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Real roots of Littlewood polynomials

Bachelor Project Mathematics – Final draft

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Abstract

Littlewood polynomials are polynomials with coefficients -1 or 1. Past research into the complex roots of these polynomials, showed that the real roots are found inside the intervals $[-2, -\frac{1}{2}] \cup [\frac{1}{2}, 2]$. It is however not clear whether the closure of the set consisting of all the real roots of these polynomials is equal to these intervals. This thesis attempts to find an answer to what this closure is, starting with the hypothesis that it is fact the intervals $[-2, -\frac{1}{2}] \cup [\frac{1}{2}, 2]$. Several iterative methods will be discussed, which find polynomials with roots increasingly close to any given number in $[-2, -\frac{1}{2}] \cup [\frac{1}{2}, 2]$. A proof will be presented which proves the hypothesis, albeit under one assumption. Finally, the validity and concequences of this assumption will be discussed.

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1 Introduction

Littlewood polynomials are named after John Edensor Littlewood, a famous British mathematician who specialised in analysis, number theory and differential equations [1]. He studied these polynomials for a period in the 1950's. In more recent years, many others have had a look at these polynomials as well: [2] [3] [4] [6]. They were often most interested in the complex roots of these polynomials, which tend to give beautiful symmetric shapes when plotted (see Figure 1).

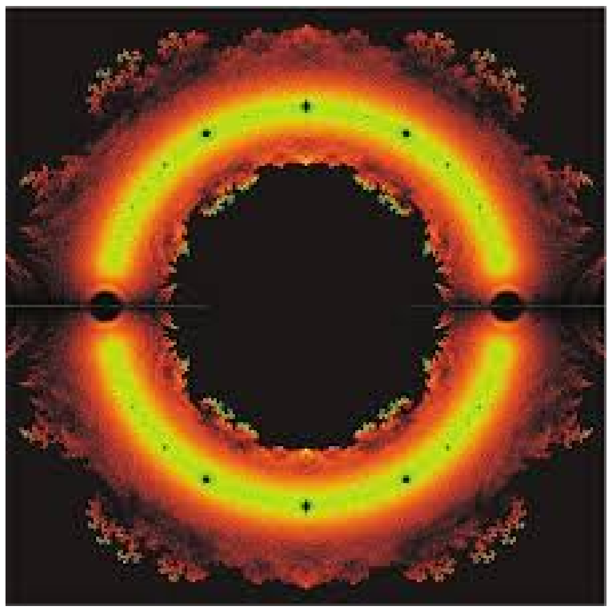

Figure 1: Plot of complex roots of Littlewood polynomials of degree $n \leq 18$.[4]

A Littlewood polynomial, l_n, is defined as follows:

$$l_n(x) = \sum_{i=0}^{n} a_i x^i,$$

where $\forall i : a_i \in \{-1, 1\}$. This means that a Littlewood polynomial can be written as:

$$l(x) = \pm 1 \pm x \pm x^2 \pm \cdots \pm x^n.$$

One of the people how studied these polynomials, was Majken Roelfszema [2]. Her 2015 Bachelor Project revolved around their complex roots. She proved multiple properties of the set containing all complex roots of these Littlewood polynomials. She was however left with an unanswered question [2]. Whilst plotting the complex roots discussed above, she observed and showed that the real roots (the roots on the horizontal real axis in Figure 1) are contained in the intervals $[-2, -\frac{1}{2}] \cup [\frac{1}{2}, 2]$. Moreover, the picture (see Figure 1) suggests that these roots are dense in the intervals.

This observation was counter intuitive with another property she found, namely that the set of Littlewood roots in $\mathbb{Q}(\sqrt{d})$ with $d \in \mathbb{Q}_{>0}$ is the set

$$\{-1, 1, \frac{1-\sqrt{5}}{2}, -\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}, -\frac{1-\sqrt{5}}{2}\}.[2]$$

To put it somewhat bluntly, D leaves a lot of 'holes' in $[-2, -\frac{1}{2}] \cup [\frac{1}{2}, 2]$. This means that all other numbers $x \in \mathbb{Q}(\sqrt{d})$ in these intervals would be limit points of some sequence of roots of certain Littlewood polynomials. M. Roelfszema managed to find a degree 55 polynomial which had a root very close to $1\frac{1}{2}$, but was unable to create a sequence out of it.[2]

2 Research question

This Bachelor Project will attempt to give an answer to the question described in the previous section:

What is closure of the set consisting of all the real roots of all Littlewood polynomials?

The hypothesis to this answer is that this closure is the set $[-2, -\frac{1}{2}] \cup [\frac{1}{2}, 2]$. To put it in formula's, this thesis will attempt to answer what $\overline{D}$ is, for

$$D = \{x \in \mathbb{R} \mid \exists n \in \mathbb{N} : l_n(x) = 0\},$$

where l_n is

$$l_n(x) = \sum_{i=0}^{n} a_i x^i,$$

and $\forall i : a_i \in \{-1, 1\}$.

The hypothesis is that $\overline{D} = [-2, -\frac{1}{2}] \cup [\frac{1}{2}, 2]$.

3 Useful properties

For the convenience of the reader, some properties proven by M. Roelfszema will be repeated in this section. They will have an important influence on the sections beyond. For some properties, the proof for complex roots can be translated identically to a proof for real roots. In these cases only the translated property will be mentioned and the proof will not be written out in full. In cases where different proofs are required when looking at real roots, a full alternative proof will be given.

Theorem 3.1. *All real zeroes of Littlewood polynomials satisfy* $|\alpha| < 2$ *and* $\alpha \neq 0$.

Proof. To prove $|\alpha| < 2$, see the proof of Lemma 2 by M. Roelfszema and assume α to be a real zero instead of a complex zero, the proof remains intact.[2] The fact that $\alpha \neq 0$ can be seen by substituting $x = 0$ into any Littlewood polynomial. For any Littlewood polynomial $l_n(x)$

$$\begin{aligned} l_n(0) &= \sum_{i=0}^{n} a_i 0^i \\ &= a_0 + a_1 0 + a_2 0^2 + \cdots + a_n 0^n \\ &= a_0 \\ &= \pm 1 \\ &\neq 0. \end{aligned}$$

□

We see that 0 can never be a zero.

Theorem 3.2. *If* α *is a real zero of a Littlewood polynomial, then so is* $\frac{1}{\alpha}$.

Proof. See the proof of Lemma 3 by M. Roelfszema and again assume α to be a real zero instead of a complex zero, the proof remains intact.[2] □

Theorem 3.3. *If* α *is a real zero of a Littlewood polynomial, so is* $-\alpha$.

Proof. This property can be seen as a simplification of Theorem 4 by M. Roelfszema, where it is proven that D is symmetric in the imaginary axis.[2] A proof which is more appropriate for real zeroes will be presented here.

Say α is a zero of a certain Littlewood polynomial, so $\exists l(x)$ such that $l(\alpha) = 0$. When writing this function out, one obtains

$$l(\alpha) = a_0 + a_1\alpha + a_2\alpha^2 + a_3\alpha^3 + \dots$$

When filling in $-\alpha$, we obtain

$$l(-\alpha) = a_0 - a_1\alpha + a_2\alpha^2 - a_3\alpha^3 + \dots$$

Say α is the zero for $l(x)$, which has the set of coefficients $\{a_0, a_1, a_2, a_3, ...\}$. Then for $l^-(x)$ with the set of coefficients $\{a_0, -a_1, a_2, -a_3, ...\}$ (i.e. where all odd coefficients are multiplied by -1) has $-\alpha$ as its zero. $l^-(x)$ is still a Littlewood polynomial, since its coefficients are still equal to ± 1. □

Theorem 3.4. *If* α *is a real zero of a Littlewood polynomial, then* $\forall n \in \mathbb{N}$ *it follows that* $\pm\sqrt[n]{|\alpha|}$ *is also a real zero of a Littlewood polynomial.*

Proof. This property is similar to Lemma 10 by M. Roelfszema.[2] The proof can be translated entirely, one only has to make sure that $\sqrt[n]{\alpha} \in \mathbb{R}$, which is not the case if n is even and $\alpha < 0$.

Theorem 3.5. *The only rational zeroes of Littlewood polynomials are* -1 *and* 1.

Proof. The proof is exactly the same as Theorem 5 by M. Roelfszema.[2] □

4 Iterative methods for rootfinding

The five theorems described in the section above form the basis for our search for an answer to the research question. In this section, certain iterative methods for finding roots of Littlewood polynomials which are increasingly close to a given number k, will be described. The theorems from the previous section will come in extremely useful.

The iterative methods below will try to find roots close to any k for which $\sqrt{2} \leq k \leq 2$. In other words, it will try to find evidence that:

$$[\sqrt{2}, 2] \overset{?}{\subseteq} \overline{D}.$$

Theorem 4.1. *For the set* D *containing all real roots of Littlewood polynomials*

$$[\sqrt{2}, 2] \subseteq \overline{D} \Rightarrow [-2, -\frac{1}{2}] \cup [\frac{1}{2}, 2] = \overline{D}$$

Proof. Assume $[\sqrt{2}, 2] \subseteq \overline{D}$. Using Theorem 3.4, we see that

$$[\sqrt{2}, 2] \subseteq \overline{D} \Rightarrow (1, 2] \subseteq \overline{D},$$

since $\forall r \in \mathbb{R}_{>0}$ it follows that $\sqrt[n]{r} \to 1$ as $n \to \infty$. Theorem 3.5 implies that 1 is always a root and can therefore be added to this, which means that

$$(1, 2] \subseteq \overline{D} \Rightarrow [1, 2] \subseteq \overline{D}.$$

Subsequently using Theorem 3.2 will allow us the expand further and add $[\frac{1}{2}, 1]$, leading to

$$[1, 2] \subseteq \overline{D} \Rightarrow [\frac{1}{2}, 2] \subseteq \overline{D}.$$

Thereafter using Theorem 3.3, the negative of $[\frac{1}{2}, 2]$ is also in the closure $\overline{D}$, which in other words means that

$$[\frac{1}{2}, 2] \subseteq \overline{D} \Rightarrow [-2, -\frac{1}{2}] \cup [\frac{1}{2}, 2] \subseteq \overline{D}.$$

Finally seeing that Theorem 3.1, 3.2 and 3.3 imply that there are no zeroes outside $[-2, -\frac{1}{2}] \cup [\frac{1}{2}, 2]$, will lead to the final implication that

$$[-2, -\frac{1}{2}] \cup [\frac{1}{2}, 2] \subseteq \overline{D} \Rightarrow [-2, -\frac{1}{2}] \cup [\frac{1}{2}, 2] = \overline{D}.$$

This proves the theorem. □

In other words, the ability to find roots arbitrarily close to any $k \in [\sqrt{2}, 2]$, will prove the hypothesis.

4.1 Method A

There are only a few possible iterative steps using Littlewood polynomials that will also produce Littlewood polynomials. We can for example never multiply a Littlewood polynomial $l_n(x)$ with a certain a if $a \neq x^n$ for some $n \in \mathbb{N}$, since the resulting polynomials will have coefficients other then -1 or 1. We can not add or subtract Littlewood polynomials either, without correcting the result to produce the right coefficients again. In fact a Littlewood polynomial can only be multiplied by a factor of it's input x. When multiplying with x^n, we have to add combinations of $\pm x^i$ for $i < n$ to complete the Littlewood polynomial.

The above implies that

$$l_{new}(x) = x \cdot l_{old}(x) \pm 1$$

will result in a new Littlewood polynomial. Looking at $l(x) = 1 + x - x^2$ for example, we see that it has a root at $\alpha \approx 1.62$ (see Figure 2). Multiplying this function by x will change the amplitude, but the resulting function will still have α as a root. When adding or subtracting 1, the function is moved either upwards or downwards, resulting in the root moving to the right or left respectively. In the case of $l(x) = 1 + x - x^2$, this will result in the root moving towards $\alpha \approx 1.84$ or $\alpha = 1$ (see Figure 2).

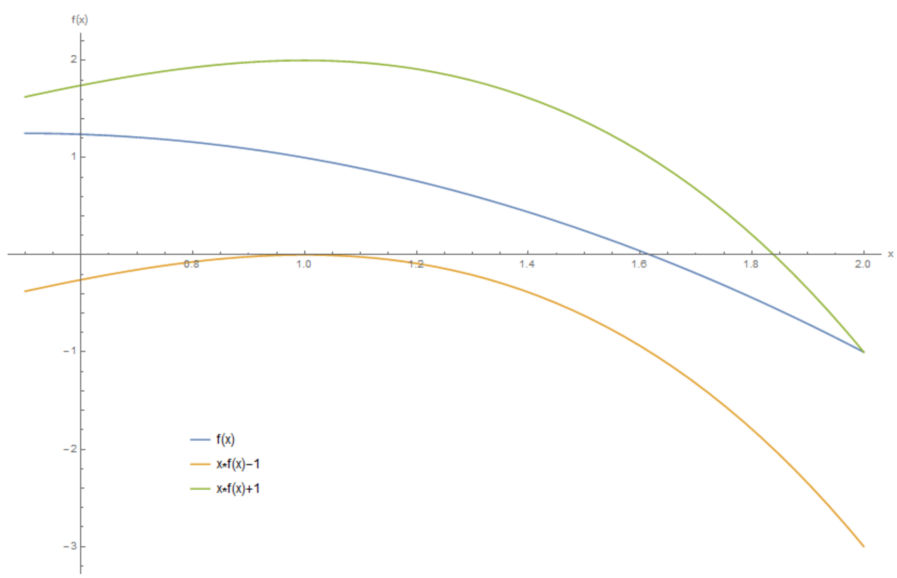

Figure 2: Graphs for $f_0(x) = 1 + x - x^2$ and its manipulations $f_\pm = x \cdot f(x) \pm 1$.

This notion is used in the first method for finding roots arbitrarily close to any given $k \in [\sqrt{2}, 2]$:

1. Start with $f_{old}(x) = 1 + x - x^2$.
2. Check whether a root α of $f_{old}(x)$ in the interval $[\sqrt{2}, 2]$ is greater or smaller than k.
3. If $\alpha < k$, then $f_{new}(x) = x \cdot f_{old}(x) + 1$.
4. If $\alpha > k$, then $f_{new}(x) = x \cdot f_{old}(x) - 1$.
5. If there is no root α in $[\sqrt{2}, 2]$, then $f_{new}(x) = x \cdot f_{old}(x) + 1$.
6. Set $f_{old}(x) = f_{new}(x)$ and return to step 2.

This method will find Littlewood polynomials with roots closer and closer to k. The graphs of the first iterations for $k = 1.5$ can be seen in Figure 3, where we can see that at each step the amplitude changes and the intersections with the x-axis move to the right or left, depending on their location with respect to $k = 1.5$. When we continue these iterations for $k = 1.5$, we find the roots given in Table 1 (rounded off to 5 decimals). After 30 iterations, all new polynomials have roots very close to 1.5. A Mathematica document based on this method can be found in Appendix A. Since it can not use the root found by 'NSolve' for evaluation, the code checks the value of $f(k)$ at each step and decides to use $+1$ or -1 accordingly. Multiple tests of the method for different values between $\sqrt{2}$ and 2, have all shown convergence to the input k.

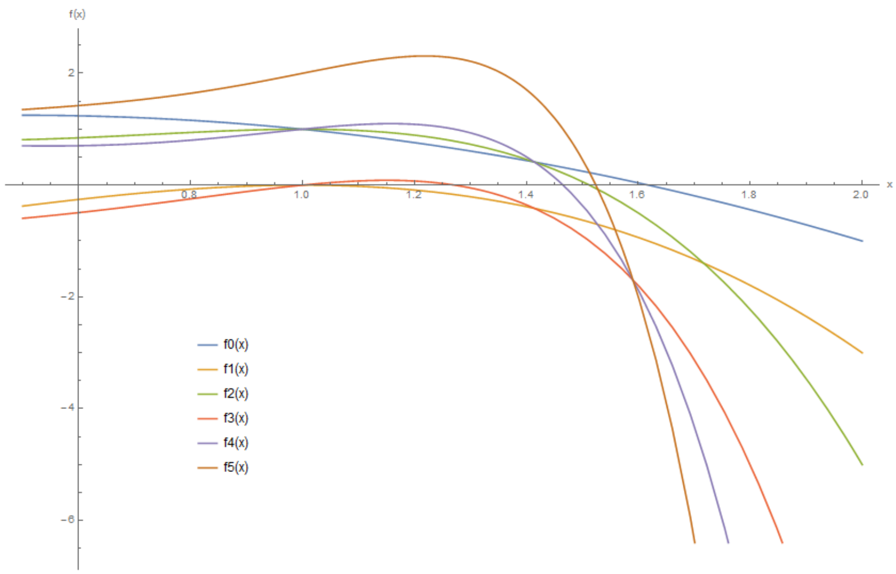

Figure 3: Graph for the first 5 iterations of Method A, with $k = 1.5$.

i	α
0	1.61803
1	-
2	1.51228
3	1.27202
4	1.46557
5	1.52412
6	1.48748
7	1.51288
8	1.49661
9	1.50771

i	α
10	1.50048
11	1.49521
12	1.49877
13	1.50103
14	1.49954
15	1.50054
16	1.49987
17	1.50032
18	1.50002
19	1.49982

i	α
20	1.49995
21	1.50004
22	1.49998
23	1.50002
24	1.5
25	1.50001
26	1.5
27	1.49999
28	1.5
29	1.5

i	α
30	1.5
31	1.5
32	1.5
33	1.5
34	1.5
35	1.5
36	1.5
37	1.5
38	1.5
39	1.5

Table 1: Roots of polynomials in $[1, 2]$ found, using Method A with $k = 1.5$.

Unfortunately we did not succeed in using this method to construct an analytical proof. An important obstacle in the attempts, was the fact that although the roots α of the polynomials get increasingly close to the target value k when viewed after sufficiently many iterations, this is not true at every step. At some steps 'transporting' the polynomial to the right or left results in the new root being further away from k, than the previous root. Trying to construct a proof using a decreasing $|\alpha - k|$ can therefore not succeed, since it is not true for any step (it is not monotonically decreasing).

Another difficulty is the fact that although $\alpha \to k$ as the iterations $i \to \infty$, that does not lead to $f(k) \to 0$. This occurs due to the fact that in order to create polynomials with a root every closer to k, the degree of the polynomial is increased along with it. Any error between f and α will be amplified by f. As the error decreases, the amplification will increase. Trying to construct a proof using for example the Cauchy sequences $|f_n(k) - f_m(k)|$ is therefore also futile.

4.2 Method B

The second iterative method for root-finding that was created, uses upper and lower bounds. In the section above it is shown that if α is the root of a Littlewood polynomial $f(x)$, then $f_+(x) = x \cdot f(x) + 1$ has a root $\alpha_+ > \alpha$ and $f_-(x) = x \cdot f(x) - 1$ has a root $\alpha_- < \alpha$. A consequence of this fact is that if all n'th degree Littlewood polynomials are 'ordered' as in Table 2, we see that if $f_{(n,i)}(x)$ has a root $\alpha_i > 1$, then if $f_{(n,i+1)}(x)$ has a root, it will have the property $\alpha_{i+1} > \alpha_i$.

	x^n	x^{n-1}	$\cdots$	x^2	x^1	x^0
$f_{(n,2^n))}(x)$	$+$	$+$	$\cdots$	$+$	$+$	$+$
$f_{(n,2^n-1)}(x)$	$+$	$+$	$\cdots$	$+$	$+$	$-$
$f_{(n,2^n-2)}(x)$	$+$	$+$	$\cdots$	$+$	$-$	$+$
$f_{(n,2^n-3)}(x)$	$+$	$+$	$\cdots$	$+$	$-$	$-$
$f_{(n,2^n-4)}(x)$	$+$	$+$	$\cdots$	$-$	$+$	$+$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$
$f_{(n,2}(x)$	$-$	$-$	$\cdots$	$-$	$-$	$+$
$f_{(n,1}(x)$	$-$	$-$	$\cdots$	$-$	$-$	$-$

Table 2: 'Ordering' of n'th degree Littlewood polynomials $f_{(n,i)}(x)$.

A consequence of this, is the fact that for $f_{(n,i)}(x)$ with root $\alpha_{(n,i)}$ as in Table 2 and

$$f_{(n,i,\pm)}(x) = xf_{(n,i)}(x) \pm 1,$$

with root $\alpha_{(n,i,\pm)}$ the following properties hold:

- $\alpha_{(n,i,-)} < \alpha_{(n,i)} < \alpha_{(n,i,+)}$
- $\alpha_{(n,i,-)} < \alpha_{(n,i+1,-)}$ if $\alpha_{(n,i)} < \alpha_{(n,i+1)}$
- $\alpha_{(n,i,+)} < \alpha_{(n,i+1,+)}$ if $\alpha_{(n,i)} < \alpha_{(n,i+1)}$.

These properties can be used to find the following relations when looking at a target value k. Assume that

$$\alpha_{(n,i)} < k < \alpha_{(n,i+1)},$$

for some $n \in \mathbb{N}$ and $i < 2n$. Then also

$$\alpha_{(n,i,-)} < k < \alpha_{(n,i+1,+)}$$

and

$$\alpha_{(n,i,+)}, \alpha_{(n,i+1,-)} \in [\alpha_{(n,i,-)}, \alpha_{(n,i+1,+)}]$$

This means that if $f_{(n,i)}(x)$ and $f_{(n,i+1)}(x)$ can be identified as a lower and upper bound for k, then $f_{(n,i,-)}(x)$ and $f_{(n,i+1,+)}(x)$ will also identify as a lower and upper bound with $f_{(n,i,+)}(x)$ and $f_{(n,i+1,-)}(x)$ potentially being even better bounds.

The properties above are used for Method B:

1. Define the polynomials $f_1(x) = -1 - x + x^2 - x^4$, $f_2(x) = 1 - x + x^2 - x^4$, $f_3(x) = -1 + x + x^2 - x^4$ and $f_4(x) = 1 + x + x^2 - x^4$.
2. Find the roots α_i of the polynomials $f_i(x)$.
3. Find for which i the target value k has the property $\alpha_i < k < \alpha_{i+1}$ or $\alpha_i > k > \alpha_{i+1}$.

4. For this i, define $f_{lower}(x) = f_i(x)$ and $f_{higher}(x) = f_{i+1}(x)$.
5. Define $f_1(x) = xf_{lower}(x)-1$, $f_2(x) = xf_{lower}(x)+1$, $f_3(x) = xf_{higher}(x)-1$ and $f_4(x) = xf_{higher}(x) + 1$.
6. Return to step 2.

This method therefore attempts to 'close in' k. Taking $k = 1.8$, the roots yielding from the first 20 steps of this method, can be seen in Table 3. Although the bounds start far way from k, in about 20 iterations both the upper and lower bound are very close to 1.8. A Mathematica document based on this method can be found in Appendix B. Similarly to method A, Mathematica can not use the root values obtained through 'NSolve' and evaluates the four functions for k at each step in order to determine which one to designate as new bounds.

i	α_a	α_b	α_c	α_d
0	-	1.51288	1.72208	1.96596
1	1.61803	1.7924	1.8832	1.96595
2	1.75488	1.82394	1.85589	1.90734
3	1.72912	1.77689	1.80709	1.83929
4	1.7648	1.78799	1.79692	1.81658
5	1.79089	1.80265	1.81142	1.82153
6	1.78737	1.79428	1.7995	1.80572
7	1.7977	1.80126	1.80403	1.80737
8	1.79669	1.79871	1.80029	1.80223
9	1.79815	1.79926	1.79974	1.80083

i	α_a	α_b	α_c	α_d
10	1.79943	1.80004	1.80053	1.80113
11	1.79926	1.7996	1.79988	1.80021
12	1.79978	1.79997	1.80012	1.80031
13	1.79992	1.80002	1.80007	1.80017
14	1.79989	1.79995	1.79999	1.80005
15	1.79998	1.80001	1.80003	1.80007
16	1.79997	1.79999	1.8	1.80002
17	1.79999	1.8	1.80001	1.80002
18	1.79999	1.8	1.8	1.80001
19	1.8	1.8	1.8	1.8

Table 3: Roots of four polynomials found using Method B with $k = 1.8$, bounds for the next iteration are given in red.

Regrettably this method is not suitable for constructing a formal proof either. The method has the same 'degree problem' and Method A. In order to define its upper and lower bounds, the degrees of the polynomials increase with every iteration. It seems that as the iterations $i \to \infty$, it follows that $\alpha_{lower} \to k$ and $\alpha_{higher} \to k$. This again does not lead to $f_{lower}(k) \to 0$ or $f_{higher}(k) \to 0$. Constructing a proof along the lines of $|f_{lower}(k) - f_{higher}(k)|$ will therefore not work.

Method B however has one quality over Method A, since multiple tests of the method for different values k all seem to show that $|\alpha_{lower} - \alpha_{higher}|$ is decreasing with every step. We would expect we have to be able to used this in the construction of a proof. Sadly this did not succeed. An important reason for this is the fact that although numerically it seems as though $|\alpha_{lower} - \alpha_{higher}|$ decreases with every step, an analytical proof for this was not found.

5 $\sqrt{2}$ is a limit point of D

The numerical methods described in the previous section add to the plausibility of the hypothesis. It is regrettable that we did not manage to turn these methods into a formal proof. This section will attempt to prove the hypothesis for one of the values in $[-2, -\frac{1}{2}] \cup [\frac{1}{2}, 2]$, namely $\sqrt{2}$. It should be noted that $\sqrt{2}$ itself is not a root of any Littlewood polynomials since Theorem 3.4 states that 2 would then also be a root while Theorem 3.5 proves that that it not the case.

Theorem 5.1. *The largest real root of* $l_n(x) = 1+x+x^2+...+x^m-x^{m+1}-x^{m+2}$ *for every n'th degree Littlewood polynomial with* $m = n-2$ *has the property* $\alpha_n \to \sqrt{2}$ *as* $n \to \infty$*.*

Proof. A root of the $l_n(x)$ is found if (using [7]):

$$\begin{aligned}
0 &= (1 + x + x^2 + ... + x^m) - x^{m+1} - x^{m+2} \\
&= \frac{x^{m+1} - 1}{x - 1} - x^{m+1} - x^{m+2} \\
&= \frac{x^{m+1} - 1 - (x-1)(x^{m+1} + x^{m+2})}{x - 1} \\
&= \frac{x^{m+1} - 1 - x^{m+2} + x^{m+1} - x^{m+3} + x^{m+2}}{x - 1} \\
&= \frac{x^{m+1} - 1 + x^{m+1} - x^{m+3}}{x - 1} \\
&= \frac{-x^{m+3} + 2x^{m+1} - 1}{x - 1}.
\end{aligned}$$

This equation will hold if the numerator equals zero and $x \neq 1$. We should try to find roots of $f_m(x) = -x^{m+3} + 2x^{m+1} - 1$ and which are not equal to 1. Quickly observe that for $x = 0 : f_m(0) = -1$. Similarly $f_m(1) = 0$. $f_m(x)$ therefore goes through the points $(0, -1)$ and $(1, 0)$ for every m.

We continue by calculating the derivative of $f_m(x)$,

$$f'_m(x) = -(m+3)x^{m+2} + (2m+2)x^m.$$

This means that $f'_m(0) = 0$ and $f'_m(1) = m - 1$. It follows that at $x = 1$ the function intersects the x-axis, but it's derivative is positive for any $m > 1$. This means the function becomes positive after the point of intersection. However, for a sufficient large x, the function value will be negative again. Especially:

$$\begin{aligned}
f_n(\sqrt{2}) &= -(\sqrt{2})^{m+3} + 2(\sqrt{2})^{m+1} - 1 \\
&= -(\sqrt{2})^2(\sqrt{2})^{m+1} + 2(\sqrt{2})^{m+1} - 1 \\
&= -2(\sqrt{2})^{m+1} + 2(\sqrt{2})^{m+1} - 1 \\
&= -1
\end{aligned}$$

This means that for some values $x > 1$, $f_n(x) > 0$, but for $x = \sqrt{2}$ $f_n(x) < 0$. Since $f_m(x)$ is continuous, the intermediate value theorem says it has at least one other root $1 \leq \alpha_m < \sqrt{2}$.[5] Also, these have to be $f_m(x)$'s largest roots, since for values larger then $x = 1$, the derivative function $f'_m(x)$ will remain positive. Therefore the function value of $f_m(x)$ will remain positive as well and no larger root can occur. These specific roots α_m have $f_m(\alpha_m) = 0$. When looking at $f_{m+1}(\alpha_m)$:

$$\begin{aligned}
f_{m+1}(\alpha_m) &= 1 + \alpha_m + \alpha_m^2 + ... + \alpha_m^m + \alpha_m^{m+1} - \alpha_m^{m+2} - \alpha_m^{m+3} \\
&= (1 + \alpha_m + \alpha_m^2 + ... + \alpha_m^m) + \alpha_m^{m+1} - \alpha_m^{m+2} - \alpha_m^{m+3} \\
&= (\alpha_m^{m+1} + \alpha_m^{m+2}) + \alpha_m^{m+1} - \alpha_m^{m+2} - \alpha_m^{m+3} \\
&= 2\alpha_m^{m+1} - \alpha_m^{m+3} \\
&= \alpha_m^{m+1}(2 - \alpha_m^2)
\end{aligned}$$

Since $1 < \alpha_m < \sqrt{2}\ \forall m$, it follows that $\alpha_m^{m+1} > 0$ and $2 - \alpha_m^2 > 0$. Therefore also $f_{m+1}(\alpha_m) = \alpha_m^{m+1}(2 - \alpha_m^2) > 0$. This in term implies that α_{m+1} for which $f_{m+1}(\alpha_m) = 0$ has to adhere to $\alpha_m < \alpha_{m+1} < \sqrt{2}$. In other words, it is increasing. Furthermore, for any $f_m(x)$ we see that

$$\begin{aligned}
f_m\left(\sqrt{2 - \left(\frac{2}{3}\right)^{\frac{m+1}{2}}}\right) &= -\left(\sqrt{2 - \left(\frac{2}{3}\right)^{\frac{m+1}{2}}}\right)^{m+3} + 2\left(\sqrt{2 - \left(\frac{2}{3}\right)^{\frac{m+1}{2}}}\right)^{m+1} - 1 \\
&= \left(-2 + \left(\frac{2}{3}\right)^{\frac{m+1}{2}}\right)\left(\sqrt{2 - \left(\frac{2}{3}\right)^{\frac{m+1}{2}}}\right)^{m+1} + 2\left(\sqrt{2 - \left(\frac{2}{3}\right)^{\frac{m+1}{2}}}\right)^{m+1} - 1 \\
&= \left(\frac{2}{3}\right)^{\frac{m+1}{2}}\left(\sqrt{2 - \left(\frac{2}{3}\right)^{\frac{m+1}{2}}}\right)^{m+1} - 1 \\
&= \left(\frac{\sqrt{2}}{\sqrt{3}}\right)^{m+1}\left(\sqrt{2 - \left(\frac{2}{3}\right)^{\frac{m+1}{2}}}\right)^{m+1} - 1 \\
&= \left(\frac{\sqrt{2}\sqrt{2 - \left(\frac{2}{3}\right)^{\frac{m+1}{2}}}}{\sqrt{3}}\right)^{m+1} - 1 \\
&= \left(\frac{\sqrt{4 - 2\left(\frac{2}{3}\right)^{\frac{m+1}{2}}}}{\sqrt{3}}\right)^{m+1} - 1.
\end{aligned}$$

For $m > 3$, $\left(\frac{2}{3}\right)^{\frac{m+1}{2}} < \frac{1}{2}$, therefore $2\left(\frac{2}{3}\right)^{\frac{m+1}{2}} < 1$. This means that

$$\begin{aligned} f_m\left(\sqrt{2-\left(\frac{2}{3}\right)^{\frac{m+1}{2}}}\right) &= \left(\frac{\sqrt{4-2\left(\frac{2}{3}\right)^{\frac{m+1}{2}}}}{\sqrt{3}}\right)^{m+1} - 1 \\ &> \left(\frac{\sqrt{4-1}}{\sqrt{3}}\right)^{m+1} - 1 \\ &> 1^{m+1} - 1 \\ &> 0. \end{aligned}$$

This finally concludes the proof, since $\forall m \in \mathbb{N}_{>3}$ it follows that $f_m(\sqrt{2}) < 0$, but $f_m\left(\sqrt{2-\left(\frac{2}{3}\right)^{\frac{m+1}{2}}}\right) > 0$. Since as $m \to \infty$, $\left(\frac{2}{3}\right)^{\frac{m+1}{2}} \to 0$, the intermediate value theorem tells us that there a sequence of roots α_m will converge to $\sqrt{2}$. This means that $\sqrt{2}$ is a limit point of the set $D = \{x \in \mathbb{R} : l_n(x) = 0\}$ and therefore $\sqrt{2} \in \overline{D}$. The reader should note that for $\alpha_m \to \sqrt{2}$,

$$l_n(x) = \frac{-x^{m+3} + 2x^{m+1} - 1}{x - 1}$$

will have a solution, since the denominator $\sqrt{2} - 1 \neq 0$. The theorem is thereby proven. □

Remark Similarly, the series of polynomials $l_n(x) = 1 + x + x^2 + \cdots + x^m - x^{m+1}$ with $m = n - 1$, will have largest roots α which will converge to 2. By Theorems 3.2, 3.3 and 3.4 this also implies that $\frac{1}{2}$, -2 and any $\sqrt[n]{2}$ are also limit points of D.

6 $\overline{D} \cap \mathbb{R}_{>0}$ is connected?

Theorem 5.1 proved that $\sqrt{2} \in \overline{D}$ and also concluded that $2 \in \overline{D}$. Having in mind that the goal is to prove that $[\sqrt{2}, 2] \subseteq \overline{D}$ (see Theorem 4.1), it is sufficient to prove that $\overline{D} \cap [\sqrt{2}, 2]$ is connected. Indeed this would imply that $\sqrt{2}$ and 2 are path-connected, which in $\mathbb{R}$ means that the entire interval $[\sqrt{2}, 2] \subseteq \overline{D}$.[5] This section will prove that for any root of a Littlewood polynomial this connectedness holds, albeit making use of one important assumption.

The proof is largely based on a proof by Thierry Bousch.[6] His proof is in French and assumes the roots to be in $\mathbb{C}$. Therefore the proof below will be a translation of the work by T. Bousch; adding, omitting, augmenting or replacing parts where necessary to translate his proof to the case where the roots are in $\mathbb{R}$.

The best illustrating reason that a proof for connectedness in $\mathbb{C}$ does not simply

imply connectedness in $\mathbb{R}$ can be seen through path-connectedness. Any connected open set in $\mathbb{R}^n$ is path-connected.[5] Since $\mathbb{C} \cong \mathbb{R}^2$, T. Bousch's proof will imply that the closure of the set of all complex roots of Littlewood polynomials is path-connected. However, this does not at all imply that the set of real roots is path-connected as well. Any two real roots α_1 and α_2 can be connected by a path in $\mathbb{C}$ without this path being contained in the real axis.

As stated before, it should be stressed that this proof has a downside, it assumes all zeroes $\alpha \neq \pm 1$ of Littlewood polynomials are simple zeroes. Unfortunately a proof for multiple zeroes, was not found. There are however strong indications that multiple zeroes do not occur. They will be discussed in the next section.

Theorem 6.1. *The closure $\overline{D}$ of the set of all real roots of Littlewood polynomials, is connected for all its elements r which are contained in $(0,1)$. Under the assumption that for these values of r, Littlewood polynomials do not have roots which are also extrema.*

Proof. We start by defining the set

$$A_r := \left\{ \sum_{i=0}^{\infty} a_i r^i \;\mid\; \text{all } a_i \in \{-1, 1\} \right\}$$

for a fixed $r \in (0,1)$. As explained by Bousch, A is then a compact set.[6] Also, define $A_r^+ = r \cdot A_r + 1$ and $A_r^- = r \cdot A_r - 1$. This implies that $A_r = A_r^+ \cup A_r^-$.

Proposition 6.2. *The following statement is true if $r \neq 0$:*

1. *A_r is connected $\Rightarrow A_r^+ \cap A_r^- \neq \emptyset$.*

The following statements are equivalent under the same conditions

2. *$A_r^+ \cap A_r^- \neq \emptyset$.*
3. *The map $\eta : \{-1,1\}^{\mathbb{N}} \to \mathbb{R}$ by $(a_i)_{i \in \mathbb{N}} \mapsto \sum_{i \in \mathbb{N}} a_i r^i$ is not injective.*
4. *There exists a sequence $(b_i)_{i \in \mathbb{N}}$ such that $b_0 = 1$, $\forall i : b_i \in \{-1, 0, 1\}$ and $\sum b_i r^i = 0$*
5. *There exists a sequence $(b_i)_{i \in \mathbb{N}} \neq (0)$ such that $\forall i : b_i \in \{-1, 0, 1\}$ and $\sum b_i r^i = 0$.*

Proof. (1) This follows from that fact that $A_r^+ \cup A_r^- = A_r$ and the fact that both A^+ and A^- or both closed and non-empty. If $A_r^+ \cap A_r^- = \emptyset$, the above would imply that they form a partition for A_r.[5] This gives a contradiction with the fact that A_r is connected.

(2 $\Rightarrow$ 3) $A_r^+ \cap A_r^- \neq \emptyset$, so $\exists \beta \in A_r$ such that $\beta \in A_r^+$ and $\beta \in A_r^-$. This means $\exists \beta^+, \beta^-$ such that $r\beta^+ + 1 = \beta$ and $r\beta^- - 1 = \beta$. It follows that η is not injective.

(3 ⇒ 5) Assume that η is not injective. Then there exists an element $a \in \mathbb{R}$ such that η maps two different $(a_i), (b_i) \in \{-1,1\}^{\mathbb{N}}$ to a. This means that η will map the sequence $(c_i) = (\frac{1}{2}(a_i - b_i))$ to 0. In other words, $\sum c_i r^i = 0$.

(5 ⇒ 4) Assume $\sum b_i r^i = 0$ for some sequence (b_i) where $\exists i : b_i \neq 0$. Find the smallest such integer l such that $b_l \neq 0$. Then:

$$\begin{aligned} 0 &= \sum b_i r^i \\ &= \sum_{0 \le i < l} b_i r^i + \sum_{j \ge l} b_j r^j \\ &= 0 + \sum_{j \ge l} b_j r^j \\ &= \sum_{j \ge 0} b_{j+l} r^{j+l} \\ &= \sum_{j \ge 0} b_{j+l} r^j r^l \\ \frac{0}{r^l} = 0 &= \sum_{j \ge 0} b_{j+l} r^j. \end{aligned}$$

The sequence (b_{j+l}) still has that $\forall j + l : b_{j+l} \in \{-1, 0, 1\}$. However now the first term $b_0 = \pm 1$. If $b_0 = 1$, the statement is proven. If $b_0 = -1$, multiply the entire sum formula with -1. $\forall j + l : b_{j+l} \in \{-1, 0, 1\}$ will still hold, as will $\sum\limits_{j \ge 0} b_{j+l} r^j = 0$, but $b_0 = 1$.

(4 ⇔ 2) (⇒) See that A_r^+ contains all real numbers which can be written as $\sum a_i r^i$ with $a_0 = 1$ (since $A_r^+ = r \cdot A_r + 1$ makes sure the number 1 is present in the sequence, which can only be achieved if $a_0 = 1$) and $\forall i : a_i \in \{-1, 1\}$. Similarly A_r^- contains all real numbers which can be written as $\sum a_i r^i$ with $a_0 = -1$ and $\forall i : a_i \in \{-1, 1\}$. Since $A_r^+ \cap A_r^- \neq \emptyset$, there will exist two sequences (b_i^+) and (b_i^-) such that $b_0^+ = 1$, $b_0^- = -1$, $\forall i : b_i^{\pm} \in \{-1, 1\}$ and $\sum b_i^+ r^i = \sum b_i^- r^i$. Then define the sequence (b_i) where $\forall i : b_i = \frac{1}{2}(b_i^+ - b_i^-)$ and see that $b_0 = 1$, $\forall i : b_i \in \{-1, 0, 1\}$ and $\sum b_i r^i = \sum b_i^+ r^i - \sum b_i^- r^i = 0$.

(⇐) When assuming that there exists a sequence (b_i) with the above properties, define (b_i^+) and (b_i^-) such that $b_i^+ - b_i^- = 2b_i$ and see that $A_r^+ \cap A_r^- \neq \emptyset$.

The observant reader will have noticed that using these implications, it is possible to go from statement 1 to statement 2 and then from statement 2 to any of the other statements. Proposition 6.2 is thereby proven. □

We should note that Proposition 6.2 states that proving that A_r is connected implies that for any $0 \neq |r| < 1$ there exists a sequence (b_i) with $b_i \in \{-1, 0, 1\}$

such that the polynomial using this (b_i) as coefficients will have r as its root.

We continue by defining the following sets:

$$\begin{aligned} N &= \{r \in (-1,1) \quad | \ A_r \ \text{ is disconnected}\} \\ C &= \{r \in (-1,1) \quad | \ A_r \ \text{ is connected}\} \\ C' &= \{r \in (-1,1) \quad | \ 0 \in A_r\}. \end{aligned}$$

Since a set is either connected or disconnected, we see that N and C are complementary. Since A_r contains the values of all Littlewood polynomials for a given r, it follows that if $\gamma \in A_r$, then also $-\gamma \in A_r$. From this we see that A_r has to be symmetric around 0. In order for A_r to be connected, it therefore has to contain 0 as well (for it would not be path-connected otherwise, which on $\mathbb{R}$ means it is not connected either). It can therefore be said that $C = C'$. From this point forward, we will therefore only refer to C.

The goal will be to prove that C is connected and through that, prove that A itself is connected as well. First we define the following sets:

$$\begin{aligned} W &= \{f(r) = \sum b_i r^i \quad | \ b_0 = 1, \ |b_i| \leq 2\} \\ H &= \{f(r) = \sum b_i r^i \quad | \ b_0 = 1, \ b_i \in \{-1,0,1\}\} \\ H' &= \{f(r) = \sum b_i r^i \quad | \ b_0 = 1, \ b_i \in \{-1,1\}\}. \end{aligned}$$

Lemma 6.3. *Take $s \in (0,1)$ and $\varepsilon > 0$ such that $s + \varepsilon < 1$. Then $\exists n_0 \in \mathbb{N}$ such that $\forall h \in H$ and $\alpha \in [0,s]$ such that $h(\alpha) = 0$, then $\forall h^* \in H$ such that the first n_0 coefficients of h and h^* are equal, there will exist an $\alpha^* \in (\alpha - \varepsilon, \alpha + \varepsilon)$ such that $h^*(\alpha^*) = 0$.*

This theorem basically states that for any Littlewood polynomial and its root α given an input $0 < r < 1$ and $\varepsilon > 0$ there will be a certain n_0, such that all Littlewood polynomials which share the same first n_0 coefficients as the original Littlewood polynomial, will have roots in the ε-neighbourhood around α. T. Bousch proves his similar Lemma 2 using Rouché's Theorem.[6] Unfortunately this theorem is heavily dependant on complex numbers without an good alternative in $\mathbb{R}$. The proof below will therefore deviate greatly from his proof. This part will assume any root of Littlewood polynomials not to be an extremum.

Proof. Assume $s \in (0,1)$ and $\varepsilon > 0$ such that $s + \varepsilon < 1$. Look at all $h \in H$ and $\alpha \in [-s,s]$ such that $h(\alpha) = 0$. Assume α is a real intersection of the x-axis (as supposed to a extremum). This implies that $\exists \varepsilon^* : 0 < \varepsilon^* < \varepsilon$ such that $h(\alpha + \varepsilon^*) = \delta_1$ and $h(\alpha - \varepsilon^*) = -\delta_2$ where $\delta_1, \delta_2 > 0$ or $\delta_1, \delta_2 < 0$. In the latter case state multiply all coefficients of h with -1 such that $h_{new}(\alpha + \varepsilon^*) = -\delta_1$ and $h_{new}(\alpha - \varepsilon^*) = \delta_2$.

Without loss of generality, assume $h(\alpha + \varepsilon^*) = \delta_1 > 0$. We see that any h^* with the first n_0 coefficients equal to h, will have the following relation to h:

$$\begin{aligned}|h(\alpha + \varepsilon^*) - h^*(\alpha + \varepsilon^*)| &= |\sum_{i=0}^{\infty} b_i(\alpha + \varepsilon^*)^i - \sum_{i=0}^{\infty} b_i^*(\alpha + \varepsilon^*)^i| \\ &= |\sum_{i=n_0}^{\infty} b_i(\alpha + \varepsilon^*)^i - \sum_{i=n_0}^{\infty} b_i^*(\alpha + \varepsilon^*)^i|.\end{aligned}$$

Now we use that this distance is greatest if the coefficients in one of the functions are all 1 and the coefficients in the other are all -1 and see that

$$\begin{aligned}|h(\alpha + \varepsilon^*) - h^*(\alpha + \varepsilon^*)| &\leq 2\sum_{i=n_0}^{\infty} (\alpha + \varepsilon^*)^i \\ &= 2\frac{(\alpha + \varepsilon^*)^{n_0} - 0}{1 - (\alpha + \varepsilon^*)}.[7]\end{aligned}$$

As long this maximal distance remains smaller than δ_1, we have that $h^*(\alpha+\varepsilon^*) > 0$. This happens if the n_0 for δ_1, denoted as $n_{(0,\delta_1)}$ adheres to

$$\begin{aligned}2\frac{(\alpha + \varepsilon^*)^{n_{(0,\delta 1)}}}{1 - (\alpha + \varepsilon^*)} &< \delta_1 \\ 2(\alpha + \varepsilon^*)^{n_{(0,\delta 1)}} &< \delta_1(1 - (\alpha + \varepsilon^*)) \\ n_{(0,\delta 1)} &> \log_{(\alpha+\varepsilon^*)}(\frac{1}{2}\delta_1(1 - (\alpha + \varepsilon^*))).\end{aligned}$$

Since $(\alpha+\varepsilon^*) < 1$, the sign at the last step flips because the base of the logarithm $(\alpha + \varepsilon^*) < 1$. This logarithm will always be defined on $\mathbb{R}$ since $(\alpha + \varepsilon^*) < 1$, because of which $(1 - (\alpha + \varepsilon^*)) > 0$ and so also $(\frac{1}{2}\delta_1(1 - (\alpha + \varepsilon^*))) > 0$. This is a necessity since if $(\frac{1}{2}\delta_1(1 - (\alpha + \varepsilon^*))) < 0$, the logarithm would only have a complex answer.

The same can be done on the 'negative side' of α, resulting in the notion that

$$n_{(0,\delta_2)} > \log_{(\alpha-\varepsilon^*)}(\frac{1}{2}\delta_2(1 - (\alpha - \varepsilon^*))).$$

Take $n_0 = \max(n_{(0,\delta_2)}, n_{(0,\delta_1)})$. We now know that for any h^* with the first n_0 coefficients equal to h, it follows that $h^*(\alpha + \varepsilon^*) = \delta_1^* > 0$ and $h^*(\alpha - \varepsilon^*) = -\delta_2^* < 0$. Since h^* is a continuous function, we can apply the intermediate value theorem and find that $\exists \alpha^* \in (\alpha - \varepsilon^*, \alpha + \varepsilon^*) \subset (\alpha - \varepsilon, \alpha + \varepsilon)$ such that $h^*(\alpha^*) = 0$.[5] □

Remark Lemma 6.3 also works if H is replaced by H'.

Now using the fact that both H and H' are compact, we are able to find a general n_0. Define

$$F = \{(f, r) \in H \times [0, s] \mid f(r) = 0\}.$$

Clearly $F \subset H \times [-s,s]$. Obviously $[-s,s]$ is a closed and bounded interval meaning it is compact, as is H due to its construction. Therefore $H \times [-s,s]$ is also compact. Since f consists of combinations of single functions f in H, along with their roots, this is a closed subset. We finally see that F is therefore itself compact, since any closed subset of a compact space is compact itself.

Due to compactness, we can take a finite subcover of this set F and find the n_0 for each of these subcovers. Afterwards taking the largest of these n_0's will give an n_0 that works for every $\alpha \in [0,s]$. We continue by defining the following reduction map:

$$\varphi : H \to H_{n_0}$$
$$\sum_{i=0}^{\infty} b_i r^i \mapsto \sum_{i=0}^{n_0} b_i r^i,$$

where H_{n_0} is the image of this map. In other words:

$$H_{n_0} = \{\sum_{i=0}^{n_0} b_i r^i |\ b_0 = 1,\ b_i \in \{-1,0,1\}\}.$$

H'_{n_0} is defined in a similar way.

Lemma 6.4. *Let $n_0 \in \mathbb{N}$ and $R, S \in H_{n_0}$ have the properties described above. Then there exists a sequence $P_0\ Q_0\ P_1\ Q_2\ \ldots\ P_{t-1}\ Q_{t-1}\ P_t$ such that the following properties hold:*

- $\forall i : P_i, Q_i \in W$
- $\forall i : P_i \in H$
- $\forall i : Q_i = nP_i$ *for some* $n \in \mathbb{N}$
- $\forall i : P_{i+1} = \varphi(Q_i)$
- $P_0 = R$ *and* $P_t = S$

Proof. Proving this lemma will mean that H is connected. We will prove this lemma by a proof founded on induction. In the first part we will prove that assuming the first four properties, there exist an $R, S \in H_{n_0}$ such that the fifth property holds. We thereby find a so called ε-chain between R and S. In the second part of the proof we will assume that the lemma is true for this R and S and proof that it then also follows for other elements of H_{n_0}.

Begin by taking an s and ε, such that there exists a root α which has $0 < \alpha < s$ and $\varepsilon < 1 - s$ (which means Lemma 6.3 can be applied). Then define n_0 to be the n_0 which is found using the method in Lemma 6.3. Subsequently take $\beta \in C$. Our goal is to create an 'ε-chain' from $R \in H_{n_0}$ to $S \in H_{n_0}$.

We can apply Lemma 6.3 to our polynomial $f \in H$ with root β and obtain

$R = \varphi(f)$, which has a root β_0 which is in the ε-neighbourhood of β. Define $S = 1$ (the constant polynomial at 1). We will construct a sequence (β_i) such that $\forall i : \beta_i$ is a root of P_i and $|\beta_i - \beta_{i+1}| < \varepsilon$. We continue until $|\beta_i| \geq s$. The reason that this is sufficient, is because every β has to be contained in $[0, s]$. Being able to find an ε-chain from inside this to outside $[0, s]$, whilst staying in C, proves all subsets for each $|s| < 1$ can be reached from another s.

Suppose $|\beta_i| < s$. β_i is a root of P_i, which means it also is a root of Q_i (since $P_i | Q_i$). Since Q_i and P_{i+1} differ only in their terms which are greater then n_0, Lemma 6.3 tells us that P_{i+1} will have a root β_{i+1} which lies in the ε-neighbourhood of β_i. Indeed we see that $|\beta_i - \beta_{i+1}| < \varepsilon$. The steps $\beta_i \to \beta_{i+1}$ will however end before β_t since we constructed $P_t = S = 1$ (in other words, P_t leaves only $a_0 = 1$ from Q_{t-1}), which never has a root. There will therefore exist an integer k such that $|\beta_k| \geq s$. This proves the first part for H_{n_0}. The proof for H'_{n_0} is similar, instead of $S = 1$, take $S = \sum_{i=0}^{n_0} r^i = \frac{r^{n_0+1}-1}{r-1}$

For the second part we assume the above holds and will prove that this means other values of R and S are also connected. We denote the coefficients of $R(x)$ by r_i and the coefficients of $S(x)$ by s_i. For R and S, their error is equal to the infimum of the smallest coefficient they don't have in common or n_0. If this infimum is equal to n_0, it means R and S share at least their first n_0 coefficients and it is clear that the above method for an ε-chain works.

Therefore assume that this is not the case and say this error is equal to $k < n_0$. This means that R and S are equal up until the k'th coefficient, so we can write

$$\begin{aligned} R(x) &= 1 + r_1 x + \cdots r_k x^k + \cdots \\ S(x) &= 1 + s_1 x + \cdots s_k x^k + \cdots , \end{aligned}$$

□

where $\forall i < k : r_i = s_i$. Their coefficients can therefore be displayed as

$$\begin{aligned} coef(R) &= (1 \;\; r_1 \;\; \cdots \;\; r_{k-1} \;\; r_k \;\; r_{k+1} \;\; r_{k+2} \;\; \cdots) \\ coef(S) &= (1 \;\; r_1 \;\; \cdots \;\; r_{k-1} \;\; s_k \;\; s_{k+1} \;\; s_{k+2} \;\; \cdots). \end{aligned}$$

We can also define the following polynomial

$$\begin{aligned} coef(R') &= (1 \;\; r_1 \;\; \cdots \;\; r_{k-1} \;\; r_k \;\; 0 \;\; 0 \;\; \cdots) \\ &= (1 \;\; r_1 \;\; \cdots \;\; r_{k-1} \;\; r_k). \end{aligned}$$

We can see that we can go from R to R' by a sequence (P_i, Q_i) as in the part above, by the assumption that this ε-chaining is possible. We would now like to be able to go from this R' to a S' with

$$coef(S') = (1 \;\; r_1 \;\; \cdots \;\; r_{k-1} \;\; s_k \;\; ? \;\; ? \;\; \cdots),$$

which has the first k terms equal as S. Since r_k and s_k are not equal, for Littlewood polynomials this implies that $|r_k - s_k| = 2$. If we define $U = \varphi(R' \times (0 - r_k)x^k)$, we see that we obtain

$$coef(U) = (1 \;\; r_1 \;\; \cdots \;\; r_{k-1} \;\; 0 \;\; ? \;\; ? \;\; \cdots),$$

which in fact also has $U \in H_{n_0}$. This means we can go from R' to U by a similar sequence (P_i, Q_i) for $k-1$. From this U we can form another sequence leading to

$$\begin{aligned} coef(U') &= (1 \;\; r_1 \;\; \cdots \;\; r_{k-1} \;\; 0 \;\; 0 \;\; 0 \;\; \cdots) \\ &= (1 \;\; r_1 \;\; \cdots \;\; r_{k-1} \;\; 0). \end{aligned}$$

Now if we define $V = \varphi(U' \times ((s_k - 0)x^k))$, leading to

$$coef(V) = (1 \;\; r_1 \;\; \cdots \;\; r_{k-1} \;\; s_k \;\; ? \;\; ? \;\; \cdots).$$

By the same argument as before, we know that there also exists a sequence from U' to V. We can finally go from the V to S by the assumption in the upper part of the proof. Lemma 6.4 is thereby proven. □

The entire Theorem 6.1 is now also proven. Since the proof above is long and complicated, we will present the reader a brief walkthrough, going backwards through the proof:

- The combination of Lemma 6.4 and Lemma 6.3 prove that there is a path between any two elements in H' via ε-chaines.
- This means H' is connected for any $0 < \alpha < 1$ where α is a root of a Littlewood polynomial.
- From this we conclude that any values between these roots α, also give an H' which is connected.
- Every α for which H' is connected, has to be contained in C due to the construction of H' and C.
- The interval of values for which H' is connected, will therefore be contained in C.
- Due to the connectedness of the values α in H', we see that these have to be the only elements of C.
- Thereby C is connected too, it is specifically the interval of values for which H' is connected.
- C contains all values $0 < r < 1$ for which $0 \in A_r$ where A_r contains all outcomes of all (up to degree ∞) combinations of Littlewood polynomials for that r.
- From this we can finally conclude that all values in C have to either be a root of some finite Littlewood polynomial, or a root of some infinite Littlewood polynomial (making it a limit point).

- The set C is therefore equal to $\overline{D}$ from the hypothesis and Theorem 6.1, when viewed for its values smaller then 1 and greater then 0.
- Since C is connected, $\overline{D}$ is connected for its elements in $(0, 1)$.

This concludes the proof for Theorem 6.1. □

7 Do extrema occur?

Under the assumption that Littlewood polynomials do not have extremum-roots for values $0 < r < 1$, Theorem 6.1 provides an involved but valid proof that the real roots of Littlewood polynomials between 0 and 1 together with all their limit points form a connected set in $\mathbb{R}$. In Section 5 we saw that 2 and thereby also $\frac{1}{2}$ has to be such a limit point. Theorem 3.1 proved there are no roots or limit point in $(0, \frac{1}{2})$. From this we conclude that $[\frac{1}{2}, 1) \subset \overline{D}$. Using the same reasoning as in Theorem 4.1, we see that $(1, 2] \subset \overline{D}$, $[\frac{1}{2}, 2] \subset \overline{D}$ and finally that $[-2, -\frac{1}{2}] \cup [\frac{1}{2}, 2] = \overline{D}$.

The question whether extrema-roots occur, is therefore vital for the usefulness of the rest of this thesis. Many different approaches, amongst which

- Attempting to use the higher-order derivative test and the assumption that α is a root, trying to prove that α can not be an extremum,
- Attempting to use the higher-order derivative test and the assumption that α is a extremum, trying to prove that α can not be a root,
- Attempting to use the higher-order derivative test and the assumption that α is a root and extremum, trying to prove that α attains wrong values,
- Attempting to use the higher-order derivative test and the difference between any n'th derivative $f^{(n)}(\alpha)$ and $f^{(n+1)}(\alpha)$,

were attempted, all to no avail.[8] Therefore a script was written to attempt to find out if it would at all occur that for a Littlewood polynomial $f(x)$ and root α, the derivative in this value $f'(\alpha)$ could be 0, ignoring for the moment that this can result in either an extremum (for which Theorem 6.1 does not apply) or a saddle (for which Theorem 6.1 does apply). The next attempt therefore became to exclude double zero's (which are values for which both the function value and it's derivative equal 0) and thereby also exclude the possibility of extrema-roots.

This resulted in the Mathematica document which can be found in Appendix C. The document creates all combinations of Littlewood polynomials up until a give degree l and evaluates for each of these polynomials whether there exist values for which both the function and its derivative equals 0. Having run this program for $l = 13$ (it takes much more computation time when l gets bigger), we see that the only values that appear to sometimes be a double zero, are -1 and 1.

The above strongly suggests that double zero's simply do not exist, except for -1 and 1, which does not interfere with our other proofs. This would mean any extremum can not be root and the assumption in the previous section proves to be a valid one. With this knowledge, the search to exclude extrema turned into the search to exclude double zero's. The obvious method was to assume a root α such that for some Littlewood polynomial

$$l_n(\alpha) = \sum_{i=0}^{n} a_i \alpha^i = 0,$$

and attempt to prove that it's derivative

$$l'_n(\alpha) = \sum_{i=0}^{n} i a_i \alpha^{i-1} \overset{?}{\neq} 0,$$

by repeatedly using the fact that in $\sum_{i=0}^{n} a_i \alpha^i = 0$. The fact that there is however no indication that $l_{n-1}(\alpha)$ is also 0, made that these attempts did not succeed. Proving that a roots with multiplicity 2 does not exist either, by checking whether

$$(x^2 - 2x\alpha + \alpha^2) \tag{1}$$

could divide any Littlewood polynomial for $\alpha \neq \pm 1$, turned intro a dead end as well.

We did however succeed in narrowing down the remaining problem even further, as can be seen by the Theorem 7.1.

Theorem 7.1. *A Littlewood polynomial $l_n(x)$ with an even degree n, does not have multiple zeroes.*

Proof. Assume there exists a Littlewood polynomial l_n and α such that both $l_n(\alpha) = 0$ and $l'_n(\alpha) = 0$. Let $f(x)$ be the minimum polynomial of α over $\mathbb{Q}$. Then the Lemma of Gauss tells us since $f(x)|l_n(x)$ and $f(x)|l'_n(x)$, that $f(x) \in \mathbb{Z}[x]$, and $l_n(x) = f(x) \cdot g(x)$ and $l_n(x) = f(x) \cdot h(x)$ where $f, g, h \in \mathbb{Z}[x]$.[9][10]

Next define $\overline{f} = f \mod 2 \in \mathbb{F}_2[x]$. Then our previous equations show: $\overline{l_n(x)} = \overline{f(x)} \cdot \overline{g(x)}$ and $\overline{l_n(x)'} = \overline{f(x)} \cdot \overline{h(x)}$. It follows that $\overline{l_n(x)}$ also has a multiple zero. Since l_n only has coefficients ± 1, $\overline{l_n}$ has coefficients $\overline{1} \in \mathbb{F}_2$. So $\overline{l_n(x)} = 1 + x + x^2 + ... + x^{2m}$ for some m, since our starting Littlewood polynomials has an even degree $n = 2m$.

In $\mathbb{F}_2[x]$ one has

$$(\overline{1+x}) \cdot \overline{l_n(x)} = \overline{1 + x^{2m+1)}}.$$

Hence the fact that $\overline{l_n(x)}$ has a multiple zero implies that also $\overline{1 + x^{(2m+1)}}$ has a multiple zero. This polynomial has derivative $\overline{(2m+1)x^{2m}} = x^{2m}$ which is clearly coprime to $\overline{1 + x^{(2m+1)}}$. This gives a contradiction, whereby the theorem is proven. □

8 Conclusion

The conclusion to the research question:

> **What is closure of the set containing all real roots of Littlewood polynomials?**

is that it is extremely plausible that the hypothesis that

$$\overline{D} = [-2, -\frac{1}{2}] \cup [\frac{1}{2}, 2]$$

is true. However, we did not succeed in finding a formal proof of this.

- Firstly, two different iterative methods seem to be able to find Littlewood polynomials arbitrarily close to any $k \in [\sqrt{2}, 2]$, which would imply that $\overline{D} = [-2, -\frac{1}{2}] \cup [\frac{1}{2}, 2]$.
- Secondly, we did find a proof for our hypothesis under the assumption that Littlewood polynomials only have non-extremum roots.
- The first indication that this assumption is valid, is that a third iterative method shows that the only double zero's that occur for Littlewood polynomials with degree smaller then 13, are located at -1 and 1, which does not interfere with our other proofs.
- Additionally we saw that double zero's can not occur for any Littlewood polynomials with even degree.

9 Suggestions for additional study

If someone would be interested in attempting to definitively solve the problem this thesis describes, we suggest attempting it along either of two directions.

9.1 Proving odd degree Littlewood polynomials can not have double zero's

The iterative methods indicate that it is not likely that Littlewood polynomials can have double zero's. Any attempt to prove connectedness under the assumption that extremum-roots exist, should therefore be avoided. It would be more probable it can be found through a proof that odd degree Littlewood polynomials can not have double zero's. If such a proof exists, It is likely makes use of ring structures on these polynomials.

9.2 The length of the intervals in method B is monotonically decreasing

A second method of solving the problem could potentially be found through the steps of Method B. The difference between the upper and lower bounds of this method seem to be decreasing at every step, using Mathematica. If this could be proven analytically, it could be argued that this provides a proof for the problem. In other to succeed in this, it is most likely necessary to find a way to find a bound for the difference between $\alpha_{(n,i)}$ and $\alpha_{(n,i,\pm)}$, preferable as a factor of the degree n of the polynomial at each step.

I would like to thank Professor J. Top for his guidance throughout this research. Without weekly meetings I would have never gotten this far.

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Appendices

A Mathematica document for Method A

```
n = {choose any targetvalue between 2 and its square root}
iterations = {choose the number of iterations for the for-loop}
f[x] = 1 + x - x^2;
f[x_]=f[x];
value[x_] := If[f[x] < 0, -1, 1];
For[i = 1, i < iterations, i++, f[x] = x*f[x] - value[n];
f[x_] = f[x]; Print[NSolve[f[x]==0 &&x>1,Reals]]];
```

B Mathematica document for Method B

```
n = {choose any targetvalue between 2 and its square root}
iterations = {choose the number of iterations for the for-loop}
d[x] = 1 + x + x^2 + x^3 - x^4;
d[x_] = d[x];
c[x] = -1 + x + x^2 + x^3 - x^4;
c[x_] = c[x];
b[x] = 1 - x + x^2 + x^3 - x^4;
b[x_] = b[x];
a[x] = -1 - x + x^2 + x^3 - x^4;
a[x_] = a[x];

NSolve[a[x] == 0 && x > 1, Reals]
NSolve[b[x] == 0 && x > 1, Reals]
NSolve[c[x] == 0 && x > 1, Reals]
NSolve[d[x] == 0 && x > 1, Reals]

For[i = 1, i < iterations, i++,
  If[a[n]*b[n] < 0, lowb = a[x]; highb = b[x]; track = 1];
  If[b[n]*c[n] < 0, lowb = b[x]; highb = c[x]; track = 2];
  If[c[n]*d[n] < 0, lowb = c[x]; highb = d[x]; track = 3];
  lowbound = lowb; highbound = highb;
  a[x] = x*lowbound - 1; a[x_] = a[x];
  b[x] = x*lowbound + 1; b[x_] = b[x];
  c[x] = x*highbound - 1; c[x_] = c[x];
  d[x] = x*highbound + 1; d[x_] = d[x];
  Print[i];
  Print[NSolve[a[x] == 0 && x > 1, Reals]];
  Print[NSolve[b[x] == 0 && x > 1, Reals]];
  Print[NSolve[c[x] == 0 && x > 1, Reals]];
  Print[NSolve[d[x] == 0 && x > 1, Reals]]];
```

C Mathematica document for Method C

```
l = {choose any degree for which this script will check
whether all Littlewood polynomials up until this degree
have double zero's};
For[n = 1, n < l + 1, n++,
 a[-1] = 0;
 For[j = 0, j < (2^n)/2, j++,
  For[i = 0, i < n, i++, a[i] = Mod[j, 2^(i + 1)];
   If[a[i] > (2^i) - 1, b[i] = 1, b[i] = -1]];
  f[x] = \!\(
\*SubsuperscriptBox[\(\[Sum]\), \(k = 0\), \(n - 1\)]\(b[k]*x^k\)\);
  f[x_] = f[x]; derf[x] = f'[x];
  Print[f[x]];
  Print[NSolve[{f[x] == 0, derf[x] == 0 && -2 < x < 2}, Reals]]]]
```