(t-rh)\\
&= \lambda\ {}_aD_t^p f(t) + \mu\ {}_aD_t^p g(t).
\end{aligned}$$

In this proof $f(t)$ and $g(t)$ are functions for which the given operator is defined and $\lambda, \mu \in \mathbb{R}$ are real constants. A similar proof can be given for the fractional integral.

A proof for the linearity of the Riemann-Liouville differintegral will also be given. Using the fractional integral given in Definition 3.2 we have

$$\begin{aligned}
{}_aD_t^{-p}\Big(\lambda f(t)+\mu g(t)\Big) &= \frac{1}{\Gamma(p)}\int_a^t (t-\tau)^{p-1}\Big(\lambda f(\tau)+\mu g(\tau)\Big)\,\mathrm{d}\tau\\
&= \lambda\frac{1}{\Gamma(p)}\int_a^t (t-\tau)^{p-1} f(\tau)\,\mathrm{d}\tau\\
&\quad + \mu\frac{1}{\Gamma(p)}\int_a^t (t-\tau)^{p-1} g(\tau)\,\mathrm{d}\tau\\
&= \lambda\ {}_aD_t^{-p} f(t) + \mu\ {}_aD_t^{-p} g(t).
\end{aligned}$$

Again, a similar proof can be given for the Rieman-Liouville derivative. For example using the linearity of Riemann-Liouville integral which we have just proved and Definition 3.3.

4.2 Zero Rule

It can be proved that if $f(t)$ is continuous for $t \geq a$ then we have

$$\lim_{p\to 0}\ {}_aD_t^{-p} f(t) = f(t).$$

The proof can be found in [2, p. 65-67]. Hence, we define

$${}_aD_t^0 f(t) = f(t). \tag{14}$$

4.3 Product Rule & Leibniz's Rule

If f and g are functions we know the derivative of their product is given by the product rule

$$(f \cdot g)' = f' \cdot g + f \cdot g'.$$

This can be generalized to

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)},$$

which is also known as the Leibniz rule. In the last expression f and g are n-times differentiable functions. If $f(\tau)$ and $g(\tau)$ and their derivatives are continuous in $[a, t]$ it can be proved that the Leibniz rule for fractional derivatives is given by the following expression

$$ {}_aD_t^p\Big(f(t)g(t)\Big) = \sum_{k=0}^{m} \binom{p}{k} f^{(k)}(t)\ {}_aD_t^{p-k} g(t), \tag{15}$$

where again the binomial coefficient is given by (8) and $m \in \mathbb{N}$ satisfies $(m \leq p < m+1)$. The proof is fairly long so it won't be given here, but can be found in [2, p. 91-97]. If we know the fractional derivative of some function, say $g(t)$ and we want to determine the fractional derivative of a function which is a product of $g(t)$ and another function, say $f(t)$, the Leibniz's rule is very helpful.

4.4 Composition

4.4.1 Fractional integration of a fractional integral

The Riemann-Liouville fractional integral given in Definition 3.2 has the following important property

$$ {}_aD_t^{-p}\Big({}_aD_t^{-q} f(t)\Big) = {}_aD_t^{-q}\Big({}_aD_t^{-p} f(t)\Big) = {}_aD_t^{-p-q} f(t), \tag{16}$$

which is called the composition rule for the Riemann-Liouville fractional integrals. Using the definition the proof is quite straightforward

$$\begin{aligned} {}_aD_t^{-p}\Big({}_aD_t^{-q} f(t)\Big) &= \frac{1}{\Gamma(p)} \int_a^t (t-\tau)^{p-1} \Big({}_aD_\tau^{-q} f(\tau)\Big)\, \mathrm{d}\tau \\ &= \frac{1}{\Gamma(p)} \int_a^t (t-\tau)^{p-1} \left(\frac{1}{\Gamma(q)} \int_a^\tau (\tau-\xi)^{q-1}\ f(\xi)\, \mathrm{d}\xi\right) \mathrm{d}\tau \\ &= \frac{1}{\Gamma(p)\Gamma(q)} \int_a^t \int_a^\tau (t-\tau)^{p-1}\ (\tau-\xi)^{q-1}\ f(\xi)\, \mathrm{d}\xi\, \mathrm{d}\tau. \end{aligned}$$

Changing the order of integration using formula (6) gives

$$ {}_aD_t^{-p}\Big({}_aD_t^{-q} f(t)\Big) = \frac{1}{\Gamma(p)\Gamma(q)} \int_a^t f(\xi) \int_\xi^t (t-\tau)^{p-1}\ (\tau-\xi)^{q-1}\, \mathrm{d}\tau\, \mathrm{d}\xi.$$

We make the substitution $\frac{\tau-\xi}{t-\xi} = \zeta$ from which it follows that $\mathrm{d}\tau = (t-\xi)\mathrm{d}\zeta$ and the new interval of integration is $[0, 1]$. Now we are able to rewrite the last expression as

$$\begin{aligned}{}_aD_t^{-p}\Big({}_aD_t^{-q}f(t)\Big) &= \frac{1}{\Gamma(p)\Gamma(q)}\int_a^t f(\xi)\Big((t-\xi)^{p+q-1}\int_0^1(1-\zeta)^{p-1}\zeta^{q-1}\,\mathrm{d}\zeta\Big)\,\mathrm{d}\xi\\ &= \frac{1}{\Gamma(p)\Gamma(q)}\int_a^t f(\xi)\Big((t-\xi)^{p+q-1}B(p,q)\Big)\,\mathrm{d}\xi,\end{aligned}$$

where in the last formula we used the Beta function given in Definition 2.2. If we now use identity (2) to express the Beta function in terms of the Gamma function we obtain

$$\begin{aligned}{}_aD_t^{-p}\Big({}_aD_t^{-q}f(t)\Big) &= \frac{1}{\Gamma(p)\Gamma(q)}\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}\int_a^t f(\xi)(t-\xi)^{p+q-1}\,\mathrm{d}\xi\\ &= \frac{1}{\Gamma(p+q)}\int_a^t (t-\xi)^{p+q-1}f(\xi)\,\mathrm{d}\xi\\ &= {}_aD_t^{-p-q}f(t).\end{aligned}$$

4.4.2 Fractional differentiation of a fractional integral

An important property of the Riemann-Liouville fractional derivative is

$${}_aD_t^{p}\Big({}_aD_t^{-q}f(t)\Big) = {}_aD_t^{p-q}f(t), \tag{17}$$

where $f(t)$ has to be continuous and if $p \geq q \geq 0$, the derivative ${}_aD_t^{p-q}f(t)$ exists. This property is called the composition rule for the Riemann-Liouville fractional derivatives. We shall prove this property, but first we need another property which actually is a special case of the previous one with $q = p$

$${}_aD_t^{p}\Big({}_aD_t^{-p}f(t)\Big) = f(t), \tag{18}$$

where $p > 0$ and $t > a$. This implies that the Riemann-Liouville fractional differentiation operator is the left inverse of the Riemann-Liouville fractional integration of the same order p. We prove this in the following way. First we consider the case $p = n \in \mathbb{N}_{\geq 1}$, then we have

$$\begin{aligned}{}_aD_t^{n}\Big({}_aD_t^{-n}f(t)\Big) &= \frac{\mathrm{d}^n}{\mathrm{d}t^n}\frac{1}{\Gamma(n)}\int_a^t (t-\tau)^{n-1}f(\tau)\,\mathrm{d}\tau\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\int_a^t f(\tau)\,\mathrm{d}\tau = f(t).\end{aligned}$$

For the non-integer case we take $k-1 \leq p < k$ and use (16) to write

$${}_aD_t^{-k}f(t) = {}_aD_t^{-(k-p)}\Big({}_aD_t^{-p}f(t)\Big).$$

Now using the definition of the Riemann-Liouville differintegral we obtain

$$\begin{aligned}{}_aD_t^{p}\Big({}_aD_t^{-p}f(t)\Big) &= \frac{\mathrm{d}^k}{\mathrm{d}t^k}\Big[{}_aD_t^{-(k-p)}\Big({}_aD_t^{-p}f(t)\Big)\Big]\\ &= \frac{\mathrm{d}^k}{\mathrm{d}t^k}\Big[{}_aD_t^{-k}f(t)\Big] = f(t).\end{aligned}$$

This completes the proof. One note has to be made. The converse of (18) is not true, so ${}_aD_t^{-p}\left({}_aD_t^{p} f(t)\right) \neq f(t)$. The proof for this can be found in [2, p. 70-71]. We won't give it here since it does not contribute to the proof of (17).

So now we are able to prove (17). We consider two cases. First we'll deal with $q \geq p \geq 0$. Then we have

$$ {}_aD_t^{p}\left({}_aD_t^{-q} f(t))\right) = {}_aD_t^{p}\left[{}_aD_t^{-p}\left({}_aD_t^{-(q-p)} f(t)\right)\right] = {}_aD_t^{p-q} f(t). $$

This follows directly from (16) and (18). Now we will consider the second case in which we have $p > q \geq 0$. Using Definition 3.3 and again (16) we see that

$$ \begin{aligned} {}_aD_t^{p}\left({}_aD_t^{-q} f(t)\right) &= \frac{\mathrm{d}^k}{\mathrm{d}t^k}\left[{}_aD_t^{-(k-p)}\left({}_aD_t^{-q} f(t)\right)\right] \\ &= \frac{\mathrm{d}^k}{\mathrm{d}t^k}\left({}_aD_t^{p-q-k} f(t)\right) = \frac{\mathrm{d}^k}{\mathrm{d}t^k}\left({}_aD_t^{-(k-(p-q))} f(t)\right) \\ &= {}_aD_t^{p-q} f(t). \end{aligned} $$

So in both cases we proved equation (17).

4.4.3 Fractional integration and differentiation of a fractional derivative

Their are two more possibilities when we're dealing with composition of differintegrals, i.e. the fractional integration of a derivative and the fractional differentiation of a fractional derivative. Both compositions are not useful contributions to this thesis so we shall not give their definitions and proofs here.

5 Examples

This section deals with some examples of fractional derivatives and integrals. First we will take a look at the Power function and thereafter explore the Exponential function and Trigonometric functions.

5.1 The Power Function

The Power function is very important in Mathematics since many functions can be derived from an infinite power series. First we will use the Riemann-Liouville fractional integral given in Definition 3.2 to compute the integral of order $p \in \mathbb{R}_{>0}$ of the power function $(t-a)^\beta$. Plugging this into the equation gives

$$ {}_aD_t^{-p}(t-a)^\beta = \frac{1}{\Gamma(p)} \int_a^t (t-\tau)^{p-1}(\tau-a)^\beta \, \mathrm{d}\tau. $$

If we make the substitution $\frac{\tau-a}{t-a} = \xi$ from which it follows that $\mathrm{d}\tau = (t-a)\mathrm{d}\xi$ and the new interval of integration is $[0,1]$, we can rewrite the last expression as

$$ \begin{aligned} {}_aD_t^{-p}(t-a)^\beta &= \frac{(t-a)^{\beta+p}}{\Gamma(p)} \int_0^1 (1-\xi)^{p-1}\xi^\beta \, \mathrm{d}\xi \\ &= \frac{(t-a)^{\beta+p}}{\Gamma(p)} B(p, \beta+1) \\ &= \frac{\Gamma(\beta+1)}{\Gamma(\beta+p+1)}(t-a)^{\beta+p}, \end{aligned} \tag{19} $$

where in the last equation we made use of (2) to write the Beta function in terms of the Gamma function. It follows that $\beta > -1$.

Next we will compute the derivative of order $r \in \mathbb{R}_{>0}$ of the same power function $(t-a)^\beta$ using the Riemann-Liouville fractional derivative given in Definition 3.3. Again filling in $f(t) = (t-a)^\beta$ gives

$$ {}_aD_t^r(t-a)^\beta = \frac{\mathrm{d}^k}{\mathrm{d}t^k}\left({}_aD_t^{-(k-r)}(t-a)^\beta \right). $$

Now we are able to use the integral of the power function we have just computed in (19). If we replace the order p by $k-r > 0$ we can rewrite the last expression as

$$ \begin{aligned} {}_aD_t^r(t-a)^\beta &= \frac{\Gamma(\beta+1)}{\Gamma(\beta+k-r+1)} \frac{\mathrm{d}^k}{\mathrm{d}t^k}(t-a)^{\beta+k-r} \\ &= \frac{\Gamma(\beta+1)}{\Gamma(\beta-r+1)}(t-a)^{\beta-r}, \end{aligned} \tag{20} $$

with $\beta > -1$.

The following two examples can clarify this using concrete numbers. First we would like to derive the half-derivative of the function $f(x) = x$, so in the last expression we set $t = x$, $a = 0$, $\beta = 1$ and $r = \frac{1}{2}$. Then we obtain

$$
\begin{aligned}
{}_aD_t^{\frac{1}{2}}(x-0)^1 &= \frac{\Gamma(1+1)}{\Gamma(1-\frac{1}{2}+1)}(x-0)^{1-\frac{1}{2}} \\
{}_aD_t^{\frac{1}{2}}x &= \frac{\Gamma(2)}{\Gamma(\frac{3}{2})}x^{\frac{1}{2}} = 2\sqrt{\frac{x}{\pi}}.
\end{aligned}
$$

In our next example we would like to know the derivative of order $\frac{3}{4}$ of the function $f(x) = x^2$, so again in formula (20) we set $t = x$, $a = 0$, but now $\beta = 2$ and $r = \frac{3}{4}$. This gives us

$$
\begin{aligned}
{}_aD_t^{\frac{3}{4}}(x-0)^2 &= \frac{\Gamma(2+1)}{\Gamma(2-\frac{3}{4}+1)}(x-0)^{2-\frac{3}{4}} \\
{}_aD_t^{\frac{3}{4}}x^2 &= \frac{\Gamma(3)}{\Gamma(2\frac{1}{4})}x^{1\frac{1}{4}} \approx 1.76522x^{1\frac{1}{4}}
\end{aligned}
$$

5.2 The Exponential Function

Another frequently used function in Mathematics is the exponential function. We shall use the Weyl fractional integral, which is formally equal to the Riemann-Liouville fractional integral given in Definition 3.2, to compute the integral of order $p \in \mathbb{R}_{>0}$ of the function $f(t) = e^{\lambda t}$, where $\lambda \in \mathbb{C}$. This Weyl differintegral, which can be found in [3, p. 80], applies to periodic functions where the integral is equal to zero over a period. If we use the Weyl differintegral we do not have to make the restriction of setting $\mathrm{Re}(\lambda) > 0$. So using the Weyl fractional integral and setting a equal to $-\infty$ gives us

$$
{}_{-\infty}D_t^{-p}e^{\lambda t} = \frac{1}{\Gamma(p)}\int_{-\infty}^{t}(t-\tau)^{p-1}e^{\lambda\tau}\,\mathrm{d}\tau.
$$

This expression can be rewritten as

$$
{}_{-\infty}D_t^{-p}e^{\lambda t} = \lambda^{1-p}\frac{1}{\Gamma(p)}\int_{-\infty}^{t}\big(\lambda(t-\tau)\big)^{p-1}e^{\lambda\tau}\,\mathrm{d}\tau.
$$

If we make the substitution $\xi = \lambda(t-\tau)$ it follows that ξ goes from $\infty \to 0$ and $-\lambda\mathrm{d}\tau = \mathrm{d}\xi$ so $\mathrm{d}\tau = -\lambda^{-1}\mathrm{d}\xi$. Now we can rewrite the last expression as

$$
\begin{aligned}
{}_{-\infty}D_t^{-p}e^{\lambda t} &= -\lambda^{1-p}\frac{1}{\Gamma(p)}\int_{\infty}^{0}\xi^{p-1}e^{\lambda t-\xi}\lambda^{-1}\mathrm{d}\xi \\
&= \lambda^{1-p}\frac{1}{\Gamma(p)}\int_{0}^{\infty}\xi^{p-1}e^{\lambda t-\xi}\lambda^{-1}\mathrm{d}\xi \\
&= \lambda^{-p}\frac{e^{\lambda t}}{\Gamma(p)}\int_{0}^{\infty}\xi^{p-1}e^{-\xi}\mathrm{d}\xi.
\end{aligned}
$$

Now using the Gamma function given in Definition 2.1 we get

$$
{}_{-\infty}D_t^{-p}e^{\lambda t} = \lambda^{-p}\frac{e^{\lambda t}}{\Gamma(p)}\Gamma(p) = \lambda^{-p}e^{\lambda t}.
$$

The fractional derivative of order $p \in \mathbb{R}_{>0}$ can be obtained in the same way but now using Definition 3.3 and is given by

$$_{-\infty}D_t^p e^{\lambda t} = \lambda^p e^{\lambda t}.$$

So actually we have

$$_{-\infty}D_t^p e^{\lambda t} = \lambda^p e^{\lambda t} \tag{21}$$

for all $p \in \mathbb{R}$.

5.3 The Trigonometric Functions

In this example we would like to explore the differintegral of the sine and cosine functions. We are able to use the last example since we can write the trigonometric functions in terms of the exponential function in the following way

$$\sin(t) = \frac{e^{it} - e^{-it}}{2i} \qquad \cos(t) = \frac{e^{it} + e^{-it}}{2}.$$

First we will explore the Weyl differintegral of order $p \in \mathbb{R}$ of the sine function

$$_{-\infty}D_t^p \sin(t) = {}_{-\infty}D_t^p \left(\frac{e^{it} - e^{-it}}{2i} \right).$$

If we now use the linearity of the Weyl differintegral, which follows directly from the linearity of the Rieman-Liouville differintegral given in Section 4.1 since they are formally equal, the last expression can be rewritten as

$$_{-\infty}D_t^p \sin(t) = \frac{1}{2i} \left({}_{-\infty}D_t^p e^{it} - {}_{-\infty}D_t^p e^{-it} \right).$$

If we now use the expression for the differintegral of the exponential function (21) given in the last example we obtain

$$\begin{aligned} _{-\infty}D_t^p \sin(t) &= \frac{1}{2i} \left(i^p e^{it} - (-i)^p e^{-it} \right) = \frac{1}{2i} \left(e^{i \frac{\pi}{2} p} e^{it} - e^{-i \frac{\pi}{2} p} e^{-it} \right) \\ &= \frac{1}{2i} \left(e^{i(t + \frac{\pi}{2} p)} - e^{-i(t + \frac{\pi}{2} p)} \right) = \sin(t + \frac{\pi}{2} p). \end{aligned}$$

The differintegral for the cosine function can be obtained in the same way and is given by

$$_{-\infty}D_t^p \cos(t) = \cos(t + \frac{\pi}{2} p).$$

6 Fractional Linear Differential Equations

Fractional differential equations are a generalization of differential equations. They can be solved by several methods of which the Laplace transform is one. We shall explore this method, but first give some basic properties of the Laplace transform, which are necessary to understand the rest of this chapter.

6.1 The Laplace Transforms of Fractional Derivatives

First the definition of the Laplace transform itself is given.

Definition 6.1. *We define the Laplace transform of a function $f(t)$ for $t \in \mathbb{R}_{\geq 0}$ and $s \in \mathbb{C}$ as the function $F(s)$ such that*

$$F(s) = L\{f(t); s\} = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t.$$

For this integral to exist we must have

$$e^{-\alpha t}|f(t)| \leq M \quad \text{for all } t > T,$$

where M and T are positive constants. The original function $f(t)$ can be recovered from the Laplace transform.

Definition 6.2. *The inverse Laplace transform $f(t)$ where $t \in \mathbb{R}_{>0}$, $s \in \mathbb{C}$ and $F(s)$ is the Laplace transform is given by*

$$f(t) = L^{-1}\{F(s); t\} = \int_{c-\infty}^{c+\infty} e^{st} F(s)\,\mathrm{d}s.$$

In Definition 6.2 $c = \mathrm{Re}(s) > c_0$ and c_0 lies in the right half plane of the absolute convergence of the Laplace integral given in Definition 6.1.

An important property of the Laplace transform is that it is a linear operator, i.e.

$$\begin{aligned} L\{f(t) + g(t); s\} &= L\{f(t); s\} + L\{g(t); s\}, \\ L\{cf(t); s\} &= cL\{f(t); s\}, \end{aligned} \tag{22}$$

where $L\{f(t); s\}$ and $L\{g(t); s\}$ have to exist and c is a constant.

For another useful property of the Laplace transform we first have to define the convolution of two functions.

Definition 6.3. *The convolution of two functions $f(t)$ and $g(t)$ is defined as*

$$(f * g)(t) = \int_0^t f(t-\tau) g(\tau)\,\mathrm{d}\tau = \int_0^t f(\tau) g(t-\tau)\,\mathrm{d}\tau.$$

If $f(t)$ and $g(t)$ are equal to zero for $t < 0$ and $F(s)$ and $G(s)$ exist, the Laplace transform of this convolution is equal to the product of the Laplace transform of those functions. This property is given in the following theorem.

Theorem 6.4. *The Laplace transform of the convolution of two functions $f(t)$ and $g(t)$ is given by*

$$L\{f(t) * g(t); s\} = F(s)G(s).$$

If we integrate the Laplace integral (Definition 6.1) by parts we obtain another necessary property.

Corollary 6.1. *The Laplace transform of the derivative of integer order n is given by*

$$L\{f^n(t); s\} = s^n F(s) - \sum_{k=0}^{n-1} s^{n-k-1} f^{(k)}(0) = s^n F(s) - \sum_{k=0}^{n-1} s^k f^{(n-k-1)}(0).$$

6.1.1 Laplace Transform of the Riemann-Liouville Differintegral

First we shall explore the Laplace transform of the Riemann-Liouville fractional integral. Using Definition 3.2 and setting the lower terminal a equal to zero we get

$${}_0D_t^{-p} f(t) = \frac{1}{\Gamma(p)} \int_0^t (t-\tau)^{p-1} f(\tau)\, \mathrm{d}\tau.$$

If we use the definition for convolution (Definition 6.3) and define the function $g(t) = t^{p-1}$, the last expression can be rewritten as

$${}_0D_t^{-p} f(t) = \frac{1}{\Gamma(p)} t^{p-1} * f(t) = \frac{1}{\Gamma(p)} g(t) * f(t) = \frac{1}{\Gamma(p)} (g * f)(t). \tag{23}$$

If we now take a look at the Laplace transform of $g(t)$ and therefore use the definition of the Laplace transform given in Definition 6.1 we have

$$G(s) = L\{g(t); s\} = L\{t^{p-1}; s\} = \int_0^\infty t^{p-1} e^{-st}\, \mathrm{d}t.$$

If we make the substitution $st = r$ it follows that $\mathrm{d}t = \frac{1}{s}\mathrm{d}r$ and we can rewrite the last expression as

$$G(s) = \frac{1}{s^p} \int_0^\infty r^{p-1} e^{-r}\, \mathrm{d}r = s^{-p} \int_0^\infty r^{p-1} e^{-r}\, \mathrm{d}r = \Gamma(p) s^{-p}, \tag{24}$$

where in the last equality we used the definition of the Gamma function given in Definition 2.1. Now it's possible to define the Laplace transform of the Riemann-Liouville fractional integral. First using (23) we get

$$L\{{}_0D_t^{-p} f(t); s\} = L\left\{\frac{1}{\Gamma(p)} (g * f)(t); s\right\}.$$

Using the Laplace transform of a convolution given in Theorem 6.4 and the linearity of of the Laplace transform (22), the last expression can be rewritten as

$$L\{{}_0D_t^{-p} f(t); s\} = \frac{1}{\Gamma(p)} G(s)F(s).$$

If we now use (24) we obtain for the Laplace transform of the Riemann-Liouville integral of order $p > 0$

$$L\{{}_0D_t^{-p} f(t); s\} = \frac{1}{\Gamma(p)} \Gamma(p) s^{-p} F(s) = s^{-p} F(s). \tag{25}$$

Next we shall explore the Laplace transform of the Riemann-Liouville fractional derivative. As suggested in [2] we shall write this fractional derivative in the following form

$$ {}_0D_t^p f(t) = g^{(n)}(t), $$

from which it follows that

$$ g(t) = {}_0D_t^{-(n-p)} f(t) = \frac{1}{\Gamma(n-p)} \int_0^t (t-\tau)^{n-p-1} f(\tau)\, \mathrm{d}\tau, \tag{26} $$

for $n-1 \leq p < n$. If we use the Laplace transform of an integer-order derivative given in Corollary 6.1 we can write

$$ L\{{}_0D_t^p f(t); s\} = L\{g^{(n)(t)}; s\} = s^n G(s) - \sum_{k=0}^{n-1} s^k g^{(n-k-1)}(0). \tag{27} $$

To rewrite this last expression we will evaluate $G(s)$ and $g^{(n-k-1)}(t)$. First we make use of the Laplace transform of the Riemann-Liouville fractional integral given in (25) to write

$$ G(s) = L\{g(t); s\} = L\{ {}_0D_t^{-(n-p)} f(t); s\} = s^{-(n-p)} F(s). \tag{28} $$

Now we will explore $g^{(n-k-1)}(t)$ by taking the $(n-k-1)^{th}$-derivative of $g(t)$ given in (26). Also using the Riemann-Liouville fractional derivative formula given in Definition 3.3 enables us to write

$$ g^{(n-k-1)}(t) = \frac{\mathrm{d}^{n-k-1}}{\mathrm{d}t^{n-k-1}} {}_0D_t^{-(n-p)} f(t) = {}_0D_t^{p-k-1} f(t). \tag{29} $$

So substituting the last two equations in (27) gives the expression for the Laplace transform of the Riemann-Liouville fractional derivative of order $p > 0$

$$ \begin{aligned} L\{{}_0D_t^p f(t); s\} &= s^n s^{-(n-p)} F(s) - \sum_{k=0}^{n-1} s^k \Big[{}_0D_t^{p-k-1} f(t) \Big]_{t=0} \\ &= s^p F(s) - \sum_{k=0}^{n-1} s^k \Big[{}_0D_t^{p-k-1} f(t) \Big]_{t=0}, \end{aligned} \tag{30} $$

for $n-1 \leq p < n$.

So using the last expression for the case $n = 1$ we obtain

$$ L\{{}_0D_t^p f(t); s\} = s^p F(s) - {}_0D_t^{p-1} f(0), \tag{31} $$

where $0 \leq p < 1$. For $n = 2$ we have $1 \leq p < 2$ and it follows from (30) that

$$ L\{{}_0D_t^p f(t); s\} = s^p F(s) - {}_0D_t^{p-1} f(0) - s\, {}_0D_t^{p-2} f(0). \tag{32} $$

We shall see that these special cases are helpful in solving some simple fractional differential equations which will be treated in the examples at the end of this chapter.

6.1.2 Laplace Transform of the Grünwald-Letnikov Fractional Derivative

In this part we will explore the Laplace transform of the Grünwald-Letnikov fractional derivative. Actually we've already done most of the work and it's basically using definitions. Again, as in the Riemann-Liouville case, we set the lower terminal a equal to zero. First we shall consider the case $0 \leq p < 1$. Using the definition of the Grünwald-Letnikov fractional derivative given in Corollary 3.1 we have

$$ {}_0D_t^p f(t) = \frac{f(0)t^{-p}}{\Gamma(1-p)} + \frac{1}{\Gamma(1-p)} \int_0^t (t-\tau)^{-p} f'(\tau)\,\mathrm{d}\tau, $$

where $f(t)$ is bounded near $t = 0$. Using the Laplace transform of the function given in (24), the laplace transform for convolutions given in Theorem 6.4 and the Laplace transform of the integer-order derivative given in Corollary 6.1 we obtain

$$ L\{{}_0D_t^p f(t); s\} = \frac{f(0)}{s^{1-p}} + \frac{1}{s^{1-p}} \Big(sF(s) - f(0) \Big) = s^p F(s). \tag{33} $$

In the case of $p > 1$ the functions in the sum of Corollary 3.1 can not be integrated in the classical sense. However, it can be proved that under the assumption that $m \leq p < m+1$ the Laplace transform of the Grünwald-Letnikov fractional derivative given in (33) still holds in the sense of generalized functions.

6.2 The Laplace Transform Method

Before we continue we also need the Laplace transform of a very important function for linear fractional differential equations consisting of two terms. We need to explore the Laplace transform of the following function

$$ L\Big\{ t^{\alpha m+\beta-1} E_{\alpha,\beta}^{(m)}(at^\alpha); s \Big\}. \tag{34} $$

If we look more closely we can see this function is a combination of the power function and the differentiated Mittag-Leffler function given in Corollary 2.1. Evaluating this Mittag-Leffler function in at^α yields

$$ E_{\alpha,\beta}^{(m)}(at^\alpha) = \sum_{k=0}^{\infty} \frac{(k+m)!}{k!} \frac{(at^\alpha)^k}{\Gamma(\alpha k + \alpha m + \beta)} = \sum_{k=0}^{\infty} \frac{(k+m)!}{k!} \frac{a^k t^{\alpha k}}{\Gamma(\alpha k + \alpha m + \beta)}. $$

Substuting this expression in (34) gives

$$ L\Big\{ t^{\alpha m+\beta-1} E_{\alpha,\beta}^{(m)}(at^\alpha); s \Big\} = L\Big\{ t^{\alpha m+\beta-1} \sum_{k=0}^{\infty} \frac{(k+m)!}{k!} \frac{a^k t^{\alpha k}}{\Gamma(\alpha k + \alpha m + \beta)}; s \Big\}. $$

Using the linearity of the Laplace transform (22) we can rewrite the last expression as

$$ L\Big\{ t^{\alpha m+\beta-1} E_{\alpha,\beta}^{(m)}(at^\alpha); s \Big\} = \sum_{k=0}^{\infty} \frac{(k+m)!a^k}{k!\,\Gamma(\alpha k + \alpha m + \beta)} L\{t^{\alpha k + \alpha m + \beta - 1}; s\} \tag{35} $$

Now we want to inspect $L\{t^{\alpha k+\alpha m+\beta-1}; s\}$ from the last equation. We've already determined the Laplace transform of the power function in (24) which gave us the following equality

$$L\{t^{p-1}; s\} = \Gamma(p)s^{-p}.$$

So in this case we have

$$L\{t^{\alpha k+\alpha m+\beta-1}; s\} = \Gamma(\alpha k + \alpha m + \beta)s^{-(\alpha k+\alpha m+\beta)} = \frac{\Gamma(\alpha k + \alpha m + \beta)}{s^{\alpha k+\alpha m+\beta}}.$$

Substituting this in (35) gives us

$$\begin{aligned} L\Big\{t^{\alpha m+\beta-1}E^{(m)}_{\alpha,\beta}(at^\alpha); s\Big\} &= \sum_{k=0}^{\infty} \frac{(k+m)!\, a^k}{k!\, \Gamma(\alpha k + \alpha m + \beta)} \frac{\Gamma(\alpha k + \alpha m + \beta)}{s^{\alpha k+\alpha m+\beta}} \\ &= \sum_{k=0}^{\infty} \frac{(k+m)! a^k}{k!\, s^{\alpha k+\alpha m+\beta}} = \sum_{k=0}^{\infty} \frac{(k+m)!}{k!} \frac{a^k}{s^{\alpha k+\alpha m+\beta}} \\ &= s^{-\alpha m-\beta} \sum_{k=0}^{\infty} \frac{(k+m)!}{k!} \left(\frac{a}{s^\alpha}\right)^k. \end{aligned} \tag{36}$$

To further rewrite the last expression we look at the series

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{(k+m)!}{k!} \left(\frac{a}{s^\alpha}\right)^k &= \sum_{k=0}^{\infty} (k+m)(k+m-1)\cdots(k+1)\left(\frac{a}{s^\alpha}\right)^k \\ &= \sum_{k=m}^{\infty} k(k-1)\cdots(k-m+1)\left(\frac{a}{s^\alpha}\right)^{k-m} \\ &= \frac{\mathrm{d}^m}{\mathrm{d}t^m} \sum_{k=m}^{\infty} \left(\frac{a}{s^\alpha}\right)^k. \end{aligned}$$

Since the first m terms disappear after differentiation we can rewrite the last expression as

$$\sum_{k=0}^{\infty} \frac{(k+m)!}{k!} \left(\frac{a}{s^\alpha}\right)^k = \frac{\mathrm{d}^m}{\mathrm{d}t^m} \sum_{k=0}^{\infty} \left(\frac{a}{s^\alpha}\right)^k = \frac{\mathrm{d}^m}{\mathrm{d}t^m} \frac{1}{1-\frac{a}{s^\alpha}} = \frac{m!}{(1-\frac{a}{s^\alpha})^{m+1}}.$$

So substituting this in (36) we finally obtain

$$L\Big\{t^{\alpha m+\beta-1}E^{(m)}_{\alpha,\beta}(at^\alpha); s\Big\} = s^{-\alpha m-\beta} \frac{m!}{(1-\frac{a}{s^\alpha})^{m+1}} = \frac{m!\, s^{\alpha-\beta}}{(s^\alpha - a)^{m+1}}. \tag{37}$$

The following table shows some special cases of expression (37) and also the Laplace transform of the Power function given in (24).

Table 1: Useful Laplace transforms

$F(s)$	$f(t) = L^{-1}\{F(s)\}$
$\frac{1}{s^\alpha}$	$\frac{t^{\alpha-1}}{\Gamma(\alpha)}$
$\frac{1}{s^\alpha - a}$	$t^{\alpha-1}E_{\alpha,\alpha}(at^\alpha)$
$\frac{s^\alpha}{s(s^\alpha+a)}$	$E_\alpha(-at^\alpha)$
$\frac{a}{s(s^\alpha+a)}$	$1 - E_\alpha(-at^\alpha)$
$\frac{1}{s^\alpha(s-a)}$	$t^\alpha E_{1,\alpha+1}(at)$
$\frac{s^{\alpha-\beta}}{s^\alpha - a}$	$t^{\beta-1}E_{\alpha,\beta}(at^\alpha)$

In Table 1 L^{-1} is the inverse Laplace transform given in Definition 6.2.

6.2.1 Examples

In this section we shall explore some examples of simple linear fractional differential equations.

Example 1 Let's say we would like to solve the fractional differential equation given by

$$ {}_0D_t^{\frac{1}{3}} f(t) = c_1 f(t), \tag{38}$$

where c_1 is a constant. Since $0 \leq p = \frac{1}{3} < 1$ we will use the Laplace transform of the Riemann-Liouville fractional derivative for $n = 1$ given in (31) to take the Laplace tansform at both sides of the last equation. If we also use the linearity of the Laplace transform (22) this gives

$$\begin{aligned} L\{ {}_0D_t^{\frac{1}{3}} f(t)\} = L\{c_1 f(t)\} = c_1 L\{f(t)\} \\ s^{\frac{1}{3}}F(s) - {}_0D_t^{\frac{1}{3}-1} f(0) = c_1 F(s) \\ s^{\frac{1}{3}}F(s) - {}_0D_t^{-\frac{2}{3}} f(0) = c_1 F(s). \end{aligned}$$

We see that ${}_0D_t^{-\frac{2}{3}} f(0)$ is the value of ${}_0D_t^{-\frac{2}{3}} f(t)$ evaluated at $t = 0$. If we assume this value exists we can set ${}_0D_t^{-\frac{2}{3}} f(0)$ equal to c_2 to obtain

$$s^{\frac{1}{3}}F(s) - c_2 = c_1 F(s).$$

If we solve this for F(s) we get

$$F(s) = \frac{c_2}{s^{\frac{1}{3}} - c_1}.$$

If we look at Table 1 we see this is a special case of (37) with $\alpha = \frac{1}{3}, \beta = \frac{1}{3}$ and $a = c_1$, so the solution is given by

$$f(t) = L^{-1}\left\{\frac{c_2}{s^{\frac{1}{3}} - c_1}\right\} = c_2 t^{\frac{1}{3}-1} E_{\frac{1}{3},\frac{1}{3}}(c_1 t^{\frac{1}{3}}) = c_2 t^{-\frac{2}{3}} E_{\frac{1}{3},\frac{1}{3}}(c_1 t^{\frac{1}{3}}).$$

In this example we assumed ${}_0D_t^{-\frac{2}{3}} f(0)$ exists and it's value is equal to c_2. To prove this assumption was correct we will first take the Laplace transform of ${}_0D_t^{-\frac{2}{3}}f(t)$ using the Laplace transform of the Riemann-Liouville integral given in (25). This gives

$$L\{{}_0D_t^{-\frac{2}{3}}f(t)\} = s^{-\frac{2}{3}}F(s).$$

Since we just found $F(s) = \frac{c_2}{s^{\frac{1}{3}}-c_1}$ we can substitute this in the last equation to get

$$L\{{}_0D_t^{-\frac{2}{3}}f(t)\} = \frac{c_2 s^{-\frac{2}{3}}}{s^{\frac{1}{3}} - c_1}.$$

Taking the inverse Laplace transform of both sides yields

$${}_0D_t^{-\frac{2}{3}}f(t) = L^{-1}\left\{\frac{c_2 s^{-\frac{2}{3}}}{s^{\frac{1}{3}} - c_1}\right\}.$$

If we take a look at Table 1 again we see that this is the value of $F(s)$ with $\alpha = \frac{1}{3}$, $\beta = 1$ and $a = c_1$, so this implies the last equation is equal to

$${}_0D_t^{-\frac{2}{3}}f(t) = c_2 t^{1-1}E_{\frac{1}{3},1}(c_1 t^{\frac{1}{3}}) = c_2 E_{\frac{1}{3},1}(c_1 t^{\frac{1}{3}}).$$

If we evaluate this expression at $t = 0$ we have

$${}_0D_t^{-\frac{2}{3}}f(0) = c_2 E_{\frac{1}{3},1}(c_1 0^{\frac{1}{3}}) = c_2 E_{\frac{1}{3},1}(0) = c_2,$$

as desired.

Example 2 Now we would like to solve the fractional differential equation given by

$${}_0D_t^{\frac{19}{12}}f(t) = 0.$$

Since $1 \leq p = \frac{19}{12} < 2$ we will use the Laplace transform of the Riemann-Liouville fractional derivative for $n = 2$ given in (32) to take the Laplace transform of both sides. This gives

$$\begin{aligned} L\{{}_0D_t^{\frac{19}{12}}\} &= 0 \\ s^{\frac{19}{12}}F(s) - {}_0D_t^{\frac{19}{12}-1}f(0) - s\,{}_0D_t^{\frac{19}{12}-2}f(0) &= 0 \\ s^{\frac{19}{12}}F(s) - {}_0D_t^{\frac{7}{12}}f(0) - s\,{}_0D_t^{-\frac{5}{12}}f(0) &= 0. \end{aligned}$$

Just as in example 1 we assume ${}_0D_t^{\frac{7}{12}}f(0)$ and ${}_0D_t^{-\frac{5}{12}}f(0)$ exist and we set them equal to c_3 and c_4 respectively. Then the last equation becomes

$$s^{\frac{19}{12}}F(s) - c_3 - c_4 s = 0.$$

If we solve this for $F(s)$ we obtain

$$F(s) = \frac{c_3}{s^{\frac{19}{12}}} + \frac{c_4 s}{s^{\frac{19}{12}}}.$$

Again using Table 1 we get the solution

$$\begin{aligned} f(t) &= L^{-1}\left\{\frac{c_3}{s^{\frac{19}{12}}}\right\} + L^{-1}\left\{\frac{c_4 s}{s^{\frac{19}{12}}}\right\} \\ &= L^{-1}\left\{\frac{c_3}{s^{\frac{19}{12}}}\right\} + L^{-1}\left\{\frac{c_4}{s^{\frac{7}{12}}}\right\} \\ &= \frac{c_3 t^{\frac{7}{12}}}{\Gamma\left(\frac{19}{12}\right)} + \frac{c_4 t^{-\frac{5}{12}}}{\Gamma\left(\frac{7}{12}\right)}. \end{aligned}$$

Example 3 In this example we will generalize the problem given in Example 1 and we are given an initial value. So we would like to solve

$$ {}_0D_t^p f(t) = c_1 f(t), \qquad f(0) = c_2 \tag{39}$$

with $0 \leq p < 1$ and c_1 a constant. Again we will use the Laplace transform of the Riemann-Liouville fractional derivative for $n = 1$ given in (31) to take the Laplace tansform at both sides of the last equation. If we also use the linearity of the Laplace transform (22) this gives

$$\begin{aligned} L\{{}_0D_t^p f(t)\} &= c_1 L\{f(t)\} \\ s^p F(s) - {}_0D_t^{p-1} f(0) &= c_1 F(s). \end{aligned}$$

Again assuming that ${}_0D_t^{p-1} f(0)$ exists and setting it equal to c_3 gives

$$s^p F(s) - c_3 = c_1 F(s).$$

If we solve the last expression for $F(s)$ we get

$$F(s) = \frac{c_3}{s^p - c_1}.$$

Making use of Table 1 we find the solution

$$f(t) = L^{-1}\left\{\frac{c_3}{s^p - c_1}\right\} = c_3 t^{p-1} E_{p,p}(c_1 t^p).$$

To find the value of c_3 we will use the initial value $f(0) = c_2$. Since we have

$$\lim_{t \to 0^+} t^{p-1} E_{p,p}(c_1 t^p) = 1,$$

it follows that

$$f(0) = c_3 \cdot 1 = c_3.$$

So the intitial value $f(0) = c_2$ is equal to c_3 and we can rewrite the solution of the fractional linear differential equation as

$$f(t) = c_2 t^{p-1} E_{p,p}(c_1 t^p).$$

7 Applications

Fractional Calculus is used in many problems, for example in engineering, physics, economics, biological processes, etc. Many models can be represented by fractional differential equations and therefore it is increasingly used in these branches. It brings new possibilities, namely fractional derivatives can describe memory effects, so it is possible to evaluate the influence of the past on the behavior of the system at present time.

One of the first to use Fractional Calculus for a problem was the Norwegian mathematician Niels Henrik Abel. In 1823 he applied it in the formulation of his solution for the Tautochrone Problem. The idea of this problem is to find the curve of a frictionless wire which lies in the (x, y)-plane such that the time required for a particle to slide down to the lowest point of the curve is independent of where the particle is placed.

Since then Fractional Calculus has been applied to many other problems such as the fractional conservation of mass, the groundwater flow problem, the fractional advection dispersion equation, time-space fractional diffusion equation models, structural damping models, acoustical wave equations for complex media, the fractional Schrödinger equation in quantum theory and many more.

Although it would be nice to discuss some of these problems, their solutions go beyond the difficulty level of this thesis. Therefore we only mentioned some models and problems and leave it to the reader to further explore these applications of Fractional Calculus if desired. We will treat one simple economic example to show how fractional calculus can be implemented in a commonly used model.

7.1 Economic example

Let's say a customer buys a product for a price €b. The customer does not pay instantly for the product, but chooses to pay off in y months. The interest rate of the seller is $r\%$ per month. The monthly payment the customer is charged is denoted by €m. If we define $f(\tau)$ to be the remaining debt at the end of the τ^{th} month, it can be shown that we have

$$f(\tau) = b(1+r)^\tau - \frac{m}{r}\big[(1+r)^\tau - 1\big]. \tag{40}$$

At $\tau = y$ the customer should have payed off his product so then we must have $f(y) = 0$. Now we are able to solve (40) for m which gives

$$m = \frac{b(1+r)^y r}{(1+r)^y - 1}. \tag{41}$$

Usually this problem can be solved using the following differential equation

$$f'(\tau) - rf(\tau) = -m. \tag{42}$$

If we want to approximate this with a fractional differential equation we rewrite the last formula and consider

$$ {}_0D_t^p f(\tau) - rf(\tau) = -m, \qquad \text{with } 0 < p \leq 1. \tag{43}$$

As we have shown in section 6.2.1 we can solve this fractional differential equation by taking the Laplace transform on both sides. So using the Laplace transform of the Riemann-Liouville fractional derivative for $n = 1$ (31) and the linearity of the Laplace transform (22) we obtain

$$L\{{}_0D_t^p f(\tau)\} - L\{rf(\tau)\} = -L\{m\}$$
$$s^p F(s) - {}_0D_t^{p-1} f(0) - rF(s) = -\frac{m}{s}.$$

As before, in section 6.2.1, we assume ${}_0D_t^{p-1} f(0)$ exists and call it c. Now we are able to solve for $F(s)$ and obtain

$$F(s) = \frac{c}{s^p - r} - \frac{m}{s(s^p - r)}.$$

Using Table 1 we take the inverse Laplace transform on both sides and get

$$f_p(\tau) = c\tau^{p-1} E_{p,p}(r\tau^p) - m\tau^p E_{p,p+1}(r\tau^p). \tag{44}$$

We have already seen that

$$\lim_{\tau \to 0^+} \tau^{p-1} E_{p,p}(r\tau^p) = 1,$$

and we also have

$$\lim_{\tau \to 0^+} \tau^p E_{p,p+1}(r\tau^p) = 0.$$

Therefore, if we evaluate expression (44) in $\tau = 0$ we get

$$f_p(0) = c.$$

Since $f_p(\tau)$ denotes the remaining debt at the end of month τ, the last expression can be seen as the debt at the beginning, which is equal to the price of the product €b. So we have $b = f_p(0) = c$ and we can rewrite (44) as

$$f_p(\tau) = b\tau^{p-1} E_{p,p}(r\tau^p) - m\tau^p E_{p,p+1}(r\tau^p). \tag{45}$$

To clarify this we shall give an example using conrete numbers.

7.1.1 Concrete example

Suppose we have a customer who wants to buy a car. This car costs €20,000. He has to pay it back in 5 years and the interest rate of the car salesman is 14% per year. This means we have the following values

$$b = 20,000; \quad r = \frac{0.14}{12}; \quad y = 60; \quad m = \frac{b(1+r)^y r}{(1+r)^y - 1} = 465.37$$

This allows us to rewrite expression (45) as

$$f_p(\tau) = 20,000\tau^{p-1} E_{p,p}\left(\frac{0.14}{12}\tau^p\right) - 465.37\tau^p E_{p,p+1}\left(\frac{0.14}{12}\tau^p\right). \tag{46}$$

Note that for $p = 1$ we have the integer order first derivative which implies we are dealing with a normal differential equation. Evaluating the last expression in $p = 1$ gives

$$f_1(\tau) = 20,000\left(1 + \frac{0.14}{12}\right)^{\tau} - \frac{465.37}{\frac{0.14}{12}}\left[\left(1 + \frac{0.14}{12}\right)^{\tau} - 1\right].$$

Indeed if we set $\tau = y = 60$ we obtain $f_1(60) \approx 0$, so the remaining debt after 60 months (5 years) is approximately zero. Values of $p \leq 1$ near 1 are a bit harder to compute, but it turns out in these cases it takes less time to pay off the car which means y is smaller. This is due to the intervals between payments becoming shorter and therefore the interest rate will be lower.

8 Conclusions

This thesis introduced the concept of Fractional Calculus; the branch of Mathematics which explores fractional integrals and derivatives. We first gave some basic techniques and functions, such as the Gamma function, the Beta function and the Mittag-Leffler function, which were necessary to understand the rest of this paper.

Thereafter we proved the construction of the Grünwald-Letnikov and the Riemann-Liouville method to define a differintegral. Therefore we used the forward difference derivative and the Cauchy formula for repeated integration respectively. Altough these differintegrals do not look the same, we saw that the Grünwald-Letnikov differintegral was a special case of the Riemann-Liouville differintegral and therefore give the same result under some special conditions. Then we checked if some basic rules of differentiation and integration still hold for these differintegrals. We proved they are both linear and gave an expression for the Leibniz rule for fractional derivatives. We also explored the composition of fractional integrals and fractional derivatives. After giving the framework of differintegrals we were able to make use of it. We explored examples of some frequently used functions, namely the Power function, the Exponential function and the Trigonometric functions.

Next we studied Fractional Linear Differential Equations. First we had to give some basics about the Laplace transform, since we were about to use this method for solving these differential equations. Then we applied the Laplace transform to the Riemann-Liouville and Grünwald-Letnikov differintegral. After evaluating the Laplace transform of a very useful function, namely a combination of the Power function and the Mittag-Leffler function, we were able to explore some simple examples.

At last we briefly discussed some applications of Fractional Calculus and examined a commonly used economic model using fractional differential equations.

This thesis did not cover everything related to Fractional Calculus. There have been many more approaches to define a differintegral. For example the Caputo, Hadamard and Miller-Ross differintegrals are also frequently used. However the Grünwald-Letnikov and the Riemann-Liouville differintegral are the most common so we decided to leave it there since the other differintegrals would not have been a very useful addition to this thesis.

In addition there are many more methods for solving fractional linear differential equations. Besides the Laplace transform we could also have used the Fourier transform, the method of reduction to a Volterra integral equation, the power series method or the transformation to an ordinary differential equation. Since the purpose was to give some brief introduction to fractional differential equations and their solutions we decided to explore only one method.

Some people advocate differintegrals should be implemented in standard Mathematics and replace the integer order derivatives and integrals. According to them they provide more possibilities since differintegrals cover derivatives and integrals of arbitrary order, and therefore also integer order derivatives and integrals. Although I agree to some extent with this, I don't think Fractional Calculus is necessary for ordinary Mathematics, since these extra possibilities are not really commonly used additions. Besides, many definitions for a differintegral exist so which one should we use in general? I also think that the formulas are pretty awkward, definitely for first year students. It would be a lot harder to compute just a simple integer order derivative or integral. Though it is a very interesting subject and definitely worth researching, I believe it should be left as an 'exotic' branch of Mathematics.

9 References

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