Binary Markov Games and Tennis Matches

Tijmen Tuinsma
University of Groningen

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First Supervisor: dr. A. van der Made (FEB)
Second Supervisor: prof. dr. E. Wit (FMNS)

Abstract

In this paper a class of games called binary Markov games will be introduced. These can be used to model tennis matches. Equilibrium strategies for the two players in a tennis match are established. Based on these strategies, both players' probabilities to win the match are calculated, as well as the expected number of points in a match. Finally the strength of the players in the longest tennis match in history can be estimated using the expected number of points in a tennis match.

1 Introduction

Tennis is a sport with a very complex scoring system. The scoring system invites to model a match of tennis using binary Markov games, a type of game defined in (Walker et al., 2011). A binary Markov game is a two-player game in which the players compete for points and the history of these points can be summarized in a score. The players only care about winning the match and are thus indifferent between a big or a close win.

The scoring system in tennis is as follows: a match consists of several 'sets'. These sets consist of a sequence of 'games'. Finally, a game consists of a sequence of points. The players compete for points, but the player to score the most points is not necessarily the winner. Every game, one player brings the ball into play (called a 'service'), the other player is the receiver. These roles switch after every game. The winner of a game is the first player to have scored at least four points while also having scored at least two more points than his opponent. The player first to win at least six games in a set, while also having won at least two more games than his opponent, wins the set. In order to win a match, a player has to win two sets (three for men in some big tournaments). A special case is when a set has reached a 6-6 score in games. Then, in most cases, the winner of the set is decided by playing a 'tiebreak': points are played until a player has won at least seven points and has won at least two more points than his opponent.

But there are a few tournaments that do not allow for a tiebreak in the final, deciding set. In this case after a 6-6 score in games is reached in the deciding set (the set of which the winner also wins the match), games are played until one of the players has a lead of two games. Probably the most famous example of this is the Wimbledon match between John Isner and Nicolas Mahut in 2010, won by Isner. The final score in the fifth set was 70-68 in games. This match currently holds the world record for most points played in a tennis match: 980. The match had to be spread out over three days, which led to several subsequent matches being postponed. In this paper it is found that matches between equal players with above average serving qualities, and with a final set that does not allow for a tiebreak, have the highest probability to become such a record-breaking match.

The goal of this paper is to calculate the probabilities to win a tennis match for any player, depending on his quality and the quality of his opponent. Also of interest is the expected number of points in a tennis match for all match types and using this, the strengths of Isner and Mahut in their record match will be estimated. To do this, first a model of tennis matches using binary Markov games will be given, as well as the equilibrium strategies for both players in this model.

The remainder of this paper is structured as follows. In section 2 the theory behind binary Markov games is explained, followed by the way they can be used to model a tennis match. Then a simple Monotonicity Condition, satisfied by our model of a tennis match, establishes equilibrium strategies for both players, presented in section 3. In the following section the probabilities for both players to win a game, set and match are derived, depending on the players' qualities and assuming they play their equilibrium strategy. In section 5 the expected number of points in a match are given, again depending on the players' qualities. Using this the qualities of Isner and Mahut in their record match can be estimated. In the final section these results will be discussed.

2 Binary Markov Games

A binary Markov game (BMG) consists of a binary scoring rule and a sequence of points games. I will first introduce the binary scoring rule, then the point game is described.

2.1 Binary Scoring Rule

The binary scoring rule of a BMG consists of a finite set of states S and two transition functions mapping S into S, where a state represents the current score of the match. The transition functions are $(\cdot)^+$ and $(\cdot)^-$. From every state $s \in S$ there are only two transitions possible: if player A wins the point, the next state is s^+. If player B wins the point, the next state is s^-. S contains two *absorbing states*. These are states in which the match has ended. In case player A has won the match, I will call this absorbing state ω_A, if player B has won the match the absorbing state is called ω_B. Thus we will assume $(\omega_A)^- = (\omega_A)^+ = \omega_A$ and $(\omega_B)^- = (\omega_B)^+ = \omega_B$. In Figure 1 the scoring rule for one game in a tennis match is depicted. Note that $\bar{g}$ and $\underline{g}$ are not absorbing states, since a tennis match lasts longer than one game. $\bar{g}$ represents the state in which player A has won the game, the state $\underline{g}$ is for the case in which player B has won the game.

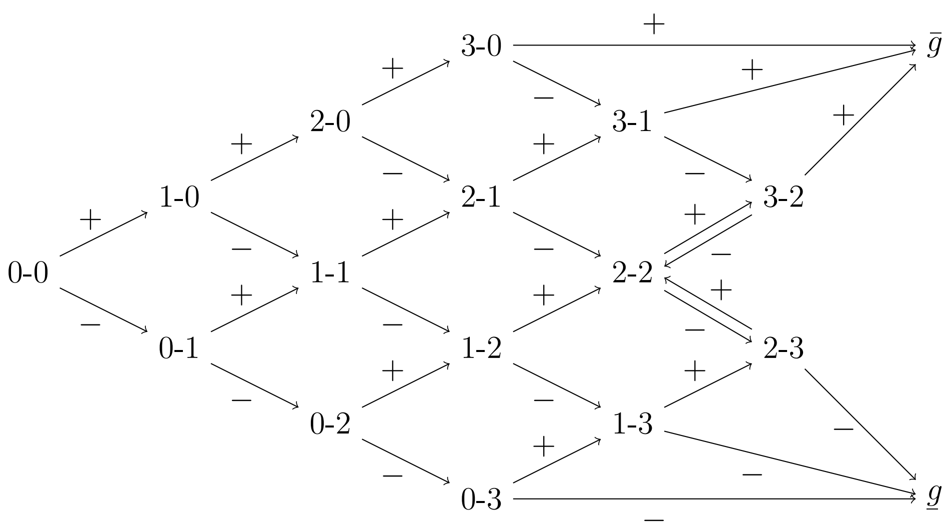

Figure 1: The Binary Scoring Rule of a Tennis Game.

While it is possible to draw the binary scoring rule for an entire tennis match, it would result in a very large diagram. It consists of repetitions of

Figure 1, where $\bar{g}$ and $\underline{g}$ would be replaced by the 0-0 node of the new game. In every state, also the score in games and sets should be summarized. At the nodes where the score is such that one of the players has won the match, ω_A or ω_B is displayed and the diagram ends.

2.2 Point Game

A point game is a two-player normal form game G_s associated with the current state s. In every state the players have finite action sets: $A(s)$ for player A and $B(s)$ for player B. For player i the game has payoff function $\pi_{s,i} : A(s) \times B(s) \to [0, 1]$, representing the probability for the player to win the point played in state s. Hence $\pi_{s,i}(a, b)$ is the probability that player i will win the point played in state s when actions a and b are chosen. This is a constant-sum game since every point must be won by one of the two players and thus $\pi_{s,A}(a, b) + \pi_{s,B}(a, b) = 1$ for all $a \in A(s)$, $b \in B(s)$.

Assume that the following point game is played at every point in a match: the serving player chooses to serve left (L) or right (R). Simultaneously, the receiving player anticipates to the left (L) or right (R). Hence in every state the action sets for both players are the same, $A(s) = B(s) = \{L, R\}$. Furthermore, it is assumed that the probabilities for both players to win the point are completely determined by these choices, as is reflected in Figure 2.

Server \ Receiver	L	R
L	$p_1, 1-p_1$	$p_2, 1-p_2$
R	$p_3, 1-p_3$	$p_4, 1-p_4$

(a) Service Game Player A

Server \ Receiver	L	R
L	$q_1, 1-q_1$	$q_2, 1-q_2$
R	$q_3, 1-q_3$	$q_4, 1-q_4$

(b) Service Game Player B

Figure 2: Two Point Games

Here, $0 < p_i, q_i < 1$ for $i = 1, 2, 3, 4$ where p_i, q_i are the probabilities that the serving player wins the point. In (Klaassen and Magnus, 2001) it is shown that although points in tennis are not independent and identically distributed (i.i.d), when you account for player strengths it provides a good approximation to assume that the points are i.i.d. This validates the assumption that independent of the score or the outcome of the last point(s), the point game of Figure 2(a) is played at every node of the binary scoring rule in which player A has service and similarly that the point game of Figure 2(b) is played at every node where player B serves. This is a binary Markov game and will be our model of a tennis match.

3 Strategies

In this section I will use game-theoretic concepts from Mas-Colell et al. (1995).

Under a reasonable assumption, the point games in Figure 2 have unique Nash equilibria in strictly mixed strategies. The assumption is that the serving player has a greater probability to win the point when the receiver anticipates incorrectly than when the receiver anticipates to the right side. This leads to the following conditions on the p_i's and q_i's:

$$\begin{aligned} &p_1 < p_2,\ p_4 < p_2,\ p_1 < p_3 \text{ and } p_4 < p_3 \\ &q_1 < q_2,\ q_4 < q_2,\ q_1 < q_3 \text{ and } q_4 < q_3. \end{aligned} \tag{1}$$

First, look at the point game played when player A has service (Figure 2(a)). Suppose that the server plays L with probability λ and R with probability $1-\lambda$. The receiver plays L with probability μ and R with probability $1-\mu$. When the serving player plays his equilibrium strategy, the receiver should be indifferent between playing L or R:

$$\lambda(1-p_1) + (1-\lambda)(1-p_3) = \lambda(1-p_2) + (1-\lambda)(1-p_4). \tag{2}$$

Hence λ is completely determined by the p_i's:

$$\lambda = \frac{p_3 - p_4}{p_2 + p_3 - p_1 - p_4}. \tag{3}$$

Note $0 < \lambda < 1$ because of (1) and hence the server's equilibrium strategy is strictly mixed. With a similar argument we can find an expression for μ if player B plays an equilibrium strategy:

$$\mu p_1 + (1-\mu)p_2 = \mu p_3 + (1-\mu)p_4, \tag{4}$$

yielding

$$\mu = \frac{p_4 - p_2}{p_1 + p_4 - p_2 - p_3}, \tag{5}$$

where also $0 < \mu < 1$ because of (1). For the point game in Figure 2(b) the same reasoning leads to similar expressions.

Now, it can be shown that if a binary Markov game satisfies a simple Monotonicity Condition, it is an equilibrium strategy in the complete match for both players to play their equilibrium strategies in every point game. This condition entails that the probability to win the match should be greater if the player wins the current point game than if the player loses the current point game. The proofs in this section roughly follow (Walker et al., 2011).

Monotonicity Condition (Monotonicity Condition) If $W_A(s)$ and $W_B(s)$ are the probabilities to win the match if the current state is s, for players A and B respectively, the match satisfies the Monotonicity Condition if $W_A(s^+) > W_A(s^-)$ and $W_B(s^-) > W_B(s^+)$, for every non-absorbing state $s \in S$.

This is an appealing condition, likely to be satisfied by binary Markov games. Note that the two inequalities stated are actually equivalent, since $\forall s \in S$, $W_B(s) = 1 - W_A(s)$.

Theorem 3.1. *A tennis match, modeled as a binary Markov game as explained in Section 2, satisfies the Monotonicity Condition.*

Proof. First, assume that for every possible set score and game score $W_A(s)$ is greater when player A wins the current game than if he loses it. Formally: assume that

$$W_A(\bar{g}) > W_A(\underline{g}). \tag{6}$$

If the states s and t are in the same column in Fnigure 1, I will write $s > t$ if player A has won more points in s than in t. Then the Monotonicity Condition is equivalent to

$$s > t \Rightarrow W_A(s) > W_A(t) \tag{7}$$

Let $v_{ss'}(a, b)$ be the probability to move from state s to s' if the players choose actions a and b. To shorten notation, I will write $v_s(a, b)$ for $v_{ss^+}(a, b)$ and drop the arguments where they are not needed. Then the probability to move from s to s^- is given by $1 - v_s$. See that

$$W_A(s) = v_s W_A(s^+) + (1 - v_s) W_A(s^-) \tag{8}$$

and $0 < v_s < 1$. This gives

$$\begin{aligned} W_A(3-2) &= v_{3-2} W_A(\bar{g}) + (1 - v_{3-2}) W_A(2-2) \\ W_A(2-2) &= v_{2-2} W_A(3-2) + (1 - v_{2-2}) W_A(2-3) \\ W_A(2-3) &= v_{2-3} W_A(2-2) + (1 - v_{2-3}) W_A(\underline{g}), \end{aligned} \tag{9}$$

where $3-2$, $2-2$ and $2-3$ represent scores in the current game. By substituting the first and third equation of (9) in the middle equation, we find

$$
\begin{aligned}
&(1-((1-v_{3-2})v_{2-2}+(1-v_{2-2})v_{2-3}))W_A(2-2)\\
&=v_{2-2}v_{3-2}W_A(\bar{g})+(1-v_{2-2})(1-v_{2-3})W_A(\underline{g})\\
&<(v_{2-2}v_{3-2}+(1-v_{2-2})(1-v_{2-3}))W_A(\bar{g})\\
&\Rightarrow W_A(2-2)<\frac{(v_{2-2}v_{3-2}+(1-v_{2-2})(1-v_{2-3}))}{(1-(1-v_{3-2})v_{2-2}+(1-v_{2-2})v_{2-3})}W_A(\bar{g})=W_A(\bar{g}),
\end{aligned}
\tag{10}
$$

where the inequality comes from (6). With a similar argument, it is found that $W_A(2-2) > W_A(\underline{g})$. Using $W_A(2-2) < W_A(\bar{g})$ in the first equation and $W_A(2-2) > W_A(\underline{g})$ in the third equation of (9), find that $W_A(3-2) > W_A(2-2) > W_A(2-3)$. Hence (7) is now satisfied by the rightmost non-absorbing column of Figure 1. From this column, we can work to the left. Consider two adjacent columns in Figure 1. Assume that the Monotonicity Condition is satisfied by the right column and that the left column contains states s and t, with $s > t$. If $s^+ \neq \bar{g}$ and $t^- \neq \underline{g}$, the right column contains s^+, s^-, t^+ and t^-. Clearly, $s^+ > s^- \geq t^+ > t^-$ and it follows from (8) and the assumption that the Monotonicity Condition is satisfied by the column containing s^+, s^-, t^+ and t^- that $W_A(s) > W_A(t)$. If $s^+ = \bar{g}$, then $W_A(s^+) > W_A(t^+)$ and since $s^- > t^-$, again $W_A(s) > W_A(t)$. The final possibility is if $t^- = \underline{g}$. We then have $W_A(s^-) > W_A(t^-)$ and $s^+ > t^+$ so in this case also $W_A(s) > W_A(t)$. We have now established that if (7) is satisfied by a non-absorbing column in Figure 1, the condition will also hold in the column to the left of it. Since we've shown that the condition holds in the rightmost non-absorbing column, it follows that (7) is satisfied in all non-absorbing columns of Figure 1.

What is left to show is that our assumption $W_A(\bar{g}) > W_A(\underline{g})$ is valid. To do this, assume that a player has a greater probability of winning the match if he wins the current set than if he loses it. Since a set is won by winning games in the same way as a game is won by winning points (except to win a set, you must win six games instead of the four points to win a game), the same argument as above gives that indeed $W_A(\bar{g}) > W_A(\underline{g})$ when winning a set leads to a greater possibility to win the match than losing a set. Finally, it is obvious that winning a set leads to a greater probability to win the match. For example, take a match where the winner is the first to win two sets. If a player loses the first set, he has to win the last two sets. But if he would have won the first set, he has to win one out of the last two sets, which clearly has a greater probability than winning two out of two sets. And if the player loses the second set, he has either lost the match or he has to win the last set. But if he would have won the second set, he either has won the match or he has to win the last set, so winning the second set also gives a greater probability to win the match. The winner of a third set is the winner of the match, so winning the third set clearly gives a greater possibility to win the match than losing the third set. This means our assumption (6) is valid.

Hence a tennis match satisfies the Monotonicity Condition. □

Now, let $\rho(a|s)$ and $\phi(b|s)$ be the probabilities assigned to actions a and b in state s. When player B is playing strategy ϕ the probabilities of moving from state s to s', depending on the action choice $a \in \{L, R\}$ by player A, are denoted by $\bar{v}_{ss'}(a, \phi)$:

$$\forall s, s' \in S, a \in \{L, R\} : \bar{v}_{ss'}(a, \phi) = \sum_{b \in \{L,R\}} \phi(b|s) v_{ss'}(a, b). \tag{11}$$

Let $W_A(s|\rho, \phi)$ denote the probability of winning the match for player A when the current state is s, given that player A and B play strategies ρ and ϕ respectively. For a strategy ρ to be a best response to ϕ, it must maximize this probability. Hence it is necessary for a best response ρ that

$$\forall s \in S : W_A(s|\rho, \phi) = \max_{a \in \{L,R\}} \sum_{s' \in \{s^+, s^-\}} \bar{v}_{ss'}(a, \phi) W_A(s'|\rho, \phi). \tag{12}$$

The following lemma tells that (12) is also sufficient for ρ to be a best response.

Lemma 3.2. *If a strategy ρ satisfies (12), it is a best response to ϕ.*

Proof. Suppose $W_A(s|\rho, \phi)$ satisfies (12). Let ρ' be an arbitrary strategy for player A, we then have

$$\begin{aligned} W_A(s|\rho, \phi) &= \max_{a \in \{L,R\}} \sum_{s' \in \{s^+, s^-\}} \sum_{b \in \{L,R\}} \phi(b|s) v_{ss'}(a, b) W_A(s'|\rho, \phi) \\ &\geq \sum_{a \in \{L,R\}} \rho'(a|s) \sum_{s' \in \{s^+, s^-\}} \sum_{b \in \{L,R\}} \phi(b|s) v_{ss'}(a, b) W_A(s'|\rho, \phi) \\ &= \sum_{s' \in \{s^+, s^-\}} \sum_{a \in \{L,R\}} \sum_{b \in \{L,R\}} \rho'(a|s) \phi(b|s) v_{ss'}(a, b) W_A(s'|\rho, \phi). \end{aligned} \tag{13}$$

Here, the inequality comes from the fact that choosing action $a \in \{L, R\}$ that maximizes player A's probability to win the match when the current state is s and player B plays ϕ will give player A a greater probability to win the match than any weighted average of the probabilities when playing L and R. Since either $s^+ = \omega_A$ or $s^+ \neq \omega_A$, this can be rewritten as

$$W_A(s|\rho,\phi) \geq \sum_{a\in\{L,R\}} \sum_{b\in\{L,R\}} \rho'(a|s)\phi(b|s)v_{s\omega_A}(a,b) + \sum_{s'\neq\omega_A} \sum_{a\in\{L,R\}} \sum_{b\in\{L,R\}} \rho'(a|s)\phi(b|s)v_{ss'}(a,b)W_A(s'|\rho,\phi). \tag{14}$$

Let (ρ'_{S_i},ρ) denote the strategy for player A if he plays ρ' in all states $s \in S_i$ and ρ in all other states. For $S_i = \{s\}$, this is equal to the right-hand side of (14), and thus $\forall s \in S$:

$$W_A(s|\rho,\phi) \geq W_A(s|(\rho'_{\{s\}},\rho),\phi). \tag{15}$$

Now denote $s \to s_i$ if s_i can occur at a later time in the match than s. Let $S_j \subset S$ denote the set of states including s, s_j and all s_i such that $s \to s_i \to s_j$. I will show that

$$W_A(s|(\rho'_{S_1},\rho),\phi) \geq W_A(s|(\rho'_{S_2},\rho),\phi). \tag{16}$$

for every $s, s_1, s_2 \in S$ such that $s \to s_1 \to s_2$.

$$\begin{aligned} W_A(s|(\rho'_{S_1},\rho),\phi) = & \sum_{\{s,s'\in S_1|\omega_A\in S_1\}} \rho'(a|s)\phi(b|s)v_{ss'}(a,b) \\ & + \sum_{\{s,s'\in S_1|\omega_A\notin S_1\}} \rho'(a|s)\phi(b|s)v_{ss'}(a,b)W_A(s_1|\rho,\phi) \\ \geq & \sum_{\{s,s'\in S_1|\omega_A\in S_1\}} \rho'(a|s)\phi(b|s)v_{ss'}(a,b) \\ & + \sum_{\{s,s'\in S_1|\omega_A\notin S_1\}} \rho'(a|s)\phi(b|s)v_{ss'} \\ & \times \Bigg(\sum_{a\in\{L,R\}} \sum_{b\in\{L,R\}} \rho'(a|s)\phi(b|s)v_{s\omega_A}(a,b) \\ & + \sum_{s'\neq\omega_A} \sum_{a\in\{L,R\}} \sum_{b\in\{L,R\}} \rho'(a|s)\phi(b|s)v_{ss'}(a,b)W_A(s'|\rho,\phi) \Bigg) \end{aligned} \tag{17}$$

Where the part between brackets comes from (14). The right-hand side of

(17) can be rewritten as

$$\sum_{\{s,s'\in S_1|\omega_A\in S_1\}} \rho'(a|s)\phi(b|s)v_{ss'}(a,b)$$
$$+ \sum_{\{s,s'\in S_1|\omega_A\notin S_1\}} \rho'(a|s)\phi(b|s)v_{ss'} \sum_{a\in\{L,R\}} \sum_{b\in\{L,R\}} \rho'(a|s)\phi(b|s)v_{s\omega_A}(a,b)$$
$$+ \sum_{\{s,s'\in S_1|\omega_A\notin S_1\}} \rho'(a|s)\phi(b|s)v_{ss'}$$
$$\times \sum_{s'\neq\omega_A} \sum_{a\in\{L,R\}} \sum_{b\in\{L,R\}} \rho'(a|s)\phi(b|s)v_{ss'}(a,b)W_A(s'|\rho,\phi) \quad (18)$$

The first term is the probability that player A will have won the match in state s_1. The second term is the probability that player A wins the match *exactly in* the state s_1^+ (and thus $s_1^+ = \omega_A$). The last term is the probability that player A will win the match in a state later than s_1^+. Hence if $S_1^+ = S_1 \cap \{s_1^+\}$, we have for $s_1^+ \in S_1^+$:

$$\begin{aligned} W_A(s|(\rho'_{S_1},\rho),\phi) \geq & \sum_{\{s,s'\in S_1^+|\omega_A\in S_1^+\}} \rho'(a|s)\phi(b|s)v_{ss'}(a,b) \\ & + \sum_{\{s,s'\in S_1^+|\omega_A\notin S_1^+\}} \rho'(a|s)\phi(b|s)v_{ss'}(a,b)W_A(s_1^+|\rho,\phi) \\ & = W_A(s|(\rho'_{S_1^+},\rho),\phi). \end{aligned} \quad (19)$$

We have now shown that (15) and (16) hold, and thus

$$W_A(s|\rho,\phi) \geq W_A(s|\rho',\phi). \quad (20)$$

This proves that ρ is indeed a best response to ϕ. □

Lemma 3.2 gives a condition on which a strategy ρ is a best response to a given strategy ϕ. It does not *give* the actual best response. I will give the definition for a strategy called minimax, which is a strategy where the maximum payoff of your opponent in minimised. As it turns out that against a minimax strategy, it is a best response to also play minimax and hence a pair of minimax strategies form a Nash equilibrium. Let $\Delta\{L,R\}$ denote the set of probability distributions over $\{L,R\}$.

Definition For each state s, a mixture $\rho_s \in \Delta\{L,R\}$, satisfying

$$\rho_s \in \arg\min_{\rho'_s\in\Delta\{L,R\}} \max_{\phi_s\in\Delta\{L,R\}} \sum_{a\in\{L,R\}} \sum_{b\in\{L,R\}} \rho'_s(a)\phi_s(b)\pi_{s,B}(a,b) \quad (21)$$

is called a minimax strategy for player A.

It has been shown that sporters like tennis and soccer players use this strategy in several real-world situations, whether consciously or not (Palacios-Huerta, 2003), (Walker and Wooders, 2001).

Theorem 3.3. *If a binary Markov game satisfies the Monotonicity Condition, any pair of minimax strategies form a Nash equilibrium.*

Proof. Suppose ρ and ϕ are minimax strategies for a binary Markov game satisfying the Monotonicity condition. By Lemma 3.2, it is sufficient to prove ρ satisfies (12). So we must show that

$$\forall s \in S : W_A(s|\rho,\phi) = \max_{a\in\{L,R\}} \sum_{s'\in\{s^+,s^-\}} \bar{v}_{ss'}(a,\phi) W_A(s'|\rho,\phi). \tag{22}$$

The right-hand side of this is equal to

$$\begin{aligned} &\max_{a\in\{L,R\}} \sum_{b\in\{L,R\}} \phi(b|s)[\pi_{s,A}(a,b)W_A(s^+|\rho,\phi) + \pi_{s,B}(a,b)W_A(s^-|\rho,\phi)] \\ &= \max_{a\in\{L,R\}} \Big[W_A(s^+|\rho,\phi) \sum_{b\in\{L,R\}} \phi(b|s)\pi_{s,A}(a,b) \\ &+ W_A(s^-|\rho,\phi) \sum_{b\in\{L,R\}} \phi(b|s)\pi_{s,B}(a,b) \Big] \end{aligned} \tag{23}$$

Since the Monotonicity Condition is satisfied, $W_A(s^+|\rho,\phi) > W_A(s^-|\rho,\phi)$ and $\pi_{s,A}(a,b) + \pi_{s,B}(a,b) = 1$, a will only maximize the expression if it maximizes $\sum_{b\in\{L,R\}} \phi(b|s)\pi_{s,A}(a,b)$. Since $\rho(\cdot|s)$ and $\phi(\cdot|s)$ are minimax strategies, for every state s, every a in the support of $\rho(\cdot|s)$ maximizes $\sum_{b\in\{L,R\}} \phi(b|s)\pi_{s,A}(a,b)$. Hence $\forall a \in \text{supp } \rho(\cdot,s)$, we can discard the max in the right-hand side of (23) to simplify it to

$$\begin{aligned} &W_A(s^+,\rho,\phi) \sum_{b\in\{L,R\}} \phi(b|s)\pi_{s,A}(a,b) + W_A(s^-,\rho,\phi) \sum_{b\in\{L,R\}} \phi(b|s)\pi_{s,B}(a,b) \\ &= W_A(s^+,\rho,\phi) \sum_{a\in\{L,R\}} \rho(a|s) \sum_{b\in\{L,R\}} \phi(b|s)\pi_{s,A}(a,b) \\ &+ W_A(s^-,\rho,\phi) \sum_{a\in\{L,R\}} \rho(a|s) \sum_{b\in\{L,R\}} \phi(b|s)\pi_{s,B}(a,b) \\ &= v_{ss^+}(\rho,\phi)W_A(s^+,\rho,\phi) + v_{ss^-}(\rho,\phi)W_A(s^-,\rho,\phi) \\ &= W_A(s,\rho,\phi). \end{aligned} \tag{24}$$

Hence (22) is proven to be true and this completes the proof. □

4 Probabilities

In this section I will first analyze single points and find the probabilities for both players to win a point. Based on these probabilities, we can also find the probabilities of winning a game, then a set, and finally the match.

4.1 Points

Assuming both players play their equilibrium strategy, we can now find the probabilities of winning the point for every player. For this, I will assume that points in a service game are independent and identically distributed (i.i.d). Although in practice points are not i.i.d, the divergence from i.i.d is small and for practical applications, it is reasonable to assume the points are i.i.d when correcting for the quality of the players (Klaassen and Magnus, 2001). Let α be the probability that the player A wins the point on service (and thus $1-\alpha$ is the probability that player B wins the point). Similarly, β is the probability that the player B wins the point on his own service. These probabilities can be calculated as

$$\begin{aligned}\alpha &= \lambda\mu p_1 + \lambda(1-\mu)p_2 + (1-\lambda)\mu p_3 + (1-\lambda)(1-\mu)p_4 \\ &= \frac{p_1p_4 - p_2p_3}{p_1 + p_4 - p_2 - p_3}\end{aligned} \tag{25}$$

and

$$\beta = \frac{q_1q_4 - q_2q_3}{q_1 + q_4 - q_2 - q_3}. \tag{26}$$

4.2 Games

Consider a game in which player A has service. We can find the probabilities of having crossed a node in a game for every node in Figure 1. For most of these nodes, these probabilities follow a binomial distribution. The exceptions are $\bar{g}$, $\underline{g}$, 2-2, 3-2 and 2-3. Hence for the other nodes, if $i+j$ points have been played, the probability of the score being i-j is $\binom{i+j}{i}\alpha^i(1-\alpha)^j$. Now, let x be the number of points played in a game and let $G_A(i-j|x=i+j)$ be the probability that the node i-j has been crossed at some point in a game in which player A has service. To shorten notation, I will leave out the condition and write $G_A(i-j)$. For the five remaining nodes, we can find these probabilities using

$$\begin{aligned}
G_A(3-2) &= \alpha G_A(2-2) + (1-\alpha)G_A(3-1) \\
G_A(2-3) &= (1-\alpha)G_A(2-2) + \alpha G_A(1-3) \\
G_A(2-2) &= \alpha(G_A(1-2) + G_A(2-3)) + (1-\alpha)(G_A(2-1) + G_A(3-2)).
\end{aligned} \tag{27}$$

We get

$$\begin{aligned}
G_A(2-2) &= \frac{6\alpha^2(1-\alpha)^2 + 8\alpha^3(1-\alpha)^3}{1-2\alpha(1-\alpha)} \\
G_A(3-2) &= \frac{10\alpha^3(1-\alpha)^2}{1-2\alpha(1-\alpha)} \\
G_A(2-3) &= \frac{10\alpha^2(1-\alpha)^3}{1-2\alpha(1-\alpha)}.
\end{aligned} \tag{28}$$

Now we can also calculate $G_A(\bar{g})$ and $G_A(\underline{g})$, since

$$\begin{aligned}
G_A(\bar{g}) &= \alpha(G_A(3-0) + G_A(3-1) + G_A(3-2)) \\
G_A(\underline{g}) &= 1 - G_A(\bar{g}).
\end{aligned} \tag{29}$$

Filling in the $G_A(3-0)$, $G_A(3-1)$, $G_A(3-2)$ found earlier we get, after some simplifying:

$$\begin{aligned}
G_A(\bar{g}) &= \frac{\alpha^4(15 - 34\alpha + 28\alpha^2 - 8\alpha^3)}{1-2\alpha(1-\alpha)} \\
G_A(\underline{g}) &= 1 - G_A(\bar{g}).
\end{aligned} \tag{30}$$

Similarly, when player B is serving these probabilities are

$$\begin{aligned}
G_B(\underline{g}) &= \frac{\beta^4(15 - 34\beta + 28\beta^2 - 8\beta^3)}{1-2\beta(1-\beta)} \\
G_B(\bar{g}) &= 1 - G_B(\underline{g}).
\end{aligned} \tag{31}$$

4.3 Sets

Using this we can also find the probabilities for both players to win a set, but now we have to keep in mind that the players switch service after every game. The final probabilities to win a set however, do not depend on the initial server as has been proven in (Riddle, 1988). In this article, it is shown that in matches of the type 'first-to-n, win-by-2' in which two types of games alternate, the probability for both players to win is independent of the initial type of game. This is the case in a tennis set, where the two types are the service games of both players. It is also the case in a tiebreak, thus also for a set with tiebreak, which is a combination of two types of matches that are first-to-n, win-by-2. Hence the final probabilities of winning a set and consequently a match are independent of the initial server. So I will assume player A starts on service in every set. His probability to win a point in his own service game is α and player B's probability to win a point in his own service game is β. Let $G_A(\bar{g})$ and $G_B(\underline{g})$ be the probabilities for the players to win their respective service games. Let y be the number of games played in the current set and $Q(i-j|y=i+j)$ be the probability that the score has been i-j in games at some time during the set. Again, to shorten notation I will leave out the condition and write $Q(i-j)$.

Divide a set in instances where an even number of games have been played and instances where an odd number of games have been played. After an even number of games both players will have served half of the games. If an odd number of games has been played, because player A started serving, he will have served one more game than player B. If the score in games is 5-j, $j \in \{1,3\}$, both players have had $\frac{5+j}{2}$ service games. Player A will only need to win his current service game to win the set. This score can only be reached if player A has 'broken' player B's service game at least $\frac{5-j}{2}$ times (a break occurs when a player manages to win a game in which his opponent had service). If player B has broken player A's service game k times, then player A has to have broken player B's service $\frac{5-j}{2}+k$ times. This gives

$$Q(5-j) = \sum_{k=0}^{j} \Bigg[\binom{\frac{5+j}{2}}{k} \binom{\frac{5+j}{2}}{\frac{5-j}{2}+k} G_A(\bar{g})^{\frac{5+j}{2}-k} G_B(\underline{g})^{j-k} \\ \times (1-G_A(\bar{g}))^k (1-G_B(\underline{g}))^{\frac{5-j}{2}+k} \Bigg] \text{ if } j=1,3. \qquad (32)$$

Now if the score is 5-j with $j \in \{0,2,4\}$, player A will only need to break player B's current service game to win the set. Player A has now had $\frac{6+j}{2}$ service games, while player B only had $\frac{4+j}{2}$ service games. Because he has won a maximum of j of these, his service has been broken at least $\frac{4-j}{2}$ times. This gives

$$Q(5-j) = \sum_{k=0}^{j} \Bigg[\binom{\frac{6+j}{2}}{k} \binom{\frac{4+j}{2}}{\frac{4-j}{2}+k} G_A(\bar{g})^{\frac{6+j}{2}-k} G_B(\underline{g})^{j-k} \\ \times (1-G_A(\bar{g}))^k (1-G_B(\underline{g}))^{\frac{4-j}{2}+k} \Bigg] \text{ if } j=0,2,4. \qquad (33)$$

Multiply these probabilities with the probabilities for player A to win his sixth game to find

$$Q(6-j) = \begin{cases} (1-G_B(\underline{g}))Q(5-j) & \text{if } j = 0,2,4, \\ G_A(\bar{g})Q(5-j) & \text{if } j = 1,3. \end{cases} \tag{34}$$

But these are not yet all scores in which A has won the set. We still need the probabilities for a 7-5 score and for a tiebreak. For this we need

$$Q(5-5) = \sum_{k=0}^{5} \binom{5}{k}^2 G_A(\bar{g})^{5-k} G_B(\underline{g})^{5-k} (1-G_A(\bar{g}))^k (1-G_B(\underline{g}))^k \tag{35}$$

which gives

$$Q(7-5) = G_A(\bar{g})(1-G_B(\underline{g}))Q(5-5). \tag{36}$$

Player A can also win the set after a tiebreak. Let $P(\tau)$ be the probability of a tiebreak occurring, x_τ the number of points played in the tiebreak and $T(i-j|x_\tau = i+j)$ the probability that the score in a tiebreak is i-j after $i+j$ points. As in the previous sections, I will leave out the condition and write $T(i-j)$.

$$P(\tau) = Q(6-6) = Q(5-5)(G_A(\bar{g})G_B(\underline{g}) + (1-G_A(\bar{g}))(1-G_B(\underline{g}))). \tag{37}$$

A tiebreak has a special order of service, namely A, B, B, A, A, B, B, A, A, etc. This leads to the following probabilities, found in a similar way as the $Q(i-j)$'s:

$$T(7-j) = \begin{cases} \sum_{k=0}^{j} \binom{\frac{6+j}{2}}{k} \binom{\frac{6+j}{2}}{\frac{6-j}{2}+k} \alpha^{\frac{6+j}{2}-k} \beta^{j-k} (1-\alpha)^k (1-\beta)^{\frac{8-j}{2}+k} & \text{if } j = 0,4, \\ \sum_{k=0}^{j} \binom{\frac{5+j}{2}}{k} \binom{\frac{7+j}{2}}{\frac{7-j}{2}+k} \alpha^{\frac{7+j}{2}-k} \beta^{j-k} (1-\alpha)^k (1-\beta)^{\frac{7-j}{2}+k} & \text{if } j = 1,5, \\ \sum_{k=0}^{j} \binom{\frac{6+j}{2}}{k} \binom{\frac{6+j}{2}}{\frac{6-j}{2}+k} \alpha^{\frac{8+j}{2}-k} \beta^{j-k} (1-\alpha)^k (1-\beta)^{\frac{6-j}{2}+k} & \text{if } j = 2, \\ \sum_{k=0}^{j} \binom{\frac{7+j}{2}}{k} \binom{\frac{5+j}{2}}{\frac{5-j}{2}+k} \alpha^{\frac{7+j}{2}-k} \beta^{j-k} (1-\alpha)^k (1-\beta)^{\frac{7-j}{2}+k} & \text{if } j = 3. \end{cases} \tag{38}$$

Also,

$$T(6-6) = \sum_{k=0}^{6} \binom{6}{k}^2 \alpha^{6-k} \beta^{6-k} (1-\alpha)^k (1-\beta)^k, \tag{39}$$

and for $i \geq 6$:

$$T([i+2]-i) = T(6-6)(\alpha\beta + (1-\alpha)(1-\beta))^{i-6}(\alpha(1-\beta)) \tag{40}$$

since to reach the score $[i+2]$-i, first a 6-6 score must have been reached, with probability $T(6-6)$. Then there must have been $i-6$ sequences of two points of which both players won one. This occurs with probability $(\alpha\beta+(1-\alpha)(1-\beta))^{i-6}$. After that, player A must have won two consecutive points, which happens with probability $\alpha(1-\beta)$. Let $\bar{t}$ be the state in which player A has won the tiebreak. We can find the probability of arriving in this state:

$$T(\bar{t}) = \sum_{m=0}^{5} T(7-m) + \sum_{m=6}^{\infty} T([m+2]-m). \tag{41}$$

Note that the infinite series in the final term of (41) is finite. Since $\sum_{i=0}^{\infty} \delta^i = \frac{1}{1-\delta}$ for $|\delta| < 1$ and $0 < \alpha\beta + (1-\alpha)(1-\beta) < 1$, we have

$$\begin{aligned}\sum_{m=6}^{\infty} T([m+2]-m) &= T(6-6)(\alpha(1-\beta)) \sum_{m=6}^{\infty} (\alpha\beta + (1-\alpha)(1-\beta))^{m-6} \\ &= \frac{T(6-6)(\alpha(1-\beta))}{1-(\alpha\beta+(1-\alpha)(1-\beta))}. \end{aligned} \tag{42}$$

Let $\bar{s}$ and $\underline{s}$ be the states in which players A and B win the set, respectively. Combining (34), (36), (37) and (41) gives the probability for player A to win a set:

$$Q(\bar{s}) = \sum_{j=0}^{4} Q(6-j) + Q(7-5) + P(\tau)T(\bar{t}). \tag{43}$$

Consequently,

$$Q(\underline{s}) = 1 - Q(\bar{s}). \tag{44}$$

For a set that does not allow for a tiebreak, the probabilities are a little different. The first two terms are the same as in (43). But because there is no tiebreak if the score 6-6 is reached, instead games are played until a two-game difference is achieved, the last term changes. A 6-6 score is reached with probability $Q(6-6)$, a two-game difference for player A happens with probability $G_A(\bar{g})G_B(\underline{g})$ and this is multiplied by the sum of the probabilities for all tied scores in between 6-6 and the final two games.

$$Q_{\text{alt}}(\bar{s}) = \sum_{j=0}^{4} Q(6-j) + Q(7-5) + Q(6-6) G_A(\bar{g}) G_B(\underline{g}) \times \sum_{i=0}^{\infty} (G_A(\bar{g}) G_B(\bar{g}) + G_A(\underline{g}) G_B(\underline{g}))^i. \tag{45}$$

Note that this last term is finite by the same argument as used in (42). So we can simplify (45) to

$$Q_{\text{alt}}(\bar{s}) = \sum_{j=0}^{4} Q(6-j) + Q(7-5) + \frac{Q(6-6) G_A(\bar{g}) G_B(\underline{g})}{1 - (G_A(\bar{g}) G_B(\bar{g}) + G_A(\underline{g}) G_B(\underline{g}))}. \tag{46}$$

4.4 Match

Using the previous sections, we can find the probabilities per player to win the match, depending on the type of match. The four match types are given in Table 1. The probabilities corresponding to each match type are as follows. For a type 1 match:

$$P_{\text{type 1}}(\omega_A) = Q(\bar{s})^2 + \binom{2}{1} Q(\bar{s})^2 Q(\underline{s}). \tag{47}$$

For a type 2 match:

$$P_{\text{type 2}}(\omega_A) = Q(\bar{s})^2 + \binom{2}{1} Q(\bar{s}) Q(\underline{s}) Q_{\text{alt}}(\bar{s}). \tag{48}$$

For a type 3 match:

$$P_{\text{type 3}}(\omega_A) = Q(\bar{s})^3 + \binom{3}{1} Q(\bar{s})^3 Q(\underline{s}) + \binom{4}{2} Q(\bar{s})^3 Q(\underline{s})^2. \tag{49}$$

For a type 4 match:

$$P_{\text{type 4}}(\omega_A) = Q(\bar{s})^3 + \binom{3}{1} Q(\bar{s})^3 Q(\underline{s}) + \binom{4}{2} Q(\bar{s})^2 Q(\underline{s})^2 Q_{\text{alt}}(\bar{s}). \tag{50}$$

And of course for $i = 1, 2, 3, 4$,

$$P_{\text{type } i}(\omega_B) = 1 - P_{\text{type } i}(\omega_A). \tag{51}$$

Type 1	First to two sets
Type 2	First to two sets, no tiebreak in third set
Type 3	First to three sets
Type 4	First to three sets, no tiebreak in fifth set

Table 1: Types of matches

Example Player A and player B are playing a tennis match. The first player to win two sets wins the match. The third set may end in a tiebreak. Suppose player A is an 'average' player, which I will define as having p_i's in his service game as found in (Walker and Wooders, 2001). But player B is slightly less talented, the q_i's are given by $q_i = p_i - 0.02$. See Figure 3. Now if both players play their equilibrium strategies, their probabilities of winning a point on service are $\alpha = 0.65$ and $\beta = 0.63$. This difference means that the probability to win the match will not be 0.5 for both players, as would be if $\alpha = \beta$. The MATLAB program given in Appendix C.1 gives, with input $\alpha = 0.65$ and $\beta = 0.63$, that $P_{match}(A) = 0.60$. So the small difference in serving quality leads to a larger difference in the players' probabilities to win the match.

Server \ Receiver	L	R
L	0.58, 0.42	0.79, 0.21
R	0.73, 0.27	0.49, 0.51

(a) Service Game Player A

Server \ Receiver	L	R
L	0.56, 0.44	0.77, 0.23
R	0.71, 0.29	0.47, 0.53

(b) Service Game Player B

Figure 3: Two Point Games

In the match between Isner and Mahut at Wimbledon in 2010, Isner won 76.2% of his service points while Mahut won 78.7% of his service points. If we take these percentages as input, hence $\alpha = 0.762$ and $\beta = 0.787$, the program calculates that Isner had a 0.356 probability of winning the match.

5 Expected Values

In this section I will calculate the expected number of points played in matches between two players of equal strength ($\alpha = \beta$). This assumption is made for two reasons: to avoid having to distinguish between sets and tiebreaks with different initial serving players, and because these matches are expected to see more points played than uneven matches.

First, the expected number of points in a game and in a tiebreak are calculated. Then we can calculate the expected number of points in a set, and finally the expected number of points per match.

5.1 Number of points per game

First we calculate the expected number of points per game, $E_{game}(X)$. Let $P(X_{game} = i)$ be the probability that the number of points in a service game equals i and calculate

$$E_{game}(X) = \sum_{i=4}^{\infty} iP(X_{game} = i) \tag{52}$$

where for $i = 4, 5, 6$, the $P(X_{game} = i)$ are easily found with

$$\begin{aligned} P(X_{game} = 4) &= \alpha G_A(3-0) + (1-\alpha)G_A(0-3), \\ P(X_{game} = 5) &= \alpha G_A(3-1) + (1-\alpha)G_A(1-3), \\ P(X_{game} = 6) &= \alpha G_A(3-2) + (1-\alpha)G_A(2-3). \end{aligned} \tag{53}$$

because a game has to be won by a 2-point difference it will, except if it ends in 4-1 or 1-4, always end after an even number of points. We will now look for a general expression for $P(X_{game} = i)$ for $i = 8, 10, 12, ...$ In this case, the score must have been 3-3 after six points. After this score was reached, all other ties up to $[\frac{1}{2}i-1]$-$[\frac{1}{2}i-1]$ have been reached too, after which one of the players wins two points in a row. Hence for $i = 8, 10, 12, ...,$

$$\begin{aligned} G_A([\frac{1}{2}i-1]-[\frac{1}{2}i-1]) &= G_A(3-3)(2\alpha(1-\alpha))^{\frac{1}{2}i-1-3} \\ &= \frac{5}{2}(2\alpha(1-\alpha))^{\frac{1}{2}i-1}. \end{aligned} \tag{54}$$

Then $P(X_{game} = i) = (\alpha^2 + (1-\alpha)^2)G_A([\frac{1}{2}i-1)] - [\frac{1}{2}i-1])$, hence

$$P(X_{game} = i) = \begin{cases} \alpha^4 + (1-\alpha)^4 & \text{if } i = 4, \\ \binom{4}{1}(\alpha^4(1-\alpha) + \alpha(1-\alpha)^4) & \text{if } i = 5, \\ \binom{5}{2}(\alpha^4(1-\alpha)^2 + \alpha^2(1-\alpha)^4) & \text{if } i = 6, \\ 0 & \text{if } i = 7, 9, 11, ..., \\ \frac{5}{2}(\alpha^2 + (1-\alpha)^2)(2\alpha(1-\alpha))^{\frac{1}{2}i-1} & \text{if } i = 8, 10, 12, ... \end{cases} \tag{55}$$

To calculate $E_{game}(X)$ note again that $\sum_{i=0}^{\infty} \delta^i = \frac{1}{1-\delta}$ for $|\delta| < 1$. Taking the derivative on both sides, we get

$$\frac{d}{d\delta}\sum_{i=0}^{\infty} \delta^i = \sum_{i=0}^{\infty} \frac{d}{d\delta}\delta^i = \sum_{i=1}^{\infty} i\delta^{i-1} = \frac{1}{(1-\delta)^2}. \tag{56}$$

So in the expression

$$\begin{aligned} E_{game}(X) &= \sum_{i=4}^{\infty} iP(X_{game} = i) \\ &= 4(\alpha^4 + (1-\alpha)^4) + 5(\binom{4}{1}(\alpha^4(1-\alpha) + \alpha(1-\alpha)^4)) \\ &\quad + 6(\binom{5}{2}(\alpha^4(1-\alpha)^2 + \alpha^2(1-\alpha)^4)) + \sum_{i=8}^{\infty} iP(X_{game_A} = i) \end{aligned} \tag{57}$$

we can apply (56) to the last term and get

$$\begin{aligned} \sum_{i=8}^{\infty} iP(X_{game_A} = i) &= \sum_{i=4}^{\infty}(2i)\frac{5}{2}(\alpha^2 + (1-\alpha)^2)(2\alpha(1-\alpha))^{\frac{1}{2}(2i)-1} \\ &= 5(\alpha^2 + (1-\alpha)^2)\sum_{i=4}^{\infty} i(2\alpha(1-\alpha))^{i-1} \\ &= 5(\alpha^2 + (1-\alpha)^2)\Big(\frac{1}{(1-2\alpha(1-\alpha))^2} \\ &\qquad - \sum_{i=1}^{3} i(2\alpha(1-\alpha))^{i-1}\Big) \\ &= \frac{160\alpha^3(1-\alpha)^3 - 240\alpha^4(1-\alpha)^4}{1-2\alpha(1-\alpha)}. \end{aligned} \tag{58}$$

This leads to, after some simplification,

$$\begin{aligned} E_{game}(X) &= \sum_{i=4}^{\infty} iP(X_{game} = i) \\ &= \frac{4(-6\alpha^6 + 18\alpha^5 - 18\alpha^4 + 6\alpha^3 + \alpha^2 - \alpha + 1)}{1-2\alpha(1-\alpha)}. \end{aligned} \tag{59}$$

Although theoretically a game can go on indefinitely, if no player wins consecutive points, this expected number of points per game is finite and attains a maximum at $\alpha = 0.5$, in which case $E_{game}(p) = 6.75$.

5.2 Number of points in a tiebreak

The probability of a tiebreak ending after exactly x points, denoted $P_\tau(X = x)$, can be computed rather easily for $x \leq 12$ using the $T(7 - j)$ found in the previous section and the $T(i - 7)$ given in Appendix B.1.2. Tiebreaks with more than 12 points can only end after an even number of points, because the winner has to achieve a 2-point difference. Suppose a tiebreak has ended after en even number of points x, with $x \geq 14$. In this case, first a 6-6 score has to be reached. This happens with probability $T(6 - 6) = \sum_{k=0}^{6} \binom{6}{k}^2 \alpha^{6-k}\beta^{6-k}(1 - \alpha)^k(1 - \beta)^k$. Then all subsequent tied scores up to $[\frac{1}{2}x - 1]$-$[\frac{1}{2}x - 1]$ must have been reached too. This happens with probability $(\alpha\beta + (1-\alpha)(1-\beta))^{\frac{1}{2}x-7}$. Finally, one of the players has to win two consecutive points, happening with probability $\alpha(1-\beta)+\beta(1-\alpha)$. Combining this three probabilities we get, for even $x \geq 14$:

$$P_\tau(X = x) = T(6 - 6)(\alpha(1 - \beta) + \beta(1 - \alpha)) \times (\alpha\beta + (1 - \alpha)(1 - \beta))^{\frac{1}{2}x-7} \tag{60}$$

and for odd $x \geq 13$, $P_\tau(X = x) = 0$. Now the expected number of points in a tiebreak is given by

$$E_\tau(X) = \sum_{x=7}^{\infty} xP_\tau(X = x). \tag{61}$$

Note that this gives rise to a similar infite series as in the previous section, and the argument using (56) tells us that the $E_\tau(X)$ is also finite.

5.3 Number of points per set

We will first find the expected number of games per set, where the tiebreak does not count as a game. There are two types of sets: a set ending in a tiebreak when the score is 6-6 and a set that does not allow for a tiebreak and ends when one of the player has won at least 6 games, with a 2-game difference. Now let $E_{set,\tau}(G)$ be the expected number of games in a set of the former type:

$$E_{set,\tau}(G) = \sum_{j=0}^{4}(6 + j)Q(6 - j) + \sum_{i=0}^{4}(6 + i)Q(i - 6) + 12Q(5 - 5) \tag{62}$$

where $Q(6 - j)$ is as in section 4.3 and $Q(i - 6)$ is found in a similar way (given in Appendix B.1.1). And thus

$$E_{set,\tau}(X) = E_{set,\tau}(G)E_{game}(X) \tag{63}$$

gives the expected number of points played for this set type before a tiebreak.

Consider the other set type: let $E_{set}(X)$ be the expected number of points played in a set that does not allow for a tiebreak. The first two terms are the same as in (62). The third term changes, because this type of set can contain more than twelve games. Now calculate $P(G = i)$ for even $i \geq 12$, the probability that the set ends after exactly i games. First the score 5-5 has to be reached, which happens with probability $Q(5-5)$. Then all subsequent tied scores up to $[\frac{1}{2}i - 1]$-$[\frac{1}{2}i - 1]$ have to be reached too, happening with probability $(G_A(\bar{g})^2 + (1 - G_A(\bar{g}))^2)^{\frac{1}{2}i-6}$. After that, one of the players has to win two consecutive games, happening with probability $2G_A(\bar{g})(1 - G_A(\bar{g}))$. So for $i \geq 12$,

$$P(G = i) = Q(5-5)2(G_A(\bar{g})(1-G_A(\bar{g}))(G_A(\bar{g})^2 + (1-G_A(\bar{g}))^2)^{\frac{1}{2}i-6}. \quad (64)$$

This gives

$$\begin{aligned} E_{set}(X) &= \sum_{j=0}^{4}(6+j)Q(6-j) + \sum_{i=0}^{4}(6+i)Q(i-6) \\ &\quad + \sum_{i=12,\text{ even}}^{\infty} iP(G_{set} = i) \\ &= \sum_{j=0}^{4}(6+j)Q(6-j) + \sum_{i=0}^{4}(6+i)Q(i-6) + Q(5-5) \\ &\quad \times 2G_A(\bar{g})(1 - G_A(\bar{g}))\sum_{i=6}^{\infty} 2i(G_A(\bar{g})^2 + (1 - G_A(\bar{g}))^2)^{i-6}. \end{aligned} \quad (65)$$

This last infinite series is finite by the same argument as used in section 5.1. Hence for this type of set the expected number of points played is given by

$$E_{set}(X) = E_{set}(G)E_{game}(X). \quad (66)$$

5.4 Number of tiebreaks in a match

The expected number of tiebreaks $E_{match}(\tau)$ depends on the type of the match. For example, for a match of type 2, $E_{match}(\tau)$ is found as follows. Only 0, 1 or 2 tiebreaks can occur in such a match. Let $P(Q = q)$ denote the probability that such a match lasts q sets. First, calculate the probabilities of a match lasting two and three sets:

$$
\begin{aligned}
P(Q=2) &= Q(\bar{s})^2 + Q(\underline{s})^2 = 2Q(\bar{s})^2 \\
P(Q=3) &= \binom{2}{1} Q(\bar{s})Q(\underline{s}) = 2Q(\bar{s})^2
\end{aligned}
\tag{67}
$$

since we assumed $\alpha = \beta$ and thus $Q(\bar{s}) = Q(\underline{s})$. Now, find the expected number of tiebreaks by

$$
\begin{aligned}
E_{match}(\tau) = 1 \times \Big(&P(Q=2)\binom{2}{1}Q(6-6)(1-Q(6-6)) \\
&+ P(Q=3)\binom{2}{1}Q(6-6)(1-Q(6-6))\Big) \\
&+ 2 \times (P(Q=2)Q(6-6)^2 + P(Q=3)Q(6-6)^2).
\end{aligned}
\tag{68}
$$

The $E_{match}(\tau)$'s for all match types can be found in similar ways and are given in Appendix B.2.2.

5.5 Number of points in a match

First, compute the expected number of sets per match $E_{match}(Q)$. This of course depends on the type of match and the probability that a match ends after q sets: $P(Q=q)$. The expected number of sets are then found by $E_{match}(Q) = \sum_{q=2}^{3} qP(Q=q)$ for matches of type 1 and 2. For matches of type 3 and 4, $E_{match}(Q) = \sum_{q=3}^{5} qP(Q=q)$. See Appendix.

Finally, we can compute the expected number of points played in a match of type 1 and 2 with

$$
E_{match}(X) = E_{match}(Q)E_{set,\tau}(X) + E_{match}(\tau)E_{\tau}(X). \tag{69}
$$

For type 3 and type 4 matches, we have

$$
\begin{aligned}
E_{match}(X) = &E_{match}(Q)E_{set,\tau}(X) + E_{match}(\tau)E_{\tau}(X) \\
&+ P(Q=5)Q(6-6) \sum_{i=12,\text{ even}}^{\infty} iP(G_{set}=i).
\end{aligned}
\tag{70}
$$

Where care has to be taken in (69) and (70) to take the $E_{match}(Q)$ and $E_{match}(\tau)$ corresponding to the right match type.

For all types of matches, $E_{match}(X)$ is plotted in Figure 4. α is taken from 0.5 to 0.8, since almost all professional tennis players have a serving quality in this interval. Also plotted is the number of points Isner and Mahut played in their match at Wimbledon in 2010. This was a match of type 4. The plot for matches of type 4 intersects the line for 980 points at $\alpha = 0.762$. This value can be taken as an estimate for both players' serving strength.

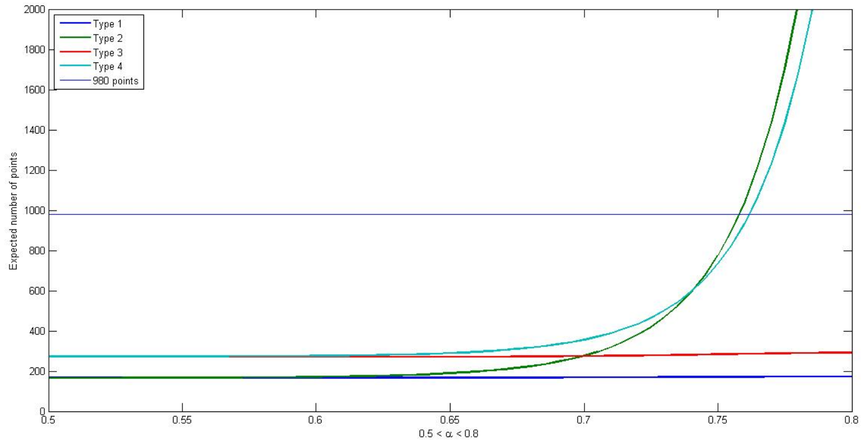

Figure 4: Expected number of points for all match types

6 Discussion

The actual statistics for the Isner-Mahut match are $\alpha_{\text{Isner}} = 0.762$ for Isner and $\beta_{\text{Mahut}} = 0.787$ for Mahut. As calculated in Section 4, this gave Isner a probability to win the match of 0.356. In general, any difference in qualities between two tennis players will lead to a greater difference between the players' probabilities to win the match. This is due to the fact that against a stronger player it is still possible to occasionally win a point, however to win a match it is important to win certain sequences of points when the other player is serving, which happens with a smaller probability. This did not stop Isner from winning the match against Mahut, which goes to show that tennis (and most other sports) has many factors influencing the outcome of the match, keeping the sport unpredictable.

The actual values for α_{Isner} and β_{Mahut} are close to our estimate of $\alpha = \beta = 0.762$. This suggests that the expected number of points might give a good estimate for the serving qualities of players with equal, above average qualities in a tennis match. Since the Isner-Mahut match is the only professional tennis match known with this number of points played, there are no other data to test this hypothesis against. The hypothesis was also tested for shorter matches (less than 500 points), here the expected number of points did not give a good estimate of the players' qualities. This is due to the fact that the graphs in Figure 4 are nearly horizontal or have a relatively small slope at smaller values. This means that at smaller values, any deviation in the expected number of points here leads to a greater difference in the estimate for the qualities of both players than it would at larger values.

On the other hand, the plots of the expected number of points per match suggest that matches of type 2 and 4, between players of equal strength playing their equilibrium strategies, have the greatest probability of becoming a record-breaking match in terms of length. This probability increases sharply as both players achieve a higher than average winning percentage (65%, as found in the example in Section 4.4) on their own service. The reason for this is simply that to end the match, one player has to break the other player's service game. The probability of this happening gets increasingly smaller when both players' qualities increase.

For matches of type 1 and 3 between players of equal strength playing their equilibrium strategies, it is found that the expected number of points played in a match does not vary much in the interval of reasonable serving strengths. Games are expected to contain the most points when $\alpha = \beta = 0.5$, but the expected number of games in these matches reaches a minimum at these values. When $\alpha = \beta$ increases, the expected number of points per game decreases but the expected number of games per match increases. They do so in such a way that this increase and decrease more or less compensate each other. Of course, in the limit where $\alpha = \beta$ goes to 1, the expected number of points for matches of type 1 and 3 will go to infinity as in the case of type 2 and 4 matches. Because neither player will manage to win a point on his opponent's service, needed to end the final tiebreak, the match will go on indefinitely. This however happens outside the region of realistic player strengths and thus it is not likely for such a match to reach a record-breaking length.

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Appendix A Symbols

S	Set of states
s, t	States
ω_i	Absorbing state in which player i has won the match
$\bar{g}, \underline{g}$	States in which player A or player B has won the current game
G_s	Normal form game associated with state s
$A(s), B(s)$	Action set for players in state s
$\pi_{s,i}(a, b)$	Probability player i to win point in state s given actions a and b
p_i, q_i	Probabilities in the normal form games played in a point game
λ, μ	Probabilities of playing L in point game
$W_i(s)$	Probability of player i to win the match if the current state is s
$v_{ss'}(a, b)$	Probability of moving from state s to s' given actions a and b
ρ, ϕ	Strategies for player A and B assigning probabilities to playing L and R
$\bar{v}_{ss'}(a, \phi)$	Probability of moving from state s to s' given action a and strategy ϕ
(ρ'_{S_i}, ρ)	Strategy of playing ρ' in all $s \in S_i$ and ρ otherwise
$\Delta A(s)$	Set of probability distributions over $A(s)$
α, β	Probabilities of winning a point for both players in their service game
$G_k(i - j)$	Probability of the score being $i - j$ at some point in the service game of player k
$Q(i - j)$	Probability of the score being $i - j$ at some point in the set
$G_i(\bar{g})$	Probability that player A wins player i's service game
$P(\tau)$	Probability of a tiebreak occurring in a set
$T(i - j)$	Probability of the score being $i - j$ at some point in the tiebreak
$T(\bar{t})$	Probability that player A wins a tiebreak
$Q(\bar{s})$	Probability that player A wins a set
$P_{\text{type } i}(\omega_j)$	Probability that player j wins a match of type i
$P(X_{game} = i)$	Probability of a game ending after i points
$P(Q = q)$	Probability of a match ending after q sets
$E_{game}(X)$	Expected number of points in a game
$E_{set}(X), E_{set,\tau}(X)$	Expected number of points in a set, for both set types

$E_{set}(G)$	Expected number of games in a set
$P_\tau(X = x)$	Probability of a tiebreak ending after x points
$E_\tau(X)$	Expected number of points in a tiebreak
$E_{match}(\tau)$	Expected number of tiebreaks in a match
$P(G = i)$	Probability that a set ends after i games
$E_{match}(\tau)$	Expected number of tiebreaks in a match
$E_{match}(Q)$	Expected number of sets in a match
$E_{match}(X)$	Expected number of points in a match

Appendix B Additional Calculations

B.1 Probabilities

B.1.1 Set

$$Q(i-6) = \begin{cases} \sum_{k=0}^{i} \Big[& \binom{\frac{6+i}{2}}{k}\binom{\frac{6+i}{2}}{\frac{6-i}{2}+k} G_B(\underline{g})^{\frac{6+i}{2}-k} G_A(\bar{g})^{i-k} \\ & \times(1-G_B(\underline{g})^k (1-G_A(\bar{g}))^{\frac{4-i}{2}+k}\Big] \text{ if } i=0,2,4. \\ \sum_{k=0}^{i} \Big[& \binom{\frac{5+i}{2}}{k}\binom{\frac{5+i}{2}}{\frac{5-i}{2}+k} G_B(\underline{g})^{\frac{5+i}{2}-k} G_A(\bar{g})^{i-k} \\ & \times(1-G_B(\underline{g})^k (1-G_A(\bar{g}))^{\frac{7-i}{2}+k}\Big] \text{ if } i=1,3. \end{cases} \tag{71}$$

B.1.2 Tiebreak

$$P_\tau(i-7) = \begin{cases} \sum_{k=0}^{i} \binom{\frac{6+i}{2}}{k}\binom{\frac{6+i}{2}}{\frac{6-i}{2}+k} \beta^{\frac{8+i}{2}-k}\alpha^{i-k}(1-\beta)^k(1-\alpha)^{\frac{6-i}{2}+k} & \text{if } i=0,4, \\ \sum_{k=0}^{i} \binom{\frac{7+i}{2}}{k}\binom{\frac{5+i}{2}}{\frac{5-i}{2}+k} \beta^{\frac{7+i}{2}-k}\alpha^{i-k}(1-\beta)^k(1-\alpha)^{\frac{7-i}{2}+k} & \text{if } i=1,5, \\ \sum_{k=0}^{i} \binom{\frac{6+i}{2}}{k}\binom{\frac{6+i}{2}}{\frac{6-i}{2}+k} \beta^{\frac{6+i}{2}-k}\alpha^{i-k}(1-\beta)^k(1-\alpha)^{\frac{8-i}{2}+k} & \text{if } i=2, \\ \sum_{k=0}^{i} \binom{\frac{5+i}{2}}{k}\binom{\frac{7+i}{2}}{\frac{7-i}{2}+k} \beta^{\frac{7+i}{2}-k}\alpha^{i-k}(1-\beta)^k(1-\alpha)^{\frac{7-i}{2}+k} & \text{if } i=3. \end{cases} \tag{72}$$

$$P_\tau(X=x) = \begin{cases} P_\tau(7-0)+P_\tau(0-7) \text{ if } x=7, \\ P_\tau(7-1)+P_\tau(1-7) \text{ if } x=8, \\ P_\tau(7-2)+P_\tau(2-7) \text{ if } x=9, \\ P_\tau(7-3)+P_\tau(3-7) \text{ if } x=10, \\ P_\tau(7-4)+P_\tau(4-7) \text{ if } x=11, \\ P_\tau(7-5)+P_\tau(5-7) \text{ if } x=12. \end{cases} \tag{73}$$

B.2 Expected values

B.2.1 Number of sets per match

Type 1 and 2:

$$E_{match}(Y) = 2(Q(\bar{s})^2 + Q(\underline{s})^2) \\ + 3(\binom{2}{1} Q(\bar{s})Q(\underline{s})) \tag{74}$$

Type 3 and 4:

$$E_{match}(Y) = 3(Q(\bar{s})^3 + Q(\underline{s})^3) \\ + 4(\binom{3}{1}(Q(\bar{s})^3 Q(\underline{s}) + Q(\underline{s})^3 Q(\bar{s}))) \\ + 5(\binom{4}{2} Q(\bar{s})^2 Q(\underline{s})^2). \tag{75}$$

B.2.2 Number of tiebreaks per match

Type 1:

$$E_{match}(\tau) = 2\binom{2}{1} Q(\bar{s})^2 Q(6-6)(1 - Q(6-6)) \\ + 2\binom{2}{1}\binom{3}{1} Q(\bar{s})^2 Q(6-6)(1 - Q(6-6))^2 \\ + 2\Big(2Q(\bar{s})^2 Q(6-6)^2 + \binom{3}{1} Q(\bar{s})^2 Q(6-6)^2 (1 - Q(6-6))\Big) \\ + 3\Big(2Q(\bar{s})^2 Q(6-6)^3\Big) \tag{76}$$

Type 2:

$$E_{match}(\tau) = 2\binom{2}{1} Q(\bar{s})^2 Q(6-6)(1 - Q(6-6)) \\ + 2\binom{2}{1} Q(\bar{s})^2 Q(6-6)(1 - Q(6-6)) \\ + 2\Big(2Q(\bar{s})^2 Q(6-6)^2 + \binom{2}{1} Q(\bar{s})^2 Q(6-6)^2\Big) \tag{77}$$

Type 3:

$$
\begin{aligned}
E_{match}(\tau) = {} & 2\binom{3}{1}Q(\bar{s})^3Q(6-6)(1-Q(6-6))^2 \\
& + 2\binom{3}{1}\binom{4}{1}Q(\bar{s})^4Q(6-6)(1-Q(6-6))^3 \\
& + \binom{4}{2}\binom{5}{1}Q(\bar{s})^4Q(6-6)(1-Q(6-6))^4 \\
& + 2\Big(2\binom{3}{2}Q(\bar{s})^3Q(6-6)^2(1-Q(6-6)) \\
& + 2\binom{3}{1}\binom{4}{2}Q(\bar{s})^4Q(6-6)^2(1-Q(6-6))^2 \\
& + \binom{4}{2}\binom{5}{2}Q(\bar{s})^4Q(6-6)^2(1-Q(6-6))^3\Big) \\
& + 3\Big(2Q(\bar{s})^3Q(6-6) + 2\binom{3}{1}\binom{4}{1}Q(\bar{s})^4Q(6-6)^3(1-Q(6-6)) \\
& + \binom{4}{2}\binom{5}{3}Q(\bar{s})^4Q(6-6)^3(1-Q(6-6))^2\Big) \\
& + 4\Big(2\binom{3}{1}Q(\bar{s})^4Q(6-6)^4 + \binom{4}{2}\binom{5}{4}Q(\bar{s})^4Q(6-6)^4(1-Q(6-6))\Big) \\
& + 5(\binom{4}{2}Q(\bar{s})^4Q(6-6)^5) \qquad (78)
\end{aligned}
$$

Type 4:

$$\begin{aligned}
E_{match}(\tau) = {} & 2\binom{3}{1}Q(\bar{s})^3Q(6-6)(1-Q(6-6))^2 \\
& + 2\binom{3}{1}\binom{4}{1}Q(\bar{s})^4Q(6-6)(1-Q(6-6))^3 \\
& + \binom{4}{2}\binom{4}{1}Q(\bar{s})^4Q(6-6)(1-Q(6-6))^3 \\
& + 2\Big(2\binom{3}{2}Q(\bar{s})^3Q(6-6)^2(1-Q(6-6)) \\
& + 2\binom{3}{1}\binom{4}{2}Q(\bar{s})^4Q(6-6)^2(1-Q(6-6))^2 \\
& + \binom{4}{2}\binom{4}{2}Q(\bar{s})^4Q(6-6)^2(1-Q(6-6))^2\Big) \\
& + 3\Big(2Q(\bar{s})^3Q(6-6) + 2\binom{3}{1}\binom{4}{1}Q(\bar{s})^4Q(6-6)^3(1-Q(6-6)) \\
& + \binom{4}{2}\binom{4}{3}Q(\bar{s})^4Q(6-6)^3(1-Q(6-6))\Big) \\
& + 4\Big(2\binom{3}{1}Q(\bar{s})^4Q(6-6)^4 + \binom{4}{2}Q(\bar{s})^4Q(6-6)^4\Big) \qquad (79)
\end{aligned}$$

Appendix C MATLAB codes

C.1 Probability of winning the match

```
function probabilities = probabilities(a,b)

% Input: a and b are the probabilities to win a point on ...
    service, for players A
% and B respectively.

% Output: a vector of length 4. The vector contains the ...
    probabilities for
% player A to win the match. The first two entries are for ...
    matches of the
% first-to-2 sets type. The last two entries are for matches ...
    of the
% first-to-3 sets type. The second and fourth entry are for ...
    matches that do
% not allow for a tiebreak in the deciding set.

% calculate the probabilities to win a service game
PgameA = (a^4*(15-34*a+28*a^2-8*a^3))/(1-2*a*(1-a));
PgameB = (b^4*(15-34*b+28*b^2-8*b^3))/(1-2*b*(1-b));

% calculate the probabilities to win a set before a tiebreak
Pset5odd = 0;
syms k
for j = [1 3]
    Pset5odd = Pset5odd + symsum(nchoosek((5+j)/2,k)*nchoosek...
        ((5+j)/2,(5-j)/2+k)*PgameA^((5+j)/2-k)*PgameB^(j-k)...
        *(1-PgameA)^(k)*(1-PgameB)^((5-j)/2+k),k,0,j);
end

Pset5even = 0;
for j = [0 2 4]
    Pset5even = Pset5even + symsum(nchoosek((6+j)/2,k)*...
        nchoosek((4+j)/2,(4-j)/2+k)*PgameA^((6+j)/2-k)*PgameB...
        ^(j-k)*(1-PgameA)^(k)*(1-PgameB)^((4-j)/2+k),k,0,j);
end

Pset6j = (1-PgameB)*Pset5even+PgameA*Pset5odd;
Pset55 = symsum(nchoosek(5,k)^2*PgameA^(5-k)*PgameB^(5-k)*(1-...
    PgameA)^(k)*(1-PgameB)^(k),k,0,5);
Pset75 = PgameA*(1-PgameB)*Pset55;

%calculate the probability for a tiebreak
Pset66 = (PgameA*PgameB+(1-PgameA)*(1-PgameB))*Pset55;
```

```
%calculate the probability to win a tiebreak
Ptie70 = a^(3)*(1-b)^(4);
Ptie71 = symsum(nchoosek(3,k)*nchoosek(4,3+k)*a^(4-k)*b^(1-k)...
    *(1-a)^(k)*(1-b)^(3+k),k,0,1);
Ptie72 = symsum(nchoosek(4,k)*nchoosek(4,2+k)*a^(5-k)*b^(2-k)...
    *(1-a)^(k)*(1-b)^(2+k),k,0,2);
Ptie73 = symsum(nchoosek(5,k)*nchoosek(4,1+k)*a^(5-k)*b^(3-k)...
    *(1-a)^(k)*(1-b)^(2+k),k,0,3);
Ptie74 = symsum(nchoosek(5,k)*nchoosek(5,1+k)*a^(5-k)*b^(4-k)...
    *(1-a)^(k)*(1-b)^(2+k),k,0,4);
Ptie75 = symsum(nchoosek(5,k)*nchoosek(6,1+k)*a^(6-k)*b^(5-k)...
    *(1-a)^(k)*(1-b)^(1+k),k,0,5);
Ptie66 = symsum(nchoosek(6,k)^(2)*a^(6-k)*b^(6-k)*(1-a)^(k)...
    *(1-b)^(k),k,0,6);

x = symsum((a*b+(1-a)*(1-b))^(k-6),k,6,Inf);

PtieA = Ptie70+Ptie71+Ptie72+Ptie73+Ptie74+Ptie75+Ptie66*a...
    *(1-b)*x;

% calculate the probability to win a set, including a ...
    tiebreak
PsetA = Pset6j + Pset75 + Pset66*PtieA;
% calculate the probability to win a set that does not allow ...
    for a tiebreak
PsetAalt = Pset6j + Pset75 + Pset66*(PgameA*(1-PgameB))*...
    symsum((PgameA*PgameB+(1-PgameA)*(1-PgameB))^k,k,0,Inf);
PsetB = 1 - PsetA;

% calculate the probability to win a match, for all 4 match ...
    types
PmatchA = double([PsetA^2 + 2*PsetA^2*PsetB; PsetA^2 + 2*...
    PsetA*PsetB*PsetAalt;
                    PsetA^3 + 3*PsetA^3*PsetB + 6*PsetA^3*PsetB...
                        ^2;
                    PsetA^3 + 3*PsetA^3*PsetB + 6*PsetA^2*PsetB...
                        ^2*PsetAalt])
```

C.2 Expected number of points in a match

```
function [expectations] = expectations(a)

% Input: a is the probability to win a point on service, for ...
    players A
% and B both.
```

```
% Output: a vector of length 4. The vector contains the ...
    expected number of points played in a match.
% The first two entries are for matches of the
% first-to-2 sets type. The last two entries are for matches ...
    of the
% first-to-3 sets type. The second and fourth entry are for ...
    matches that do
% not allow for a tiebreak in the deciding set.

% Calculate the expected number of points in a game
ExpPperGame = 4*(-6*a^6+18*a^5-18*a^4+6*a^3+a^2-a+1)/(2*a...
    ^2-2*a+1);

% Calculate the expected number of games per set
PgameA = (a^4*(15-34*a+28*a^2-8*a^3))/(1-2*a*(1-a));
PgameB = PgameA;
ExpGamesPerSet = 0;
syms k
for j = [0 2 4]
    ExpGamesPerSet = ExpGamesPerSet + (6+j)*(1-PgameB)*symsum...
        (nchoosek((6+j)/2,k)*nchoosek((4+j)/2,(4-j)/2+k)*...
        PgameA^((6+j)/2-k)*PgameB^(j-k)*(1-PgameA)^(k)*(1-...
        PgameB)^((4-j)/2+k),k,0,j);
end

for j = [1 3]
    ExpGamesPerSet = ExpGamesPerSet + (6+j)*PgameA*symsum(...
        nchoosek((5+j)/2,k)*nchoosek((5+j)/2,(5-j)/2+k)*PgameA...
        ^((5+j)/2-k)*PgameB^(j-k)*(1-PgameA)^(k)*(1-PgameB)...
        ^((5-j)/2+k),k,0,j);
end

for j = [0 2 4]
    ExpGamesPerSet = ExpGamesPerSet + (6+j)*symsum(nchoosek...
        ((4+j)/2,k)*nchoosek((6+j)/2,(6-j)/2+k)*PgameA^((6+j)...
        /2-k)*PgameB^(j-k)*(1-PgameA)^(k)*(1-PgameB)^((6-j)/2+...
        k),k,0,j);
end

for j = [1 3]
    ExpGamesPerSet = ExpGamesPerSet + (6+j)*symsum(nchoosek...
        ((5+j)/2,k)*nchoosek((5+j)/2,(5-j)/2+k)*PgameA^((5+j)...
        /2-k)*PgameB^(j-k)*(1-PgameA)^(k)*(1-PgameB)^((7-j)/2+...
        k),k,0,j);
end

Pset55 = symsum(nchoosek(5,k)^2*PgameA^(5-k)*PgameB^(5-k)*(1-...
    PgameA)^(k)*(1-PgameB)^(k),k,0,5);
Pset66 = Pset55*(PgameA*PgameB+(1-PgameA)*(1-PgameB));
```

```
ExpGamesPerSet = ExpGamesPerSet + 12*Pset55;

% Calculate the expected number of points in a set before a ...
    tiebreak
ExpPperSet = ExpGamesPerSet*ExpPperGame;

% Calculate the expected number of sets per match for every ...
    match type
Pset = 0.5;
ExpSetsPerMatch = [2*(2*Pset^2)+3*(2*Pset^2);2*(2*Pset^2)...
    +3*(2*Pset^2);3*(2*Pset^3)+4*(6*Pset^4)+5*(6*Pset^4);3*(2*...
    Pset^3)+4*(6*Pset^4)+5*(6*Pset^4)];

% Calculate the probabilities of a tiebreak ending after ...
    7,8,9,10,11 or 12
% points
Ptie7 = a^3*(1-a)^4 + (1-a)^3*a^4;
Ptie8 = 7*a^5*(1-a)^3 + 7*a^3*(1-a)^5;
Ptie9 = 6*a^7*(1-a)^2 + 16*a^5*(1-a)^4 + 6*a^3*(1-a)^6 + 6*a...
    ^6*(1-a)^3 + 16*a^4*(1-a)^5 + 6*a^2*(1-a)^7;
Ptie10 = 14*a^8*(1-a)^2 + 70*a^6*(1-a)^4 + 70*a^4*(1-a)^6 + ...
    14*a^2*(1-a)^8;
Ptie11 = 5*a^9*(1-a)^2 + 50*a^7*(1-a)^4 + 100*a^5*(1-a)^6 + ...
    50*a^3*(1-a)^8 + 5*a*(1-a)^10 + 5*a^10*(1-a) + 50*a^8*(1-a...
    )^3 + 100*a^6*(1-a)^5 + 50*a^4*(1-a)^7 + 5*a^2*(1-a)^9;
Ptie12 = 7*a^11*(1-a) + 105*a^9*(1-a)^3 + 350*a^7*(1-a)^5 + ...
    350*a^5*(1-a)^7 + 105*a^3*(1-a)^9 + 7*a*(1-a)^11;

Ptie66 = symsum(nchoosek(6,k)^(2)*a^(6-k)*a^(6-k)*(1-a)^(k)...
    *(1-a)^(k),k,0,6);

ExpTie = symsum(2*k*(a^2+(1-a)^2)^(k-7),k,7,Inf);

% Calculate the expected number of points in a tiebreak
ExpPperTie = 7*Ptie7+8*Ptie8+9*Ptie9+10*Ptie10+11*Ptie11+12*...
    Ptie12+Ptie66*2*(a*(1-a))*ExpTie;

% Calculate the expected number of tiebreaks for every match ...
    type
ExpTiePerMatch = [0;0;0;0];
ExpTiePerMatch(1) = (2*Pset^2*2*Pset66*(1-Pset66)+2*2*Pset...
    ^2*3*Pset66*(1-Pset66)^2)+2*2*Pset^2*Pset66^2*(1+3*(1-...
    Pset66)) + 3*2*Pset^2*Pset66^3;
ExpTiePerMatch(2) = (2*Pset^2*2*Pset66*(1-Pset66)+2*Pset^2*2*...
    Pset66*(1-Pset66))+2*2*Pset^2*Pset66^2*2;
ExpTiePerMatch(3) = (2*Pset^3*3*(1-Pset66)^2*Pset66+6*Pset...
    ^4*4*(1-Pset66)^3*Pset66+6*Pset^4*5*(1-Pset66)^4*Pset66)...
    +2*(2*Pset^3*3*(1-Pset66)*Pset66^2+6*Pset^4*6*(1-Pset66)...
```

```
    ^2*Pset66^2+6*Pset^4*10*(1-Pset66)^3*Pset66^2)+3*(2*Pset...
    ^3*Pset66^3+6*Pset^4*4*(1-Pset66)*Pset66^3+6*Pset^4*10*(1-...
    Pset66)^2*Pset66^3)+4*(6*Pset^4*Pset66^4+6*Pset^4*10*...
    Pset66^4*(1-Pset66))+5*6*Pset^4*Pset66^5;
ExpTiePerMatch(4) = (2*Pset^3*3*(1-Pset66)^2*Pset66+6*Pset...
    ^4*4*(1-Pset66)^3*Pset66+6*Pset^4*4*(1-Pset66)^3*Pset66)...
    +2*(2*Pset^3*3*(1-Pset66)*Pset66^2+6*Pset^4*6*(1-Pset66)...
    ^2*Pset66^2+6*Pset^4*6*(1-Pset66)^2*Pset66^2)+3*(2*Pset^3*...
    Pset66^3+6*Pset^4*4*(1-Pset66)*Pset66^3+6*Pset^4*4*(1-...
    Pset66)*Pset66^3)+4*(6*Pset^4*Pset66^4+6*Pset^4*Pset66^4);

% Calculate the expected number of points in a match
ExpPointsPerMatch = ExpSetsPerMatch*ExpPperSet + ExpPperTie*...
    ExpTiePerMatch;

% Calculate the extra expected number of points for match ...
    types 2 and 4
ExpGamesAfter66 = (PgameA^2+PgameB^2)*symsum((2*k+2)*(PgameA*...
    PgameB+(1-PgameA)*(1-PgameB))^k,k,0,Inf);
ExpPointsPerMatch(2) = ExpPointsPerMatch(2) + 2*Pset^2*Pset66...
    *ExpGamesAfter66*ExpPperGame;
ExpPointsPerMatch(4) = ExpPointsPerMatch(4) + 6*Pset^4*Pset66...
    *ExpGamesAfter66*ExpPperGame;

% Gives the vector containing the expected number of points ...
    for every match
% type
[expectations] = double(ExpPointsPerMatch);
```