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Dynamics of a solar sail

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Contents

1 Abstract

Abstract

In this paper we will discuss the dynamics of a solar sail. At first we will look at the derivation of the solar sail models and the dynamics of the models under certain circumstances. Then the augmented Hill problem and its dynamics are discussed. Finally the process of reduction to the centre manifold is explained and used to compute the dynamics numerically.

2 Derivation of solar sail models

This paper deals with the dynamics of solar sailing. A solar sail is a space probe which travels through space with the use of solar radiation. The solar radiation is absorbed or reflected by large mirrors which ultimately produces an acceleration. This way of transportation is very convenient because of the fact that the solar radiation does not depend on time. The focus in this paper will first be on the dynamics of the solar sail under the solar radiation pressure and in the gravitational fields of the Earth and the Sun (Restricted Three Body Problem). Then the case with a nearby asteroid will be discussed. In this case the augmented Hill problem can be used as a model because the asteroid has a low mass and the sun is at a sufficient large distance. To create a model for these system it is convenient to express it into a Hamiltonian function and to reduce it to an centre manifold. This will be done by writing the Hamiltonian as a power expansion and to erase certain parts of it. With the Hamiltonian function written as a power series it is more suitable to approach the solutions numerically.

Assume that the solar sail is a flat surface. For the most part the solar radiation is reflected by the sail which gives the sail a force in to the normal direction. However in a non-ideal situation the sail absorbs photons as well, this will create a force in the opposite direction of the solar radiation. Assume that the solar sail is at a distance r from the sun. Then the Solar Radiation Pressure is

$$P = \frac{S_0}{c}\left(\frac{r_0}{r_{sp}}\right)^2 = 4.6\frac{\mu N}{m^2}\left(\frac{r_0}{r_{sp}}\right)^2 \tag{1}$$

Where c is the speed of light constant, $S_0 = 1368\dfrac{W}{m^2}$ is the solar constant and where r_0 is 1 astronomical unit [3]. In an ideal situation (Figure 1) where the solar sail reflects every photon the force of the solar sail becomes

$$\vec{F} = 2PA\cos^2\alpha\vec{\mathbf{n}}$$

With A the area of the solar sail, P is as in (1) and $\vec{\mathbf{n}} = (\cos\alpha\cos\delta, \sin\alpha\cos\delta, \sin\delta)$. Because $\vec{F} = m\vec{a}$ one can define the acceleration of the solar sail which is oriented radially to the sun as follows:

$$\vec{a} = p\beta\frac{m_s}{r_{ps}^2}\cos^2\alpha\cos^2\delta\vec{\mathbf{n}} + 0.5(1-p)\beta\frac{m_s}{r_{ps}^2}\vec{r_s}$$

Where p is the reflectivity coefficient which determines the quality of reflection or absorption of the photons, 0 and 1 correspond respectively to full absorption and reflection. Another important parameter is the lighting number which is the ratio between the sun radiation pressure acting on the solar sail, which is perpendicular to the sun, and the sun's gravitational acceleration. If β is 1 the sale will be directed perpendicular to the sun and the radiation and the gravitational force of the sun will be equal [3]. Assume that the solar sail is fully reflecting, then our acceleration becomes

$$\vec{a} = \beta \frac{m_s}{r_{ps}^2} \cos^2 \alpha \cos^2 \delta \vec{\mathbf{n}}$$

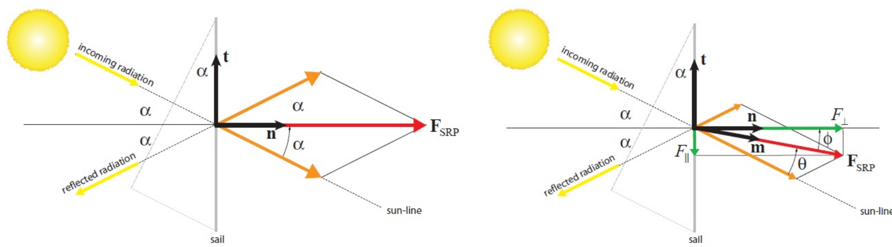

Figure 1: Left: Ideal situation where $\mathbf{n} = \mathbf{m}$ right: non-ideal situation[3]

2.1 Motion of the solar sail

Again assume that the solar sail is fully reflecting as well as that the Earth and the Sun are point masses which move circularly around the centre. The period of the orbits is 2π and the distance between the Earth and the Sun as well as the total mass is set 1. Because the solar sail is influenced by the gravitational force of the Sun and the Earth we can express it in a rotating frame. One has:

$$\ddot{\mathbf{r}} + 2\mathbf{\Omega} \times \dot{\mathbf{r}} + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) = \mathbf{a} + \nabla \Pi(\mathbf{r})$$

Where,

$$\Pi(\mathbf{r}) = \frac{1-\mu}{|\mathbf{r_{ps}}|} + \frac{\mu}{|\mathbf{r_{pe}}|}$$

We denote $\mathbf{a}$ and $\mathbf{r}$ respectively as the acceleration and position vector of the probe and $\mathbf{\Omega}$ is the angular velocity of the frame. $\Pi(\mathbf{r})$ is the gravitational potential. Note that μ is the ratio of the masses of Earth and the Sun. The positions $\mathbf{r_{ps}}, \mathbf{r_{pe}}$ are given by the following equations where $\{\mathbf{v_x}, \mathbf{v_y}, \mathbf{v_z}\}$ are the unit vectors along the 3 axis [6].

$$\begin{aligned} \mathbf{r_{ps}} &= (x+\mu)\mathbf{v_x} + y\mathbf{v_y} + z\mathbf{v_z} \\ \mathbf{r_{pe}} &= (x-1+\mu)\mathbf{v_x} + y\mathbf{v_y} + z\mathbf{v_z} \end{aligned}$$

As one can notice , the angular velocity is perpendicular to the plane. This gives the following 3 equations of motion where $\alpha, \delta \in [-\pi/2, \pi/2]$ and $\kappa = \beta \frac{1-\mu}{r_{ps}^2} \cos^2 \alpha \cos^2 \delta$.

$$\ddot{x} = 2\dot{y} + x - (1-\mu)\frac{x-\mu}{r_{ps}^3} - \mu\frac{x+1-\mu}{r_{pe}^3} + \kappa\cos(\phi(x,y)+\alpha)\cos\psi(x,y,z)+\delta)$$
$$\ddot{y} = -2\dot{x} + y - \left(\frac{1-\mu}{r_{ps}^3} + \frac{\mu}{r_{pe}^3}\right)y + \kappa\sin(\phi(x,y)+\alpha)\cos(\psi(x,y,z)+\delta)$$
$$\ddot{z} = -\left(\frac{1-\mu}{r_{ps}^3} + \frac{\mu}{r_{pe}^3}\right)z + \kappa\sin(\psi(x,y,z)+\delta) \tag{2}$$

2.2 Equilibrium points

To find the equilibrium points of this system, one has to nullify (2) and analyse the 3 equations. Because we assumed that the solar sail is perpendicular to the radiation of the sun κ is zero. Lets assume that $z = 0$, then $y \neq 0$. On the other hand if $z \neq 0$ then the expression inside the brackets should be 0 and hence $y = 0$. If $y \neq 0$ then we see that $r_{ps} = r_{pe} = 1$, from this one can see in figure 2 that both equilibrium points $\{L_1, L_2\}$ are at the same distance from both masses. Setting $y = z = 0$, one can see that the first equation in (3) gives three more equilibrium points $\{L_3, L_4, L_5\}$.

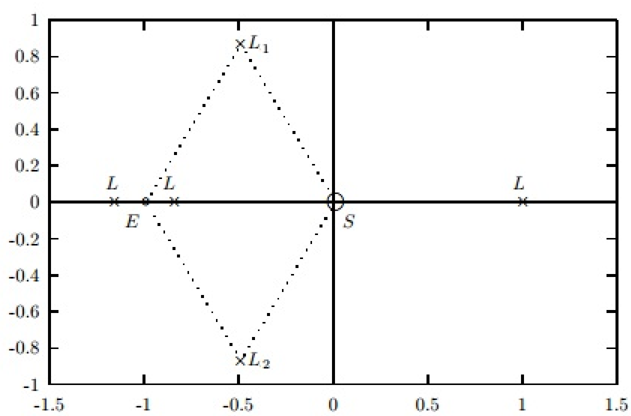

Figure 2: Equilibrium points of the RTB-problem

3 Augmented Hill problem

Assume that the Earth, a massive body, is in a fixed position and assume that the sun, a massive body as well, is at such a distance that it produces a uniform gravitational field around the Earth. The classical Hill problem deals with the attraction of two massive bodies, described as above, on a smaller body in comparison e.g. an asteroid. Here the position of the solar sail is defined by (x, y, z) and the distance by $r = \sqrt{x^2 + y^2 + z^2}$. Now consider a rotating reference frame centered on the small body. Normalizing the units of distance and time gives $L = (\mu_{sb}/\mu_{sun})^{1/3}R$ and $T = 1/\omega$ where μ_{sun} and μ_{sb} are the gravitational parameters of the Sun[1] and the small body[2] respectively, R is the distance between asteroid and the Sun and where $\omega = \sqrt{\mu_{sun}/R^3}$ is the angular velocity of the sun. The normalised lightness number is $\beta(\mu_{sb}\omega^4)^{-1/3}$, knowing what β, ω and P are we can rewrite the lighting number as $K(A/m){\mu_{sb}}^{-1/3}$ [4]. With the normalised units in mind the 3 equations of motions for a solar sail become:

$$\begin{aligned} \ddot{x} - 2\dot{y} &= \frac{\partial \Pi}{\partial x} + a_x \\ \ddot{y} + 2\dot{x} &= \frac{\partial \Pi}{\partial y} + a_y \\ \ddot{z} &= \frac{\partial \Pi}{\partial z} + a_z \end{aligned} \tag{3}$$

where $\Pi(x, y, z) = \dfrac{1}{r} + \dfrac{1}{2}(3x^2 - z^2)$ is as before. Working this out gives the following:

$$\begin{aligned} \ddot{x} - 2\dot{y} &= -\frac{x}{r^3} + 3x + a_x \\ \ddot{y} + 2\dot{x} &= -\frac{y}{r^3} + a_y \\ \ddot{z} &= -\frac{z}{r^3} - z + a_z \end{aligned} \tag{4}$$

[1] $\mu_{sun} = 1.327 \times 10^{11} \mathrm{km}^3/\mathrm{s}^2$, where μ_{sun} is the product of the gravitational constant and mass of the Sun [4].

[2] The gravitational parameter μ_{sb} depends on the asteroid, e.g. $\mu_{ceres} = 63.2\ \mathrm{km}^3/\mathrm{s}^2$ [4].

Because the Sun is sufficiently large and at a sufficient distance, it produces a uniform force field. The direction of the sun is constant and the acceleration of the solar sail are defined respectively by $r_s = (1, 0, 0)$ and $\vec{a} = (a_x, a_y, a_z)$, where $\vec{a}$ is as follows.

$$\begin{aligned} a_x &= p\beta\frac{m_s}{r_{ps}^2}\cos^3\alpha\cos^3\delta + \frac{1}{2}(1-p)\beta\frac{m_s}{r_{ps}^2}\vec{r_s} \\ a_y &= p\beta\frac{m_s}{r_{ps}^2}\cos^2\alpha\cos^3\delta\sin\alpha \\ a_z &= p\beta\frac{m_s}{r_{ps}^2}\cos^2\alpha\cos^2\delta\sin\delta \end{aligned} \tag{5}$$

δ, α are still the same angles as before. The equations of motion can be described now. Before describing this system as a Hamiltonian function the definition is given.

Definition *A Hamiltonian H is a function of $2n$ coordinates $q = (q_1, q_2, \ldots, q_n)$ and $p = (p_1, p_2, \ldots, p_n)$ that satisfies the following relations where $1 \leq i \leq n$*

$$\begin{aligned} \dot{p}_i &= -\frac{\partial H}{\partial q_i} \\ \dot{q}_i &= \frac{\partial H}{\partial p_i} \end{aligned}$$

System (4) can be nicely written as a Hamiltonian system H.

$$\begin{aligned} P_x &= \dot{x} - y \\ P_y &= \dot{y} + x \\ P_z &= \dot{z} \end{aligned}$$

$$H = \frac{1}{2}(P_x^2 + P_y^2 + P_z^2) + yP_x - xP_y - \frac{1}{2}(2x^2 - y^2 - z^2) - \frac{1}{r} - a_x x - a_y y - a_z z \tag{6}$$

Setting $\beta = 0$ one can see that (4) has a couple of equilibrium points. Setting everything to 0 it is seen that $y = z = 0$ which gives that $3x = x/r^3$, where $r^3 = \pm x^3$. Both equilibrium points $L_1 = (3^{-1/3}, 0, 0)$, $L_2 = (-3^{-1/3}, 0, 0)$ are located at the same distance from the asteroid in opposite directions. In figure 3, one can see how the equilibrium point x_s behaves for different values of β

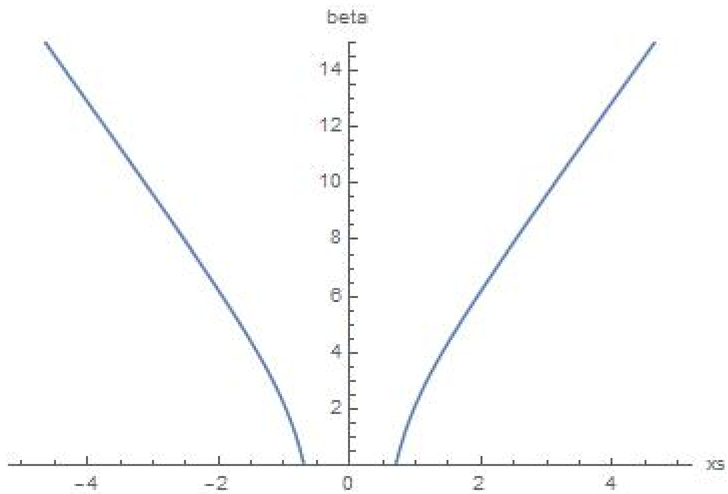

Figure 3: Behaviour of the equilibrium point

3.1 Linearization

After making a power expansion around $t = 0$ for (6), H_2 is obtained. It is essential to to linearize H_2 at the equilibrium points. In order to do that, the partial derivatives are needed and are as follows:

$$\begin{aligned}
\frac{\partial H}{\partial P_x} &= P_x + y \\
\frac{\partial H}{\partial P_y} &= P_y - x \\
\frac{\partial H}{\partial P_z} &= P_z \\
\frac{\partial H}{\partial y} &= P_x + y + \frac{y}{\sqrt{(x^2+y^2+z^2)^3}} \\
\frac{\partial H}{\partial x} &= -P_y - 2x + \frac{x}{\sqrt{(x^2+y^2+z^2)^3}} - a_x \\
\frac{\partial H}{\partial z} &= z + \frac{z}{\sqrt{(x^2+y^2+z^2)^3}}
\end{aligned} \tag{7}$$

Setting the system (7) to zero, one finds that $P_x = P_z = z = y = 0$. Ignoring constant terms, the following expression for $H_2(x, y)$ is derived (see Appendix for the full computation). Where $x_s = 1.36$ is the solution for the equilibrium point for $(\beta = 5, \rho = 0.85, \alpha = 0, \delta = 0)$:

$$H_2(x,y) = \frac{P_x^2}{2} + \frac{P_y^2}{2} + \frac{P_z^2}{2} - xP_y - x^2 - \frac{x^2}{x_s^3} - x_s^2 + P_x y + \frac{y^2}{2} + \frac{y^2}{2x_s^2} + \frac{z^2}{2} + \frac{z^2}{2x_s^3} \quad (8)$$

Computing the linearization at the equilibrium point of $H_2(x, y)$ and setting $1 + 1/x_s^3 = s$, one gets the following matrix:

$$X = \begin{pmatrix} 0 & -1 & 0 & 2s & 0 & 0 \\ 1 & 0 & 0 & 0 & -s & 0 \\ 0 & 0 & 0 & 0 & 0 & -s \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Now let V be the space spanned by the eigenvectors of X and let ID_+, ID_- be arrays containing the positive and negative eigenvalues. Now one can construct a symplectic basis for the eigenvectors of this Hamiltonian system. Let J be a symplectic 6×6 matrix of the following form

$$\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

Computing the eigenvalues of the Hamiltonian matrix X gives 3 pairs of eigenvalues $(\lambda_1, -\lambda_1, \lambda_2, -\lambda_2, \lambda_3, -\lambda_3)$. V is spanned by two bases $U, W \in \mathbb{R}^{n\times 2n}$, i.e. $V = U \bigoplus W$ where they correspond respectively with the positive and negative eigenvalues.

$$U = \lambda_1 U_1 + \lambda_2 U_2 + \lambda_3 U_3$$
$$W = -\lambda_1 W_1 - \lambda_2 W_2 - \lambda_3 W_3$$

In order to construct the symplectic bases, K is defined:

$$K = (\begin{bmatrix}\lambda_1 U_1 & \lambda_2 U_2 & \lambda_3 U_3\end{bmatrix} J \begin{bmatrix}\lambda_{-1} W_1 & \lambda_{-2} W_2 & \lambda_{-3} W_3\end{bmatrix})^{-1}$$

And the symplectic basis is written as a 6×6 matrix:

$$B = \begin{bmatrix}\lambda_1 U_1 & \lambda_2 U_2 & \lambda_3 U_3 & K^T(\lambda_{-1} W_1 & \lambda_{-2} U_2 & \lambda_{-3} W_3)\end{bmatrix}$$

Now to be certain that the final transformation of coordinates is real, one has to be sure that the symplectic pairs U, W have the same length. This process is given in the Appendix. Now because the canonical transformation matrix B is obtained, the change of coordinates can be carried out. Making the power expansion to the third degree of the original system and ignoring constant terms gives the following:

$$H_3(x, y) = \frac{x^3}{x_s^4} - \frac{3xy^2}{2x_s^4} - \frac{3xz^2}{2x_s^4}$$

Putting (8) through the canonical transformation gives the following expression

$$H_2(q, p) = \lambda_1 q_1 p_1 + i\omega_1 q_2 p_2 + i\omega_2 q_3 p_3 \tag{9}$$

The equation which is derived has a complex form. In order to evade this problem it is suitable to transform the system into real coordinates. The transformation $(q, p) \to (x, y)$ is given by the following equations for $i = 2, 3$:

$$q_i = \frac{x_i - iy_i}{\sqrt{2}}$$
$$p_i = \frac{y_i - ix_i}{\sqrt{2}}$$

As one can observe, q_i, ip_i are each other's conjugates. Because q_1, p_1 are real, its transformations are trivial i.e. $q_1 = x_1$ and $p_1 = y_1$.

4 Centre manifold

The splitting of the centre manifold from the hyperbolic one in (8) is done by following a certain programme which includes normalizing transformations on the power expansion of the Hamiltonian system at the equilibrium point. Lets assume that the origin is the equilibrium point of the Hamiltonian system which is written as a power expansion. The Hamiltonian system can now be expressed as follows:

$$H(q,p) = \sum_{n \geq 2} H_n(q,p)$$

Assume that $H_2 = \lambda_1 q_1 p_1 + i\omega_1 q_2 p_2 + i\omega_2 q_3 p_3$ and that the rest of the H_n's are homogeneous polynomials. Note that it is not necessary for the linear part of the Hamiltonian system to be diagonal. But then if the degree of the system increase, so will the dimensions of the linear system and there will be computational errors. The linear system can be solved if and only if $H_2 \neq 0$ which is always true because $\lambda \in \mathbb{R}/\{0\}$ [2].

4.1 The Lie series method

Before going into the Lie Series method some definitions are given.

Definition *A transformation of coordinates is canonical if it preserves the form of Hamilton's equations.*

Let $x = (q,p)$ and let J be the following $2n \times 2n$ block matrix where I_n is the identity matrix:

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

Hamilton's equations can now be expressed as follows:

$$\dot{\mathbf{x}} = J \nabla H$$

Let $y : x_i \mapsto y_i(x)$ be a transformation of coordinates. Then the following holds

$$\dot{y_i} = \sum_{j=1}^{2n} \frac{\partial y_i}{\partial x_j} \dot{x_j} = \sum_{j=1}^{2n} \frac{\partial y_i}{x_j} \left(J \frac{\partial H}{\partial \mathbf{x}} \right)_j$$

and hence:

$$\dot{\mathbf{y}} = \left(N J N^T \right) \nabla H$$

Where $(N)_{ij} = \partial y_i / \partial x_j$ is the Jacobian of y. The Hamiltonian form is preserved if

$$N J N^T = J$$

Hence such a transformation is canonical if the above equation is satisfied Such a matrix N is called symplectic,

Definition *Poisson brackets give a binary operation on two functions $f(q,p), g(q,p)$ defined by:*

$$\{f,g\} = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q}$$

One can see that $\{f,g\} = -\{g,f\}$ and that the following relations imply that the coordinates (q,p) are canonical:

$$\begin{aligned}\{q_i, q_j\} &= 0\\ \{p_i, p_j\} &= 0\\ \{q_i, p_j\} &= \delta_{ij}\end{aligned}$$

The last equation can be put into a theorem which shall be proved:

Theorem 4.1 *The operation carried out by the Poisson brackets is invariant under canonical transformations and conversely if*

$$\begin{aligned}\boldsymbol{Q}\colon\ q_i &\mapsto Q_i(q,p)\\ \boldsymbol{P}\colon\ p_i &\mapsto P_i(q,p)\end{aligned}$$

is a transformation such that the following hold, then the transformation is canonical:

$$\begin{aligned}\{Q_i, Q_j\} &= 0\\ \{P_i, P_j\} &= 0\\ \{Q_i, P_j\} &= \delta_{ij}\end{aligned}$$

Proof Consider two functions $f(q,p)$ and $g(q,p)$ be two random functions. Then their Poisson brackets can be expressed as follows:

$$\begin{aligned}\{f,g\} &= \sum_{i=1}^{n}\left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}\right)\\ &= \frac{\partial f}{\partial \mathbf{x}} J (\frac{\partial g}{\partial \mathbf{x}})^T\end{aligned}$$

Let $\mathbf{y(x)}$ be a transformation with the Jacobian N, then the following holds:

$$\frac{\partial f}{\partial \mathbf{x}} = \frac{\partial f}{\partial \mathbf{y}} N$$

Putting this together yields:

$$\begin{aligned}\{f,g\} &= \frac{\partial f}{\mathbf{x}} N J N^T (\frac{\partial g}{\partial \mathbf{x}})^T\\ &= \frac{\partial f}{\partial \mathbf{y}} N J (\frac{\partial f}{\partial \mathbf{y}} N)^T\\ &= \frac{\partial f}{\partial \mathbf{y}} N J N^T (\frac{\partial f}{\partial \mathbf{y}})^T\\ &= \frac{\partial f}{\partial \mathbf{y}} J (\frac{\partial f}{\partial \mathbf{y}})^T\end{aligned}$$

This concludes that operations under the Poisson brackets are invariant. To prove the last part the Jacobian is written as follows:

$$N = \begin{pmatrix} \frac{\partial \mathbf{Q}}{\partial q} & \frac{\partial \mathbf{Q}}{\partial p} \\ \frac{\partial \mathbf{P}}{\partial q} & \frac{\partial \mathbf{P}}{\partial p} \end{pmatrix}$$

Then the following can be computed where $(\{Q,P\})_{ij} = \{Q_i, P_j\}$,

$$NJN^T = \begin{pmatrix} \{Q,Q\} & \{Q,P\} \\ \{P,Q\} & \{P,P\} \end{pmatrix}$$

Hence $NJN^T = J$ and the transformation is canonical [5]. $\square$

The Hamiltonian form is very practical because they give the opportunity to work on a single function instead of a system. To get the normal form one has to get through a procedure. Let $(q_0(t), p_0(t))$ be the orbits of a Hamiltonian system $L(q,p)$ with k degrees of freedom. Then $\forall f \in C^\infty$ the following holds:

$$\begin{aligned} \frac{d}{dt} f(q_0(t), p_0(t)) &= \frac{\partial f}{\partial q_0(t)} \frac{\partial q_0(t)}{\partial t} + \frac{\partial f}{\partial p_0(t)} \frac{\partial p_0(t)}{\partial t} \\ &= \frac{\partial f}{\partial q_0(t)} \dot{q_0} + \frac{\partial f}{\partial p_0(t)} \dot{p_0} \\ &= \frac{\partial f}{\partial q_0(t)} \frac{\partial L}{\partial p_0(t)} - \frac{\partial f}{\partial p_0(t)} \frac{\partial L}{\partial p_0(t)} \\ &= \{f, L\}(q_0(t), p_0(t)) \end{aligned} \tag{10}$$

Now with the Taylor series, around $t = 0$, in mind and using (10) as replacement of the derivatives the power expansion of H can be denoted as follows for the time 1 flow:

$$\begin{aligned} H(q(t), p(t)) = H(q(0), p(0)) &+ \frac{\partial H(q(0), p(0))}{\partial t}(q(t), p(t)) + \\ &\frac{1}{2!} \frac{\partial^2 H(q(0), p(0))}{\partial t^2}(q(t), p(t))^2 + \frac{1}{3!} \frac{\partial^3 H(q(0), p(0))}{\partial t^3}(q(t), p(t))^3 \\ &= H(q(0), p(0)) + \{H, L\}(q(0), p(0))(q(t), p(t)) + \\ &\frac{1}{2!} \frac{\partial \{H, L\}(q(0), p(0))}{\partial t}(q(t), p(t))^2 + \frac{1}{3!} \frac{\partial^2 \{H, L\}(q(0), p(0))}{\partial t^2}(q(t), p(t))^3 \\ &= H(q(0), p(0)) + \{H, L\}(q(0), p(0))(q(t), p(t)) + \\ &\frac{1}{2!}\{\{H, L\}, L\}(q(0), p(0))(q(t), p(t))^2 + \frac{1}{3!}\{\{\{H, L\}, L\}, L\}(q(0), p(0))(q(t), p(t))^3 \end{aligned}$$

Deleting the $(q(0), p(0))$ terms gives the following equation

$$\hat{H} \equiv H + \{H, L\} + \frac{1}{2!}\{\{H, L\}, L\} + \frac{1}{3!}\{\{\{H, L\}, L\}, L\} + \dots,$$

Because this expression can easily be computed, it is very convenient for practical purposes and can give high order approximations. Let's assume that L and G are two homogeneous polynomials with $deg(L) = r$ and $deg(G) = k$. Then looking closely at the definition of the Poisson brackets, one can see that $deg(\{L, G\}) = r + k - 2$. Choosing a generating homogeneous polynomial of degree 3 L_3 to get rid of the monomials of degree 3, the transformed Hamiltonian of third, fourth, fifth etc. degree can be expressed as follows:

$$\begin{aligned} \hat{H}_3 &= H_3 + \{H_2, L_3\} \\ \hat{H}_4 &= H_4 + \{H_2, L_4\} + \{H_3, L_3\} + \frac{1}{2!}\{\{H_2, L_3\}, L_3\} \end{aligned} \tag{11}$$

Because the central directions needs to be uncoupled from the hyperbolic ones, it is not interesting to compute the complete normal form. To have a great radius of convergence it is necessary to stay close to the original form, thus the number of elimination of the monomials should stay low. In order to do so consider the following monomial $q_1^{k_{q_1}} q_2^{k_{q_2}} q_3^{k_{q_3}} p_1^{k_{p_1}} p_2^{k_{p_2}} p_3^{k_{p_3}}$ which is written shortly as $q^{k_q} p^{k_p}$, where $k_q = (k_{q_1}, k_{q_2}, k_{q_3})$ and $k_p = (k_{p_1}, k_{p_2}, k_{p_3})$ are integer vectors. The first component of the integer vector k_q cannot be equal to the first component of the integer vector k_p i.e. $k_{q_1} \neq k_{p_1}$. Let A: $\mathbb{C}^6 \longrightarrow \mathbb{C}^6$ be an operator such that it only preserves the terms where $k_{q_1} = k_{p_1}$. Using the fact that H and L are homogeneous polynomials we can write the following:

$$\begin{aligned} H_3(q, p) &= \sum_{|k_q|+|k_p|=3} h_{k_q, k_p} q^{k_q} p^{k_p} \\ L_3(q, p) &= \sum_{|k_q|+|k_p|=3} g_{k_q, k_p} q^{k_q} p^{k_p} \end{aligned}$$

Where h_{k_q,k_p} and g_{k_q,k_p} are coefficients of the polynomial. The normalized H_3 is then expressed as follows:

$$\begin{aligned} \hat{H}_3 &= AH_3 \\ &= H_3 + \{H_2, L_3\} \end{aligned}$$

This gives the following where I is the identity operator,

$$\{H_2, L_3\} = -(I - A)H_3$$

Now one can write the following:

$$\begin{aligned}\{H_2, L_3\} &= \sum_{|k_q|+|k_p|=3} < k_p - k_q, \psi > g_{k_q,k_p} q^{k_q} p^{k_p} \\ &= -(I-A) \sum_{|k_q|+|k_p|=3} h_{k_q,k_p} q^{k_q} p^{k_p} \\ &= \sum_{\substack{|k_q|+|k_p|=3 \\ k_{q_1}\neq k_{p_1}}} -h_{k_q,k_p} q^{k_q} p^{k_p}\end{aligned} \tag{12}$$

Where $\psi = (\psi_1, \psi_2, \psi_3)$ and each element is defined as follows:

$$\begin{aligned}\psi_1 &= \lambda_1 \\ \psi_2 &= i\omega_1 \\ \psi_3 &= i\omega_2\end{aligned}$$

So one obtains the following:

$$L_3(q,p) = (I-A) \sum_{|k_q|+|k_p|=3} \frac{-h_{k_p,k_q}}{< k_p - k_q, \psi >} q^{k_q} p^{k_p}$$

$$L_3(q,p) = \sum_{\substack{|k_q|+|k_p|=3 \\ k_{q_1}\neq k_{p_1}}} \frac{-h_{k_p,k_q}}{< k_p - k_q, \psi >} q^{k_q} p^{k_p}$$

Implementing the obtained generating function in (11) will give $\hat{H}_3$. However up to order 3 the normal form can be obtained without following these steps and because in this case H_4 is not computed, the paper will show an alternative way of deriving $\hat{H}_3$.

Now because $n \geq 3$ the transformed Hamiltonian can be expressed as follows.

$$\hat{H}(q,p) = H_2(q,p) + \hat{H}_3(q,p) + \hat{H}_4(q,p)+,\ldots,+\hat{H}_N(q,p)$$

Because the monomial $q^{k_p}p^{k_p}$ is eliminated such that the first factor of k_p is different from k_q the homogeneous polynomial $\hat{H}_3$ depends on q_1p_1

$$\hat{H}_3(q,p) = \hat{H}_3(q_1p_1, q_2, p_2, q_3, p_3)$$

Now $\hat{H}_3(q,p)$ can be written as follows where $c_i \in \mathbb{R}$ for $i = 1, 2, \ldots, 12$:

$$\begin{aligned} \hat{H}_3(q,p) &= c_1 i p_2^3 + c_2 i p_2 p_3^2 + c_3 i p_1 p_2 q_1 - c_4 p_2^2 q_2 + c_5 p_3^2 q_2 \\ +c_6 p_1 q_1 q_2 &+ c_7 i p_2 q_2^2 - c_8 q_2^3 - c_9 p_2 p_3 q_3 + c_{10} i p_3 q_2 q_3 - c_{11} i p_2 q_3^2 - c_{12} q_2 q_3^2 \end{aligned}$$

This can be done until a finite order N where $\hat{H}_N$ depends on q_1p_1 as well. Then a Hamiltonian system of the following form can be seen.

$$\bar{H}(q,p) = \bar{H}_N(q,p) + R_{N+1}(q,p)$$

Where R_{N+1} is called the remainder of order $N+1$. In order to approximate the dynamics near the origin, the remainder is dropped [5]. Next using $I_1 = q_1p_1$, $\bar{q} = (q_2, q_3)$ and $\bar{p} = (p_2, p_3)$ and setting $I_1 = 0$ one restricts the Hamiltonian system to an invariant model.
Applying the real transformation to $H_2(q,p)$ and $\hat{H}_3(q,p)$ gives the following, where $a_i, b_i \in \mathbb{R}$ for $i = 1, 2, 3, 4, 5$:

$$\begin{aligned} H_2(x,y) &= a_1x_2^2 + a_2x_3^2 + a_3x_1y_1 + a_4y_2^2 + a_5y_3^2 \\ \hat{H}_3(x,y) &= b_1x_2^3 + b_2x_1x_2y_1 + b_3x_2y_2^2 + b_4x_2y_3^2 \end{aligned}$$

This gives the final transformed Hamiltonian $\bar{H}_N(0, \bar{q}, \bar{p})$ the following form

$$\begin{aligned} \bar{H}_3(x,y) &= H_2(x,y) + \hat{H}_3(x,y) \\ &= a_1x_2^2 + a_2x_3^2 + a_3x_1y_1 + a_4y_2^2 + a_5y_3^2 \\ &+ b_1x_2^3 + b_2x_1x_2y_1 + b_3x_2y_2^2 + b_4x_2y_3^2 \end{aligned}$$

Filling in the coefficients, one gets the following:

$$\begin{aligned} \bar{H}_3(x,y) &= H_2(x,y) + \hat{H}_3(x,y) \\ &= 0.626x_2^2 + 0.591x_3^2 + 0.626y_2^2 + 0.591y_3^2 \\ &- 0.066x_2^3 + 0.451x_2y_2^2 + 0.225x_2y_3^2 \end{aligned}$$

5 Poincaré sections

Now the dynamics of the reduced centre manifold are shown. Let (x_h, y_h) and (x_v, y_v) relate respectively to the horizontal and vertical oscillations such that $(x_h, y_h) = (q_2, p_2)$ and $(x_v, y_v) = (q_3, p_3)$. Fixing $\rho = 0.85$ and $\beta = 5$, which corresponds to a small sail near a large asteroid [7], gives 1.36 as a solution for x_s. Now consider the Poincaré section $x_v = 0$, then the reduced Hamiltonian becomes as follows:

$$\bar{H}_3(0, x_h, y_h, x_v = 0, y_v) = a_1 x_h^2 + a_4 y_v^2 + a_5 y_v^2 + b_1 x_h^3 + b_3 x_h y_h^2 + b_4 x_2 y_v^2$$

Selecting the energy level such that $\bar{H}_3(0, x_h, y_h, x_v, y_v = 0) = 0.4$ gives a graph in (x_h, y_h) coordinates such that only the values $x_v > 0$ are computed where it intersects with $y_v = 0$ as one can see in figure 4.

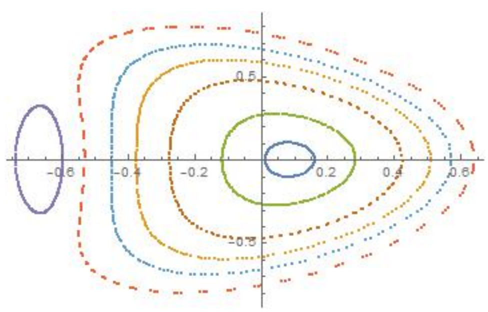

Figure 4: $\bar{H}_3 = 0.4$, $\alpha = \delta = 0$, Poincaré section $y_v = 0$

In figure 5 the Hamiltonian at the same energy level sliced at $x_h = 0$ is shown. It is seen that the curves, for both figures, are invariant. The symmetric form comes from the fact that δ is taken as zero.

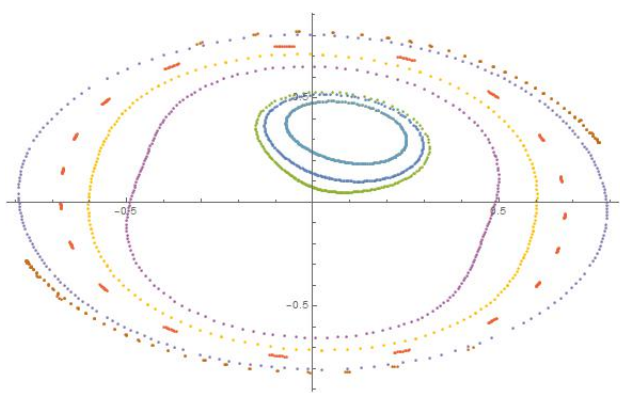

Figure 5: $\bar{H}_3 = 0.4$, $\alpha = \delta = 0$, Poincaré section $x_h = 0$

Now fixing the energy level as 0.8 and computing the Poincaré section at $y_v = 0$, one can see at figure 6 that new elliptic points appeared inside the greater orbit. These orbits are called Halo orbits which are situated around the Langrangian points. The practical use of these orbits is to station a probe to keep a continuous monitoring between 2 bodies. Situating the probe in a alternative position could bring noise from the third body to the antenna's on earth, because the third body lies in the path between the antenna and the probe. Slicing figure 6 at $x_h = 0$ gives orbits of the same structure as in figure 5. However this time Halo orbits appear.

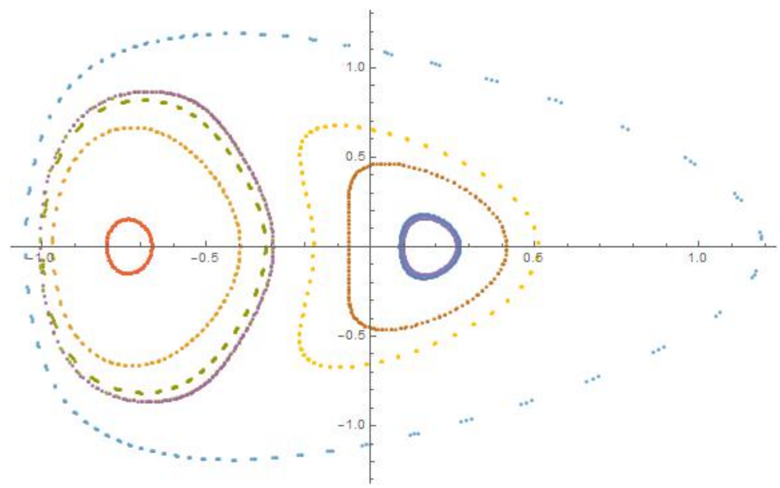

Figure 6: $\bar{H}_3 = 0.8$, $\alpha = \delta = 0$, Poincaré section $y_v = 0$

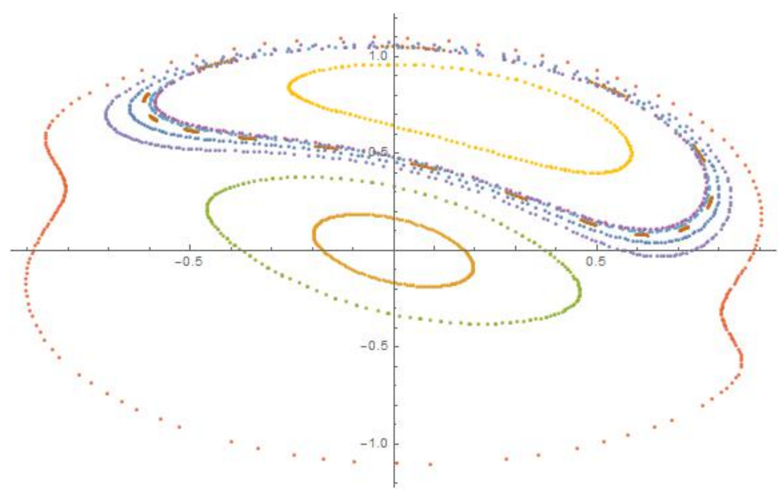

Figure 7: $\bar{H}_3 = 0.8$, $\alpha = \delta = 0$, Poincaré section $x_h = 0$

In the next 2 graphs (Figure 8 and 9) the Poincaré sections $y_v = 0$ and $x_h = 0$ are computed where the energy level is fixed at a low constant 0.02. It is seen that for a low constant the orbits have an almost perfect elliptic form.

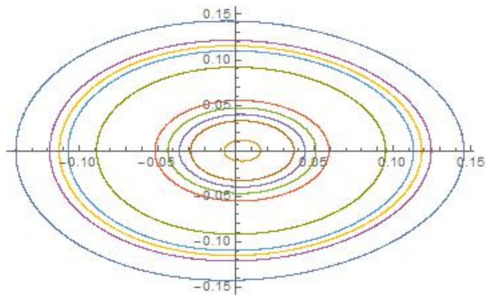

Figure 8: $\bar{H}_3 = 0.02$, $\alpha = \delta = 0$, Poincaré section $y_v = 0$

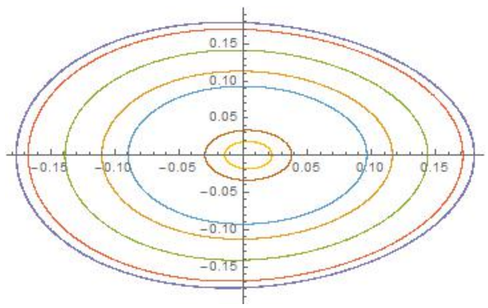

Figure 9: $\bar{H}_3 = 0.02$, $\alpha = \delta = 0$, Poincaré section $x_h = 0$

Now the Hamiltonian is fixed at a higher energy level $\bar{H}_3 = 1.2$ and sliced again at both $y_v = 0$ and $x_h = 0$ as one can see in figure 10 and 11. The Poincaré sections are similar to the ones for $\bar{H}_3 = 0.8$ in the sense that the same periods are observed as well that the same halo orbits appear with a larger periods.

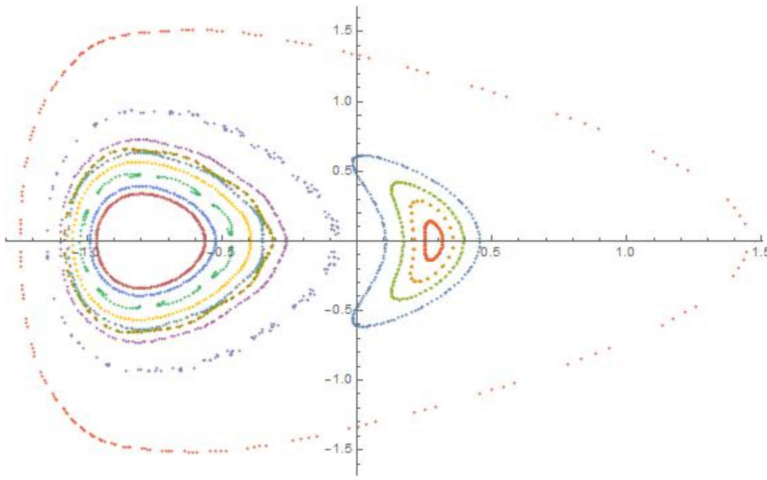

Figure 10: $\bar{H}_3 = 1.2$, $\alpha = \delta = 0$, Poincaré section $y_v = 0$

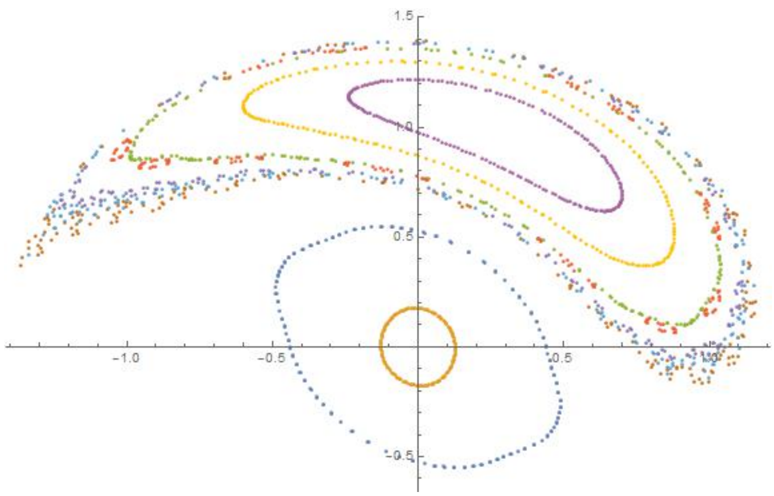

Figure 11: $\bar{H}_3 = 1.2$, $\alpha = \delta = 0$, Poincaré section $x_h = 0$

6 Conclusion

The paper showed in the beginning the derivation of a solar sail as a part of the Earth-Sun system. Then the paper used the augmented Hill problem to describe the motion of the solar sail nearby a small asteroid within the Sun's orbit. The motion is given as a Hamiltonian system which, at the end, is reduced to a central manifold. This is done by a normalization process. After obtaining the reduced form the dynamics of the orbits are described in a graph.

7 References

[1] Z.E. Musielak, B Quarles. The three-body problem, *Reports on Progress in Physics, Volume 77, Issue 6*, 30-31 2012

[2] Angel Jorba, A methodology for the numerical computation of normal forms,centre manifolds and first integrals of Hamiltonian systems, *Experimental Mathematics Volume 8, 1999 - Issue 2*, 182-190, 1999

[3] Bernd Dachwald and Wolfgang Seboldt, Potential solar sail degradation effects on trajectory and attitude control,*AIAA Guidance, Navigation, and Control Conference and Exhibit, San Francisco*, 3-5, 2005

[4] Ariadna Farrés, Angel Jorba and Josep-Maria Mondela, Orbital dynamics for a non-perfectly reflecting solar sail close to an asteroid, *2nd IAA Coinference on Dynamics and Control of Space Systems,Rome*, 2-4, 2009

[5] Brian Tu,Applications of canonical transformations in hamiltonian mechanics,*Cornell University Database*, 2-4, 2014

[6] L. Rios-Reyes abd D.J. Scheeres, Robust solar sail trajectory control for large pre-launch modeling errors,*American Institute of Aeronautics and Astronautics*, 2-6, 2005

[7] A.Farres and A. Jorba, Dynamics, Geometry and Solar Sails,*Indagationes Mathematicae June 2016*, 1-15, 2016

A Appendix

A.1 Computing quadratic and cubic part

The Hamiltonian function is described as follows

```
H=1/2(px^2+py^2+pz^2)+ypx-xpy-1/2(2x^2-y^2-z^2)
-1/Sqrt[x^2+y^2+z^2]
```

Finding the equilibrium point for x_s

```
[in] FullSimplify[Nsolve[x^3-4.625x+1=0,x]]
[out] {{x->-0.41296},{x->0.592},{x->1.36}}
```

Making a second degree power series

```
[in]Expand[simplify[Normal[Series[H/.{x->xs+X,py->ps+Py}
,{X,0,2},{y,0,2},
{z,0,2},{px,0,2},{Py,0,2},{pz,0,2}]],xs>0]]
```

Deleting the higher and lower order terms gives H_2

```
H2 = 1/2(px^2+Py^2+pz^2+y^2+z^2)+pxy-xPy+1/2xs^3(z^2+y^2)
/.{Py->py,X->x}
```

Computing the linearization at the equilibrium of H_2

```
L={
Table[D[D[H2,px],u],{u,{x,y,z,px,py,pz}}],
Table[D[D[H2,py],u],{u,{x,y,z,px,py,pz}}],
Table[D[D[H2,pz],u],{u,{x,y,z,px,py,pz}}],
Table[D[-D[H2,x],u],{u,{x,y,z,px,py,pz}}],
Table[D[-D[H2,y],u],{u,{x,y,z,px,py,pz}}],
Table[D[-D[H2,z],u],{u,{x,y,z,px,py,pz}}],
```

Computing the eigenvalues for the Hessian of H2

```
[in] FullSimplify[Eigenvalues[L]/.xs->1.36‘]
```

Making a third degree power series

```
Expand[simplify[Normal[Series[H/.{x->xs+X,py->ps+Py}
,{X,0,3},{y,0,3},
{z,0,3},{px,0,3},{Py,0,3},{pz,0,3}]],xs>0]]
```

Canceling out the higher and lower order terms gives

```
H3= x^3/xs^4-3xy^2/2xs^4-3xz^2/2xs^4
```

A.2 Diagonalization of the quadratic and cubic part

In this section the coordinates are slightly changed $(x, P_x, y, P_y, z, P_z) = (x_1, y_1, x_2, y_2, z_1, z_2)$
Creating a symplectic 2nx2n matrix

```
SymplecticMatrix[n_] := ArrayFlatten[{
{ConstantArray[0, {n, n}], IdentityMatrix[n]},
{-IdentityMatrix[n], ConstantArray[0, {n, n}]}
}]

SymplecticBasis[basisU_, basisW_] := Module[{J, n, K},
(* basisU and basisW are nx(2n) square matrices,
each row is a basis vector of U and W respectively *)
n = Length[basisU];
J = SymplecticMatrix[n];
K = Inverse[basisU.J.Transpose[basisW]];
Join[basisU, Transpose[K].basisW]
]
```

Constructing diagonalization matrix such that the symplectic pairs have the same length

```
SymplecticDiagonalization[A_, idx1_, idx2_]
:= Module[{e, v, B, s, n},
{e, v} = Eigensystem[A];
(* if A is Hamiltonian then its eigenvalues come in pairs \
\[Lambda], -\[Lambda] *)
B = SymplecticBasis[v[[idx1]], v[[idx2]]];
Inverse[Transpose[B]].A.Transpose[B];
(* scale vectors so that symplectic pairs have the same length *)
n = Length[B]/2;
Do[
s = Sqrt[Norm[B[[k]]]/Norm[B[[k + n]]]];
B[[k]] = B[[k]]/s;
B[[k + n]] = B[[k + n]] s;
, {k, 1, n}];
Transpose[B]
]
```

Redefining H_2 in new coordinates

```
H2 = (y1^2 + y2^2)/2 - w^2 x1^2
+ 1/2 w^2 x2^2 - x1 y2 + x2 y1 +
1/2 (y3^2 + w^2 x3^2);
xsval = xs -> 136/100;
xsvalN = N[xsval, 20];
wval = w -> Simplify[(Sqrt[1 + 1/xs^3]
/. xs -> 136/100)];
wvalN = N[wval, 20]
```

Defining Poisson brackets

```
pb6[F_, G_] :=
D[F, x1] D[G, y1] - D[F, y1] D[G, x1] + D[F, x2] D[G, y2] -
D[F, y2] D[G, x2] + D[F, x3] D[G, y3] - D[F, y3] D[G, x3]
```

Creating the hessian of H_2

```
vf[H_, pb_, vars_] := Table[D[pb[u, H], v],
{u, vars}, {v, vars}]
AA0 = vf[H2, pb6, {x1, x2, x3, y1, y2, y3}] /. wval
```

Constructing transformation matrix such that the *qp* pairs correspond to the right eigenvalues

```
MM = Chop[N[SymplecticDiagonalization[AA0, {6, 1, 3}
, {5, 2, 4}], 20]]
```

Define Transformation function

```
trule = MapThread[
Rule, {{x1, x2, x3, y1, y2, y3}
, MM.{q1, q2, q3, p1, p2, p3}}]
```

Transforming quadratic and qubic part

```
H2pq = Chop[Expand[H2 /. wval /. trule]]
H3pq = Chop[Expand[x1^3 - (3 x1 x2^2)/(2)
- (3 x1 x3^2)/(2) /. trule]]
```

Defining transformation to real coordinate system

```
realRule = {
q1 -> x1, q2 -> (x2 - I y2)/
(Sqrt[2]), q3 -> (x3 - I y3)/(Sqrt[2]),
p1 -> y1, p2 -> (y2 - I x2)/
(Sqrt[2]), p3 -> (y3 - I x3)/(Sqrt[2])
};
```

Applying to the quadratic part

```
Chop[Expand[H2pq /. realRule], 10^-8]
```

Applying to the cubic part

```
Chop[Expand[H3pq /. realRule], 10^-8]
```

Normalizing cubic part

```
N3pq = Chop[
Expand[Select[H3pq, (Exponent[#, q1]
== Exponent[#, p1]) &]]]
```

Transforming normalized cubic part back to real coordinates

```
Chop[Expand[N3pq/xs^4 /. realRule /. xsvalN], 10^-8]
```

Defining transformed H

```
H[x2_, y2_, x3_, y3_] :=
0.626 x2^2 + 0.591 x3^2 + 0.626 y2^2
+ 0.591 y3^2 - 0.066 x2^3 +
0.451 x2 y2^2 + 0.225 x2 y3^2
```

Plotting a poincare section

```
Show[##, FrameLabel -> {"x2", "y2"}] &@
Table[ContourPlot[
H[x2, y2, x3, 0] == 0.4, {x2, 1, -1},
{y2, 1, -1}], {x3, -7, 7,
0.1}]
```

Defining the function as a system of PDE's

```
abc = {Derivative[1][x2][t] ==
1.252 y2[t] + 0.902 x2[t] y2[t],
Derivative[1][y2][t] == -1.252 x2[t]
+ 0.198 x2[t]^2 -
0.451 y2[t]^2 + 0.225 y3[t]^2,
Derivative[1][x3][t] == 1.182 y3[t] +
0.450 x2[t] y3[t],
Derivative[1][y3][t] == -1.182 x3[t]};
```

Defining the Poincare section

```
psect[{x02_, y02_, x03_, y03_}] :=
Reap[NDSolve[{abc, x2[0] == x02,
y2[0] == y02, x3[0] == x03,
y3[0] == y03,
WhenEvent[y3[t] == 0, Sow[{x2[t],
y2[t]}]]}, {}, {t, 0, 1000},
MaxSteps -> \[Infinity]]][[-1, 1]]
```

Plotting the poincare section $y_v = 0$ with the initial conditions for $H_{CM} = 0.8$

```
abcdata =
Map[psect, {{0.1, 0.1, 1.154037274292669, 0}
, {0.5, 0.5,
0.8617408500960113, 0}, {0.01, 1, 0.5354244106243065, 0}
, {1, 0.0001, 0.637252973578802, 0},
{0.1, 0.0001, 1.15894665496336,
0}, {0.2, 0.4, 1.057481604324925, 0},
{0.001, 1.1, 0.26656194242090897, 0},
{0.5, 0.1, 1.0432564856039253,
0}, {0.02, 1, 0.5279499796846946`, 0}, {0.03, 0.9,
0.6900539251592948`, 0}}];
ListPlot[abcdata, ImageSize -> Large]
```

Plotting the poincare section $y_v = 0$ with the initial conditions for $H_{CM} = 0.4$

```
abcdata =
Map[psect, {{0.1, 0.1, 0.809310249240813, 0}, {0.5, 0.1,
0.6415334315715956`, 0}, {0.4, 0.5, 0.4163819670331428`,
0}, {0.8, 0.1, 0.19848836523502253`, 0}}];
ListPlot[abcdata, ImageSize -> Medium]
```