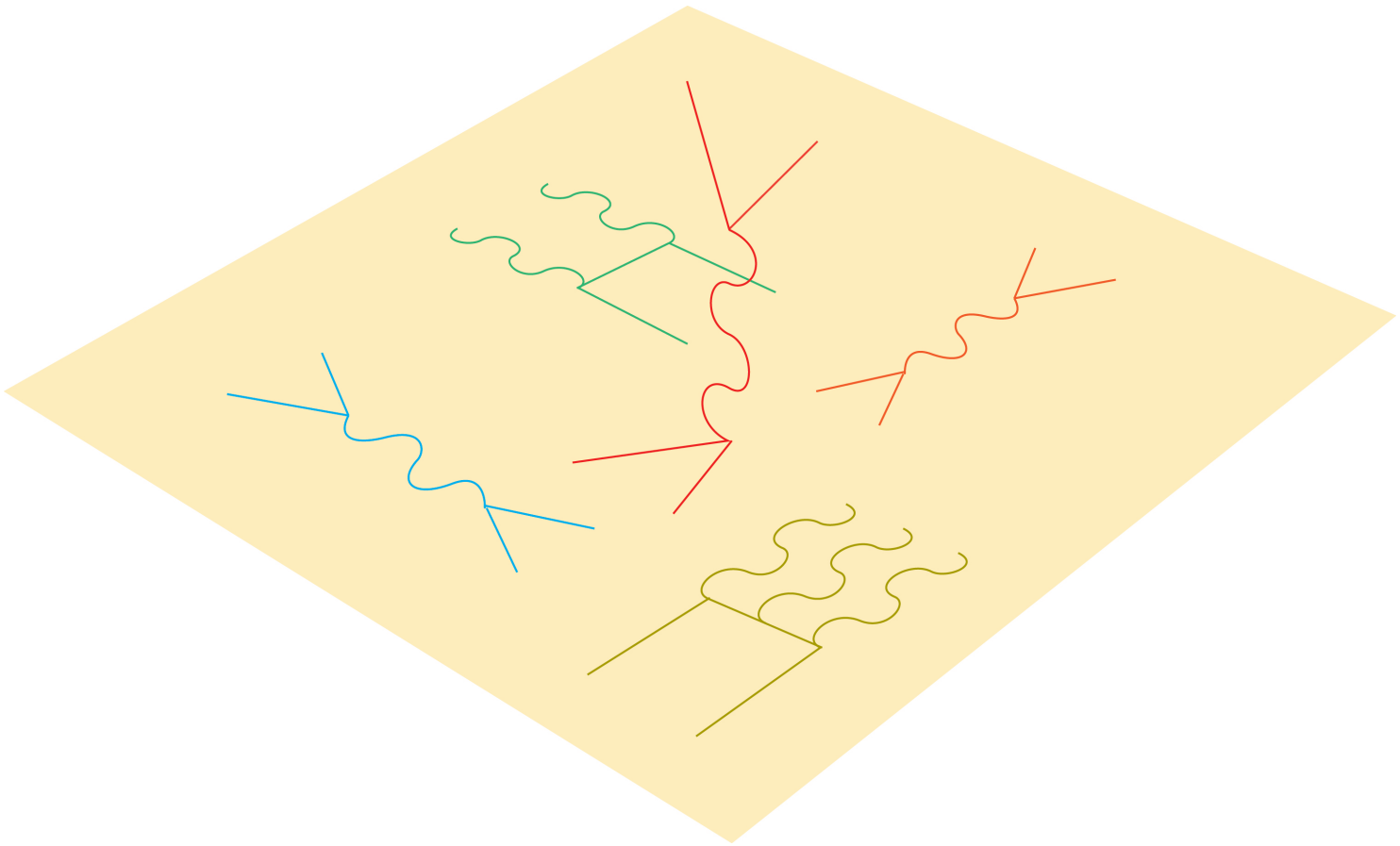
university of groningen

faculty of mathematics and natural sciences

Disappearance of positronium into extra dimensions

Bachelor thesis (theoretical) physics



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Abstract

In this bachelor thesis, it is discussed whether the search for invisible decay of positronium could in the near future be used to test models with extra dimensions. To this end, the energy spectrum and decay modes of positronium are discussed. An introduction to different models with extra dimensions is given and it is discussed how positronium could disappear due to the existence of extra dimensions. Limits from other processes on this disappearance rate are acquired and compared to experimental sensitivity which is realistic to obtain in the near future. It is concluded that it is very unlikely that this search for extra dimensions will be viable in the near future because there is too large a discrepancy between the limits and the aspired sensitivity.

Contents

Introduction

All our everyday experiences take place in an arena of three space-like dimensions and one time-like dimension. It thus seems obvious to assume nature is limited to exactly these four dimensions. One can, however, wonder why this is so. After all, it is mathematically possible to describe spaces with any amount of dimensions (whether Euclidean, Minkowski or something else); it does not seem our amount of dimensions is a priori preferred. Perhaps there are other worlds with a different amount of dimensions, and mankind developed in one of the few worlds with four dimensions because this amount is most favorable for life. Maybe our four dimensions are only a result of randomness. Another possibility is that the world we live in is not limited to this amount of dimensions at all, but there are *extra dimensions* hidden from our view.

Kaluza and Klein first (the earliest publication dates from 1921) took seriously the possibility of there being a fifth dimension. Klein proposed this fifth dimension is unlike the other four in the sense that it is *compact*, which means it is only very small and folded back onto itself. This compactness ensures that the dimension remains invisible at the macroscopic scales we are usually concerned with. They found that when general relativity was applied in these five dimensions, a new field arises which could account for electrodynamics. Thus, the electromagnetic and gravitational forces would be unified as resultants of the geometry of spacetime. While Kaluza-Klein theory has been somewhat abandoned now, it has more recently been discovered that hidden extra dimensions could account for the hierarchy problem, i.e. the weakness of gravity relative to the other fundamental forces.

In these newer theories, (only) compactness does not do the trick of hiding extra dimensions. Rather, it is proposed that our world is a so-called *brane world*, a hypersurface in a higher-dimensional reality. It has even been shown by Randall and Sundrum that the existence of an infinite, non-compact fifth dimension is not in conflict with our observations. In such models, ordinary matter is constrained to the brane world in one way or another. But would it not be possible for this matter to, in certain circumstances, leave this brane and thus simply disappear from our world?

Indeed, it is (at least in some scenarios) possible for particles to disappear into an extra dimension through the phenomenon of quantum tunneling. Such tunneling rates can be quantified and provide a very interesting possibility of empirically testing models proposing extra dimensions. What would constitute a good candidate for such experiments? First, we observe that there are many

conservation laws known which are experimentally well-established, such as conservation of charge. Thus, we can in a model-independent way guess that those particles which would violate as few conservation laws as possible by disappearing are also the most likely ones to do so. Furthermore, if a particle also has other decay modes we better know a lot about these in order to ever be able to distinguish the disappearance into extra dimensions.

What seems to be the perfect candidate is *positronium*. For good reason, this exotic atom, which is the lightest atom known, has recently been very popular for searching for extra dimensions (for other searches for physics beyond the standard model too, like the search for mirror matter and millicharged particles.) Positronium is composed of an electron and its antiparticle, a positron. It is a very clean system governed solely by quantum electrodynamics and thus allows for very precise theoretical description. Deviation from this description could thus be well detectable. Also, because it is composed of two antiparticles, its quantum numbers closely resemble those of the vacuum, allowing it to disappear while violating a minimum of conservation laws.

Positronium in its ground state comes in two flavors. One is parapositronium, in which the spin angular momenta of the electron and positron are opposite to each other, giving the atom a total spin angular momentum of zero. In orthopositronium, the spins are aligned, giving it a total of one reduced Planck constant. Positronium is unstable as the electron and positron can annihilate, making it decay into predominantly photons. Decay into one photon is prohibited, as this process would violate conservation of energy and momentum. Parapositronium is capable of decaying into any even number of photons, because such a system of photons is even under charge conjugation, just as parapositronium. Orthopositronium, however, is uneven under this operation and can thus only decay into an uneven number of photons. Due to the amount of nodes found in the Feynman diagram of such a decay, and since each node effectively adds a factor of the fine-structure constant to the decay rate, the more photons are involved the longer the lifetime is. Thus, parapositronium decays mostly into two photons and lives much shorter than orthopositronium, which decays mostly into three photons. Since it is easier to detect a deviant decay mode if the system is more stable, it is orthopositronium with which experiments are generally involved.

In this bachelor thesis, I will examine the possibility of empirically testing the existence of extra dimensions using positronium. Particularly, I will discuss whether current or near-future experiments would be capable of either detecting disappearance of positronium into extra dimensions or otherwise imposing conditions on theories involving extra dimensions by detecting no such disappearance. To this end, the first three chapters will provide a detailed analysis of the positronium system (in particular of its energy levels and decay modes). In the last and fourth chapter, an overview of theories with extra dimensions will be provided. By combining theoretical predictions with available observational data, limits on the disappearance of positronium will be set, to the end of answering the question of whether a search for direct detection in the near future could go beyond these limits.

Chapter 1

Structure of positronium

In this chapter, it will be examined what positronium is. In the first section, a general description of the system will be given. In Section 1.2, it will be discussed in the context of classical mechanics how best to describe a two body system such as positronium. Next, in Section 1.3, a quantum-mechanical description will be given by finding the wave function in a way analogous to how it is done for the system of hydrogen. Here, the gross structure of the energy levels will also be obtained, corrections to which will be made in Chapter 2.

1.1 What is positronium?

Positronium (often denoted Ps) is the bound state of two particles, namely an electron and a positron. The electron is an elementary spin-$\frac{1}{2}$ particle, which is called a lepton. Three generations of leptons are known, which are electron, muon and tauon. Each lepton generation contains charged and neutral leptons (which are known as neutrinos). The electron is the lightest of all charged leptons. It carries one negative unit of elementary charge.

The positron is the antiparticle of the electron, which means that it is much like an electron with opposite charge. Thus, the positron carries one positive unit of elementary charge. Furthermore, particles and antiparticles are able to annihilate. In this process, both particle and antiparticle 'disappear' and their energy is converted into new particles (mostly photons in the case of electrons and positrons).

From the opposite charge, it becomes immediately clear how both particles can be bound together: they are pulled towards one another by a simple Coulomb force. Thus, if their energy is low enough, they will not be able to escape each other and must remain in the bound state of positronium, which is called an exotic atom. This system is very much reminiscent of what is probably the most studied system in all of quantum mechanics, namely the hydrogen atom. Hydrogen is made up out of one positively-charged proton and one negatively-charged electron. Although the proton is not an elementary particle itself (rather, it is made up of quarks), it happens to possess the exact same charge as the positron and also has the same amount of spin angular momen-

tum. It is thus not surprising that solving for both systems proceeds on similar grounds.

There are, however, two major differences between the systems. A first is the mass. While positron and electron have the same exact mass and thus form a quite symmetrical system, the proton mass is much larger than the electron's (by a factor of about 1836). The result is that certain effects in positronium will be much more important than in hydrogen, such as the spin-spin coupling discussed in Section 2.4. Secondly, the electron and positron are each other's antiparticle and can thus annihilate. Because of this, the positron is very short-lived (lifetimes are below 10^{-6} seconds), while the hydrogen atom is stable. The decay of positronium will be discussed in detail in Chapter 3.

1.2 Central potential

In this section, I will discuss how best to describe the positronium system. In order to do this, quantum mechanics will be put aside for a moment, and classical mechanics and electrodynamics will be considered to find the degrees of freedom with which best to express a two body system like positronium. Furthermore, a Hamiltonian will be determined as a function of these degrees of freedom. Then, in Section 1.3, quantization of positronium using this description will be considered.

First, consider the potential that binds two charged particles. This is a simple Coulomb potential, which is given by

$$U(r) = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r}, \tag{1.1}$$

where r is the separation between the two charged particles, q_1 is the charge of the first particle and q_2 is the charge of the second particle.

To find a complete Hamiltonian (from which further dynamics can be determined), the kinetic energy must be determined. First define $\mathbf{r}_1$ to be the position vector of the first particle and $\mathbf{r}_2$ to be the position vector of the second particle. The kinetic energy is then given by

$$T = \frac{1}{2}m_1|\frac{d\mathbf{r}_1}{dt}|^2 + \frac{1}{2}m_2|\frac{d\mathbf{r}_2}{dt}|^2 \equiv \frac{1}{2}m_1|\dot{\mathbf{r}}_1|^2 + \frac{1}{2}m_2|\dot{\mathbf{r}}_2|^2. \tag{1.2}$$

The Hamiltonian is now given by $H = T + U$. The problem can be further simplified by considering the center-of-mass frame of the system. The position vector of the center of mass is defined by $\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1+m_2}$. It can then be easily checked that one can rewrite the kinetic energy as

$$\begin{aligned} T &= \frac{1}{2}(m_1+m_2)|\dot{\mathbf{R}}|^2 + \frac{1}{2}m_1|\dot{\mathbf{r}}_1 - \dot{\mathbf{R}}|^2 + \frac{1}{2}m_2|\dot{\mathbf{r}}_2 - \dot{\mathbf{R}}|^2 \\ &\equiv T_{CM} + T_{int}, \end{aligned} \tag{1.3}$$

which means that the kinetic energy splits up in a term corresponding to the velocity of the center of mass and a term corresponding to velocity relative to

the center of mass (internal energy). This then allows us to easily describe the system in the center-of-mass frame. This frame corresponds to $\mathbf{R} = 0$ and is illustrated in Figure 1.1. If we then, however, want to switch to another laboratory frame, we can simply add the energy corresponding to the center of mass.

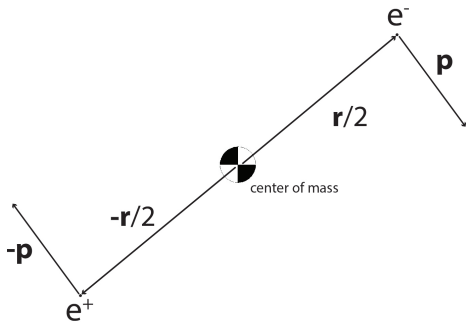


Figure 1.1: The electron and positron in the center-of-mass frame. It can be clearly seen that the system can be described by $\mathbf{r}$ and $\mathbf{p}$ only, thus allowing for a description in terms of a single fictitious particle.

Setting $\mathbf{R} = 0$ gives us the familiar Hamiltonian

$$H = T + U = \frac{1}{2}m_1|\dot{\mathbf{r}}_1|^2 + \frac{1}{2}m_2|\dot{\mathbf{r}}_2|^2 + U(r), \tag{1.4}$$

but now with the extra condition that $m_1\mathbf{r}_1 + m_2\mathbf{r}_2 = 0$. This allows for a very lucrative description of the system. Let $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$. Then we can rewrite the kinetic energy in terms of $\mathbf{r}$ only (using $\mathbf{r}_1 = \frac{m_2}{m_1+m_2}\mathbf{r}$ and $\mathbf{r}_2 = -\frac{m_1}{m_1+m_2}\mathbf{r}$), giving $T = \frac{1}{2}\frac{m_1m_2}{m_1+m_2}|\dot{\mathbf{r}}|^2$. The Hamiltonian then becomes

$$H = \frac{1}{2}\mu|\dot{\mathbf{r}}|^2 + U(r), \tag{1.5}$$

where μ is the *reduced mass* of the system, given by $\mu \equiv \frac{m_1m_2}{m_1+m_2}$. Furthermore, $r = |\mathbf{r}|$, so we have obtained a Hamiltonian which only depends on one coordinate vector. Thus, when the center of mass has a known position and velocity (and is not subjected to external forces) the problem reduces effectively to a *one-body problem*, as we can always just consider the problem from the center-of-mass frame and later on easily translate the problem to the laboratory frame.

But what is this 'one body'? It is neither of the two interacting particles. Rather, it is a fictitious particle with position vector $\mathbf{r} = \mathbf{r}_1/2 = -\mathbf{r}_2/2$ and momentum $\mathbf{p} = \mu\dot{\mathbf{r}} = \mathbf{p}_1 = -\mathbf{p}_2$, being pulled to the origin by the potential. This case is known as a *central force problem*, which is well known.

So what now are the Hamiltonians we obtain for hydrogen and positronium? First of all, both contain the same potential energy term, as the charges involved are the same. Let e denote the elementary charge. Then in each case we find

$$U(r) = \frac{1}{4\pi\epsilon_0}\frac{-e^2}{r}. \tag{1.6}$$

The kinetic term depends on the reduced mass of the systems. For hydrogen, we have $\mu = \frac{m_e m_p}{m_e+m_p} = \frac{m_e}{1+\frac{m_e}{m_p}}$. As mentioned before, $\frac{m_e}{m_p} \simeq \frac{1}{1836}$ which is a quite small number. Thus, $\mu \simeq m_e$ for hydrogen. This means that the reduced one-body problem for hydrogen is approximately equal to the original problem when neglecting the kinetic energy of the proton. That is, one can obtain a good approximation of how hydrogen works simply by assuming that the proton is motionless, focussing only on the electron. This is exactly how the hydrogen atom is usually treated. More accurate results for the energy levels, however, can be found by replacing the electron mass by the reduced mass of the system.

For positronium, things are different. Here, the approximation of one of the two particles being motionless would be totally nonsensical as both are equally heavy. Instead, a reduced mass is found of $\mu = \frac{m_e^2}{2m_e} = \frac{1}{2}m_e$. The one-body problem thus really concerns a fictitious particle. However, this particle can be solved for similarly to the electron in the hydrogen atom (as long as no perturbations to this Hamiltonian are considered), only with a different mass.

1.3 Wave functions

To find the quantum-mechanical wave function and energy levels of a two body system as described above, the (time-independent) Schrödinger equation,

$$i\hbar\frac{\partial\Psi}{\partial t} = H\Psi, \tag{1.7}$$

must be solved for the Hamiltonian

$$H = \frac{p^2}{2\mu} + U(r), \tag{1.8}$$

where $p = |\mathbf{p}| = \mu|\dot{\mathbf{r}}|$. This momentum can be written in operator form, by writing for each component $p_j \to \frac{\hbar}{i}\frac{\partial}{\partial j}$ (with $j = x, y, z$) [11, page 17], as

$$\mathbf{p} \to \frac{\hbar}{i}\nabla. \tag{1.9}$$

We then get a time derivative of Ψ which must always be equal to a spatial derivative of Ψ. This must mean that both are constant! When we label the functions by this separation constant and call it E_n (which will obtain the interpretation of the energy), we find the solutions must be of the form

$$\Psi_n(\mathbf{r}, t) = \psi_n(\mathbf{r})e^{-iE_n t/\hbar}, \tag{1.10}$$

where the solutions ψ_n must satisfy the time-independent Schrödinger equation,

$$-\frac{\hbar^2}{2\mu}\nabla^2\psi + U\psi = E\psi. \tag{1.11}$$

Solving this equation is a bit tedious and has been done already in a lot of literature (e.g. [11, chapter 4]), and thus I will only present here solutions to the problem. The usual approach is to apply separation of variables and write

$$\psi(r, \theta, \phi) = R(r)Y(\theta, \phi). \tag{1.12}$$

The solutions to the angular part are then given by the (in)famous *spherical harmonics*

$$Y_\ell^m(\theta,\phi) = \epsilon\sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-|m|)!}{(\ell+|m|)!}}e^{im\phi}P_\ell^m(\cos\theta), \tag{1.13}$$

where P_ℓ^m is the associated Legendre function

$$P_\ell^m(x) \equiv (1-x^2)^{|m|/2}\left(\frac{d}{dx}\right)^{|m|}P_\ell(x), \tag{1.14}$$

and P_ℓ is the Legendre polynomial, defined by Rodrigues formula

$$P_\ell(x) \equiv \frac{1}{2^\ell!}\left(\frac{d}{dx}\right)^\ell(x^2-1)^\ell. \tag{1.15}$$

Furthermore, $\epsilon = (-1)^m$ for $m \geq 0$ and $\epsilon = 1$ for $m \leq 0$. The full wave function is given by

$$\psi_{n\ell m} = \sqrt{\left(\frac{2}{na}\right)^3\frac{(n-l-1)!}{2n[(n_\ell)!]^3}}e^{-r/na}\left(\frac{2r}{na}\right)^\ell\left[L_{n-l-1}^{2\ell+1}(2r/na)\right]Y_\ell^m(\theta,\phi), \tag{1.16}$$

where the part before the spherical harmonic constitutes the radial part. Here, L is the associated Laquerre polynomial defined by

$$L_{q-p}^p(x) \equiv (-1)^p\left(\frac{d}{dx}\right)^p L_q(x), \tag{1.17}$$

with L_q the qth Laguerre polynomial

$$L_q(x) \equiv e^x\left(\frac{d}{dx}\right)^q(e^{-x}x^q). \tag{1.18}$$

More interesting is a, which is the *Bohr radius* defined by

$$a \equiv \frac{4\pi\epsilon_0\hbar^2}{\mu e^2} = \frac{\hbar}{\mu c\alpha} \tag{1.19}$$

where α is the *fine structure constant*

$$\alpha \equiv \frac{1}{\hbar c}\frac{e^2}{4\pi\epsilon_0} \simeq \frac{1}{137}. \tag{1.20}$$

What, then, are the labels n, ℓ and m? We already encountered n as a label for the energy. It is called the *principal quantum number* and it indicates of which energy the wave function is an eigenstate. It can only take positive (nonzero) integer values. The energies can be found to yield

$$E_n = -\frac{1}{2n^2}\mu c^2\alpha^2 \equiv -\frac{1}{2n^2}\text{Ry}, \tag{1.21}$$

where Ry is the Rydberg unit of energy,

$$1\text{Ry} \simeq 13.6\text{ eV}. \tag{1.22}$$

This predicts the energy spectrum of hydrogen (and to smaller degree of positronium) to good approximation, although we will find many corrections can be made in chapter 2. The number ℓ is called the *azimuthal quantum number* and is related to the angular momentum of the system (more accurately: of the fictitious particle in the reduced problem, $\mathbf{L} \equiv \mathbf{r} \times \mathbf{p}$). It is restricted to positive (including zero) integer values smaller than n, so $\ell = 0, 1, 2, ..., n-1$. The states are angular momentum eigenstates with eigenvalues

$$\mathbf{L}^2 |\psi_{n\ell m}\rangle = \hbar^2 \ell(\ell+1) |\psi_{n\ell m}\rangle . \tag{1.23}$$

The last quantum number, m, is called the *magnetic quantum number* and indicates to some extent the direction of the angular momentum. It can take any integer value with magnitude smaller or equal to ℓ, so $m = \ell, \ell-1, ... -\ell+1, -\ell$. If we take the quantization axis to be the z-axis, we find that the wave functions are eigenstates for the projection of the angular momentum on this axis with eigenvalues

$$L_z |\psi_{n\ell m}\rangle = \hbar m |\psi_{n\ell m}\rangle . \tag{1.24}$$

Thus, our general solution is

$$|\Psi(\mathbf{r}, t)\rangle = \sum_{n,\ell,m} c_{n,\ell,m} |\psi_{n\ell m}(\mathbf{r})\rangle e^{-iE_n t/\hbar}, \tag{1.25}$$

with normalization condition $\sum_{n,\ell,m} c_{n,\ell,m} = 1$.

However, we must not forget that our system has *two* degrees of freedom. After all, if we would describe the system as two separate particles, we would need a product of two wave functions. The wave function now found describes the virtual particle, which is a complete description of the system only when combined with the wave function of the center of mass, $\Psi(\mathbf{R}, \mathbf{t})$. The wave function of the system then becomes

$$|\Psi(\mathbf{R}, \mathbf{r}, t)\rangle = |\Psi(\mathbf{R}, t)\rangle |\Psi(\mathbf{r}, t)\rangle |S_1(t)\rangle |S_2(t)\rangle , \tag{1.26}$$

where I have also included the spin wave functions of the two particles ($|S_i\rangle$ is the spin wave function of particle i). What is the form of $\Psi(\mathbf{R}, t)$? This is simply the wave function of a particle with mass $M = m_1 + m_2$. If there are no internal forces acting on the system, this particle will be free, and thus the wave function will be of the form

$$\Psi(\mathbf{R}, t) = \frac{1}{(\sqrt{2\pi}\hbar)^3} \int f(\mathbf{p}) e^{i(\mathbf{p}\cdot\mathbf{R} - Et)/\hbar} d^3p. \tag{1.27}$$

Chapter 2

Energy corrections

2.1 Perturbation theory

First-order perturbation theory helps us tackling a problem where the Hamiltonian takes the form $H = H^0 + H'$, where H^0 is a Hamiltonian for which we can solve the wave functions and energies analytically and H' is a small perturbation due to which H cannot (easily) be solved for. The most important result (for current purposes) is a method to give a first-order correction to the energy of a wave function due to the perturbation: this is simply the expectation value of the perturbation,

$$E_n^1 = \langle \psi_n^0 | \, H' \, | \psi_n^0 \rangle \, . \tag{2.1}$$

Here, E_n^1 is the first-order correction to the energy of eigenstate n of the original Hamiltonian H^0. Since only first-order corrections will be treated in this thesis, this correction will henceforth simply be written

$$E' = \langle H' \rangle \, , \tag{2.2}$$

where the subscript of E' will be reserved to label what perturbation the correction corresponds to. The corrected energy of the original energy eigenstate $|\psi_n^0\rangle$ can be obtained from $E = E^0 + E'$.

2.2 Relativistic correction

A first correction to the energy of both hydrogen and positronium can be found by not assuming a classical energy-momentum relation but considering the relativistic energy of the constituents. In order to do this, we need to reconsider the steps taken in Section 1.2. Here, a Hamiltonian corresponding to an easier problem, of a particle with reduced mass μ in a central potential, was obtained. However, in order to get here, I made use of the kinetic energy relation $T_i = \frac{1}{2} m_i |\dot{\mathbf{r}}_i|^2 = \frac{\mathbf{p}_i^2}{2m_i}$ for particle i (where $\mathbf{p}_i = m_i \dot{\mathbf{r}}_i$ is the momentum). In the relativistic case, however, the total energy (not regarding potential energy) is given by $E_i = \sqrt{\mathbf{p}_i^2 c^2 + m_i^2 c^4}$, where c is the velocity of light in vacuum. The rest mass energy of a particle is given by $m_i c^2$, giving us the kinetic energy

relation

$$T_i = \sqrt{\mathbf{p}_i^2 c^2 + m_i^2 c^4} - m_i c^2 = m_i c^2 \left[\sqrt{1 + (\frac{p_i}{mc})^2} - 1 \right], \tag{2.3}$$

with $p_i = |\mathbf{p}_i|$.

This is very hard to solve. So we take the physicist's ordinary approach, and take a Taylor series so we can only take into account lower-order terms, which are the terms that matter most. When we take a Taylor series of the function $f(x) = \sqrt{1-x}$, we find

$$f(x) = f(0) + f'(0) \times x + \frac{1}{2} f''(0) \times x^2 + \mathcal{O}(3) = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \mathcal{O}(3), \tag{2.4}$$

where $\mathcal{O}(3)$ represents terms of order 3 or higher in x. When we apply this to Eq. (2.3) (replacing x with $(\frac{p_i}{m_i c})^2$) we obtain

$$T_i = \frac{p_i^2}{2m_i} - \frac{p_i^4}{8m_i^3 c^3} + \mathcal{O}(6), \tag{2.5}$$

where $\mathcal{O}(6)$ represents terms of order six and higher in $\frac{p_i}{m_i c}$. We find a first term which is equal to the kinetic energy used in the classical case. This is as we would expect, as this should be obtained in the classical limit $p_i \to 0$.

Now, however, we approximate the relativistic energy a bit better by keeping one extra term and neglecting all $\mathcal{O}(6)$ terms, which must be relatively small as $\frac{p_i}{m_i c} = \frac{v_i}{c} < 1$. We can then write our two-body system's total kinetic energy as

$$T = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} - \frac{p_1^4}{8m_1^3 c^3} - \frac{p_2^4}{8m_2^3 c^3} = T^0 + T'. \tag{2.6}$$

T^0 is simply the kinetic energy as we had it in Section 1.2, and we already found we can rewrite this as $T_0 = \frac{1}{2}\frac{p^2}{2\mu}$ in the center-of-mass frame with $\mathbf{r} = \mathbf{r_1} - \mathbf{r_2}$, $\mu = \frac{m_1 m_2}{m_1 + m_2}$ and $p = \mu|\dot{\mathbf{r}}|$. As our Hamiltonian is $H = T + U = T^0 + U + T'$ and the potential remains unaltered, we find our perturbation to be

$$H'_{rel} = -\frac{p_1^4}{8m_1^3 c^3} - \frac{p_2^4}{8m_2^3 c^3}. \tag{2.7}$$

Can we also rewrite this in terms of $\mathbf{r}$ so we get some sort of reduced problem again? As we are working in the center-of-mass frame, we have $m_1\mathbf{r}_1 + m_2\mathbf{r}_2 = 0$, which gives $\mathbf{p} = \mathbf{p}_1 = -\mathbf{p}_2$. Thus, $p_1 = p_2 = p$. This allows us to write

$$H'_{rel} = -\frac{1}{8c^3}\frac{(m_1^3 + m_2^3)p^4}{m_1^3 m_2^3} = -\frac{1}{8c^3}p^4 / \frac{m_1^3 m_2^3}{m_1^3 + m_2^3}, \tag{2.8}$$

where we see we divide by a mass factor reminiscent of the reduced mass. In the case of hydrogen, $m_1 = m_e$ and $m_2 = m_p$ with $m_p \gg m_e$ and the mass factor becomes $m_e^3/(1 + m_e^3/m_p^3) \simeq m_e^3$. As for hydrogen $\mu = m_e$, we simply get a term with p^4/μ^3. However, we do not obtain the same result for positronium with simply the different reduced mass. When we fill in $m_1 = m_2 = m_e$, we find

the mass term to be $\frac{1}{2}m_e^3 = 4\mu^3$. Thus, this differs form the hydrogen case by a factor 4 in terms of reduced masses. The perturbations to the Hamiltonians then are

$$H'_{rel} = \begin{cases} \frac{p^4}{8\mu^3c^3} = \frac{p^4}{8m_e^3c^3} & \text{for hydrogen,} \\ \frac{p^4}{32\mu^3c^3} = \frac{p^4}{4m_e^3c^3} & \text{for positronium.} \end{cases} \tag{2.9}$$

Now we want to apply perturbation theory to find the energy E'_{rel}. To this end we must take the expectation value of p^4. $\langle p^4 \rangle = \langle p^2\psi^0 | p^2\psi^0 \rangle$. Since these states are eigenstates of H^0, they are also eigenstates of p^2 with eigenvalues $E - U$. Thus, $\langle p^4 \rangle = \langle (E-U)^2 \rangle = (E^2 - 2E\langle U \rangle + \langle U^2 \rangle)$. For the potential we can write $U = \mu c^2 \alpha^2 a/r$ where α and a are the fine structure constant (Eq. (1.20)) and the Bohr radius (Eq. (1.19)) respectively. It can be shown that $\langle \frac{1}{r} \rangle = \frac{1}{n^2 a}$ and $\langle \frac{1}{r^2} \rangle = \frac{1}{(\ell + \frac{1}{2})n^3 a^2}$ [11, Eqs. (55) and (56)]. When furthermore filling in $E = E_n = -\frac{1}{2}\mu c^2 \alpha^2 \frac{1}{n^2}$, this gives

$$E'_{rel} = -\left(\frac{4n}{\ell + \frac{1}{2}} - 3 \right) \times \begin{cases} \frac{(E_n)^2}{2\mu c^2} = \frac{m_e c^2 \alpha^4}{8n^4} & \text{for hydrogen,} \\ \frac{(E_n)^2}{8\mu c^2} = \frac{m_e c^2 \alpha^4}{64n^4} & \text{for positronium.} \end{cases} \tag{2.10}$$

2.3 Spin-orbit interaction

A next energy correction is due to *spin-orbit coupling*. Here, the spin angular momentum of one particle couples to the orbital angular momentum of the other through magnetic interaction. The energy particle 2 acquires due to the magnetic field generated by the orbit of particle 1 is

$$H = -\boldsymbol{\mu}_2 \cdot \mathbf{B}_1, \tag{2.11}$$

where $\boldsymbol{\mu}_i$ is the magnetic moment of particle i and $\mathbf{B}_i$ is the magnetic field generated by the orbit of particle i. Let's consider the problem from the rest frame of particle 2. In this frame, the position of particle 1 is simply given by $\mathbf{r} = \mathbf{r_1} - \mathbf{r_2}$ (since $\mathbf{r_2} = 0$). The magnitude of the magnetic field is given by the Biot-Savart law, $|\mathbf{B}_1| = \frac{\mu_0 I}{2r}$. Here, μ_0 is the magnetic permeability of free space and is I the magnitude of the effective current generated by particle 1. If each particle has charge $q = \pm e$, then this current is simply $\frac{e}{T}$ where T is the orbit period. If we assume the orbit is circular, we can write $T = \frac{2\pi r}{|\dot{\mathbf{r}}|}$.

Combining this now gives $|\mathbf{B}_1| = \frac{\mu_0 e |\dot{\mathbf{r}}|}{4\pi r^2}$. Next, we notice that $|\mathbf{L}| = |\mathbf{r} \times \mathbf{p}| = \mu r |\dot{\mathbf{r}}|$ for circular orbits. Furthermore, we can infer from the right-hand rule for magnetic fields that the field must be in the same direction as $\mathbf{L}_1$ if the charge is positive and in the opposite direction if the charge is negative. $\mathbf{L}_1$ in turn points in the same direction as $\mathbf{L}$ since $\mathbf{L}_1 = \mathbf{r} \times (m_1 \dot{\mathbf{r}}) = \frac{m_1}{\mu}\mathbf{L}$. Putting this all together allows us to write

$$\mathbf{B}_1 = \frac{q_1}{4\pi\epsilon_0} \frac{1}{\mu c^2 r^3} \mathbf{L}, \tag{2.12}$$

where the identity $c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$, with ϵ_0 the electric permittivity of free space, was used to eliminate μ_0.

The magnetic dipole moment of a classical particle is given by $\mu = (\frac{q}{2m})\mathbf{S}$. However, in reality particles deviate from this value through multiplication by the appropriate g-factor (a result provided by quantum electrodynamics). For electrons and positrons the g-factor is the Landé g-factor and is approximately 2. For a proton, it is about 5.59. Thus,

$$\boldsymbol{\mu}_i = \frac{g_i q_i}{2m_i}\mathbf{S_i}, \tag{2.13}$$

which gives Hamiltonian (using $q_1 q_2 = -e^2$)

$$H = \frac{e^2}{4\pi\epsilon_0}\frac{g_i}{2m_i\mu c^2 r^3}\mathbf{S}_1 \cdot \mathbf{L} = \frac{\mu}{m_1}\frac{g_i}{2}\mu c^2\alpha^4\left(\frac{a}{r}\right)^3\frac{\mathbf{L}\cdot\mathbf{S_1}}{\hbar^2}. \tag{2.14}$$

However, I have been using the rest frame of one of the particles in the system, which is certainly not an inertial frame! There is a price which must be paid for this, which comes in the form of *Thomas precession*. In an inertial reference frame, the rest frame of an accelerating particle rotates with angular frequency

$$\boldsymbol{\omega}_T = \frac{1}{2c^2}\mathbf{a}\times\mathbf{v}, \tag{2.15}$$

where $\mathbf{a}$ is the acceleration of the particle and $\mathbf{v}$ is its velocity. I will now calculate the correction that must be made for the energy found for the magnetic dipole of particle 2 in its own rest frame, which is as stated in Eq. (2.14). The correction is then given by

$$V_T = \boldsymbol{\omega}_T \cdot \mathbf{S_2}. \tag{2.16}$$

Consider the inertial center-of-mass frame. Here, $\mathbf{r}_2 = -\frac{m_1}{m_1+m_2}\mathbf{r}$. Thus, the Thomas precession is given by

$$\boldsymbol{\omega}_T = \frac{1}{2c^2}\left(\frac{m_1}{m_1+m_2}\right)^2\ddot{\mathbf{r}}\times\dot{\mathbf{r}}. \tag{2.17}$$

$\ddot{\mathbf{r}}$ is the acceleration of the fictitious particle in the reduced problem described in Section 1.2. This can be easily found from the force generated by the Coulomb potential divided by the reduced mass, $\ddot{\mathbf{r}} = -\frac{1}{\mu}\frac{dU(r)}{dr}\hat{r} = -\frac{1}{\mu}\frac{e^2}{4\pi\epsilon_0}\frac{\hat{r}}{r^2} = -\frac{1}{\mu}\frac{e^2}{4\pi\epsilon_0}\frac{1}{r^2}\mathbf{r}$. Then, since $\mathbf{L} = \mu(\mathbf{r}\times\dot{\mathbf{r}})$,

$$\boldsymbol{\omega}_T = -\frac{1}{2c^2}\frac{e^2}{4\pi\epsilon_0}\frac{1}{r^3}\mathbf{L}\left(\frac{m_1}{(m_1+m_2)\mu}\right)^2 = -\frac{1}{2}\left(\frac{\mu}{m_2}\right)^2\mu c^2\alpha^4\left(\frac{a}{r}\right)^3\frac{\mathbf{L}}{\hbar^2}, \tag{2.18}$$

which quickly gives the energy correction due to the Thomas precession

$$V_T = -\frac{1}{2}\left(\frac{\mu}{m_2}\right)^2\mu c^2\alpha^4\left(\frac{a}{r}\right)^3\frac{\mathbf{L}\cdot\mathbf{S}_2}{\hbar^2}. \tag{2.19}$$

Now we are almost there. To find the correction to the Hamiltonian due to the spin-orbit coupling, we need to add all contributions together. This thus involves both particles. If the above steps would be taken to find the energy of particle 2 in the rest frame of particle 1, and next to find the Thomas precession correction, the only difference would be that we would have to consider $\mathbf{r_2} = -\mathbf{r}$. But since the results are only dependent on $r = |\mathbf{r}_1| = |\mathbf{r}_2|$ and the magnitudes

of the derivatives of $\mathbf{r}$, this would make no difference at all (as it should be after all, as we also could simply switch the labels of the particles). It is thus that we find that

$$\begin{aligned} H'_{SO} &= \frac{\mu}{m_1}\frac{g_1}{2}\mu c^2\alpha^4\left(\frac{a}{r}\right)^3\frac{\mathbf{L}\cdot\mathbf{S_1}}{\hbar^2} - \frac{1}{2}\left(\frac{\mu}{m_2}\right)^2\mu c^2\alpha^4\left(\frac{a}{r}\right)^3\frac{\mathbf{L}\cdot\mathbf{S}_2}{\hbar^2} \\ &\quad + \frac{\mu}{m_2}\frac{g_2}{2}\mu c^2\alpha^4\left(\frac{a}{r}\right)^3\frac{\mathbf{L}\cdot\mathbf{S_2}}{\hbar^2} - \frac{1}{2}\left(\frac{\mu}{m_1}\right)^2\mu c^2\alpha^4\left(\frac{a}{r}\right)^3\frac{\mathbf{L}\cdot\mathbf{S}_1}{\hbar^2} \\ &= \left(\frac{\mu}{m_1}\frac{g_1}{2} - \frac{1}{2}\left(\frac{\mu}{m_1}\right)^2\right)\mu c^2\alpha^4\left(\frac{a}{r}\right)^3\frac{\mathbf{S_1}\cdot\mathbf{L}}{\hbar^2} \\ &\quad + \left(\frac{\mu}{m_2}\frac{g_2}{2} - \frac{1}{2}\left(\frac{\mu}{m_2}\right)^2\right)\mu c^2\alpha^4\left(\frac{a}{r}\right)^3\frac{\mathbf{S_2}\cdot\mathbf{L}}{\hbar^2}. \end{aligned} \tag{2.20}$$

For hydrogen, $\mu = m_1 = m_e$ and $m_2 = m_p$. Making again the approximation $m_p \gg m_e$ and neglect the $\frac{m_e}{m_p}$ terms, we only keep the first half of our Hamiltonian and get (filling in $g_e = 2$)

$$H'_{SO} = \frac{1}{2}\mu c^2\alpha^4\left(\frac{a}{r}\right)^3\frac{\mathbf{S}_{e^-}\cdot\mathbf{L}}{\hbar^2} = \frac{1}{2}m_e c^2\alpha^4\left(\frac{a}{r}\right)^3\frac{\mathbf{S}_{e^-}\cdot\mathbf{L}}{\hbar^2} \qquad \text{for hydrogen.} \tag{2.21}$$

For positronium, we can add the terms together since $m_1 = m_2 = m_e$ and $g_1 = g_2 = g_e = 2$. Thus, we find

$$\begin{aligned} H'_{SO} &= \left(\frac{\mu}{m_e} - \frac{1}{2}\left(\frac{\mu}{m_e}\right)^2\right)\mu c^2\alpha^4\left(\frac{a}{r}\right)^3\frac{(\mathbf{S_{e^-}} + \mathbf{S_{e^+}})\cdot\mathbf{L}}{\hbar^2} \\ &= \frac{3}{8}\mu c^2\alpha^4\left(\frac{a}{r}\right)^3\frac{(\mathbf{S_{e^-}} + \mathbf{S_{e^+}})\cdot\mathbf{L}}{\hbar^2} \\ &= \frac{3}{16}m_e c^2\alpha^4\left(\frac{a}{r}\right)^3\frac{(\mathbf{S_{e^-}} + \mathbf{S_{e^+}})\cdot\mathbf{L}}{\hbar^2} \qquad \text{for positronium.} \end{aligned} \tag{2.22}$$

Now, calculating the first order energy correction corresponding to this perturbation is relatively straightforward. When we take the expectation value of H'_{SO}, we find the product of a bunch of constants and the terms $\langle\frac{1}{r^3}\rangle$ and $\langle\mathbf{S}\cdot\mathbf{L}\rangle$, where $\mathbf{S} = \mathbf{S}_{e^-}$ for hydrogen and $\mathbf{S} = \mathbf{S_{e^-}} + \mathbf{S_{e^+}}$ for positronium. The expectation values can be untangled like this because the wave function factorizes into a radial part, which contains all the r dependence, and the rest, which determines the orbital angular momentum through the spherical harmonics and contains the spin wave functions. The first term can be found to yield

$$\langle\frac{1}{r^3}\rangle = \frac{1}{\ell(\ell+1/2)(\ell+1)n^3a^3}, \tag{2.23}$$

which I will not derive here, but can be found in literature elsewhere, e.g. [11, page 276].

Because the Hamiltonian now contains a term which couples the spin and orbital angular momentum, they are individually no longer conserved and do not give a 'good' quantum number. Their sum, $\mathbf{J} = \mathbf{L} + \mathbf{S}$, however, is. Since $\mathbf{J}^2 = \mathbf{L}^2 + \mathbf{S}^2 + 2\mathbf{L}\cdot\mathbf{S}$, we can write

$$\langle\mathbf{S}\cdot\mathbf{L}\rangle = \frac{1}{2}(\langle\mathbf{J}^2\rangle - \langle\mathbf{L}^2\rangle - \langle\mathbf{S}^2\rangle) = \frac{1}{2}\hbar^2(j(j+1) - \ell(\ell+1) - s(s+1)). \tag{2.24}$$

Let us first consider the case for hydrogen. Here, the spin angular momentum is only the angular momentum of the electron, and thus $s = \frac{1}{2}$. Since $j = (\ell + s), (\ell + s - 1), ..., |\ell - s|$, this simply gives $j = \ell \pm \frac{1}{2}$. Combining gives

$$\begin{aligned} E'_{SO} =& \frac{1}{4n^3} m_e c^2 \alpha^4 \frac{(\ell \pm 1/2)(\ell \pm 1/2 + 1) - \ell(\ell+1) - 3/4}{\ell(\ell + 1/2)(\ell + 1)} \\ =& \frac{1}{4n^3} m_e c^2 \alpha^4 \frac{\pm(\ell + 1/2) - 1/2}{\ell(\ell + 1/2)(\ell + 1)} \qquad \text{for hydrogen.} \end{aligned} \tag{2.25}$$

This can be combined with the relativistic corrections to give the *fine structure*, $E'_{fs} = E'_{rel} + E'_{SO}$. Filling in our results gives

$$\begin{aligned} E'_{fs} =& \frac{1}{4n^4} m_e c^2 \alpha^4 \left(n \frac{\pm(\ell + 1/2) - 1/2 - 2\ell(\ell+1)}{\ell(\ell + 1/2)(\ell + 1)} - \frac{3}{2} \right) \\ =& \frac{1}{2n^4} m_e c^2 \alpha^4 \times \begin{cases} (-\frac{n}{\ell+1} + \frac{3}{4}) & \text{for } j = \ell + 1/2 \\ (-\frac{n}{\ell} + \frac{3}{4}) & \text{for } j = \ell - 1/2 \end{cases} \\ =& \frac{1}{2n^4} m_e c^2 \alpha^4 \left(-\frac{n}{j + 1/2} + \frac{3}{4} \right) \qquad \text{for hydrogen.} \end{aligned} \tag{2.26}$$

For positronium, the spin angular momentum makes things slightly more complicated because we need to consider both electron and positron. Both have spin-$\frac{1}{2}$, and thus compose into $\mathbf{S}$ such that $s = 1$, which is orthopositronium, or $s = 0$, which is parapositronium. Then, for orthopositronium, $j = \ell + 1$, ℓ, $\ell - 1$ and for parapositronium $j = \ell$. It is easily verified that $E'_{SO} = 0$ for the para-state. For the ortho-state, the three different configurations can be worked out quite easily which yields the following energies:

$$E'_{SO} = \frac{3}{16n^3} m_e c^2 \alpha^4 \times \begin{cases} \frac{1}{(\ell+1/2)(\ell+1)} & \text{for ortho, } j = \ell + 1, \\ 0 & \text{for para } (j = \ell), \\ \frac{-1}{\ell(\ell+1/2)(\ell+1)} & \text{for ortho, } j = \ell, \\ \frac{-1}{\ell(\ell+1/2)} & \text{for ortho, } j = \ell - 1. \end{cases} \tag{2.27}$$

Just as with hydrogen, the fine structure can be defined as $E'_{fs} = E'_{rel} + E'_{SO}$. However, while with hydrogen combining the two terms allows to write the fine structure energy in the very nice form of one formula depending on j, such simplification cannot be made for positronium and thus the addition will not be performed explicitly here.

2.4 Spin-spin interaction

The spin angular momenta of the particles in the system do not only couple to the orbital angular momenta of their counterpart, they also couple to each other. This is known as *spin-spin coupling*, and the effect on the energy is often called the *hyperfine structure*. We will proceed in a way similar to when calculating the spin-orbit corrections. Each particle has a magnetic dipole moment $\mu_i = \frac{g_i q}{2m_i}$. Let us consider the problem from the rest frame of particle 2. In this frame, the position of particle 1 is given by $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, and it can be found by classical

electrodynamics [11, page 285] that the magnetic field generated by particle 2 in this frame at the position of particle 1 is given by

$$\mathbf{B}_2 = \frac{\mu_0}{4\pi r^3}[3(\boldsymbol{\mu}_2 \times \hat{r})\hat{r} - \boldsymbol{\mu}_2] + \frac{2\mu_0}{3}\boldsymbol{\mu}_2\delta^3(\mathbf{r}). \tag{2.28}$$

The energy particle 1 acquires due to this field is then simply

$$H'_{SS} = -\boldsymbol{\mu}_1 \cdot \mathbf{B}_2 = \frac{\mu_0 g_1 g_2 e^2}{4m_1 m_2}\left(\frac{1}{4\pi}\frac{\hbar^2}{4}\frac{S_{12}}{r^3} + \frac{2}{3}\mathbf{S}_1 \cdot \mathbf{S}_2\delta^3(\mathbf{r})\right), \tag{2.29}$$

where the tensor operator S_{12} is defined by

$$S_{12} \equiv 3(\boldsymbol{\sigma}_1 \cdot \hat{r})(\boldsymbol{\sigma}_2 \cdot \hat{r}) - \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2. \tag{2.30}$$

The $\boldsymbol{\sigma}$'s are Pauli vectors as defined in Appendix A. These operators are related to the spin operators by $\mathbf{S}_i = \frac{\hbar}{2}\boldsymbol{\sigma}_i$. Thus,

$$S_{12} = \frac{4}{\hbar^2}\left(3(\mathbf{S}_1 \cdot \hat{r})(\mathbf{S}_2 \cdot \hat{r}) - \mathbf{S}_1 \cdot \mathbf{S}_2\right). \tag{2.31}$$

This allows us to calculate the energy obtained due to the spin-spin interaction from $E'_{SS} = \langle H'_{SS}\rangle$. Note that the energy of the system thus followed from asymmetrical considerations, where only the energy of one particle was calculated due to the magnetic field of the other. Should we not also take into account the opposite effect, and thus find an energy twice as large? The answer is no. This can be seen by noting that the above calculations can also be used to yield the force which particle 1 experiences due to particle 2 (we picked the particle 2 rest frame because here the effect of dipole 2 is easily calculated as a pure magnetic field; would the particle also move, an electric dipole would be picked up, which would complicate things).

Newton's third law then guarantees us that particle 1 experiences exactly the same force in the direction of particle 2, but this does not matter for the energy the system as a whole has. We can imagine calculating the energy by either keeping particle 2 fixed and integrate the force particle 1 experiences as it falls inwards or allowing both to move, which means that both particles cover half the distance considered in the other case, but the total force which is integrated is the double, yielding the same result.

Taking expectation values and eliminating μ_0 in favour of ϵ_0 gives us

$$\begin{aligned} E'_{SS} &= \frac{g_1 g_2 e^2}{4 \times 4\pi\epsilon_0 m_1 m_2 c^2}\left(\frac{\hbar^2}{4}\langle S_{12}\rangle\left\langle\frac{1}{r^3}\right\rangle + \frac{8\pi}{3}\langle \mathbf{S}_1 \cdot \mathbf{S}_2\rangle\left\langle\frac{1}{r^3}\right\rangle\right) \\ &= \frac{\mu^2}{m_1 m_2}\frac{g_1 g_2}{4\hbar^2}\mu c^2\alpha^4 a^3\left(\frac{\hbar^2}{4}\langle S_{12}\rangle\left\langle\frac{1}{r^3}\right\rangle + \frac{8\pi}{3}\langle \mathbf{S}_1 \cdot \mathbf{S}_2\rangle\left\langle\frac{1}{r^3}\right\rangle\right). \end{aligned} \tag{2.32}$$

It can be shown that $S_{12} = 0$ for any spherically symmetrical wave function, i.e. for $\ell = 0$ [11, page 186]. Also, it is easily seen that $\langle\delta^3(\mathbf{r})\rangle = |\psi(0)|^2$. For the spin dot product, we use a similar strategy as for the dot product in Section 2.3: $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$, $\mathbf{S}^2 = \mathbf{S}_1^2 + \mathbf{S}_2^2 + 2\mathbf{S}_1 \cdot \mathbf{S}_2 \rightarrow \langle \mathbf{S}_1 \cdot \mathbf{S}_2\rangle = \frac{1}{2}(\langle\mathbf{S}^2\rangle - \langle\mathbf{S}_1^2\rangle - \langle\mathbf{S}_2^2\rangle) = \frac{1}{2}\langle\mathbf{S}^2\rangle - \frac{3}{4}\hbar^2$. Thus,

$$E'_{SS} = \frac{\mu^2}{m_1 m_2} \frac{g_1 g_2}{4\hbar^2} \mu c^2 \alpha^4 a^3 \frac{8\pi}{3} |\psi(0)|^2 (\frac{1}{2} \langle \mathbf{S}^2 \rangle - \frac{3}{4}\hbar^2)$$
$$= \frac{8\pi}{3} \frac{\mu^2}{m_1 m_2} \frac{g_1 g_2}{4} \mu c^2 \alpha^4 a^3 |\psi(0)|^2 \times \begin{cases} +\frac{1}{4} & \text{for triplet } (s=1) \\ -\frac{3}{4} & \text{for singlet } (s=0) \end{cases} \quad \text{for } \ell = 0. \tag{2.33}$$

Let us consider here the ground state, $n = 1$, $\ell = 0$. Then, $\psi = \frac{1}{\sqrt{\pi a^3}} e^{-r/a}$ and thus

$$E'_{SS} = \frac{8}{3} \frac{\mu^2}{m_1 m_2} \frac{g_1 g_2}{4} \mu c^2 \alpha^4 \times \begin{cases} +\frac{1}{4} & \text{for triplet } (s=1) \\ -\frac{3}{4} & \text{for singlet } (s=0) \end{cases} \quad \text{for the ground state.} \tag{2.34}$$

Filling this in gives

$$E'_{SS} = \frac{4}{3} \frac{m_e}{m_p} g_p m_e c^2 \alpha^4 \times \begin{cases} +\frac{1}{4} & \text{for triplet } (s=1) \\ -\frac{3}{4} & \text{for singlet } (s=0) \end{cases} \quad \text{for hydrogen ground state,} \tag{2.35}$$

and

$$E'_{SS} = \frac{1}{3} m_e c^2 \alpha^4 \times \begin{cases} +\frac{1}{4} & \text{for ortho } (s=1) \\ -\frac{3}{4} & \text{for para } (s=0) \end{cases} \quad \text{for positronium ground state.} \tag{2.36}$$

I will discuss the hyperfine structure for one other situation, namely when for the angular momentum $j = \ell$ holds. In that case, $\langle S_{12} \rangle = 2$ for $s = 1$ [14, Eq. (2.11)]. Furthermore, it is then shown in [22, Eq. (6.31)] that

$$S_{12}^2 = \frac{4}{\hbar^2} \mathbf{S}^2 - 2S_{12}. \tag{2.37}$$

From this equation, it is quickly found that $S_{12} |\psi\rangle = 2 |\psi\rangle$ is indeed a solution for $s = 1$. It also shows that $S_{12} |\psi\rangle = 0$ for $s = 0$. Thus we have

$$\langle S_{12} \rangle = \begin{cases} 0 & \text{for para } (s=0), \\ 2 & \text{for ortho } (s=1). \end{cases} \tag{2.38}$$

Also, $\ell \neq 0$ in this situation since if ℓ would be 0 and $s = 1$, then it would follow $j = 1$ and the condition $j = \ell$ can thus impossibly hold. Now it happens to hold that $\psi(0) = 0$ for $\ell \neq 0$. Thus, of the two terms in the Hamiltonian (Eq. (2.29)), we always need to consider only one! The last piece of the puzzle then is provided by Eq. (2.23). Filling everything in gives

$$E'_{SS} = \begin{cases} \frac{\mu^2}{m_1 m_2} \frac{g_1 g_2}{4} \mu c^2 \alpha^4 \frac{1}{2n^3 \ell(\ell+1/2)(\ell+1)} & \text{for } s = 1 \\ 0 & \text{for } s = 0 \end{cases} \quad \text{for } j = \ell, \tag{2.39}$$

and thus for the two different systems

$$E'_{SS} = \frac{1}{4} m_e c^2 \alpha^4 \frac{g_p}{n^3 \ell(\ell+1/2)(\ell+1)} \quad \text{for hydrogen, } j = \ell \text{ and } s = 1, \tag{2.40}$$

and

$$E'_{SS} = \frac{1}{16} m_e c^2 \alpha^4 \frac{1}{n^3 \ell(\ell+1/2)(\ell+1)} \quad \text{for orthopositronium with } j = \ell. \tag{2.41}$$

2.5 Virtual annihilation

As will be discussed in Chapter 3, positronium can decay into a number of photons when the electron and positron annihilate (which is possible because they are antiparticles, which is of course not the case for hydrogen). For orthopositronium in the ground state, all quantum numbers would be conserved if it decayed into one photon: just like a photon it has a total angular momentum of one $\hbar$, and as will be discussed in Chapter 3, it also has the right charge conjugacy quantum number (which is -1 for a single photon and $(-1)^{\ell+s}$ for positronium.) Obviously, for parapositronium decay into one photon is forbidden.

There is, however, one problem with one-photon decay: energy and momentum cannot be *both* properly conserved in the process. This can be most easily seen by considering the center-of-mass frame of positronium. In this frame, the total momentum is zero (see Section 1.2). Momentum can thus only be conserved if the emitted photon also has zero momentum. But this is impossible for a photon! The conclusion is that the photon must be *virtual*. A virtual particle exploits the energy-time uncertainty relation (see [11, page 116]) to ignore it's proper energy-momentum relation for a short time, allowing it to exist even though it would seem impossible. The argument can be made more formal. The relativistic energy-momentum relation is $E^2 = \mathbf{p}^2c^2 + m^2c^4$, where E is the particle's energy, $\mathbf{p}$ is it's momentum and m is its invariant mass. The energy of the photon is then

$$E_\gamma = \sqrt{\mathbf{p}_{e^-}^2 c^2 + m_e^2 c^4} + \sqrt{\mathbf{p}_{e^+}^2 c^2 + m_e^2 c^4}, \tag{2.42}$$

and its momentum is

$$\mathbf{P}_\gamma = \mathbf{p}_{e^-} + \mathbf{p}_{e^+}, \tag{2.43}$$

where $\mathbf{p}_{e^\pm}$ are the momenta of the positron and electron before annihilation. A real photon is massless, so the condition for a real photon becomes $E_\gamma = \sqrt{\mathbf{P}_\gamma^2 c^2}$ or

$$\sqrt{\mathbf{p}_{e^-}^2 c^2 + m_e^2 c^4} + \sqrt{\mathbf{p}_{e^+}^2 c^2 + m_e^2 c^4} = \sqrt{(\mathbf{p}_{e^-} + \mathbf{p}_{e^+})^2}c. \tag{2.44}$$

By picking the right frames it is now quickly seen that this equation cannot be satisfied. For example, looking once more at the problem from the center-of-mass frame, we see that the right-hand side becomes zero. The left hand side, however, cannot possibly be zero because of the rest masses. Alternatively, one might pick the rest frame of the electron such that $\mathbf{p}_{e^-} = \mathbf{0}$. It is then found that the equation is not satisfied since $\sqrt{\mathbf{p}_{e^+}^2 c^2 + m_e^2 c^4} + m_e c^2 \geq \sqrt{\mathbf{p}_{e^+}^2 c^2}$, whatever the momentum.

The photon is thus virtual and short lived, which means that it must revert to positronium. In the final state, the electron and positron have momentum $\mathbf{p}'_{e^\pm}$. The process is summarized by the Feynman diagram in Figure 2.1. To find the energy contribution, we must determine the 'potential' between the particles due to virtual annihilation. This can be done by calculating the S-matrix element for the process both through relativistic quantum mechanics, using the Feynman diagram, and through normal quantum mechanics with an

interaction potential. The form of the potential follows from comparing the two. As will be discussed in Section 3.1, the S matrix is defined by

$$\langle f|\, S\, |i\rangle = \lim_{t\to\infty} \langle f|i(t)\rangle\,, \tag{2.45}$$

where $|f\rangle$ is the final state and $|i\rangle$ is the initial state.

We will first do the calculation using relativistic quantum mechanics. In general, the S matrix can be written in the form

$$\langle f|\, S\, |i\rangle = \delta_{fi} + i\mathcal{A}(2\pi)^4\delta(E_f - E_i)\delta^{(3)}(\mathbf{P}_f - \mathbf{P}_i). \tag{2.46}$$

Here, $\mathbf{P}$ is the total momentum of a state and E is the energy. $\mathcal{A}$ is the amplitude of the process. It is this amplitude which can be found using the Feynman diagram using the Feynman rules appropriate for quantum electrodynamics (QED). The values which these rules assign to the various parts of the diagram can be found in the figure. Note that here natural units ($\hbar = c = \epsilon_0 = 1$) are used, as is conventional in relativistic quantum mechanics. Here, $p_{e^\pm}$ are the four-vector momenta of the electron and positron initially. Using the convention for the flat metric $\eta = diag(1,-1,-1,-1)$, this means $(p_{e^-} + p_{e^+})^2 = (E_{e^-} + E_{e^+})^2 - (\mathbf{p_{e^-}} + \mathbf{p_{e^+}})^2$. Furthermore, $u_{m_s}(\mathbf{p})$ is a positive frequency solution spinor with spin s and momentum $\mathbf{p}$, $v_{m_s}(\mathbf{p})$ is a negative solution spinor and $\bar{u}_{m_s}(\mathbf{p})$ and $\bar{v}_{m_s}(\mathbf{p})$ are their Dirac adjoints. The initial azimuthal quantum numbers are given by $m_{se^\pm}$, the final by $m'_{se^\pm}$.

To now calculate the amplitude, we must simply put all the expressions in the diagram together, giving

$$i\mathcal{A} = ie^2\bar{v}_{m_{se^+}}(\mathbf{p}_{e^+})\gamma^\mu u_{m_{se^-}}(\mathbf{p}_{e^-})\frac{g_{\mu\nu}}{(p_{e^-}+p_{e^+})^2}\bar{u}_{m'_{se^-}}(\mathbf{p}'_{e^-})\gamma^\nu v_{m'_{se^+}}(\mathbf{p}'_{e^+}). \tag{2.47}$$

As we want to find the effective potential in non-relativistic quantum mechanics, we must consider the classical limit. We then have $g_{\mu\nu} = \eta_{\mu\nu}$ (flat spacetime), $E_{e^\pm} \simeq m_e$ and $|\mathbf{p}_{e^\pm}| \ll E_{e^\pm}$ (non-relativistic) so that $\frac{g_{\mu\nu}}{(p_{e^-}+p_{e^+})^2} \simeq \frac{\eta_{\mu\nu}}{(2m_e)^2}$. Furthermore, using the Dirac representation of the Clifford algebra, we have for the electron spinor

$$\begin{aligned} u_{m_{se^-}}(\mathbf{p}_{e^-}) &= \sqrt{E+m_e}\begin{pmatrix}\chi_{m_{se^-}}\\ \frac{\boldsymbol{\sigma}\cdot\mathbf{p}_{e^-}}{E+m_e}\chi_{m_{se^-}}\end{pmatrix} \\ &\simeq \sqrt{2m_e}\begin{pmatrix}\chi_{m_{se^-}}\\ 0\end{pmatrix} = \sqrt{2m_e}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}. \end{aligned} \tag{2.48}$$

Similarly, we find for the positron

$$\begin{aligned} v_{m_{se^+}}(\mathbf{p}'_{e^+}) &= -\sqrt{E+m_e}\begin{pmatrix}\frac{\boldsymbol{\sigma}\cdot\mathbf{p}'_{e^+}}{E+m_e}\epsilon\chi_{m_{se^+}}\\ \epsilon\chi_{m_{se^+}}\end{pmatrix} \\ &\simeq -\sqrt{2m_e}\begin{pmatrix}0\\ \epsilon\chi_{m_{se^+}}\end{pmatrix} = \sqrt{2m_e}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}. \end{aligned} \tag{2.49}$$

Here χ_s is the Pauli spinor. Furthermore, ϵ is the 2×2 Levi-Civita symbol and $\boldsymbol{\sigma}$ is a vector operator with the Pauli matrices as components. These are all defined in Appendix A.

We are here interested in the ortho-state of positronium. This means the total spin angular momentum is one, and the magnetic quantum number $M = -1, 0, 1$. We can pick our reference frame such that $M = 1$, which corresponds (using Clebsch-Gordan coefficients) to both electron and positron having spin up. From conservation of angular momentum, it follows that this holds both initially and finally. Thus, $m_{se^-} = m'_{se^-} = m_{se^+} = m'_{se^+} = 1/2$. The Dirac adjoints can be found from $\bar{u} = u^\dagger \gamma^0$. By using the explicit form of γ^0 (which is given, just as of the other γ matrices, in Eq. (A.2)) , we find $\bar{u}_{m'_{se^-}}(\mathbf{p}'_{e^-}) = (u_{m_{se^-}}(\mathbf{p}_{e^-}))^T$ and $\bar{v}_{m_{se^+}}(\mathbf{p}_{e^+}) = -(v_{m_{se^+}'}(\mathbf{p}'_{e^+}))^T$. By now also explicitly filling in the other γ matrices, we can evaluate the amplitude and find

$$i\mathcal{A} = ie^2 \times (-2) = -2ie^2. \tag{2.50}$$

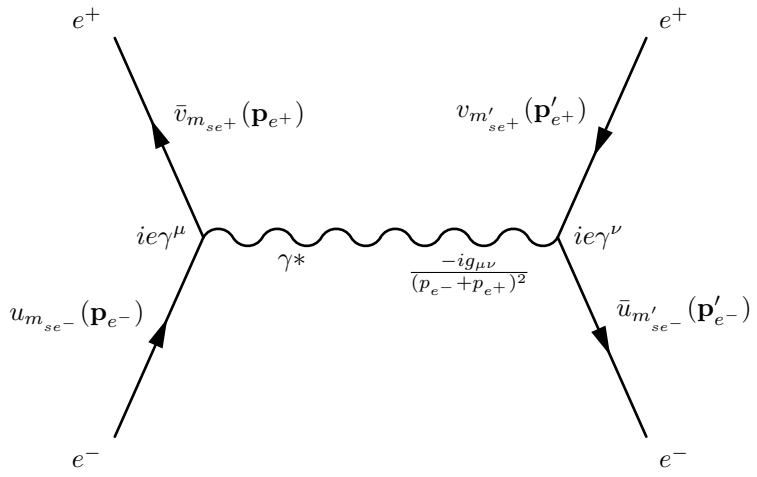


Figure 2.1: Feynman diagram for positronium virtual annihilation through a virtual photon.

In order to find the S matrix through ordinary quantum mechanics, we make use of the interaction (or Dirac) picture in which an interaction Hamiltonian H_I is split from the rest of the Hamiltonian, $H = H_0 + H_I$. In this picture, most of the time dependence is in the operators (like in the Heisenberg picture), governed by H_0. The rest of the time dependence is the part governed by H_I and is in the states. The relations between operators and states in the interaction picture and the standard Schrödinger picture is given by

$$\begin{aligned} |\psi(t)\rangle_{int} &= e^{iH_0 t} |\psi(t)\rangle_S \,, \\ \mathcal{O}_{int}(t) &= e^{iH_0 t} \mathcal{O}_S e^{-iH_0 t}. \end{aligned} \tag{2.51}$$

In the rest of this section, unless specified differently, all operators and states are in the interaction picture. By filling the above relations in into the Schrödinger

equation, we find

$$i\frac{d\left|\phi\right\rangle}{dt} = H_I(t)\left|\phi\right\rangle. \tag{2.52}$$

Dyson's formula, which can be derived from Eq. (2.52), tells us how to solve for $|i(t)\rangle$. When we take the Born approximation of this formula, we find [13, Eq. (5.10)]

$$\left\langle f\right| S\left|i\right\rangle = \delta_{fi} - i\int_{-\infty}^{+\infty} dt\left\langle f\right| H_I(t)\left|i\right\rangle. \tag{2.53}$$

In our situation, we take H_0 to be the Hamiltonian which is obtained by ignoring the virtual annihilation effect (which is thus represented by H_I). We assume the interaction Hamiltonian has, in the Schrödinger picture, the form of a potential which only depends on relative position $\mathbf{r}$ between the electron and positron. We then have, from Eq. (2.51),

$$H_I(t) = e^{iH_0}V(\mathbf{r})e^{-iH_0}. \tag{2.54}$$

The initial state is an electron with momentum $\mathbf{p}_{e^-}$ and a positron with momentum $\mathbf{p}_{e^+}$, and we end with momenta $\mathbf{p}'_{e^-}$ and $\mathbf{p}'_{e^+}$. This allows for the description developed in Chapter 1 with

$$\begin{aligned} \left|i\right\rangle &= \left|\mathbf{p}\right\rangle\left|\mathbf{P}\right\rangle_{CoM} \text{ and} \\ \left|f\right\rangle &= \left|\mathbf{p'}\right\rangle\left|\mathbf{P'}\right\rangle_{CoM}, \end{aligned} \tag{2.55}$$

in which $|\mathbf{p}\rangle$ is a momentum eigenstate of the fictitious particle in the center-of-mass frame, and $|\mathbf{P}\rangle_{CoM}$ is a momentum eigenstate of the center of mass.

We are now in a position to compare the S matrix found using regular quantum mechanics and Eq. (2.53) with the one found from the amplitude above. I will now perform the quantum-mechanical computation.

$$\begin{aligned} &\left\langle\mathbf{P}'\right|_{CoM}\left\langle\mathbf{p}'\right| S\left|\mathbf{p}\right\rangle\left|\mathbf{P}\right\rangle_{CoM} \\ &= \delta_{fi} - i\int_{-\infty}^{+\infty} dt\left\langle\mathbf{P}'\right|_{CoM}\left\langle\mathbf{p}'\right| H_I(t)\left|\mathbf{p}\right\rangle\left|\mathbf{P}\right\rangle_{CoM} \\ &= \delta_{fi} - i\int_{-\infty}^{+\infty} dt\left\langle\mathbf{P}'\right|_{CoM}\left\langle\mathbf{p}'\right| e^{iH_0}Ve^{-iH_0}\left|\mathbf{p}\right\rangle\left|\mathbf{P}\right\rangle_{CoM} \\ &= \delta_{fi} - i\int_{-\infty}^{+\infty} dt\, e^{(E-E')t}\left\langle\mathbf{P}'\right|_{CoM}\left\langle\mathbf{p}'\right| V\left|\mathbf{p}\right\rangle\left|\mathbf{P}\right\rangle_{CoM} \\ &= \delta_{fi} - i2\pi\delta(E-E')\left\langle\mathbf{P}'\right|_{CoM}\left\langle\mathbf{p}'\right| V\left|\mathbf{p}\right\rangle\left|\mathbf{P}\right\rangle_{CoM} \\ &= \delta_{fi} - i2\pi\delta(E-E')\left\langle\mathbf{P}'_{CoM}|\mathbf{P}_{CoM}\right\rangle\left\langle\mathbf{p}'\right| V\left|\mathbf{p}\right\rangle \\ &= \delta_{fi} - i(2\pi)^4\delta(E-E')\delta^{(3)}(\mathbf{P}-\mathbf{P}')\left\langle\mathbf{p}'\right| V\left|\mathbf{p}\right\rangle. \end{aligned} \tag{2.56}$$

Because V only depends on the relative position between the electron and positron, and thus not on the center-of-mass wave function, we could make the step $\left\langle\mathbf{P}'\right|_{CoM}\left\langle\mathbf{p}'\right| V\left|\mathbf{p}\right\rangle\left|\mathbf{P}\right\rangle_{CoM} = \left\langle\mathbf{P}'_{CoM}|\mathbf{P}_{CoM}\right\rangle\left\langle\mathbf{p}'\right| V\left|\mathbf{p}\right\rangle$. By definition, the total momentum of the system is the center-of-mass momentum, thus we have $\mathbf{P_i} = \mathbf{P}$ and $\mathbf{P_f} = \mathbf{P'}$. Furthermore, $E = E_i$ and $E' = E_f$. The S matrices from Eqs. (2.46) and (2.56) can thus be directly compared, giving

$$-i\left\langle f\right| V\left|i\right\rangle = \frac{i\mathcal{A}}{(2m_e)^2} \rightarrow \left\langle\mathbf{p}'\right| V\left|\mathbf{p}\right\rangle = \frac{e^2}{2m_e^2}. \tag{2.57}$$

(where the relative vector $(2m_e)^2$ arises in the comparison due to relativistic normalization of states).

A momentum eigenstate $|\mathbf{p}\rangle$ can be written as a superposition of position eigenstates through a Fourier transform (this relation follows from the form of the momentum operator):

$$|\Psi\rangle = \int \phi(\mathbf{q}) \, |\mathbf{q}\rangle \, d^3q = \int \psi(\mathbf{r}) \, |\mathbf{r}\rangle \, d^3r, \tag{2.58}$$

$$\psi(\mathbf{r}) = \frac{1}{(2\pi\hbar)^{3/2}} \int \phi(\mathbf{q}) e^{i\mathbf{q}\cdot\mathbf{r}/\hbar} d^3q. \tag{2.59}$$

When $|\Psi\rangle = |\mathbf{p}\rangle$, it is easily seen that $\phi(\mathbf{q}) = \delta^{(3)}(\mathbf{q} - \mathbf{p})$. It thus follows that

$$\psi(\mathbf{r}) = \frac{1}{(2\pi\hbar)^{3/2}} e^{i\mathbf{p}\cdot\mathbf{r}/\hbar} \to |\mathbf{p}\rangle = \int \frac{1}{(2\pi\hbar)^{3/2}} e^{i\mathbf{p}\cdot\mathbf{r}/\hbar} \, |\mathbf{r}\rangle \, d^3r. \tag{2.60}$$

This allows us to calculate the matrix element (using $V \, |\mathbf{r}\rangle = V(\mathbf{r}) \, |\mathbf{r}\rangle$)

$$\begin{aligned}
\langle \mathbf{p}' | \, V \, | \mathbf{p} \rangle &= \frac{1}{(2\pi\hbar)^3} \int \int V(\mathbf{r}) e^{i(\mathbf{p}\cdot\mathbf{r} - \mathbf{p}'\cdot\mathbf{s})/\hbar} \, \langle \mathbf{s} | \mathbf{r} \rangle \, d^3r d^3s \\
&= \frac{1}{(2\pi\hbar)^3} \int \int V(\mathbf{r}) e^{i(\mathbf{p}\cdot\mathbf{r} - \mathbf{p}'\cdot\mathbf{s})/\hbar} \delta^{(3)}(\mathbf{r} - \mathbf{s}) d^3r d^3s \\
&= \frac{1}{(2\pi\hbar)^3} \int V(\mathbf{r}) e^{i(\mathbf{p} - \mathbf{p}')\cdot\mathbf{r}/\hbar} d^3r \\
&\equiv \frac{1}{(2\pi\hbar)^3} \int V(\mathbf{r}) e^{i\mathbf{q}\cdot\mathbf{r}/\hbar} d^3r = \frac{e^2}{2m_e^2}.
\end{aligned} \tag{2.61}$$

This is simply a Fourier transform of $V(\mathbf{r})$: the whole thing would be equal to $\tilde{V}(\mathbf{q})$ where $\mathbf{q} = \mathbf{p} - \mathbf{p}'$. Thus, to invert the equation and obtain $V(\mathbf{r})$, we must simply take the inverse Fourier transform of the right-hand side and find

$$V(\mathbf{r}) = \frac{1}{(2\pi\hbar)^3} \int \frac{e^2}{2m_e^2} e^{\mathbf{q}\cdot\mathbf{r}/\hbar} d^3q = \frac{e^2}{2m_e^2} \delta^{(3)}(\mathbf{r}). \tag{2.62}$$

The potential is a delta function, just as we would have expected. After all, annihilation only takes place if the electron and positron meet each other. Now, in the Schrödinger picture, $H = H_0 + H_I = H_0 + V(r)$. We see we can simply apply perturbation theory and find

$$E'_{ann} = \langle V \rangle = \frac{e^2}{2m_e^2} \langle \delta^{(3)}(\mathbf{r}) \rangle = \frac{e^2}{2m_e^2} |\psi(0)|^2 = \frac{e^2}{2m_e^2} \times \frac{1}{\pi a^3} = \frac{1}{4} \alpha^4 m_e c^2. \tag{2.63}$$

Since virtual annihilation is impossible in the para-state, this gives the final result (after converting from natural units to regular units)

$$E'_{ann} = \begin{cases} \frac{1}{4}\alpha^4 m_e c^2 & \text{for orthopositronium ground state,} \\ 0 & \text{for parapositronium ground state.} \end{cases} \tag{2.64}$$

2.6 Spectrum of positronium

A very interesting transition in positronium is the transition of $s = 1$ to $s = 0$, i.e. from orthopositronium to parapositronium. I will calculate this for two distinct situations: first for the ground state $n = 1$, second for any excited state (any value of n) from $j = \ell, s = 1$ to $j = \ell, s = 0$.

The first case is easily calculated. It is simply the sum of the energy differences in Eqs (2.36) (spin-spin coupling) and (2.64) (virtual annihilation). This gives

$$\Delta E = \frac{7}{12} m_e c^2 \alpha^4 \simeq 8.45 \times 10^{-4} \text{ eV} \qquad \text{for ortho} \to \text{para}, \, n = 1, \tag{2.65}$$

which corresponds to a frequency (from $E = h\nu$)

$$\nu \simeq 204 \, \text{GHz} \tag{2.66}$$

which is in close agreement with both much more elaborate full theoretical calculation and the most recent experiments [9, Eqs (8), (9), (11)]: they are all equal up to five significant digits. Clearly, the spin-spin interaction as calculated here is truly the dominant contribution to this splitting.

The second case is a bit harder. First of all, we now need to take the spin-orbit coupling into account, since the spin-orbit energy depends on s and is only present when the orbital angular momentum ℓ is non-zero. This energy can be read off from Eq. (2.27),

$$\Delta E_{SO} = -\frac{3}{8n^3} \frac{m_e c^2 \alpha^4}{j(j+1/2)(j+1)} \qquad \text{for ortho} \to \text{para}, \, j = \ell. \tag{2.67}$$

We can do the same for the spin-spin energy. When $s = 0$, the spin-spin energy is zero because the tensor S_{12} becomes 0. Thus, the energy difference is simply given by Eq. (2.41),

$$\Delta E_{SS} = \frac{1}{4n^3} \frac{m_e c^2 \alpha^4}{j(j+1/2)(j+1)} \qquad \text{for ortho} \to \text{para}, \, j = \ell. \tag{2.68}$$

When combined this yields

$$\Delta E_{coupling} = -\frac{1}{8n^3} \frac{m_e c^2 \alpha^4}{j(j+1/2)(j+1)} \qquad \text{for ortho} \to \text{para}, \, j = \ell. \tag{2.69}$$

For most transitions, we do not need to take virtual annihilation energy into account. The only $j = \ell$ states which conserve the charge conjugacy quantum number when decaying into one photon (which is the case when $(-1)^{\ell+s} = -1$, see Section 3.2) and angular momentum (i.e. $j = 1$) are orthopositronium in the $\ell = 0$ state and parapositronium in the $\ell = 1$ state (see Section 3.2). Taking this into account gives

$$\Delta E = \\ -\frac{1}{8n^3} \frac{m_e c^2 \alpha^4}{j(j+1/2)(j+1)} \begin{cases} +\frac{1}{4} m_e c^2 \alpha^4 & \text{for } j = 0 \\ +0 & \text{for } j > 1 \\ -\frac{1}{4} m_e c^2 \alpha^4 & \text{for } j = 1 \end{cases} \qquad \text{for ortho} \to \text{para}, \, j = \ell. \tag{2.70}$$

We can calculate this energy e.g. for the first excited state, $n = 2$, with $j = 1$. It can then be found to yield

$$\begin{aligned} \Delta E &\simeq -\,3.70 \times 10^{-4}\ \text{eV} \\ \rightarrow \Delta\nu &\simeq -\,89.4\ \text{GHz} \qquad \text{for ortho} \rightarrow \text{para},\ j = \ell = 1,\ n = 2, \end{aligned} \tag{2.71}$$

while for higher exited states the energy splitting quickly becomes smaller due to higher n and the possibility that virtual annihilation no longer contributes, e.g.

$$\begin{aligned} \Delta E &\simeq -\,3.93 \times 10^{-7}\ \text{eV} \\ \rightarrow \Delta\nu &\simeq -\,0.0950\ \text{GHz} \qquad \text{for ortho} \rightarrow \text{para},\ j = \ell = 2,\ n = 3. \end{aligned} \tag{2.72}$$

The energy levels for $n = 1, 2$ are summarized in the spectrum drawn in Figure 2.2, with calculated energy differences between states indicated by arrows.

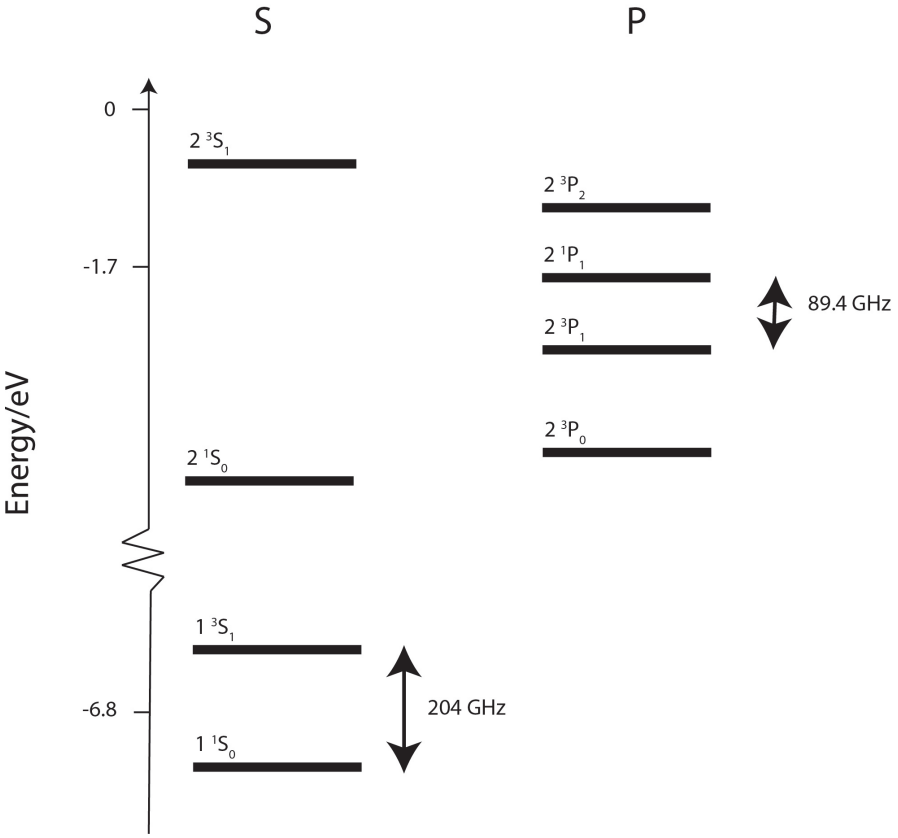


Figure 2.2: Spectrum of positronium for $n = 1\,, 2$.

Chapter 3

Decay of positronium

In this chapter the decay of positronium through annihilation will be examined in order to estimate the lifetime of positronium according to quantum electrodynamics (QED). This will allow for comparison with experiment and thus to judge whether the disappearance of positronium into an extra dimension might actually be taking place. In Section 3.1 I will discuss the general machinery used to calculate decay probabilities and through these lifetimes. Then, in Section 3.2 I will discuss what the dominant decay modes are for positronium, which I will use in Section 3.3 to calculate the lifetime of parapositronium explicitly. Throughout the chapter, natural units ($\hbar = c = \epsilon_0 = 1$) will be used.

3.1 Transition probability

Of first concern to us are transitions between different states in general. Let the state of a system at time t be $|t\rangle$. Now define the initial state, way before any interaction or transition occurs, to be

$$|i\rangle \equiv \lim_{t\to-\infty} |t\rangle \,. \tag{3.1}$$

We are now interested in the probability amplitude of finding the system in a final state $|f\rangle$ after all interaction has taken place. The S (scattering) matrix is defined as the operator

$$S\,|i\rangle = \lim_{t\to+\infty} |t\rangle \,, \tag{3.2}$$

i.e. it gives the state to which the system converges by time evolution. The probability amplitude is then simply given by

$$\lim_{t\to+\infty} \langle f|t\rangle = \langle f|\, S\,|i\rangle = S_{fi}. \tag{3.3}$$

It is a matrix element of S.

Now, it can be shown that these matrix elements can always be written in the form [13, page 87]

$$S_{fi} = \delta_{fi} + i(2\pi)^4\delta^{(4)}(P_f - P_i)\mathcal{A}_{fi}, \tag{3.4}$$

where $\mathcal{A}_{fi}$ is called the amplitude of the transition. It is also known as the T matrix ($\mathcal{A}_{fi} = T_{fi} = \langle f | T | i \rangle$). It is $i\mathcal{A}_{fi}$ which can be easily calculated through the use of Feynman diagrams. In a Feynman diagram, it is graphically shown what the initial and final states are and what is the process through which the transition takes place. Through Feynman rules, an expression can obtained to correspond to the diagram. The amplitude is now obtained by summing the expressions of all Feynman diagrams which connect the initial and final states through allowed processes. Usually, the more vertices a diagram has, the smaller its expression becomes. Thus, one often only needs to consider the simplest Feynman diagrams for a good approximation, as will be done in this thesis.

Additionally, the S matrix contains a delta function with the initial and final total four-momentum. This represents the fact that energy and momentum must be conserved by the process. Let us now look more specifically at the decay of positronium, where an electron and a positron together annihilate to create m new particles, i.e. the process

$$e^-(p_{e^-}, s_{e^-}) + e^-(p_{e^-}, s_{e^-}) \to a_1(p_1) + a_2(p_2) + ... + a_m(p_m). \tag{3.5}$$

We can use this to calculate the transition probability to a certain infinitesimal volume in final state space [13, Eq. (5.19)]

$$dw = \prod_{j=1}^{m} \frac{d^3 p_j}{(2\pi)^3 2p_j^0} |S_{fi}|^2. \tag{3.6}$$

This will involve squaring the Dirac delta function. This seems to be problematic, but can be dealt with by what is called Fermi's trick [13, page 84]. We can write the square of the S matrix as

$$\begin{aligned} |S_{fi}|^2 &= ((2\pi)^4 \delta^{(4)}(P_f - P_i))^2 |\mathcal{A}_{fi}|^2 \\ &= \int d^4x e^{i(P_f - P_i)x} (2\pi)^4 \delta^{(4)}(P_f - P_i) |\mathcal{A}_{fi}|^2 \\ &= \left(\int d^4x \right) (2\pi)^4 \delta^{(4)}(P_f - P_i) |\mathcal{A}_{fi}|^2. \end{aligned} \tag{3.7}$$

Here, the integral is taken to be over all spacetime. However, in reality we are never concerned with this but are limited to finite space volumes and finite time intervals. Indeed, the thing which is rather of interest to us is the transition probability per time interval and probabilities per unit volume. This allows us to replace the integral over all spacetime with an integral over the volume of spacetime of interest to us,

$$\int d^4x \to TV, \tag{3.8}$$

where T is the time interval and V is the normalization volume. This can be combined to find

$$d\Gamma \equiv \frac{dw}{T} = V(2\pi)^4 \delta^{(4)}(p_{e^-} + p_{e^+} - p_1 - p_2 - ... - p_m) \prod_{j=1}^{m} \frac{d^3 p_i}{(2\pi)^3 2p_i^0} |A_{fi}|^2, \tag{3.9}$$

where Γ is the transition rate.

It will also be useful to present a general expression for the positronium wave function in terms of the creation operators for electron ($a^\dagger_{m_s}$) and positron ($b^\dagger_{m'_s}$) (where m_s is the azimuthal quantum number for the electron and m'_s for the positron). This is simply obtained by convolving the positronium momentum eigenstate (using the Lorentz-invariant measure) with the wave function in momentum space. If the positronium has angular momentum quantum number S and azimuthal quantum number M_S, this eigenstate is given by $\sum_{m_s,m'_s} a^\dagger_{m_s}(\mathbf{p})b^\dagger_{m'_s}(-\mathbf{p})\left\langle SM_s|1/2,m_s;1/2,m'_s\right\rangle$. Here, $\left\langle SM_s|1/2,m_s;1/2m'_s\right\rangle$ are Clebsch-Gordan coefficients. Thus we obtain

$$|P_s(S,M_S)\rangle = A\int\frac{d^3p}{(2\pi)^3}\phi(\mathbf{p})\sum_{m_s,m'_s} a^\dagger_{m_s}(\mathbf{p})b^\dagger_{m'_s}(-\mathbf{p})\left\langle SM_s|1/2,m_s;1/2,m'_s\right\rangle|0\rangle\,. \tag{3.10}$$

Here, A is a normalization constant. In order for the wave function to be normalized, we require

$$\begin{aligned}
&\langle P_s(S,M_S)|P_s(S,M_S)\rangle = A^2\int\frac{d^3pd^3p'}{(2\pi)^6}\phi^*(\mathbf{p}')\phi(\mathbf{p})\\
&\langle 0|\left[\sum_{n_s,n'_s} a_{n_s}(\mathbf{p}')b_{n'_s}(-\mathbf{p}')\left\langle SM_s|1/2,n_s;1/2n'_s\right\rangle\right]\\
&\left[\sum_{m_s,m'_s} a^\dagger_{m_s}(\mathbf{p})b^\dagger_{m'_s}(-\mathbf{p})\left\langle SM_s|1/2,m_s;1/2m'_s\right\rangle\right]|0\rangle\\
&=A^2\int\frac{d^3pd^3p'}{(2\pi)^6}\phi^*(\mathbf{p}')\phi(\mathbf{p})\langle 0|\sum_{m_s,m'_s} a_{m_s}(\mathbf{p}')b_{m'_s}(-\mathbf{p}')a^\dagger_{m_s}(\mathbf{p})b^\dagger_{m'_s}(-\mathbf{p})\\
&\quad|\left\langle SM_s|1/2,m_s;1/2m'_s\right\rangle|^2|0\rangle\\
&=A^2\int\frac{d^3pd^3p'}{(2\pi)^6}\phi^*(\mathbf{p}')\phi(\mathbf{p})(2\pi)^6[\delta^{(3)}(\mathbf{p}-\mathbf{p}')]^2\sum_{m_s,m'_s}|\left\langle SM_s|1/2,m_s;1/2m'_s\right\rangle|^2\\
&=A^2V\int\frac{d^3p}{(2\pi)^3}|\phi(\mathbf{p})|^2 = A^2V.
\end{aligned} \tag{3.11}$$

Thus, the state is normalized to one for $A=\frac{1}{\sqrt{V}}$. Here, use has been made of the normalization of the Clebsch-Gordan coefficients, of the vacuum and of the momentum wave function. Furthermore, use has been made of the anti-commutation relations

$$\{a_{m_s}(\mathbf{p}),a_{m'_s}(\mathbf{p}')\} = \{b_{m_s}(\mathbf{p}),b_{m'_s}(\mathbf{p}')\} = (2\pi)^2\delta^{m_s,m'_s}\delta^{(3)}(\mathbf{p}-\mathbf{p}'), \tag{3.12}$$

while all other anti-commutators vanish. We can also rewrite the wave function in terms of relativistically normalized electron-positron states. The raising operators are here defined such that they create a normalized state, $a^\dagger|0\rangle = \frac{1}{2E}|e^-\rangle$. Thus,

$$\begin{aligned}
|P_s(S,M_S)\rangle =&\frac{1}{\sqrt{V}}\int\frac{d^3p}{(2\pi)^3 2p^0}\phi(\mathbf{p})\\
&\sum_{m_s,m'_s}|e^-(\mathbf{p},m_s)e^+(-\mathbf{p},m'_s)\rangle\left\langle SM_s|1/2,m_s;1/2,m'_s\right\rangle.
\end{aligned} \tag{3.13}$$

3.2 Decay modes

So how can positronium decay? When the electron and positron annihilate, a system of new particles is created such that the new system has all the same quantum numbers as the positronium before decay. It can be calculated that decay into a number of photons is by far the dominant decay mode. What number of photons is created depends on the state positronium is in. First of all, we know it is impossible for positronium to decay into a single photon since this process cannot conserve both momentum and energy (see Section 2.5).

Into what number of photons can positronium then decay? Let us consider further conservation laws. It is easiest to verify which processes are allowed for by considering the charge conjugation operator $\hat{C}$. Since $\hat{C}^2 |\phi\rangle = |\phi\rangle$, it only has eigenvalues ± 1. Positronium as well as a collection of photons is uncharged and thus an eigenstate of $\hat{C}$. Denote the eigenvalue for positronium C_{Ps} and for the photons C_γ. Decay is then only possible if $C_{Ps} = C_\gamma$.

When applying charge conjugation to positronium, this simply interchanges the electron and the positron. For the spatial wave function, this means $|\phi(\mathbf{r})\rangle \to |\phi(-\mathbf{r})\rangle$. This is equivalent to $\theta \to \pi - \theta$ and $\phi \to \phi + \pi$. Filling this in into the spherical harmonics (see Section 1.3) gives $|\phi(-\mathbf{r})\rangle = (-1)^\ell |\phi(\mathbf{r})\rangle$. To see the full effect of the transformation, we apply the operator to the full relativistic wave function of positronium and find

$$\frac{1}{\sqrt{V}} \int \frac{d^3p}{(2\pi)^3} (-1)^\ell \phi(\mathbf{p}) \sum_{m_s, m_s'} b^\dagger_{m_s}(\mathbf{p}) a^\dagger_{m_s'}(-\mathbf{p}) \langle SM_s | 1/2, m_s; 1/2, m_s' \rangle |0\rangle \,. \tag{3.14}$$

The original function can be re-obtained by first interchanging the a and b operators, giving a factor -1, and then interchanging m_s and m_s'. Because of the general relation

$$\langle SM_s | s, m_s; s', m_s' \rangle = (-1)^{S-s-s'} \langle SM_s | s', m_s'; s, m_s \rangle \,, \tag{3.15}$$

this picks up an additional factor $(-1)^{S+1}$. Putting together all the factors gives

$$C_{P_s} = (-1)^{\ell+S}. \tag{3.16}$$

Because for photons the charge conjugation eigenvalue is -1, for a system of n photons, the eigenvalue is given by [10, page 129]

$$C_\gamma = (-1)^n. \tag{3.17}$$

Of especial interest here is the decay of the ground state of positronium. For this state $\ell = 0$, while $s = 1$ for orthopositronium and $s = 0$ for parapositronium. This immediately reveals that parapositronium can decay into any even number of photons and orthopositronium can decay into any odd number of photons (except one). The fewer photons it decay into, the fewer vertices we can expect to occur in the corresponding Feynman diagram and the more likely a transition becomes. Thus, for parapositronium, the dominant decay mode is into two photons, while for orthopositronium it is into three photons. Also, this leads us to expect orthopositronium to be longer lived. The dominant decay modes are illustrated by the Feynman diagrams in Figures 3.1 and 3.2.

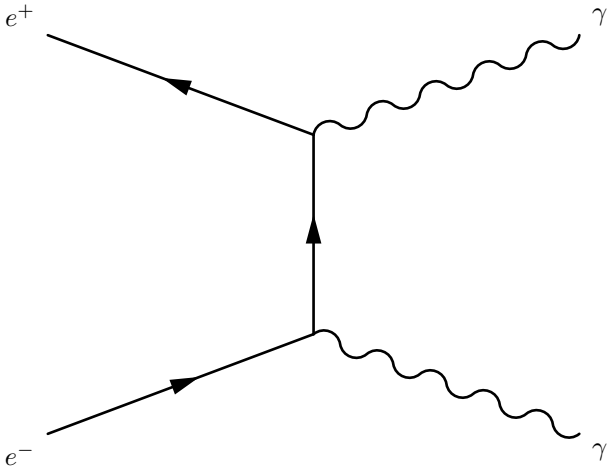


Figure 3.1: Feynman diagram for the decay of parapositronium into two photons.

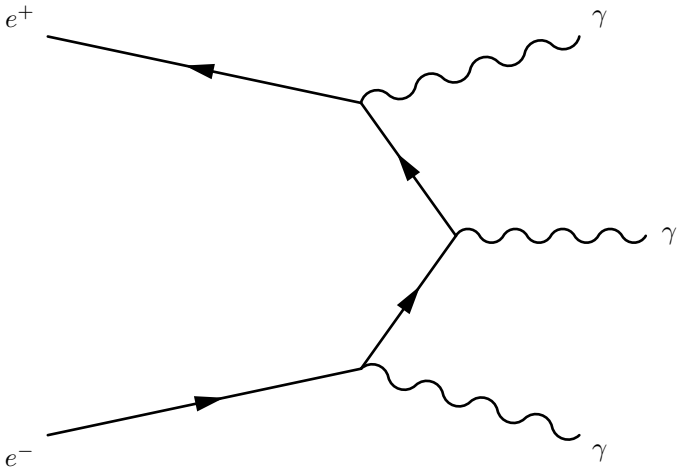


Figure 3.2: Feynman diagram for the decay of orthopositronium into three photons.

3.3 Lifetime

The decay mode which is easiest to calculate is that of ground-state parapositronium into two photons. Our first step will be to write down the wave function for this state. Simply filling in Eq. (3.13) gives

$$\begin{aligned}|1^1S_0\rangle =& \frac{1}{\sqrt{V}}\int \frac{d^3p}{(2\pi)^3 2p^0}\phi(\mathbf{p})\frac{1}{\sqrt{2}}\Big(|e^-(\mathbf{p},+\frac{1}{2})e^+(-\mathbf{p},-\frac{1}{2})\rangle \\ & - |e^-(\mathbf{p},-\frac{1}{2})e^+(-\mathbf{p},+\frac{1}{2})\rangle\Big).\end{aligned} \tag{3.18}$$

To find the decay rate of this state, we must find the S matrix for its decay into two photons. That is, we must find

$$\begin{aligned}\langle\gamma(k_1,\epsilon_1)\gamma(k_2,\epsilon_2)|\,S\,|1^1S_0\rangle = & \\ & \frac{1}{\sqrt{V}}\int \frac{d^3p}{(2\pi)^3 2p^0}\phi(\mathbf{p})\frac{1}{\sqrt{2}} \\ & \Big(\langle\gamma(k_1,\epsilon_1)\gamma(k_2,\epsilon_2)|\,S\,|e^-(\mathbf{p},+\frac{1}{2})e^+(-\mathbf{p},-\frac{1}{2})\rangle - \\ & \langle\gamma(k_1,\epsilon_1)\gamma(k_2,\epsilon_2)|\,S\,|e^-(\mathbf{p},-\frac{1}{2})e^+(-\mathbf{p},+\frac{1}{2})\rangle\Big),\end{aligned} \tag{3.19}$$

where $\gamma(k,\epsilon)$ is a photon with four-momentum k and polarization vector ϵ. Thus, what we need to know is the S-matrix element

$$\langle\gamma(k_1,\epsilon_1)\gamma(k_2,\epsilon_2)|\,S\,|e^-(\mathbf{p}_{e^-},m_s)e^+(-\mathbf{p},m_s')\rangle\,.$$

To find this, we can use the Feynman rules for QED, which have been inscribed into the Feynman diagram in Figure 3.3. Note that these are in fact two different processes summarized by one diagram: one corresponds to photon one having four-momentum k_1 and the other k_2, the other corresponds to the same situation with the momenta swapped. By already adding these contributions, it will later on be easier to integrate over al final states.

Putting it all together gives

$$\begin{aligned}&\langle\gamma(k_1,\epsilon_1)\gamma(k_2,\epsilon_2)|\,S\,|e^-(\mathbf{p}_{e^-},m_s)e^+(\mathbf{p}_{e^+},m_s')\rangle = \\ &(2\pi)^4\delta^{(4)}(k_1+k_2-p_{e^-}-p_{e^+})(-ie^2)\epsilon_1^\mu\epsilon_2^\nu\bar{v}_{m_s'}(\mathbf{p}_{e^+}) \\ &\Big(\gamma_\nu\frac{\not{p}_{e^-}-\not{k}_1+m_e}{(p_{e^-}-k_1)^2-m_e^2}\gamma_\mu+\gamma_\mu\frac{\not{p}_{e^-}-\not{k}_2+m_e}{(p_{e^-}-k_2)^2-m_e^2}\gamma_\nu\Big)u_{m_s}(\mathbf{p}_{e^-}).\end{aligned} \tag{3.20}$$

As we are looking at the ground state, it is reasonable to assume the momenta are small and thus make a non-relativistic approximation just as in Section 2.5. We thus take

$$p_{e^-}\simeq p_{e^+}\simeq\begin{pmatrix}m_e\\0\\0\\0\end{pmatrix}, \tag{3.21}$$

$$u_{m_s}(\mathbf{p}_{e^-})\simeq\sqrt{2m_e}\begin{pmatrix}\chi_{m_s}\\0\end{pmatrix}, \tag{3.22}$$

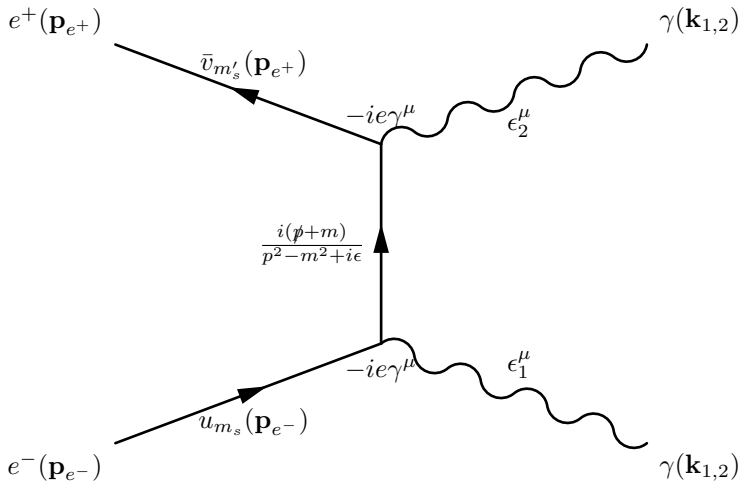


Figure 3.3: Feynman diagram for the decay of parapositronium into two photons.

and

$$\bar{v}_{m_s}(\mathbf{p}_{e^+}) \simeq \begin{pmatrix} 0 \\ \epsilon\chi_{m'_s} \end{pmatrix}, \tag{3.23}$$

where the Pauli spinors and Levi-Civita ϵ ($= \epsilon_{m_s,m'_s}$) symbol are defined as in equations A.3 and A.4. Filling it all in and working this through gives

$$\begin{aligned} &\langle\gamma(k_1,\epsilon_1)\gamma(k_2,\epsilon_2)|\, S\,|e^-(\mathbf{p}_{e^-},m_s)e^+(\mathbf{p}_{e^+},m'_s)\rangle \simeq \\ &\frac{4\alpha}{m_e}(2\pi)^5\delta(k_1^0+k_2^0-2m_e)\delta^{(3)}(\mathbf{k_1}+\mathbf{k_2})(\boldsymbol{\epsilon_1}\times\boldsymbol{\epsilon_2})\cdot\mathbf{k_1}\epsilon_{m_s,m'_s}. \end{aligned} \tag{3.24}$$

This allows calculation of the S-matrix element for parapositronium

$$\begin{aligned} &\langle\gamma(k_1,\epsilon_1)\gamma(k_2,\epsilon_2)|\, S\,|1^1S_0\rangle = \\ &\sqrt{\frac{2}{V}}\frac{2\alpha}{m_e^2}(2\pi)^2\delta(k_1^0+k_2^0-2m_e)\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2) \\ &(\boldsymbol{\epsilon_1}\times\boldsymbol{\epsilon_2})\cdot\mathbf{k}_1\int d^3p\phi(\mathbf{p}). \end{aligned} \tag{3.25}$$

From Eq. (2.59) it follows that $\int d^3p\,\phi(\mathbf{p}) = \int d^3p\,\phi(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{0}} = \psi(\mathbf{0})(2\pi)^3$. Putting it all into Eq. (3.6) and applying Fermi's trick to deal with the squared delta functions, we find

$$\begin{aligned} \Gamma(1^1S_0\to 2\gamma) =& \frac{1}{2}\int d^3k_1d^3k_2\delta(k_1^0+k_2^0-2m_e)\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2) \\ &\frac{2\alpha^2}{m_e^6}\sum_{spins}|(\boldsymbol{\epsilon_1}\times\boldsymbol{\epsilon_2})\cdot\mathbf{k}_1|^2|\psi(0)|^2. \end{aligned} \tag{3.26}$$

The factor 1/2 is because the final-state photons are indistinguishable: it prevents us from counting them all twice. It can be shown (see [13, page 168]) that

$$\sum_{spins}|(\boldsymbol{\epsilon_1}\times\boldsymbol{\epsilon_2})\cdot\mathbf{k}_1|^2 = 2|\mathbf{k_1}|^2 = 2(k_1^0)^2. \tag{3.27}$$

This allows us to evaluate the integral as

$$\begin{aligned}\Gamma(1^1S_0 \to 2\gamma) =& \frac{1}{2}\int d^3k_1 \delta(2k_1^0 - 2m_e)\frac{2\alpha^2}{m_e^6}2(k_1^0)^2|\psi(0)|^2 \\ =& \frac{\alpha^2}{m_e^4}|\psi(0)|^2 \int_{|\mathbf{k_1}|=m_e} d^2k_1 = \frac{4\pi\alpha^2}{m_e^2}|\psi(0)|^2 \\ =& \frac{1}{2}\alpha^5 m_e \simeq 8.03 \times 10^9 \text{ s}^{-1},\end{aligned} \tag{3.28}$$

where the value of the wave function from Eq. (1.16) has been used. This is the final decay rate for the state. This can easily be converted into a lifetime (not taking into account all other decay channels)

$$\tau(1^1S_0 \to 2\gamma) = \frac{1}{\Gamma(1^1S_0 \to 2\gamma)} \simeq 1.24 \times 10^{-10} \text{ s}. \tag{3.29}$$

It is also possible to calculate the decay rate for orthopositronium, using similar methods. Only the integral over all final states becomes much more complicated due to the extra photon that is emitted. It is, however, immediately clear from the Feynman diagram that the rate should be smaller due to the presence of an extra vertex. The vertex carries a factor e. When squared, as is done when finding the decay rate, this becomes a factor $e^2 \sim \alpha$ in natural units. Thus, we roughly expect the decay rate to smaller by a factor α and the life time to be larger by a factor $1/\alpha \simeq 137$. Additional differences arise because the final state space, over which is integrated, differs as there are more degrees of freedom involved. Explicit calculation gives [13, eq 13.38]

$$\Gamma(1^3S_1 \to 3\gamma) = \frac{2(\pi^2 - 9)}{9\pi}\alpha^6 m_e \simeq 7.21 \times 10^6 \text{s}^{-1}, \tag{3.30}$$

$$\tau(1^3S_1 \to 3\gamma) \simeq 1.39 \times 10^{-7} \text{ s}. \tag{3.31}$$

Chapter 4

Extra dimensions

In this chapter, I will try to determine the possibility of experimentally testing, through the decay of positronium, models which include extra dimensions. By extra dimensions, I mean dimensions other than the three space-like and one time-like dimensions we experience in everyday life. In Section 4.1, I will introduce the concepts necessary to discuss extra dimensions. Then in Section 4.2, the most important models will be presented and motivated. In Section 4.3, I will discuss the possibility and probability of particles disappearing into extra dimension. This will be applied to positronium in Section 4.4, and upper limits on the branching ratio of this invisible process will be obtained. The last section, 4.5, will be dedicated to current and planned experimental efforts. Here, I will conclude whether it is viable to search for the existence of extra dimensions through the decay of positronium or not.

4.1 Compactification and brane worlds

The everyday world as we experience it evidently consists out of four dimensions, of which there are three space-like and one time-like. Nevertheless, there are physical theories around according to which the world is actually composed of more dimensions, not least of which the different versions of string theory (e.g. M-theory requires eleven dimensions for its mathematical description). The number of dimensions in a model is often written as $(4+n)$, where n is the number of extra dimensions conjectured. Models imposing such extra dimensions need to explain how it is possible that these dimensions are hidden from us. There are two main strategies utilized to this end by different models.

One way to deal with extra dimensions is to *compactify* them. While our regular four spacetime dimensions are, as far as we know, infinite in extent, a compactified dimension is not. Rather, the dimension closes on itself, as if it had been folded into a circle. A compactified dimension z is characterized by its compactification radius R. z then runs from $-\pi R$ to πR, where the points $z = -\pi R$ and $z = \pi R$ are identified. An alternative description is possible through the 'angle' by $z = \phi R$. Now, if z is an extra and compact dimension, it becomes hidden at scales much larger than R. But at scales much smaller than R, z appears to be a non-compact extra dimension. This is best illustrated by

considering two space-like dimensions, of which one is compactified. This space then can be represented by the surface of an infinite cylinder with radius R, where the compactified dimension is along the circumference. If we look at the cylinder from far away, it appears to us as a one-dimensional line. However, when we take a very close look, we only see a more or less flat surface and we could hardly tell which direction is along the compactified dimension and which is not.

The second method of hiding extra dimensions does not necessarily require them to be small. Rather, we can imagine ourselves to be living in a four-dimensional subspace, a *brane world*, of a $(4+n)$-dimensional reality. All fields of the standard model must then be confined to this brane in some way, so that ordinary matter is localized on it. Gravity, however, cannot be so easily confined as it is not a field embedded in spacetime but rather the dynamics of spacetime itself (more on this in Section 4.2). One way a brane could be constituted is by a topological defect, a domain wall separating two classical vacua of a scalar field. This would create a $(4+n-1)$-dimensional hypersurface. As our brane world must be a four-dimensional, it is less easy to create the brane world if $n > 1$, but it might result from the overlap between sufficient hypersurfaces whatever the value of n. Matter will experience a potential well at the topological defect due to interaction with the scalar field, thus effecting localization.

4.2 Models with extra dimensions and their motivation

In this section, some models of extra dimensions are described and motivated. They will be treated chronologically.

4.2.1 Kaluza-Klein theory

The first important model to contemplate the existence of extra dimensions is *Kaluza-Klein* (KK) theory. It was first proposed by Kaluza in 1919 as a generalization of general relativity to five dimensions, and further worked out by Klein who introduced the concept of compactification in its current form. The theory is thus (4+1)-dimensional, where the extra dimension (denoted z) is compact and very small. Its compactification radius is of the order of the Planck length (10^{-33} cm) [15, chapter 2], or at least smaller than 10^{-17} cm [19, page 2].

KK theory has some very interesting consequences. Consider a $(4+1)$-dimensional free massless scalar particle in a flat metric. Its wave function is required to solve the five-dimensional generalization of the Klein-Gordon equation,

$$\left(\partial_\mu \partial^\mu - \frac{\partial^2}{\partial z^2}\right)\psi = 0. \tag{4.1}$$

This is solved by the eigenfunctions

$$\psi = e^{ip_\mu x^\mu} e^{ip_z z}, \tag{4.2}$$

where p_μ is interpreted as regular four-momentum and x_μ is the four-vector containing the regular four-dimensional spacetime coordinates. Because the z

dimension is compact, the wave function is subject to the condition $\psi(x_\mu, z) = \psi(x_\mu, z + 2\pi R)$. This quantizes the z momentum because only wavelengths which fit an integer times within the compactification circumference are allowed for. The solution is $p_z = n/R$, where n is an integer (zero included). Now, from Eq. (4.1) we find

$$p_\mu p^\mu = \frac{n^2}{R^2}, \tag{4.3}$$

that is, from our four-dimensional perspective, it seems the particle has a rest energy, i.e. it is *massive*. While it may not have a five-dimensional mass, the particle effectively does have a four-dimensional mass. The different excitation levels (different values of n) are called *Kaluza-Klein modes*, and each would be observed by us as a different particle, each with a distinct mass given by

$$m_n = \frac{|n|}{R}. \tag{4.4}$$

Massless KK modes are refered to as zero modes. Similar arguments can be made for different kinds of fields and for particles which already are massive (through e.g. the Higgs mechanism). So KK theory predicts a Kaluza-Klein tower of particles corresponding to the particles we know. But since we never observed such partners, the theory can only hold if we do not have access to energies in the order of $1/R$ so that we are only concerned with $n = 0$ modes. This allows us to infer that R must be very small indeed.

Another interesting consequence of the theory, which was one of its main motivations, is what happens when we apply a KK decomposition, as above, to the graviton (the quantum of the gravitational tensor field). From a four-dimensional perspective, we do not only obtain general relativity, but we also find the vector field of electromagnetism (the 'graviphoton') and an additional scalar field associated with fluctuations in the compactification radius (the 'radion'). This is a nice unification of forces! However, the presence of a massless radion alters the four-dimensional gravitational predictions in a way which heavily conflicts with data [18, page 22]. Therefore, it should acquire mass through some mechanism, which is equivalent to stabilizing the compactification radius.

4.2.2 ADD model

The Arkani-Hamed - Dimopoulos - Dvali (ADD) model, also known as the model with large extra dimensions, was first proposed in 1998 [2] and applies both of the methods of hiding extra dimensions discussed in Section 4.1. In this $(4+n)$-dimensional mode, there is room for many extra dimensions which do not need to be as small as in KK theory, since standard model matter is confined to a very thin brane and thus for these particles no KK tower is predicted at accessible energies. The main motivation for this model and a renewed interest in the possibility of extra dimensions is to solve the *hierarchy problem*.

The hierarchy problem concerns the observed hierarchy between the strength of on one hand the electroweak interaction and on the other hand gravitational force. In natural units, both strengths can be expressed using a mass scale. For gravitational interaction, this is the Planck mass $M_4 \equiv \frac{1}{\sqrt{G_4}}$, where G

is Newton's constant. The 4 indicates that we are talking about the effective values at macroscopic scales in four dimensions. For the electroweak interaction, this scale is $m_{EW} \equiv \frac{1}{\sqrt{G_F}}$, where G_F is Fermi's constant. Comparing the observational values of the two reveals the hierarchy[2, page 1],

$$, m_{EW}/M_4 \sim 10^{-17} \tag{4.5}$$

which means gravity is very weak compared to the other forces. It has been the effort of much theoretical physics (e.g. supersymmetry) to explain or do away with this hierarchy. The solution offered by the ADD model is that there is no hierarchy at all. While the Planck mass as we observe it may be very large, it is possible that the more fundamental mass scale M_{4+n} is actually of the same order as $m_{EW} \simeq 1$ TeV. The relation between the different Ms can be obtained through classical considerations.

According to the Newtonian view of gravity, all mass generates a gravitational field $\mathbf{g}(\mathbf{r})$, which is the gravitational acceleration of any mass at position $\mathbf{r}$. The divergence of this field is proportional to the mass density ρ: $\boldsymbol{\nabla} \cdot \mathbf{g}(\mathbf{r}) = -4\pi\rho(\mathbf{r})G_4$ (Gauss' law for gravity). Consider a point mass m_1 at the origin. Applying Gauss's theorem then gives

$$\iint\limits_{|\mathbf{r}'|=r} \mathbf{g}(\mathbf{r}') \cdot d\mathbf{a} = \iiint\limits_{|\mathbf{r}'|=0}^{|\mathbf{r}'|=r} -4\pi\rho(\mathbf{r}')G_4 dV, \tag{4.6}$$

which, by considering spherical symmetry, becomes

$$\mathbf{g}(\mathbf{r}) = -\frac{G_4 m_1}{r^2}\hat{r}. \tag{4.7}$$

This reduces to Newton's force law between two masses,

$$\mathbf{F}(\mathbf{r}) = m_2\mathbf{g}(\mathbf{r}) = -\frac{G_4 m_1 m_2}{r^2}\hat{r}. \tag{4.8}$$

This argument is easily generalized to $(4+n)$ dimensions. We then have, using spacial vectors with $(3+n)$ components, $\partial^A g^A(x) \sim -\rho(x)G_{4+n}$. Here, A runs from 1 to $3+n$ and is summed over. The right hand side of Eq. (4.6) remains m_1 for a point mass while the left hand side becomes an integral over a $(2+n)$ dimensional spherical shell with radius r, which is proportional to r^{2+n}. We thus find a force law of the form

$$F(x) \sim -\frac{G_{4+n} m_1 m_2}{r^{2+n}}\hat{x}, \tag{4.9}$$

with $\hat{x} = x/\sqrt{x^A x^A}$. Now, if the n extra dimensions have compactification radius R, gravity will behave like this as long as $r \ll R$. But at $r \gg R$, the surface integral over the extra dimensions is limited to R. There is no more space left for the gravitational field lines to diverge in these dimensions. They are diluted over a volume R^n. This is illustrated for the case of one infinite and one compact extra dimension in Fig. 4.1 using field lines of the gravitational

force field. Considering the effect mathematically, the left hand side of Eq. (4.6) becomes (roughly)

$$\int_{x^4=0}^{x^4=R} ... \int_{x^{3+n}=0}^{x^{3+n}=R} \iint_{|x|=r} g^A(x) da^A = \int_{x^4=0}^{x^4=R} ... \int_{x^{3+n}=0}^{x^{3+n}=R} \iint_{|\mathbf{r}'|=r} \mathbf{g}(\mathbf{r}) d\mathbf{a} dx^4 ... dx^{3+n}$$
$$= R^n \iint_{|\mathbf{r}'|=r} \mathbf{g}(\mathbf{r}) d\mathbf{a}, \tag{4.10}$$

and we obtain force law

$$F(\mathbf{r}) \sim -\frac{G_{4+n} m_1 m_2}{R^n r^2} \hat{r}. \tag{4.11}$$

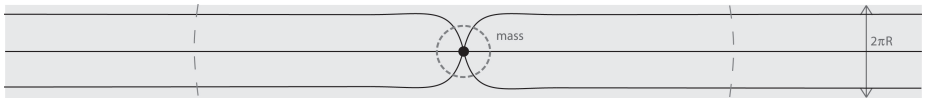


Figure 4.1: The transition between microscopic and macroscopic gravity for two dimensions where one is compact. The dot represents a mass and the lines originating from it the gravitational field lines. On small scale, the line density (and thus the force) falls off according to the two-dimensional $1/r$ law (see first circle). On large scale, the lines have no more space to diverge and become straight, diluting one-dimensionally like $1/r^0$ (see second 'circle').

If we equal this formula to the macroscopically confirmed law of Eq. (4.8), we find

$$G_4 = \frac{G_{4+n}}{R^n}. \tag{4.12}$$

By definition, $G^4 = \frac{1}{M_4^2}$. We similarly want to define any G in terms of a mass scale. Because the mass dimensions in Eq. (4.12) must be equal on each side, and distance has mass dimension $[R] = 1$, this can only be done through

$$G_{4+n} \equiv \frac{1}{M_{4+n}^{2+n}}. \tag{4.13}$$

Thus we obtain the important relation

$$M_4^2 \sim M_{4+n}^{2+n} R^n. \tag{4.14}$$

This allows for a solution to the hierarchy problem by setting the most fundamental scale for gravity to $M_{4+n} \sim m_{EW}$. Macroscopic gravity becomes as we know it on the condition that

$$R \sim m_{EW}^{-1} \left(\frac{M_4}{m_{EW}} \right)^{\frac{2}{n}} \sim 10^{\frac{32}{n} - 17} \text{cm}. \tag{4.15}$$

This reveals that if the amount of extra dimensions is not very large, they can be quite big. One way of detecting these extra dimensions would be by observing the transition of $F \propto r^{-2}$ to $F \propto r^{-(n+2)}$ behavior of gravity at the

scale of R. If $n = 1$, R would be of solar system scale and thus this possibility must be excluded as we have not yet observed such a deviation from Newtonian gravity. For $n = 2$, R would be in the range of 100 μm to 1 mm. This is excluded by measurement of the gravitational force up to 25 μm [20]. Furthermore, it has been shown through analyzing gamma ray data from neutron stars that $n = 2$ and $n = 3$ are excluded if all dimensions have the same compactification radius [5]. It is possible to envision scenarios where this is indeed not the case, but this can lead to complex topological structures (equally large dimensions have the simple topology of a torus).

4.2.3 Randall-Sundrum model

In the above models, it has been assumed that $(4+n)$-dimensional spacetime is simply the product space of a four-dimensional spacetime and an n-dimensional spacetime. The Randall-Sundrum (RS) model is a model with (4+1) dimensions which examines other possibilities. The model comes in two versions. The first, RS1, was proposed in 1999 [17], and the second, RS2, in a quick follow-up paper later that year [16]. I will here only consider $(4 + 1)$-dimensional RS models.

In general, the metric of spacetime is given by

$$ds^2 = g_{MN} dx^M dx^N, \tag{4.16}$$

where M and N run from 0 to 4 and $x^4 = z = R\phi$, the coordinate along the extra dimension with compactification radius R. In the case of a flat metric, where g_{MN} is the five-dimensional generalization of $\eta_{\mu\nu}$ as defined in Eq. (A.1), this becomes

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu - R^2 d\phi^2. \tag{4.17}$$

In this factorizable metric, there is no dependence on the absolute value of any of the coordinates. Instead now consider

$$ds^2 = a^2(\phi) \eta_{\mu\nu} dx^\mu dx^\nu - R^2 d\phi^2, \tag{4.18}$$

where μ runs from 0 to 3. Here the *warp factor* is introduced,

$$a(\phi) = e^{-kR|\phi|}, \tag{4.19}$$

where k is a constant of the order of the Planck scale. It is determined by the brane potential energy and the bulk cosmological constant. If we assume four-dimensional Poincaré invariance, this is the solution of the Einstein equations from general relativity for a setup containing two brane worlds. It is this setup which corresponds to the RS1 model.

We consider each brane to have a certain energy density, which is also called brane tension. This must of course be the case, as our brane world contains a lot of it (in the form of (dark) matter and dark energy). One of the branes has a positive tension and is located at $z = 0$, while the other has a negative tension and is located at $z = \pi$. It is this brane which is our visible world according to the model. When the tension is neglected, an ADD-like model is obtained. Due to the compactification and the symmetry this setup introduces, there is an effective identification between z and $-z$. The spacetime geometry we obtain is

that of two pieces of five-dimensional anti-de Sitter space glued together back to back (with anti-de Sitter radius $1/k$).

A very interesting consequence of the warp factor is the possibility to solve the hierarchy problem without the need to invoke many or large extra dimensions. Let us consider both M_{EW} and M_4 at the visible brane. What happens when a particle moves from the positive tension brane to the negative tension brane? Both space and time are stretched, increasing any wave length and decreasing any frequency. This has the general effect of *red shifting* all energies, including masses, by a factor $a(\phi)$. If we consider masses at the positive tension brane to be fundamental, this means that for the masses on the negative tension brane we have

$$m = e^{-kR\pi} m_0. \tag{4.20}$$

This scaling also holds for m_{EW}, which is related to the mass of the Higgs particle.

For gravity, another scaling relation holds. From Eq. (4.14), we expect $M_4^2 = M_5^3 R$, where M_4 is the four-dimensional effective Planck mass at the negative tension brane. Due to the warp factor, however, there is an extra factor involved (which can be found by considering the gravitational action from general relativity, see [17, Eq. (15)]), giving

$$M_4^2 = M_5^3 R \int_{-\pi}^{+\pi} d\phi \, a^2(\phi) = \frac{M_5^3}{k}[1 - e^{-2kR\pi}]. \tag{4.21}$$

Now we can try to solve the hierarchy problem by simply setting $M_5 = m_{EW,0} = e^{-kR\pi} m_{EW}$. We see that we need $kR \simeq 10$ to solve this. This solution allows to set $k \sim 1/R \sim M_5$, minimizing hierarchies (while in the ADD model, the compactification scale $1/R^n \ll m_{EW}$ introduces a new hierarchy) and introducing only a very small extra dimension. In this case, $M_4 \simeq M_5$; the Planck mass hardly depends on at which brane you are. This is because the gravitational interaction involves an integral over the extra dimension and thus becomes dominated by the high-energy region. The electroweak energy scale however is red shifted, and it is because of this that we would be observing an hierarchy between the two.

What we find are two branes, one positive tension brane where particle masses are of the order of the Planck mass $M_4 \simeq m_{EW,0}$ and one negative where masses are of order $m_{EW} \simeq e^{-kR\pi} M_4 \simeq 10^{-15} M_4$, i.e. of TeV order. Because of this the first brane is often called the Planck brane and the second the TeV brane. Note that on the Planck brane, there is no hierarchy problem while on the TeV brane gravity is relatively weak (as m_{EW} is red shifted and the gravitating masses are a lot smaller). The situation also allows for an alternate interpretation where m_{EW} is the fundamental scale and gravity is weak because the graviton zero modes are localized at the Planck brane (their wave function is of the form of the warp factor). Their KK modes have a splitting of order $1/R \simeq m_{EW}$ and are thus not of interest to low energy physics. This interpretation can be realized by a change of coordinates $x_\mu \to e^{kR\pi} x_\mu$, in which no rescaling of masses is required between the branes. The different interpretations can be understood by considering that we use gravity (i.e. the Planck mass) to

measure masses. If at the TeV brane all masses measure less, did the masses become lighter or did gravity become weaker? The moral of the story is that only the magnitude of the hierarchy (Eq. (4.5)) is an observable.

It is also possible in this model to take the limit $R \to \infty$, which means z is no longer compactified. This model of five infinite dimensions, in which we live on the positive tension brane (the negative tension brane has been removed), is RS2. In contrast to RS1, RS2 does not provide a solution to the hierarchy problem, but it importantly shows us that our observations do not exclude an additional infinite dimension. The zero mode of the graviton has in this model too a wave function with z-dependence of the form of the warp factor, which means that it is still normalizable and localized at the brane. The exchange of this mode results in $1/r^2$ gravity. Additionally, there is now a continuum spectrum of KK graviton modes as the quantization energy $1/R$ from Eq. (4.4) now goes to zero. Their wave function has complicated z-dependence composed of Bessel functions (see [19, Eq. (46)]), but is proportional to $\sqrt{m}$ at the brane. Thus, small mass modes couple only weakly while large mass modes are not produced in low energy physics. The Yukawa potential the KK modes give rise to provides a $1/r^3$ gravitational correction which is negligible at distances exceeding $1/k \sim 10^{-19}$ m.

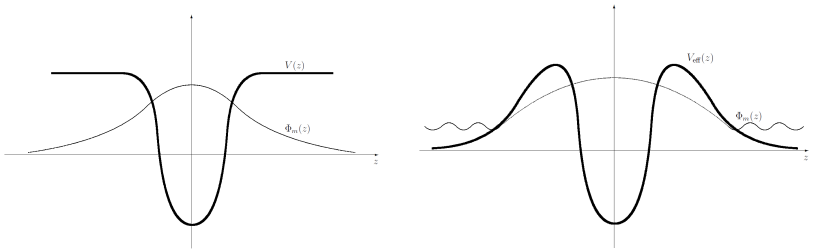
4.3 Tunneling into extra dimensions

In a model with extra dimensions, one important question is whether particles would be able to travel along such an extra dimension and disappear from view. In KK theory, this is impossible due to the compactness of the extra dimension; the compactification radius is much smaller than the Compton wave length of particles and thus there is nowhere for them to travel to. In the ADD model, extra dimensions are bigger and such disappearance would certainly be possible if particles manage to escape the brane. Interestingly, particles could orbit around the extra dimensions, appearing and disappearing again, leaving behind continuous 'fireworks'.

Randall and Sundrum in [17] and [16] originally only identified the possibility of graviton KK modes leaving the brane in their models, but it was found in [6] for scalar and fermion fields that while their zero modes are fully localized, any state with nonzero four-dimensional mass is only metastable when trapped at a domain wall. This is the case only in the presence of a warp factor, which has the effect of increasing energy away from the brane, thus allowing particles which do not have the energy to overcome locally the potential barrier of the domain wall to be free at large z. They can therefore tunnel away from the brane and move towards infinite z as a free particle, as is illustrated in Fig. 4.2.

If all these massive particles can just suddenly tunnel away according to the model, why do we not reject it on the grounds that we do not observe it happening all the time? This is because the 'decay' mode is extremely suppressed for small masses. It is shown in [6] that for scalars the decay width is $\Gamma = (\pi/16)m(m/k)^2$, while for fermions the width is suppressed exponentially strong. Also, there are many scenarios where charge universality implies that

charged particles cannot leave the brane at all, thus further limiting our expectations. Additionally, most particles we encounter everyday are in bound states, thus making it energetically unfavorable to tunnel away for individual particles and further suppressing this disappearance.



(a) Without gravity. (b) With gravity.

Figure 4.2: Wave function of massive scalar particle in (effective) potential with and without gravitational warp factor (Figs. 8 and 9 from [19]).

One problem with keeping our world on a brane is the localization of gauge fields, as this is not possible using a domain wall. There is a limited amount of field-theoretic mechanisms which would be able to do the trick. A very attractive option is to invoke the brane's gravity itself to localize particles. One problem is that although the scalar and gravitational fields are naturally localized on a positive tension brane with decreasing warp factor, fermions are localized only on a negative tension brane with increasing warp factor. Clearly, gravity alone will not do the trick and to also bind fermions to our brane, an additional mechanism such as a domain wall is still needed. But a positive brane is able to localize vector fields. This is however only possible is we go beyond the RS2 model and additionally impose the existence of compactified, warped dimensions. We obtain a $(4 + n + 1)$-dimensional model in which z again represents the infinite dimension and the compactified dimensions are represented by 'angular' coordinates $-\pi < \theta^i \leq \pi$ with compactification radii R^i. The metric thus becomes

$$ds^2 = a^2(z)\left(\eta_{\mu\nu}dx^\mu dx^\nu - \delta_{ij}R^i d\theta^i R^j d\theta^j\right) - dz^2, \tag{4.22}$$

with again $a(z) = e^{-k|z|}$.

The measure according to which eigenfunctions of fields must be normalized in the z dimension is then $\int dz a^n(z)$. This allows for normalization of gauge fields only if $n \geq 1$. In this case, our spin 1 field is localized. By a change of coordinates and field redefinition, the eigenfunction equation of both spin 1 and spin 2 fields ([7] and [16] respectively) can be formulated in terms of a regular quantum mechanical problem in a volcano potential such as in Fig. 4.2, with a dirac delta function at the origin. This potential allows for a zero mode which has energy eigenvalue 0. But just as we saw before, there also is a continuum of KK modes with nonzero energy, which correspond to massive particles on the brane and which are metastable with respect to disappearance into the z dimensions. Thus, while photons are usually massless, we expect a

virtual photon with mass to be able to tunnel away from the brane. The decay width is then similar to the scalar case [7, Eq. (8)]:

$$\Gamma = c_n m (m/k)^n, \tag{4.23}$$

where c_n is the numerical coefficient given by $c_n = (\pi n)/(2^{n+1}\Gamma[n/2+1]^2)$. In the second expression, Γ denotes the gamma function.

4.4 Disappearing positronium

In this section, the $(4+n+1)$-dimensional model from the last section will be assumed. An interesting possibility of probing disappearance into extra dimensions is offered by the process of electron-positron annihilation. When the two antiparticles are brought together, most of their quantum numbers sum up to zero. Thus, disappearance of the particles or (part of) their decay products will likely violate a minimum amount of conservation laws on the brane. Since suppression of processes violating conservation laws is generally observationally well-established, any higher dimensional theory should not (significantly) lift this suppression. Thus, it can model-independently be expected that if disappearance into extra dimensions is possible, electron-positron annihilation offers one of the better opportunities for observing this.

One approach would be to let the two particles collide at high energy in an accelerator. The signature of extra dimensions would then mainly be the process $e^- + e^+ \to \gamma +$ nothing, where 'nothing' is a photon which acquired momentum in the direction of the infinite extra dimensions and thus leaves the brane. Note that this violates conservation of higher-dimensional momentum. Momentum conservation follows from translational symmetry, but due to the warp factor this symmetry is not present and thus we should not expect such conservation law to hold. The signal, however, will be very hard to spot among the $e^+ + e^- \to \gamma + \nu\bar{\nu}$ standard model background unless $n \geq 5$ [3].

Alternatively, we can take a low-energy approach and use positronium to search for extra dimensions, which is the focus of this thesis. A great advantage of using positronium is that it is a very simple system which can be described by quantum electrodynamics alone. The standard model allows for very accurate calculation of its properties such as its decay width, which allows a search for invisible decay. Also, while the signal in an accelerator only becomes sizable for larger n, positronium decay offers the best chance of seeing disappearance if n is low. Because orthopositronium is relatively long-lived (see Section 3.3), the contribution of decay into extra dimensions will be relatively large and it will be easiest to detect the invisible decay mode using this state of positronium.

How can positronium tunnel into extra dimensions? The electron and positron cannot be expected to disappear themselves as they are in a bound state and thus the process would be energetically unfavorable. There is, however, one process which is clearly accessible even at low energies. It is through virtual annihilation, discussed in Section 2.5. In this process, the electron and positron annihilate to create a single photon. To conserve four-momentum, the photon must be massive. It is a virtual photon which is short-lived. Such a massive

photon however does have a finite probability to tunnel away from the brane. Its decay width, $\Gamma(\gamma^* \to \text{extra dim})$, is given by Eq. (4.23). This can be used to find the decay rate of orthopositronium into extra dimensions.

What we want to calculate is the decay rate $\Gamma(\text{o-Ps}) \to \gamma^* \to \text{extra dim}) \equiv \Gamma$. Let N_1 be the number of orthopositronium atoms, N_2 the amount of virtual photons, and N_3 the amount of particles that have decayed into the extra dimension. Then, by definition,

$$\frac{dN_3}{dt} = \Gamma N_1. \tag{4.24}$$

The only part of the path into the extra dimension we know about, however, is how many of the virtual photons leave the brane:

$$\frac{dN_3}{dt} = \Gamma(\gamma^* \to \text{extra dim}) N_2 \equiv \Gamma_\gamma N_2. \tag{4.25}$$

So, in order to calculate the decay rate, we need to know what the population of virtual photons is. Since we do not want Γ to be a function of time, it is best to assume the positronium and photons have been allowed to get into equilibrium such that $N_2 = \eta N_1$. In order for this to be static, it must be assumed that Γ_γ is much smaller than the other transition rates (which will turn out to be fair). If we find η, we have $\Gamma = \eta\Gamma_\gamma$.

One approach to calculate η is as a ratio of lifetimes,

$$\eta = \frac{\tau(\gamma^* \to \text{extra dim})}{\tau(1^3S_1 \to \gamma^*)} = \Gamma(1^3S_1 \to \gamma^*)\tau(\gamma^* \to \text{extra dim}). \tag{4.26}$$

The lifetime of the virtual photon can be estimated using the energy-time uncertainty relation, $\Delta E \times \tau \geq 1/2$. Since $\Delta E = m$, we can use $\tau \sim 1/m$. The decay rate of orthopositronium into a virtual photon is not quite so straightforward since it is not allowed to have a virtual particle as a final state in a Feynman diagram. However, a hand-waving argument can be made by looking at the virtual annihilation process in Fig. 2.1. If we calculate the rate of this process (analogues to the approach in Chapter 3), we find $\Gamma \sim m\alpha^5$, where three powers of α are due to the positronium wave function and two due to the vertices. However, if we 'cut off' the part of the Feynman diagram where the photon reverts to positronium, we lose one vertex and thus effectively one power of α. It thus seems plausible that $\Gamma \sim m\alpha^4$. Indeed, comparing to [8, Eq. (10)] gives (although without motivation, so the result might be considered not extremely reliable) $\eta = \alpha^4/4$.

Thus, we find

$$\Gamma = \frac{c_n}{4}\alpha^4 m \left(\frac{m}{k}\right)^n. \tag{4.27}$$

Here, m is the virtual-photon mass and thus, by conservation of four-momentum, the positronium mass $\simeq 2m_e$. When using Eq. (3.30) for $\Gamma(1^3S_1 \to 3\gamma)$ and assuming $\Gamma_{tot}(1^3S_1) \simeq \Gamma(1^3S_1 \to 3\gamma)$, where $\Gamma_{tot}(1^3S_1)$ is the sum of all decay rates of orthopositronium, we find the result for the branching ratio of the decay

into extra dimensions,

$$\begin{aligned}\mathrm{Br}(1^3S_1 \to \text{extra dim}) &= \frac{\Gamma}{\Gamma_{tot}(1^3S_1)} \simeq \frac{\Gamma}{\Gamma(1^3S_1 \to 3\gamma)} \\ &= c_n \frac{9\pi}{4(\pi^2-9)} \alpha^{-2} \left(\frac{m}{k}\right)^n \equiv 1.52 \times 10^5 \times c_n \left(\frac{m}{k}\right)^n .\end{aligned} \tag{4.28}$$

To acquire quantitative estimates of the branching ratio, we must put limits on the constant k. Although an infinite extra dimension does not solve the hierarchy problem, it is attractive to at least not introduce yet a new hierarchy and let $1/k \simeq m_{EW}$. Since $m \simeq 1$ MeV and $m_{EW} \simeq 1$ TeV, this gives

$$\mathrm{Br}(1^3S_1 \to \text{extra dim}) \simeq 1.52 c_n \times 10^{5-6n}. \tag{4.29}$$

An empirical bound can be obtained by analyzing data from LEP [1] on the decay of the Z boson, as is done in [8]. The rate of the invisible decay modes of Z can be obtained by subtracting from the total decay rate the visible modes, i.e. the leptonic and hadronic decay rates. From the invisible decay rate, the standard model background of decay into neutrinos must be subtracted. If we assume that the only invisible decay beyond the standard model is due to extra dimensions we find

$$\Gamma(Z \to \text{extra dim}) = -2.7 \pm 1.5 \text{ MeV}. \tag{4.30}$$

If we assume this decay rate must be positive and renormalize the probability distribution accordingly, and additionally assume the decay is in accordance with Eq. (4.23), we find with 95% confidence level

$$\Gamma(Z \to \text{extra dim}) = c_n m_z \left(\frac{m_z}{k}\right)^n < 2.0 \text{ MeV}. \tag{4.31}$$

From this, we can find a lower limit on k and an upper limit on $\mathrm{Br}(1^3S_1 \to$ extra dim). For k we find

$$k > 9.1\sqrt{c_n} \times 10^{4.66/n-2} \text{ TeV}, \tag{4.32}$$

the graph of which is shown in Fig. 4.3. We see that $k \sim 1$TeV is not allowed for $n < 3$. Using Eq. (4.28), we can also use this result to set an upper limit on the branching ratio of the disappearance of positronium,

$$\text{Z boson:} \qquad \mathrm{Br}(1^3S_1 \to \text{extra dim}) \lesssim 3.33 \times 10^{-4.95n}. \tag{4.33}$$

Other, for low n much more stringent, conditions are obtained from analyzing the astrophysical implications the tunneling effect would have. In [7], red giant (RG) stars, horizontal branch (HB) stars and the cooling process of supernova SN1987A are examined. They show that in order for k to be indeed of the TeV order, it is required that $n \geq 4$. What is more, each analysis leads directly to bounds on the branching ratio:

$$\begin{aligned}&\text{RG:} \qquad \mathrm{Br}(1^3S_1 \to \text{extra dim}) \lesssim 2 \times 10^{-24+1.75n}, \\ &\text{SN:} \qquad \mathrm{Br}(1^3S_1 \to \text{extra dim}) \lesssim 2 \times 10^{-15+1.48n}, \\ &\text{HB:} \qquad \mathrm{Br}(1^3S_1 \to \text{extra dim}) \lesssim 2 \times 10^{-25+2.8n}.\end{aligned} \tag{4.34}$$

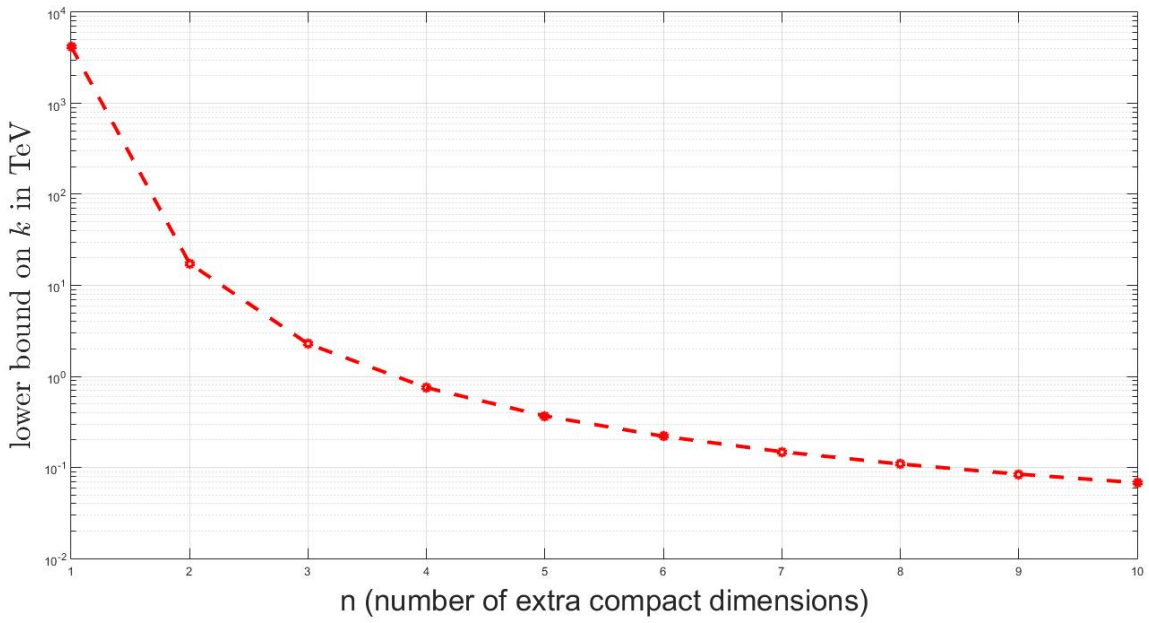


Figure 4.3: Lower limit on k from Z boson invisible decay.

4.5 Experimental status and outlook

Now that we have quantified the disappearance of positronium due to extra dimensions, we are in a position to judge whether the search for this process is a viable one. We must take a look at the current experimental limits, see how far they could be improved in the (near) future and check in what measure it could be possible to detect the tunneling into an extra dimension. I will finish by giving my opinion on the subject.

The best current experimental bound on the branching ratio is

$$\text{Current experimental limit: } \mathrm{Br}(1^3S_1 \to \text{extra dim}) < 2.8 \times 10^{-6}. \qquad (4.35)$$

This bound was obtained by the Tokyo group [12] and already dates from 1993. It was acquired using a 4π calorimeter. Positrons originate form a radioactive ^{22}Na source and pass through scintillator material, where they create a light pulse which is registered by a photomultiplier. This, together with the detection of a 1275-KeV photon emitted in the β^+ decay, functions to tag the radioactive-decay event. Next, the positrons are stopped in a target of silica aerogel, where they can interact with the electrons available in the material to either annihilate or form positronium and subsequently decay. This is all hermetically surrounded by thick CsI counters, monitoring the energy emitted by the annihilation processes. An invisible orthopositronium decay would manifest itself as a trigger signal with no energy deposition, a gap of $\sim$ 1MeV. They have been unable to detect such an event. This is converted to Eq. (4.35), which is at 90% confidence level, using estimates of the orthopositronium formation

probability.

There are efforts being made to increase the experimental sensitivity and thus hopefully detect the disappearance of positronium. A nice overview of these setups is provided in [9]. One such effort is a collaboration at Lawrence Berkely and Lawrence Livermore National Laboratories, which uses liquid organic scintillator in an attempt to reach 10^{-9} sensitivity [21]. Another experimental opportunity is offered by the low-energy positron/electron storage ring LEPTA, which could be able to create slow orthopositronium beams. However, the sensitivity in the branching ration would only achieve $\sim 10^{-6}$.

Another attempt is made by a group in Zürich. It consists of a 4π calorimeter similar to the one described above, only with lower photon detection inefficiency. Also, the experiment aims to detect a more pure orthopositronium annihilation spectrum by subtracting the spectrum obtained when the calorimeter is filled with air, in which, due to paramagnetic O_2 which can exchange spin angular momentum with positronium, there will be much less orthopositronium present. It is hoped the experiment can reach branching ratios of $\sim 10^{-8}$ [4]. The same Zürich group also works on an experiment which involves a pulsed slow positron beam impinging on a target within a 4π calorimeter, which aims at a sensitivity of 10^{-7}.

Summarized, the current experimental bound on the branching ratio is 2.6×10^{-6}. An optimistic estimate of the sensitivity in the branching ratio in the near future is $\sim 10^{-9}$. This will either result in the discovery of disappearing positronium, or further increase the experimental bounds on such processes. Is it likely that this higher sensitivity will allow us to observe the process of virtual annihilation, followed by disappearance into an infinite extra dimensions? And will it otherwise provide useful limits to the theory? To answer this, we must compare it to the limits set in Section 4.4. This is done graphically in Fig. 4.4.

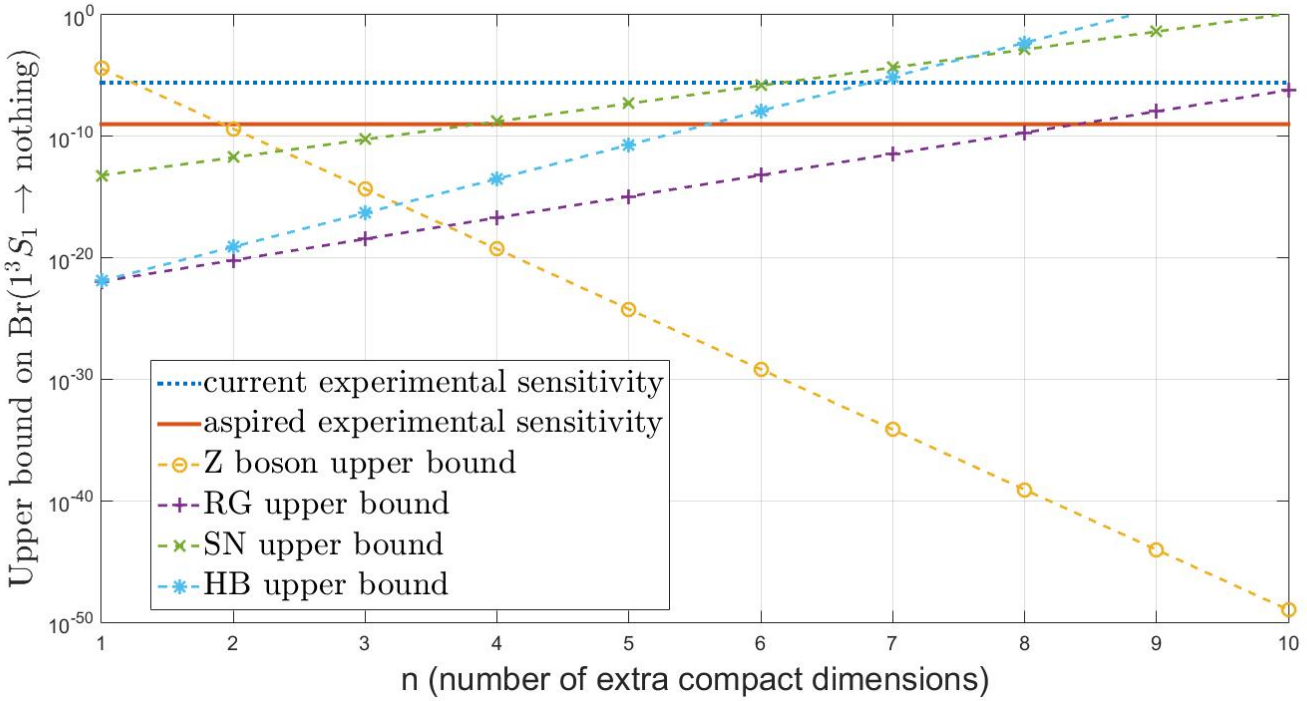
If we trust all the upper bounds set on the branching ratio, we need only look at the most stringent one. When, for each value of n, we discard all but the smallest upper limit, we arrive at Fig. 4.4b. This is not a pretty picture. It is shown in [9] that if $k \simeq 17$ TeV, which is the Z boson limit for $n = 2$, we might just be able to detect decay into an extra dimension when experiments are able to probe branching ratios of $\sim 10^{-9} - 10^{-8}$. But if we take into account also the astrophysical evidence, the discrepancy between current experiment and the upper limit of the branching ratio is not three but rather thirteen orders of magnitude (at $n = 3$, where the resultant limit is the least stringent). As a sensitivity of $\sim 10^{-19}$ appears to be currently very far from our reach, I can only conclude that there is not much hope of detecting the presence of extra dimensions through the decay of positronium. Also, whatever conditions such an experiment could impose on the theories has already been imposed more strongly by the other upper limits.

Note how the resultant upper bound is small for all n, because the astrophysical considerations supply a small limit for small n (but not at all for large n) while the evidence from the Z boson decay only provides a stringent limit at larger n (the turning point is $n = 3$). There thus might still be hope if one of the

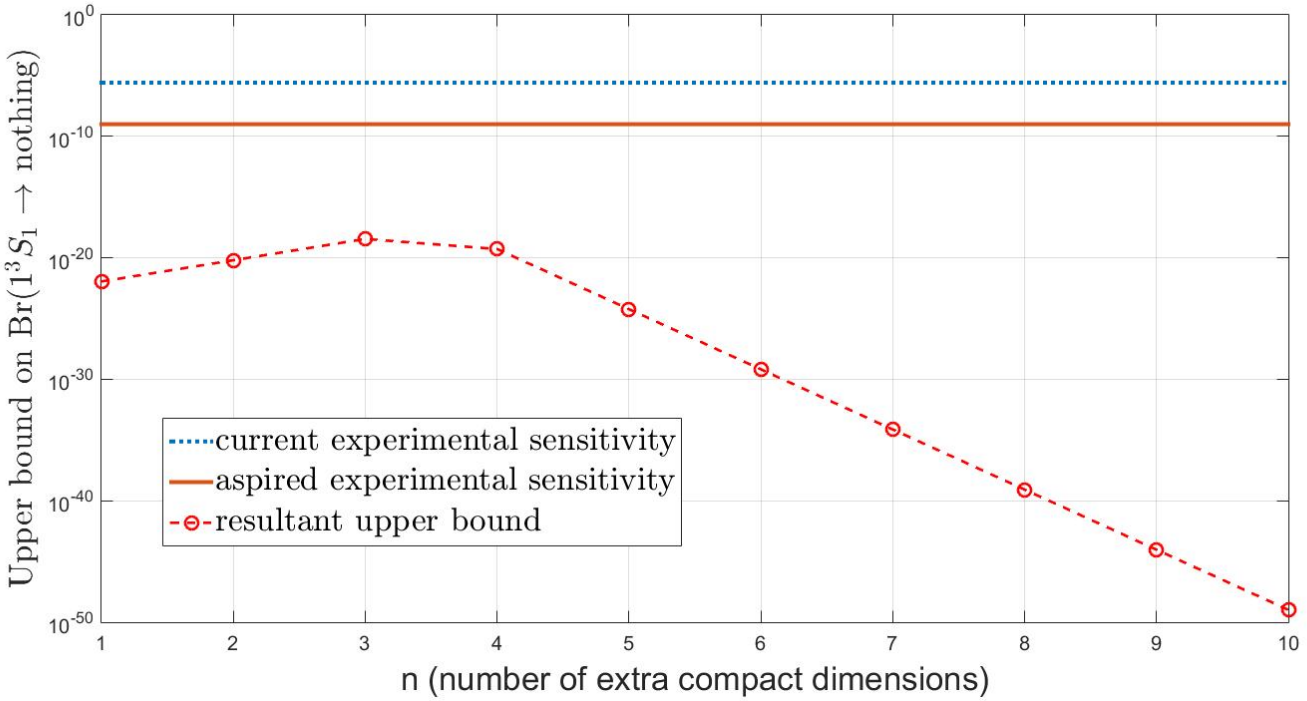
two limits proves incorrect. I think the method through which the Z boson limit was acquired is not very trustworthy, as the data suggested the decay rate into extra dimensions to be negative. I believe this to be an indication of insufficient accuracy in either the measurement of the visible and total decay rates or in the calculation of the standard model neutrino background, as the discrepancy can certainly not be explained by invoking extra dimensions. If anything, it suggests that the decay rate into extra dimensions is (a lot) smaller than the accuracy as the contribution was unable to make the missing invisible decay rate positive. As this accuracy is 1.6 MeV, I think the constraint $\Gamma < 2.0$ MeV is not at all stringent enough.

It seems the Z boson provides evidence *against* the existence of extra dimensions or for a very large value of k, suppressing the branching ratio even more than was now assumed. Alternatively, there may turn out to have been a systematic error involved responsible for the negative decay rate. In this case, the limit it provides must be rejected and hope reemerges for $n \geq 9$. This however seems unlikely. The only remaining hope is that the astrophysical considerations are off. Astrophysics is not known for its large accuracy, but the current discrepancy is so many orders of magnitude that whatever corrections will be made, it is unlikely that extra dimensions will become accessible in the near future if n is small. Furthermore, there will be a standard model background decay of positronium into neutrinos, which is also an invisible form of decay in the experiments described above. This process is expected to have a branching ratio of $\simeq 6 \times 10^{-18}$ [9, Eq. (19)]. So even if we ever get beyond the 10^{-19} experimental sensitivity, the invisible decay mode will be dominated by this background. Thus, simply detecting missing energy (which is the current experimental goal) will not be enough. The detection of disappearance into extra dimensions would not be as independent as when it would be accessible with less sensitivity because accurate theory of the neutrino decay mode would be important, providing further challenge and probably less reliability.

In conclusion, positronium seems completely unfit for testing the existence of extra dimensions. The experimental sensitivity in the branching ratio may be increased three orders of magnitude in the near future, which is certainly a great improvement, but it would still be at least ten orders of magnitude short of the sensitivity at which we might detect extra dimensions. It is however interesting to see our chances of detecting extra dimensions through positronium are almost best for the case of four extra dimensions, which is also the first value of n at which k is allowed to be of order 1 TeV according to both the bounds provided by astrophysics [7] and Z boson decay (see Figure 4.3). $K \sim 1$ TeV is plausible (yet far from certain) since it would prevent the introduction of a new hierarchy, and thus it seems we might be in luck. Still, an experiment capable of probing the existence of extra dimensions through the disappearance of positronium seems currently far from realistic.



(a) The five different upper bounds acquired in Section 4.4.



(b) The resultant upper bound from combining the five upper bounds.

Figure 4.4: Experimental sensitivity and upper bounds.

Conclusion

In this bachelor thesis, I have tried to answer the question of whether the direct search for disappearing positronium provides a viable experimental method of probing the existence of extra dimensions. To this end, I have studied the properties of positronium, in particular its energy spectrum and decay modes. I compared the experimental sensitivity with which disappearing positronium could be measured to upper limits in the disappearance rate obtained by assuming a model with one infinite extra dimension (of the Randall-Sundrum type) and $n \geq 1$ compact extra dimensions and using data on Z boson decay and stellar cooling. An optimistic estimate is that sensitivity may be improved by three orders of magnitude in the near future, while the limits suggest that an improvement of at least thirteen orders of magnitude is required in order to obtain new evidence which could either confirm the model or put extra conditions on it. From this, I conclude that it does not seem realistic that experiments will be able to detect positronium disappearing into extra dimensions anytime soon.

Appendix A

Conventions

In this appendix, various symbols which are used in this thesis are defined. These definitions reflect the conventions in accordance with which this thesis is written.

For the flat metric, the mostly-minus convention is used,

$$\eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{A.1}$$

In this thesis, the Dirac representation of the Clifford algebra is used. This means the γ matrices are defined by

$$\begin{gathered} \gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \\ \gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \end{gathered} \tag{A.2}$$

The Pauli spinors are defined by

$$\chi_s = \begin{cases} \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \text{for } s = +1/2, \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \text{for } s = -1/2. \end{cases} \tag{A.3}$$

The 2×2 Levi-Civita symbol is

$$\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{A.4}$$

$\boldsymbol{\sigma}$ is the Pauli vector defined by

$$\boldsymbol{\sigma} = \sigma_1 \hat{x} + \sigma_2 \hat{y} + \sigma_3 \hat{z}, \tag{A.5}$$

where the Pauli matrices are defined by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.6}$$

It is an operator related to the quantum mechanical spin operator of a Fermion by

$$\mathbf{S} = \frac{\hbar}{2}\boldsymbol{\sigma}. \tag{A.7}$$

The derivative four-vector ∂_μ is defined by

$$\partial_\mu = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right) \tag{A.8}$$

where x_μ is the four-vector containing spacetime coordinates.

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