The Lyapunov Exponent Test and the 0-1 Test for Chaos compared

The Lyapunov Exponent Test and the $0 - 1$ Test for Chaos compared

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Abstract

In this paper we will discuss two methods to measure chaos for dynamical systems; the Lyapunov Exponent test and the $0-1$ test. The Lyapunov Exponent test requires phase space reconstruction and has been used for a longer time, where the $0 - 1$ test is quite new and works directly with the time series. To make a comparison, we will use the logistic map, $f_a(x) = ax(1 - x)$, to show advantages and disadvantages of the two methods. In chapter 1, we will introduce the notion of chaos and see why the logistic map is a very good example when discussing chaos. After this chapter, we will introduce the two methods to distinguish between regular, i.e. periodic, dynamics and chaotic dynamics. Next to that, we will see the implementation of the tests with regard to the logistic map. A comparison between the two tests can be read in the last chapter. Here we see that, although the $0 - 1$ test seemed a better test at the start, the Lyapunov Exponent test is much easier to understand and to implement, provided that the map f is known explicitly. It is also able to determine the bifurcation points and the super attractive points, if present, whereas the $0 - 1$ test is not able to find those. However if a phase space reconstruction is not possible, the $0-1$ test can still be used and is therefore a more general test to measure chaos.

Contents

1 Introduction

In this bachelor thesis we will analyze dynamical systems. There are different types of dynamical systems. An *n'th-order autonomous time-continuous system* is defined as follows:

$$\frac{dx}{dt} = f(x), \qquad x(t_0) = x_0 \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state at time t, and $f : \mathbb{R}^n \to \mathbb{R}^n$ is called the vector field. The solution to (1) is written as $\phi_t(x_0)$ and is called the flow.

A dynamical system does not always have to be autonomous. An *n'th-order non-autonomous time-continuous dynamical system* is defined by:

$$\frac{dx}{dt} = f(x, t), \qquad x(t_0) = x_0. \tag{2}$$

Here the vector field does depend on time, unlike the autonomous case. The solution to (2) is then written as $\phi_t(x_0, t_0)$.

We can also have time-discrete systems. Such systems are defined as

$$x_{k+1} = P(x_k), \qquad k = 0, 1, 2, \dots \tag{3}$$

A good way of illustrating the behavior of a discrete dynamical system is by making a bifurcation diagram. Such a diagram is created by choosing a random initial condition, iterating this initial condition 200 times and then plotting the value of the 100 next iterates.

In this paper we will consider two tests that can measure whether or not a dynamical system displays regular dynamics or chaotic dynamics. The logistic map, a time-discrete dynamical system, is used to explain when a system is chaotic and to show how both tests work.

2 Chaos

It is often interesting to consider the long term behavior of a dynamical system. Some systems behave regularly, that is, they are periodic. When a system is periodic, it is relatively easy to predict what will happen after a while when you know some points in the beginning. However, not all systems are periodic and for these we see that there is something 'chaotic' going on. Without knowing the exact definition of chaos, one has obviously a vision in mind about what chaos looks like. We often imagine chaos to be something where there is precisely no structure at all; we observe full randomness. Even if there is some pattern, the pattern won't be of much interest. However, this is not true; where there is chaos, there are extremely beautiful things arising.

2.1 The logistic map

Let us consider the dynamical system that describes population growth. The logistic model is the most basic model and is described by the following equation:

$$N'(t) = \frac{a}{K}N(t)(K - N(T)).$$

In this equation, $N(t)$ is the number of animals in a population, a is the maximum rate of population growth and K represents a sort of "ideal" population or "carrying capacity", which is basically the maximum size of the population. By a straightforward change in variables, defining $x(t) = \frac{N(t)}{k}$, $0 \leq x(0) \leq 1$, we get the *logistic equation*

$$x'(t) = ax(1 - x).$$

For $a < 0$, x converges to 0, for $a > 0$, x converges to 1 and for $a = 0$ the size of the population is constant. This is about the easiest and most straightforward result one can get from a dynamical system.

However, it is sometimes more obvious to look at the growth of a population in steps of years for example, instead of looking at the continuous model. This is where the logistic map comes along:

$$x_{n+1} := f_a(x_n) = ax_n(1 - x_n) \qquad \text{for } 0 \leq x_0 \leq 1.$$

This discrete time equivalent of the logistic equation has some very peculiar features and will be the example we will use most throughout this paper.

This logistic map seems very easy and basic, but the solutions turn out to be not. Let us consider the bifurcation diagram as shown in Figure 1. This diagram is created by choosing a random initial condition, iterating this initial condition 200 times and then plotting the value of the next 100 iterates. We see that for $1 \leq x \leq 3$, there is just one stable fixed point. But when a is made bigger than 3, a period doubling bifurcation occurs. This keeps happening until at some point there is only chaos. This point is called the Feigenbaum point and is measured to be at $a = s_\infty = 3.5699456\ldots$. For $s_\infty < a \leq 4$, the chaos appears to be bounded. A special case is $a = 4$: Here we observe chaos on the whole interval $[0, 1]$.
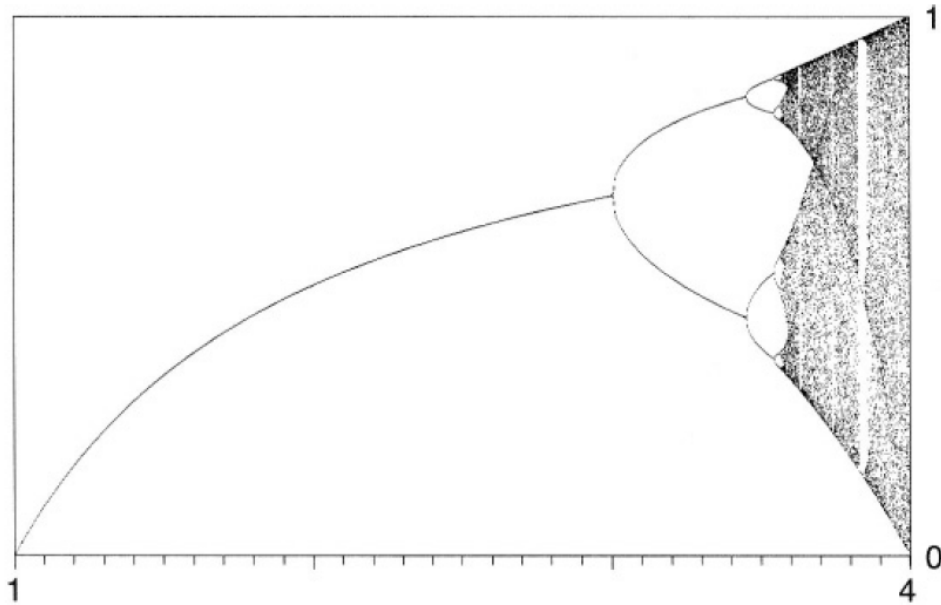
**Figure 1** – The bifurcation diagram of the logistic map.

2.2 When is a system chaotic?

But when exactly is a system chaotic? The most important condition is that the system depends sensitively on the initial values. But there are other conditions as well. If the system has periodic orbits, then they should be dense. We also need the system to be transitive. In the following definition we will state these along with their formal definition.

Definition 2.1. *A system $f(x)$ on the metric space (X, d) is chaotic if the following three statements hold:*

1. *The periodic orbits are dense. Formally:*
   $\forall x \in X,\ \forall \varepsilon > 0,\ \exists p \in X\ \&\ \exists n > 0$ *such that* $f^n(p) = p$ *and* $d(x, p) < \varepsilon$;
2. *The system is transitive. Formally:*
   For every two open subsets $U_1, U_2 \subset X$, there is an $n > 0$ such that $f^n(U_1) \cap U_2 \neq \emptyset$;
3. *The system is sensitive dependent on the initial values. Formally:*
   $\exists \beta$ *such that* $\forall x_0 \in X,\ \forall \varepsilon > 0,\ \exists x_1\ \&\ \exists n > 0$ *such that* $d(x_0, x_1) < \varepsilon$ *and* $d(f^n(x_0), f^n(x_1)) \geq \beta$.

It is not always easy to use only this definition. When it is too difficult to prove that there is chaos with the previous statements, we can use other things as well. One of the methods is using an equivalence with another system for which you càn proof chaos by using the definition only. When two systems are equivalent and we can prove that there is chaos for one of the two, then the other system must also be chaotic[3].

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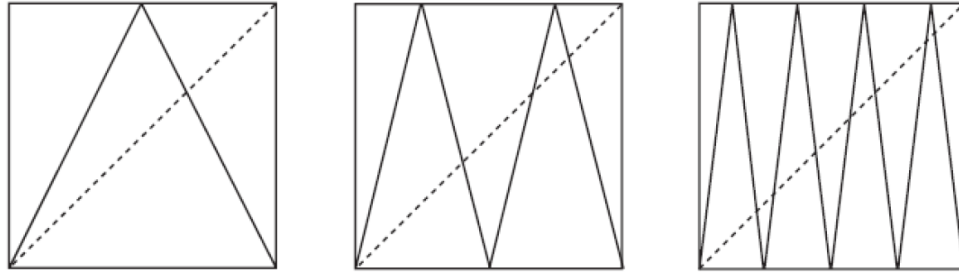
Figure 2 – The first, second and third iterate of the tent map

Let's see this in practice for the logistic map where $a = 4$. This is a special case of the logistic map since for this map $f_4(x)$ we observe chaos on the interval $[0, 1]$. It is very difficult to prove that there is chaos for $f_4(x)$ by using the formal definition, but we can prove that there is chaos for another map which turns out to be equivalent to $f_4(x)$. This map is called the tent map:

$$(x) = \begin{cases} 2x & \text{if } 0 \le x < \frac{1}{2} \\ 2 - 2x & \text{if } \frac{1}{2} \le x \le 1 \end{cases} \tag{4}$$

In Figure 2 we can see the first, second and third iterate of the tent map, which is useful to understand the proof of our statement that the tent map is chaotic.

Proof. We can prove that the three statements of the definition hold for the tent map:
1. (Density of periodic points) T^n maps each interval $[\frac{k-1}{2^n}, \frac{k}{2^n}]$ to $[0, 1]$ for $k = 1, ..., 2^n$. Therefore, T^n intersects the line $y = x$ once in each interval. As a result, each interval contains a fixed point of T^n or equivalently, a periodic point of T of period n. Therefore, periodic points of T are dense in $[0, 1]$.
2. (Transitivity) Let U_1 and U_2 be open sub-intervals of $[0, 1]$. For n sufficiently large and for some k, U_1 contains an interval of the form $[\frac{k}{2^n}, \frac{k+1}{2^n}]$. Therefore, T^n maps U_1 to $[0, 1]$ which contains U_2. This means that the tent map is transitive.
3. (Sensitive dependence on initial conditions) Let $x_0 \in [0, 1]$. We will show that a sensitivity constant $\beta = 0.5$ works. As in (2), any open interval U of the form $[\frac{k}{2^n}, \frac{k+1}{2^n}]$ around x_0 is mapped by T^n to $[0, 1]$ for some sufficiently large n. Therefore, there exists $y_0 \in U$ such that $|f^n(x_0) - f^n(y_0)| \ge 0.5 = \beta$. So the tent map has a sensitive dependence on the initial conditions.

$\square$

This proves the fact that the tent map is chaotic. So we only have to formulate a conjugacy with $f_4(x) = 4x(1 - x)$ in order to state that $f_4(x)$ is chaotic. Two systems f and g are conjugate when $f \circ h = h \circ g$. Indeed, there exists a conjugacy

via $h(x) = \sin^2(\frac{\pi x}{2}) = \frac{1}{2}(1 - \cos(\pi x))$ since

$$
\begin{aligned}
h(T(x)) &= \tfrac{1}{2}(1 - \cos(2\pi x)) \\
&= 1 - \cos^2(\pi x) \\
&= 1 + \cos(\pi x) - \cos(\pi x) - \cos^2(\pi x) \\
&= 4(\tfrac{1}{2} - \tfrac{1}{2}\cos(\pi x))(1 - \tfrac{1}{2} + \tfrac{1}{2}\cos(\pi x)) \\
&= 4(\tfrac{1}{2}(1 - \cos(\pi x)))(1 - \tfrac{1}{2}(1 - \cos(\pi x))) \\
&= f_4(h(x))
\end{aligned}
$$

We used $T(x) = 2x$, but the same is true for $T(x) = 2 - 2x$. We have now shown that the logistic map for $a = 4$ is conjugate with the tent map, which is chaotic on $[0, 1]$, so we can conclude that the logistic map for $a = 4$ is also chaotic on $[0, 1]$. We can also see this intuitively when looking at the first two iterates of the logistic map (Figure 3) and compare these to the first two iterates of the tent map (Figure 2).
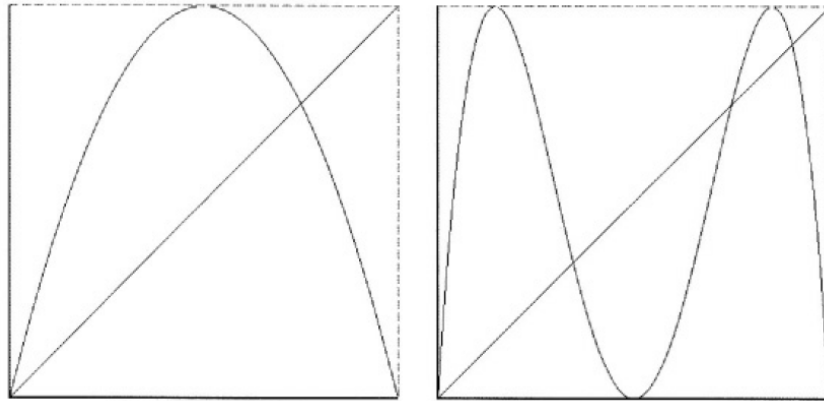
**Figure 3** – The first and second iterate of $f(x) = 4x(1 - x)$

The resemblance is striking and this shows that we can prove that there is chaos in a similar manner for the logistic map as for the tent map.

2.3 Superattractive points

Let us go back to the bifurcation diagram (Figure 1) of the logistic map. The stable, or attractive, points are plotted in this diagram. These are the stable fixed points of $f_a(x) = ax(1 - x)$. To calculate the fixed points of a map, you have to solve $f_a(x) = x$, which gives you two fixed points: $p_0 = 0$ and $p_1 = 1 - \frac{1}{a}$.

A fixed point x^* is stable, or attractive, if $|f'(x^*)| < 1$ and it is unstable, or repelling, if $|f'(x^*)| > 1$. If $|f'(x^*)| = 1$, then the stability of the fixed point depends $|f''(x^*)|$. This can be explained by looking at the Taylor series of f_a around the point x^*. Besides, we notice that derivatives of order higher than two vanish, since we are
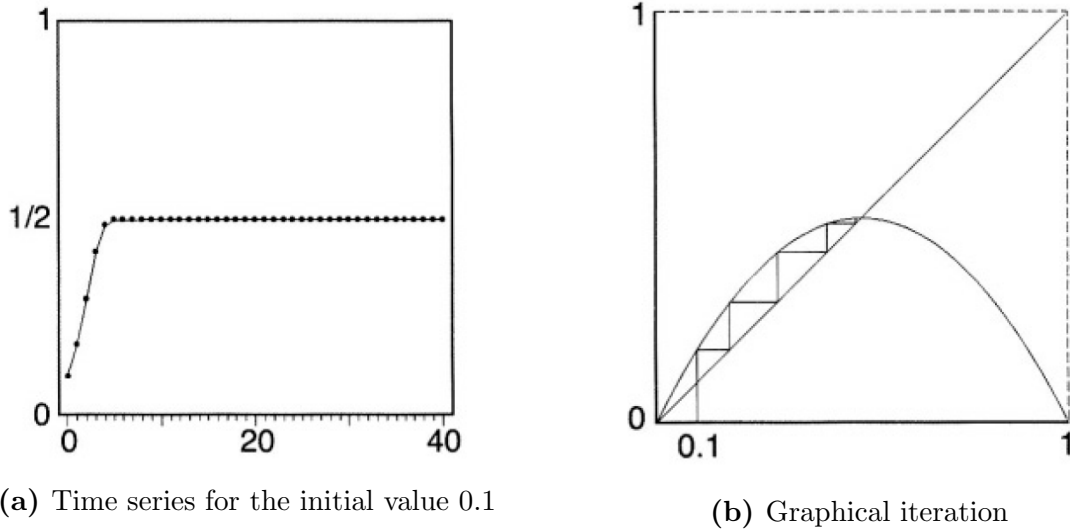
(a) Time series for the initial value 0.1

(b) Graphical iteration

Figure 4 – Convergence to super attractive fixed point $x^* = 0.5$ with $a = 2$

dealing with a quadratic function.

$$f_a(x^*) = x^*$$
$$f_a(x) = x^* + f_a'(x^*)(x - x^*) + f_a''(x^*)(x - x^*)^2$$
$$x_{n+1} = x^* + f_a'(x^*)(x_n - x^*) - 2a(x_n - x^*)^2$$
$$|x_{n+1} - x^*| = |f_a'(x^*)||x_n - x^*| + |2a||x_n - x^*|^2$$

When $|f'(x^*)| \neq 0$, we can neglect higher order terms, because $|x_n - x^*|$ will be smaller than 1. Hence we can see that for $|f'(x^*)| < 1$, the distance to the fixed point is decreasing, while if $|f'(x^*)| > 1$, it is increasing. When $|f'(x^*)| = 1$, the second order derivative determines whether this distance is increasing or decreasing.

For the logistic map, we can now calculate the stability of its fixed points p_0 and p_1. Since $f'(x) = a(1 - 2x)$, we see that $p_0 = 0$ is stable when $a < 1$ and unstable when $a > 1$. For p_1 we calculate $f'(p_1) = 2 - a$, so we can conclude that p_a is stable when $|2 - a| < 1$, i.e. when $a \in (1, 3)$, and unstable elsewhere. We thus have a bifurcation at $a = 1$.

After $a = 3$ a 2-cycle arises. To calculate this 2-cycle one has to compute the fixed points of $f_a^2(x)$. It is calculated to be stable for $3 < a < 1 + \sqrt{6}$. When that 2-cycle is not stable anymore, one computes the fixed points of $f_a^4(x)$. This will give us a 4-cycle for $f_a(x)$, which has again an even smaller range of stability. This continues until we arrive at the Feigenbaum point.

But something interesting happens when the fixed point is actually the top of the parabola of $f_a(x)$, as seen in Figure 4b. When this is the case, the fixed point is approach much faster. Indeed, we say that a fixed point x^* is *super attractive* if $f'(x^*) = 0$. Since the parabola has it maximum at $x_{\max} = 0.5$, we have to solve the equation $0.5 = 0.5a(1 - 0.5)$.

(a) The widest window

(b) Zooming in reveals more self-similarity
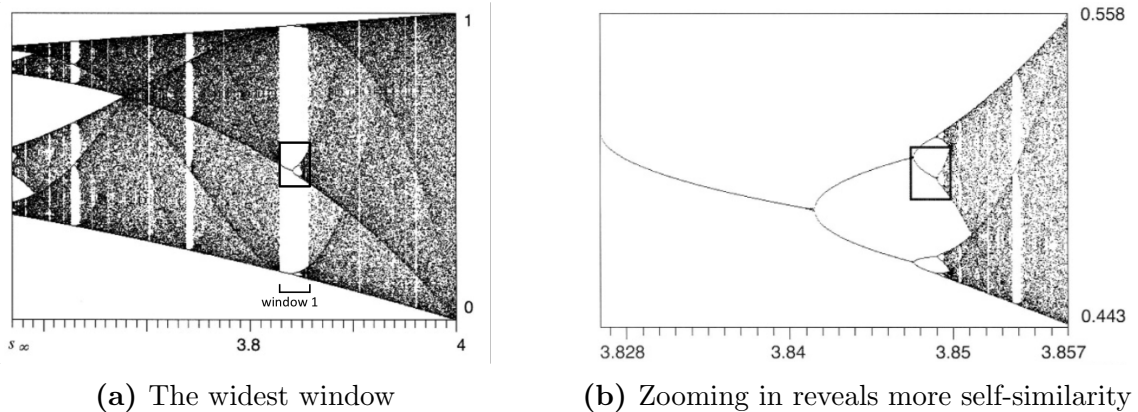
Figure 5 – The widest window of the bifurcation diagram of the logistic map

The solution for this equation is $a = 2$. To see what really happens here, we take a look at Figure 4. We see that for $a = 2$, the fixed point is approached much faster than for $a = 1.75$ or $a = 2.75$ for example! We can again explain this by looking at the Taylor series: at this point $|f'(x^*)| = 0$, so the first order term vanishes! That means we're left with only the second order term and hence the convergence is quadratic (which is much faster). We call the value of a for which quadratic convergence to the super attractive fixed point $x^* = 0.5$ occurs a *super attractive parameter* and name it $s_1 = 2$. This super attractive parameter is not the only super attractive parameter. In each part where there is a new cycle, there is one such parameter. For $1 < a < 3$ we have $s_1 = 2$, for $3 < a < 1 + \sqrt{6}$ the super attractive point is the solution of $f_a^2(0.5) = 0.5$, which is $s_2 = 1 + \sqrt{5}$. In each segment we find one super attractive parameter.

2.4   Windows in the chaotic part

Once a is past the Feigenbaum point, we observe chaos. However when we take a closer look to that part of the bifurcation diagram (Figure 5a), we see that there are some gaps where there seems to be no chaos. Let us take a closer look to those gaps.

Figure 5b shows that the gap reveals a bifurcation diagram which is very similar to the entire bifurcation diagram. This leads to the idea that there are similar dynamics here. There are three periodic attractors before the period doubling starts and these turn out to be the fixed points of $f_a^3(x)$. This window is not the only one; there are uncountable many windows. We won't go into much detail and we refer the reader to [6] for more information on the windows.

3   Tests for chaos

As seen in the previous chapter, it is not always easy to use the definition only to check for chaos. It is therefore essential to come up with other tests that are easier to use. In this section, we will consider two tests, the Lyapunov Exponent test and the $1 - 0$ test, which can both be used to determine the dynamics of a system.

3.1   The Lyapunov Exponent

One of the most used tests is the Lyapunov Exponent test, since it is easy to implement if the map f is known explicitly.

3.1.1   Description of the test

In this paper we will focus on discrete time systems, since we use the discrete logistic map to show the workings of the tests. The Lyapunov Exponent test for one-dimensional maps is based on the average exponential growth for n iterations.

To see this, we begin with a starting condition x_0, and add a perturbation ε, such that $x_0 + \varepsilon$ is the perturbed starting condition. The error after n iterations is then $f^n(x_0 + \varepsilon) - f^n(x_0)$, and the relative error $\frac{f^n(x_0+\varepsilon)-f^n(x_0)}{\varepsilon}$. It is of course interesting what will happen when the perturbation is infinitesimally small. When a system is regular, the relative error won't be too high. But when a system is chaotic, it is sensitive dependent on initial conditions and therefore the relative error after n iterations will be very big. Since we want to look at an infinitesimally small error we might as well take the limit where $\varepsilon \to 0$: $\lim_{\varepsilon \to 0} \frac{f^n(x_0+\varepsilon)-f^n(x_0)}{\varepsilon}$. This is actually the derivative of f^n evaluated at x_0; $(\frac{d}{dx}f^n(x))|_{x=x_0}$. We know that $f^n(x_0) = f^{n-1}(f(x_0)) = f^{n-1}(x_1)$ etc., so we can use the chain rule to obtain

$$\left(\frac{d}{dx} f^n(x) \right) \Big|_{x=x_0} = f'(x_{n-1})f'(x_{n-2}) \cdots f'(x_0),$$

which is the product of the local growth factors. A growth factor smaller than 1 corresponds to contraction, whereas a growth factor greater than 1 shows expansion. The average exponential growth factor for n iterates is then

$$\frac{1}{n}(\ln |f'(x_{n-1})| + \cdots + \ln |f'(x_0)|).$$

The logarithm of a value higher than 1 will be positive and the logarithm of a value lower than 1 will be negative. The Lyapunov Exponent is than the limit of this for $n \to \infty$, which gives us the following definition:

Definition 3.1. *The* Lyapunov Exponent *of a discrete time system* $x_{n+1} = f(x_n)$ *is given by*

$$\lambda = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln |f'(x_i)|.$$

Lyapunov Exponents are used to measure chaos. This depends on the sign of λ as follows:

- $\lambda > 0$, $\{x_n\}$ shows chaotic behavior;
- $\lambda < 0$, $\{x_n\}$ shows periodic behavior;
- $\lambda = 0$, a bifurcation occurs.

3.1.2 The algorithm

In the script below we see how a Lyapunov Exponent can be calculated using a computer. As an input, we plug in the function, the derivative of the function, the initial value and the number of iteration we wish.

```
function lambda=lyapunov(f,df,x0,iter)
s = 0;
x = x0;
for i = 1:200
    x = f(x);
end
for i = 1:iter
    s = s + log(abs(df(x)));
    lambda = s/i;
    x = f(x);
end
```

Since we will be using this script only for the logistic map in this paper, the script was slightly altered such that one can plug in the value of a as well.

```
function lambda=lyapunov(f,df,a,x0,iter)
s = 0;
x = x0;
for i = 1:200
    x = f(x,a);
end
for i = 1:iter
    s = s + log(abs(df(x,a)));
    lambda = s/i;
    x = f(x,a);
end
```

According to the script which can be found in Appendix A we can see that in order to get the error between the actual value ($\ln(2)$, see below for the confirmation of this) and the numerical limit smaller then 10^{-6}, we need 365 iterates. For $a = 4$ and 365

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iterations, we find that $\lambda = 0.6932\ldots$ This value corresponds to the value of $\ln(2)$. For the logistic map, we can confirm this very easily. We have already seen that the logistic map for $a = 4$ is conjugate with the tent map. The Lyapunov Exponent for the tent map is easily calculated (recall that the tent map is given in subsection 2.2, equation (4)) We know that $|T'(x)| = 2$, $\forall x \neq \frac{1}{2}$. The Lyapunov Exponent is now determined as follows:

$$\lambda = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln |T'(x_i)|$$

$$= \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln(2)$$

$$= \lim_{n \to \infty} \frac{1}{n} \cdot n \cdot \ln(2)$$

$$= \ln(2)$$

How does this relate to the logistic map for $a = 4$? In subsection 2.2 we stated that there exists a conjugacy via $h(x) = \sin^2(\frac{\pi x}{2})$ for the logistic map f_4 and the tent map $T(x)$. If we take $\phi(x)$ to be the inverse of $h(x)$, where $\phi(x) = \frac{2}{\pi} \arcsin(\sqrt{x})$, we see that

$$\phi(f_4(x_n) = T(\phi(x_n))$$
$$\Rightarrow \phi'(f_4(x_n)) \cdot f_4'(x_n) = T'(\phi(x_n)) \cdot \phi'(x_n)$$
$$\Rightarrow f_4'(x_n) = T'(\phi(x_n)) \cdot \frac{\phi'(x_n)}{\phi'(f_4(x_n))}.$$

The Lyapunov Exponent of $f_4(x)$ is then calculated as follows:

$$\lambda = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln |f_4'(x_i)|$$

$$= \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln \left| T'(\phi(x_i)) \cdot \frac{\phi'(x_i)}{\phi'(f_4(x_i))} \right|$$

$$= \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln |T'(\phi(x_i))| + \ln |\phi'(x_i)| - \ln |\phi'(x_{i+1})|$$

$$= \lim_{n \to \infty} \frac{1}{n} \ln |\phi'(x_0)| - \lim_{n \to \infty} \frac{1}{n} \ln |\phi'(x_n)| + \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln |T'(\phi(x_i))|$$

$$= \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \ln |T'(\phi(x_i))| \qquad (*)$$

$$= \ln(2).$$

$(*)$ NOTE: It is clear that $\lim_{n \to \infty} \frac{1}{n} \ln |\phi'(x_0)| = 0$ when $x_0 \neq 0$ and $x_0 \neq 1$, since $\ln |\phi'(x_0)|$ is constant. However, it is not at all obvious that $\lim_{n \to \infty} \frac{1}{n} \ln |\phi'(x_n)| = 0$! In almost all cases this will indeed be 0, but there are certain requirements for this limit to exist. See [1] for more details on this.
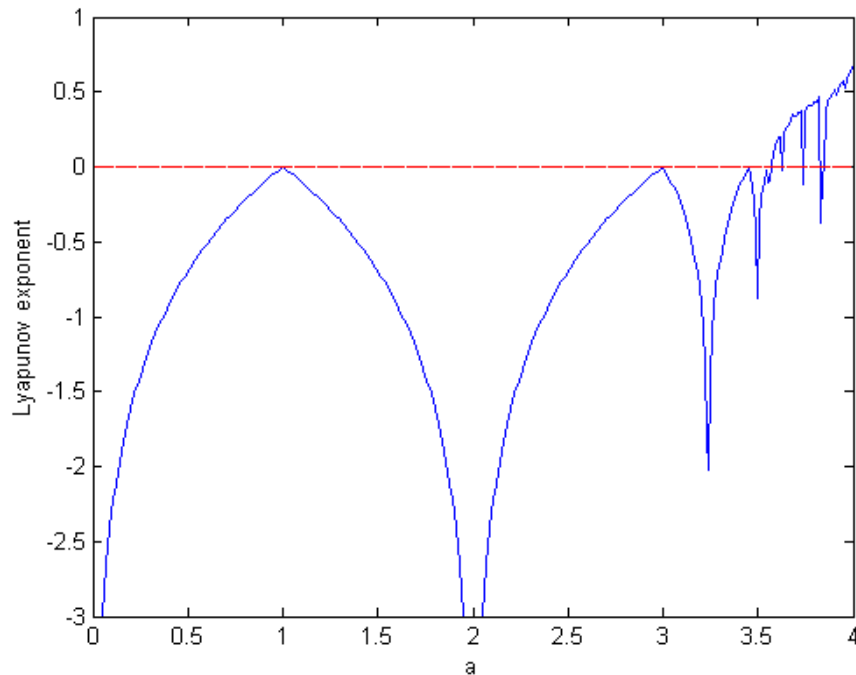
Figure 6 – Plot of the Lyapunov Exponent versus a for the logistic map.

3.1.3 Bifurcations, super attractive points and the windows

In Figure 6 we see three very interesting things. First of all, we observe for $0 < a < 3.57$ some Lyapunov Exponents to be 0. This corresponds to the bifurcations occurring for those values of a. For example at $a = 3$, where there is a period doubling bifurcation and from where a 2-cycle appears.

We see also that for some values of a the Lyapunov Exponent is in fact $-\infty$. These are characteristic for the super attractive points we have seen in subsection 2.3. This is not hard to see. When we take for example $s_1 = 2$, we have the fixed point to be $x^* = 0.5$. For this fixed point $|f'(x^*)| = 0$ holds, so when $\lim_{i \to \infty} x_i = x^* = 0.5$ (and this value is approached very fast), then $\lim_{i \to \infty} \ln |f'(x_i)| = -\infty$. It is easily seen that the Lyapunov Exponent for $a = 2$ is then $\lambda = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln |f'(x_i)| = -\infty$. The same happens for the other super attractive points.

The third interesting fact arises after the Feigenbaum point. From this point, there is chaos, but not everywhere, as seen in subsection 2.4. These windows are clearly seen in Figure 6.

3.2 The $0-1$ test

Where the Lyapunov Exponent test needs phase space reconstruction in order to determine whether a dynamical system is chaotic, for the $0-1$ test we only need to know the time series $\phi(n)$ for $n = 1, 2, \ldots$ of the dynamical system. Basically, this test provides a 2-dimensional system derived from $\phi(n)$ for which we can define the mean square displacement $M(n)$. The growth rate of $M(n)$ will give us knowledge about the dynamics of the system we started with.

3.2.1 Description of the test

The $0-1$ test uses the time series $\phi(n)$ for $n = 1, 2, \ldots$ to drive the 2-dimensional system

$$
\begin{aligned}
p(n+1) &= p(n) + \phi(n+1)\cos(cn), \\
q(n+1) &= q(n) + \phi(n+1)\sin(cn),
\end{aligned}
\tag{5}
$$

where $c \in (0, 2\pi)$ is fixed. The mean square displacement of this 2-dimensional system is given as follows:

$$
M(n) = \lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} ([p(j+n) - p(j)]^2 + [q(j+n) - q(j)]^2), \quad n = 1, 2, 3, \ldots.
$$

It's growth rate is then:

$$
K = \lim_{n\to\infty} \frac{\log(M(n))}{\log(n)}.
$$

In general, K takes either the value $K = 0$ or $K = 1$, where $K = 0$ means that the system is regular and $K = 1$ means that the system is chaotic [2]

The 2-dimensional system as seen in (5) is bounded when the time series represent regular dynamics. The mean square displacement $M(n)$ is then bounded as well and returns $K = 0$ as a growth rate. When the system we consider is chaotic, the 2-dimensional system as seen in (5) behaves approximately like a 2-dimensional Brownian motion, which can be seen as the random motion of a particle in a fluid as a result from collision with other particles. The mean square displacement of such a diffusive Brownian motion grows linearly, and will give us the growth rate $K = 1$. All in all we see the following cases:

Underlying dynamics	Dynamics of p(n) and q(n)	M(n)	K
regular	bounded	bounded	0
chaotic	diffusive	linear	1

3.2.2 The algorithm

The algorithm for this test is slightly more involved than the algorithm used for the Lyapunov Exponent test. First we have to solve the 2-dimensional system (5). We

obtain:

$$p_c(n) = \sum_{j=1}^{n} \phi(j) \cos(jc), \quad q_c(n) = \sum_{j=1}^{n} \phi(j) \sin(jc).$$

The following script provides two arrays p and q, where the i'th entry of p is equal to $p_c(i)$ and similar for q:

```
function [p,q]=pq(phi,c,n,N)
co = zeros(n+N,1);
si = zeros(n+N,1);
p = zeros(n+N,1);
q = zeros(n+N,1);

for i = 1:N+n
    co(i) = cos(i*c);
    si(i) = sin(i*c);
end

p(1) = phi(1)*co(1);
q(1) = phi(1)*si(1);

for i = 2:N+n
    p(i) = p(i-1)+phi(i)*co(i);
    q(i) = q(i-1)+phi(i)*si(i);
end
```

The following function calculates the growth rate K for a fixed value of n. In this script, we can additionally plug in the value of a for the logistic map. This script also plots p versus q. These plots show very clearly the difference between regular and chaotic dynamics. in 7, we can see these plots for the logistic map for two different values of a. The script for $\phi(n) = ax_n(1 - x_n)$ can be found in Appendix A.

```
function K=searchK(a,n,N,x0,c)
s = 0;
phi = phiscript(a,n,N,x0);
[p,q] = pq(phi,c,n,N);
plot(p,q)

for i = 1:N
    s = s + ((p(i+n)-p(i))^2 + (q(i+n)-q(i))^2);
end

M = s/N;
K = log(M)/log(n);
```
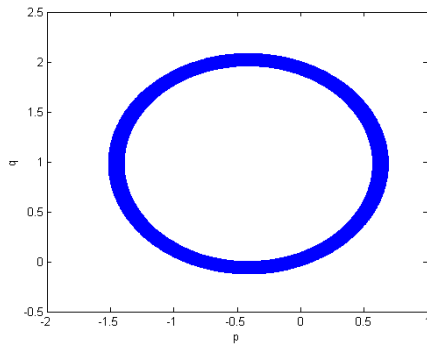
(a) Regular dynamics at $a = 3.55$
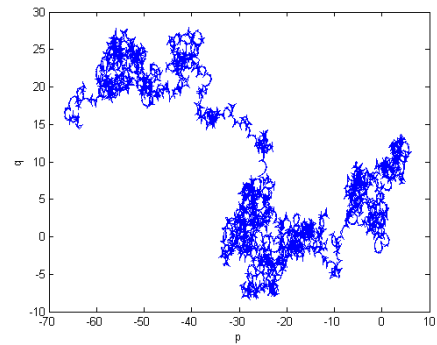
(b) Chaotic dynamics at $a = 3.97$

Figure 7 – Plot of p versus q for the logistic map.
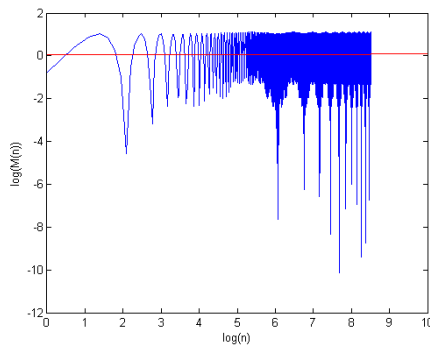
(a) The slope of the red line is approx. 0, which corresponds to the regular dynamics of $a = 3.55$.
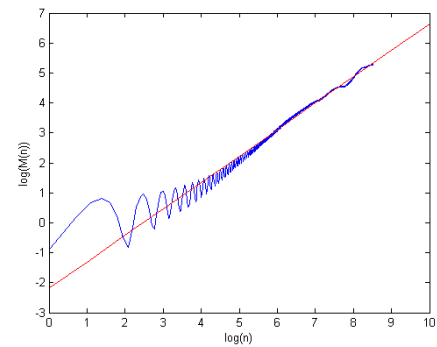
(b) The slope of the red line is approx. 1 here, which corresponds to the chaotic dynamics at $a = 3.97$.

Figure 8 – Plot of $\log(M(n))$ versus $\log(n)$ for the logistic map. The red line is the fitted straight line which determines the value of K by its slope.

However, when we just stop at some random value of n, we might stop at an inconvenient value of M. In Figure 9a and Figure 9c the value of K was calculated by this method and plotted versus different values of c. One can see that it does not always turn out right. Therefore, it is better to determine K by fitting a straight line to the graph of $\log(M(n))$ versus $\log(n)$. The following script uses the previous to calculate M for different values of n, after which it plots $\log(M(n))$ versus $\log(n)$ together with the fitted straight line. In Figure 8 we see these plots for the logistic map. 8a corresponds to $a = 3.55$, i.e. regular dynamics, and 8b corresponds to the chaotic dynamics at $a = 3.97$. Finally, the script returns the value of the slope of the straight line, which will be the value of K.

```
function Kslope=plotMn(a,N,x0,c,stapgrootte)

i = 1;
groot = ceil(N/stapgrootte);
Marray = zeros(groot,1);
Narray = zeros(groot,1);


for n = 1:stapgrootte:N
    [K,M] = findK(a,n,N,x0,c);
    Marray(i) = log(M);
    Narray(i) = log(n);
    i = i+1;
end

coefs = polyfit(Narray,Marray,1);
plot(Narray,Marray,'b',0:10,polyval(coefs,0:10),'r')
Kslope=coefs(1);
```

In Figure 8 we used $N = 5000$, $x_0 = 0.7$ and $c = 0.8$. For $a = 3.55$, the slope corresponding to the value of K is then $0.0036\ldots$, and for $a = 3.97$ it is $0.8831\ldots$. These values are very close to the expected values of 0 and 1, so we can see that this test is highly accurate. Going back to Figure 9, we see now that the method of the fitted straight line is, especially for the regular case, much better than when the first method is used. For the chaotic case, Figure 9c and 9d, we see that the second method shows more deviations than when the first method is used, but the mean is closer to 1. In [2] another method to compute K was presented, which will again be better and will give an even more accurate value for K than the method with the fitted straight line.

### 3.2.3   Bifurcations, super attractive points and the windows

In Figure 10 we see the value of K for different values of a, starting at $a = 3.3$. The parts where there are regular dynamics are clearly seen in this graph, K is zero for all values of a below the Feigenbaum point. The windows in the chaotic part where there is no chaos are also visible. Where for the Lyapunov Exponent test one was also able to see the bifurcations and the super attractive points, here this is not the case. The test does what it is made for and provides no extra's.
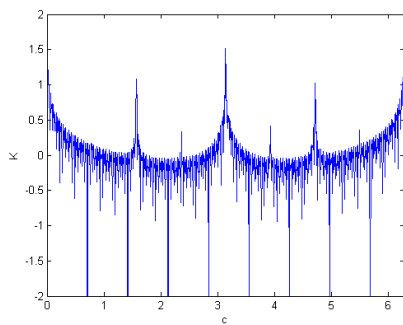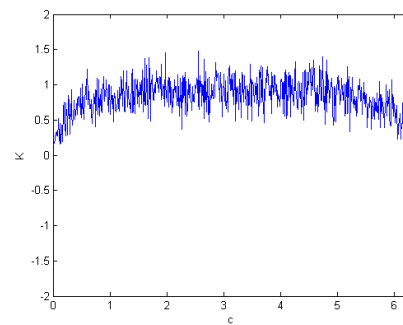
(a) Plot for $a = 3.55$, corresponding to regular dynamics, using the first method.
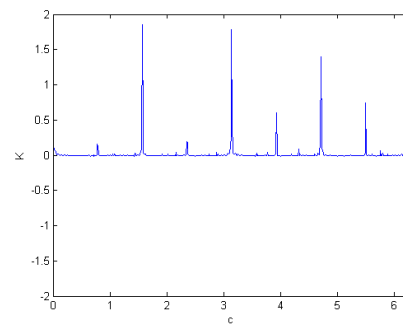
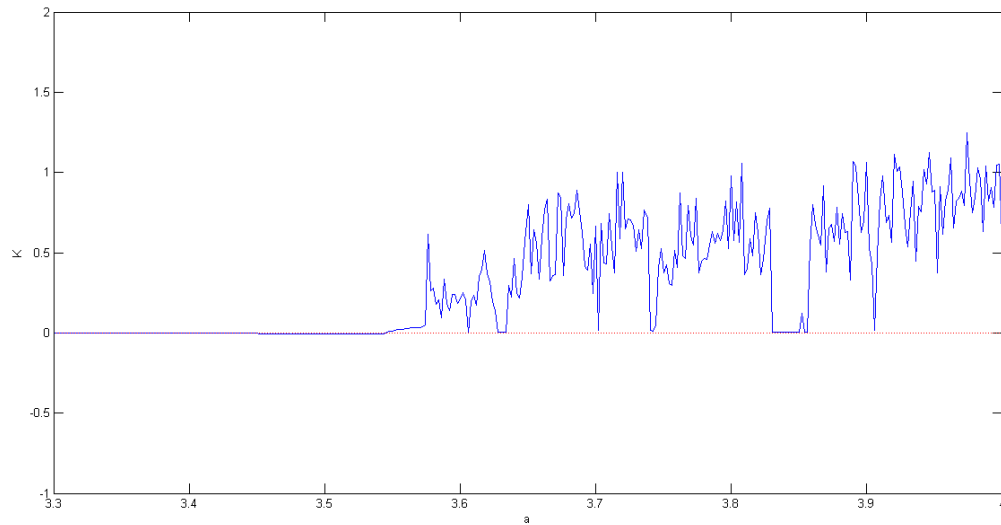**(b)** Plot for $a = 3.55$, corresponding to regular dynamics, using the slope of the red line.
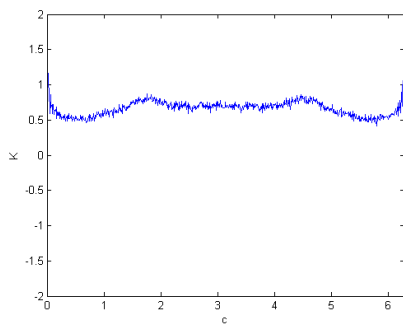
(c) Plot for $a = 3.97$, corresponding to chaotic dynamics, using the first method.

(d) Plot for $a = 3.97$, corresponding to chaotic dynamics, using the slope of the red line.

Figure 9 – Plot of K versus c for the logistic map.

Figure 10 – Plot of K versus a for the logistic map for $3.3 \leq a \leq 4$

4 Comparison

Now that we have studied the two tests, the Lyapunov Exponent test and the $0 - 1$ test, we are ready to compare the two. Both tests have clear advantages and disadvantages. We will compare the test for ease of understanding the methods, ease of implementation and computation time, generality, accuracy and insight in the dynamics.

For the Lyapunov Exponent test, it is very clear to understand how this test was arrived at (subsection 3.1.1). It does not involve very difficult mathematics and it seems like a logical way to measure chaos. The $0 - 1$ test on the other hand is much more complex. The reasons to construct a two-dimensional system from the time series are not very clear and the proofs for why the test indeed works involve more advanced mathematics. This works highly in favor of the Lyapunov Exponent test.

Also when we look at the implementation of the two tests, we see that the script that was used for the Lyapunov Exponent test is much smaller than the scripts that were used for the $0 - 1$ test. Not only do we need a longer and more involved script, we also have some variables on which the test depends, namely c, n and N. The fact that you need to plot a fitted straight line through $\log(M(n))$ versus $\log(n)$ makes it even more complex. We also measured the computation time for both tests and concluded that the computation time for the $0 - 1$ test is significantly longer than the time needed to compute Lyapunov Exponents.

However, the $0 - 1$ test is way more general than the Lyapunov Exponent test. We did not see this very clearly since we only used the logistic map to show the workings of the tests, but we do observe that for the Lyapunov Exponent test we needed the map $f_a(x)$ explicitly, whereas the $0 - 1$ test only needs the time series. This means that even when we only have a data set of successive x_i's, we can still

determine whether we are dealing with a chaotic system or a regular system. This is the main advantage of the $0 - 1$ test and probably the reason why it is invented.

Both test are accurate. For the parts where the logistic map is chaotic, no test returns that the system is regular, and vice versa. We can see this for the Lyapunov Exponent test and for the $0 - 1$ test in Figure 6 respectively Figure 10. Both test therefore return correct answers to the question *"Is this system chaotic or not?"*.

The Lyapunov Exponent gives a lot of extra information about the dynamics of the system than the $0-1$ test does. Of course, the $0-1$ test does its job and returns a 0 for regular and a 1 for chaotic dynamics, but that is all. The Lyapunov Exponent also shows the bifurcation points of the dynamical system (here the Lyapunov Exponent is 0) and the super attractive points (where the Lyapunov Exponent is $-\infty$). These extra's are a very nice aspect of the Lyapunov Exponent test.

We can conclude that when a phase space reconstruction is possible, the Lyapunov Exponent test has some major advantages compared to the $0 - 1$ test; but when this is not possible, we can still use the $0 - 1$ test and the accuracy was maintained.

5   Conclusion

In this paper we have examined the notion of chaos for dynamical systems. The main criterion for a system to be chaotic is that it is sensitive dependent on initial conditions. This criterion can be tested in two ways, by using the Lyapunov Exponent test and by using the $0-1$ test. We have used the logistic map to show how both tests work. Although the $0 - 1$ test is much more general and is suitable for the analysis of many different dynamical systems, including experimental data, the Lyapunov Exponent test appears to be more elegant in understanding and in use, and has some beneficial extra's, for example it's ability to show super attractive fixed points. If a phase space reconstruction is possible, the Lyapunov Exponent test is highly favorable, but if a phase space reconstruction is not possible, the $0 - 1$ test can be used.

References

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A   Matlab codes

The script that was used for computing how many iterations had to be used in order to get the error smaller that 10^{-6}:

```
a = 4;
i = 1;
s = 0;
x = 0.7;
lambda = 0;
err = 1000;
tol = 10^-6;

for j=1:200
    x = a*x*(1-x);
end

while err>tol
    s = s + log(abs(a*(1-2*x)));
    lambda = s/i;
    x = a*x*(1-x);
    i = i+1;
    err = abs(lambda-log(2));
end
lim
i
```

The following function makes an array in which $\phi(n) = ax_n(1 - x_n)$:

```
function phi=phiscript(a,n,N,x0)
phi = zeros(n+N,1);
x = x0;

for i = 1:200
    x = a*x*(1-x);
end

for i = 1:N+n
    phi(i) = x;
    x = a*x*(1-x);
end
```

The following script was used to plot K versus a for a given value of c as seen in Figure 10:

```
function a=differenta(c,N,x0,nsteps,a0,a1,stapgrootte2)
Karray = zeros(nsteps,1);
Aarray = zeros(nsteps,1);

for i = 1:nsteps
    a = a0+i/nsteps*(a1-a0);
    Kslope = plotMn(a,N,x0,c,stapgrootte2);
    Karray(i) = Kslope;
    Aarray(i) = a;
end

plot(Aarray,Karray,'b',Aarray,0,'r')
axis([3.3 4 -1 2])
```

The following script was used to produce Figure 6, where we see the Lyapunov Exponent plotted versus a for the logistic map:

```
function lyap=lya(f,df,x0,iter,stapgrootte)
lyap = ceil(4/stapgrootte);
Aarray = zeros(lyap,1);
Larray = zeros(lyap,1);
i = 1;

for a = 0:stapgrootte:4
    lim = lyapunov(f,df,a,x0,iter);
    Aarray(i) = a;
    Larray(i) = lim;
    i = i+1;
end

plot(Aarray,Larray,'b',Aarray,0,'r')
axis([0 4 -3 1])
```

The next script plots c versus K as seen in Figure 9:

```
function kenc=kc(a,N,x0,stapgrootte1,stapgrootte2)
i = 1;
f = 2*pi;
kenc = ceil(f/stapgrootte1);
Karray = zeros(kenc,1);
Carray = zeros(kenc,1);

for c = 0:stapgrootte1:f
    Kslope=plotMn(a,N,x0,c,stapgrootte2);
    Karray(i) = Kslope;
    Carray(i) = c;
    i = i+1;
end

plot(Carray,Karray)
axis([0 2*pi -2 2])
```