Rijksuniversiteit Groningen

Bachelor thesis in mathematics

Fibre Bundles: Trivial or Not?

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July 12, 2016

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Abstract

When are fibre bundles globally isomorphic to a product space? We develop some theory to answer this question for vector bundles and principal bundles and consider examples such as the Möbius band, the Klein Bottle, and the Hopf fibration.

Chapter 1

Introduction to Bundles

In this chapter we develop some general theory and give criteria for the triviality of vector bundles and principal bundles.

1.1 Fibre Bundles

Definition 1.1.1 (Fibre Bundle)**.** A *fibre bundle* is a four-tuple (E, B, π, F) consisting of manifolds[1] (E, B, F) and a smooth surjection $\pi : E \to B$ such that the following conditions are satisfied.

1. Every $x \in E$ has a neighborhood $U_x \subset B$ such that there is a diffeomorphism $\phi : \pi^{-1}(U_x) \to U_x \times F$. The neighborhood U_x and the diffeomorphism ϕ constitute a *local trivialisation*;

2. If we let $\mathrm{proj}_1 : B \times F \to B$ be the map $\mathrm{proj}_1(a, b) = a$ then the diffeomorphism ϕ of condition 1 satisfies $\mathrm{proj}_1 \circ \phi = \pi$. This corresponds to the commutativity of the diagram seen below.

$$\begin{array}{ccc} \pi^{-1}(U_x) & \xrightarrow{\pi} & U_x \subset B \\ {\scriptstyle\phi}\downarrow & \nearrow_{\mathrm{proj}_1} & \\ U_x \times F & & \end{array}$$

The spaces E, B and F are called the **total** space, **base** space and **fibre**, respectively. The set $\pi^{-1}(b)$ is called the fibre at $b \in B$.

Definition 1.1.1 says that the total space E of a fibre bundle (E, B, π, F) is locally isomorphic to $B \times F$. If we view topology as the study of properties

[1]Manifold will mean *smooth (real and C^∞) manifold* from now on.

that are invariant under homeomorphisms and ignore the C^∞ structure, then we can say that $\pi^{-1}(U_x)$ is topologically "the same" as the product space $U_x \times F$.

Remark 1.1.1 (Notation). The fibre bundle (E, B, π, F) is often represented schematically as

$$F \to E \xrightarrow{\pi} B.$$

Example 1.1.1 (Product Bundle)**.** If we let $E = B \times F$, then $(E, B, \mathrm{proj}_1, F)$ is a fibre bundle. Specifically, the identity i_d serves as the diffeomorphism of Definition 1.1.1. In this case, a single diffeomorphism works for the whole of E. This is called a *global trivialisation*.

Definition 1.1.2 (Trivial)**.** We call a fibre bundle that admits a global trivialisation a *trivial fibre bundle*.

Definition 1.1.3 (Section)**.** A section $\sigma : B \to E$ over a fibre bundle (E, B, π, F) is a smooth right inverse of $\pi : E \to B$. Specifically,

$$\pi(\sigma(x)) = x$$

for all $x \in B$.

Example 1.1.2 (Sections of the product bundle)**.** A section of the product bundle is a smooth function $\sigma : B \to B \times F$ such that $\mathrm{proj}_1 \circ \sigma = i_d$. Specifically, if we write

$$\sigma : a \mapsto (b, c),$$
$$\mathrm{proj}_1 : (b, c) \mapsto b,$$

we see that σ can be any smooth function of the form $a \mapsto (a, c)$. Hence sections of the product bundle are graphs of smooth functions.

1.2 Vector Bundles

Definition 1.2.1 (Vector Bundle)**.** A vector bundle over the field $\mathbb{F}$ is a fibre bundle (E, B, π, F) satisfying the following conditions.

1. The fibre F is a k-dimensional vector space over $\mathbb{F}$. We call k the rank of the vector bundle.[2]

2. The map $v \mapsto \phi(x, v)$ for $v \in F$ is a linear isomorphism between F and $\pi^{-1}(x)$.

[2]Some authors do not require k to be constant. The rank of the vector bundle is then not always well-defined, unless it is assumed that the base space is connected.

Remark 1.2.1. A vector bundle over $\mathbb{R}$ is called a *real* vector bundle. Likewise, if $\mathbb{F} = \mathbb{C}$ we call the vector bundle *complex*.

The tangent and cotangent bundles of differentiable manifolds form an important example of vector bundles.

Definition 1.2.2 (Tangent Bundle)**.** Let M be a differentiable manifold and suppose we write T_xM for the tangent space of M at the point $x \in M$. Then the tangent bundle TM of M is defined

$$TM = \bigsqcup_{x \in M} T_xM = \bigcup_{x \in M} \{(x, y) \mid y \in T_xM\}.$$

Remark 1.2.2. A section of the tangent bundle TM is a vector field on M.

Definition 1.2.3 (Cotangent Bundle)**.** Let M be a differentiable manifold and suppose we write T_x^*M for the cotangent space of M at the point $x \in M$. Then the cotangent bundle T^*M of M is defined

$$T^*M = \bigsqcup_{x \in M} T_x^*M = \bigcup_{x \in M} \{(x, y) \mid y \in T_x^*M\}.$$

Remark 1.2.3. A section of the cotangent bundle TM is a differential one-form on M.

Theorem 1.2.1 (TM as vector bundle)**.** Suppose M is a k-dimensional manifold. Then $(TM, M, \mathrm{proj}_1, \mathbb{R}^k)$ is a vector bundle.

Proof. Let $\{x_1, \ldots, x_n\}$ be local coordinates for the open neighborhood U_x containing x. We recall that

$$T_xM = \mathrm{span}_{\mathbb{R}} \left\{ \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k} \right\}.$$

Hence for $(a_1, \ldots, a_k) \in \mathbb{R}^k$ the bijection

$$(a_1, a_2, \ldots, a_k) \leftrightarrow a_1 \frac{\partial}{\partial x_1} + \ldots + \frac{\partial}{\partial x_k},$$

is an isomorphism between the vector spaces T_xM and $\mathbb{R}^k$.
We now let $\phi : TM \to M \times \mathbb{R}^n$ be the map

$$\phi \left(x, a_1 \frac{\partial}{\partial x_1} + \ldots + a_k \frac{\partial}{\partial x_k} \right) = (x, a_1, \ldots, a_k).$$

We note that both ϕ and ϕ^{-1} are smooth. This means that ϕ is a diffeomorphism between TM and $M \times \mathbb{R}^n$. Moreover, we have $\mathrm{proj}_1 \circ \phi = i_d$. Finally, $v \mapsto \phi(x, v)$ is a linear isomorphism since for $v = \sum v_i \frac{\partial}{\partial x_i}$ and $w = \sum w_i \frac{\partial}{\partial x_i}$ we have that $\phi(x, v) + \phi(x, w) = (x, v_1 + w_1, \ldots, v_k + w_k) = \phi(x, v + w)$. □

Remark 1.2.4. The cotangent bundle is a vector bundle. We will show later that the tangent bundle is isomorphic to the cotangent bundle.

Definition 1.2.4 (Structure group)**.** Suppose that (U_i, ϕ_i) and (U_j, ϕ_j) are trivialisations of the vector bundle (E, B, π, F) such that $U_i \cap U_j \neq \emptyset$. Then the composite functions

$$\phi_i^{-1} \circ \phi_j : (U_i \cap U_j) \times F \to (U_i \cap U_j) \times F$$

are of the form

$$\phi_i^{-1} \circ \phi_j(x, y) = (x, \Psi_{ij}(x)y).$$

Here $\Psi_{ij} : (U_i \cap U_j) \to \mathrm{GL}_k$ are called *coordinate transformations*. If the maps $\Psi_{ij}(x)$ all belong to a subgroup $G \subset \mathrm{GL}_k$, then we call G the *structure group* of the vector bundle.

More generally, if the transition functions of a *fibre* bundle are well-defined and members of a group G, then we call G the structure group.

1.3 Principal Bundles

A principal bundle is a bundle for which the fibre is the structure group.

Definition 1.3.1 (Principal Bundle)**.** A *Principal Bundle* is a fibre bundle (E, B, π, F), where the fibre F is a Lie group equipped with a smooth right group action on E such that the following conditions are satisfied.

1. The group action of F on E is free and transitive on the fibres $\pi^{-1}(b)$ for $b \in B$;

2. The orbits of F in E are identified with B:

$$B = E/G.$$

Remark 1.3.1. Let (E, B, π, F) be a principal bundle and $\star$ its group action. Given some fixed $y \in E$, we can write any x that lies in the fibre of y uniquely in the form $x = y \star g$ for some $g \in F$.

Chapter 2

Bundle Morphisms and Triviality

2.1 Fibre Bundles

A fibre bundle morphism is a map between two fibre bundles that 'respects' fibers.

Definition 2.1.1 (Fibre Bundle morphism)**.** Let (E, B, π, F) and (E', B', π', F') be fibre bundles. Then a fibre bundle morphism is a pair (g, f) of smooth maps

$$g : E \to E', \quad f : B \to B',$$

such that $\pi' \circ g = f \circ \pi$. This condition means that the following diagram is commutative.

$$\begin{array}{ccc} E & \xrightarrow{g} & E' \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} \\ B & \xrightarrow{f} & B' \end{array}$$

If a fibre bundle morphism has an inverse that is also a bundle morphism we speak about a *fibre bundle isomorphism.*

Theorem 2.1.1. The condition $\pi' g = f\pi$ implies

$$g\pi^{-1}(b) \subset (\pi')^{-1} f(b).$$

That is, the fibre at the point $b \in B$ is mapped to the fibre at $f(b) \in B'$.

Proof. We first note that

$$(\pi')^{-1} f(b) = \{x' \in E' \mid \pi'(x) = f(b)\}$$

and

$$g(\pi^{-1}b) = \{g(x) \mid x \in E \text{ such that } \pi(x) = b\}.$$

Now suppose that $y = g(x) \in g\pi^{-1}(b)$. Then $y \in g(E) \subset E'$. Moreover, we have that

$$\pi'(y) = \pi'(g(x)) = f(\pi(x)) = f(b).$$

It follows that $y \in \pi^{-1}(f(b))$. $\square$

Remark 2.1.1. (compare Definition 1.1.2.) A fibre bundle (E, B, π, F) is trivial if and only if it is isomorphic to the product $(B \times F, B, \mathrm{proj}_1, F)$. The fibre bundle isomorphism (g, f) gives us a diffeomorphism $g : E \to B \times F$. For the converse we note that we can take i_d and the diffeomorphism ϕ of the global trivialisation to obtain a fibre bundle isomorphism (ϕ, i_d).

2.2 Vector Bundles

Definition 2.2.1 (Vector bundle morphism)**.** Let $V = (E, B, \pi, F)$ and $W = (E', B', \pi', F')$ be vector bundles. Then a vector bundle morphism between V and W is a pair (g, f) of smooth maps

$$g : E \to E', \quad f : B \to B',$$

such that $\pi' \circ g = f \circ \pi$ and the map

$$\pi^{-1}(b) \mapsto (\pi')^{-1}(f(b)),$$

is linear. See the diagram below.

$$\begin{array}{ccc} E & \xrightarrow{g} & E' \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} \\ B & \xrightarrow{f} & B' \end{array}$$

If a vector bundle morphism has an inverse which is also a vector bundle morphism, we speak about a *vector bundle isomorphism*.

Remark 2.2.1. Just like every vector bundle is a fibre bundle, every vector bundle morphism is a fibre bundle morphism. The extra requirements on a vector bundle morphism are such that the linear structure is preserved.

Theorem 2.2.1. *TM is isomorphic to T^*M.*

Proof. Let $\langle \cdot, \cdot \rangle_x$ be a Riemannian metric on M. We construct a vector bundle isomorphism (g, f). TM and T^*M have the same base space, so we let $f = i_d$. For $(x, v) \in TM$, the map

$$g(x, v) = (x, \langle v, \cdot \rangle_x)$$

is a diffeomorphism. □

Theorem 2.2.2 (Triviality of Vector Bundles)**.** A vector bundle $(E, B, \pi, \mathbb{R}^n)$ is trivial if and only if there exist n linearly independent sections $s_1, \ldots, s_n$, $s_j : B \to E$ such that $\{s_1, \ldots, s_n\}$ is a basis for the fibre $\pi^{-1}(p)$ for every $p \in B$.

Proof. Suppose $(E, B, \pi, \mathbb{R}^n)$ has n sections $\{s_1, \ldots, s_n\}$ that are everywhere independent. Then let $g : B \times \mathbb{R}^n \to E$ be the map

$$(p, x_1, \ldots, x_n) \mapsto x_1 s_1 + \ldots + x_n s_n.$$

The diagram below now describes a vector bundle isomorphism.

$$\begin{array}{ccc} B \times \mathbb{R}^n & \xrightarrow{\quad g \quad} & E \\ \Big\downarrow{\scriptstyle \mathrm{proj}_1} & & \Big\downarrow{\scriptstyle \pi} \\ B & \xrightarrow{\quad i_d \quad} & B \end{array}$$

□

Remark 2.2.2. Since the zero vector does not span anything, the sections $\{s_1, \ldots, s_n\}$ of Theorem 2.2.2 have to be non-zero everywhere.

2.3 Principal Bundles

Definition 2.3.1 (Principal Bundle Morphism)**.** Let $V = (E, B, \pi, F)$ and $W = (E', B', \pi', F)$ be two principal bundles. Then a *principal bundle morphism* from V to W is a fibre bundle morphism (g, f) with $g : E \to E'$ and $f : B \to B'$ such that g is F-equivariant. That is, given any $h \in F$ we have that

$$g(x) \star h = g(x * h),$$

where $\star$ is the action of F on E' and $*$ is the action of F on E.

Theorem 2.3.1 (Triviality of Principal Bundles)**.** A principal bundle (E, B, π, F) is trivial if and only if there exists a smooth section $s : B \to E$.

Proof. Suppose that $s : B \to E$ is a smooth section for the principal bundle (E, B, π, F). Denote the action of F on E by $\star$. We will show that

$$(b, g) \leftrightarrow g \star s(b)$$

defines a diffeomorphism $B \times F \to E$.
If $g \star s(b) = g' \star s(b')$ then $s(b')$ is in the orbit of $s(b)$. The orbits of F in E are identified with B, so $s(b')$ and $s(b)$ then have the same base point $b = b'$. This shows that the identification given above is injective. Because $\star$ is transitive, it is also surjective.
Given $x \in E$ we can recover (b, g) explicitly as $(b, g) = \pi(g^{-1} \star x)$. The four maps $s, \star$, π, and $g \mapsto g^{-1}$ are smooth, so we have constructed a diffeomorphism. □

Remark 2.3.1. A vector space contains a zero element. Therefore, a vector bundle always has a global section, namely the zero section $\sigma(x) = (x, 0)$. It follows through Theorem 2.3.1 that any vector bundle that is also a principal bundle must be trivial.

Chapter 3

Examples of Bundles and Their Triviality

In this chapter we consider the tangent bundles $T\mathbb{S}^1$ and $T\mathbb{S}^2$, showing that $T\mathbb{S}^1$ is trivial while $T\mathbb{S}^2$ is not.
We then construct the Möbius Band as an example of a vector bundle, the Klein Bottle as an example of a line bundle and the Hopf Fibration as an example of a non-trivial principal circle bundle.

3.1 Tangent Bundles of $\mathbb{S}^1$ and $\mathbb{S}^2$

Example 3.1.1 ($T\mathbb{S}^1$ is Trivial)**.** We will show that the map $F : T\mathbb{S}^1 \to \mathbb{S}^1 \times \mathbb{R}$ given by

$$F(x, v) = (x, v/ix)$$

is a diffeomorphism.
We first parametrise $\mathbb{S}^1$ by $t \mapsto e^{it}$ for $t \in \mathbb{R}$. Using this parametrisation, the tangent space at $x \in \mathbb{S}^1$ is given by

$$T_x\mathbb{S}^1 = \left[ie^{it}\right]_{\mathbb{R}}.$$

Now let F be as above. For $(x, v) = (e^{it_x}, \alpha ie^{it_x}) \in T\mathbb{S}^1$ we have

$$F(x, v) = F(e^{it_x}, \alpha ie^{it_x}) = (e^{it_x}, \alpha ie^{it_x}/ie^{it_x}) = (x, \alpha) \in \mathbb{S}^1 \times \mathbb{R}.$$

Hence, F is a map from $T\mathbb{S}^1$ into $\mathbb{S}^1 \times \mathbb{R}$. Moreover, F is smooth since $x \neq 0$ on $\mathbb{S}^1$. Finally, the inverse map $F^{-1} : \mathbb{S}^1 \times \mathbb{R} \to T\mathbb{S}^1$ is given by

$$F^{-1}(x, \alpha) = (x, \alpha ix)$$

and is also smooth.

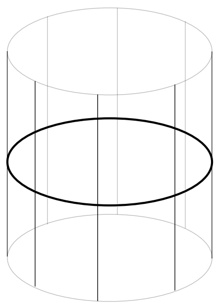

Figure 3.1: The tangent bundle $T\mathbb{S}^1$. The vertical lines represent the tangent spaces attached disjointly to $\mathbb{S}^1$, represented by the black circle.

Remark 3.1.1. We can view $T\mathbb{S}^1$ as a principal bundle with fibre $(\mathbb{R}, +)$. Specifically, $\star$ defined for $\beta \in \mathbb{R}$ and $(x, v) \in T\mathbb{S}^1$ by

$$(x, v) \star \beta = (x, v + \beta i x)$$

provides a natural group action of $(\mathbb{R}, +)$ on $T\mathbb{S}^1$. The action $\star$ is *free* on the fibre since $v + \beta_1 e^{it} = v + \beta_2 e^{it}$ implies $\beta_1 = \beta_2$. Moreover, $\star$ is also *transitive*; See Figure 3.2.

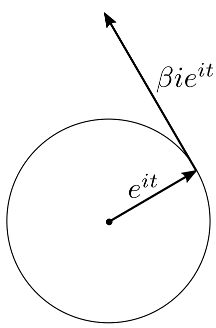

Figure 3.2: The action of β adds a tangent vector of length β. Since we can reach any vector in the tangent space at e^{it}, the action is transitive.

It follows from Remark 2.3.1 that $T\mathbb{S}^1$ is trivial.

Remark 3.1.2. We can view $T\mathbb{S}^1$ as a vector bundle. The map

$$x \mapsto ix,$$

is a global section. It follows from Theorem 2.2.2 that $T\mathbb{S}^1$ is trivial.

Example 3.1.2 ($T\mathbb{S}^2$)**.** The Hairy Ball Theorem (see for example Theorem 10.15 of [14]) states that a nowhere zero section on $T\mathbb{S}^2$ does not exist. It follows from Remark 2.2.2 that the tangent bundle $T\mathbb{S}^2$ is non-trivial.

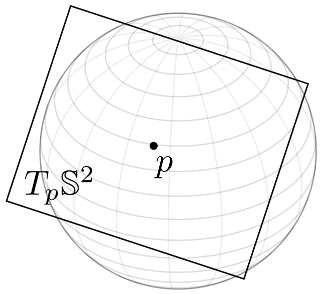

Figure 3.3: The tangent space of the two-sphere.

Remark 3.1.3. It was shown by Adams[1] that $T\mathbb{S}^n$ is trivial only for $n = 1, 3, 7$.

3.2 The Möbius Band

Construction. The cylinder $\mathbb{S}^1 \times \mathbb{R}$ is trivial. We use the cylinder to construct a non-trivial vector bundle called the *Möbius Band*.

Let $\sim$ be the equivalence relation $(p, x) \sim (p + 2\pi, -x)$ on $\mathbb{S}^1 \times \mathbb{R}$. The Möbius band **Mo** is defined as the quotient space

$$\mathbf{Mo} = (\mathbb{S}^1 \times \mathbb{R})/\sim .$$

See Figure 3.4.

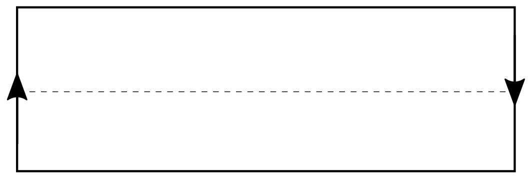

Figure 3.4: Construction of the Möbius band. Points on the left side of the rectangle are identified, through reflection in the dashed line, with points on the right side of the rectangle.

Vector Bundle. The indentification $\sim$ has no effect on subsets of **Mo** shorter than 2π. See Figure 3.5. Furthermore, the fibre $\mathbb{R}$ is a vector space and the fibres are isomorphic to $\mathbb{R}$.

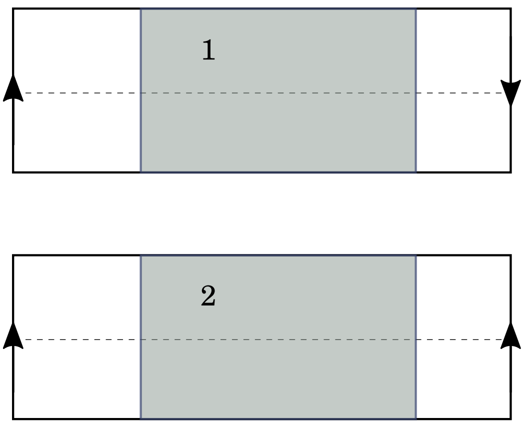

Figure 3.5: Rectangle **1** is a subset of **Mo**. Rectangle **2** is a subset of $\mathbb{S}^1 \times \mathbb{R}$.

Not Trivial. By definition, sections $\sigma : \mathbb{S}^1 \to \mathbf{Mo}$ are of the form

$$\sigma(x) = (x, f(x)),$$

where $f : \mathbb{S}^1 \to \mathbb{R}$. For σ to be continuous, f has to satisfy $f(2\pi) = -f(0)$. See Figure 3.6. By the intermediate value theorem, there exists some $\zeta \in [0, 2\pi]$ such that $f(\zeta) = 0$. See Figure 3.6. It follows from Remark 2.2.2 that the Möbius Band is not trivial.

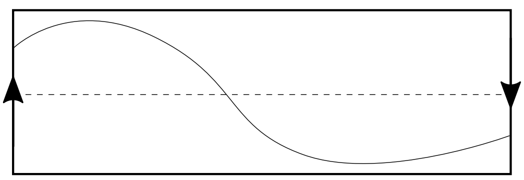

Figure 3.6: Every section of the Möbius Band intersects the zero section, which is represented by the dashed line.

Not Orientable.

Theorem 3.2.1. If two vector bundles are isomorphic and one of them is orientable, then so is the other.

Proof. See for example paragraph 38 of [6]. □

The Möbius band is not orientable. See Figure 3.7.

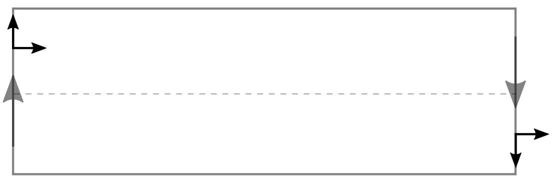

Figure 3.7: The identification $\sim$ is orientation-reversing.

By contrast, the cylinder $\mathbb{S}^1 \times \mathbb{R}$ is orientable. It follows from Theorem 3.2.1 that **Mo** is not homeomorphic to the cylinder, and hence not trivial.

3.3 The Klein Bottle

Construction. The Torus $\mathbb{S}^1 \times \mathbb{S}^1$ is trivial. We will use the Torus to construct a non-trivial circle bundle called the *Klein Bottle*.

Let $\sim_2$ be the equivalence relation

$$\begin{aligned} (0, y) &\sim_2 (2\pi, 2\pi - y), \\ (x, 0) &\sim_2 (x, 2\pi). \end{aligned} \tag{3.1}$$

We define the Klein bottle **Kl** as the product space

$$\mathbf{Kl} = \mathbb{S}^1 \times \mathbb{S}^1 / \sim_2 .$$

See Figure 3.8.

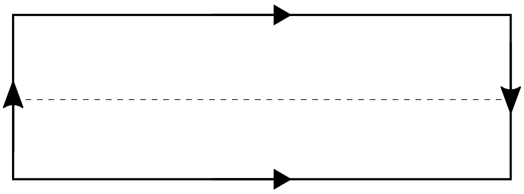

Figure 3.8: By identifying the top of a rectangle with its bottom, a cylinder is obtained. If we identify points on the sides with their reflections in the dashed line we obtain a Klein Bottle.

We claim that the Klein Bottle is a circle bundle. The reasoning is the same as for the Möbius Band.

Not Orientable and Not Trivial. It was shown in Section 3.2 that the Möbius band is not orientable. The Möbius band is an open subset of the Klein Bottle, see Figure 3.9. It follows that the Klein bottle is not orientable.

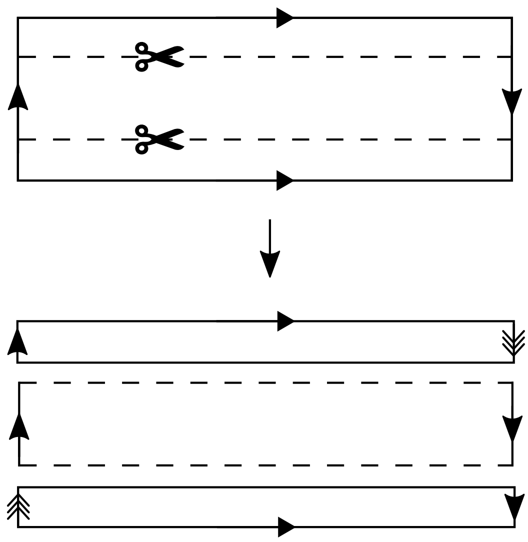

Figure 3.9: The Möbius strip is an open subset of the Klein Bottle.

Because the Torus $\mathbb{S}^1 \times \mathbb{S}^1$ is orientable we can conclude from Theorem 3.2.1 that the Klein Bottle is not homeomorphic to the Torus, and hence not trivial.

3.4 The Hopf Fibration

The Hopf Fibration $(\mathbb{S}^3, \mathbb{S}^2, \pi_h, \mathbb{S}^1)$ is a way to view $\mathbb{S}^3$ as a principal circle bundle over $\mathbb{S}^2$.

Construction. We identify $\mathbb{R}^4$ with $\mathbb{C}^2$. We can now describe $\mathbb{S}^3$ as

$$\mathbb{S}^3 = \left\{(z_1, z_2) \in \mathbb{C}^2 \;\middle|\; |z_1|^2 + |z_2|^2 = 1\right\}.$$

We also identify $\mathbb{R}^3$ with $\mathbb{C} \times \mathbb{R}$ to describe $\mathbb{S}^2$ as

$$\mathbb{S}^2 = \left\{(z, x) \in \mathbb{C} \times \mathbb{R} \;\middle|\; |z|^2 + x^2 = 1\right\}.$$

The Hopf projection $\pi_h : \mathbb{S}^3 \to \mathbb{S}^2$ is given by

$$\pi_h(z_1, z_2) = (2z_1 z_2^*, |z_1|^2 - |z_2|^2),$$

where z_2^* is the complex conjugate of z_2. We note that π_h indeed maps into $\mathbb{S}^2$ because for $(z_1, z_2) \in \mathbb{S}^3$ we have

$$\begin{aligned} |\pi_h(z_1, z_2)| &= (2z_1 z_2^*)(2z_1 z_2^*)^* + (|z_1|^2 - |z_2|^2)^2 \\ &= 4|z_2|^2|z_1|^2 + |z_1|^4 + |z_2|^4 - 2|z_1|^2|z_2|^2 \\ &= (|z_1|^2 + |z_2|^2)^2 = 1. \end{aligned}$$

Principal Circle Bundle. We claim that

$$(z_1, z_2) \star \lambda = (\lambda z_1, \lambda z_2)$$

provides an action of $\mathbb{S}^1 \ni \lambda$ on $\mathbb{S}^3 \ni (z_1, z_2)$. To see this, write

$$z_1 = r_1 e^{i\theta_1} \quad z_2 = r_2 e^{i\theta_2},$$

where $r_1^2 + r_2^2 = 1$. We have for $\lambda \in \mathbb{S}^1$ that

$$\pi_h(\lambda(z_1, z_2)) = (2z_1 z_2^*, |z_1|^2 - |z_2|^2) = \pi_h(z_1, z_2),$$

since the factor λ cancels in both components of π_h. Conversely, if $(w_1, w_2) = (r_3 e^{i\theta_3}, r_4 e^{i\theta_4}) \in \mathbb{S}^3$ is such that

$$\pi_h(z_1, z_2) = \pi_h(w_1, w_2),$$

then $r_1 r_2 = r_3 r_4$ from the equality in the first coordinate and $r_1^2 - r_2^2 = r_3^2 - r_4^2$ from the equality in the second coordinate. It follows that $r_1 = r_3$ and $r_2 = r_4$. Since $z_1 z_2^* = w_1 w_2^*$ we also have that $e^{i(\theta_1 - \theta_2)} = e^{i(\theta_3 - \theta_4)}$. It follows that $(w_1, w_2) = (\lambda z_1, \lambda z_2)$ for some $\lambda \in \mathbb{S}^3$.

This makes the Hopf Fibration a principal bundle with fibre $\mathbb{S}^1$.

Triviality. Let $\Pi_1(X)$ denote the fundamental group of X. Theorem A.2.3 says that if (E, B, π, F) is trivial, then $\Pi_1(E) \cong \Pi_1(B) \times \Pi_1(F)$. We will show that $\Pi_1(\mathbb{S}^3) \ncong \Pi_1(\mathbb{S}^2) \times \Pi_1(\mathbb{S}^1)$.

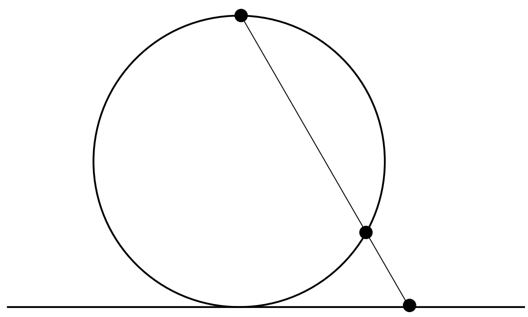

Figure 3.10: Stereographic projection of $\mathbb{S}^1$ on $\mathbb{R}$. In general, we can project $\mathbb{S}^n$ onto $\mathbb{R}^n$.

Lemma 3.4.1. $\Pi_1(\mathbb{S}^3)$ is the trivial group.

Proof. We use Seifert-van Kampen, Theorem A.2.4.

Let x and y be any two antipodal points on $\mathbb{S}^3$. We define U and V by

$$U = \mathbb{S}^3 - \{x\} \quad \text{and} \quad V = \mathbb{S}^3 - \{y\}.$$

We can project U onto $\mathbb{R}^3$ through stereographic projection from x.

This provides a diffeomorphism between U and $\mathbb{R}^3$. Because $\Pi_1(\mathbb{R}^3)$ is trivial it follows that $\Pi_1(U)$ is trivial, see Theorem A.2.2. The same reasoning shows that $\Pi_1(V)$ is trivial.

We now show that $U \cap V = \mathbb{S}^3 - \{x, y\}$ is path-connected. We stereographically project $\mathbb{S}^3$ into $\mathbb{R}^3$ at x. The point x is already missing from this projection. Removing y also removes a single point from $\mathbb{R}^3$. We can easily construct paths in $\mathbb{R}^3$ that avoid this missing point.

It follows from Theorem A.2.4 that $\Pi_1(\mathbb{S}^3)$ is the trivial group. □

Lemma 3.4.2. $\Pi_1(\mathbb{S}^2) \times \Pi_1(\mathbb{S}^1)$ is not the trivial group.

Proof. A loop that winds around the circle once can not be continuously deformed to the constant loop. See Figure 3.11. This shows that $\Pi_1(\mathbb{S}^1)$ is not trivial. It follows that $\Pi_1(\mathbb{S}^2) \times \Pi_1(\mathbb{S}^1)$ is not trivial.

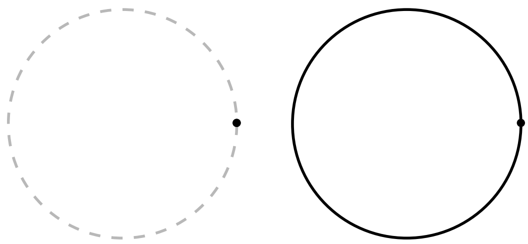

Figure 3.11: It is impossible to continuously deform the loop displayed on the right to the constant loop displayed on the left.

□

Remark 3.4.1 ($\Pi_1(\mathbb{S}^1) = \mathbb{Z}$). The elements of $\Pi_1(\mathbb{S}^1)$ can be identified with the number of times a loop wraps around the circle.

Since $\Pi_1(\mathbb{R}^3)$ is trivial and $\Pi_1(\mathbb{S}^2) \times \Pi_1(\mathbb{S}^1)$ is not trivial we have $\Pi_1(\mathbb{S}^3) \ncong \Pi_1(\mathbb{S}^2) \times \Pi_1(\mathbb{S}^1)$. We conclude that the Hopf fibration is not trivial.

Chapter 4

Conclusion

In chapter 2 criteria for triviality of vector bundles and principal bundles were given. In chapter 3 these were applied to several examples. It was found that $T\mathbb{S}^n$ is trivial only for $n = 1, 3, 7$. The Möbius band was shown to be a non-trivial vector bundle, and the Klein bottle was shown to be a non-trivial circle bundle. Finally, the fundamental group was used to show that the Hopf fibration is a non-trivial principal bundle.

Appendix A

A.1 Group Actions

Definition A.1.1 (Right group action)**.** If $(G, \star)$ is a group and X is a set, then a right group action is a map $\sigma : X \times G \to X$ satisfying the following conditions.

1. If e is the identity of G, then $\sigma(x, e) = x$.
2. If g and h are elements of G, then

$$\sigma(\sigma(x, g), h) = \sigma(x, g \star h).$$

Remark A.1.1. It is common to write xg for $\sigma(x, g)$. The conditions above then read 1) $xe = x$ and 2) $(xg)h = x(g \star h)$.

Definition A.1.2 (Free Action)**.** Suppose $(G, \star)$ is a group, and g and h are elements of G. A right group action of $(G, \star)$ on $X \ni x$ is *free* if $xh = xg$ implies that $h = g$.

Definition A.1.3 (Transitive Action)**.** A group action $\star$ of G on X is transitive if for every pair $x, y \in X$ there exists a $g \in G$ such that $x \star g = y$.

A.2 The Fundamental Group

Definition A.2.1 (Loop)**.** Let X be a topological space with x_0 X. A *loop* at x_0 is a continuous function $f : [0, 1] \to X$ such that $f(0) = x_0 = f(1)$. See Figure A.1.

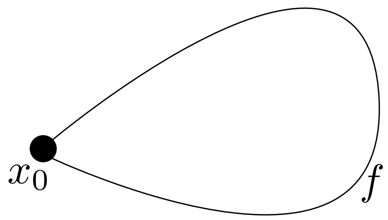

Figure A.1: A loop at the point x_0.

Definition A.2.2 (Homotopy)**.** Let $f : X \to Y$ and $g : X \to Y$ be two continuous functions between to topological spaces X, Y. A homotopy h is a continuous function $h : X \times [0, 1] \to Y$ such that if $x \in X$, then

$$h(x, 0) = f(x) \quad \text{and} \quad h(x, 1) = g(x).$$

Two loops are called homotopy-equivalent if there exists a homotopy between them.

Definition A.2.3 (The Fundamental Group)**.** Let X be a topological space and $x_0 \in X$ a point. Let F be the set of all loops at x_0. The *fundamental group of* X *at* x_0, denoted $\pi_1(X, x_0)$ is the group

$$\pi_1(X, x_0) = F/h,$$

where h identifies two loops if they are homotopy-equivalent.

Remark A.2.1. The fundamental group $\Pi_1(X, x_0)$ of a path-connected space X is independent of the base point x_0.

Theorem A.2.1. Let X and Y be topological spaces with $x_0 \in X$ and $y_0 \in Y$. Then

$$\Pi_1(X \times Y, (x_0, y_0)) = \Pi_1(X, x_0) \times \Pi_1(Y, y_0),$$

where $X \times Y$ is the Cartesian product and $\Pi_1(X, x_0) \times \Pi_2(Y, y_0)$ is a direct product of groups.

Theorem A.2.2 (Induced isomorphism)**.** If X is homeomorphic to Y, then $\Pi_1(X)$ is isomorphic to $\Pi_1(Y)$.

Theorem A.2.3. If (E, B, π, F) is a trivial bundle, then $\Pi_1(E) \cong \Pi_1(B) \times \Pi_1(F)$.

Proof. Combine Theorem A.2.1 and A.2.2. □

Theorem A.2.4 (Van Kampen)**.** Let $X = U_1 \cup U_2$ be the union of two open and path-connected sets U_1, U_2 such that $U_1 \cap U_2$ is path-connected. Let $i_{12} : \Pi_1(U_1 \cap U_2) \to \Pi_1(U_1)$ be the homomorphism induced by the inclusion $U_1 \cap U_2 \hookrightarrow U_1$ and define i_{21} analogously. Then

$$\Pi_1(X) \cong \Pi_1(U_1) * \Pi_1(U_1)/N,$$

where N is the normal subgroup generated by all elements of the form $i_{\alpha\beta}(\omega)i_{\alpha\beta}^{-1}(\omega)$ for $\omega \in \Pi_1(U_1 \cap U_2)$. Here $*$ denotes the free product.

Proof. See Theorem 1.20 of Hatcher [15]. □

Remark A.2.2. If both $\Pi_1(U_1) = e_1$ and $\Pi_2(U_2) = e_2$ are trivial, then $\Pi_1(U_1) * \Pi_1(U_2)$ contains only the reduced element e_1e_2. It then follows that $\Pi_1(X)$ is trivial without considering N.

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