The Goldstone theorem and spontaneous breaking of conformal symmetry

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July 19, 2016

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Abstract

In this paper it will be shown that the spontaneous breaking of conformal symmetry gives rise to a Goldstone boson, the Dilaton. Then we will take our discussion to the quantum level. Conformal symmetry can only be recovered at a fixed point of the Renormalization Group flow.

Contents

1 Introduction

Symmetries have always played an important role explaining physical phenomena. The principle of relativity for example follows from the invariance of a physical system under a Lorentz transformation which forms the Lorentz group. This can be extended with time and space translations to obtain the Poincare group. Many other symmetry groups are commonplace in physics. The Lagrangian of the standard model is also constrained under a product of symmetry groups, $U(1) \times SU(2) \times SU(3)$ to be precise.

What we will focus on in this paper, however, is conformal symmetry. We will consider classically the case where conformal symmetry is spontaneously broken and gives rise to a Goldstone boson, the Dilaton. Then we will investigate how this generalizes to a quantum field theory.

This thesis will start by giving a general explanation about symmetry in physics in section 2. In section 3 we will give a general proof of Goldstone's theorem. In section 4 we will give a quick introduction into dimensional analysis. Then we will derive the generators of the conformal group and their physical meanings in section 5. In section 6 we will consider what happens when conformal symmetry is spontaneously broken. We will see how these results generalize to the quantum level in section 7. We conclude in section 8.

2 Symmetries in physics

The symmetry of a theory is an transformation on the fields that leaves its dynamics unchanged, i.e. the physics does not change. We start clasically by considering the implications of symmetries via Noether's theorem. Consider a Lagrangian $L(\phi, \partial_\mu \phi)$, function of its fields and their first derivatives, and a continuous infinitesimal transformation of the fields $\phi \to \phi + \delta\phi$ such that $\delta L = \partial_\mu F^\mu$. This is called a symmetry since it does not change the action of the theory.
let us now consider an arbitrary continuous infinitesimal transformation of the Lagrangian, where we used $\delta(\partial_\mu \phi) = \partial_\mu(\delta\phi)$:

$$\delta L = \frac{\partial L}{\partial \phi}\delta\phi + \frac{\partial L}{\partial(\partial_\mu \phi)}\partial_\mu(\delta\phi) \tag{1}$$

Integrating by parts and using the fact that the equations of motions are satisfied we find that this reduces to:

$$\delta L = \partial_\mu\left(\frac{\partial L}{\partial(\partial_\mu \phi)}\delta\phi\right) \tag{2}$$

but if $\delta\phi$ corresponds to a symmetry then we have by definition $\delta L = \partial_\mu F^\mu$. It follows from this that there exists a conserved current, i.e. $\partial_\mu j^\mu = 0$:

$$j^\mu = \frac{\partial L}{\partial(\partial_\mu \phi)}\delta\phi - F^\mu \tag{3}$$

This argument applies to each generator of the symmetry group, so that there is a conserved current for each exact generator.

2.1 Global and local symmetries

It is important to make a distinction between two types of symmetries. On the one hand we have global symmetry transformations which are independent of space-time coordinates.

On the other hand we have local symmetries. The field transformation parameters here do depend on space-time coordinates. Such symmetries are allowed to transform differently at every space-time coordinate as long as it eventually yields an invariant Lagrangian.

As an example consider a U(1) transformation of a complex scalar field $\phi(x)$. A global transformation could be written as $\phi(x) \to e^{i\alpha}\phi(x)$. Note that here α does not depend on spacetime coordinates. If we however make it space-time dependent, i.e. $\alpha(x)$, then we get a local symmetry transformation $\phi(x) \to e^{i\alpha(x)}\phi(x)$ as long as the Lagrangian is invariant up to a total derivative.

2.2 Breaking a symmetry

2.2.1 Explicit symmetry breaking

We will now consider how symmetries can be broken in nature. There are two ways of achieving this: explicit and spontaneous symmetry breaking. Explicit breaking is realized by manually inserting some symmetry breaking term in the theory. Consider the Lagrangian for a harmonic oscillator with frequency ω:

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2 \tag{4}$$

which is invariant under $x \to -x$. When we now however insert a displacement λx into the Lagrangian it will no longer be invariant under spatial inversion:

$$L' = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2 + \lambda^3 x \tag{5}$$

In this example the manually inserted term thus caused an explicit breaking of symmetry.

2.2.2 Spontaneous symmetry breaking

With this type of symmetry breaking there is no term introduced in the Lagrangian but it is a mere consequence of the transformation properties of the ground state of the system. Spontaneous symmetry breaking occurs when the ground state of a system is degenerate. Consider a Lagrangian whose vacuum state has $U(1)$ symmetry, see figure 1. There are infinitely many states with the same energy when varying the angle θ around the origin.. By choosing a specific angle the symmetry is broken.

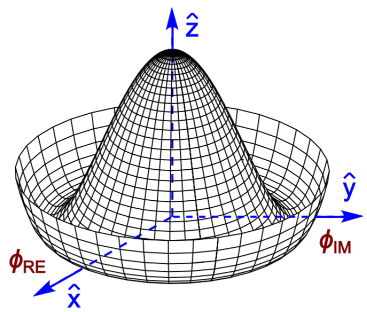


Figure 1: U(1) potential well symmetric about its z-axis.

3 Dimensional analysis

In our considerations it will be useful to have an understanding of dimensional analysis and what is meant by terms such as dimensionful couplings. In quantum field theory it is convenient to work in natural units $\hbar = c = 1$. This ensures that all quantities will be measured in units of the energy(mass) raised to some power. Mass and momentum thus have dimensionality one, $[m] = 1$ and length of course has inverse dimensionality $[x^\mu] = -1$. Now we know that the action must be dimensionless:

$$S = \int d^4 x L \tag{6}$$

From which we deduce that $[L] = 4$, in four dimensions. We can now easily derive the dimensions of the fields.

To find the mass dimension of the scalar field we consider the Klein-Gordon Lagrangian:

$$L_{KG} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 \tag{7}$$

Thus from $[L] = 4$, $[m^2] = 2$, $[\partial_\mu] = 1$ it follows that a scalar fields has $[\phi] = 1$, in four dimensions. All bosonic fields carry E^{+1} in four dimensions since their kinetic terms are quadratic in ∂_μ.

The dimensionality of the spinor field is determined from the free Dirac Lagrangian:

$$L_{Dirac} = \overline{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi \tag{8}$$

Using the same analysis we find that $[\overline{\Psi}\Psi] = 3$ and thus $[\Psi] = [\overline{\Psi}] = 3/2$ for the Dirac field.

Next we can identify two types of coupling constants. On one side we have the dimensionless coupling constants such as the fine-structure constant. On the other hand we have dimensionful couplings such as in the Klein-Gordon equation above. These terms will explicitly break scale invariance as we will see.

4 Goldstone's theorem

If one has a degenerate ground state with non-zero vacuum expectation value of the order parameter then spontaneous symmetry breaking occurs. If we now consider a continuous symmetry which is spontaneously broken then a degree of freedom exists associated to transformations along the bottom of the potential. Since the second derivative of the potential along the bottom of the well vanishes we expect this degree of freedom to be massless. This result is quantified in the Goldstone theorem which will now be derived. The outline of this proof will be according to Guralnik, Hagen, Kibble [2].

We start with the assumption that there is a continuous global symmetry. Noether's theorem implies that there is some conserved current associated with each exact generator of the symmetry group:

$$\partial_\mu j^\mu(x) = 0 \tag{9}$$

We define a time dependent generator (charge):

$$Q_R(t) \equiv \int_{|z|<R} d^3x j^0(\mathbf{x}, t) \tag{10}$$

Where $|\mathbf{z}|$ =R is the radius of the surrounding sphere with surface $\sigma(R)$. We will consider an arbitrary combination of the field operators $A(t)$ and take the commutator of $A(t)$ with the space integral of (9) which is zero by construction.

$$\int_{|x|<R} [\partial_\mu j^\mu(x), A(t)] = 0 \tag{11}$$

Evaluating the integral separately over its time and space indices we obtain:

$$[\partial_0 Q_r(t), A(t)] + \int_{|z|<R} [\nabla j(x), A(t)] = 0 \tag{12}$$

Using Stokes' theorem on the second term implies:

$$[\partial_0 Q_r(t), A(t)] + [\int_{\sigma(R)} d\boldsymbol{\sigma} \cdot \boldsymbol{j}, A(t)] = 0 \tag{13}$$

If the second term vanishes for sufficiently large R

$$\lim_{R\to\infty} [\int_{\sigma(R)} d\boldsymbol{\sigma} \cdot \boldsymbol{j}, A(t)] = 0 \tag{14}$$

then the first term of (13) also vanishes in this limit:

$$\lim_{R\to\infty} \partial_0 [Q_R(t), A(t)] = 0 \tag{15}$$

hence by integrating with respect to time to some constant B:

$$\lim_{R\to\infty} [Q_R(t), A(t)] = B \tag{16}$$

with $dB/dt = 0$. The commutator of a symmetry generator with $A(t)$ yields:

$$[Q(t), A(t)] = -i\frac{\delta A(t)}{\delta\alpha} \tag{17}$$

Now we know that if the VEV $\langle 0|\delta A(t)|0\rangle$ vanishes then there is clearly no symmetry transformation that leads to a state with equal energy and this state must thus be non-degenerate. A non-vanishing value, on the contrary, implies a degenerate ground state. Now evaluating (16) at the ground state and imposing that it spontaneously breaks symmetry we get:

$$\lim_{R\to\infty} \langle 0|[Q_R(t), A]|0\rangle = \langle 0|B|0\rangle \neq 0 \tag{18}$$

Now we use Hilbert's completeness relation $1 = \sum_n |n\rangle\langle n|$ to introduce intermediate states and obtain:

$$\lim_{R\to\infty} \sum_n [\langle 0| Q_R(t) |n\rangle \langle n| A |0\rangle - \langle 0| A |n\rangle \langle n| Q_R(t) |0\rangle] = \langle 0|B|0\rangle \neq 0 \tag{19}$$

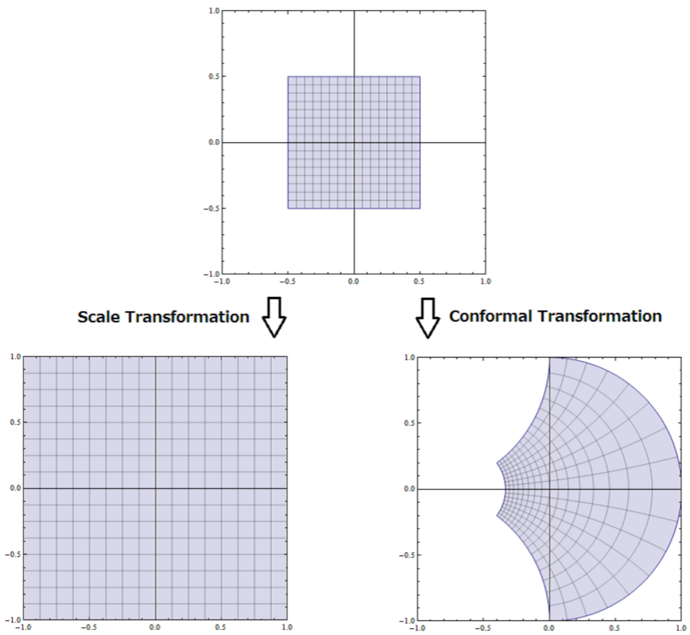
Next we assume that $j^0(x)$ is a local operator and is translationally invariant $j^0(x) = e^{-iPx} j^0(0) e^{iPx}$. Since we assume that the vacuum is also translationally invariant it follows that $e^{iPx}|0\rangle = |0\rangle$. Using these assumptions (19) becomes:

$$\lim_{R\to\infty} \sum_n \int_R d^3x [\langle 0| j^0(0) |n\rangle \langle n| A |0\rangle e^{ip_n x} - \langle 0| A(t) |n\rangle \langle n| j^0(0) |0\rangle e^{-ip_n x}] \tag{20}$$

Now we use that $\int_R d^3x e^{\pm i\vec{p}_n\cdot\vec{x}} = (2\pi)^3\delta^3(\vec{p}_n)$ to find:

$$\lim_{R\to\infty} \sum_n (2\pi)^3\delta^3(\vec{p}_n)[\langle 0| j^0(0) |n\rangle \langle n| A |0\rangle e^{ip_n^0 x^0} - \langle 0| A |n\rangle \langle n| j^0(0) |0\rangle e^{-ip_n^0 x^0}] = \langle 0|B|0\rangle \tag{21}$$

Now we know that B is time independent $dB/dx^0 = 0$ and this equation holds for all x^0. Thus in order to be consistent and guarantee that $\langle 0|B|0\rangle \neq 0$ the left hand side of (21) must vanish except when $p_n^0|_{\vec{p}_n\to 0} = 0$ since this implicitly takes out all dependence on x^0 and the Dirac delta function is non-zero. This means that when the symmetry is broken with there exists a state with vanishing energy for vanishing momentum. This state is therefore associated to a particle with zero mass.



5 Conformal invariance

In this section we will derive the generators and symmetries of the conformal group for a classical field theory following A.N. Schellekens [3]. We will consider the line element $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ and find all conformal transformations, i.e. all transformations that conserve the angle between two vectors.

5.1 Deriving conformal transformations

We start by considering a Weyl transformation, which is a rescaling of the metric:

$$g_{\mu\nu}(x) \to \Omega(x) g_{\mu\nu}(x) \tag{22}$$

A conformal transformation is a coordinate transformation $x \to x'$ that transforms the metric as if it were a Weyl transformation:

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \Omega(x) g_{\mu\nu}(x) \tag{23}$$

The symmetry transformations of the conformal group are found by investigating the infinitesimal coordinate transformation $x^\mu \to x^\mu + \epsilon^\mu$ such that the line element transforms as:

$$ds^2 \to ds^2 + (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) dx^\mu dx^\nu \tag{24}$$

For this to satisfy (23), $(\partial_\mu \epsilon_\nu + \partial_\mu \epsilon_\nu)$ has to be proportional to the metric. We should find some function $K(x)$ such that:

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = K(x)\eta_{\mu\nu} \tag{25}$$

$K(x)$ can easily be found by tracing both sides with the metric. This gives:

$$\begin{aligned} \eta^{\mu\nu}(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) &= K(x)\eta_{\mu\nu}\eta^{\mu\nu} \\ 2\partial^\mu \epsilon_\mu &= K(x)d \end{aligned} \tag{26}$$

Thus we find:

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d}(\partial \cdot \epsilon)\eta_{\mu\nu} \tag{27}$$

We would like to know what order our solution must be in x. To figure this out we first contract (27) with $\partial^\mu \partial^\nu$ to obtain:

$$(1 - \frac{1}{d})\Box\partial \cdot \epsilon = 0 \to \Box\partial \cdot \epsilon = 0 \tag{28}$$

Then we contract with $\partial_\rho \partial^\nu$ to find:

$$\Box\partial_\rho \epsilon_\mu + (1 - \frac{2}{d})\partial_\rho \partial_\mu \partial \cdot \epsilon = 0 \tag{29}$$

Next we take the sum of this equation plus the sum with its indices interchanged $\rho \leftrightarrow \mu$ to find:

$$(1 - \frac{2}{d})\partial_\rho \partial_\mu \partial \cdot \epsilon = 0 \tag{30}$$

From this equation we can derive that $\partial_\rho \partial_\mu \partial \cdot \epsilon = 0$ when $d \neq 2$. Now to determine if the solution can have terms higher then second order we take the uncontracted derivative $\partial_\rho \partial_\sigma$. We next define $\Delta_{\rho\sigma\mu\nu} \equiv \partial_\rho \partial_\sigma \partial_\mu \epsilon_\nu$. This function is symmetric in the first three indices. Also taking this same uncontracted derivative onto (27) we derive

$$\Delta_{\rho\sigma\mu\nu} = -\Delta_{\rho\sigma\nu\mu} \tag{31}$$

All other forms are symmetric permutations of these forms. Since these only hold whenever Δ is zero we know there can be no solutions higher than second order in x. This follows from the fact that we can define a higher order Δ by taking the product with n uncontracted derivatives. These will however always be symmetric in the first $n + 1$ indices and so will also be zero. With this in mind we make an ansatz for the solution:

$$\epsilon^\mu = \alpha^\mu + \beta^\mu_\nu x^\nu + \gamma^\nu_{\nu\rho} x^\nu x^\rho \tag{32}$$

Substituting into (27) and grouping terms of equal order we find that the conditions turn out to be:

$$\begin{aligned} \beta_{\mu\nu} + \beta_{\nu\mu} &= \frac{2}{d}\beta^\rho_\rho g_{\mu\nu} \\ \gamma_{\mu\nu\sigma} + \gamma_{\nu\mu\sigma} &= \frac{2}{d}\gamma^\rho_{\rho\sigma} g_{\mu\nu} \end{aligned} \tag{33}$$

$\beta_{\mu\nu}$ can be splitted into its symmetric and anti-symmetric part resulting in:

$$\beta_{\mu\nu} = \omega_{\mu\nu} + S_{\mu\nu} \tag{34}$$

5.2 Conformal group

From this discussion we can recognize the following transformations of the conformal group:

1. **Rotations**: $\epsilon^{\mu} = w^{\mu}_{\nu} x^{\nu}$ These are the Lorentz rotations.
2. **Translations**: $\epsilon^{\mu} = \alpha^{\mu}$ These are the translations which together with the Lorentz rotations form the Poincare group.
3. **Scale transformations**: $\epsilon^{\mu} = \sigma x^{\mu}$ this transformation scales every vector with a factor σ.
4. **Special conformal transformations**: $\epsilon^{\mu} = b^{\mu} x^2 - 2x^{\mu} b \cdot x$ This transformation does not seem very intuitive. The conformal group however also has a discrete symmetry, with no infinitesimal form, namely space-time inversion $x^u \to x^{\mu}/x^2$. But since at the origin space-time inversion is not defined it can not generally transform to the identity element. If we instead consider a composite transformation where we first apply the inversion, followed by a translation and then again an inversion we have an composite transformation that is connected to the identity. This is exactly what this transformation does.

The generators of these infinitesimal forms are:

$$\begin{aligned} P_{\mu} &= -i\partial_{\mu} \\ M_{\mu\nu} &= i(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu}) \\ D &= -ix^{\mu}\partial_{\mu} \\ K_{\mu} &= i(x^2\partial_{\mu} - 2x_{\mu}x^{\nu})\partial_{\nu} \end{aligned} \tag{35}$$

where D is the generator of the scale transformations and K_{μ} of the special conformal transformations. One can show that it is a group, i.e. all commutators generate an element of the group, since the the conformal algebra is closed by:

$$\begin{aligned} i[P^{\mu}, P^{\nu}] &= 0 \\ i[M^{\mu\nu}, P^{\lambda}] &= g^{\mu\lambda}P^{\nu} - g^{\nu\lambda}P^{\mu} \\ i[M^{\mu\nu}, M^{\rho\sigma}] &= g^{\mu\rho}M^{\nu\sigma} - g^{\nu\rho}M^{\mu\sigma} - g^{\mu\sigma}M^{\nu\rho} + g^{\nu\sigma}M^{\mu\rho} \\ i[D, P^{\mu}] &= P^{\mu} \\ i[D, K^{\mu}] &= -K^{\mu} \\ i[M^{\mu\nu}, K^{\lambda}] &= \eta^{\mu\lambda}K^{\nu} - \eta^{\nu\lambda}K^{\mu} \\ i[P^{\mu}, K_{\nu}] &= -2\eta^{\mu\nu}D + 2M^{\mu\nu} \\ i[D, D] &= i[D, M^{\mu\nu}] = i[K^{\mu}, K^{\nu}] = 0 \end{aligned} \tag{36}$$

5.3 Traceless energy-momentum tensor

Due to translational invariance we can always define a energy-momentum tensor $T^{\mu\nu}$. Furthermore there is always a way to make this tensor a symmetric tensor by addition of a total derivative such that we get a tensor $\theta^{\mu\nu} = \theta^{\nu\mu}$. In addition it can be shown [10] that due to invariance under scale and special conformal transformations this tensor can be further improved to be traceless and conserved. We thus have the following conditions for the EM tensor in a conformal theory:

$$\partial_\mu \theta^{\mu\nu} = 0, \theta_{\mu\nu} = \theta_{\nu\mu}, \eta_{\mu\nu}\theta^{\mu\nu} = 0 \tag{37}$$

Using Noether's theorem we furthermore find there exists a current:

$$j_\mu = \theta_{\mu\nu}\epsilon^\nu \tag{38}$$

ϵ^ν is (32) that satisfies the conformal group.

6 Spontaneous breaking of conformal symmetry

We would like to see what happens conformal symmetry is spontaneouslu broken, as is done in [6]. Following the Goldstone proof we would like to find an expression for $\langle 0| [\theta^{\mu\nu}, \phi] |0\rangle$ since, as we will shortly see, we can write all generators of the conformal group in terms of $\theta^{\mu\nu}$. Then we introduce the spontaneously broken symmetry assumption. We use the conditions (37) to express the generators of the conformal group in terms of the energy momentum tensor. From (38) we know that $\theta_{\mu\nu}\epsilon^\nu$ is conserved and for every solution we have a corresponding conformal transformation. We thus have a conserved charge $Q = \int d^3x \theta^{0\nu}\epsilon_\nu$ for every solution ϵ^ν that satisfies (27). The conserved charges are then generators of the conformal group. Inserting the conformal transformations in Q and noting that these are conserved for every parameter we find that:

$$\begin{aligned} P^\mu &= \int d^3x \theta^{0\mu} \\ M^{\mu\nu} &= \int d^3x (x^\mu \theta^{0\nu} - x^\nu \theta^{0\mu}) \\ D &= \int d^3x \theta^{0\mu} x_\mu \\ K^\mu &= \int d^3x \theta^{0\nu}(2x_\nu x^\mu - \delta^\mu_\nu x^2) \end{aligned} \tag{39}$$

We also present the commutation relations of the original generators with a local field $\phi(x)$.

$$\begin{aligned} i[P^\mu, \phi(x)] &= \partial^\mu \phi(x) \\ i[M^{\mu\nu}, \phi(x)] &= (x^\mu \partial^\nu - x^\nu \partial^\mu + \Sigma^{\mu\nu})\phi(x) \\ i[D, \phi(x)] &= (x \cdot \partial + d)\phi(x) \\ i[K^\mu, \phi(x)] &= (2x^\mu x^\nu - \eta^{\mu\nu})\partial_\nu \phi(x) + 2x_\nu(\eta^{\mu\nu} d - \Sigma^{\nu\mu})\phi(x) \end{aligned} \tag{40}$$

Where d is the dimension of $\phi(x)$ and $\Sigma^{\mu\nu}$ the spin matrix of the field as will be explained in section 6. We do not want to spontaneously break the Poincaré group and thus in order to assure that the VEV of its commutation relations vanish we need have to assume that $\phi(x)$ is a scalar field and we have to set $\Sigma^{\mu\nu}$ to zero. Now we will derive the Kallen-Lehmann spectral representation [9] of the two-point function between $\phi(x)$ and $\theta^{\mu\nu}$. We start off by, as in the proof of the Goldstone theorem, inserting a complete set of states $\int \frac{d^3p}{(2\pi)^3} |p,\lambda\rangle \langle p,\lambda|$ to obtain:

$$\begin{aligned}\langle 0| [\theta^{\mu\nu}(x), \phi(y)] |0\rangle = \sum_\lambda \int [\frac{d^3p}{(2\pi)^3} \frac{1}{2E_p(\lambda)} \langle 0| \theta^{\mu\nu}(x) |p,\lambda\rangle \langle p,\lambda| \phi(y) |0\rangle \\ - \langle 0| \phi(y) |p,\lambda\rangle \langle p,\lambda| \theta^{\mu\nu}(x) |0\rangle]\end{aligned} \tag{41}$$

Now because the vacuum is translationally invariant we find:

$$\begin{aligned}\langle 0| [\theta^{\mu\nu}(x), \phi(y)] |0\rangle = \sum_\lambda \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p(\lambda)} (\langle 0| \theta^{\mu\nu}(0) |p,\lambda\rangle \langle p,\lambda| \phi(0) |0\rangle e^{-ip(x-y)} \\ - \langle 0| \phi(0) |p,\lambda\rangle \langle p,\lambda| \theta^{\mu\nu}(0) |0\rangle e^{ip(x-y)})\end{aligned} \tag{42}$$

Now lets define two invariant functions:

$$\Delta^{\pm}(x-y; m_\lambda^2) = i \int \frac{d^3p}{(2\pi)^3 2E_p(\lambda)} e^{\mp ip(x-y)} \tag{43}$$

This function has the following properties, for a computation see the appendix:

$$\begin{aligned}(\Box + m^2)\Delta^{\pm}(x-y; m^2) = 0 \\ \partial_0 \Delta^{\pm}(x-y; m^2)|_{x^0=y^0} = \pm i \frac{1}{2} \delta^3(x-y)\end{aligned} \tag{44}$$

Thus now we can express (42) as a sum of $\Delta^{\pm}(x-y; m_\lambda^2)$ to obtain:

$$\begin{aligned}\langle 0| [\theta^{\mu\nu}(x), \phi(y)] |0\rangle = \sum_\lambda \Delta^{+}(x-y; m_\lambda^2) \langle 0| \theta^{\mu\nu}(0) |p,\lambda\rangle \langle p,\lambda| \phi(0) |0\rangle - \\ \Delta^{-}(x-y; m_\lambda^2) \langle 0| \phi(0) |p,\lambda\rangle \langle p,\lambda| \theta^{\mu\nu}(0) |0\rangle\end{aligned} \tag{45}$$

We put this in the Kallen-Lehmann representation [9]. $\rho^{\mu\nu}(m^2)$ contains the dirac delta $\delta(m_\lambda^2 - m^2)$. We have to give Lorentz indices to the spectral function $\rho^{\mu\nu}(m^2)$ because they are present on the l.h.s. . The expression for the spectral representation of the two-point function is:

$$\langle 0| [\theta^{\mu\nu}(x), \phi(y)] |0\rangle = \int_0^\infty dm^2 \rho_+^{\mu\nu}(m^2)\Delta^{+}(x-y; m^2) - \rho_-^{\mu\nu}(m^2)\Delta^{-}(x-y; m^2) \tag{46}$$

There are only two possible forms which have two Lorentz indices, namely; two uncontracted derivatives and the metric. $\rho_\pm^{\mu\nu}(m^2)$ must thus be of the following generic form:

$$\rho_\pm^{\mu\nu} = (A\partial^\mu\partial^\nu + B\eta^{\mu\nu})\rho \tag{47}$$

However we also know from (37) that (41) must vanish when contracted with ∂_μ. Hence we find that $B = -\partial^2 A = \Box A$ and the form should be (where we set A to 1):

$$\rho^{\mu\nu}_{\pm} = (\partial^\mu \partial^\nu + \eta^{\mu\nu}\Box)\rho \tag{48}$$

We now have the final Kallen-Lehman spectral representation where we retrieved the spectral function $\rho(m^2)$ independent of the lorentz indices:

$$\begin{aligned} \langle 0| [\theta^{\mu\nu}(x), \phi(y)] |0\rangle = \int_0^\infty dm^2 (\partial^\mu \partial^\nu + \eta^{\mu\nu}\Box)[\rho_+(m^2)\Delta^+(x-y; m^2) - \\ \rho_-(m^2)\Delta^-(x-y; m^2)] \end{aligned} \tag{49}$$

Next we use the fact that the energy-momentum tensor is traceless to find:

$$\begin{aligned} \langle 0| [\eta_{\mu\nu}\theta^{\mu\nu}(x), \phi(y)] |0\rangle &= -3 \int_0^\infty dm^2 [\rho_+(m^2)\Box\Delta^+(x-y; m^2) - \\ &\rho_-(m^2)\Box\Delta^-(x-y; m^2)] \\ &= 3 \int_0^\infty dm^2 m^2 [\rho_+(m^2)\Delta^+(x-y; m^2) - \\ &\rho_-(m^2)\Delta^-(x-y; m^2)] = 0 \end{aligned} \tag{50}$$

Where we used the properties (44) to get to the second line. This function has to vanish for all spacetime coordinates and thus the integrand must be zero. I.e $m^2\rho_\pm(m^2) = 0$ and this has to vanish when $m^2 \neq 0$ which is true when:

$$\rho_\pm(m^2) = c_\pm \delta(m^2) \tag{51}$$

Where $c_+ \neq -c_-$ except $c_\pm = 0$ since $\rho(m^2) \geq 0$. Inserting (51) in (49) we find that because of the integral over the dirac delta it will only be evaluated at $m^2 = 0$:

$$\langle 0| [\theta^{\mu\nu}(x), \phi(y)] |0\rangle = i\frac{1}{2}(c_+ - c_-)\partial^\mu \partial^\nu D(x-y) \tag{52}$$

With $D(x-y)$ the massless propogator. Now we would like to find an expression for the not yet defined c. We start off by evaluating all possible equal time forms of the two point function using the properties (44):

$$\begin{aligned} \langle 0| [\theta^{0\nu}(x), \phi(y)] |0\rangle &= 0 \\ \langle 0| [\theta^{0i}(x), \phi(y)] |0\rangle &= \frac{1}{2}(c_+ + c_-)\partial_i \delta^3(x-y) \end{aligned} \tag{53}$$

Next we evaluate the vacuum expectation values of the conformal generators with the scalar field. This follows by using (39) to compute the two point functions.

$$\begin{aligned} i\langle 0| [P^\mu, \phi(y)] |0\rangle &= 0 \\ i\langle 0| [M^{\mu\nu}, \phi(y)] |0\rangle &= 0 \\ i\langle 0| [D, \phi(y)] |0\rangle &= 3\frac{1}{2}(c_+ + c_-) \\ i\langle 0| [K^\mu, \phi(y)] |0\rangle &= 6\frac{1}{2}(c_+ + c_-)y^\mu \end{aligned} \tag{54}$$

As we expect the vacuum expectation value for the Poincaré group vanishes. Additionally the scale and special conformal generators get an vacuum expectation value when $c \neq 0$ and thus spontaneously break symmetry. Referring back to (46) and comparing their vacuum expectation values we find that:

$$\begin{aligned} i\langle 0|\,[D,\phi(y)]\,|0\rangle &= d\,\langle 0|\,\phi(y)\,|0\rangle \\ i\langle 0|\,[K^\mu,\phi(y)]\,|0\rangle &= 2y^\mu d\,\langle 0|\,\phi(y)\,|0\rangle \end{aligned} \tag{55}$$

Comparing these equations to (54) we find that these equations are statisfied for:

$$\frac{1}{2}(c_+ + c_-) = \frac{d}{3}\langle 0|\,\phi(y)\,|0\rangle \tag{56}$$

Inserting our newly found expression for into the expression for the two point function (52) we find:

$$i\langle 0|\,[\theta^{\mu\nu},\phi(y)]\,|0\rangle = -\frac{d}{3}\langle 0|\,\phi(y)\,|0\rangle\,\partial^\mu\partial^\nu D(x-y) \tag{57}$$

In fact we can find an explicit expression for $\theta^{\mu\nu}$:

$$\theta^{\mu\nu}(x) = \frac{d}{3}\langle 0|\phi(0)|0\rangle\,\partial^\mu\partial^\nu\phi(x) \tag{58}$$

Thus as soon as the scalar field has a non-vanishing vacuum expectation value conformal invariance is spontaneously broken. Goldstone's theorem then implies that there should appear a massless Goldstone boson ϕ. This is also seen by considering (58) and noting that when the expectation value is non-zero the scalar field needs to be massless in order to leave the EM tensor traceless.

Goldstone's theorem furthermore implies that for every broken generator there should appear a Goldstone boson in the spectrum. With the spontaneous breaking of conformal invariance both the dilatational generator as well as the special conformal generator are broken. One would expect five Goldstone bosons for every broken generator. However on the r.h.s. of (57) there does not appear a vector boson which one could reasonably expect because the special conformal generator is broken. Since it is not there one only has a single Goldstone boson. This boson is generally referred to as the Dilaton.

7 Implications at the quantum level

Now we would like to look at what the implications are of conformal symmetry and how spontaneous symmetry breaking could occur. Our previous discussion remained in the classical regime. In this section we will consider changes at the quantum level and see how we can describe a conformal theory. We will find that in most cases we get a so-called trace anomaly, which only disappears at a fixed point of the renormalization group.

7.1 Renormalization group and anomalous dimensions

The renormalization group is used to describe how physical systems behave at different energy scales. An at first hand surprising result is that higher energy processes seem to have almost no influence on lower energy processes. Processes seem to be dominated by the dynamics at comparable energy scales. A physical explanation was given by Kenneth Wilson, who suggested that all parameters of a field theory are scale dependent and their evolution with the change of energy scale can be described by the Renormalization Group equations.

We will look at the Callan-Symanzik equation (Renormalization Group equations) which is obtained in a natural way by considering how the couplings and fields of a n-point correlation function change when going to a different energy scale[9]. It is used to evaluate the evolution of n-point correlation functions, i.e. describe the renormalization. We will not derive it rigorously but just want to give some intuitive understanding into the so-called β and γ functions.

We start by considering the n-point correlation function with its bare fields ,ϕ_0, and coupling constants:

$$\langle 0| T\phi_0(x_1)\phi_0(x_2)...\phi_0(x_n) |0\rangle \tag{59}$$

We can now go to any other scale by rescaling the fields $\phi = Z^{1/2}\phi_0$ such that any renormalized correlation function can be expressed as follows:

$$\langle 0| T\phi(x_1)\phi(x_2)...\phi(x_n) |0\rangle = Z^{-n/2} \langle 0| T\phi_0(x_1)\phi_0(x_2)...\phi_0(x_n) |0\rangle \tag{60}$$

From now on we will write the l.h.s. of (60) as $G^{(n)}(x_1, x_2...x_n)$. Next we will consider the effects of performing an infinitesimal shift in the scale $M \to M+\delta M$ such that the coupling constants and fields transform as follows:

$$\begin{aligned} \lambda &\to \lambda + \delta\lambda \\ \phi &\to (1+\delta\eta)\phi \end{aligned} \tag{61}$$

Thus shifting the n fields of the correlation function then simply yields

$$G^{(n)} \to (1+n\delta\eta)G^{(n)} \tag{62}$$

Since the correlation function depends only on M and λ we can also write the variation $\delta G^{(n)}$ as:

$$\delta G^{(n)} = \frac{\partial G^{(n)}}{\partial M}\delta M + \frac{\partial G^{(n)}}{\partial \lambda}\delta\lambda = n\delta\eta G^{(n)} \tag{63}$$

Multiplying this equation by $M/\delta M$ we get:

$$[M\frac{\partial}{\partial M} + \beta\frac{\partial}{\partial\lambda} + n\gamma]G^{(n)} = 0 \tag{64}$$

This is called the Callan-Symanzik equation. β describes how a coupling constant changes under a shift in the energy. γ indicates the appearance of anomalous dimensions. When they are present fields scale according to $\Delta = \Delta_0 + \gamma$. Under a scale transformation $x \to \lambda x$ operators get a factor λ^{Δ}.

7.2 Trace anomaly

We will consider quantum field theories which are classically conformally invariant. However due to quantum corrections it can happen that $\theta^\mu_\mu \neq 0$. Explicit breaking of conformal symmetry can occur in different ways. The first is due to the appearance of a scale under renormalization. The second way is that Weyl invariance is broken when the theory is embedded in curved spacetime.

It is derived in [11] that when going to the quantum case the introduction of a renormalization scale introduces a trace anomaly:

$$T^\mu_\mu = \sum_j \beta_j O_j(x) + \text{mass terms} \tag{65}$$

Where $O_j(x)$ are renormalized operators of the theory. We thus find that full conformal invariance is only restored when β_j is zero and the theory is massless. We call a point where β_j is zero a fixed point of the renormalization group. This makes sense since a theory that has couplings which depend on the energy scale is not even scale invariant.

Conformal invariant quantum field theories only exist at fixed points of the renormalization group. We would however also like to investigate what happens under a rescaling $x_j \to \lambda x_j$, with $\beta \neq 0$, where $\lambda \to 0$ corresponds to a small distance, the UV limit and $\lambda \to \infty$ to large distances, the IR. If we assume the Callan-Symanzik equation (64) than for a massless theory at scale μ:

$$(\beta(g)\frac{\partial}{\partial g} + \lambda\frac{\partial}{\partial \lambda} + n(d_\phi + \gamma(g))G_n(\lambda x_j, g, \mu) = 0 \tag{66}$$

thus one can see that the canonical dimensions get additional contributions from the anomalous dimensions. This is a numerical deviation from the standard canonical dimensions and results in a different rescaling. If we can show that the theory has a fixed point with $g^{\mathrm{UV}}(\lambda)$ or $g^{\mathrm{IR}}(\lambda)$ then under the right conditions [11] for the anomalous dimensions one finds that:

$$G_n(\lambda x_j, g, \mu) \propto_{\lambda \to 0} \lambda^{-n(d_\phi + \gamma(g^{\mathrm{UV}}(t))} \tag{67}$$

where for a IR fixed point $\lambda \to \infty$. Under uniform scaling the non-conformally invariant QFT asymptotically scales like a conformal QFT when flowing to an UV or IR fixed point. The asymptotic freedom of QCD is a prime example where at high energies the theory behaves asymptotically free as a CFT. Since the theory will always be an interacting theory it will however never be at the trivial UV fixed point where $g = 0$, i.e. the couplings just vanish.

Additionally when considering a theory in curved space-time one gets additional terms in the trace of the EM tensor as can be seen in [8].

8 Discussion

In this thesis we started by proving Goldstone's theorem for a classical field theory and subsequently deriving the conformal group. Next we showed that any classical conformally invariant theory where conformal symmetry is spontaneously broken necessarily has a Dilaton in its spectrum. Subsequently, we went from the classical to the quantum level. We briefly touched upon the renormalization group and derived the Callan-Symanzik equation to get an intuition for the renormalization of quantum field theories and the role of β and γ functions. Then we showed that an exact conformal symmetry of the quantum theory only exists at a fixed point of the RG where beta functions vanish. Furthermore we commented on known theories, such as QCD, where conformal invariance can only be asymptotically realized in the free and massless limit.

We have not investigated whether it is possible to realize the spontaneous breaking of conformal symmetry in the quantum case. It would be interesting to investigate whether it is possible to flow from a fixed point due to spontaneous symmetry breaking. It would be particularly interesting to see if this can happen when considering a quantum theory at a UV or IR fixed point non-trivial, i.e. interacting, fixed point. This becomes interesting when the Standard Model of particle interactions is unified with gravity.

Acknowledgments
I would like to thank Elisabetta Pallante for her guidance and many hours spent discussing with me. Furthermore I would like to thank Marco Boers for taking time out to help me when I got stuck.

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