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Generalized Coherent States

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Abstract

In quantum physics coherent states are quantum states which have properties that closely resemble classical description. This thesis studies the notion of generalized coherent states, also called Gilmore-Perelomov coherent states. Firstly, we consider the coherent states associated to the harmonic oscillator. These can be described as the set of states resulting from a representation of the Weyl-Heisenberg group acting on the ground state of the harmonic oscillator. Gilmore-Perelomov coherent states generalize this idea; for an arbitrary Lie group we can define generalized coherent states as the states resulting from the action of an irreducible, unitary representation of a Lie group on some fixed state. If the fixed state is chosen to have minimal uncertainty, we obtain a set of coherent states that have minimal uncertainty and are closest to classical in this sense. As examples we examine the coherent states resulting from representations of the Lie groups $SU(2)$ and $SU(1,1)$.

Contents

1 Introduction

Coherent states are a type of quantum states. Coherent states were first introduced by Schrödinger in 1926 when he described certain states of the harmonic oscillator. Glauber used these states in 1963 for his quantum mechanical description of coherent laser light, and coined the term coherent state. These states have since been generalized in various ways, while the coherent states of the harmonic oscillator are now known as the canonical coherent states. As we shall see, coherent states in physics are often used to describe a set of states that have minimum uncertainty and in a way are closest to a classical description of the phenomenon it represents.[4]

In this thesis we will mainly study the notion of generalized coherent states introduced by Gilmore and Perelomov, that uses the representation of Lie groups in the space in which our states are defined. In the next section we will start with a some basics of quantum mechanics. Section 3 concerns the canonical coherent states as originally defined. In the fourth section, we will look at the Weyl-Heisenberg group and the Schrödinger representation, which generates the canonical coherent states. In section five we will define generalized coherent states for arbitrary Lie groups. Sections six and seven are meant to illustrate this definition. We will first study spin coherent states, the coherent states associated with the Lie group $SU(2)$. Finally, we will examine some of the coherent states of $SU(1,1)$, which are often referred to as pseudo-spin coherent states.

1.1 Motivation for Physics

The motivation to generalize the concept of canonical coherent states is simple; we would like to find states for general quantum systems that have similar properties, and are thus in some way "closest to classical" themselves. Such states may provide us with a natural way to connect classical with quantum description, and can be useful in applications where minimum uncertainty is desirable. Areas of application are various, and include quantum optics, quantum information theory and condensed matter physics.[7]

1.2 Motivation for Mathematics

The notion of generalized coherent states is a mathematical concept which gives rise to many different applications, both inside and outside the world of physics. It defines a wealth of possible systems, which have mathematical properties that are interesting even without any applications in mind. Therefore, the exploration of these individual systems, as well as the properties they share, will be well worth our time.

2 Quantum Mechanical Preliminaries

Here we will recall some basics of quantum mechanics. This section is mainly based on [1] and [3].

2.1 Basics

We will first recall some basic notions of quantum mechanics. In quantum mechanics we describe a particle in n dimensions with a wave function in $L^2(\mathbb{R}^n)$, the Hilbert space of square integrable functions from $\mathbb{R}^n \to \mathbb{C}$, i.e.

$$f : \mathbb{R}^n \to \mathbb{C}$$

for which

$$\left(\int_{\mathbb{R}^n} |f^2|(x) d\mu \right)^{\frac{1}{2}}$$

is finite, with μ the Lebesgue measure. The inner product is defined by

$$\langle f|g\rangle = \int_{\mathbb{R}^n} f^*(x)g(x)d\mu. \tag{1}$$

We describe the position and momentum of the particle using the position and momentum operators,

$$\hat{Q} = (\hat{Q}_1, ..., \hat{Q}_n),\ Q_j : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n),\ \hat{Q}_j f = x_j f \tag{2}$$

and

$$\hat{P} = (\hat{P}_1, ..., \hat{P}_n),\ \hat{P}_k : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n),\ \hat{P}_k f = -i\hbar \frac{\partial f}{\partial x_k}. \tag{3}$$

$\hat{Q}$ and $\hat{P}$ defined in this manner satisfy the commutation relations

$$[\hat{Q}_j, \hat{P}_k] = \delta_{jk} i\hbar \tag{4}$$

on the domain of $[\hat{Q}, \hat{P}]$. These commutation relations are often referred to as the canonical commutation relations.

We will use bra-ket notation, where we denote a state in our Hilbert space by $|\psi\rangle$ and the inner product of $|\psi\rangle$ and $|\chi\rangle$ by $\langle\psi|\chi\rangle$. The uncertainty of an operator $\hat{O}$ on our Hilbert space $\mathcal{H}$ with respect to the state $|\psi\rangle$ is defined in the following way:

$$\Delta_\psi \hat{O} = \sqrt{\langle \hat{O}^2\rangle_\psi - \langle \hat{O}\rangle_\psi^2} \tag{5}$$

With the uncertainty defined in this manner, the (generalized) Heisenberg uncertainty principle holds: For $\hat{A}$ and $\hat{B}$ self-adjoint operators on $\mathcal{H}$, we have

$$(\Delta_\psi \hat{A})(\Delta_\psi \hat{B}) \geq \frac{1}{2} |\langle i[\hat{A}, \hat{B}]\rangle_\psi| \tag{6}$$

for any $|\psi\rangle \in \mathcal{H}$. For the position and momentum operator this means that

$$\Delta_\psi \hat{Q}_j \Delta_\psi \hat{P}_k \geq \delta_{jk}\frac{\hbar}{2}. \tag{7}$$

To describe the dynamics of a quantum mechanical system, we use the Hamiltonian operator $\hat{H}$, related to the total energy of the system. The Hamiltonian, together with the famous Schrödinger equation

$$\hat{H}|\psi(t)\rangle = i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle, \tag{8}$$

then gives us the dynamics of the system. When the Hamiltonian operator is itself independent of time, we can solve the time-independent Schrödinger equation

$$\hat{H}|\psi\rangle = E|\psi\rangle \tag{9}$$

to find the so-called energy eigenstates or stationary states $|E\rangle$ of the system, with energy eigenvalues E. These states will then evolve in time according to

$$|E(t)\rangle = \exp(-\frac{iEt}{\hbar})|E\rangle. \tag{10}$$

We can construct a linear combination of the energy eigenstates

$$|\psi\rangle = \sum_{n=1}^{\infty} C_n|E_n\rangle, \tag{11}$$

where we have labeled the energy eigenstates $|E_n\rangle$ by $n \in \mathbb{N}$, and the $C_n \in \mathbb{C}$ are complex coefficients. This state will then evolve in time as

$$|\psi(t)\rangle = \exp(\frac{-i\hat{H}t}{\hbar})|\psi\rangle = \sum_{n=1}^{\infty} C_n \exp(\frac{-iE_nt}{\hbar})|E_n\rangle = \sum_{n=1}^{\infty} C_n|E_n(t)\rangle \tag{12}$$

and hence it forms a solution to the time-dependent Schrödinger equation.

2.2 The Harmonic Oscillator

Now, a quick reminder on the (one-dimensional) harmonic oscillator. Its Hamiltonian is given by:

$$\hat{H} = \frac{1}{2m}(\hat{P}^2 + (m\omega\hat{Q})^2). \tag{13}$$

To obtain the energy eigenstates of the harmonic oscillator, one has to solve the time-independent Schrödinger equation for this Hamiltonian. To do this, we rewrite the Hamiltonian as:

$$\hat{H} = \hbar\omega(\hat{a}_+\hat{a}_- + \frac{1}{2}), \tag{14}$$

where we have introduced the raising operator $\hat{a}_+$ and the lowering operator $\hat{a}_-$, defined by:

$$\hat{a}_+ = \frac{1}{\sqrt{2\hbar m\omega}}(-i\hat{P} + m\omega\hat{Q}) \tag{15}$$

and

$$\hat{a}_- = \frac{1}{\sqrt{2\hbar m\omega}}(i\hat{P} + m\omega\hat{Q}) \tag{16}$$

respectively. Hence, the operators are adjoint, i.e. $\hat{a}_-^\dagger = \hat{a}_+$.
Using these raising and lowering operators, it can be shown that the normalized energy eigenstates are given by

$$|n\rangle = \frac{1}{\sqrt{n!}}(\hat{a}_+)^n|0\rangle, \tag{17}$$

where the ground state $|0\rangle$ can be given by:

$$(\frac{m\omega}{\pi\hbar})^{\frac{1}{4}}e^{-\frac{m\omega}{2\hbar}x^2}, \tag{18}$$

and the energy by:

$$E_n = \hbar\omega(n + \frac{1}{2}). \tag{19}$$

Note also that

$$\hat{a}_-|n\rangle = \sqrt{n}|n-1\rangle. \tag{20}$$

The general solution of the time-dependent Schrödinger equation can then be formed by the linear combinations of these eigenstates.

3 Canonical Coherent States

The simplest examples of coherent states stem from the one-dimensional (quantum) harmonic oscillator, and are called canonical coherent states. References used throughout this section are [1] and [7].

3.1 Canonical Coherent States

The coherent states of the harmonic oscillator are defined as the eigenstates of the lowering operator:

$$\hat{a}_-|\alpha\rangle = \alpha|\alpha\rangle, \tag{21}$$

with α some complex number, and $|\alpha\rangle$ the eigenstate corresponding to α. Note that the ground state $|0\rangle$ is a coherent state, since $\hat{a}_-|0\rangle = 0$. They are minimum uncertainty states, in the sense that

$$\Delta\hat{Q}\Delta\hat{P} = \frac{\hbar}{2}. \tag{22}$$

Proof. To obtain $\Delta\hat{Q}$ and $\Delta\hat{P}$, we must calculate the expectation values $\langle\hat{Q}\rangle$, $\langle\hat{Q}^2\rangle$, $\langle\hat{P}\rangle$ and $\langle\hat{P}^2\rangle$. To this end, we will write $\hat{Q}$ and $\hat{P}$ in terms of the raising and lowering operators:

$$\hat{Q} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}_+ + \hat{a}_-), \tag{23}$$

$$\hat{P} = i\sqrt{\frac{\hbar m\omega}{2}}(\hat{a}_+ - \hat{a}_-). \tag{24}$$

This allows us to calculate

$$\begin{aligned}
\langle\hat{Q}\rangle = \langle\alpha|\hat{Q}|\alpha\rangle &= \sqrt{\frac{\hbar}{2m\omega}}\langle\alpha|\hat{a}_+ + \hat{a}_-|\alpha\rangle \\
&= \sqrt{\frac{\hbar}{2m\omega}}(\langle\alpha|\hat{a}_+|\alpha\rangle + \langle\alpha|\hat{a}_-|\alpha\rangle) \\
&= \sqrt{\frac{\hbar}{2m\omega}}(\alpha^*\langle\alpha|\alpha\rangle + \alpha\langle\alpha|\alpha\rangle) \\
&= \sqrt{\frac{\hbar}{2m\omega}}(\alpha^* + \alpha),
\end{aligned}$$

where we used that $\hat{a}_+ = \hat{a}_-^\dagger$. Similarly, we can calculate that

$$\begin{aligned}
\langle\hat{Q}^2\rangle &= \frac{\hbar}{2m\omega}((\alpha^* + \alpha)^2 + 1)), \\
\langle\hat{P}\rangle &= i\sqrt{\frac{\hbar m\omega}{2}}(\alpha^* - \alpha), \\
\langle\hat{P}^2\rangle &= -\frac{\hbar m\omega}{2}((\alpha^* - \alpha)^2 - 1).
\end{aligned}$$

So now, using these values, we see that

$$\Delta\hat{Q}\Delta\hat{P} = \sqrt{\langle\hat{Q}^2\rangle - \langle\hat{Q}\rangle^2}\sqrt{\langle\hat{P}^2\rangle - \langle P\rangle^2}$$
$$= \sqrt{\frac{\hbar}{2m\omega}}\sqrt{\frac{\hbar m\omega}{2}} = \frac{\hbar}{2}.$$

□

We can go further and describe the coherent states in more detail. Since the energy eigenstates form a complete, orthogonal set, we can write any coherent state $|\alpha\rangle$ as

$$|\alpha\rangle = \sum_{n=0}^{\infty} C_n|n\rangle, \tag{25}$$

with the coefficients C_n determined by

$$C_n = \langle n|\alpha\rangle. \tag{26}$$

Now, using that $|\alpha\rangle$ is an eigenstate of $\hat{a}_-$, we see that

$$\begin{aligned} C_n &= \langle n|\alpha\rangle \\ &= \frac{1}{\alpha}\langle n|\hat{a}_-\alpha\rangle \\ &= \frac{1}{\alpha}\langle\hat{a}_+n|\alpha\rangle \\ &= \frac{\sqrt{n+1}}{\alpha}\langle n+1|\alpha\rangle = \frac{\sqrt{n+1}}{\alpha}C_{n+1}, \end{aligned}$$

and hence

$$C_n = \frac{\alpha^n}{\sqrt{n!}}C_0.$$

To determine C_0, we will normalize the state by setting

$$\langle\alpha|\alpha\rangle = \sum_{n=0}^{\infty}\frac{(\alpha^n)^*(\alpha)^n}{n!}|C_0|^2 = e^{-|\alpha|^2}|C_0|^2 = 1,$$

which allows us to take

$$C_0 = e^{-\frac{|\alpha|^2}{2}}.$$

So in general, a coherent state of the harmonic oscillator is a state

$$|\alpha\rangle = \sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}e^{-\frac{|\alpha|^2}{2}}|n\rangle \tag{27}$$

for some $\alpha \in \mathbb{C}$. We will now show that a coherent state will remain a coherent state in time. We will write

$$|n(t)\rangle = e^{-\frac{i}{\hbar}E_n t}|n\rangle \tag{28}$$

for the time dependent energy eigenstates of the harmonic oscillator. A coherent state will then evolve in time according to

$$\begin{aligned}|\alpha(t)\rangle &= \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} e^{-\frac{|\alpha|^2}{2}} |n(t)\rangle \\ &= \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} e^{-\frac{|\alpha|^2}{2}} e^{-\frac{i}{\hbar}E_n t}|n\rangle. \end{aligned} \tag{29}$$

So if we apply $\hat{a}_-$ to $|\alpha(t)\rangle$, we get

$$\begin{aligned}\hat{a}_-|\alpha(t)\rangle &= \sum_{n=1}^{\infty} \frac{\alpha^n}{\sqrt{n!}} e^{-\frac{|\alpha|^2}{2}} e^{-\frac{i}{\hbar}E_n t}\sqrt{n}|n-1\rangle \\ &= \sum_{n=0}^{\infty} \frac{\alpha^{n+1}}{\sqrt{(n+1)!}} e^{-\frac{|\alpha|^2}{2}} e^{-\frac{i}{\hbar}E_{n+1} t}\sqrt{n+1}|n\rangle \\ &= \sum_{n=0}^{\infty} \alpha e^{-\frac{i}{\hbar}(E_{n+1}-E_n)t}\left(\frac{\alpha^n}{\sqrt{n!}} e^{-\frac{|\alpha|^2}{2}} e^{-\frac{i}{\hbar}E_n t}|n\rangle\right) \\ &= \alpha e^{-i\omega t}|\alpha(t)\rangle, \end{aligned} \tag{30}$$

which means the coherent state at time t forms a coherent state with eigenvalue $\alpha(t) := \alpha e^{-i\omega t}$. So when the coherent state evolves in time, it will remain a coherent state, though its eigenvalue will rotate in the complex plane.

We will go over some properties of coherent states. The first thing we will note is that the coherent states $|\alpha\rangle$ are continuous functions of the label α.

Proof. The function $|\cdot\rangle : \alpha \in \mathbb{C} \to |\alpha\rangle \in \mathcal{H}$ is continuous if for every sequence of points $\alpha_n \in \mathbb{C}$ that has $\lim_{n\to\infty} \alpha_n = \alpha$, $\lim_{n\to\infty} |\alpha_n\rangle = |\alpha\rangle$. We have that

$$\begin{aligned}\langle\alpha_1|\alpha_2\rangle &= \sum_{n,m=0}^{\infty} \langle m|\frac{C_{1,m}^*}{\sqrt{m!}}\frac{C_{2,n}}{\sqrt{n!}}|n\rangle \\ &= \sum_{n=0}^{\infty} \frac{C_{1,n}^* C_{2,n}}{n!} \\ &= \sum_{n=0}^{\infty} \frac{(\alpha_1^*)^n \alpha_2^n}{n!}\ \exp(-\frac{1}{2}|\alpha_1|^2)\exp(-\frac{1}{2}|\alpha_2|^2) \\ &= \exp(\alpha_1^*\alpha_2 - \frac{1}{2}|\alpha_1|^2 - \frac{1}{2}|\alpha_2|^2) \\ &= \exp(-\frac{1}{2}|\alpha_1 - \alpha_2|^2 + \mathrm{Im}(\alpha_2^*\alpha_1)). \end{aligned}$$

Now we calculate

$$\begin{aligned}
\||\alpha\rangle - |\alpha_n\rangle\| &= \langle\alpha|\alpha\rangle - \langle\alpha|\alpha_n\rangle - \langle\alpha_n|\alpha\rangle + \langle\alpha_n|\alpha_n\rangle \\
&= 2 - \exp(-\frac{1}{2}|\alpha - \alpha_n|^2 + i\ \mathrm{Im}(\alpha_n^*\alpha)) \\
&\quad - \exp(-\frac{1}{2}|\alpha_n - \alpha|^2 + i\ \mathrm{Im}(\alpha^*\alpha_n)) \\
&= 2 - \exp(-\frac{1}{2}|\alpha - \alpha_n|^2)(\exp(i\ \mathrm{Im}(\alpha_n^*\alpha)) + \exp(i\ \mathrm{Im}(\alpha^*\alpha_n))) \\
&= 2 - 2\ \exp(-\frac{1}{2}|\alpha - \alpha_n|^2)\cos(\mathrm{Im}(\alpha_n^*\alpha)),
\end{aligned}$$

which goes to zero when α_n goes to zero, since $\mathrm{Im}(\alpha^*\alpha_n) \to \mathrm{Im}(\alpha^*\alpha) = \mathrm{Im}(|\alpha|^2) = 0$, while the exponential term goes to one. So $\lim_{n\to\infty}\ |\alpha_n\rangle = |\alpha\rangle$. □

With our expression (29) for $|\alpha(t)\rangle$, it can be calculated that

$$\langle\alpha(t)|\hat{Q}|\alpha(t)\rangle = \langle\alpha|\hat{Q}|\alpha\rangle\cos(\omega t) + \frac{1}{m\omega}\langle\alpha|\hat{P}|\alpha\rangle\sin(\omega t) \tag{31}$$

$$\langle\alpha(t)|\hat{P}|\alpha(t)\rangle = \langle\alpha|\hat{P}|\alpha\rangle\cos(\omega t) - m\omega\langle\alpha|\hat{Q}|\alpha\rangle\sin(\omega t) \tag{32}$$

Hence the expectation values of position and momentum oscillate in phase space in accordance with the classical harmonic oscillator. We can also determine the probability density $\||\alpha(t)\rangle\|^2$. It is given by

$$\sqrt{\frac{m\omega}{\hbar\pi}}\exp(-\frac{m\omega}{\hbar}(x - \langle\alpha(t)|\hat{Q}|\alpha(t)\rangle)^2). \tag{33}$$

So the probability density of the wave function remains the same in time relative to its center, $\langle\alpha(t)|\hat{Q}|\alpha(t)\rangle$, which oscillates in phase space in accordance with the classical harmonic oscillator.

4 The Weyl-Heisenberg Group

The canonical coherent states derived in the previous can be defined differently, using the Weyl-Heisenberg translation operator, which is connected to a Lie group called the Weyl-Heisenberg group. This section follows the exposition in [3]

4.1 The Weyl-Heisenberg Translation Operator

On the intersection of the domains of $\hat{Q}$ and $\hat{P}$ we will define the operators $\hat{L}(z)$:

$$\hat{L}(z) = p \cdot \hat{Q} - q \cdot \hat{P}, \tag{34}$$

for every $z = (q, p) \in \mathbb{R}^n \times \mathbb{R}^n$, where the dot stands for the scalar product, defined by

$$r \cdot \hat{O} \equiv \sum_{j=1}^{n} r_j \hat{O}_j \tag{35}$$

for any $r \in \mathbb{R}^n$ and operator $\hat{O} = (\hat{O}_1, ..., \hat{O}_n)$ on $L^2(\mathbb{R}^n)$.
Since $\hat{Q}$ and $\hat{P}$ are self-adjoint, $\hat{L}(z)$ is self-adjoint and hence the generator of a unitary operator $\hat{T}(z)$, called the Weyl-Heisenberg translation operator:

$$\hat{T}(z) = \exp(\frac{i}{\hbar}\hat{L}(z)). \tag{36}$$

It is referred to as a translation operator because it translates a state in phase space by a shift $z = (q, p)$.

We will study this operator in more detail. We will make use of the consequence of the general Baker-Campbell-Hausdorff formula

$$\exp(\hat{A} + \hat{B}) = \exp(-\frac{1}{2}[\hat{A}, \hat{B}]) \exp(\hat{A}) \exp(\hat{B}), \tag{37}$$

for operators $\hat{A}$ and $\hat{B}$ satisfying $[\hat{A}, [\hat{A}, \hat{B}]] = [\hat{B}, [\hat{A}, \hat{B}]] = 0$ (and some other basic conditions, see [3]).
Using this formula we can show that composition of two Weyl-Heisenberg trans-

lations is given by

$$
\begin{aligned}
\hat{T}(z)\hat{T}(\tilde{z}) &= \exp(\frac{i}{\hbar}\hat{L}(z))\exp(\frac{i}{\hbar}\hat{L}(\tilde{z})) \\
&= \exp(\frac{i}{\hbar}(p\cdot\hat{Q}-q\cdot\hat{P}))\exp(\frac{i}{\hbar}(\tilde{p}\cdot\hat{Q}-\tilde{q}\cdot\hat{P})) \\
&= \exp(\frac{1}{2}[\frac{i}{\hbar}(p\cdot\hat{Q}-q\cdot\hat{P}),\frac{i}{\hbar}(\tilde{p}\cdot\hat{Q}-\tilde{q}\cdot\hat{P})])\hat{T}(z+\tilde{z}) \\
&= \exp(-\frac{1}{2\hbar^2}([p\cdot\hat{Q},-\tilde{q}\cdot\hat{P}]+[-q\cdot\hat{P},-\tilde{q}\cdot\hat{P}]+ \\
&\qquad [-q\cdot\hat{P},\tilde{p}\cdot\hat{Q}]+[p\cdot\hat{Q},\tilde{p}\cdot\hat{Q}]))\hat{T}(z+\tilde{z}) \\
&= \exp(-\frac{i}{2\hbar}(q\cdot\tilde{p}-p\cdot\tilde{q}))\hat{T}(z+\tilde{z}),
\end{aligned} \tag{38}
$$

where $z=(q,p)$ and $\tilde{z}=(\tilde{q},\tilde{p})$. So two consecutive translations z and $\tilde{z}$ correspond to a translation of $z+\tilde{z}$, with an additional phase factor. Also note that

$$\hat{T}(z)\hat{T}(\tilde{z}) = \exp(-\frac{i}{\hbar}\sigma(z,\tilde{z}))\hat{T}(\tilde{z})\hat{T}(z), \tag{39}$$

where $\sigma(z,\tilde{z})$ denotes the symplectic product

$$\sigma(z,\tilde{z}) = q\cdot\tilde{p}-p\cdot\tilde{q}. \tag{40}$$

Another result, not proven here, is that, for any $z=(q,p)\in\mathbb{R}^n\times\mathbb{R}^n$, we have

$$\hat{T}(z)\begin{pmatrix}\hat{Q}\\ \hat{P}\end{pmatrix}\hat{T}(z)^{-1} = \begin{pmatrix}\hat{Q}-q\\ \hat{P}-p\end{pmatrix}. \tag{41}$$

$\hat{T}(z)$ is a translation operator in the sense that it shifts a state in phase space; if we have a state $u(x)\in L^2(\mathbb{R}^n)$ and a $z=(q,p)$, then

$$(\hat{T}(z)u)(x) = \exp(-\frac{i}{2\hbar}q\cdot p)\exp(\frac{i}{\hbar}x\cdot p)u(x-q) \tag{42}$$

and for its Fourier transform, we have

$$(\mathcal{F}(\hat{T}(z)u)(\xi) = \exp(\frac{i}{2\hbar}q\cdot p)\exp(-\frac{i}{\hbar}q\cdot\xi)\mathcal{F}(u)(\xi-p). \tag{43}$$

So the domain of the state u as a function of position is shifted by q, and as a function of momentum by p, aside from changing the state by a phase factor.

4.2 Canonical Coherent States Revisited

The Weyl-Heisenberg translation operator gives us an alternate way of looking at the canonical coherent states. As we saw in section 3, the ground state $|0\rangle$ of the harmonic oscillator is a canonical coherent state. We can obtain all canonical coherent states by taking this ground state and shifting it around in phase space by the translation operator $\hat{T}(z)$:

$$|z\rangle = \hat{T}(z)|0\rangle. \tag{44}$$

The states $|z\rangle$ and the states $|\alpha\rangle$ defined earlier are related by:

$$\alpha = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega q + ip), \tag{45}$$

for $z = (q, p) \in \mathbb{R}^2$.

Proof. From equation (41) we can deduce that

$$\begin{aligned}
\hat{T}(z)\hat{a}_-\hat{T}(z)^{-1} &= \hat{T}(z)\frac{1}{\sqrt{2\hbar m\omega}}(m\omega\hat{Q} - i\hat{P})\hat{T}(z)^{-1} \\
&= \frac{1}{\sqrt{2\hbar m\omega}}(m\omega(\hat{Q} - q) - i(\hat{P} - p)) \\
&= \hat{a}_- - \alpha.
\end{aligned}$$

Now,

$$(\hat{a}_- - \alpha)|z\rangle = \hat{T}(z)\hat{a}_-\hat{T}^{-1}(z)|z\rangle = \hat{T}(z)\hat{a}_-|0\rangle = 0.$$

So $\hat{a}_-|z\rangle = \alpha|z\rangle$, and we can produce any $\alpha \in \mathbb{C}$ from (q, p). Therefore all canonical coherent states correspond to the ground state shifted by some Weyl-Heisenberg translation. □

4.3 The Weyl-Heisenberg Algebra

We will now look at the Weyl-Heisenberg group and its corresponding Weyl-Heisenberg algebra. We will see later that the Weyl-Heisenberg translation operators can be used to form an irreducible, unitary representation of the Weyl-Heisenberg group. Working in dimension n, we will first define the Weyl-Heisenberg algebra, $\mathfrak{h}_n$, as the $2n+1$ dimensional real vector space generated by elements $I, P_1, \ldots, P_n, Q_1, \ldots, Q_n$, satisfying the commutation relations

$$[Q_j, Q_k] = [P_j, P_k] = [I, Q_j] = [I, P_k] = 0, \ [Q_j, P_k] = \delta_{jk}\hbar I \tag{46}$$

for $j, k \in \{1, ...n\}$. It is the Lie algebra corresponding to the set of operators $\hat{I}$, $i\hat{Q}$ and $i\hat{P}$, where $\hat{I} = -i\hat{\mathbb{1}}$ is $-i$ times the identity operator. We can write any element W of $\mathfrak{h}$ as

$$W = aI + \sum_j b_j Q_j + \sum_k c_k P_k. \tag{47}$$

for $a, b_j, c_k \in \mathbb{R}$. However, another set of coordinates will be more beneficial to us. We will express W as

$$W = \frac{t}{2\hbar}I + \frac{1}{\hbar}(p \cdot Q - q \cdot P), \tag{48}$$

with $Q = (Q_1, ..., Q_n)$, $P = (P_1, ..., P_n)$, $z = (q, p) \in \mathbb{R}^n \times \mathbb{R}^n$ and $t \in \mathbb{R}$. t, p and q form a coordinate system in this way.

We can form a Lie algebra representation of the Weyl-Heisenberg algebra by simply letting

$$I \mapsto -i\hat{\mathbb{1}}, \quad Q \mapsto i\hat{Q} \quad P \mapsto i\hat{P}, \tag{49}$$

which is called the Schrödinger representation. Every W hence corresponds to an operator

$$\hat{W} = -\frac{it}{2\hbar}\hat{\mathbb{1}} + \frac{i}{\hbar}(p \cdot \hat{Q} - q \cdot \hat{P}) = -\frac{it}{2\hbar}\hat{\mathbb{1}} + \frac{i}{\hbar}\hat{L}(z). \tag{50}$$

[8]

4.4 The Weyl-Heisenberg Group and its Coherent States

The Weyl-Heisenberg group $\mathscr{H}_n$ is defined as the simply connected Lie group associated to the Weyl-Heisenberg algebra $\mathfrak{h}_n$ by the exponential map. It is convenient to use the coordinates t and $z = (p, q)$ coming from $\mathfrak{h}_n$. In these coordinates, the group operation on $\mathscr{H}$ (denoted by juxtaposition) can be written as

$$(t, z)(\tilde{t}, \tilde{z}) = (t + \tilde{t} + \sigma(z, \tilde{z}), z + \tilde{z}). \tag{51}$$

The Schrodinger representation on $\mathfrak{h}_n$ lifts to a unitary, irreducible representation on $\mathscr{H}_n$, and is defined by

$$\rho(t, z) = \exp(-\frac{it}{2\hbar}\hat{\mathbb{1}})\exp(\frac{i}{\hbar}\hat{L}(z)) = \exp(-\frac{it}{2\hbar})\hat{T}(z). \tag{52}$$

The operator $\frac{i}{\hbar}\hat{L}(z)$ lifts to the Weyl Heisenberg translation operator $\hat{T}(z)$, while the operator $\hat{I}$ lifts to the operator $\exp(-\frac{it}{2\hbar})\hat{\mathbb{1}}$, which seems to relate to the fact that vectors in $L^2(\mathbb{R}^n)$ correspond to the same state if they differ only by a phase factor. We will note that the Schrödinger representation is not just an irreducible representation of $\mathscr{H}_n$, but it is, up to conjugation with a unitary operator, the unique irreducible representation of $\mathscr{H}_n$, as stated by the Stone-Von Neumann theorem[14].

If we now again look back on the canonical coherent states, we can place them in an even different context; the canonical coherent states correspond to the set of states $\{|\psi\rangle = \rho(g)|\psi_0\rangle,\ g \in \mathscr{H}_1\}$, i.e. they correspond to elements of the Weyl-Heisenberg group acting on the ground state through the Schrödinger representation. It is this same construction that we can use to define a more general notion of coherent states for irreducible, unitary representations of arbitrary Lie groups, as we will show in the next section.

5 Generalized Coherent States

We have seen that canonical coherent states can be described as the set of vectors resulting from a unitary, irreducible representation of the Weyl-Heisenberg group. We can apply this process to an arbitrary Lie group and define the notion of a generalized coherent state. This definition of generalized coherent states is due to R. Gilmore [13] and A.M. Perelomov [12], and hence these states are also referred to as Gilmore-Perelomov coherent states. Though the general concept is the same, Gilmore used a slightly different definition than Perelomov, as explained later this section. We will follow here the introduction of generalized coherent states by Perelomov as in [4].

5.1 Generalized Coherent States

For the following discussion it will be important to make the distinction between the elements (vectors) of our Hilbert space and their corresponding states. In quantum mechanics, we say that two vectors ψ_1 and ψ_2 in $\mathcal{H}$ correspond to the same state if they only differ by a phase factor, $\psi_1 = e^{i\phi}\psi_2$. So when we talk about a state in Hilbert space, what we are really talking about is a one-dimensional subspace of this Hilbert space. Often, this distinction is glossed over, and we denote both by the ket $|\psi\rangle$. In this section however, we will choose to denote the vectors in the Hilbert space $\mathcal{H}$ by $\psi \in \mathcal{H}$ and their corresponding state by $|\psi\rangle$.

Let us now define the notion of generalized coherent states.

Definition 1. *Suppose we have a Lie group G with a unitary, irreducible representation T acting on some Hilbert space $\mathcal{H}$. If we take a fixed vector ψ_0, we define the coherent state system $\{T, \psi_0\}$ to be the set of vectors $\psi \in \mathcal{H}$ such that $\psi = T(g)\psi_0$ for some $g \in G$. Generalized coherent states are defined as the states $|\psi\rangle$ corresponding to these vectors in $\mathcal{H}$.*

We would like to parametrize our generalized coherent states in some manner. Note that we can not just parametrize them by elements of G, since it may be that $|T(g_1)\psi_0\rangle = |T(g_2)\psi_0\rangle$, i.e. two group elements may correspond to the same state. If we look at two vectors $T(g_1)\psi_0$ and $T(g_2)\psi_0$, we see that they correspond to the same state if and only if $T(g_2^{-1}g_1)\psi_0 = e^{i\phi}\psi_0$ for some $\phi \in [0, 2\pi)$. We will define H as the subgroup of G such that

$$H = \left\{h \in G : \; T(h)\psi_0 = e^{i\phi}\psi_0 \text{ for some } \phi \in [0, 2\pi)\right\}. \tag{53}$$

This is called the isotropy subgroup of G, and corresponds to those elements $g \in G$ such that $|T(g)\psi_0\rangle = |\psi_0\rangle$. If g_1 and g_2 belong to the same left coset class in G/H, then their corresponding vectors differ only by a phase and hence define the same state. So we can parametrize the set of coherent states by the elements of the coset space G/H.

5.2 Selection of the Fixed State

We will go through some properties of all generalized coherent states. Note that in the definition of a generalized coherent state we have made no requirements on the state $|\psi_0\rangle$. In physics, it is often preferred to consider the coherent states of a state that is closest to classical states in some sense, but in principle we could take any state in our Hilbert space and generate a system of coherent states.

To find a state that is closest to classical, we will need to have some generalization of what it means for a state to have minimal uncertainty. For compact, simple Lie groups, R. Delbourgo and J.R. Fox state in [9] that an appropriate measure is the dispersion of the Casimir operator $\hat{C}$, an operator that commutes with all elements of the Lie algebra, and is defined in the following way:

$$\hat{C} = g^{jk}\hat{A}_j\hat{A}_k, \tag{54}$$

where A_j stands for the jth generator of the Lie algebra and g^{jk} is the Killing-Cartan metric. The dispersion itself is defined as

$$\Delta_\psi\hat{C} = \langle\hat{C}\rangle_\psi - g^{jk}\langle\hat{A}_j\rangle_\psi\langle\hat{A}_k\rangle_\psi. \tag{55}$$

The states closest to classical states are considered to be the states for which the dispersion of the Casimir operator is minimal. For this uncertainty measure, all generalized coherent states will have minimum uncertainty if the fixed state $|\psi_0\rangle$ does.
Perelomov suggests a construction that specifies a state which has minimal uncertainty in this sense, which seems to be more widely applicable than to just semi-simple Lie groups (for example for the Weyl-Heisenberg group, see again [4]). Suppose we have a Lie algebra $\mathfrak{g}$ of a Lie group G with an irreducible representation T defined on it. We consider the complexification of the Lie algebra $\mathfrak{g}$, which we will denote by $\mathfrak{g}^c$. The representation T defines a Lie algebra representation $\mathfrak{T}$ on $\mathfrak{g}$ (and hence $\mathfrak{g}^c$). Hence it is possible to define an analog of the isotropy subgroup in $\mathfrak{g}^c$, which we will call the isotropy subalgebra $\mathfrak{h}$:

$$\mathfrak{h} = \{h \in \mathfrak{h},\ \mathfrak{T}(h)|\psi_0\rangle = \lambda_h|\psi_0\rangle\}\,. \tag{56}$$

Perelomov states that ground states $|\psi_0\rangle$ for which the isotropy subalgebra is maximal lead to a coherent state system that is closest to classical states, i.e. minimize the uncertainty of the Casimir operator.

5.3 Difference Between Definitions of Gilmore and Perelomov

As mentioned, the concept of generalized coherent states introduced by Gilmore differs slightly from the definition of Perelomov. Both use the concept of a unitary, irreducible representation acting on a fixed state in Hilbert space. However, while Perelomov uses an arbitrary Lie group, Gilmore takes a finite dimensional dynamical symmetry group as his starting point. This requires less structure on the group, though it excludes the possibility of infinite dimensional Lie groups. Also, while Perelomov allows for choosing any fixed state in his definition (though he suggests suitable candidates), Gilmore instructs us to choose an extremal state that can be annihilated by a maximal subset of the algebra of the dynamical symmetry group.[10]

To illustrate the concept of generalized coherent states, we will examine the coherent states coming from the irreducible, unitary representations of the Lie groups $SU(2)$ and $SU(1,1)$.

6 Spin Coherent States

We will now take a look at the coherent states of $SU(2)$, which in physics are often referred to as spin coherent states. This section again closely follows [3].

6.1 $SU(2)$

The group $SU(n)$ is called the special unitary group of dimension n, and is defined as the group of $n \times n$ unitary matrices with complex coefficients with determinant 1. For dimension 2, it consists of the matrices

$$A = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}, \; |a|^2 + |b|^2 = 1, \tag{57}$$

with $a, b \in \mathbb{C}$. Its Lie algebra can be shown to be the real vector space of 2×2 complex anti-Hermitian matrices with vanishing trace:

$$\mathfrak{su}(2) = \left\{ x \in \mathfrak{gl}(2, \mathbb{C}) | x^\dagger + x = 0, Tr(x) = 0 \right\}, \tag{58}$$

where $\mathfrak{gl}(2, \mathbb{C})$ is the vector space of 2×2 invertible matrices with complex coefficients (the Lie algebra of the general linear group $GL(2, \mathbb{C})$).
As generators for $\mathfrak{su}(2)$ we can take J_1, J_2 and J_3, with the Lie product

$$[J_p, J_q] = \epsilon_{pqr} K_r, \tag{59}$$

where ϵ_{pqr} denotes the Levi-Cevita symbol of dimension 3.
$SU(2)$ forms a compact, simply connected Lie group. $SU(2)$ is a double cover of $SO(3)$, the special orthogonal group in 3 dimensions, and hence we have $\mathfrak{so}(3) \cong \mathfrak{su}(2)$.

Since $SU(2)$ is compact, all its irreducible representations (irreps) are unitary and finite dimensional. In case of $SU(2)$, there is exactly one non-trivial irrep for each dimension n (in the sense that all others are equivalent). In physics each such irrep is denoted by its so-called quantum number j, and the irrep $T^{(j)}$ forms the $n = 2j + 1$-dimensional irrep of $SU(2)$, where $2j \in \mathbb{N}$. The vector space it works on, $V^{(j)}$ can be provided with an orthonormal basis $|j, m\rangle$, where m is chosen to run over the values $\{-j, -j+1, ..., j-1, j\}$.

The Lie product on $\mathfrak{su}(2)$ in combination with the irreps translate J_1, J_2 and J_3 into operators satisfying same commutation relations (59). We will also define the operator $\hat{J}^2 = J_1^2 + \hat{J}_2^2 + \hat{J}_3^2$, which forms the Casimir operator of $\mathfrak{su}(2)$, and the ladder operators

$$\hat{J}_+ = \hat{J}_1 + i\hat{J}_2, \quad \hat{J}_- = \hat{J}_1 - i\hat{J}_2, \tag{60}$$

all of which do not correspond to elements of $\mathfrak{su}(2)$. The orthonormal basis $|j, m\rangle$ is defined such that, when $\hat{J}_3$ and $\hat{J}^2$ are applied to a state $|j, m\rangle$, we have

$$J_3 |j, m\rangle = m |j, m\rangle, \tag{61}$$

$$J^2 |j, m\rangle = j(j+1) |j, m\rangle, \tag{62}$$

i.e. the basis states form eigenstates of operators $\hat{J}_3$ and $\hat{J}^2$. The ladder operators $\hat{J}_+$ and $\hat{J}_-$ respectively raise and lower the value of m:

$$\hat{J}_+|j,m\rangle = \sqrt{(j-m)(j+m-1)}|j,m+1\rangle, \tag{63}$$
$$\hat{J}_-|j,m\rangle = \sqrt{(j+m)(j-m+1)}|j,m-1\rangle, \tag{64}$$

while $\hat{J}_+$ annihilates the $|j,j\rangle$ state and $\hat{J}_-$ annihilates $|j,-j\rangle$ state.

6.2 Spin Coherent States

We can define coherent states for each irrep of $SU(2)$ in the manner described in section 5. Firstly, we note that the states $|j,-j\rangle$ and $|j,j\rangle$ are the minimum uncertainty states of $SU(2)$.

Proof. We will use the generalized Heisenberg uncertainty principle as stated in (6) in section 2. We have:

$$\Delta_\psi J_k = \langle \hat{J}_k^2\rangle_\psi - \langle \hat{J}_k\rangle_\psi^2, \tag{65}$$

where $k = 1,2,3$, and

$$(\Delta_\psi \hat{J}_1)(\Delta_\psi \hat{J}_2) \geq \frac{1}{2}|\langle \hat{J}_3\rangle_\psi|. \tag{66}$$

We want to see if $|j,\pm j\rangle$ minimizes this inequality. If we denote $\psi_m = |j,m\rangle$, then for $\hat{J}_1$, $\hat{J}_2$ and $\hat{J}_3$ we have

$$\begin{aligned}(\Delta_{\psi_m}\hat{J}_1)(\Delta_{\psi_m}\hat{J}_2) &= \sqrt{\langle \hat{J}_1^2\rangle_{\psi_m} - \langle \hat{J}_1\rangle_{\psi_m}^2}\sqrt{\langle \hat{J}_2^2\rangle_{\psi_m} - \langle \hat{J}_2\rangle_{\psi_m}^2} \\ &= \langle \hat{J}_1^2\rangle_{\psi_m}\langle \hat{J}_2^2\rangle_{\psi_m} = \frac{1}{2}(j(j+1)-m^2)\end{aligned}$$

which can be calculated directly by expressing $\hat{J}_1$ and $\hat{J}_2$ in ladder operators and using equations (63) and (64). So for $|j,-j\rangle$ and $|j,j\rangle$ we get

$$\Delta_{\psi_{\pm j}}\hat{J}_1\Delta_{\psi_{\pm j}}\hat{J}_2 = \frac{1}{2}j = \frac{1}{2}|\langle \hat{J}_3\rangle_{\psi_{\pm j}}|$$

and hence minimum uncertainty. □

The uncertainty requirement (55) mentioned in section 5 leads to the same condition on the states, as

$$\Delta_{\psi_m}\hat{J}^2 = \langle \hat{J}^2\rangle - g^{jk}\langle \hat{J}_j\rangle_{\psi_m}\langle \hat{J}_k\rangle_{\psi_m} = j(j+1) - m^2. \tag{67}$$

We will choose $|j,-j\rangle$ as our fixed vector. Our spin coherent states thus form the set of states

$$\left\{T^{(j)}(g)|j,-j\rangle,\ g \in SU(2)\right\}. \tag{68}$$

We want to find a parametrization for these states. With the canonical coherent states we saw that we could parametrize the states $|\alpha\rangle$ by the complex plane $\mathbb{C}$. Here we will see that spin coherent states can be parametrized by the sphere $\mathbb{S}^2$.

In section 5, we saw that we can find a parametrization by considering the quotient space G/H, where H denotes the isotropy subgroup of the Lie group G with respect to the representation. Let us examine the isotropy subgroup H of $SU(2)$ under the irrep $T^{(j)}$,

$$H = \left\{ g \in SU(2), \exists \phi \in [0, 2\pi),\ T^{(j)}(g)|j, -j\rangle = e^{i\phi}|j, -j\rangle \right\}. \tag{69}$$

So H consists of the elements of $SU(2)$ whose j-dimensional representation only changes $|j, -j\rangle$ by a phase factor. It can be shown that H forms the group of diagonal matrices

$$H = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^* \end{pmatrix},\ \lambda = e^{i\phi},\ \phi \in [0, 2\pi) \right\}. \tag{70}$$

Hence H is isomorphic to $U(1)$. Now, $\tilde{g} \to |g\rangle$ defines a bijection from $SU(2)/H$ to the set of coherent states, where $\tilde{g}$ is the element resulting from the canonical map from $SU(2)$ to $SU(2)/H$. We will denote $X \equiv SU(2)/H$, and

$$X_0 \equiv X \setminus \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}, \tag{71}$$

where $\left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$ denotes the equivalence class of this matrix in $SU(2)/H$. We claim that X_0 is homeomorphic to the set[1]

$$\left\{ \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha \end{pmatrix},\ \alpha \in \mathbb{R},\ \alpha > 0,\ \beta = \beta_1 + i\beta_2 \in \mathbb{C},\ \alpha^2 + \beta_1^2 + \beta_2^2 = 1 \right\}. \tag{72}$$

Proof. We will denote by $g(a, b)$ the element $\begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \in SU(2)$ for $a, b \in \mathbb{C}$ with $|a|^2 + |b|^2 = 1$. Firstly, note that for every b, $g(0, b)$ is in the coset of $g(0, 1)$, since

$$\begin{pmatrix} b & 0 \\ 0 & b^* \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & b \\ -b^* & 0 \end{pmatrix}.$$

Hence none of these matrices need to be considered. If $a \neq 0$ then we have the decomposition

$$\begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} = \begin{pmatrix} |a| & \frac{|a|}{a^*} b \\ -\frac{|a|}{a} b^* & |a| \end{pmatrix} \begin{pmatrix} \frac{a}{|a|} & 0 \\ 0 & \frac{a^*}{|a|} \end{pmatrix}.$$

[1] Note that [3] contains an error here; it states the condition on α as $\alpha \neq 0$ instead of $\alpha > 0$.

So if we define $\alpha = |a|$, $\beta = \frac{|a|}{a^*}b$ and $\lambda = \frac{a}{|a|}$, we get

$$g(a,b) = g(\alpha,\beta)g(\lambda,0).$$

Since $|\lambda| = 1$, $g(\lambda,0)$ is an element of H, and the elements of $SU(2)/H$ can be parametrized by all possible α and β; α needs to be real and larger than 0, while β can be any complex number such that $\alpha^2 + |\beta|^2 = 1$. So the proof is completed. □

The set X thus consists of the open half-sphere, together with one point, $(\alpha,\beta) = (0,1)$, on the boundary. We can represent any of our elements (α,β) in polar coordinates θ and ϕ:

$$(\alpha,\beta) = (\cos(\theta), \sin(\theta)e^{i\phi}).$$

The points of the half-sphere consist of all points for which $\theta \in [0, \frac{\pi}{2})$, $\phi \in [0, 2\pi)$, while the point on the boundary is given by $(\theta,\phi) = (\frac{\pi}{2}, 0)$. We now consider the homeomorphism defined by $\theta \to 2\theta$ for all elements but the north pole, $(\alpha,\beta) = (1,0)$, which is kept fixed. It maps the open half sphere to the sphere missing the south pole, while the boundary point itself is mapped to the south pole, completing the sphere. Thus X, and hence our coherent states, can be parametrized by elements of the sphere $\mathbb{S}^2$.

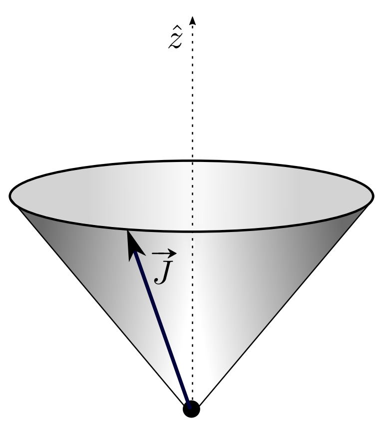

Figure 1: The cone traced out by $\vec{J}$.

Our results in this section can be understood intuitively through a familiar picture from quantum mechanics. Figure 1 shows the cone that is traced out by the "angular momentum vector" $\vec{J}$. For the state $|j,m\rangle$, the z-component of $\vec{J}$ is given by $m\hbar$, while the length of $\vec{J}$ is given by $\sqrt{j(j+1)}\hbar$. The wider the cone, the larger the uncertainty in the x and y components of $\vec{J}$. Hence, the $\vec{J}$ corresponding to the $|j,-j\rangle$ and $|j,j\rangle$ states, which lie closest to the z-axis, have minimum uncertainty. Furthermore, the direction of our z-axis in space is arbitrary. This corresponds to the fact that we can parametrize the spin coherent states by means of $\mathbb{S}^2$.

6.3 Comparison with Canonical Coherent States

If we now compare the spin coherent states to the canonical coherent states, we can note the following: though both sets of coherent states can be described by the action of a group on a minimum uncertainty state, of these two only the canonical coherent states can be described as the eigenstates of a lowering operator, for if we apply such a lowering operator to a state

$$|\psi\rangle = \sum_{m=-j}^{j} c_m |j, m\rangle$$

we lower all of the basis states, while annihilating the $|j, -j\rangle$ state. Hence, the coefficient of the basis vector with largest m and $c_m \neq 0$ will be zero after application of the lowering operator, which means the state could never be an eigenvector of the lowering operator. For the canonical coherent states this is not a problem, as there is no such highest state; for any $|n\rangle$ there is the state $|n+1\rangle$ such that $\hat{a}_-|n+1\rangle = \sqrt{n+1}|n\rangle$.
An interesting fact, which we will not explore here, is that as we let j go to infinity in what is referred to as the high spin limit, we transition into the canonical coherent states.

7 Pseudo-Spin Coherent States

We will examine here the coherent states of the Lie group $SU(1,1)$, also called pseudo-spin coherent states. References used for this section are [3], [5], [6] and [11].

7.1 $SU(1,1)$

$SU(1,1)$ is the group of matrices of the form

$$\begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix}, \ |\alpha|^2 - |\beta|^2 = 1. \tag{73}$$

Its Lie algebra $\mathfrak{su}(1,1)$ can be generated by vectors K_0, K_1 and K_2, satisfying commutation relations

$$[K_0, K_1] = iK_2, \ [K_1, K_2] = -iK_0, \ [K_2, K_0] = iK_1. \tag{74}$$

Note that the Lie algebra of $SU(1,1)$ can be obtained from the Lie algebra of $SU(2)$ by letting adding factors of i to the commutation relations of two of the generators, while adding $-i$ to the remaining one. The Casimir operator of $SU(1,1)$ is defined by

$$\hat{C} = \hat{K}_0^2 - \hat{K}_1^2 - \hat{K}_2^2. \tag{75}$$

$SU(1,1)$ is not a compact Lie group, and in fact has no finite dimensional unitary irreps except for the trivial representation. It can be shown that the irreps of $SU(1,1)$ come in three different kinds: its discrete series, its principal series and its complementary series. For the discrete series, the irreps can be parametrized by (half-)integers, like the irreps of $SU(2)$. The principal and complementary series can be parametrized by a real parameter.

7.2 The Pseudo-Sphere and the Poincaré Disk

We start with the vector space $\mathbb{R}^3$, and define on it the inner product $\langle x, y \rangle_{\mathcal{M}}$ defined by

$$\langle x, y \rangle_{\mathcal{M}} = x_1 y_1 + x_2 y_2 - x_0 y_0, \tag{76}$$

for $x = (x_0, x_1, x_2)$ and $y = (y_0, y_1, y_2)$. This space is often denoted $\mathbb{R}^{1,2}$. The set of points satisfying $\|x\|_{\mathcal{M}} = -1$ defines a hyperboloid with two symmetric sheets. We choose the upper one to define the pseudo-sphere $P\mathbb{S}^2$:

$$P\mathbb{S}^2 = \left\{ x \in \mathbb{R}^{1,2}, \ x_1^2 + x_2^2 - x_0^2 = -1, \ x_0 > 0 \right\}. \tag{77}$$

This surface in $\mathbb{R}^{1,2}$ can be parametrized by the so-called pseudo-polar coordinates, (τ, ϕ), defined by

$$x_0 = \cosh(\tau) \quad x_1 = \sinh(\tau)\cos(\phi) \quad x_2 = \sinh(\tau)\sin(\phi), \tag{78}$$

with $\tau \in [0, \infty)$, $\phi \in [0, 2\pi)$. We can project the pseudo-sphere on the unit disk $\mathbb{D}$ by drawing through each point of $P\mathbb{S}$ a line going through the point $(0, 0, -1)$, and taking its intersection with the (x_1, x_2) plane as the projection of the point. The resulting open unit disk in the complex plane with the metric defined on it by the projection from the pseudo-sphere is called the Poincaré disk $\mathbb{D}$.

7.3 Pseudo-Spin Coherent states

The discrete series itself consists of a negative series and a positive series and two singleton representations. We will consider here the positive series and the singleton representations. For the positive series, the irreps can be labeled by a number $k \in \left\{\frac{1}{2}, 1, \frac{3}{2}, 2, \dots\right\}$. Similar to $SU(2)$, we can define a basis of the representation space consisting of eigenvectors $|k, m\rangle$ of both operators $\hat{K}_0$ and $\hat{C}$,

$$\hat{K}_0|k, m\rangle = (k + m)|k, m\rangle \tag{79}$$

$$\hat{C}|k, m\rangle = k(k - 1)|k, m\rangle \tag{80}$$

where m is an integer larger or equal to zero. We can also define the ladder operators

$$\hat{K}_+ = \hat{K}_1 + i\hat{K}_2, \quad \hat{K}_- = \hat{K}_1 - i\hat{K}_2. \tag{81}$$

The coherent states of the discrete series can be presented as

$$|\zeta, k\rangle = (1 - |\zeta|)^2 \sum_{m=0}^{\infty} \sqrt{\frac{\Gamma(2k + m)}{m!\Gamma(2k)}} \zeta^m |k, m\rangle, \tag{82}$$

where ζ is a number in the open unit disk of the complex plane, $\{\zeta \in \mathbb{C},\ |\zeta| < 1\}$. The isotropy subgroup for these representations of $SU(1, 1)$ is again $U(1)$. We have seen that the spin coherent states could be parametrized by means of the sphere, $SU(2)/U(1) \cong \mathbb{S}^2$. Similarly, we can parametrize $SU(1, 1)$ by means of the pseudo-sphere: $SU(1, 1)/U(1) \cong P\mathbb{S}^2$.

The singleton representations are two discrete representations corresponding to $k = \frac{1}{4}$ and $k = \frac{3}{4}$ respectively. We can obtain a realization of these representations in terms of the ladder operators of the harmonic oscillator, by letting

$$\hat{K}_+ = \frac{1}{2}\hat{a}_+^2, \quad \hat{K}_- = \frac{1}{2}\hat{a}_-^2, \quad \hat{K}_0 = \frac{1}{4}(\hat{a}_-\hat{a}_+ + \hat{a}_+\hat{a}_-). \tag{83}$$

The Casimir operator in this case corresponds to

$$\hat{C} = k(k - 1)I = -3/16I \tag{84}$$

The states

$$|2n\rangle = \frac{(\hat{a}_+)^{2n}}{\sqrt{2n}}|0\rangle \tag{85}$$

form a basis for the unitary representation with $k = \frac{1}{4}$, while the states

$$|2n+1\rangle = \frac{(\hat{a}_+)^{2n+1}}{\sqrt{2n+1}}|0\rangle \tag{86}$$

do the same for $k = \frac{3}{4}$. Hence the singleton representations together give us the stationary states of the harmonic oscillator. The Weyl-Heisenberg translation operator is not in this representation of the group $SU(1,1)$, though it is an element of its so-called super-Lie group of $SU(1,1)$, generated by extending the Lie algebra $\mathfrak{su}(1,1)$ to a super Lie-algebra (for some references, see [6]).

8 Conclusion

Canonical coherent states are defined as the eigenstates of the lowering operator connected to the harmonic oscillator. They can also be described as shifts of the ground state $|0\rangle$ of the harmonic oscillator in phase space by the Weyl-Heisenberg translation operator. This translation operator can be used to form an irreducible, unitary representation ρ of the Weyl-Heisenberg group H, called the Schrödinger representation. The canonical coherent states then correspond to the states $\rho(g)|0\rangle$ for all elements $g \in H$ of the Weyl-Heisenberg group.

We can generalize this concept by defining generalized coherent states to be the states formed by applying some irreducible, unitary representation T of an arbitrary Lie group G to some fixed state $|\psi\rangle$ for all elements $g \in G$. These states can be parametrized by the quotient space G/H of G with the isotropy subgroup H of G with respect to the representation T. The choice of a fixed state can be arbitrary, but in physics a state with minimum uncertainty is preferred. For semi-simple Lie groups we define the uncertainty by use of the Casimir operator $\hat{C}$, by $\Delta_\psi \hat{C} = \langle \hat{C} \rangle_\psi - g^{jk} \langle \hat{A}_j \rangle_\psi \langle \hat{A}_k \rangle_\psi$. If the uncertainty of the Casimir operator is minimal in this sense, the generalized coherent states will all have minimum uncertainty.

Spin coherent states, the coherent states of $SU(2)$, can be found by considering the jth representation acting on the $|j, -j\rangle$ state for all elements of $SU(2)$, and can be parametrized by the $SU(2)/U(1) \cong \mathbb{S}^2$. The coherent states of the positive discrete series of $SU(1,1)$ can be parametrized by $SU(1,1)/U(1) \cong P\mathbb{S}^2$, where $P\mathbb{S}^2$ is the Pseudo-sphere, while the two singleton representations of $SU(1,1)$ give us the even and odd stationary states of the harmonic oscillator, respectively.

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