university of groningen
faculty of mathematics and natural sciences

The output regulation problem for linear multi-agent systems

Corina van der Lei
s2161079

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Supervisor: Prof. Dr. H.L. Trentelman
Second supervisor: Dr. A.E. Sterk
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Abstract

In this thesis we consider the output regulation problem for linear multi-agent systems, with and without uncertainty. We assume that the reference signals and disturbances are generated as outputs of some linear time-invariant autonomous system. The uncertainty of the agents appears in two different ways, namely as additive perturbation and multiplicative perturbation. For both, with and without uncertainty a dynamic state feedback and dynamic output feedback protocol is built and necessary and sufficient condition for the existence are given.

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Chapter 1

Introduction

Motivated by the appealing and fruitful research of distributed dynamical systems, this thesis is a study of the output regulation problem for multi-agent systems. Problems about consensus and formation of multi-agent systems have been studied in many publications in the recent years. In the basic problem, the systems are linear and all information is exactly known. In particular, not all information is measurable or available in communication, hence we not only consider state feedback but also output feedback. So, we will study linear multi-agent systems with uncertainty. In that case not all the information about the system is exactly known, but we accept a bounded uncertainty. We will start this thesis with the output regulation problem for linear multi-agent systems without uncertainty. Later we will move on to linear multi-agent systems with uncertainty.

The system consists of two types of subsystems. The first is the exosystem and the second are N agents. In this problem not all of the N agents can access the exogenous signal. Furthermore the subsystems are interconnected by a directed graph. Thus, we will study the output regulation problem for a directed dynamical network of interconnected linear systems with a bounded uncertainty. The questions that came up is, is the output regulation problem solvable for linear multi-agent systems with a bounded uncertainty? If so, what is the limit of the uncertainty? And how can we build a controller such that the controlled system is output regulated?

1.1 Multi-agent systems

A multi-agent system is a system composed of multiple interacting intelligent agents within a specific environment. Multi-agent systems are used to solve problems that are (too) difficult or impossible for an individual agent or single system to solve. A intelligent agents is an agent which observes through sensors and acts upon an environment, using actuators, and directs its activities towards achieving goals. Topics where multi-agent systems research may deliver a good approach include fire-control systems, disaster response, modelling social structures, manned/unmanned flights and guidance systems.

The topic of multi-agents systems has drawn a lot of attention because many benefits can be obtained by replacing the very complex, single systems by a (large) group of small, single systems. The question that came up was 'how can we control these multi-agent systems?'. The first idea was to built a powerful central controller that is available to control the entire group of subsystems. However, this is only an extension of the traditional method for complex, single systems and is it against the idea of using small, simple agents. Besides, the control commands have to be send to all other agents from the central controller, which requires a lot of communication channels. Driven by studying schools and swarms in nature, the idea arises to look at local communication/interconnections. This local communication between animals makes sure that the school or swarm stays together. Therefore, local communication seems a powerful mean to control the entire system. In such a cooperative and distributed environment, the multi-agent systems can be built. In addition to reducing the complexity of the large, single systems, distributed control of multi-agent systems can bring more advantages, namely flexibility, robustness an scalability. This depends of course on the design of the distributed controller, but are interesting benefits.

Nowadays, many distributed coordination problems in multi-agent systems are studied, such as consensus/synchronization, formation control, distributed optimization, distributed estimation and intelligent coordination. The problem we will study in this thesis is the output regulation problem. [4] [9]

1.2 Output regulation problem

The output regulation problem deals with asymptotic tracking of reference signals and/or asymptotic rejection of undesired disturbance in the output of a dynamical system. The main difference between the output regulation problem and the conventional tracking (and disturbance rejection) problem is that in the output regulation problem the reference signals (and disturbances) are not completely unknown, but are elements of some function class. In this thesis we assume that these reference signals (and disturbances) are generated as outputs of some linear time-invariant autonomous system. This system is called the exosystem. One can incorporate the equations of the exosystem into the equations of the control system, then the requirement is that the output of the new, aggregated system converges to zero, regardless of the initial state. [2] [5]

1.3 Mathematical problem formulation

1.3.1 The model

In this problem we consider a multi-agent system, which consists of two types of subsystems: N agents and an exosystem.

We consider a multi-agent system of N agents with an underlying directed graph $\mathcal{G}$ (more about this can be found in section 2.1). This directed graph tells us how the agents are interconnected. The dynamic of agent i is given by

$$\begin{cases} \dot{x}_i &= Ax_i + Bu_i \\ z_i &= Hx_i, \end{cases} \tag{1.1}$$

where $x_i \in \mathbb{R}^n$ is the state, $z_i \in \mathbb{R}^q$ is the output and $u_i \in \mathbb{R}^m$ the control input of agent i.

The exosystem is described by the following dynamics

$$\begin{cases} \dot{w} = Sw \\ z = Rw, \end{cases} \tag{1.2}$$

where $z \in \mathbb{R}^q$ is the output and $w \in \mathbb{R}^t$ the state of the exosystem. We assume that matrix S has all its eigenvalues on the imaginary axis. For a given initial state of the exosystem, $z(t)$ is the reference signal that the agents 1.1 are required to track. In this sense, the exosystem can be seen as a *leader*.

Since it is intended that z_i will follow the output z of the exosystem, we define the tracking error for agent i by

$$e_i = z_i - z \tag{1.3}$$

$$= Hx_i - Rw, \tag{1.4}$$

where $e_i \in \mathbb{R}^q$.

1.3.2 The problem

Definition 1 (Linear distributed output regulation problem)**.** *Given the plant 1.1, the exosystem 1.2 and the directed graph $\mathcal{G}$, find a distributed control law such that for any initial condition $x_i(0)$ and $w(0)$, the tracking errors e_i satisfy*

$$\lim_{t\to\infty} e_i(t) = 0, \ \ i = 1, ..., N.$$

Thus we have to design a controller such that the output z_i of the closed-loop system tracks the reference signal z of the exosystem. In other words the tracking errors e_i have to converge to 0. The directed graph $\mathcal{G}$ represents the interconnection between the agents and the the exosystem.

This problem is more difficult than the normal regulation problem, because each agent has to collect information in a distributed way from it neighbors. [8]

Chapter 2

Preliminaries

We start this thesis with some useful notation, basic facts and relevant results from control theory.

2.1 Graph theory

2.1.1 Graphs

First, we will introduce some basic knowledge on graph theory. A directed graph, or digraph is denoted by $\mathcal{G} = (V, \mathcal{E})$. Where $\mathcal{V} = \{1, 2, ..., N\}$ is a set of nodes; $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ represents the set of edges. An edge from node i to node j is denoted by $(i, j) \in \mathcal{E}$. In this case we call node j a neighbor of node i. Let $\mathcal{N}_i$ be the neighbor set of node i that consists of all the neighbors of node i and is a subset of $\mathcal{V}$. Now we consider a sequence of edges of the form $(i_1, i_2), (i_2, i_3), (i_3, i_4), ..., (i_{k-2}, i_{k-1}), (i_{k-1}, i_k)$. This set of edges is called a path of $\mathcal{G}$ from i_1 to i_k. Then we also say that i_k is reachable from i_1. A node j which is reachable from every other node i is called globally reachable.

When we talk about a directed graph, we use the term child and parent. An edge points from a parent to a child. A directed tree is a directed graph, where every node has exactly one parent, except one node, called the root. The root has no parent and from the root every node is reachable. A spanning tree of a digraph is a directed tree formed by edges that connect all the nodes of the graph. We call a graph $\mathcal{G}_s = (V_s, E_s)$ a subgraph of $\mathcal{G}$ if $\mathcal{V}_s \subseteq \mathcal{V}$ and $\mathcal{E}_s \subseteq \mathcal{E} \cap (\mathcal{V}_s \times \mathcal{V}_s)$. A subgraph $\mathcal{G}_s = (\mathcal{V}_s, \mathcal{E}_s)$ of $\mathcal{G}$ is called a directed spanning tree of $\mathcal{G}$ if $\mathcal{G}_s$ is a directed tree and $\mathcal{V} = \mathcal{V}_s$. Thus the digraph $\mathcal{G}$ contains a directed spanning tree if a directed spanning tree is a subgraph of $\mathcal{G}$. Therefore we can conclude that the digraph $\mathcal{G}$ contains only a directed spanning tree if $\mathcal{G}$ has at least one node which can reach every other node. [1]

Let be $\overline{\mathcal{G}} = (\overline{\mathcal{V}}, \overline{\mathcal{E}})$ a diagraph with $\overline{\mathcal{V}} = \{0, 1, ..., N\}$ where 0 is associated with the leader and $1, .., N$ with the N subsystems, and $(i, j) \in \overline{\mathcal{E}}$ if and only if there exists an edge from i to j. Further we define $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = 1, .., N$ and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. So $\mathcal{G}$ is a subgraph of $\overline{\mathcal{G}}$. [1]

2.1.2 Weighted adjacency and Laplacian matrices

Now we will introduce the weighted adjacency matrix of a digraph $\mathcal{G}$. This is a nonnegative matrix $\mathcal{A} = [a_{ij}] \in R^{N\times N}$, where $a_{ii} = 0$ and $a_{ij} > 0 \iff (j,i) \in \mathcal{E}$. It is also possible if we have a matrix $\mathcal{A} \in R^{N\times N}$ satisfying $a_{ii} = 0$ and $a_{ij} \geq 0$, to create a digraph $\mathcal{G}$ such that $\mathcal{A}$ is the weighted adjacency matrix of $\mathcal{G}$. The Laplacian of a digraph $\mathcal{G}$ is denoted by $L = [l_{ij}] \in R^{N\times N}$, where $l_{ii} = \sum_{j=1}^{N} a_{ij}$ and $l_{ij} = -a_{ij}$ if $i \neq j$. The Laplacian is also often written as $L = D - \mathcal{A}$. The matrix D is defined as $D = diag\{d_1, .., d_N\} \in R^{N\times N}$, where $d_i = \sum_{j=1}^{N} a_{ij}$. [1]

2.2 Mathematical control theory

2.2.1 Notation

Before we state the following lemma's, we will introduce the notation and definition of an annihilator of a matrix and the H_∞-norm of a matrix.

Annihilator of a matrix

Definition 2. *Let M be an $n \times m$ matrix of rank m with $m < n$. Then there exists a matrix $M^\perp$ with $n - m$ rows and n columns and rank $n - m$ such that $M^\perp M = 0$. Any such $M^\perp$ is called an annihilator of M.*

H_∞-norm

Definition 3. *We consider the linear system:*

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \tag{2.1}$$

where $A \in R^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ and $D \in R^{p\times m}$. The transfer matrix of this system is given by $G(s) = C(sI - A)^{-1}B + D$. Let $G(s)$ be proper and assume that $\sigma(A) \in \mathbb{C}^-$. Then $G(s)$ is well defined for all $s = i\omega$. In fact for all $\omega \in \mathbb{R}$ we can consider the complex matrix $G(i\omega)$. The operator norm $||G(i\omega)||$ is equal to $\sigma_1(G(i\omega))$, the largest singular value of $G(i\omega)$. We now define the H_∞-norm of $G(s)$ as

$$||G||_\infty := \sup_{\omega\in\mathbb{R}} ||G(i\omega)||.$$

[7]

2.2.2 Schur complement lemma

Lemma 1. *Let M be a symmetric matrix partitioned into blocks:*

$$M = \begin{pmatrix} M_1 & M_2 \\ M_2^T & M_3 \end{pmatrix}.$$

Assume that M_3 is positive (negative) definite. Then the following properties are equivalent:

(i) *M is positive (negative) definite;*

(ii) *The Schur complement of M_3 in M, defined as the matrix $M_1 - M_2 M_3^{-1} M_2^T$, is positive (negative) definite.*

A similar statement fold for M_1 and its Schur complement. [3]

2.2.3 Finsler's lemma

Lemma 2. *Let $x \in \mathbb{R}^n$, $M_1 \in \mathbb{R}^{n\times n}$ be symmetric, and $M_2 \in \mathbb{R}^{m\times n}$ such that rank(M_2)$< m$. Then the following statements are equivalent:*

(i) *$x^T M_1 x < 0$ for all $x \neq 0$ such that $M_2 x = \mathbf{0}$;*

(ii) *$M_2^{\perp} M_1 (M_2^{\perp})^T < \mathbf{0}$;*

(iii) *$\exists \mu \in \mathbb{R}$ such that $M_1 - \mu M_2 M_2^T < \mathbf{0}$;*

(iv) *$\exists M_3 \in \mathbb{R}^{n\times m}$ such that $M_1 + M_3 M_2 + M_2^T M_3^T < \mathbf{0}$.*

Note that in the above $M_3 = -\frac{1}{2}\mu M_2^T$ is one feasible solution. [6]

2.2.4 Bounded real lemma

We consider the linear system:

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \tag{2.2}$$

where $A \in R^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ and $D \in R^{p\times m}$. The transfer matrix of this system is given by $G(s) = C(sI - A)^{-1}B + D$. The bounded real lemma gives necessary and sufficient conditions under which A is Hurwitz and the H_∞-norm of the transfer matrix $G(s)$ is strictly less than a given $\gamma > 0$.

Theorem 1. *Let $\gamma > 0$. Then the following statements are equivalent:*

(i) *A is Hurwitz and $||G(s)||_\infty < \gamma$;*

(ii) *$\gamma^2 I - D^T D > 0$ and there exists $Y > 0$ such that*

$$YA - A^T Y + (YB + C^T D)(\gamma^2 I - D^T D)^{-1}(YB + C^T D)^T + C^T C < 0; \tag{2.3}$$

(iii) *There exist $Y > 0$ such that*

$$\begin{pmatrix} YA + A^T Y & YB & C^T \\ B^T Y & -\gamma^2 I & D^T \\ C & D & -I \end{pmatrix} < 0. \tag{2.4}$$

Proof. (ii) $\iff$ (iii): Using the Schur complement we have that (ii) is equivalent to

$$\begin{pmatrix} YA + A^TY + C^TC & YB + C^TD \\ B^TY + D^TC & D^TD - \gamma^2 I \end{pmatrix} < 0$$

subsequently this is equivalent to

$$\begin{pmatrix} YAA^T & YB \\ B^TY & -\gamma^2 I \end{pmatrix} + \begin{pmatrix} C & D \end{pmatrix}^T \begin{pmatrix} C & D \end{pmatrix} < 0$$

equivalently

$$\begin{pmatrix} YA + A^TY & YB & C^T \\ B^TY & -\gamma^2 I & D^T \\ C & D & -I \end{pmatrix} < 0.$$

(ii) $\rightarrow$ (i): First define $Q := -(YA + A^T + (YB + C^TD)(\gamma^2 I - D^TD)^{-1}(YB + C^TD)^T + C^TC)$. By 2.3, we have $Q > 0$. Now we have to prove two things: (1) A is Hurwitz and (2) $||G(s)||_\infty < 0$. First we focus on proving that A is Hurwitz. Let λ be an eigenvalue of A with eigenvector $v \neq 0$. We now take the equality from above

$$YA + A^TY = -(YB + C^TD)(\gamma^2 I - D^TD)^{-1}(YB + C^TD)^T - C^TC - Q.$$

And premultiplying with v^* and postmultiplying with v, using that $Av = \lambda v$, $v^*A = \bar{\lambda} v^*A$ and the notation $||x||_R^2 = x^*Rx$

$$2 \cdot Re(\lambda) v^*Yv = -||Cv||^2 - ||(B^TY + D^TC)v||^2_{\gamma^2 I - D^TD} - v^*Qv. \tag{2.5}$$

By (2.5) we get $Re(\lambda) \leq 0$. But if we assume $Re(\lambda) = 0$, we have $v^*Qv = 0$ and thus $v = 0$. This is a contradiction, thus $Re(\lambda) < 0$. With this, statement (1) that A is Hurwitz is proven.
Next we have to prove that $||G||_\infty < \gamma$. Therefore let $0 \neq u \in \mathcal{L}_2(\mathbb{R}^+, \mathbb{R}^m)$ and take $x(0) = 0$, then $y \in \mathcal{L}_2(\mathbb{R}^+, \mathbb{R}^p)$. Let $x(t)$ be the corresponding state trajectory and consider $x^T(t)Yx(t)$. We have

$$\frac{d}{dt}(x^TYx) - \gamma^2 ||u||^2 + ||y||^2 \tag{2.6}$$
$$= (Ax + Bu)^TYx + x^TY(Ax + Bu) - \gamma^2 u^Tu + x^TC^TCx + \tag{2.7}$$
$$x^TC^TDu + u^TD^TCx + u^TD^TDu \tag{2.8}$$
$$= \begin{pmatrix} x \\ u \end{pmatrix}^T \begin{pmatrix} A^TY + YA + C^TC & YB + C^TD \\ B^TY + D^TC & D^TD - \gamma^2 I \end{pmatrix} \begin{pmatrix} x \\ u. \end{pmatrix} \tag{2.9}$$

From the matrix in the middle we know that this is negative definite, so there exists $\epsilon > 0$ small enough such that adding $\begin{pmatrix} 0 & 0 \\ 0 & \epsilon^2 I \end{pmatrix}$ to the original matrix, the result is still negative definite. Therefore (2.9) is equal to

$$\begin{pmatrix} x \\ u \end{pmatrix}^T \begin{pmatrix} A^TY + YA + C^TC & YB + C^TD \\ B^TY + D^TC & D^TD - (\gamma^2 - \epsilon^2)I \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} - \begin{pmatrix} x \\ u \end{pmatrix}^T \begin{pmatrix} 0 & 0 \\ 0 & \epsilon^2 I \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix}.$$

Thus we get

$$\frac{d}{dt}(x^TYx) - \gamma^2||u||^2 + ||y||^2 + \epsilon^2||u||^2 = \begin{pmatrix} x \\ u \end{pmatrix}^T \begin{pmatrix} A^TY + YA + C^TC & YB + C^TD \\ B^TY + D^TC & D^TD - (\gamma^2 - \epsilon^2)I \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \leq 0,$$

or

$$\frac{d}{dt}(x^TYx) - (\gamma^2 - \epsilon^2)||u||^2 + ||y||^2 \leq 0.$$

This holds for all $t \geq 0$. Integrating from 0 tot ∞ and noting that $x(0) = 0$ and ${}_{t\to\infty}x(t) = 0$, we obtain

$$\int_0^\infty \frac{d}{dt}(x^TYx) \leq \int_0^\infty (\gamma^2 - \epsilon^2)||u||^2 dt - \int_0^\infty ||y||^2 dt.$$

So,

$$x(\infty)^TYx(\infty) - x(0)^TTx(0) \leq (\gamma^2 - \epsilon^2)||u||_2^2 - ||y||_2^2.$$

Thus,

$$||y||_2 \leq \sqrt{\gamma^2 - \epsilon^2}||u||_2.$$

This holds for all $0 \neq u \in \mathcal{L}_2(\mathbb{R}^+, \mathbb{R}^m)$, so

$$\max\left\{\frac{||y||_2}{||u||_2}|0 \neq u \in \mathcal{L}_2(\mathbb{R}^+, \mathbb{R}^m)\right\} \leq \sqrt{\gamma^2 - \epsilon^2}.$$

This implies that $||G||_\infty \leq \sqrt{\gamma^2 - \epsilon^2} < \gamma$. The proof of the converse (i) to (ii) is more involved and will be omitted here. [7] □

2.2.5 Notes on the LMI $BXC + (BXC)^T + Q < 0$

Consider the following linear matrix inequality (LMI)

$$BXC + (BXC)^T + Q < 0. \tag{2.10}$$

Let $B \in \mathbb{R}^{n\times m}$ have rank m, let $C \in \mathbb{R}^{k\times n}$ have rank k, and let $Q \in \mathbb{R}^{n\times n}$ be symmetric. Furthermore $X \in \mathbb{R}^{m\times k}$ is the unknown.

Theorem 2. *The following statements are equivalent:*

(i) *The inequality (2.10) has a solution X.*

(ii) $B^{\perp}QB^{\perp T} < 0$ *and* $C^{T\perp}QC^{T\perp T} < 0$.

In that case a solution X is given by

$$X = -R^{-1}B^T\Phi C^T(C\Phi C^T)^{-1},$$

where $R > 0$ is such that

$$\Phi := (BR^{-1}B^T - Q)^{-1} > 0.$$

Proof. (i) $\Rightarrow$ (ii): Assume that if 2.10 has a solution X, then $B^{\perp}(BXC + C^TX^TB^T + Q)B^{\perp T} \leq 0$, so we obtain $B^{\perp}QB^{\perp T} \leq 0$. For the same reason it holds that $C^{T\perp}QC^{T\perp T} \leq 0$. Assume now $x^TB^{\perp}QB^{\perp T}x = 0$. Since we have that $x^TB^{\perp}QB^{\perp T}x = x^TB^{\perp}(BXC + C^TX^TB^T + Q)B^{\perp T}x$ and $BXC + (BXC)^T + Q < 0$, this yield $B^{\perp T}x = 0$. Now we have that $B^{\perp T}$ has linearly independent columns, thus $x = 0$. Therefore $B^{\perp}QB^{\perp T} < 0$. In the same way we show that $C^{T\perp}QC^{T\perp T} < 0$.

(ii) $\Rightarrow$ (i): We start with $B^{\perp}QB^{\perp T} < 0$. By Finsler's lemma there exists $r \neq 0$ such that

$$\frac{1}{r}BB^T - Q > 0.$$

Define $R := rI$. Then $BR^{-1}B^T - Q > 0$. Therefore also

$$\Phi := (BR^{-1}B^T - Q)^{-1} > 0.$$

Since C has linearly indenpendent rows, also $C^T > 0$, thus $(C\Phi C^T)^{-1}$ exists. Define now $X := R^{-1}B^T\Phi C^T(C\Phi C^T)^{-1}$. We will proof that this X satisfies 2.2.5. Consider the matrix

$$T := \begin{pmatrix} C^{T\perp} \\ C\Phi \end{pmatrix}.$$

We claim that T is square. Indeed, $C^{T\perp}$ has $n-k$ rows. Furthermore $C\Phi$ has k rows and n columns, which proves the claim. We now prove that T is nonsingular. Let $x \in \mathbb{R}^n$ and put $Tx = 0$. Then $C^{T\perp}x = 0$ and $C\Phi x = 0$. Thus $x \in imC^T$, so there is a vector v such that $x = C^Tv$. This implies $C\Phi C^Tv = 0$, where $v = 0$, so $x = 0$.

Now we show that with the given X it holds that

$$T(BXC + (BXC)^T + Q)T^T < 0. \tag{2.11}$$

First note that

$$C^{T\perp}(BXC + (BXC)^T + Q)C^{T\perp T} = C^{T\perp}QC^{T\perp T} < 0.$$

Also,

$$\begin{aligned}
C\Phi(-BR^{-1}B^T\Phi C^T(C\Phi C^T)^{-1}C - C^T(C\Phi C^T)^{-1}C\Phi BR^{-1}B^T + Q)\Phi C^T \\
= -C\Phi BR^{-1}B^T\Phi C^T - C\Phi C^T + C\Phi Q\Phi C^T \\
\leq -C\Phi C^T \\
< 0.
\end{aligned}$$

Finally

$$\begin{aligned}
C^{T\perp}(-BR^{-1}B^T\Phi C^T(C\Phi C^T)^{-1}C - C^T(C\Phi C^T)^{-1}C\Phi BR^{-1}B^T + Q)^T \\
= -C^{T\perp}(BR^{-1}B^T - Q)\Phi C^T \\
= C^{T\perp}C^T \\
= 0.
\end{aligned}$$

In fact, we now show that

$$T(BXC + (BXC)^T + Q)T^T = \begin{pmatrix} C^{T\perp}QC^{T\perp T} & 0 \\ 0 & -C\Phi C^T \end{pmatrix} < 0.$$

From this we can conclude that also

$$BXC + (BXC)^T + Q < 0.$$

With this statements we conclude the proof. [7] □

Chapter 3

Output regulation of systems without uncertainty

3.1 Solvability

In order to study the problem defined in definition 1, we do the following assumption. Assume that the entry a_{01} in the adjacency matrix is equal to 1 and $a_{0j} = 0$ for $j \neq 1$. Thus only agent 1 is connected to the exosystem, i.e node 0. In other words only agent 1 receives the output signal z of the exosystem. It is not necessary that specific node 1 is connected to the exosystem, but it is neceassy that at least one node is connected to the exosystem.

All other agents are interconnected according to the directed graph $\mathcal{G}$. Thus not all agents are connected to all other agents, but we assume that the directed graph contains a directed spanning tree. Easily said, this means that all agents are directly or indirectly connected to the exosystem. Each agents has to collect his information in a distributed way.

To control the whole network, each agent has his own controller according to a particular protocol. We have two ways to control the interconnection of the plant and the exosystem, namely by dynamic state feedback or dynamic output feedback protocols. Dynamic state feedback protocols have the following form An example of such a directed graph is show in figure 3.1.

$$\left\{ \begin{array}{ll} \dot{w}_1 & = Sw_1 + T(z - Rw_1) \\ \dot{w}_i & = Sw_i + \sum_{j=1}^{N} a_{ij}(w_j - w_i) \text{ for } i = 2, ..., N \end{array} \right. \tag{3.1}$$

$$\left\{ \; u_i = Fx_i + Kw_i, \text{ for } i = 1, .., N, \right. \tag{3.2}$$

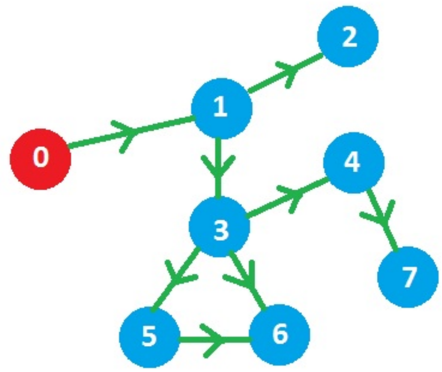

Figure 3.1: A directed graph with 7 agents and node 0 as leader.

where $w_i(t) \in \mathbb{R}^s$ is the internal state of the dynamic protocol for agent i. Furthermore T is such that $S - TR$ is Hurwitz and the matrices F and K will be determined later.

The second possibility is to control the network by means of dynamic output feedback protocols of the form

$$\left\{ \begin{array}{ll} \dot{w}_1 & = Sw_1 + T(z - Rw_1) \\ \dot{w}_i & = Sw_i + \sum_{j=1}^{N} a_{ij}(w_j - w_i) \text{ for } i = 2, ..., N \end{array} \right. \tag{3.3}$$

$$\left\{ \begin{array}{ll} \dot{v}_i & = A_c v_i + B_c(z_i - Rw_i) \\ u_i & = C_c v_i + D_c z_i + Kw_i, \text{ for } i = 1, ..., N, \end{array} \right. \tag{3.4}$$

where $w_i(t) \in \mathbb{R}^s$ and $v_i(t) \in \mathbb{R}^{n_c}$ are the internal states of the dynamic protocol for agent i. Furthermore T is such that again $S - TR$ is Hurwitz. A_c, B_c, C_c, D_c and K are the gain matrices, which we have to determine later. Further we assume that H of the plant has full row rank.

Before we will solve the main problem, formulated in chapter 1.3, we will give the following lemma.

Lemma 3. *In (3.1) and exactly similar (3.3), we have that $w_i(t) - w(t) \to \mathbf{0}$ as $t \to \infty$ exponentially for all $i = 1, ..., N$, for all initial conditions on the exosystem and the protocol.*

Proof. Define $\tilde{w}_i = w_i - w$, $\forall\ i = 1, ..., N$. Then we have

$$\begin{aligned} \dot{\tilde{w}}_1 &= \dot{w}_1 - \dot{w} \\ &= Sw_1 + Tz - TRw_1 - Sw \\ &= Sw_1 + TRw - TRw_1 - Sw \\ &= (S - TR)(w_1 - w) \\ &= (S - TR)\tilde{w}_1, \end{aligned}$$

where $S - TR$ is Hurwitz. Thus $\tilde{w_1}$ tends to $\mathbf{0}$ exponentially, due to corollary 2.11 of [2].

Now we look at the situation $i = 2, ..., N$ and evaluate $\dot{\tilde{w_i}}$,

$$\begin{aligned}\dot{\tilde{w_i}} &= \dot{w}_i - \dot{w} \\ &= Sw_i + \sum_{j=1}^{N} a_{ij}(w_j - w_i) - Sw \\ &= S(w_i - w) + \sum_{j=1}^{N} a_{ij}(w_j - w_i) \\ &= S\tilde{w_i} + \sum_{j=1}^{N} a_{ij}(\tilde{w_j} - \tilde{w_i}).\end{aligned}$$

Now we create the vector $\tilde{w} = [\tilde{w_2}^T, ..., \tilde{w_N}^T]$ and write the Laplacian L of $\mathcal{G}$ as $L = \begin{bmatrix} l_{11} & l_{12} \\ l_{21} & \tilde{\mathcal{L}} \end{bmatrix}$. Because only agent 1 is connected with the node 0 and does not use the relative information of the other nodes, we know that $l_{11} = 0$ and $l_{12} = \mathbf{0}$. So $\tilde{L} = \begin{bmatrix} 0 & \mathbf{0} \\ l_{21} & \tilde{\mathcal{L}} \end{bmatrix}$ is the Laplacian which describes the interconnection relation for the above protocols. Thus we have

$$\dot{\tilde{w}} = \big(I_{N-1} \otimes S - \tilde{\mathcal{L}} \otimes I_s\big) - l_{21} \otimes \tilde{w_1}. \tag{3.5}$$

It is easy to see that $\sum_{j=1}^{N} l_{ij} = 0$ for all i, thus we have that $\tilde{L}$ has a unique zero eigenvalue and the others have strictly positive real parts, see appendix A. Therefore we have that $-\tilde{\mathcal{L}}$ is Hurwitz. Let $v(t) = (I_{N-1} \otimes e^{-St})\tilde{w}$. Then we get

$$\begin{aligned}\dot{v} &= -(I_{N-1} \otimes Se^{-St})\tilde{w} + (I_{N-1} \otimes e^{-St})[\big(I_{N-1} \otimes S - \tilde{\mathcal{L}} \otimes I_s\big) - l_{21} \otimes \tilde{w_1}] \\ &= -(\tilde{\mathcal{L}} \otimes e^{-St})\tilde{w} - l_{21} \otimes (e^{-St}\tilde{w_1}) \\ &= -(\tilde{\mathcal{L}} \otimes I_s)v - l_{21} \otimes (e^{-St}\tilde{w_1}).\end{aligned}$$

Now we will evaluate these expression. Since S has all its eigenvalues on the imaginary axis and $\tilde{w_1}$ goes to $\mathbf{0}$ exponentially, we can conclude that also the term $e^{-St}\tilde{w_1}$ goes to $\mathbf{0}$ exponentially. Thus the last term vanishes. Also the first term vanishes, because $-(\tilde{\mathcal{L}} \otimes I_s)$ is Hurwitz and we can repeat the reasoning we did for $\tilde{w_1}$. Therefore v tends to zero, as well as $\tilde{w}$. With this statement the proof is finished. [9] □

3.2 Dynamic state feedback

In this section, we will study the dynamic state feedback protocol

$$\begin{cases} \dot{w}_1 &= Sw_1 + T(z - Rw_1) \\ \dot{w}_i &= Sw_i + \sum_{j=1}^{N} a_{ij}(w_j - w_i) \text{ for } i = 2, ..., N \end{cases} \tag{3.6}$$

$$\left\{ \begin{array}{l} u_i = Fx_i + Kw_i, \text{ for } i = 1,..,N, \end{array} \right. \tag{3.7}$$

for linear multi-agent systems without uncertainty. We will give necessary conditions for the existence of such a protocol that makes sure that the network is output regulated. This means that the control law is such that the system matrix of the network is Hurwitz, for $w \equiv 0$ and the tracking errors e_i converges to 0, see definition 1. This section also gives explicitly how to build such a protocol.

Theorem 3. *Let* Π *and* Γ *be a solution pair to the regulator equations*

$$\left\{ \begin{array}{ll} \Pi S & = A\Pi + B\Gamma \\ \boldsymbol{0} & = H\Pi - R. \end{array} \right. \tag{3.8}$$

If (A, B) *is stabilizable, i.e. there exists* F *such that* $(A+BF)$ *is Hurwitz, then the network of nodes 1.2 and 1.1 with protocol 3.6-3.7, where* $K = \Gamma - F\Pi$*, is output regulated.*

Proof. Take $\tilde{x}_i = x_i - \Pi w_i$, $i = 1,..,N$, where Π together with Γ satisfies (3.8). Consequently we get,

$$\begin{aligned}
\dot{\tilde{x}}_1 &= \dot{x}_1 - \Pi \dot{w}_1 \\
&= Ax_1 + Bu_1 - \Pi S w_1 - \Pi T(z - Rw_1) \\
&= Ax_1 + BFx_1 + BKw_1 - A\Pi w_1 - B\Gamma w_1 - \Pi T(z - Rw_1) \\
&= (A + BF)x_1 + (A + BF)\Pi w_1 - \Pi T(z - Rw_1) \\
&= (A + BF)\tilde{x}_1 - \Pi T(z - Rw_1).
\end{aligned}$$

And for $i = 2, ..., N$,

$$\begin{aligned}
\dot{\tilde{x}}_i &= \dot{x}_i - \Pi \dot{w}_i \\
&= Ax_i + B_{ui} - \Gamma S w_i - \Pi \sum_{j=1}^{N} a_{ij}(w_j - w_i) \\
&= (A + BF)x_i + B\Gamma w_i - BF\Pi w_i - A\Pi w_i - B\Gamma w_i - \Pi \sum_{j=1}^{N} a_{ij}(w_j - w_i) \\
&= (A + BF)x_i + (A + BF)\Gamma w_i - \Pi \sum_{j=1}^{N} a_{ij}(w_j - w_i) \\
&= (A + BF)\tilde{x}_i - \Pi \sum_{j=1}^{N} a_{ij}(w_j - w_i), \text{ i=2,...,N.}
\end{aligned}$$

Now we denote $\Sigma_1 = \Pi T(z - Rw_1)$ and $\Sigma_i = \Pi \sum_{j=1}^{N} a_{ij}(w_j - w_i)$ for $i = 2, ..., N$. Now we can construct a general form for $i = 1, ..., N$,

$$\dot{\tilde{x}}_i = (A + BF)\tilde{x}_i - \Sigma_i. \tag{3.9}$$

with $A+BF$ is Hurwitz. Because $\Sigma_i(t) \to \mathbf{0}$ as $t \to$ exponentially, we now take Σ_i as output of a globally exponentially stable linear system with input $\mathbf{0}$. We can see that $\tilde{x}_i(t) \to \mathbf{0}$ as $t \to \infty$, see corollary (3.22) of [2]. Together with the result of lemma 3 that $w_i(t) - w(t) \to \mathbf{0}$ as $t \to \infty$ exponentially, we can conclude that het network is output regulated. □

The following question is: How can we compute a suitable F for a given system, and an associated $K = \Gamma - F\Pi$? To answer this question we will introduce first a new lemma.

Lemma 4. *If (A, B) is stabilizable, then there exist F such that there exist $P > 0$ such that*

$$(A+BF)^T P + P(A+BF) < 0, \tag{3.10}$$

i.e. there exists $P > 0$ that solves the Lyapunov inequality.

Note that the converse is also true, but that part is not interesting for our reasoning.

Proof. Because (A, B) is stabilizable, we know that there exist F such that $A+BF$ is Hurwitz. The candidate for P we will test now is the following:

$$P = \int_0^\infty e^{(A+BF)^T t} e^{(A+BF)t} dt$$

The claim is that this P solves the Lyapunov inequality:

$$(A+BF)^T P + P(A+BF) < 0.$$

To check this we plug P into equation (3.10):

$$(A+BF)^T P + P(A+BF)$$

$$\begin{aligned}
&= \int_0^\infty [(A+BF)^T e^{(A+BF)^T t} e^{(A+BF)t} + e^{(A+BF)^T t} e^{(A+BF)t}(A+BF)] dt \\
&= \int_0^\infty [\frac{d}{dt} e^{(A+BF)^T} t e^{(A+BF)t}] dt \\
&= e^{(A+BF)^T t} e^{(A+BF)t} |_0^\infty \\
&= -I \\
&< 0.
\end{aligned}$$

We conclude that F and P satisfy (3.10). □

The following lemma gives a necessary and sufficient condition for the existence of such a pair F and P that satisfy (3.10). This condition is in terms of solvability of a linear matrix inequality. It also gives explicit formulas to compute a suitable F and P.

Lemma 5. *There exist F and matrix $P > \mathbf{0}$ such that inequality*

$$P(A + BF) + (A + BF)^T P < \mathbf{0} \tag{3.11}$$

holds if and only if there exists a matrix $Q > \mathbf{0}$ such that

$$B^{\perp}(QA^T + AQ)(B^{\perp})^T < \mathbf{0}.$$

In this case, a suitable P is given by $P = Q^{-1}$ and a suitable F is given by $F = \mu B^T Q^{-1}$, where μ is a real number that statisfies the inequality $QA^T + AQ + 2\mu BB^T < \mathbf{0}$.

Proof. $\Leftarrow$: Let $Q = P^{-1}$. Then we get

$$QA^T + AQ + QF^T B^T + BFQ < \mathbf{0}.$$

When we premultiply with $B^{\perp}$ and postmultiply with $(B^{\perp})^T$, we get

$$B^{\perp}(QA^T + AQ)(B^{\perp})^T + B^{\perp}QF^T B^T (B^{\perp})^T + B^{\perp}BFQ(B^{\perp})^T < \mathbf{0},$$

which yields $B^{\perp}(QA^T + AQ)(B^{\perp})^T < \mathbf{0}$.

$\Rightarrow$: For the 'if' part we use Finsler's lemma, which tells us that if $x^T(QA^T + AQ)x < \mathbf{0}$ for all x such that $Bx = 0$ there exists a real μ such that $AQ^T + AQ + 2\mu BB^T < \mathbf{0}$. Let $P = Q^{-1}$ and $F := \mu B^T Q^{-1}$. Then we have that

$$\begin{aligned}
&(A + BF)^T P + P(A + BF) \\
&= (A + \mu BB^T Q^{-1})^T Q^{-1} + Q^{-1}(A + \mu BB^T Q^{-1}) \\
&= A^T Q^{-1} + Q^{-1}A + 2\mu Q^{-1} BB^T Q^{-1} < \mathbf{0}.
\end{aligned}$$

□

So, in order to obtain a cooperative output regulation protocol, one has to

1. Compute a solution pair (Π, Γ) to the linear matrix equations

$$\begin{cases} \Pi S &= A\Pi + B\Gamma \\ \mathbf{0} &= H\Pi - R \end{cases}$$

2. Compute $Q > \mathbf{0}$ such that the linear matrix inequality $B^{\perp}(QA^T + AQ)(B^{\perp})^T < \mathbf{0}$ holds;
3. Compute a μ such that the linear matrix inequality $QA^T + AQ + 2\mu BB^T < \mathbf{0}$ holds;
4. Compute $F = \mu B^T Q^{-1}$;
5. Compute $K = \Gamma - F\Pi$.

[9]

3.3 Dynamic output feedback

In this section, we will discuss the design of a dynamic output feedback protocol of the form

$$\begin{cases} \dot{w}_1 &= Sw_1 + T(z - Rw_1) \\ \dot{w}_i &= Sw_i + \sum_{j=1}^{N} a_{ij}(w_j - w_i) \text{ for } i = 2, ..., N \end{cases} \tag{3.12}$$

$$\begin{cases} \dot{v}_i &= A_c v_i + B_c(z_i - Rw_i) \\ u_i &= C_c v_i + D_c z_i + K w_i, \text{ for } i = 1, ..., N. \end{cases} \tag{3.13}$$

Theorem 4. *Let Π and Γ be a solution pair to the regulator equations*

$$\begin{cases} \Pi S &= A\Pi + B\Gamma \\ \boldsymbol{0} &= H\Pi - R. \end{cases} \tag{3.14}$$

If (A, B) is stabilizable and (H, A) is detectable, i.e. there exists F such that $(A_f + B_f F H_f)$ is Hurwitz (see appendix B for the proof), where

$$A_f = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, B_f = \begin{bmatrix} B & 0 \\ 0 & I \end{bmatrix}, H_f = \begin{bmatrix} H & 0 \\ 0 & I \end{bmatrix}, F = \begin{bmatrix} D_c & C_c \\ B_c & A_c \end{bmatrix},$$

then the network of nodes 1.2, 1.1 with protocol 3.12-3.13, where $K = \Gamma - D_c H\Pi$, is output regulated.

Proof. Let $\tilde{x}_i = x_i - \Pi w_i$, $i = 1, ..., N$, where Π together with Γ satisfies (3.14). Consequently we get for $i = 1$:

$$\begin{aligned} \dot{\tilde{x}}_1 &= \dot{x}_1 - \Pi \dot{w}_1 \\ &= Ax_1 + BC_c v_1 + BD_c H x_1 + B(\Gamma - D_c H\Gamma)w_1 - \Pi S w_1 - \Pi T(z - Rw_1) \\ &= Ax_1 + B\Gamma w_1 - \Pi S w_1 + BD_c H x_1 - BD_c H\Pi w_1 + BC_c v_1 - \Pi T(z - Rw_1) \\ &= (A + BD_c H)\tilde{x}_1 + BC_c v_1 - \Pi T(z - Rw_1). \end{aligned}$$

For $i = 2, .., N$ we get

$$\begin{aligned} \dot{\tilde{x}}_i &= \dot{x}_i - \Gamma \dot{w}_i \\ &= Ax_i + BC_c v_i + BD_c H x_i + B(\Gamma - D_c H\Gamma)w_i - \Pi S w_i - \Pi \sum_{j=1}^{N} a_{ij}(w_j - w_i) \\ &= Ax_i + B\Gamma w_i - \Pi S w_i + BD_c H x_i - BD_c H\Pi w_i + BC_c v_i - \Pi \sum_{j=1}^{N} {}_{ij}(w_j - w_i) \\ &= (A + BD_c H)\tilde{x}_i + BC_c v_i - \Pi \sum_{j=1}^{N} a_{ij}(w_j - w_i). \end{aligned}$$

As before we denote $\Sigma_1 = \Pi T(z - Rw_1)$ and $\Sigma_i = \sum_{j=1}^{N} a_{ij}(w_j - w_i)$, for $i = 2, ..., N$. In short we want to evaluate the equation

$$\dot{\tilde{x}}_i = (A + BD_cH)\tilde{x}_i + BC_cv_i - \Sigma_i,$$

for $i = 1, ..., N$. From lemma 3 we know that $\Sigma_i(t) \to \mathbf{0}$ as $t \to \infty$ exponentially for all i. Furthermore we have for all i that

$$\begin{aligned}\dot{v}_i &= A_cv_i + B_cz_i - B_cRw_i \\ &= A_cv_i + B_cHx_i - B_cH\Pi w_i \\ &= A_cv_i + B_cH\tilde{x}_i.\end{aligned}$$

Combining this we get

$$\begin{pmatrix}\dot{\tilde{x}}_i \\ \dot{v}_i\end{pmatrix} = \begin{pmatrix}A + BD_cH & B_C \\ B_cH & A_c\end{pmatrix}\begin{pmatrix}\tilde{x}_i \\ v_i\end{pmatrix} + \begin{pmatrix}-I \\ 0\end{pmatrix}\Sigma_i.$$

Since there exists F such that $(A_f + B_fFH_f)$ is Hurwitz, we can also say that there exists $F = \begin{pmatrix}D_c & C_c \\ B_c & A_c\end{pmatrix}$ such that $\begin{pmatrix}A + BD_cH & B_C \\ B_cH & A_c\end{pmatrix}$ is Hurwitz. With this statement, the proof is completed. □

Using the same reasoning as in lemma 4 we see that if (A, B) is stabilizable and (H, A) is detectable, then there exists F such that there exists $P > 0$ such that

$$(A_f + B_fFH_f)^TP + P(A_f + B_fFH_f) < 0. \tag{3.15}$$

With this fact we come to a new theorem.

Theorem 5. *There exist F and $P > 0$ such that inequality*

$$(A_f + B_fFH_f)^TP + P(A_f + B_fFH_f) < 0 \tag{3.16}$$

holds if and only if there exists matrices $X > 0$ and $Y > 0$ such that $XY = I$,

$$B_f^{\perp}(A_fX + XA_f^T)B_f^{\perp T} < 0, \tag{3.17}$$

$$(H_f^T)^{\perp}(YA_f + A_f^TY)(H_f^T)^{\perp T} < 0. \tag{3.18}$$

In this case, a suitable P is given by $P = X^{-1}$ and a suitable F is given by

$$F = -rB_f^T\Theta_x^{-1}XH_f^T(H_fX\Theta_x^{-1}XH_f^T)^{-1}, \tag{3.19}$$

where Θ_x is determined by choosing a positive real number r such that

$$\Theta_x = rB_fB_f^T - A_fX - XA_f^T > 0.$$

Proof. ⇒: Define $Y = X^{-1}$ and $X = P^{-1}$. Then we have $X > \mathbf{0}$, $Y > \mathbf{0}$ and $XY = I$. Obviously, with use of Finsler's lemma, inequality (3.17) and (3.18) holds.
⇐: Now we want to prove that inequality (3.16) holds. If we choose F such as is stated in (3.19), this is the same as proving that

$$\begin{aligned}(A_f - rB_fB_f^TP)^TP + P(A_f - rB_fB_f^TP) < 0 \\ A_f^TP + PA_f - 2rPB_fB_f^TP < 0.\end{aligned}$$

Rewriting this and using the fact that $P = X^{-1}$, we get

$$XA_f^T + A_fX - 2rB_fB_f^T < 0. \tag{3.20}$$

When we prove that above inequality holds, the theorem is proven. Since we have that

$$B_f^{\perp}(A_fX + XA_f^T)B_f^{\perp T} < 0,$$

Finsler's lemma tells us that $\exists \mu \in \mathbb{R}$ such that $A_fX + XA_f^T - \mu B_fB_f^T < 0$. When we choose μ as $2r$, we have proven that equation 3.20 holds and with this the whole proof is completed. □

There is an essential variable left we do not know yet, namely the dimension n_c of the protocol state space. The following theorem tells us how to choose this dimension n_c.

Theorem 6. *Let n_c be a nonnegative integer. There exist matrices $X > 0$, $Y > 0$ of size $(n + n_c) \times (n + n_c)$ such that the conditions of theorem 5 holds if and only if there exists matrices X_p, Y_p of size $n \times n$ such that $X_p > 0$, $Y_p > 0$,*

$$B^{\perp}(AX_p + X_pA^T)B^{\perp T} < 0, \tag{3.21}$$

$$(H^T)^{\perp}(Y_pA + AY_p)(H^T)^{\perp T} < 0, \tag{3.22}$$

$$\begin{bmatrix} X_p & I \\ I & Y_p \end{bmatrix} \geq 0, \tag{3.23}$$

$$rank \begin{bmatrix} X_p & I \\ I & Y_p \end{bmatrix} \leq n + n_c. \tag{3.24}$$

Proof. ⇒: Assume that there exist $X > \mathbf{0}$, $Y > \mathbf{0}$ of size $(n + n_c) \times (n + n_c)$ such that $XY = I$, 3.17 and 3.18 holds. Take the partitions

$$X = \begin{bmatrix} X_p & X_{pc} \\ X_{pc}^T & X_c \end{bmatrix}, \; Y = \begin{bmatrix} Y_p & Y_{pc} \\ Y_{pc}^T & Y_c \end{bmatrix}.$$

Furthermore note that

$$B_f^{\perp} = \begin{bmatrix} B_{\perp} & \mathbf{0} \end{bmatrix}, H_f^{T\perp} = \begin{bmatrix} H^{T\perp} & \mathbf{0} \end{bmatrix}, \begin{bmatrix} B_f \\ \mathbf{0} \end{bmatrix}^{\perp} = \begin{bmatrix} B_f^{\perp} & \mathbf{0} \\ \mathbf{0} & I \end{bmatrix}, \begin{bmatrix} H_f \\ \mathbf{0} \end{bmatrix}^{\perp} = \begin{bmatrix} H_f^{\perp} & \mathbf{0} \\ \mathbf{0} & I \end{bmatrix}.$$

In this way we obtain (3.21) and (3.22). The fact that $XY = I$ tells us that $X_pY_p + X_{pc}Y_{pc}^T = I$ and $X_pY_{pc}{}_{X_{pc}}Y_c = \mathbf{0}$. Therefore we get

$$Y_p - X_p^{-1} = Y_{pc}Y_c^{-1}Y_{pc}^T \geq \mathbf{0}. \tag{3.25}$$

Using Schur complement, this is equal to

$$\begin{bmatrix} X_p & I \\ I & Y_p \end{bmatrix} \geq \mathbf{0},$$

with this statement equation (3.23) is proven. Moreover we have that,

$$\begin{aligned} rank \begin{bmatrix} X_p & I \\ I & Y_p \end{bmatrix} &= rank(X_p) + rank(Y_p - X_p^{-1}) \\ &= rank(X_p) + rank(Y_{pc}Y_c^{-1}Y_{pc}^T) \leq n + n_c. \end{aligned}$$

So also (3.24) holds.
$\Leftarrow$: For this direction of the proof it is important that we choose the parts of the matrices X and Y is a specific way. Therefore let Y_{pc} and $Y_c > \mathbf{0}$ be such that they satisfy (3.25) while $X_p > \mathbf{0}$ and $Y_p > \mathbf{0}$ satisfy (3.21), (3.22) and (3.23). Besides that, n_c is chosen such that (3.24) is satisfied. It can be checked that the matrices X and Y given by

$$Y = \begin{bmatrix} Y_p & Y_{pc} \\ Y_{pc}^T & Y_c \end{bmatrix}, X = Y^{-1},$$

satisfy the conditions of theorem 5. □

So, in order to obtain a cooperative output regulation protocol, one has to

1. Compute a solution pair (Π, Γ) to

$$\begin{cases} \Pi S &= A\Pi + B\Gamma \\ \mathbf{0} &= H\Pi - R; \end{cases}$$

2. Compute $X_p > \mathbf{0}$ and $Y_p > \mathbf{0}$ such that (3.21), (3.22) and (3.23);
3. Choose n_c as $n_c = rank \begin{bmatrix} X_p & I \\ I & Y_p \end{bmatrix} - n$;
4. Define A_f, B_f, C_f, E_f and H_f;
5. Choose Y_{pc} and $Y_c > \mathbf{0}$ satisfying (3.25), consequently we have Y and X;
6. Compute $r > 0$ such that $\Theta_x > \mathbf{0}$;

7. Compute $F = -rB_f^T \Theta_X^{-1} X H_f^T (H_f X \Theta_x^{-1} X H_f^T)^{-1}$;
8. Partition F as $\begin{bmatrix} D_c & C_c \\ B_c & A_c \end{bmatrix}$;
9. Compute $K = \Gamma - D_c H \Pi$.

[9]

Chapter 4

Output regulation of systems with uncertainty

4.1 Types of uncertainty

In the previous chapter we had a linear system without uncertainty. In this chapter we consider again a multi-agent system with N agents and a leader. The idea is to add some uncertainty to the dynamics of each agent. There are different types of uncertainty. In this thesis we will check *additive perturbations* and *multiplicative perturbations* of the agent transfer matrices.

The transfer matrix for the original agent i,

$$\begin{cases} \dot{x}_i &= Ax_i + Bu_i \\ z_i &= Hx_i \end{cases}, \tag{4.1}$$

is given by $G(s) = H(sI - A)^{-1}B$.

First we assume that the error is additive, i.e. the system describes the dynamics of agent i with transfer matrix $G + \Delta_i$, with $||\Delta_i||_\infty \leq \gamma$. The perturbed agent dynamics can be written as

$$\begin{cases} \dot{x}_i &= Ax_i + Bu_i \\ y_i &= u_i \\ z_i &= Hx_i + d_i \\ d_i &= \Delta_i y_i. \end{cases} \tag{4.2}$$

In addition to the additive error used in the previous case, it is often useful to describe uncertainty via a relative instead of an absolute error. This is achieved by looking at the multiplicative error. In this case the perturbed agent dynamics has the following form

$$\begin{cases} \dot{x}_i &= Ax_i + Bu_i + Bd_i \\ y_i &= u_i \\ z_i &= Hx_i \\ d_i &= \Delta_i y_i. \end{cases} \tag{4.3}$$

Both representations (4.2) and (4.3) can be seen as special cases of:

$$\begin{cases} \dot{x}_i &= Ax_i + Bu_i + Ed_i \\ y_i &= Cx_i + Du_i \\ z_i &= Hx_i + Jd_i \\ d_i &= \Delta_i y_i. \end{cases} \tag{4.4}$$

i.e. for additive perturbations it holds that $E = 0, C = 0, D = I, J = I$ and for multiplicative perturbations: $E = B, C = 0, D = I, J = 0$. Thus if we will succeed to solve the problem for the general case, we have done it automatically for both: additive and multiplicative perturbations. [2]

For completeness, we mention that the exosystem does not change.

$$\begin{cases} \dot{w} = Sw \\ z = Rw. \end{cases} \tag{4.5}$$

4.2 Solvability

Now we are wondering if also in the case with uncertainty, a matrix F and K exist such that the coupled system is output regulated in the sense of definition 1 with help of the protocols 3.6-3.7 and 3.12-3.13. Besides the necessary conditions for the existence of such a protocol, we will also explicitly give how to build such a controller.

From lemma 3 we know that $w_i(t) - w(t) \to \mathbf{0}$ as $t \to \infty$ exponentially for all $i = 1, .., N$ for all initial conditions on the exosystem and the protocols. This lemma also holds in the case of additive and multiplicative perturbations.

4.3 Dynamic state feedback with uncertainty

First, we will study the dynamic state feedback protocol 3.6-3.7. Later on, the dynamic output feedback protocol 3.12-3.13 will be mentioned.

Theorem 7. *Let* (Π, Γ) *be a solution pair to the regulator equations*

$$\begin{cases} \Pi S &= A\Pi + B\Gamma \\ \mathbf{0} &= C\Pi + D\Gamma \\ \mathbf{0} &= H\Pi - R. \end{cases} \tag{4.6}$$

Let $\gamma > 0$ *be a real number. If there exist* $P > 0$ *and* F *such that*

$$\begin{pmatrix} P(A+BF) + (A+BF)^T P + (C+DF)^T(C+DF) & PE \\ E^T P & -\frac{1}{\gamma^2} I \end{pmatrix} < 0,$$

then the network of nodes 4.4-4.5 with protocol 3.6-3.7, where $K = \Gamma - F\Pi$, is output regulated.

Proof. Let $\tilde{x}_i = x_i - \Pi w_i$, $i = 1, ..., N$, where Γ and Π satisfy (4.6). We get

$$\dot{\tilde{x}}_1 = (A + BF)\tilde{x}_1 - Ed_1 - \Pi T(z - Rw_1), \tag{4.7}$$

$$\dot{\tilde{x}}_i = (A + BF)\tilde{x}_i - Ed_i - \Pi \sum_{j=1}^{N} a_{ij}(w_j - w_i). \tag{4.8}$$

Also this time we denote $\Sigma_1 = \Pi T(z - Rw_1)$ and $\Sigma_i = \Pi \sum_{j=1}^{N} a_{ij}(w_j - w_i)$, $i = 2, ..., N$. By lemma 3, $\Sigma_i \to \mathbf{0}$ as $t \to \infty$ exponentially for $i = 1, .., N$.

Now we have that

$$\begin{pmatrix} P(A + BF) + (A + BF)^T P + (C + DF)^T (C + DF) & PE \\ E^T P & -\frac{1}{\gamma^2} I \end{pmatrix} < 0, \tag{4.9}$$

for some matrix $P > 0$ and F. Due to the bounded real lemma (see section 2.2.4) this is equivalent to saying that the closed-loop system is internally stable and the H_∞ from d to y is strictly less than $\frac{1}{\gamma}$, i.e.

$$||(C + DF)(sI - A - BF)^{-1}E||_\infty < \frac{1}{\gamma}.$$

equivalently: the closed-loop system is internally stable for all internally stable systems Δ_i with $||\Delta_i||_\infty \leq \gamma$. [2]

Now we get:

$$\begin{cases} \dot{\tilde{x}}_i & = (A + BF)\tilde{x}_i + Ed_i \\ y_i & = (C + DF)\tilde{x}_i \\ z_i & = Hx_i + Jd_i \\ d_i & = \Delta_i y_i. \end{cases} \tag{4.10}$$

Combining this with the fact that $\Sigma_i \to 0$ as $t \to \infty$ and $||(C + DF)(sI - A - BF)^{-1}E||_\infty < \frac{1}{\gamma}$, we can conclude that the system is internally stable. Thus $\tilde{x}_i \to 0$ as $t \to \infty$. From this fact and with the knowledge that d_i goes to zero, we get

$$\begin{aligned} x_i - \Pi w_i &\to 0 \\ Hx_i - H\Pi w_i &\to 0 \\ z_i - Jd_i - Rw_i &\to 0 \\ z_i - z - J_{di} &\to 0 \\ z_i &\to z. \end{aligned}$$

With this statement the proof is completed. □

Thus, theorem 7 gives us necessary conditions for the existence of a dynamic state feedback protocol such that the network is output regulated. In the next lemma we will introduce in an explicit way a prescription how to build a protocol.

Lemma 6. *There exist F and $P > 0$ such that (4.9) holds if and only if there exists a matrix $X > 0$ such that*

$$\begin{pmatrix} B \\ D \end{pmatrix}^{\perp} \begin{pmatrix} AX + XA^T + \gamma^2 EE^T & AC^T \\ CX & -I \end{pmatrix} \begin{pmatrix} B^T & D^T \end{pmatrix}^{\perp} < 0. \tag{4.11}$$

In that case a solution F is given by

$$F = -\mu \begin{pmatrix} B^T & D^T \end{pmatrix} \Phi \begin{pmatrix} X \\ 0 \end{pmatrix} \left[\begin{pmatrix} X & 0 \end{pmatrix} \Phi \begin{pmatrix} X \\ 0 \end{pmatrix} \right]^{-1},$$

where μ is such that

$$\Phi^{-1} := \begin{pmatrix} XA^T + AX + EE^T & XC^T \\ CX & -\frac{1}{\gamma^2} I \end{pmatrix} - \mu \begin{pmatrix} B \\ D \end{pmatrix} \begin{pmatrix} B^T & D^T \end{pmatrix} < 0.$$

Proof. $\Rightarrow$: We start this proof with the fact that there exist F and $P > 0$ such that

$$\begin{pmatrix} P(A+BF) + (A+BF)^T P + (C+DF)^T (C+DF) & PE \\ E^T P & -\frac{1}{\gamma^2} I \end{pmatrix} < 0,$$

Due to Schur complement lemma this is equivalent to

$$P(A+BF) + (A+BF)^T P + (C+DF)^T (C+DF) + \gamma^2 PEE^T P < 0.$$

Premultiplying with X and postmultiplying with X and define $X = P^{-1}$, we get

$$(A+BF)X + X(A+BF)^T + X(C+DF)^T (C+DF)X + \gamma^2 EE^T < 0.$$

Again apply Schur complement lemma, this is equivalent to

$$\begin{pmatrix} (A+BF)X + X(A+BF)^T + \gamma^2 EE^T & X(C+DF)^T \\ (C+DF)X & -I \end{pmatrix} < 0$$

$$\begin{pmatrix} AX + XA^T + \gamma^2 EE^T & XC^T \\ CX & -I \end{pmatrix} + \begin{pmatrix} BFX + XF^T B^T & XF^T D^T \\ DFX & 0 \end{pmatrix} < 0$$

$$\begin{pmatrix} AX + XA^T + \gamma^2 EE^T & XC^T \\ CX & -I \end{pmatrix} + \begin{pmatrix} XF^T \\ 0 \end{pmatrix} \begin{pmatrix} B^T & D^T \end{pmatrix} + \begin{pmatrix} B \\ D \end{pmatrix} \begin{pmatrix} FX & 0 \end{pmatrix} < 0.$$

Now Finsler's lemma tells us that there exist $\mu \in \mathbb{R}$ such that

$$\begin{pmatrix} AX + XA^T + \gamma^2 EE^T & XC^T \\ CX & -I \end{pmatrix} - \mu \begin{pmatrix} B \\ D \end{pmatrix} \begin{pmatrix} B^T & D^T \end{pmatrix} < 0.$$

Which is equivalent to saying that there exist X such that

$$\begin{pmatrix} B \\ D \end{pmatrix}^{\perp} \begin{pmatrix} AX + XA^T + \gamma^2 EE^T & XC^T \\ CX & -I \end{pmatrix} \begin{pmatrix} B^T & D^T \end{pmatrix}^{\perp}.$$

This statement completes this side of the proof.
$\Leftarrow$: Assume $X > 0$ and from above we see that the statement

$$\begin{pmatrix} P(A+BF) + (A+BF)^T P + (C+DF)^T(C+DF) & PE \\ E^T P & -\frac{1}{\gamma^2} I \end{pmatrix} < 0,$$

is equivalent to

$$\begin{pmatrix} XA^T + AX + EE^T & X^T C \\ CX & -\frac{1}{\gamma^2} I \end{pmatrix} + \begin{pmatrix} B \\ D \end{pmatrix} F \begin{pmatrix} X & 0 \end{pmatrix} + \left[\begin{pmatrix} B \\ D \end{pmatrix} F \begin{pmatrix} X & 0 \end{pmatrix} \right]^T < 0.$$

This is an LMI of the same form as $BXC + [BXC]^T + Q < 0$ in section 2.2.5. This theorem says that there exists a solution F if holds that

(1) $$\begin{pmatrix} B \\ D \end{pmatrix}^{\perp} \begin{pmatrix} AX + XA^T + EE^T & XC^T \\ CX & -\frac{1}{\gamma^2} I \end{pmatrix} \begin{pmatrix} B \\ D \end{pmatrix}^{\perp T} < 0\ ;$$

(2) $$\begin{pmatrix} X \\ 0 \end{pmatrix}^{\perp} \begin{pmatrix} AX + XA^T + EE^T & XC^T \\ CX & -\frac{1}{\gamma^2} I \end{pmatrix} \begin{pmatrix} X \\ 0 \end{pmatrix}^{\perp T} < 0.$$

Since X is nonsingular, we can check that $\begin{pmatrix} X \\ 0 \end{pmatrix}^{\perp} = \begin{pmatrix} 0 & I \end{pmatrix}$. Therefore requirement (2) is equivalent to say that $-\frac{1}{\gamma^2} < 0$, and that is always true. Besides that (1) is always true because it is our starting point. Therefore the statements is proven in both directions. $\square$

Now we have proven that there exist F and $P > 0$ such that (4.9) holds if and only if there exists a matrix $X > 0$ such that 4.11 holds. But the question that arises now is: How do we compute such F and $P > 0$. To say something about this, again use the theorem of section 2.2.5. By combining all the previous, we have the following roadmap.

So, in order to obtain a cooperative output regulation protocol, one has to

1. Compute a solution pair (Π, Γ) to

$$\begin{cases} \Pi S &= A\Pi + B\Gamma \\ \mathbf{0} &= C\Pi + D\Gamma \\ \mathbf{0} &= H\Pi - R. \end{cases}$$

2. Compute a $\mu > 0$ such that

$$\Phi^{-1} := \begin{pmatrix} XA^T + AX + EE^T & XC^T \\ CX & -\frac{1}{\gamma^2} I \end{pmatrix} - \mu \begin{pmatrix} B \\ D \end{pmatrix} \begin{pmatrix} B^T & D^T \end{pmatrix} < 0.$$

3. Compute $F = -\mu \begin{pmatrix} B^T & D^T \end{pmatrix} \Phi \begin{pmatrix} X \\ 0 \end{pmatrix} \left[\begin{pmatrix} X & 0 \end{pmatrix} \Phi \begin{pmatrix} X \\ 0 \end{pmatrix} \right]^{-1}$

4. Compute $K = \Gamma - F\Pi$.

4.4 Dynamic output feedback with uncertainty

Theorem 8. *Let Π and Γ be a solution pair to the regulator equations*

$$\begin{cases} \Pi S &= A\Pi + B\Gamma \\ \mathbf{0} &= C\Pi + D\Gamma \\ \mathbf{0} &= H\Pi - R. \end{cases} \tag{4.12}$$

If there exist F and a $P > 0$ such that

$$\begin{pmatrix} P(A_f + B_f F H_f) + (A_f + B_f F H_f)^T P + (C_f + D_f F)^T (C_f + D_f F) & * \\ (E_f + B_f F J_f)^T P + (D_f F J_f)^T (C_f + D_f F H_f) & ** \end{pmatrix} < 0, \tag{4.13}$$

where

$$* = P(E_f + B_f F J_f) + (C_f F H_f)^T D_f F J_f$$
$$** = (D_f F J_f)^T (D_f F J_f) - \frac{1}{\gamma^2} I.$$

and

$$A_f = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, B_f = \begin{bmatrix} B & 0 \\ 0 & I \end{bmatrix}, C_f = \begin{bmatrix} C & 0 \end{bmatrix}, D_f = \begin{bmatrix} D & 0 \end{bmatrix}, \tag{4.14}$$

$$E_f = \begin{bmatrix} E \\ 0 \end{bmatrix}, H_f = \begin{bmatrix} H & 0 \\ 0 & I \end{bmatrix}, J_f = \begin{bmatrix} J \\ 0 \end{bmatrix}, F = \begin{bmatrix} D_c & C_c \\ B_c & A_c \end{bmatrix}. \tag{4.15}$$

then the network of nodes 4.4-4.5 with protocol 3.12-3.13, where $K = \Gamma - F\Pi$, is output regulated.

Proof. Let $\tilde{x}_i = x_i - \Pi w_i$, $i = 1, ..., N$, where Γ and Π satisfy 4.12. Combining this we get

$$\begin{pmatrix} \dot{\tilde{x}}_i \\ \dot{v}_i \end{pmatrix} = \begin{pmatrix} A + BD_c H & B_c \\ B_c H & A_c \end{pmatrix} \begin{pmatrix} \tilde{x}_i \\ v_i \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} \Sigma_i \tag{4.16}$$

$$= (A_f + B_f F H_f) \begin{pmatrix} x \\ w \end{pmatrix} + (E_f + B_f F J_f) d_i + \Sigma_i, \tag{4.17}$$

where $\Sigma_1 = \Pi T(z - Rw_1)$ and $\Sigma_i = \sum_{j=1}^{N} a_{ij}(w_j - w_i)$, for $i = 2, ..., N$. Lemma 3 tells us that $\Sigma_1 \to \mathbf{0}$ as $t \to \infty$ exponentially for all i. Furthermore we have

$$
\begin{aligned}
y_i &= Cx_i + Du_i && (4.18)\\
&= Cx_i + D(C_c v_i + D_c z_i) && (4.19)\\
&= Cx_i + DC_c v_i + DD_c(Hx_i + Jd_i) && (4.20)\\
&= (C + DD_c H)x_i + DC_c w_i + DD_c J d_i && (4.21)\\
&= (C_f + D_f F H_f)\begin{pmatrix} x \\ w \end{pmatrix} + (D_f F J_f)d_i. && (4.22)
\end{aligned}
$$

From inequality (4.13) and the bounded real lemma (see section 2.2.4) we can say that this statement is equivalent to the problem: find a controller such that the closed-loop system is internally stable and the H_∞ norm from d to y is stricly less than $\frac{1}{\gamma}$, i.e.

$$||(C_f + D)fFH_f)(sI - A_f - B_f F H_f)^{-1}(E_f + B_f F J_f) + (D_f F J_f)|| < \frac{1}{\gamma}.$$

From this fact we can conclude that the system mentioned above (4.16-4.18) is internally stable and thus that $\tilde{x_i}$ goes to 0. For the same reasoning as in section 4.3 we can conclude that according to the conditions in theory 8 the system is output regulated. □

We do not mention here how to build such a controller. For the approach, we refer you to section 3.3 about dynamic output feedback without uncertainty.

Chapter 5

Conclusions and future research

In this thesis we have systematically studied the output regulation problem for linear multi-agent systems, with and without uncertainty. This uncertainty appears in two different types: additive perturbation and multiplicative perturbation. The protocols we designed can be divided into dynamic state feedback en dynamic output feedback. The roadmaps at the end of each section give a straight forward manner to build a controller.

The lemma's mentioned in the preliminaries are fundamental knowledge to prove many of the theorems. Furthermore the main content is structured as follows

Protocol - System	... without uncertainty	... with uncertainty
Dynamical state feedback	Section 3.2	Section 4.3
Dynamical output feedback	Section 3.3	Section 4.4

From this we can deduce some topics for further research. Possible topics are for example the following three topics.

- Beside the dynamical state feedback and dynamical output feedback protocols, it is possible to study static relative state feedback and static relative output feedback protocols.
- It is also possible to consider nonlinear systems instead of linear systems.
- Besides the output regulation problem, there are several other distributed control problems, like the synchronization problem, formation control and distributed optimalization. These problem can also by studies with perturbed agents.

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Appendix A

Eigenvalues of the Laplacian

The Laplacian L of a digraph $\mathcal{G}$ is a matrix with some special properties. In this subsection we will note some important ones. Therefore we assume that only node 1 is connected with node 0, the leader.

Lemma 7. *The Laplacian L of $\mathcal{G}$ has a simple zero eigenvalue.*

Proof. Because of the assumption that only node 1 is connected with node 0 and node 1 does not use relative information from the other nodes, we know that L has the following form $L = \begin{bmatrix} 0 & \mathbf{0} \\ l_{21} & \tilde{\mathcal{L}} \end{bmatrix}$. Therefor zero is an eigenvalue of L with eigenvector $\mathbf{1}$, because $\sum_{j=1}^{N} l_{ij} = 0$ for all i. □

Lemma 8. *All other eigenvalues of the Laplacian L of $\mathcal{G}$ have strictly positive real parts.*

For this proof we use Gershgorin disc theorem, which reads as follows.

Theorem 9 (Gershgorin)**.** *Let $A = [a_{ij}]$ and let $R_i(A) \equiv \sum_{j=1, j\neq i}^{n} |a_{ij}|$, $1 \leq i \leq n$ denoted the deleted absolute row sums of A. Then all eigenvalues of A are located in the union of n disc*

$$Ger(A) \equiv \bigcup_{i=1}^{n} \{z \in \mathbb{R}^2 : |z - a_{ii}| \leq R_i(A)\}$$

Further we will use the term *propery SC*.

Definition 4. *A matrix $A = [a_{ij}]$ is said to have property SC if for every pair of distinct integers p, q with $1 \leq p, q \leq n$ ther is a sequence of distinct integers $p = k_1, k_2, ..., k_{m-1}, k_m = q$, $1 \leq m \leq n$, such that all of the matrix entries $a_{k_1k_2}, a_{k_2k_3}, ..., a_{k_{m-1}k_m}$ are nonzero.*

With these new definition we can formulate an essential lemma.

Lemma 9. *Let $A = [a_{ij}] \in M_n$ and suppose that λ is an eigenvalue of A that is a boundary point of $Ger(A)$. If A has property SC, then*

1. *Every Gershgorin circle passes through* λ *and*

2. *If* $Ax = \lambda x$ *and* $x = [x_i] \neq 0$*, then* $|x_i| = |x_j|$ *for all* $i, j = 1, ..., n$.

A proof of this lemma can be found in [3].

Proof of lemma 8. We first use the Gergorin disc theorem, which tells us that all the eigenvalues of L are located in the union of N discs:

$$Ger(L) \equiv \bigcup_{i=1}^{N} \{z \in \mathbb{R}^2 : |z - \sum_{j \in \mathcal{N}_i} a_{ij}| \leq \sum_{i \neq j} |a_{ij}|\}.$$

Thus all eigenvalues of L are zero or have positive real part.

□

Appendix B

Proof regarding section 3.3

Lemma 10. *If (A, B) is stabilizable and (H, A) is detectable, then there exists F such that $(A_f + B_f F H_f)$ is Hurwitz, where*

$$A_f = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, B_f = \begin{bmatrix} B & 0 \\ 0 & I \end{bmatrix}, H_f = \begin{bmatrix} H & 0 \\ 0 & I \end{bmatrix}, F = \begin{bmatrix} D_c & C_c \\ B_c & A_c \end{bmatrix}.$$

Also qe note that the converse is true, but that part is not interesting for our reasoning.

Proof. Because (A, B) is stabilizable, we know there exists G such that $(A+BG)$ is Hurwitz. From the fact that (H, A) is detectable, we know that there exists R such that $(A + RH)$ is Hurwitz. If we now choose F as

$$F = \begin{bmatrix} D_c & C_c \\ B_c & A_c \end{bmatrix} = \begin{bmatrix} 0 & G \\ -R & A + BG + RH \end{bmatrix},$$

we get

$$A_f + B_f F H_f = \begin{bmatrix} A & BG \\ -RH & A + BG + RH \end{bmatrix}.$$

Because we want to say something about the eigenvalues of this matrix, we use the fact that for any nonsingular matrix M, $\sigma[A_f + B_f F H_f] = \sigma[M^{-1}(A_f + B_f F H_f)M]$. Now we choose $M = \begin{bmatrix} I & 0 \\ I & I \end{bmatrix}$ and thus $M^{-1} = \begin{bmatrix} I & 0 \\ -I & 0 \end{bmatrix}$. Therefore we get

$$M^{-1}(A_f + B_f F H_f)M = \begin{bmatrix} A + BG & BF \\ 0 & A + RM \end{bmatrix}$$

From this we can conclude that $\sigma[A_f + B_f F H_f] = \sigma[M^{-1}(A_f + B_f F H_f)M] = \sigma[A + BG] \cup \sigma[A + RH]$. Since $\sigma[A + BG] \subset \mathbb{C}^-$ and $\sigma[A + RH] \subset \mathbb{C}^-$, also holds that $\sigma[A_f + B_f F H_f] \subset \mathbb{C}^-$. Thus there exists F such that $(A_f + B_f F H_f)$ is Hurwitz. □