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# Delay Differential Equations

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## Abstract

This thesis introduces the concept of differential equations with delay, the so-called delay differential equations (DDEs). These differential equations depend on past history, and are therefore used in many models because they are more realistic than models independent of past history.

In this thesis, simple cases and linear systems of DDEs with a single delay will be discussed. We will look at proofs of existence and uniqueness, numerical and analytic solutions, and the stability of the steady-state solutions. This thesis ends with a discussion of the delayed logistic equation.

# Contents

# Chapter 1

# Introduction

Systems of ordinary or partial differential equations are independent of previous states or rates. Systems of differential equations that are dependent on previous states are called systems of delay differential equations (DDEs). In this thesis, these differential equations, also referred to as retarded functional differential equations (RFDEs), will be analyzed.

Models including delay differential equations exist, among other things, in biology, economics, and mechanics. An example of a DDE in the biology field is the Mackey-Glass equation for the density of certain blood cells. The delay in this differential equation is given by the time between initiation of cellular production in the bone marrow and release of mature cells into the blood.

Delay differential equations were introduced to create more realistic models since many processes depend on past history. A limitation of DDEs is that they only depend on past states and not past rates. Differential equations dependent on past rates are the so-called neutral delay differential equations (NDDEs), but these differential equations will not be discussed in this thesis. Furthermore, DDEs dependent on multiple past states will also not be discussed in this thesis.

In the next chapter, Chapter 2, simple cases of DDEs will be analyzed. In Chapter 3, linear systems with delay will be discussed. For these systems and the simple cases, we will look at analytic and numerical solutions. Furthermore, we will discuss the stability of the steady-state solutions. For the simple cases, we will also look at proofs of existence and uniqueness of solutions. This thesis ends with a discussion of an application: the delayed logistic equation, in Chapter 4.

# Chapter 2

# Simple Cases

## 2.1 Simplest DDE

### 2.1.1 Example

The simplest example of a DDE is given by

$$x'(t) = -x(t-\tau) \quad \text{for} \quad t \geq 0, \tag{2.1}$$

where $\tau > 0$ is called the delay. Suppose the initial condition for (2.1) is given by

$$x(t) = 1 \quad \text{for} \quad t \in [-\tau, 0]. \tag{2.2}$$

Following the procedure called the method of steps described in [5, p.13-14], the solution $x(t)$ for $t \in [(n-1)\tau, n\tau]$, $n \in \mathbb{N}$, can be determined in the following way.

For $t \in [0, \tau]$, it follows that $t - \tau \in [-\tau, 0]$. Therefore,

$$x'(t) = -x(t-\tau) = -1.$$

From this, we can conclude that

$$x(t) = x(0) + \int_0^t (-1)\, ds = 1 - t, \quad t \in [0, \tau]. \tag{2.3}$$

Similarly, we can show that

$$x'(t) = -x(t-\tau) = -[1-(t-\tau)], \quad t \in [\tau, 2\tau].$$

Therefore,

$$\begin{aligned} x(t) &= x(\tau) + \int_\tau^t -[1-(s-\tau)]\, ds \\ &= 1 - \tau + [-s + \frac{1}{2}(s-\tau)^2]_{s=\tau}^{s=t} \\ &= 1 - t + \frac{1}{2}(t-\tau)^2, \quad t \in [\tau, 2\tau]. \end{aligned} \tag{2.4}$$

By induction, it can be proven that

$$x(t) = 1 + \sum_{k=1}^{n} (-1)^k \frac{[t-(k-1)\tau]^k}{k!}, \quad t \in [(n-1)\tau, n\tau], \quad n \in \mathbb{N}. \tag{2.5}$$

The solution $x(t)$ is unique. This will be proven in the next section.

Let $f(a-)$ and $f(a+)$ denote $\lim_{s\to a^-} f(s)$ and $\lim_{s\to a^+} f(s)$ for a function $f$ and constant $a$, respectively. From the method of steps done above, it can immediately be seen that $x'(0-) = 0$ and $x'(0+) = -1$. Therefore, $x$ is not differentiable at $t = 0$. In fact, $x^{(n-1)}$ is not differentiable at $t = (n-1)\tau$ for all $n \in \mathbb{N}$, since $x^{(n)}((n-1)\tau-) = 0$ and $x^{(n)}((n-1)\tau+) = (-1)^n$.

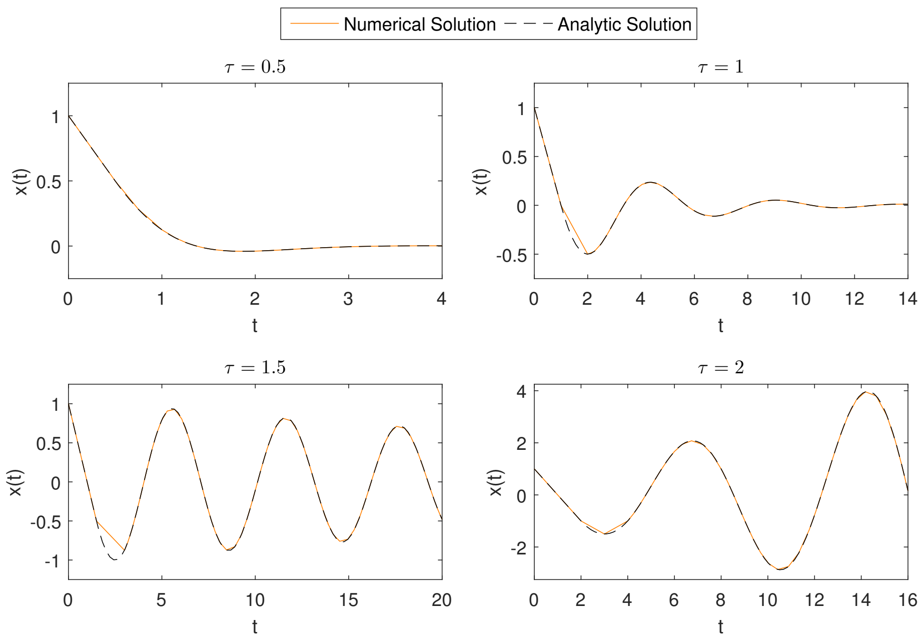


Figure 2.1: Numerical and analytic solutions of (2.1) with initial condition (2.2) for different values of $\tau$.

See Figure 2.1 for plots of numerical and analytic solutions of the initial value problem (2.1), (2.2), for some values of $\tau$. The analytic solution in this figure is given by (2.5). The Matlab codes used for computing the plots in this chapter can be found in Appendix A.1.

Using the package DDE23 in Matlab, plots can be made with on the horizontal axis the time $t$ and on the vertical axis the (numerical) solution $x(t)$ for different values of $\tau$. Comparing both solutions, we can conclude that the numerical solution approximates the analytic solution quite well. Furthermore, the accuracy increases as $t$ increases.

From Figure 2.1, we can also conclude that the solution $x(t)$ for $\tau = 0.5$ is starting to oscillate. The solutions oscillate around the steady-state solution of (2.1) given by $x \equiv 0$. Also, the oscillations increase as $\tau$ increases. According to the plots in this figure, at least until $\tau = 1.5$, $x = 0$ is stable. When $\tau = 2$,

$x = 0$ is no longer a stable solution. Later in this chapter, it will be proven that $x(t)$ oscillates for values of $\tau > 1/e \approx 0.37$, and $x = 0$ is stable when $\tau < \pi/2$ and unstable when $\tau > \pi/2 \approx 1.57$.

### 2.1.2 Existence and Uniqueness

Let $\tau \geq 0$ be a constant in $J = [\xi, \xi + a]$, where $\xi \geq 0$, and $a > 0$. The equation

$$x'(t) = f(t, x(t-\tau)) \quad \text{for} \quad t \in J \tag{2.6}$$

is called a delay differential equation, where $\tau > 0$ is called the delay. An initial condition for (2.6) is given by

$$x(t) = \phi(t) \quad \text{for} \quad t \in J_- = [\xi - \tau, \xi], \tag{2.7}$$

where $\phi$ is a given continuous function.

**Theorem 2.1.** *We consider the initial value problem* (2.6), (2.7), *where $f$ is continuous in the strip $S = J \times \mathbb{R}$, $\phi$ is continuous in $J_-$, and $\tau > 0$ is a constant in $J$. Then, there exists exactly one solution.*

The proof of this theorem is based on the proof of Theorem 7.V. in [6], and the method of steps used in the previous section.

*Proof.* Let

$$x_n'(t) = f(t, x_n(t-\tau)) \quad \text{for} \quad t \in [\xi + (n-1)\tau, \xi + n\tau],$$

where $n \in \mathbb{N}$.

For $t \in [\xi, \xi + \tau]$, it follows that $t - \tau \in [\xi - \tau, \xi]$. Therefore,

$$\begin{aligned} &x_1'(t) = f(t, x_1(t-\tau)) = f(t, \phi(t-\tau)), \quad \text{and} \\ &x_1(\xi) = \phi(\xi). \end{aligned}$$

From this, we can conclude that

$$\begin{aligned} x_1(t) &= x_1(\xi) + \int_\xi^t f(s, x_1(s-\tau))\, ds \\ &= \phi(\xi) + \int_\xi^t f(s, \phi(s-\tau))\, ds. \end{aligned} \tag{2.8}$$

Hence, $x_1$ is uniquely defined for $t \in [\xi, \xi + \tau]$. Similarly, we can show that

$$\begin{aligned} &x_2'(t) = f(t, x_2(t-\tau)) = f(t, x_1(t-\tau)), \quad t \in [\xi + \tau, \xi + 2\tau], \quad \text{and} \\ &x_2(\xi + \tau) = x_1(\xi + \tau). \end{aligned}$$

Therefore,

$$\begin{aligned} x_2(t) &= x_2(\xi + \tau) + \int_{\xi+\tau}^t f(s, x_2(s-\tau))\, ds \\ &= x_1(\xi + \tau) + \int_{\xi+\tau}^t f(s, x_1(s-\tau))\, ds. \end{aligned} \tag{2.9}$$

So $x_2$ is uniquely defined for $t \in [\xi + \tau, \xi + 2\tau]$. We can conclude that $x_n$ is defined by $x_{n-1}$ for all $n \in \mathbb{N}$. Therefore, $x_n$ is uniquely defined for $t \in [\xi + (n-1)\tau, \xi + n\tau]$, $n \in \mathbb{N}$. From this, we can conclude that $x$, defined by

$$x(t) = \begin{cases} \phi(t) & \text{for} \quad t \in J_- = [\xi - \tau, \xi], \\ \phi(\xi) + \displaystyle\int_\xi^t f(s, x(s-\tau))\, ds & \text{for} \quad t \in J = [\xi, \xi + a], \end{cases} \tag{2.10}$$

is well-defined for all values $t \in J_- \cup J = [\xi - \tau, \xi + a]$. Furthermore, $x(t)$ is a solution of the initial value problem (2.6), (2.7). From the definition of $x(t)$, we can conclude that $x(t)$ is the only solution. $\square$

### 2.1.3 General Equation

The general equation of the simplest DDE is given by

$$x'(t) = -\alpha x(t - \tau), \tag{2.11}$$

where $\alpha$ is a constant, and $\tau > 0$ is the delay. The case $\alpha > 0$ corresponds to negative feedback, and the case $\alpha < 0$ corresponds to positive feedback.

By scaling this DDE, the number of parameters can be reduced, which makes solving easier. We will determine a DDE with $U(\mu) = x(t)$, where the delay is the constant 1. For the scaling, we set $\mu := \eta t$, $\eta > 0$. Then,

$$\frac{dU}{d\mu} = \frac{dx}{\eta\, dt} = -\alpha\eta^{-1}x(t-\tau) = -\alpha\eta^{-1}U(\eta t - \eta\tau) = -\alpha\eta^{-1}U(\mu - \eta\tau).$$

If we set $\eta := 1/\tau$ and $\beta := \alpha\tau$, then

$$\frac{dU}{d\mu} = -\beta U(\mu - 1). \tag{2.12}$$

We introduce the following linear operator, defined on the differentiable functions:

$$L(U) = \frac{dU}{d\mu} + \beta U(\mu - 1).$$

We seek (complex) values of $\lambda$ such that $U(\mu) = e^{\lambda\mu}$ is a solution of (2.12). Filling this expression in for $L(U)$, we get

$$L(e^{\lambda\mu}) = \lambda e^{\lambda\mu} + \beta e^{\lambda(\mu-1)} = e^{\lambda\mu}[\lambda + \beta e^{-\lambda}]. \tag{2.13}$$

We want to solve the characteristic equation

$$h(\lambda) \equiv \lambda + \beta e^{-\lambda} = 0. \tag{2.14}$$

Then, $L(e^{\lambda\mu})$ is the zero function, hence $U(\mu) = e^{\lambda\mu}$ is a solution of (2.12).

To solve (2.14), we set $\lambda := x + iy$. Considering real and imaginary parts, we get the system

$$\begin{aligned} x &= -\beta e^{-x}\cos(y) \\ y &= \beta e^{-x}\sin(y). \end{aligned} \tag{2.15}$$

We call $\lambda \in \mathbb{C}$ a root of (2.12) of order $l$, where $l \geq 1$, if

$$h(\lambda) = h'(\lambda) = h''(\lambda) = \cdots = h^{(l-1)}(\lambda) = 0, \quad h^{(l)}(\lambda) \neq 0.$$

**Lemma 2.1.** *$\mu^j e^{\lambda\mu}$, $j = 0, 1, \dots, k$ are solutions of* (2.12) *if and only if $\lambda$ is a root of order at least $k+1$ of $h$.*

The proof of this lemma is taken from [5, p.17].

*Proof.* Differentiating (2.13) $k$ times with respect to $\lambda$ and using that this $k$th-derivative commutes with $L$ we find, by Leibniz' rule for the derivative of a product, that

$$L(\mu^k e^{\lambda\mu}) = (\frac{\partial}{\partial\lambda})^k [e^{\lambda\mu} h(\lambda)] = e^{\lambda\mu}[\sum_{j=0}^{k} C_j^k h^{(j)}(\lambda)\mu^{k-j}]$$

where $C_j^k = k!/j!(k-j)!$ are the binomial coefficients. The result follows immediately from this observation. □

As $h$ is an analytic function of the complex variable $\lambda$ it has the following elementary properties. See Appendix A in [5].

(A) The set of roots can have no accumulation point in $\mathbb{C}$; therefore, for each $R > 0$, the set of roots satisfying $|\lambda| \leq R$ is finite. It follows that the set of roots is a countable (possible finite) set.

(B) If the set of roots is infinite, denoted by $\{\lambda_n\}_{n=1}^{\infty}$, then $|\lambda_n| \to \infty$. Because $|\beta|e^{-\Re(\lambda_n)} = |\lambda_n|$, it follows that $\Re(\lambda_n) \to -\infty$. Consequently, for each $a \in \mathbb{R}$, $\Re(\lambda) \geq a$ for at most finitely many roots.

(C) If $\lambda$ is a root, then it is a root of finite order.

(D) If $\lambda$ is a root, so is its conjugate $\bar{\lambda}$.

**Lemma 2.2.** *The following hold.*

1. *If $\beta < 0$, then there is exactly one real root and it is positive.*
2. *If $0 < \beta < 1/e$, then there are exactly two real roots $x_1 < x_2$, both negative. $x_1 \to -\infty$ and $x_2 \to 0$ as $\beta \to 0$.*
3. *If $\beta = 1/e$, then there is a single real root of order two, namely $\lambda = -1$.*
4. *If $\beta > 1/e$, then there are no real roots.*

*Proof.* For (1): $h$ is an increasing function that crosses the $x$-axis, hence there is exactly one real root. Because $\beta < 0$, it follows that $\lambda = -\beta e^{-\lambda} > 0$.

For (2): $h$ has a minimum below the $x$-axis, hence there are exactly two real roots. Because $\beta > 0$, it follows that $\lambda = -\beta e^{-\lambda} < 0$. The last assertion follows from the following:

$$\beta = -\lambda e^{\lambda} \to 0 \qquad \text{if and only if} \qquad \lambda \to 0 \quad \text{or} \quad \lambda \to -\infty.$$

For (3): $h$ has a minimum on the $x$-axis, hence there is exactly one real root. Filling $\beta = 1/e$ in for $h$ gives the root $\lambda = -1$. Furthermore, $h'(-1) = 0$ and $h''(-1) = 1$, hence the order of $\lambda = -1$ is two.

For (4): $h$ has a minimum above the $x$-axis. □

The next result summarizes important information concerning the roots in the case $\beta > 0$.

**Proposition 2.2.** *The following hold for* (2.14)*.*

1. *If* $0 < \beta < \pi/2$*, then there exists* $\delta > 0$ *such that* $\Re(\lambda) \le -\delta$ *for all roots.*
2. *If* $\beta = \pi/2$*, then* $\lambda = \pm i\pi/2$ *are roots of order one.*
3. *If* $\beta > \pi/2$*, then there are roots* $\lambda = x \pm iy$ *with* $x > 0$*,* $y \in (\pi/2, \pi)$*.*

The proof of this proposition is taken from [5, p.18-20].

*Proof.* Because $\beta > 0$, if there is a root $x + iy$ with $x \ge 0$ and $y > 0$ of (2.15), then $\cos(y) \le 0 < \sin(y)$. So $y \in S \equiv \bigcup_{n=0}^{\infty}\{[\pi/2, \pi) + 2n\pi\}$. Furthermore,

$$\frac{\sin(y)}{y} = \frac{e^x}{\beta}$$

must hold. As

$$\frac{d}{dy}\frac{\sin(y)}{y} = \frac{y\cos(y) - \sin(y)}{y^2} < 0, \quad y \in S$$

and $\sin(y)/y = 2/\pi$ when $y = \pi/2$, we conclude that $\sin(y)/y \le 2/\pi$ for $y \in S$. Therefore,

$$\frac{1}{\beta} \le \frac{e^x}{\beta} = \frac{\sin(y)}{y} \le \frac{2}{\pi}.$$

So it follows that $\beta \ge \pi/2$. Thus, if $\beta < \pi/2$, then $\Re(\lambda) < 0$ for every root $\lambda$. This and the last assertion in (B) above proves (1).

The second assertion follows immediately from computation.

Let's now turn to the final assertion (3). If we write our root as $\lambda = re^{i\theta}$, then (2.14) becomes

$$r[\cos(\theta - \pi) + i\sin(\theta - \pi)] = \beta e^{-x}[\cos(-y) + i\sin(-y)].$$

Equivalently,

$$r = \beta e^{-x} \quad \text{and} \quad \theta - \pi = -y + 2k\pi$$

for some integer $k$. Let's search for a root in the first quadrant on the ray through the origin making angle $\theta \in (0, \pi/2)$ with the positive $x$-axis. Then, taking $k = 0$, $y(\theta) = \pi - \theta > 0$ and so $x(\theta) > 0$ is determined by trigonometry because $\tan(\theta) = y/x$.

We claim that:

$$\begin{aligned} x(\theta) &= (\pi - \theta)\cot(\theta) \\ y(\theta) &= \pi - \theta, \quad 0 < \theta < \pi/2 \\ \beta(\theta) &= \frac{\pi - \theta}{\sin(\theta)} e^{x(\theta)} \end{aligned} \tag{2.16}$$

is a one-parameter family of solutions of (2.14) satisfying $x > 0$ and $\pi/2 < y < \pi$.

Clearly, $x(\theta)$, $y(\theta)$, $\beta(\theta)$ depend continuously on $\theta \in (0, \pi/2)$. Also,

$$x(\theta) \to +\infty, \quad y(\theta) \to \pi, \quad \beta(\theta) \to +\infty, \quad \theta \to 0$$

and

$$x(\theta) \to 0, \quad y(\theta) \to \pi/2, \quad \beta(\theta) \to \pi/2, \quad \theta \to \pi/2.$$

Inasmuch as $\beta(\theta)$ is strictly decreasing on $(0, \pi/2)$ it follows that the range of $\beta$ is $(\pi/2, \infty)$. □

From Theorem 4.3 in [5], we can conclude that the stability of the steady-state solution of a delay differential equation is determined in the same way as for an ordinary differential equation. So the steady-state solution of (2.11) given by $x \equiv 0$ is asymptotically stable if $\Re(\lambda) < 0$ for all roots $\lambda$ of (2.14), and this solution is unstable if there is a root $\lambda$ with positive real part. Using this result, the following corollary follows immediately from Proposition 2.2 and Lemma 2.2.

**Corollary 2.3.** *The following hold for* (2.11).

1. *If* $\alpha < 0$*, then* $x = 0$ *is unstable.*
2. *If* $0 < \alpha\tau < \pi/2$*, then* $x = 0$ *is asymptotically stable.*
3. *If* $\alpha\tau = \pi/2$*, then* $x = \sin(\pi\mu/2)$ *and* $x = \cos(\pi\mu/2)$ *are solutions.*
4. *If* $\alpha\tau > \pi/2$*, then* $x = 0$ *is unstable.*

The next result summarizes important information concerning oscillations.

**Theorem 2.4.** *For every real* $\alpha$ *and* $\tau > 0$*, the following are equivalent.*

1. *Every solution of* (2.11) *is oscillatory.*
2. $\alpha\tau > 1/e$.

*Proof.* For every real $\lambda$, $U(\mu) = e^{\lambda\mu}$ is either a monotonic or constant function. Therefore, $x(t) = U(\mu)$ is not oscillatory for real $\lambda$. So $x(t)$ is oscillatory if and only if $\lambda$ is a complex (not real) root. By Lemma 2.2, $\lambda$ is a complex (not real) root if and only if $\beta = \alpha\tau > 1/e$. This proves the theorem. □

## 2.2 Scalar DDE

### 2.2.1 General Equation

The homogeneous equation of the scalar DDE is given by

$$x'(t) = ax(t) + bx(t - \tau), \tag{2.17}$$

where $a, b$ are constants, and $\tau > 0$ is the delay. The nonhomogeneous equation will be discussed in the next chapter.

We seek a nontrivial solution of (2.17) of the form

$$x(t) = e^{\lambda t}c, \quad c \neq 0,$$

where $\lambda$ is complex and $c$ is a constant.

The characteristic equation we want to solve is given by

$$h(\lambda) \equiv \lambda - a - be^{-\lambda\tau} = 0. \tag{2.18}$$

This expression can be simplified by multiplying the right-hand side by $\tau$ and setting

$$z := \lambda\tau, \quad \alpha := a\tau, \quad \beta := b\tau.$$

We get

$$h(z) \equiv z - \alpha - \beta e^{-z} = 0. \tag{2.19}$$

Solving this characteristic equation can be done in a similar way as the characteristic equation for the simplest equation in the previous section.

We can conclude the following about the stability of the steady-state solution given by $x \equiv 0$.

**Theorem 2.5.** *The following hold for* (2.17).

1. *If* $a + b > 0$*, then* $x = 0$ *is unstable.*
2. *If* $a + b < 0$ *and* $b \geq a$*, then* $x = 0$ *is asymptotically stable.*
3. *If* $a + b < 0$ *and* $b < a$*, then there exists* $\tau^* > 0$ *such that* $x = 0$ *is asymptotically stable for* $0 < \tau < \tau^*$ *and unstable for* $\tau > \tau^*$*.*

The proof of this theorem can be found in [5, p.53-54].

### 2.2.2 Existence and Uniqueness

Let $\tau > 0$ be a constant in $J = [\xi, \xi + a]$, where $\xi \geq 0$, and $a > 0$. Let's consider the delay differential equation

$$x'(t) = f(t, x(t), x(t - \tau)) \quad \text{for} \quad t \in J \tag{2.20}$$

with a single delay $\tau > 0$. Assume that $f(t, x, y)$ and $f_x(t, x, y)$ are continuous on $\mathbb{R}^3$. An initial condition for (2.20) is given by

$$x(t) = \phi(t) \quad \text{for} \quad t \in J_- = [\xi - \tau, \xi], \tag{2.21}$$

where $\phi$ is a given continuous function on $\mathbb{R}$.

Equation (2.20) can be solved by the method of steps as follows. For $t \in [\xi, \xi + \tau]$, $x(t)$ must satisfy the initial value problem for the ODE:

$$\begin{aligned} x_1'(t) &= f(t, x_1(t), \phi(t - \tau)), \\ x_1(\xi) &= \phi(\xi). \end{aligned}$$

As $g(t, x_1) \equiv f(t, x_1, \phi(t - \tau))$ and $g_{x_1}(t, x_1)$ are continuous, a unique solution of this ODE is guaranteed by standard existence results from ODE theory.

For $t \in [\xi + \tau, \xi + 2\tau]$, $x(t)$ must satisfy the initial value problem for the ODE:

$$\begin{aligned} x_2'(t) &= f(t, x_2(t), x_1(t - \tau)), \\ x_2(\xi + \tau) &= x_1(\xi + \tau). \end{aligned}$$

Again, standard existence results for such problems guarantee the existence of a unique solution. We may repeat this process to extend the solution to $[\xi + 2\tau, \xi + 3\tau]$, and so on.

Following this procedure, a unique solution of the initial value problem (2.20), (2.21) can be determined.

# Chapter 3

# Linear Systems

## 3.1 Preliminaries

A DDE with a single delay is of the form

$$x'(t) = f(t, x(t), x(t-\tau)) \quad \text{for} \quad t \geq \xi, \tag{3.1}$$

where $f$ is a given continuous function, $\xi \geq 0$ is a constant, and $\tau > 0$ is the delay. An initial condition for (3.1) is given by

$$x(t) = \phi(t) \quad \text{for} \quad t \in [\xi - \tau, \xi], \tag{3.2}$$

where $\phi$ is a given continuous function.

We are interested in determining the state of the system (3.1), (3.2) at time $t \geq \xi \geq 0$. The state of a system at time $t \geq 0$ includes all information needed to determine the state of the system at future times $s \geq t$. Therefore, the state of the system (3.1), (3.2) at time $t \geq \xi$ includes $x(\eta)$ for all $\eta \in [t-\tau, t]$. Hence, we conclude that this state, which we denote by $x_t$, is given by

$$x_t(\theta) := x(t+\theta) \quad \text{for} \quad -\tau \leq \theta \leq 0. \tag{3.3}$$

In this chapter, we will look at linear systems with delay. We will discuss linear DDEs of the form

$$x'(t) = L(x_t) \quad \text{for} \quad t \geq \xi, \tag{3.4}$$

where $\xi \geq 0$ is a constant, $x_t$ is defined as (3.3), and $L$ is the map $C \to \mathbb{C}^n$ where $C = C([-\tau, 0], \mathbb{C}^n)$. An initial condition for (3.4) is of the form

$$x_\xi = \phi, \tag{3.5}$$

where $x_\xi$ is defined as (3.3), and $\phi$ is a given continuous function in $C$.

The map $L$ is linear if it satisfies

$$L(a\phi + b\psi) = aL(\phi) + bL(\psi), \quad \phi, \psi \in C, \quad a, b \in \mathbb{C}.$$

In the next section, an example of the following linear DDE will be discussed:

$$x'(t) = Ax(t) + Bx(t-\tau), \tag{3.6}$$

where $A, B$ are $n \times n$ matrices, and $\tau > 0$ is the delay. This equation can be rewritten in the form of (3.4) by defining the map $L$ as

$$L(y) = Ay(0) + By(-\tau),$$

because then, using (3.3),

$$L(x_t) = Ax_t(0) + Bx_t(-\tau) = Ax(t) + Bx(t - \tau),$$

which is precisely (3.6).

We introduce the Laplace transform:

$$F(s) := \int_0^\infty e^{-st} f(t)\, dt, \tag{3.7}$$

where $f(t)$ is a function on $[0, \infty)$. The domain of $F(s)$ consists of all the values for which the integral in (3.7) exists. The Laplace transform of $f$ is denoted by both $F$ and $\mathfrak{L}\{f\}$. We let $f(t)$ be an exponentially bounded function of order $\alpha$; that is,

$$|f(t)| \leq Me^{\alpha t} \quad \text{for all } t \geq T$$

for some positive constants $M, T$. Then, if $f(t)$ is also piecewise continuous on $[0, \infty)$, then $\mathfrak{L}\{f\}(s)$ exists for $s > \alpha$.

The key property of the Laplace transform that we exploit is that the transform of a convolution is the product of the transforms. If $f(t)$ and $g(t)$ are piecewise continuous on $[0, \infty)$, then their convolution, denoted by $f * g$, is defined by

$$(f * g)(t) := \int_0^t f(s)g(t - s)\, ds = \int_0^t f(t - s)g(s)\, ds. \tag{3.8}$$

The Laplace transform satisfies

$$\mathfrak{L}\{f * g\}(s) = F(s)G(s),$$

where $F = \mathfrak{L}\{f\}$ and $G = \mathfrak{L}\{g\}$.

## 3.2 Linear DDE

### 3.2.1 Example

In this section, we will discuss the following homogeneous linear DDE:

$$x'(t) = Ax(t) + Bx(t - \tau) \quad \text{for} \quad t \geq 0, \tag{3.9}$$

where $A = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and $\tau > 0$ is the delay. The initial condition for (3.9) is given by

$$\phi(t) := x(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{for} \quad t \in [-\tau, 0]. \tag{3.10}$$

We will start with determining the eigenvalues and the corresponding eigenvectors of matrix $A$. The characteristic equation is given by

$$h(\lambda) \equiv \det(A - \lambda I) = \lambda^2 - 3\lambda + 2 = 0. \tag{3.11}$$

Therefore, the eigenvalues of $A$ are $\lambda_1 = 1$ and $\lambda_2 = 2$.

From

$$(A - \lambda_1 I)v_1 = \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix} v_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

it follows that an eigenvector corresponding to $\lambda_1$ is given by $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. From

$$(A - \lambda_2 I)v_2 = \begin{bmatrix} -2 & 1 \\ -2 & 1 \end{bmatrix} v_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

it follows that an eigenvector corresponding to $\lambda_2$ is given by $v_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Therefore, we can conclude that the solution of

$$x'(t) = Ax(t)$$

is given by

$$x_h(t) = c_1 e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \tag{3.12}$$

where $c_1, c_2$ are constants.

For $t \in [0, \tau]$,

$$\begin{aligned} x'(t) &= Ax(t) + B\phi(t - \tau) \\ &= \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \end{aligned} \tag{3.13}$$

A particular solution of (3.13) is of the form

$$x_p(t) = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix},$$

where $d_1, d_2$ are constants.

Substituting $x_p(t)$ for $x(t)$ in (3.13) gives

$$x_p'(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} d_2 + 1 \\ -d_1 + 3d_2 + 1 \end{bmatrix}.$$

Therefore, $d_1 = -2$ and $d_2 = -1$. So the particular solution of (3.13) is given by

$$x_p(t) = \begin{bmatrix} -2 \\ -1 \end{bmatrix} = -\begin{bmatrix} 2 \\ 1 \end{bmatrix}. \tag{3.14}$$

Hence, the general solution of (3.13) is of the form

$$x(t) = x_h(t) + x_p(t) = c_1 e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

where $c_1, c_2$ are constants. Because $x(0) = \phi(0)$, it follows that

$$x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 - 2 \\ c_1 + 2c_2 - 1 \end{bmatrix}.$$

Therefore, $c_1 = 4$ and $c_2 = -1$. So the general solution of (3.9), (3.10), for $t \in [0, \tau]$, is given by

$$x(t) = x_h(t) + x_p(t) = 4e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} - e^{2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix}. \tag{3.15}$$

In a similar way, general solutions of the initial value problem (3.9), (3.10) defined on other intervals of the form $[(n-1)\tau, n\tau]$, $n \in \mathbb{N}$, can be determined.

### 3.2.2 General Equation

The general equation of the nonhomogeneous linear DDE is given by

$$x'(t) = Ax(t) + Bx(t-\tau) + f(t) \quad \text{for} \quad t \geq 0, \tag{3.16}$$

where $A, B$ are $n \times n$ matrices, $\tau > 0$ is the delay, and $f$ is a given continuous function. The initial condition for (3.16) is of the form

$$x(t) = \phi(t) \quad \text{for} \quad t \in [-\tau, 0], \tag{3.17}$$

where $\phi$ is a given continuous function.

Applying the Laplace transform to the left-hand side of (3.16), we obtain

$$\begin{aligned}
\mathfrak{L}\{x'\}(s) &= \int_0^\infty e^{-st} x'(t)\, dt \\
&= [e^{-st}x(t)]_{t=0}^{t=\infty} + s \int_0^\infty e^{-st} x(t)\, dt \\
&= sX(s) - \phi(0).
\end{aligned}$$

Applying the Laplace transform to the right-hand side of (3.16), we obtain

$$\begin{aligned}
sX(s) - \phi(0) &= \int_0^\infty e^{-st}[Ax(t) + Bx(t-\tau) + f(t)]\, dt \\
&= A \int_0^\infty e^{-st} x(t)\, dt + B \int_0^\infty e^{-st} x(t-\tau)\, dt + \int_0^\infty e^{-st} f(t)\, dt \\
&= AX(s) + B\left[\int_0^\tau e^{-st}\phi(t-\tau)\, dt + \int_\tau^\infty e^{-st} x(t-\tau)\, dt\right] + F(s) \\
&= AX(s) + B\left[\int_0^\tau e^{-st}\phi(t-\tau)\, dt + \int_0^\infty e^{-s(t+\tau)} x(t)\, dt\right] + F(s) \\
&= AX(s) + B\left[\int_0^\tau e^{-st}\phi(t-\tau)\, dt + e^{-s\tau}X(s)\right] + F(s) \\
&= [A + e^{-s\tau}B]X(s) + B\int_0^\tau e^{-st}\phi(t-\tau)\, dt + F(s) \\
&= [A + e^{-s\tau}B]X(s) + B\Phi(s) + F(s),
\end{aligned}$$

where $\Phi = \mathfrak{L}\{\phi(\cdot - \tau)\}$, and where we have extended $\phi$ to $[-\tau, \infty)$ by making it zero for $t > 0$. This leads to

$$X(s) = K(s)[\phi(0) + B\Phi(s) + F(s)], \tag{3.18}$$

where

$$K(s) = (sI - A - e^{-s\tau}B)^{-1}$$

is a matrix-valued function.

In order to make use of the convolution result, we need to know the inverse transform $k$ of $K$. In view of the calculations above, we see that $k$ is a solution of (3.16), (3.17), with $f = \mathbf{0}$, for the initial data

$$\xi(\theta) = \begin{cases} I & \text{for} \quad \theta = 0, \\ \mathbf{0} & \text{for} \quad \theta \in [-\tau, 0). \end{cases} \tag{3.19}$$

In spite of the discontinuity of $\xi$ at zero, the method of steps readily establishes that the solution $k$ exists for $t \geq 0$. The matrix function $k$ is called the fundamental matrix solution of (3.16), (3.17).

**Lemma 3.1.** *We may express the solution of the initial value problem* (3.16), (3.17) *as*

$$x(t) := x(t; \phi, f) = x(t; \phi, 0) + x(t; 0, f). \tag{3.20}$$

*Proof.* For $t \in [0, \tau]$, the following hold:

$$\begin{aligned} x'(t; \phi, f) &= Ax(t; \phi, f) + B\phi(t - \tau) + f(t); \\ x'(t; \phi, 0) &= Ax(t; \phi, 0) + B\phi(t - \tau); \\ x'(t; 0, f) &= Ax(t; 0, f) + f(t). \end{aligned}$$

So, for $t \in [0, \tau]$,

$$x'(t; \phi, 0) + x'(t; 0, f) = A[x(t; \phi, 0) + x(t; 0, f)] + B\phi(t - \tau) + f(t).$$

Hence,

$$x(t) := x(t; \phi, f) = x(t; \phi, 0) + x(t; 0, f) \quad \text{for} \quad t \in [0, \tau]. \tag{3.21}$$

We can conclude from (3.21) that (3.20) holds for all $t \geq 0$, by uniqueness of the solutions $x(t; \phi, f), x(t; \phi, 0)$, and $x(t; 0, f)$ for all $t \geq 0$. □

## 3.3 General Results

In this chapter, we are interested in solving the linear DDE:

$$x'(t) = L(x_t), \tag{3.4}$$

where $L$ is the map $C \to \mathbb{C}^n$ where $C = C([-\tau, 0], \mathbb{C}^n)$, and where $x_t$ is defined as

$$x_t(\theta) := x(t + \theta) \quad \text{for} \quad -\tau \leq \theta \leq 0. \tag{3.3}$$

We seek a nontrivial solution of (3.4) of the form

$$x(t) = e^{\lambda t} v, \quad v \neq 0,$$

where $\lambda$ and the elements of the vector $v$ are complex. We will use the notation $\exp_\lambda$, where $\exp_\lambda(\theta) = e^{\lambda\theta}$. Then,

$$x_t(\theta) = x(t + \theta) = e^{\lambda(t+\theta)} v = e^{\lambda t} e^{\lambda\theta} v = e^{\lambda t} \exp_\lambda(\theta) v.$$

Therefore, $x_t = e^{\lambda t}(\exp_\lambda)v$.

Using that $L$ is a linear map and that $e^{\lambda t} \in \mathbb{C}$ for all $\lambda, t$; for $x(t) = e^{\lambda t}v$ to be a solution of (3.4), we must have

$$x'(t) = \lambda e^{\lambda t}v = L(x_t) = L(e^{\lambda t}(\exp_\lambda)v) = e^{\lambda t}L((\exp_\lambda)v).$$

Dividing both sides of this equation by $e^{\lambda t}$, we get

$$\lambda v = L((\exp_\lambda)v).$$

Let $\{e_1, e_2, \ldots, e_n\}$ be the standard basis for $\mathbb{C}^n$. Then, we can write $v = \sum_j v_j e_j$. Using this notation, we get $L((\exp_\lambda)v) = \sum_j v_j L((\exp_\lambda)e_j)$, by linearity of $L$ and the fact that $v_j \in \mathbb{C}$ for all $j$.

Define $L_\lambda$ to be the $n \times n$ matrix

$$L_\lambda = (L((\exp_\lambda)e_1) \,|\, L((\exp_\lambda)e_2) \,|\, \cdots \,|\, L((\exp_\lambda)e_n)) = (L_i((\exp_\lambda)e_j)),$$

where $L_i(\phi)$ is element $i$ of $L(\phi)$. Then,

$$\begin{aligned}
\lambda v &= L((\exp_\lambda)v) \\
&= \sum_j v_j L((\exp_\lambda)e_j) \\
&= (L((\exp_\lambda)e_1) \,|\, L((\exp_\lambda)e_2) \,|\, \cdots \,|\, L((\exp_\lambda)e_n)) \cdot (v_1\, v_2\, \cdots\, v_n)^T \\
&= L_\lambda v.
\end{aligned}$$

Therefore, $x(t) = e^{\lambda t}v$ is a nontrivial solution of (3.4) if and only if $\lambda$ is a solution of the characteristic equation:

$$\det(\lambda I - L_\lambda) = 0, \tag{3.22}$$

and the vector $v$ belongs to the null space of $\lambda I - L_\lambda$.

Let's return to the DDE discussed before:

$$x'(t) = Ax(t) + Bx(t - \tau), \tag{3.6}$$

where $A, B$ are $n \times n$ matrices, and $\tau > 0$ is the delay. Rewriting this DDE in the form of (3.4), see Section 3.1, the map $L$ is defined as

$$L(y) = Ay(0) + By(-\tau).$$

For this DDE, the $n \times n$ matrix $L_\lambda$ is given by

$$\begin{aligned}
L_\lambda &= (L_i((\exp_\lambda)e_j)) \\
&= ((A\exp_\lambda(0)e_j + B\exp_\lambda(-\tau)e_j)_i) \\
&= ((Ae_j + e^{-\lambda\tau}Be_j)_i) \\
&= (Ae_1 + e^{-\lambda\tau}Be_1 \,|\, Ae_2 + e^{-\lambda\tau}Be_2 \,|\, \cdots \,|\, Ae_n + e^{-\lambda\tau}Be_n) \\
&= (A + e^{-\lambda\tau}B)\,(e_1\, e_2\, \cdots\, e_n) \\
&= (A + e^{-\lambda\tau}B)I \\
&= A + e^{-\lambda\tau}B.
\end{aligned}$$

Therefore, the characteristic equation is given by

$$\det(\lambda I - L_\lambda) = \det(\lambda I - A - e^{-\lambda\tau}B) = 0. \tag{3.23}$$

Solving (3.23) for $\lambda$ and determining the vector $v$ from the null space of $\lambda I - A - e^{-\lambda\tau}B$, we can conclude that $x(t) = e^{\lambda t}v$ is a nontrivial solution of (3.6).

# Chapter 4

# Application

## 4.1 Delayed Logistic Equation

The logistic equation is given by

$$x'(t) = ax\left(1 - \frac{x}{K}\right), \tag{4.1}$$

where $a$ is the growth rate and $K$ is the carrying capacity of the ecosystem. According to the logistic equation, the growth rate of a population is directly proportional to the current population and the availability of resources in the ecosystem.

In this section, we will discuss the delayed logistic equation:

$$x'(t) = ax(t)[1 - bx(t - \tau)] \quad \text{for} \quad t \geq 0, \tag{4.2}$$

where $a, b > 0$ are constants with $b \equiv 1/K$, and $\tau > 0$ is the delay. As initial condition for (4.2), we consider

$$x(t) = 0.01 \quad \text{for} \quad t \in [-\tau, 0]. \tag{4.3}$$

We denote the solution of the initial value problem (4.2), (4.3) by $x(t)$.

We start with determining the equilibrium points of (4.2), which we denote by $x^*$. Because $x^*$ is constant, it follows that $x^* = x(t) = x(t-\tau)$. Substituting this result in (4.2), we get

$$ax^*[1 - bx^*] = 0.$$

Therefore, the equilibrium points of (4.2) are given by $x^* = 0$ and $x^* = 1/b$.

Following the procedure described in [4], we can linearize (4.2) in the following way. Let $p(t)$ be a small variation in the population such that higher powers of $p$ may be neglected. Substituting $x(t) \equiv x^* + p(t)$ in (4.2) gives the DDE:

$$p'(t) = a[x^* + p(t)][1 - bx^* - bp(t - \tau)] \quad \text{for} \quad t \geq 0. \tag{4.4}$$

We will use the same initial condition for (4.4) as for (4.2):

$$p(t) = 0.01 \quad \text{for} \quad t \in [-\tau, 0]. \tag{4.5}$$

Substituting $x^* = 0$ in (4.4) gives

$$p'(t) = ap(t)[1 - bp(t - \tau)].$$

Since we may neglect higher powers of $p$, we end up with the following ODE:

$$p'(t) = ap(t). \tag{4.6}$$

Therefore,

$$p(t) = ce^{at},$$

where $c$ is a constant. Taking the initial condition (4.5) into account, the solution of the initial value problem (4.4), (4.5), for $x^* = 0$, is given by

$$p_1(t) = 0.01e^{at}. \tag{4.7}$$

Since $a > 0$, it follows that $x^* = 0$ is unstable.

Substituting $x^* = 1/b$ in (4.4) gives

$$p'(t) = a[1/b + p(t)][-bp(t - \tau)].$$

Since we may neglect higher powers of $p$, we end up with the following DDE:

$$p'(t) = -ap(t - \tau). \tag{4.8}$$

This differential equation is already discussed in Section 2.1.3. The equilibrium point of (4.8) is given by $p^* = 0$. Because $p^* \neq x^*$ and we are interested in having $p^* = x^*$, we will add a certain constant to (4.8) such that $p^* = x^*$ holds. This constant has to be equal to $a/b$, because then, we have the following DDE:

$$p'(t) = -a[p(t - \tau) - 1/b], \tag{4.9}$$

where the equilibrium point is given by $p^* = x^* = 1/b$. Instead of (4.5), we consider (4.7) as initial condition for (4.9). The reasoning behind this becomes clear by looking at the plots of Figure 4.1. We denote the solution of the initial value problem (4.9), (4.7) by $p_2(t)$.

Using the package DDE23 in Matlab, plots of the numerical solution $x(t)$ for different values of $\tau, a$, and $b$ can be made. Because changing the value of $b$ does not affect the shapes of the plots, we will only consider one value for $b$.

See Figure 4.1 for the plots of $x(t), p_1(t)$, and $p_2(t)$ for some values of $\tau, a$, and $b = 0.5$. Because determining $p_2(t)$ analytically is quite difficult, the numerical solution $p_2(t)$ is included in these plots. The Matlab codes used for computing the plots in this chapter can be found in Appendix A.2.

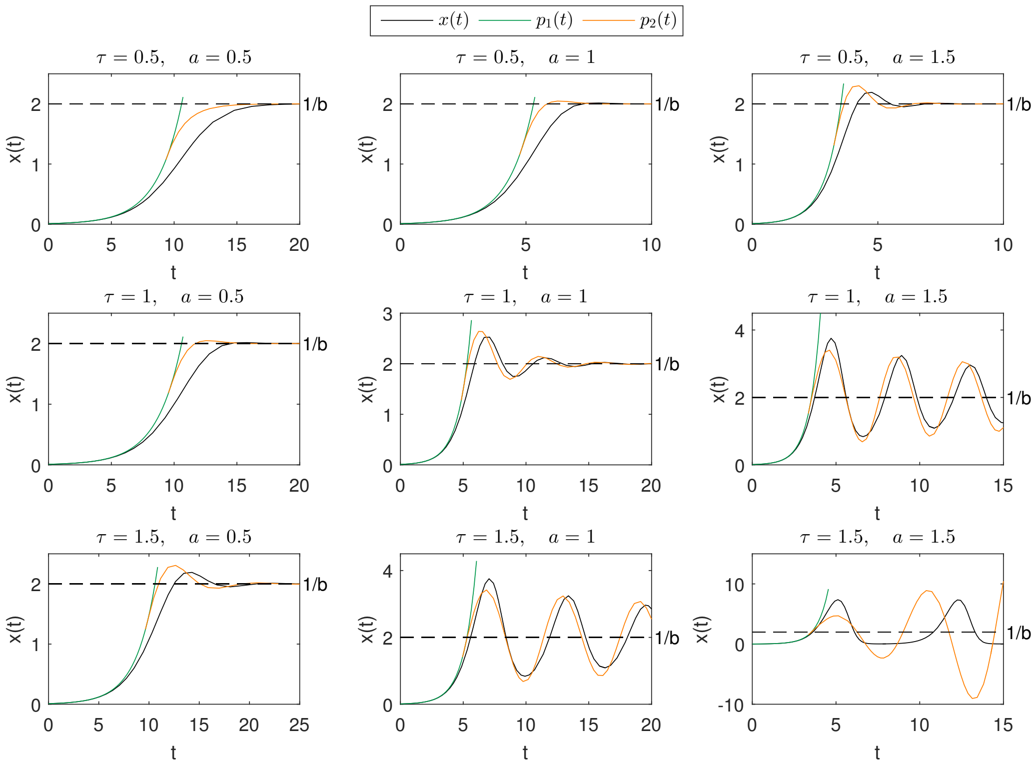


Figure 4.1: Plots of $x(t)$, $p_1(t)$, and $p_2(t)$, for $b = 0.5$, and different values of $a$ and $\tau$.

From the plots in Figure 4.1, we can conclude that $p_1(t)$ and $p_2(t)$ approximate $x(t)$ very well, except for the last case. For this case, $a\tau = (1.5)^2 = 2.25 > \pi/2$. By Corollary 2.3, it follows that $x^* = 1/b$ is unstable. From the plot of this case, we can conclude that $x(t)$ contains complex (not real) values, and $p_1(t)$ and $p_2(t)$ only contain real values. Therefore, $x(t)$ cannot be approximated by $p_1(t)$ and $p_2(t)$. For the other cases, we can conclude from the plots, and Corollary 2.3, that $x^* = 1/b$ is stable.

Furthermore, we can conclude from these plots that the cases where $a\tau$ has the same value, the graphs look similar. For example, consider the cases $\tau = 0.5, a = 1$, and $\tau = 1, a = 0.5$. Then, the plots of these cases look the same, but the time intervals are different. The same holds for the other cases with the same value of $a\tau$.

# Appendix A

# Matlab Codes

## A.1 Chapter 2: Simple Cases

**Figure 2.1**

```
function [dxdt] = SCDDE1dde (t,x,Z)
% x'(t) = -x(t-tau) for t>=0
% tau>0 is the delay
xlag = Z(:,1);
dxdt = [-xlag(1)];
end
```

```
function [s] = SCDDE1hist (t)
% Initial data is given by
% x(t) = 1 for -tau<=t<=0
s = [1];
end
```

```
%Analytic Solution
n = ...;
tau = ...;
t0 = 0;
tend = ...;
t = linspace(t0,tend);
x = zeros(length(t));
for i=1:n
    for j=1:length(t)
        if ((i-1)*tau <= t(j)) && (t(j) <= i*tau)
            A = zeros(i);
            for k=1:i
                A(k) = (-1)^k*((t(j)-(k-1)*tau)^k)/...
                 factorial(k);
            end
        x(j) = 1 + sum(A);
        end
```

```
        end
end

%Numerical Solution
sol = dde23(@SCDDE1dde,[tau],@SCDDE1hist,[t0,tend]);

%Figure
figure
plot(sol.x,sol.y,'Color',[255,128,0]./255); hold on
plot(t,x,'--k');
```

## A.2 Chapter 4: Application

**Figure 4.1**

```
function [dxdt] = APPDDE41dde (t,x,Z)
% x'(t) = ax(t)[1-bx(t-tau)] for t>=0
% tau>0 is the delay
a = ...;
b = 0.5;
xlag = Z(:,1);
dxdt = a*x(1)*[1-b*xlag(1)];
end
```

```
function [s] = APPDDE41hist (t)
% Initial data is given by
% x(t) = 0.01 for -tau<=t<=0
s = [0.01];
end
```

```
function [dp2dt] = APPDDE42dde (t,p2,Z)
% p2'(t) = -a[p2(t-tau)-1/b] for t>=t02
% tau>0 is the delay
a = ...;
b = 0.5;
p2lag = Z(:,1);
dp2dt = -a*[p2lag(1)-1/b];
end
```

```
function [s] = APPDDE42hist (t)
% Initial data is given by
% p1(t) = 0.01exp(at) for t02-tau<=t<=t02
a = ...;
b = 0.5;
s = zeros(length(t));
    for i=1:length(t)
```

```
        s(i) = 0.01*exp(a*t(i));
    end
end
```

```
a = ...;
b = 0.5;
tau = ...;
t01 = 0;
t02 = ...;
tend = ...;

% x(t)
sol = dde23(@APPDDE41dde,[tau],@APPDDE41hist,[t01,tend]);

% p1(t)
t = linspace(t01,tend);
p1 = zeros(length(t));
p1(1) = 0.01*exp(a*t(1));
for i=1:length(t)-1
    if (p1(i) <= max(sol.y))
        p1(i+1) = 0.01*exp(a*t(i+1));
    else
        break
    end
end

% p2(t)
sol2 = dde23(@APPDDE42dde,[tau],@APPDDE42hist,[t02,tend]);

%Figure
figure
plot(sol.x,sol.y,'k'); hold on
plot(t(1:i),p1(1:i),'Color',[0,153,76]./255); hold on
plot(sol2.x,sol2.y,'Color',[255,128,0]./255); hold on
plot(sol.x,1/b*ones(length(sol.x)),'--k')
text(tend+0.01*tend,2.01,'1/b')
```

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