university of groningen
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Skin surfaces in Möbius geometry

Master's thesis

22nd February, 2017

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Abstract

Studying the geometry of Van der Waals surfaces of molecules gives rise to the question: "Can we find smooth, continuous surfaces to ‘wrap around’ a certain set of spheres?". The theory of skin surfaces gives a simple algorithm to find this kind of surfaces. Underlying this relatively simple algorithm, however, are some quite interesting geometric properties, most of which can be related to orthogonality of sets of spheres. A natural way to work with orthogonal spheres is in the Möbius (or conformal) geometry. In this thesis we show how to view these skin surfaces inside the Möbius space directly.

Contents

1 Introduction: Interpolating spheres

1.1 Motivation and goal

Surfaces play an important role in a number of applications, ranging from natural sciences to computational geometry. For mathematical purposes, surfaces in $\mathbb{R}^n$ are usually relatively easy to describe, either explicitly, by graphing a function: $f : \mathbb{R}^{n-1} \to \mathbb{R}$, or implicitly, as the zero set of a suitable function: $g : \mathbb{R}^n \to \mathbb{R}$. In natural sciences however, one is often looking at approximations, given as a point set or patches of simpler shapes.

One way to approximate surfaces using simple shapes is using unions of spheres. In “Deformable Smooth Surface Design” [3], H. Edelsbrunner defines a class of surfaces, *skin surfaces*, formed by a set of spheres and smooth patches blending them. Computationally, skin surfaces have a few useful properties. They require little memory, as a sphere can be given by a centre and radius, and large complexity can be generated from relatively small inputs. A generalized scheme is given in [1], where *envelope surfaces* are introduced. Both methods define spheres centred ‘between’ the input spheres, such that the shape around the union of these spheres (the *envelope*), is a smooth surface.

An example of a practical application of skin surfaces is when the atoms in a molecule are taken as the input spheres, and the resulting skin surface as an approximation of the shape of the Van der Waals surfaces (see figure 1.1). As skin surfaces can be efficiently deformed, they can be useful for modelling the folding of proteins. In addition, the topology of the skin can be determined quickly, making it useful for, for example the docking of proteins.

A few of the defining characteristics of these skin surfaces are, below the surface, statements about orthogonal sets of spheres. These sets of spheres are bounded sections of orthogonal flats of spheres. These sets correspond to *convex hulls* in the space of weighted points, $\mathbb{R}^n \times \mathbb{R}$.

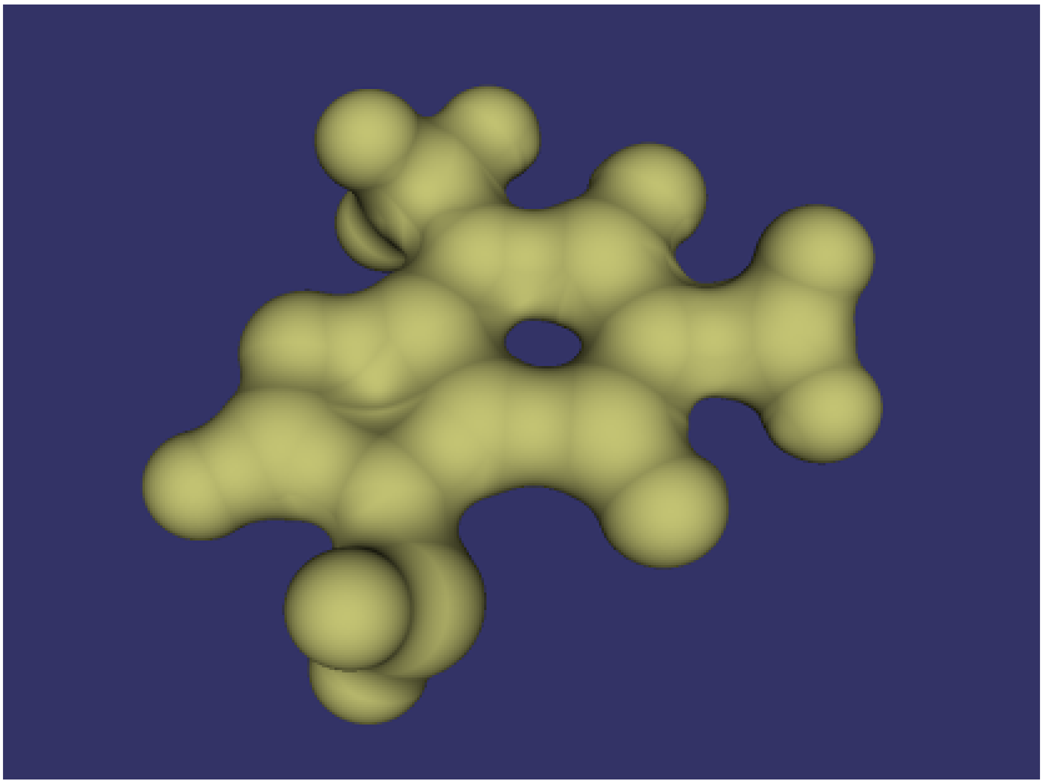

Figure 1.1: A caffeine molecule, described by a skin surface. This image is from [1].

The Möbius geometry is a model for describing spheres, hyperplanes and points of the real vector space $\mathbb{R}^n$, by representing them as points of the projective space $\mathbb{P}^{n+1}$. The aforementioned orthogonality of spheres is naturally described in Möbius geometry, therefore the goal of this thesis is:

Describe the skin surface, and its properties using Möbius geometry.

1.2 Outline and Results

The first step towards reaching our goal requires us to properly understand both skin surfaces and Möbius geometry. This is done in sections 2 and 3 respectively. The skin is introduced as in Edelsbrunner's [3], using the construction of the *space of weighted points*, $\mathbb{R}^n \times \mathbb{R}$. Here flats of spheres can be viewed as affine hulls. To find skin surfaces, we:

1. Take a convex hull of weighted points.
2. Shrink these represented spheres by multiplying their radii.
3. Take the envelope, the boundary of the union of the resulting set of spheres.

Lastly, we demonstrate two of the key properties of skin, decomposability and symmetry, and note their relation to orthogonal sets of spheres.

The Möbius space is introduced in the classical way, using a stereographic projection and an embedding into the projective space. This identifies spheres of $\mathbb{R}^n$ with points of the projective space $\mathbb{P}^{n+1}$, called the Möbius space. A quadratic form on this space describes, among other properties, the relation of orthogonality of spheres.

After introducing both the skin surface and the Möbius geometry, they are brought together in section 4. One of the key properties of skin surfaces can be viewed in the Möbius space naturally: The flats, or affine hulls, of spheres are represented by subspaces of the Möbius space. Introducing a concept of 'relative convexity' on the Möbius space allows us to find the associated convex hulls as well.

The usual method for shrinking spheres is point-wise, where each radius is simply multiplied by a factor s. Similar to how flats of spheres are represented by subspaces in the Möbius space, when shrunk, these shrunk flats can be shown to be be represented by a very restricted type of quadric. The notion of relative convexity can be extended to these quadrics, allowing us to find shrunk convex hulls of spheres in the Möbius space directly.

Having described the first two steps of finding the skin surface (the shrunk convex hulls) in the Möbius space, we leave the third step, taking the envelope, for later. In section 5, the current description of shrunk convex hulls in the Möbius space allows us to translate certain questions on skin surfaces in $\mathbb{R}^n$ into questions about quadrics in $\mathbb{P}^{n+1}$. We will use this to find, among others, the set of all possible skin surfaces around a given set of spheres.

The symmetry of skin surfaces is the fact that we can describe the surface from the in- and outside using shrunk flats of spheres. As said, these descriptions rely on orthogonal flats of spheres, which can be described in the Möbius space using a quadratic form. Section 6 uses a construction similar to relative convexity to describe, for any convex hull of spheres the set of spheres with the same envelope. This gives rise to a duality between two complexes, describing the skin surface from the in- and outside respectively. Using the duality and the decomposability of skin surfaces gives for each patch of the decomposition of $\mathbb{R}^n$ a pair of sets of spheres, where the dimensions of these two sets are k and $n-k$ respectively. As finding the envelope of a set of spheres is easier if the parameter space is of a lower dimension, this allows us to, for example, reduce finding the envelope of a dimension 2 set of spheres in $\mathbb{R}^3$ to one of dimension 1.

Finally, the last step used in defining the skin surface, finding the envelope, is viewed in the Möbius space. This changes the problem of finding an envelope of spheres (quadratic equations) in $\mathbb{R}^n$ to a problem of finding the envelope of planes (linear equations) in $\mathbb{P}^{n+1}$.

Additionally, the appendices contain among others an introduction to projective space, orthogonality and quadratic forms. Occasionally the main thesis will refer to this appendix, mostly for notation or small results. However, a lot of these concepts are assumed to be known, and therefore not in the main text.

2 The skin surface

As said, the subject of this thesis is describing surfaces in terms of sphere geometry. We will first define envelopes of sets of spheres. Using this, we can define the skin surface. This skin surface has a decomposition into quadratic patches, which is introduced using the (weighted) Voronoi- and Delauney complexes. At the end of the section the extended skin surface is introduced, which is more suited for approximation purposes than the normal skin surface.

2.1 Envelopes

Recall that we can determine a surface S in $\mathbb{R}^n$ implicitly, i.e. as the zero set of some C^1 function, $F : \mathbb{R}^n \to \mathbb{R}$. By taking $Z = \{x \in \mathbb{R}^n : F(x) \leq 0\}$, i.e. the set of points where F is negative, we can designate one side of this surface as an 'interior', of which S is the boundary. For a parametrized family of surfaces, $S(t)$, the envelope is the boundary of the union over t of these interiors. More formally:

Definition 1. Let $F_\mu : \mathbb{R}^n \to \mathbb{R}$ be a family of functions for parameter $\mu \in C$ where $C \subset \mathbb{R}^d$ (for some d), such that $F : (x, \mu) \mapsto F_\mu(x)$ is C^1. The *envelope* of the family is the boundary of $\cup Z_\mu$.

A point x is on this boundary if there is a parameter μ_0 such that $F_{\mu_0}(x) = 0$, and $F_\mu(x) \geq 0$ for all μ. This means that μ_0 is a global minimum of $\mu \mapsto F(x, \mu)$. Therefore the envelope is a subset of the *discriminant set*, D_F:

$$D_F = \{x \in \mathbb{R}^n : F(x, \mu) = 0, \nabla_\mu F(x, \mu) = 0, \text{ for some } \mu \in C\}$$

Examples of envelopes can be found in figure 2.1.

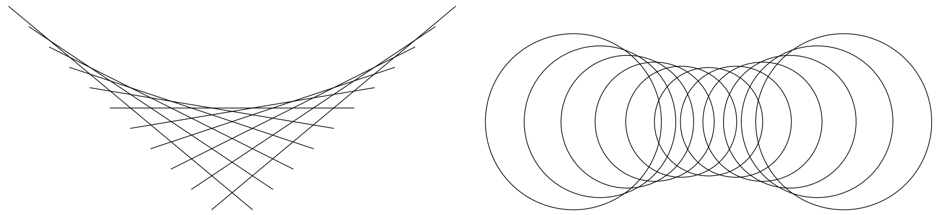

Figure 2.1: On the left, a parametrized set of lines. On the right a parametrized set of spheres. Both are visualized by showing only a few values. The envelopes show as boundaries or as the intersections of infinitesimally close (with respect to the parameter) shapes.

2.2 Introduction to skin Surfaces

In [3] a method of constructing surfaces is introduced. The construction of these *skin surfaces* gives us a piecewise quadratic shape based on a finite set of weighted input points, which we denote as $\mathcal{P}$, and a global *shrink parameter*, called s.

The following useful properties of these skin surfaces are stated (and proven) in [3]:

Decomposability:	A skin surface in $\mathbb{R}^n$ consists of a finite number of degree 2 patches.
Symmetry:	A skin surface can be defined from the inside as well as the outside.
Smoothness:	A non-degenerate skin surface is everywhere tangent continuous.
Deformability:	The changes in topology, based on changes of input can be found easily.
Continuity:	The skin varies continuously on the input of weighted points.
Universality:	Every orientable surface has a skin representation.
Constructibility:	There are fast algorithms for finding the skin.

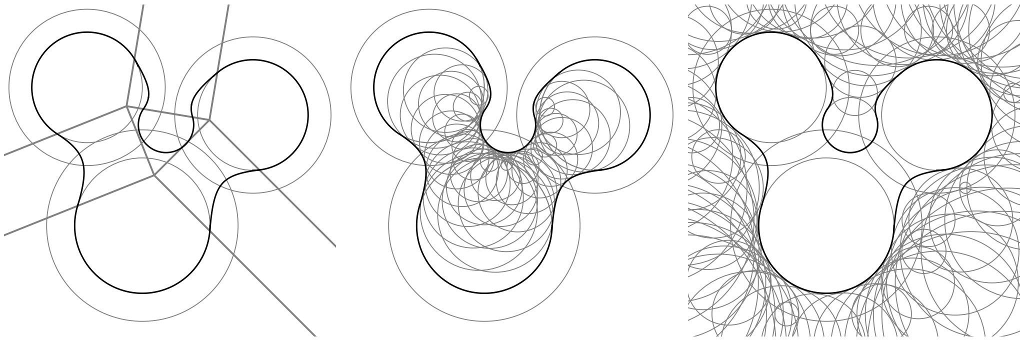

Figure 2.2: On the left, a decomposed skin curve, middle and right show the same skin defined from the in- and outside.

Economy:	Complicated surfaces can be approximated by a small input.

The first two of these properties (shown in figure 2.2) turn out to be strongly connected to orthogonal sets of spheres. Therefore, we will consider these in more detail after the initial definition of Skin surfaces. The definition of skin surface arises from the non-commutativity of two actions, shrinking and taking linear combinations of spheres. To properly state the definition, we will first need to define these two concepts.

Definition 2. Let $F, G : \mathbb{R}^n \to \mathbb{R}$ be such that the sets $\{x \in \mathbb{R}^n : F(x) = 0\}$ and $\{x \in \mathbb{R}^n : G(x) = 0\}$ are spheres (F and G determine spheres implicitly). We can identify F and G with these spheres, allowing us to take linear combinations. Define the corresponding *pencil* of spheres as the set of spheres determined by the set of functions $\{H_a : a \in \mathbb{R}\}$ where:

$$\begin{aligned} H_a : \ \mathbb{R}^n &\to \mathbb{R} \\ x &\mapsto a \cdot F(x) + (1-a) \cdot G(x) \end{aligned}$$

The higher dimensional analogue of pencils are called *flats* of spheres.

To work more easily with these pencils of spheres we first create a framework, the *space of weighted points*, such that lines in this space correspond to pencils of spheres. An identification of weighted points in $\mathbb{R}^n \times \mathbb{R}$ and spheres in $\mathbb{R}^n$ can be derived from the so-called *power distance*, between weighted points, given by:

$$\begin{aligned} \pi : \quad (\mathbb{R}^n \times \mathbb{R}) \times (\mathbb{R}^n \times \mathbb{R}) &\to \quad \mathbb{R} \\ (z_0, w_0), (z_1, w_1) &\mapsto \quad \|z_0 - z_1\|^2 - w_0 - w_1 \end{aligned} \tag{1}$$

For a chosen weighted point $\hat{p}$, the set of $\{x \in \mathbb{R}^n : \pi(\hat{p}, (x, 0)) = 0\}$ can be recognised to be a sphere of centre z, and radius $\sqrt{w}$. This not only gives us a definition for a sphere, it also gives a way to test whether a given point is in the sphere. This generalizes to the fact that two weighted points have power distance 0 if and only if their corresponding spheres intersect orthogonally: The formula $\|z_0 - z_1\|^2 - w_0 - w_1 = 0$ is a simple reformulation of the Pythagorean Theorem.

We are defining $\mathbb{R}^n \times \mathbb{R}$ to be a vector space such that lines correspond to pencils of spheres, for this we need operations on the set. Finding a line through a given set of points can be done by taking the affine hull (see definition 17 from appendix A). Hence, we choose not to view $\mathbb{R}^n \times \mathbb{R}$ as the vector space $\mathbb{R}^{n+1}$ directly. Instead, a slightly different set of operations is chosen on $\mathbb{R}^n \times \mathbb{R}$. This is not entirely arbitrary, it is chosen such that:

$$\pi(a\hat{p} + (1 - a)\hat{q}, \hat{r}) = a\pi(\hat{p}, \hat{r}) + (1 - a)\pi(\hat{q}, \hat{r})$$

In other words, this gives a correspondence between lines in $\mathbb{R}^n \times \mathbb{R}$ and pencils of spheres.

The set of weighted points, $\mathbb{R}^n \times \mathbb{R}$ can be made a vector space with the above property, using a bijection $\Pi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^{n+1}$, where $\mathbb{R}^{n+1}$ is the usual real vector space. This bijection is defined:

$$
\begin{aligned}
\Pi: \quad \mathbb{R}^n \times \mathbb{R} &\rightarrow \mathbb{R}^{n+1} \\
(z, w) &\mapsto (z, \|z\|^2 - w) \\
&= (z_0, ..., z_n, m)
\end{aligned} \tag{2}
$$

This means that we define addition and scalar multiplication for weighted points $\hat{p}$, $\hat{q} \in \mathbb{R}^n \times \mathbb{R}$ and scalar $a \in \mathbb{R}$ as:

$$
\hat{p} + \hat{q} = \Pi^{-1}(\Pi(\hat{p}) + \Pi(\hat{q})) \qquad \text{and } a \cdot \hat{p} = \Pi^{-1}(a \cdot \Pi(\hat{p}))
$$

By viewing weighted points as spheres, it makes sense to define *shrinking of weighted points*, with *shrink factor* s, as the action that simply multiplies the weight with s. We can write the image of this action, $\hat{p} \mapsto \hat{p}^s$ as a linear combination in our new vector space, by:

$$
\begin{aligned}
\hat{p} = (z, w) \mapsto \hat{p}^s &= (z, s \cdot w) \\
&= s(z, w) + (1 - s)(z, 0)
\end{aligned}
$$

Sets of spheres can be shrunk by shrinking point-wise. For a set of spheres $\mathcal{X} \subset \mathbb{R}^n \times \mathbb{R}$, the shrunk set is denoted $\mathcal{X}^s$. Note that this is only shrinking in the strict sense of the word for $s < 1$, however, we wil use the same terminology for inflation ($s > 1$). The union of spheres over all shrink factors $s \leq 1$, viewed in $\mathbb{R}$ is called the *upwards closure* of a weighted point $\hat{p}$, ucl$\hat{p}$. Similar to how $\hat{p}$ corresponds to a sphere in $\mathbb{R}^n$, the upwards closure can naturally be identified with the corresponding ball in $\mathbb{R}^n$. Again, the upwards closure of a set of spheres $\mathcal{X}$, corresponds to the set of points 'inside' any sphere of $\mathcal{X}$.

In this resulting vector space, we can take convex and affine hulls, allowing us to define the *$s-$body* of a set of spheres $\mathcal{P}$ as the union of the upwards closure of the shrunk convex hull. Finally the *$s-$skin* is defined as the boundary of the body.

Definition 3. For set of weighted points (or spheres) $\mathcal{P} \subset \mathbb{R}^n \times \mathbb{R}$, we define:

$$\begin{aligned} \mathrm{bdy}^s(\mathcal{P}) &= \mathrm{ucl}(\mathrm{conv}\mathcal{P})^s \quad \subset \mathbb{R}^n \\ \mathrm{skn}^s(\mathcal{P}) &= \partial \mathrm{bdy}^s(\mathcal{P}) \end{aligned}$$

In figure 2.3 an example for $|\mathcal{P}| = 2$ is given, with the intermediate steps shown.

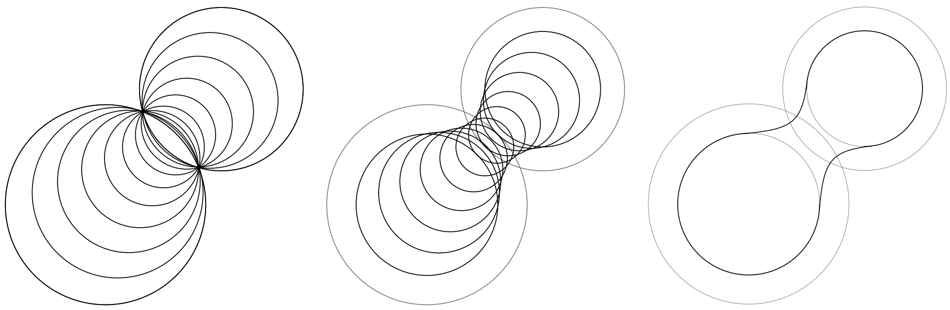

Figure 2.3: The intermediate steps for constructing a skin, based on two spheres. The convex hull is taken in the first image, which is shrunk with factor $s = \frac{1}{2}$ in the second. Taking the boundaries of the union of spheres results in the skin.

This can also be viewed as an envelope, using the framework of definition 1, by taking C the set of centres of conv$\mathcal{P}$, given as elements $\mu = (\mu_1, ..., \mu_n)$. Then take $w_\mu = \max\{w : (\mu, w) \in \mathrm{conv}\mathcal{P}\}$, i.e. the largest weight in the convex hull corresponding to this centre. Finally, take functions $F_\mu(x) = F(x, \mu)$ such that:

$$\begin{aligned} F_\mu(x_1, ..., x_n) &= (x_1 - \mu_1)^2 + ... + (x_n - \mu_n)^2 - s \cdot w_\mu \\ &= \pi((\mu, s \cdot w_\mu), (x, 0)) \end{aligned} \tag{3}$$

Then, the zero set of F_μ is precisely the sphere corresponding to $(\mu, s \cdot w_\mu)$, and the interior set Z_μ is the corresponding ball. Hence the skin is the envelope of this family of functions.

Example 1: The skin from figure 2.4 is the skin of the points with centres $(\pm 1, 0)$ and weights 2 and 1 respectively. We can parametrize the convex hull, by taking (μ, w_μ), with $-1 \leq \mu \leq 1$, and $w_\mu = \mu^2 + \frac{\mu+1}{2}$. This makes the family of functions equal to:

$$F_\mu(x, y) = (x - \mu)^2 + y^2 - s \cdot (\mu^2 + \tfrac{\mu+1}{2})$$

The tangency condition $\frac{\partial F(x,\mu)}{\partial \mu} =$ gives $\mu = \frac{1}{1-s}\left(x + \frac{s}{4}\right)$. Recall that μ is in the interval bounded by -1 and 1, and for these boundary values the skin is a patch of the input sphere. Substitution of this μ, gives us an equation for the discriminant set:

$$D_F = \left\{ (x, y) \in \mathbb{R}^2 : (x, y) = \left(\tfrac{\mu}{2} - \tfrac{1}{8}, \pm \tfrac{1}{8} \sqrt{15 + 8\mu + 16\mu^2} \right) \right\}$$

In figure 2.4, the skin is shown to be two patches of spheres, corresponding to $\mu = \pm 1$, and consists of the discriminant set for intermediate μ.

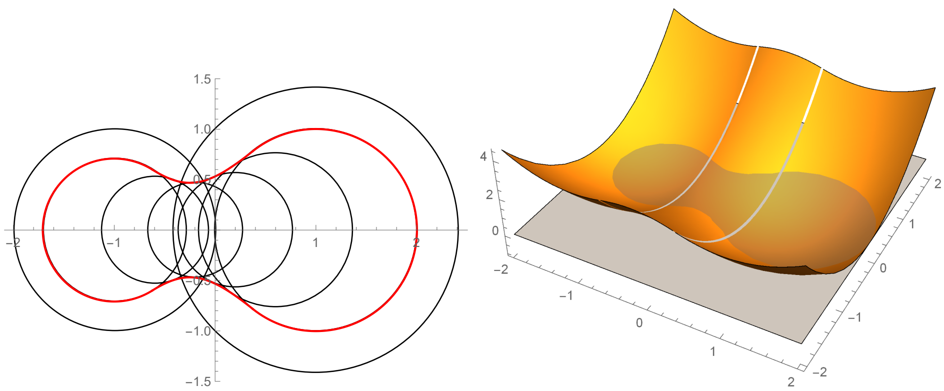

Figure 2.4: On the left, the skin surface (a skin curve in this case) of the two large spheres for shrink factor $s = \frac{1}{2}$ in red, some spheres of conv$\mathcal{P}$ are shown inside. On the right, the minimum of $F_\mu(x, y)$ over μ, as a function of x, y. The intersection with the grey plane is the envelope.

2.3 Complexes in $\mathbb{R}^n$

The (weighted) Voronoi diagram is a partitioning of the space into convex polyhedra, used in a large number of scientific fields. In the study of skin surfaces these, together with the Delauney complex play a large role. These two determine the quadratic patches a skin surface is comprised of. Furthermore, these complexes are orthogonal, giving room for a generalization to the Möbius space.

We start with defining a *weighted Voronoi cell*, $V_{\hat{p}}$. This is, for set of spheres $\mathcal{P} \subset \mathbb{R}^n \times \mathbb{R}$, given as the set of points $x \in \mathbb{R}^n$ closer to $\hat{p}$ than any other point of $\mathcal{P}$, with respect to the power distance, $\pi(\hat{p}, (x, 0))$ (defined in equation 1). More concisely, for a subset $\mathcal{X} \subset \mathcal{P}$,

Definition 4. For $\mathcal{X} \subset \mathcal{P} \subset \mathbb{R}^n \times \mathbb{R}$, the Voronoi cell $V_{\mathcal{X}}$ is defined:

$$V_{\mathcal{X}} = \{x \in \mathbb{R}^n : \pi(\hat{p}, (x, 0)) \leq \pi(\hat{q}, (x, 0)), \text{ for all } \hat{p} \in \mathcal{X}, \hat{q} \in \mathcal{P}\}$$

In particular, if $x \in V_{\mathcal{X}}$, then there is a sphere $\hat{q}$, orthogonal to all spheres in $\mathcal{X}$, and with negative power distance to $\mathcal{P} \setminus \mathcal{X}$. Two spheres with negative power distance are called *further than orthogonal*, making the Voronoi cells the set of centres of spheres, orthogonal to $\mathcal{X}$, and further than orthogonal to all other spheres in $\mathcal{P}$.

Note that Voronoi cells can be empty (see for example figure 2.5, for p_1, p_4). A weighted point is called *hidden* if it's Voronoi cell is empty. For non-empty Voronoi cells $V_{\mathcal{X}}$, we define the Delauney cell $\delta_{\mathcal{X}}$ as the convex hull of the centres in $\mathbb{R}^n$.

$$\delta_{\mathcal{X}} = \{x \in \mathbb{R}^n : \exists w \in \mathbb{R} \text{ such that } (x, w) \in \text{conv}\mathcal{X}\}$$

For l points in $\mathbb{R}^n \times \mathbb{R}$ in general position, the Voronoi cell is an $n+1-l$-dimensional polyhedron and the Delauney cell is l dimensional. Non-zero

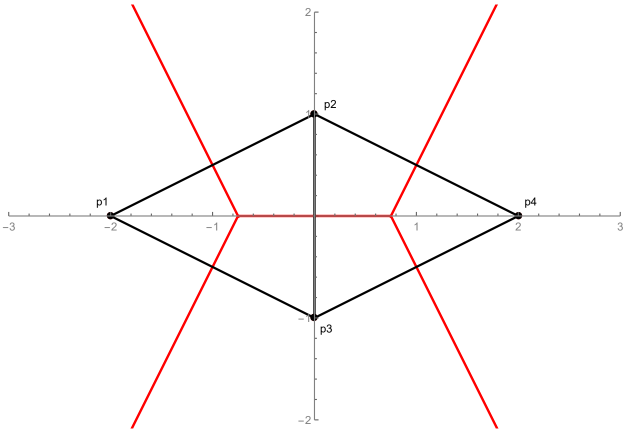

Figure 2.5: The Voronoi complex (in red) and Delauney complex (in black) of 4 points, centred at $(\pm 2, 0), (0, \pm 1)$, carrying the same weight.

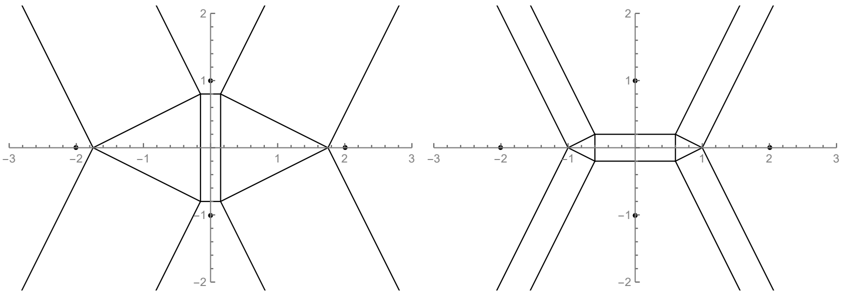

Figure 2.6: The mixed complex for the situation of figure 2.5 for small and large shrink factor s respectively.

multiplication does not change the dimension of these polyhedrons, which is why we can define the full dimensional *mixed cells*. The $s-$mixed cell of a subset $\mathcal{X}$ and shrink factor $0 < s < 1$, $\mu_{\mathcal{X}}^s$, is defined as the Minkowski-sum:

$$\mu_{\mathcal{X}}^s = (1 - s) \cdot \delta_{\mathcal{X}} \oplus s \cdot V_{\mathcal{X}}$$

The two sets of $s-$mixed cells for the situation of figure 2.5 can be found in 2.6. Using the dimensions of the Voronoi- and Delauney cells, it is obvious that for $s \neq 0$ and $s \neq 1$ the $s-$mixed cell is a full dimensional polyhedron in $\mathbb{R}^n$ for any $\mathcal{X} \subset \mathcal{P}$.

2.4 Properties of skin surfaces

Recall the statement of decomposability of skin surfaces. The definition of the mixed complex allows us to state a lemma, regarding this decomposition:

Lemma 1. (Stated as equation 2.8 in [3]) The skin surface of a set $\mathcal{P}$ is decomposed over regions of its mixed complex, and in fact:

$$\begin{aligned} \mathrm{skn}^s\mathcal{P} &= \bigcup_{\mathcal{X}\subset\mathcal{P}} \mu^s_{\mathcal{X}} \cap \mathrm{skn}^s\mathcal{X} \\ &= \bigcup_{\mathcal{X}\subset\mathcal{P}} \mu^s_{\mathcal{X}} \cap \mathrm{env}(\mathrm{aff}\mathcal{X})^s \end{aligned}$$

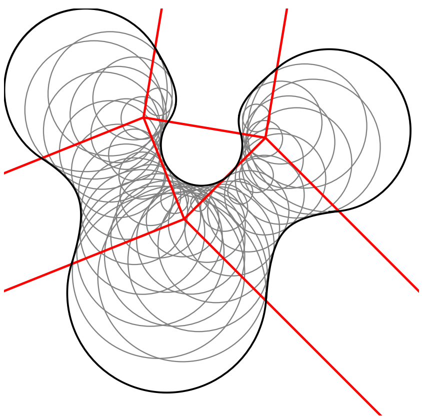

Figure 2.7: The mixed complex and shrunk convex hull for three spheres. As lemma 1 states, in the mixed cell corresponding to $\mathcal{X} \subset \mathcal{P}$, only the spheres in $\mathcal{X}$ determine the skin.

> **Note:** The second equality is not an entirely trivial statement, the usual skin is an envelope of a shrunk *convex* hull. However, what this implicitly states, is that all points of aff$\mathcal{X}$ that contribute to this skin are in the convex hull.

A corollary is that, if the points in $\mathcal{P}$ are in general position and $|\mathcal{X}| > n + 1$, then the corresponding Voronoi (and hence mixed-) cell is empty. Therefore these subsets do not contribute to the skin, and it suffices to check

$\mathcal{X}$ with $|\mathcal{X}| \leq n+1$. Furthermore, for example in the example of 2.5, only two subsets of 3 spheres need be considered.

Another stated property of the skin is symmetry, or being defined as an envelope of spheres from the in- and outside. Using the decomposition from lemma 1, it suffices to find a set of spheres to define the envelope of $(\mathrm{aff}\mathcal{X})^s$ from the outside. See figure 2.8 for an example of this. For a more formal statement of this fact, we state the following lemma:

Lemma 2. (Lemma 6 from Edelsbrunner [3]) Let $F = \mathrm{aff}\mathcal{X}$ and let G be the set of all spheres intersecting orthogonally with spheres in $\mathcal{X}$. In terms of the power distance this can be written:

$$G = \{\hat{q} : \pi(\hat{q}, \hat{p}) = 0 \quad \forall \hat{p} \in \mathcal{X}\}$$

Furthermore, let $s, t > 0$ and $s + t = 1$. Then:

$$\begin{aligned} \mathrm{ucl}F^s \cup \mathrm{ucl}G^t &= \mathbb{R}^n \\ \mathrm{ucl}F^s \cap \mathrm{ucl}G^t &= \mathrm{env}F^s \\ &= \mathrm{env}G^t \end{aligned}$$

The two lemmas stated in this section already show the relation between skin surfaces and orthogonality. The Voronoi cells are centres of orthogonal complements, and using lemma 2, the envelope of a shrunk flat of spheres can be written as the envelope of it's shrunk orthogonal complement.

2.5 Aside: The extended skin surface

The usual application for skin surfaces is in approximation of surfaces. This can be done by finding a number of spheres tangent to the surface, and then calculating the skin surface. A problem with this skin surface is the fact that the original input spheres are shrunk, and therefore no longer touch

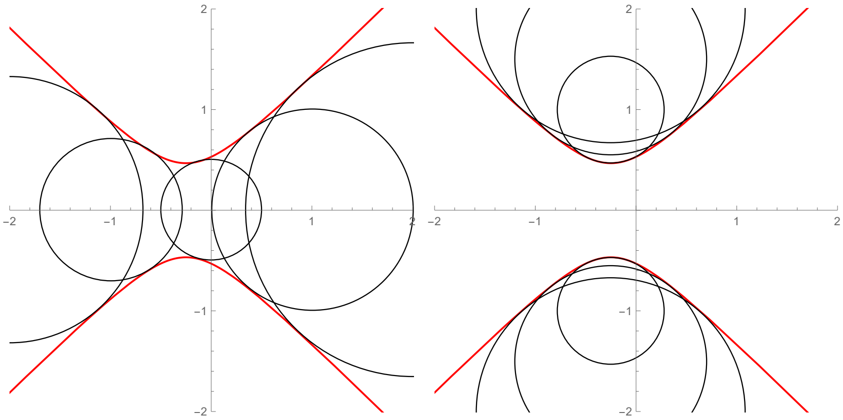

Figure 2.8: The envelope of a shrunk (1 dimensional) affine hull of spheres and its shrunk (1 dimensional) orthogonal complement. Clipped to a mixed cell this describes a patch of a skin curve.

the original surface. This can be fixed by inflating the input spheres before taking the convex hull.

Definition 5. For $s \neq 0$, the *extended skin surface* of a set of spheres $\mathcal{P}$ is given

$$\mathrm{eskn}^s \mathcal{P} = \mathrm{skn}^s\left(\mathcal{P}^{1/s}\right)$$

Note that this always wraps around the original spheres, as shrinking is multiplicative and point-wise. A comparison of the normal skin and the extended skin is found in figure 2.9.

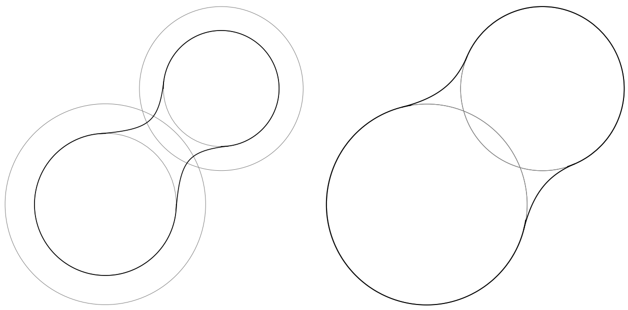

Figure 2.9: On the left, the skin of figure 2.3. On the right the corresponding extended skin.

3 Möbius geometry

Recall the decomposition of skin surfaces (see lemma 1). This used the mixed complex, which was found using the mutually orthogonal Voronoi- and Delauney cells. Furthermore, the symmetric property also suggests that orthogonality plays a large role in the underlying structure of the skin surface.

Orthogonal spheres are the natural invariant of the Möbius geometry, making this geometry a natural space to try and view them. This section will first (briefly) introduce the Möbius geometry. For a more thorough introduction of this space, see section 2.2 of Cecil's 'Lie sphere geometry' [2].

The Möbius geometry is a model, where we identify the set of *generalized spheres* with elements of the projective space $\mathbb{P}^{n+1}$ (Definition 20). The set of generalized spheres is a formalization of the normal set of spheres with the intuitive idea that planes in $\mathbb{R}^n$ are simply 'infinitely large' spheres, that points are spheres with radius equal to zero, and that spheres are allowed have negative squared radius. These 'negative' spheres contain no points in $\mathbb{R}^n$, but are found by taking, for example $x^2+y^2+1=0$ as implicit definition.

To speak about orthogonality of these generalized spheres requires an extension on the statement in section 2.2, where two spheres were called orthogonal if they have power distance 0. The same definition holds for

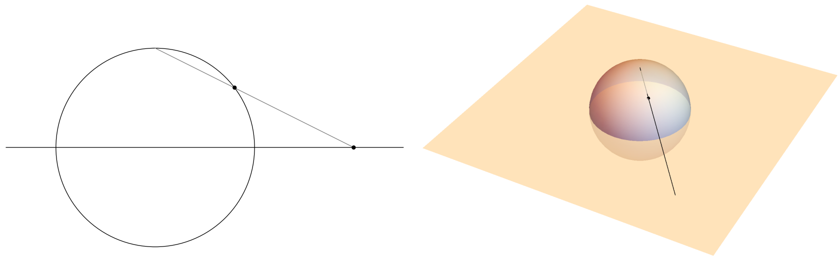

Figure 3.1: The stereographic projection σ, identifying points of $\mathbb{R}^n$ (the equatorial plane) with points on the sphere $S^n \subset \mathbb{R}^{n+1}$. Shown for $n = 1$ and $n = 2$.

negative- and point spheres. For point spheres this means that a sphere is called orthogonal to a point if it contains the point. A plane and a sphere intersect orthogonally if and only if the plane contains the centre of the sphere.

The identification of these generalized spheres with points is found using a composition of stereographic projection of $\mathbb{R}^n$ onto $S^n \subset \mathbb{R}^{n+1}$ (see figure 3.1) and embedding the result into $\mathbb{P}^{n+1}$. We denote this stereographic projection by σ and write τ for the natural embedding into the projective space, these maps can be made explicit:

$$\begin{array}{rrcl} \sigma: & \mathbb{R}^n & \to & S^n \setminus (-1, 0, ..., 0) \\ & x & \mapsto & \left(\frac{1 - x \cdot x}{1 + x \cdot x}, \frac{2x}{1 + x \cdot x}\right) \\ \tau: & \mathbb{R}^{n+1} & \to & \mathbb{P}^{n+1} \\ & x' & \mapsto & [1 : x'] \end{array}$$

For projective points $\xi = [x_0 : ... : x_{n+1}]$ and $\nu = [y_0 : ... : y_{n+1}]$, we define a symmetric bilinear form of signature $(n+1, 1)$ on $\mathbb{P}^{n+1}$ (For forms see definition 22, and for signature example 8):

$$(\xi, \nu) = -x_0 y_0 + x_1 y_1 + \ldots + x_{n+1} y_{n+1}$$

Note that this form is not entirely well defined on $\mathbb{P}^{n+1}$, as it is not

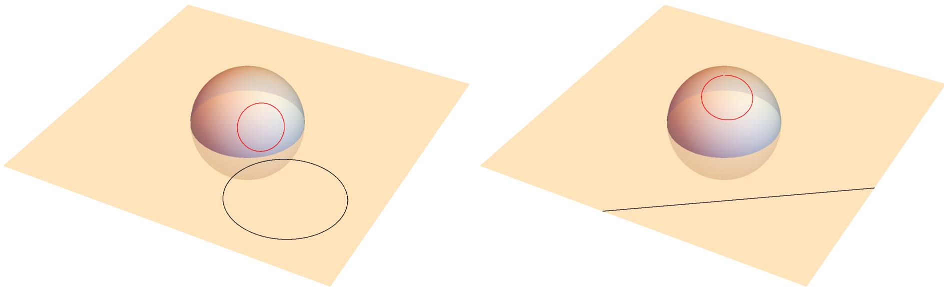

Figure 3.2: The stereographic projection σ of a sphere and a plane in $\mathbb{R}^n$ onto S^n, here shown for $n = 2$.

invariant under scaling of ξ and ν. However, the sign of the form is, and therefore any bilinear, quadratic form on a projective space over the real numbers decomposes the space (see example 12). Classically, these are called the sets of lightlike, spacelike and timelike vectors respectively.

$$
\begin{aligned}
M_0 &= \{\xi \in \mathbb{P}^{n+1} : (\xi, \xi) = 0\} \\
M_> &= \{\xi \in \mathbb{P}^{n+1} : (\xi, \xi) > 0\} \\
M_< &= \{\xi \in \mathbb{P}^{n+1} : (\xi, \xi) < 0\}
\end{aligned}
$$

Using this notation, M_0 is simply the homogenization of the equation of the unit sphere. Thus, the map τ is the natural bijection between S^n and M_0, which gives the set M_0 its usual name of 'the Möbius sphere'. Furthermore, the composition $\tau\sigma$ is a bijection between $\mathbb{R}^n$ and $M_0 \setminus [-1 : 1 : 0 : \ldots : 0]$. This missing point is called the 'improper point', which corresponds to the centre of projection, which can be viewed as a 'point at infinity' of $\mathbb{R}^n$.

So far, only the image of σ of the points of $\mathbb{R}^n$ has been considered. The strength of Möbius geometry, however, is in working with spheres and planes of $\mathbb{R}^n$, not as unions of points, but as elements of the same space. This identification follows from two facts:

1. The stereographic projection σ give a bijection of spheres and planes in $\mathbb{R}^n$ to spheres on S^n (see figure 3.2).

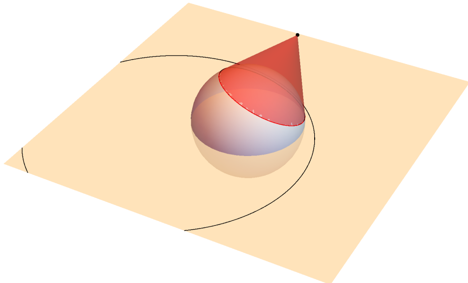

Figure 3.3: An illustration of the identification of points of $\mathbb{P}^{n+1}$ (shown here with the first coordinate scaled to 1, to reveal M_0 as a sphere) with spheres on S^n. In this case $n = 2$, the same holds for higher dimensions.

2. In $\mathbb{P}^{n+1}$, the tangent cone of any sphere on M_0 has a unique apex (see figure 3.3).

Identifying spheres of S^n with this apex gives a bijection between $M_>$ and the set of all spheres on S^n, and therefore all non-point, positive generalized spheres.

As we will be working with the Möbius geometry, it will be useful to have the explicit points representing certain generalized spheres. As before, a sphere in $\mathbb{R}^n$ is given as a weighted point (z, w), where the weight is given as the radius squared. A hyperplane of $\mathbb{R}^n$ can be given by parameters (N, h) such that the set is equal to $\{x \in \mathbb{R}^n : x \cdot N = h\}$. A hyperplane is determined uniquely by taking $|N| = 1$. This allows us to state:

Definition 6. The explicit embedding of generalized spheres, using the parameters as given above, can be written:

$$\begin{aligned} \phi : \qquad \{\text{Generalized spheres in } \mathbb{R}^n\} &\longrightarrow \mathbb{P}^{n+1} \\ \text{sphere } (z, w) &\longmapsto \left[\tfrac{1+z\cdot z-w}{2} : \tfrac{1-z\cdot z+w}{2} : z\right] \\ \text{plane } (N, h) &\longmapsto [h : -h : N] \end{aligned}$$

Recall that $[-1 : 1 : 0 : ... : 0]$ was called the 'improper point', and corresponds to the centre of projection, or a point 'at infinity' of $\mathbb{R}^n$. Furthermore,

$\mathcal{H}_P = [-1:1:0:...:0]^\perp$ is the 'hyperplane of planes'. The definition of the form gives us $\mathcal{H}_P = \{[x_0 : ... : x_{n+1}] : x_0 + x_1 = 0\}$. This explicit form directly gives us bijections:

$$\begin{aligned}
\text{points in } \mathbb{R}^n &\leftrightarrow M_0 \setminus \{[-1:1:0:...:0]\} \\
\text{the centre of projection} &\leftrightarrow [-1:1:\ldots:0] \\
\text{planes in } \mathbb{R}^n &\leftrightarrow \mathcal{H}_P \setminus \{[-1:1:0:...:0]\} \\
\text{positive spheres in } \mathbb{R}^n &\leftrightarrow M_> \setminus \mathcal{H}_P \\
\text{negative spheres in } \mathbb{R}^n &\leftrightarrow M_<
\end{aligned}$$

For a point $\xi_0 \in \mathbb{P}^{n+1}$ on the Möbius sphere M_0, the set $\xi_0^\perp = \{\nu \in \mathbb{P}^{n+1} : (\nu, \xi) = 0\}$ is simply the tangent plane with respect to M_0. If ξ_0 is on a sphere $S \subset M_0$, the apex μ of the tangent cone of S is in each of these tangent planes, and hence $S \subset \mu^\perp$. Therefore $\mu^\perp$ is the plane through S in $\mathbb{P}^{n+1}$.

Furthermore, using the explicit forms immediately reveals that this is not merely a coincidence. Some calculations reveal the following corollary:

Corollary 3. Let $\mu, \nu \in \mathbb{P}^{n+1}$, then $(\mu, \nu) = 0$ if and only if the generalized spheres corresponding to μ, ν are orthogonal.

Finally, the explicit forms allows us to give an inverse, ϕ^{-1}, which maps a point of $\mathbb{P}^{n+1} \setminus \{[-1:1:0:\ldots:0]\}$ to a generalized sphere in $\mathbb{R}^n$. This can most easily be given in two parts:

Corollary 4. An inverse of ϕ can be given by:

$$\begin{aligned}
\phi^{-1}\Big|_{\mathcal{H}_P} : \quad \mathcal{H}_P \setminus \{[-1:1:0:\ldots:0]\} &\to \{\text{planes in } \mathbb{R}^n\} \\
\xi = [x_0 : x_1 : \vec{x}] &\mapsto \left\{v \in \mathbb{R}^n : v \cdot \tfrac{\vec{x}}{\vec{x}\cdot\vec{x}} = \tfrac{x_0}{\vec{x}\cdot\vec{x}}\right\}
\end{aligned}$$

$$\begin{aligned}
\phi^{-1}\Big|_{\mathcal{H}_P^c} : \quad \mathbb{P}^{n+1} \setminus \mathcal{H}_P &\to \{\text{spheres in } \mathbb{R}^n\} \\
\xi = [x_0 : x_1 : ... : x_{n+1}] &\mapsto \left\{v \in \mathbb{R}^n : \textstyle\sum_{i=1}^n \left(v_i - \frac{x_{i+1}}{x_0+x_1}\right)^2 = \frac{(\xi,\xi)}{(x_0+x_1)^2}\right\}
\end{aligned}$$

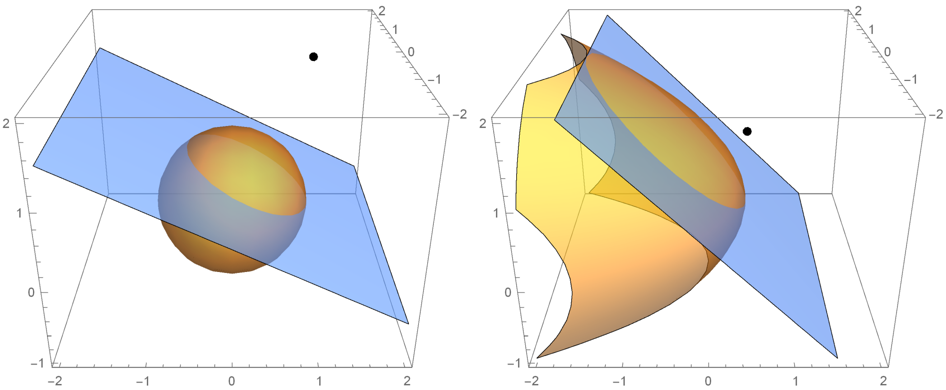

Figure 3.4: On the left is the Möbius space, viewed by scaling x_0 to 1, and plotting $\left(\frac{x_1}{x_0}, \frac{x_2}{x_0}, \frac{x_3}{x_0}\right)$. The sphere shown is the Möbius sphere, and the black point is $[1:1:0:2]$, the representative of the sphere in $\mathbb{R}^2$ of centre $(0,1)$ and weight 1. The blue plane is its orthogonal complement, which intersects the Möbius sphere in a circle. In particular in $[1:1:0:0]$, the image of the origin of $\mathbb{R}^2$. On the right $x_0 + x_1$ is scaled to 1 instead of x_0.

4 Shrunk flats in the Möbius geometry

Now that both skin surfaces and Möbius geometry are introduced, we will first introduce another property of the Möbius geometry: Flats of spheres are represented by subspaces of the Möbius space. Recall that the skin surface is given as the envelope of a shrunk convex hull of spheres, where the convex hull is a certain subset of a flat of spheres.

This section will view these shrunk convex hulls of spheres in the Möbius space. Shrinking subspaces of the Möbius space will be done in terms of quadrics. Hence shrunk convex hulls are are represented by subsets of these quadrics. More interestingly, the reverse is introduced: A construction is given to find shrunk convex hulls inside the Möbius space directly. Hence, given a set of points Σ in $\mathbb{P}^{n+1}$, we can find a subset of $\mathbb{P}^{n+1}$ corresponding to the shrunk convex hull of the spheres represented by Σ.

4.1 Flats in the Möbius geometry

To show that flats of spheres are represented by subspaces of the Möbius space, we use the space of weighted points as an intermediate step. Recall that in $\mathbb{R}^n \times \mathbb{R}$, pencils of spheres were given as lines, and flats as affine combinations. The isomorphism Π (equation 2) of the space of weighted points to $\mathbb{R}^{n+1}$ defines the operations on $\mathbb{R}^n \times \mathbb{R}$. We can also embed these weighted points directly into the Möbius space, by interpreting them as spheres. Therefore we can write bijections ϕ and ψ explicitly as:

$$\begin{tikzcd}
\mathbb{R}^n \times \mathbb{R} \arrow[rr, "\phi"] \arrow[d, "\Pi"] \arrow[dr, dash, dashed] & & \mathbb{P}^{n+1} \setminus \mathcal{H}_P \arrow[dr, dash, dashed] & \\
\mathbb{R}^{n+1} \arrow[urr, "\psi"] \arrow[dr, dash, dashed] & (z_p, w_p) \arrow[rr, mapsto] \arrow[d, mapsto] & & \left[\frac{1+\|z_p\|^2-w_p}{2} : \frac{1-\|z_p\|^2+w_p}{2} : z_p\right] \\
& (z_p, \|z_p\|^2 - w_p) \arrow[urr, mapsto] & &
\end{tikzcd}$$

This diagram commutes. In fact, ψ and ϕ preserves certain linear combinations.

Lemma 5. The maps ϕ and ψ are affine functions, that is, they map affine subspaces of $\mathbb{R}^n \times \mathbb{R}$ or $\mathbb{R}^{n+1}$ to subspaces of $\mathbb{P}^{n+1}$. In fact $\psi(\text{aff}\mathcal{P}) = \text{span}(\psi\mathcal{P})$, and hence, flats of spheres are represented by subspaces of the Möbius space.

> **Note:** Note that $\text{aff}\mathcal{P} \subset \mathbb{R}^n \times \mathbb{R}$ can not contain any planes of $\mathbb{R}^n$, whereas any subspace of the Möbius space does. To be more precise: The map ψ maps $\text{aff}\mathcal{P}$ to a Zariski open subset of $\text{span}(\psi\mathcal{P})$: The set $\text{span}(\psi\mathcal{P}) \cap \mathbb{P}^{n+1} \setminus \mathcal{H}_P$. However, in the limit the flat does contain planes, which correspond to $\text{span}(\psi\mathcal{P}) \cap \mathcal{H}_P$. See figure 4.1 for an example.

Proof. It suffices to prove the statement for ψ. By the definition of ψ, we can write an intermediate map:

$$\begin{aligned} \psi' : \quad \mathbb{R}^{n+1} &\to \mathbb{R}^{n+2} \\ (z_p, m_p) &\mapsto \left(\tfrac{1+m_p}{2} : \tfrac{1-m_p}{2} : z_p \right) \end{aligned}$$

Taking $q : \mathbb{R}^{n+2} \to \mathbb{P}^{n+1}$ as the quotient map, it is obvious that $\psi = q \circ \psi'$. Recall that a subspace of $\mathbb{P}^{n+1}$ is simply the image of a subspace under the map q. Under q, the image of a subspace and an affine hull, not containing 0 is the same. Any such flat $\mathcal{X} \subset \mathbb{R}^{n+1}$, can be given a basis $\mathcal{P} = \{(z_i, m_i)\} \subset \mathcal{X}$ such that $\mathcal{X} = \text{aff}\mathcal{P}$. Hence any $x \in \mathcal{X}$ can be written $x = \sum \gamma_i (z_i, m_i)$, for $\sum \gamma_i = 1$.

$$\begin{aligned} \psi'(x) &= \psi'(\textstyle\sum \gamma_i z_i, \sum \gamma_i m_i) \\ &= \left(\tfrac{1+\sum \gamma_i m_i}{2}, \tfrac{1-\sum \gamma_i m_i}{2}, \textstyle\sum \gamma_i z_i \right) \\ &= \left(\tfrac{\sum \gamma_i + \sum \gamma_i m_i}{2}, \tfrac{\sum \gamma_i - \sum \gamma_i m_i}{2}, \textstyle\sum \gamma_i z_i \right) \\ &= \textstyle\sum \gamma_i \left(\tfrac{1+m_i}{2}, \tfrac{1-m_i}{2}, z_i \right) \\ &= \textstyle\sum \gamma_i \psi'(z_i, m_i) \end{aligned}$$

Hence x maps to an affine combination in $\mathbb{R}^{n+2}$, which q maps to a projective subspace. □

This means that, if the set $\Sigma \subset \mathbb{P}^{n+1}$ represents the spheres in $\mathcal{P} \subset \mathbb{R}^n \times \mathbb{R}$, it is possible to find the image under ϕ of aff$\mathcal{P}$ without having to map 'back and forth'. Recall that the skin was given as an envelope of, not a shrunk affine hull, but of a shrunk convex hull. As such, we are interested in the image of conv$\mathcal{P}$ under ϕ. A convex hull is naturally a subset of the corresponding affine hull, therefore, we know the image of this convex hull is a subset of a projective subspace of $\mathbb{P}^{n+1}$.

The problem that arises: Convexity is not well defined in projective space. Any two points ξ, ν in the projective space are connected by, not one, but two straight line segments on the projective line (their span) connecting them.

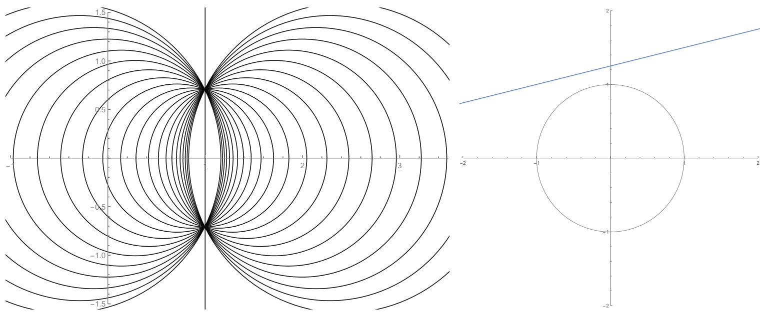

Figure 4.1: On the left, some spheres in a pencil of spheres in $\mathbb{R}^n$. On the right the same pencil represented as the blue line in the Möbius space $\mathbb{P}^3$, which is viewed by scaling $x_0 = 1$ and an intersection with the plane $x_4 = 0$. The sphere is the Möbius sphere. The point on the right where $x_1 = -1$ represents the line in the left figure.

For a hyperplane $\mathcal{H}$, if not both of $\xi, \nu \in \mathcal{H}$, their span is not contained in $\mathcal{H}$. Hence the span intersects $\mathcal{H}$ in a single point, allowing us to distinguish the segments.

Definition 7. We say a subset of $\mathbb{P}^{n+1}$ is *convex relative to* a hyperplane $\mathcal{H}$ if it is convex in $\mathbb{P}^{n+1} \setminus \mathcal{H} \cong \mathbb{R}^{n+1}$. This allows us to define the convex hull of a set Σ, with respect to $\mathcal{H}$, as the intersection of all convex sets containing Σ.

These convex sets are naturally a subset of the span of Σ. As it urns out, using the fact that a convex hull of spheres does not contain any planes allows us to view the image under ϕ of a convex hull of spheres as such a relatively convex set. This is shown in the following lemma.

Lemma 6. For set of spheres $\mathcal{P} \subset \mathbb{R}^n \times \mathbb{R}$, with convex hull $\text{conv}\mathcal{P}$ as used for the skin surface:

$$\phi(\text{conv}\mathcal{P}) = \text{conv}_{\mathcal{H}}(\phi(\mathcal{P}))$$

where the $\text{conv}_{\mathcal{H}}$ is the convex hull relative to $\mathcal{H}_P = [-1 : 1 : 0 : \ldots : 0]^{\perp}$.

Proof. We already know that the image of the affine hull is a projective

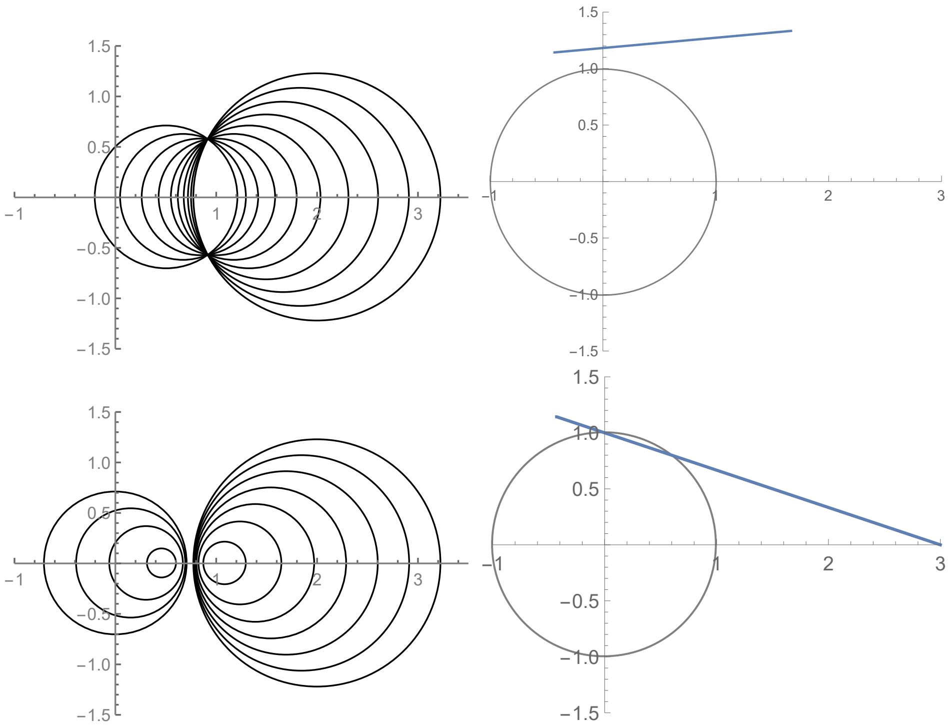

Figure 4.2: Two convex hulls of spheres. These convex hulls are viewed (on the left) as spheres in $\mathbb{R}^2$, and on the right in a projection of $\mathbb{P}^3$ into $\mathbb{R}^2$ given by $x_0 \neq 0, x_3 = 0$. The grey sphere in the right hand side is the projection of the Möbius sphere. Here $\mathcal{H}_P$ is the set where $x_1 = -1$.

subspace. The map ϕ is continuous the when viewed as a map $\mathbb{R}^n \times \mathbb{R} \to \mathbb{P}^{n+1} \setminus \mathcal{H}_P$, hence it preserves convexity. Instinctively, this also makes sense: of the two possible convex hulls for 2 points, this takes the one not containing any $\xi \in \mathbb{P}^{n+1}$ that is a representative of a hyperplane in $\mathbb{R}^n$. $\square$

This allows us to take the convex hull of a set of spheres, without needing to look in $\mathbb{R}^n$. This means that for a set of points in $\mathbb{P}^{n+1}$, we can find the relatively convex hull as a subset of the span. An example of this is found in 4.2.

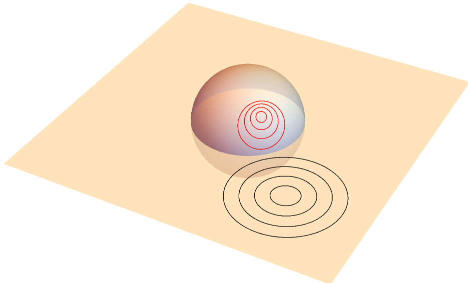

Figure 4.3: Stereographic projection of concentric spheres. Note that the projections on the Möbius sphere are not concentric.

4.2 Shrinking subspaces of the Möbius space

Recall that we are trying to view skin surfaces in the Möbius space directly. The previous section was used to find the convex hull of a set of spheres in $\mathbb{R}^n$ in the Möbius space directly. As the skin surface is found by taking envelopes of shrunk convex hulls, a similar definition is needed for shrinking. However, a complication that arises when shrinking spheres in the Möbius space is a result of the stereographic projection, used in defining this space (see section 3). Figure 4.3 illustrates this: Shrinking spheres deals with concentric spheres in $\mathbb{R}^n$, which do not necessarily map to concentric spheres on $S^n \subset \mathbb{R}^{n+1}$.

In lemma 5 it is shown that flats of spheres can be represented in the Möbius space by subspaces. In the following section we will not only shrink individual spheres, but instead shrink these subspaces of $\mathbb{P}^{n+1}$. These shrunk subspaces are given by quadrics of a specific form. On these quadrics we define a notion of convexity similar to the one defined on projective spaces (as done in lemma 6). This notion is such that a convex subset of quadric $Q \subset \mathbb{P}^{n+1}$, represents a shrunk convex hull of spheres in $\mathbb{R}^n \times \mathbb{R}$.

Using this formalism it is possible to construct a set of points in $\mathbb{P}^{n+1}$ that represents a shrunk convex hull of weighted points directly, without using the underlying space $\mathbb{R}^n \times \mathbb{R}$. This means there is no need to, for example, write the projective points in a specific form or do explicit calculations to define these shrunk convex hulls. Furthermore, recall that the skin surface is the envelope of the shrunk convex hull. Therefore we are one step closer to describing the skin in Möbius space directly.

The process of shrinking maps a sphere $\hat{p} = (z, w)$ to the sphere with the same centre and s times the weight, $\hat{p}^s = (z, s \cdot w)$. In the Möbius space we therefore define a map such that:

$$\begin{aligned} \phi(\hat{p}) = \quad & \left[\tfrac{1+\|z\|^2-w}{2} : \tfrac{1-\|z\|^2+w}{2} : z\right] \\ \mapsto \quad & \left[\tfrac{1+\|z\|^2-s\cdot w}{2} : \tfrac{1-\|z\|^2+s\cdot w}{2} : z\right] = \phi(\hat{p}^s) \end{aligned} \tag{4}$$

If $\xi \in \mathbb{P}^{n+1}$ represents $\hat{p}$, and we write ξ^s for the representative of $\hat{p}^s$, then for changing s, the ξ^s move on a projective line through $[-1 : 1 : 0 : ... : 0]$. However, the points of the Möbius space are not always given in the form of equation 4. Furthermore, we're interested in shrinking subspaces of $\mathbb{P}^{n+1}$.

As shrinking is a point-wise process, we know that for sets of spheres A and B, $(A \cap B)^s = A^s \cap B^s$. Any subspace can be found by intersecting a set of hyperplanes, therefore to shrink subspaces of the Möbius space it suffices to be able to shrink hyperplanes of $\mathbb{P}^{n+1}$. Furthermore, any hyperplane can be uniquely written as $\xi^\perp$, as the quadratic form induces a bijection between hyperplanes and points of $\mathbb{P}^{n+1}$ (see definition 25).

For a final assumption, assume a subspace $A \subset \mathbb{P}^{n+1}$ that represents a shrunk affine hull of the set of spheres $\mathcal{P}$. If this set of spheres is not in general position, $A^\perp \not\subset \mathcal{H}_P$. Any set of elements $\Lambda = \{\lambda_i\}$ that form a basis of $A^\perp$ can be chosen to write $A = \cap_i \lambda_i^\perp$, allowing us to choose all $\lambda_i \notin \mathcal{H}_P$.

Therefore, to be able to shrink representatives of affine hull of spheres, it suffices to be able to shrink $\xi^\perp \subset \mathbb{P}^{n+1}$, for $\xi \notin \mathcal{H}_P$.

From solving the explicit equations, using the shrinking map from equation 4 and the explicit map of definition 6, a quadric (projective quadrics are defined 27) arises. We will first give it explicitly, after which we give the theorem we require it for.

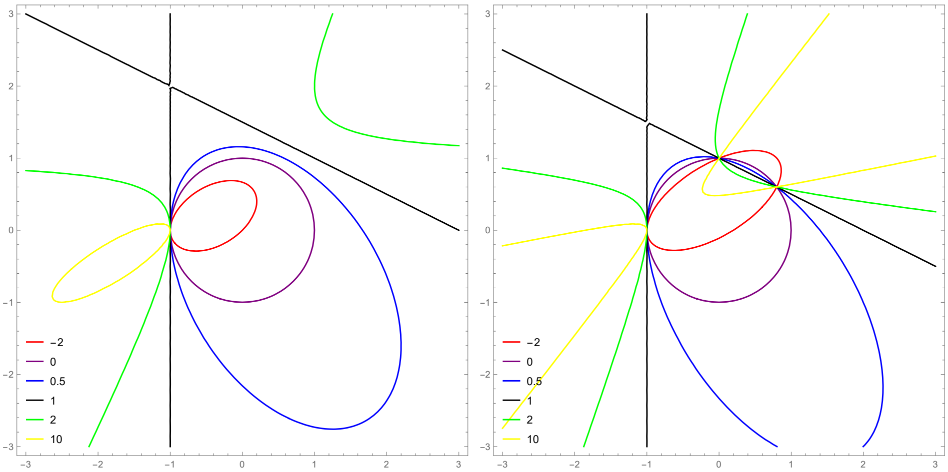

Figure 4.4: $Q^s(\xi)$ for a few values of s, visualized by scaling x_0 to 1. Left for an affine hull containing only spheres of positive radius, the affine hull on the right also contains spheres of negative radius (see figure 4.2 for similar cases).

Definition 8. For $\xi = [a_0 : \ldots : a_{n+1}] \in \mathbb{P}^{n+1}$, and $s \in \mathbb{R}$, then let $Q^s(\xi)$ be the projective quadric:

$$Q^s(\xi) = \{[x_0 : \ldots : x_{n+1}] \in \mathbb{P}^{n+1} : (a_0+a_1)(1-s)\,(x,x)+2s(x_0+x_1)\,(a,x) = 0\}$$

A low dimensional example of this quadric can be found in figure 4.4. This describes the general shape of the quadric $Q^s(\xi)$ as well.

Theorem 7. Let $\xi \notin \mathcal{H}_P$ such that $\xi^\perp$ represents aff$\mathcal{P}$ for set of spheres $\mathcal{P}$ (note that aff$\mathcal{P}$ is codimension 1) and let $s \in \mathbb{R}$. Then the elements in $Q^s(\xi) \setminus \mathcal{H}_P$, precisely represent the spheres in $(\text{aff}\mathcal{P})^s$. In fact, let ϕ be the embedding of weighted points into the Möbius space from definition 6, then:

$$Q^s(\xi) = \begin{cases} \phi(\text{aff}\mathcal{P}) \cup \mathcal{H}_P & \text{if } s = 1 \\ \phi\left((\text{aff}\mathcal{P})^s\right) \cup [-1 : 1 : 0 : \ldots : 0] & \text{otherwise} \end{cases}$$

Note: This means that if aff$\mathcal{P} \subset \mathbb{R}^n \times \mathbb{R}$ is of lower dimension than

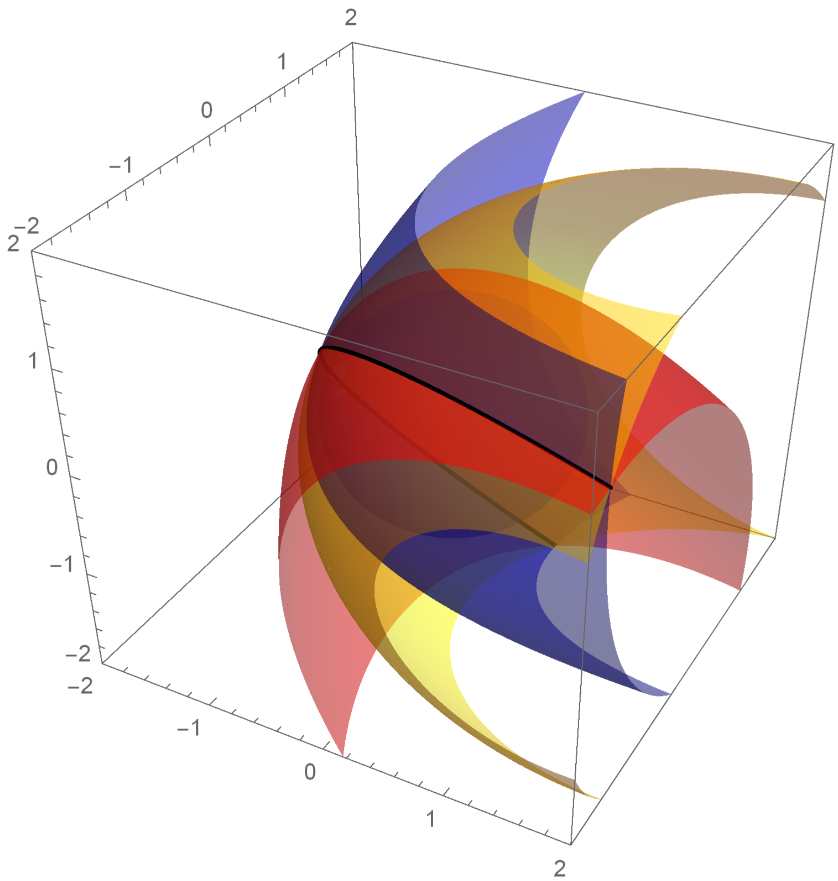

Figure 4.5: Three quadrics representing shrunk 2-dimensional affine hulls $Q^s(\lambda)$ where λ are on a line in $\mathbb{P}^{n+1}$. Note that any two intersect in the same set, a shrunk 1-dimensional affine hull.

codimension 1, and is represented by $\bigcap_{\lambda \in \Lambda} \lambda^{\perp}$ for $\Lambda \subset \mathbb{P}^{n+1}$, then, for $s \neq 0$, $(\text{aff}\mathcal{P})^s$ is represented by $\bigcap_{\lambda \in \Lambda} Q^s(\lambda)$. This representation by Λ is not unique, as any basis Λ' of $\text{span}\Lambda$ also represents $\text{aff}\mathcal{P}$. In corollary 11 we will prove that for $s \neq 0$, the intersection of quadrics is independent on the choice of Λ, Λ'. An example is given in figure 4.5.

Proof. The first statement is immediate using $s = 1$. Now assume $s \neq 1$. Let $\nu = [x_0 : \ldots : x_{n+1}] \in Q^s(\xi)$. Suppose $\nu \in \mathcal{H}_P$, then $x_0 + x_1 = 0$. As $\xi \notin \mathcal{H}_P$, we have $a_0 + a_1 \neq 0$, thus $\nu \in Q^s(\xi)$ implies $(x, x) = 0$. This means

ν represents a point-sphere and hence $\nu \in M_0$. Recall that $M_0 \cap \mathcal{H}_P$ is solely the improper point $[-1 : 1 : 0 : ... : 1]$.

We still need to prove $Q^s(\xi) \setminus \mathcal{H}_P = \phi\left((\text{aff}\mathcal{P})^s\right)$. This will be proven by proving inclusion both ways:

- In $\mathbb{P}^{n+1} \setminus \mathcal{H}_P$ we can, without loss of generality, take the sum of the first two coordinates equal to 1. Let $\nu = [x_0 : ... : x_{n+1}] \in Q^s(\xi) \setminus \mathcal{H}_P$. As $[-1 : 1 : 0 : \ldots : 0]$ is not on $\xi^\perp$, there is a μ of the form $[x_0 - t : x_1 + t : x_2 : \ldots : x_{n+1}]$ on the line through $[-1 : 1 : 0 : \ldots : 0]$ and ν, such that $(\xi, \mu) = 0$. Note that this means that $\mu \in \xi^\perp$. The inverse of ϕ on this piece of $\mathbb{P}^{n+1}$ shows that μ represents the sphere of centre $(x_2, ..., x_{n+1})$ and weight (μ, μ). Some further calculation reveals $(\nu, \nu) = s \cdot (\mu, \mu)$, and thus $\nu \in \xi^s$.
- Let $\nu = \phi(\hat{p}^s)$ such that $\phi(\hat{p})$ is orthogonal to ξ, then:

  $$(\xi, \nu) = \tfrac{1}{2}(a_0 + a_1)(1 - s)w_p$$

  Using this equality reveals $\nu \in Q^s(\xi)$.

Therefore $Q^s(\xi) \setminus \mathcal{H}_P = \phi\left((\text{aff}\mathcal{P})^s\right)$. □

As the form of $Q^s(\xi)$ is already homogeneous in both variables, we homogenize the $Q^s(\xi)$ with regard to s. Writing $\mathbb{P}^1 = \mathbb{R} \cup \{\infty\}$, we can use $s = \infty$ to denote shrinking towards infinity:

$$Q^\infty(\xi) = \{[x_0 : ... : x_{n+1}] \in \mathbb{P}^{n+1} : -(a_0 + a_1)(x, x) + 2(x_0 + x_1)(a, x) = 0\}$$

4.3 Shrunk convex hulls: Convexity on quadrics

Theorem 7 gives an explicit quadric denoting a shrunk affine hull. However, for the construction of the skin surface, we require shrunk convex hulls. Recall that, to find the skin of a set of spheres $\mathcal{P}$, it suffices to find convex hulls of subsets of at most $n+1$ weighted points, as for more points the corresponding mixed cells are empty. In lemma 6, the convex hull of $\mathcal{P}$, for $|\mathcal{P}| \leq n+1$ was mapped to the Möbius space as the convex hull of $\phi(\mathcal{P})$ in $\mathbb{P}^{n+1} \setminus \mathcal{H}$. This was denoted $\mathrm{conv}_{\mathcal{H}}(\phi\mathcal{P})$.

We define $C_{\mathcal{P}} \subset \mathbb{P}^{n+1}$ as the set of all representatives of spheres centred on the convex hull of $\mathcal{P}$. This is precisely the set containing all lines through $[-1 : 1 : 0 : \ldots : 0]$ and any $\nu \in \mathrm{conv}_{\mathcal{H}}(\phi\mathcal{P})$. In other words, this is the projective cone with apex $[-1 : 1 : 0 : \ldots : 0]$ and basis $\mathrm{conv}_{\mathcal{H}}(\phi\mathcal{P})$ in $\mathbb{P}^{n+1}$.

Corollary 8. Let $\mathcal{P} \subset \mathbb{R}^n \times \mathbb{R}$ be a set of spheres, with at most $n+1$ elements. Let $\xi^{\perp} \subset \mathbb{P}^{n+1}$ represent $\mathrm{aff}(\phi\mathcal{P})$, then:

$$\phi\left((\mathrm{conv}\mathcal{P})^s\right) = C_{\mathcal{P}} \cap Q^s(\xi) \setminus \{[-1 : 1 : 0 : ... : 0]\}$$

Note that, when using this definition, $C_{\mathcal{P}} = C_{\mathcal{P}^s}$. Therefore, given a set of points on a quadric $Q^s(\xi)$, the shrunk convex hull on $Q^s(\xi)$ can be found by taking the projective cone of $\mathrm{conv}_{\mathcal{H}}(\mathcal{P}^s)$ instead of $\mathrm{conv}_{\mathcal{H}}(\mathcal{P})$. This allows us to define a notion of convexity on the quadrics $Q^s(\xi)$, by the following equivalent statements:

Definition 9. A subset $S \subset Q^s(\xi) \subset \mathbb{P}^{n+1}$ is called convex in $Q^s(\xi)$ if (the following are equivalent):

- The subset S is equal to $C_{\mathcal{P}} \cap Q^s(\xi)$ for some $\mathcal{P} \subset \mathbb{R}^n \times \mathbb{R}$.
- The projective cone C_S is equal to $C_{\mathcal{P}}$ for some $\mathcal{P} \subset \mathbb{R}^n \times \mathbb{R}$.
- The projection of S, from $[-1 : 1 : 0 : ... : 0]$ onto $\xi^{\perp}$, is relatively convex.
- The projection of S, from $[-1 : 1 : 0 : ... : 0]$ onto M_0, is convex when $M_0 \setminus [-1 : 1 : 0 : ... : 0]$ is viewed as $\mathbb{R}^n$ via inverse stereographic projection.

Using this definition, it is clear that any convex subset of a quadric $Q^s(\xi)$ corresponds to a convex hull of spheres in $\mathbb{R}^n$.

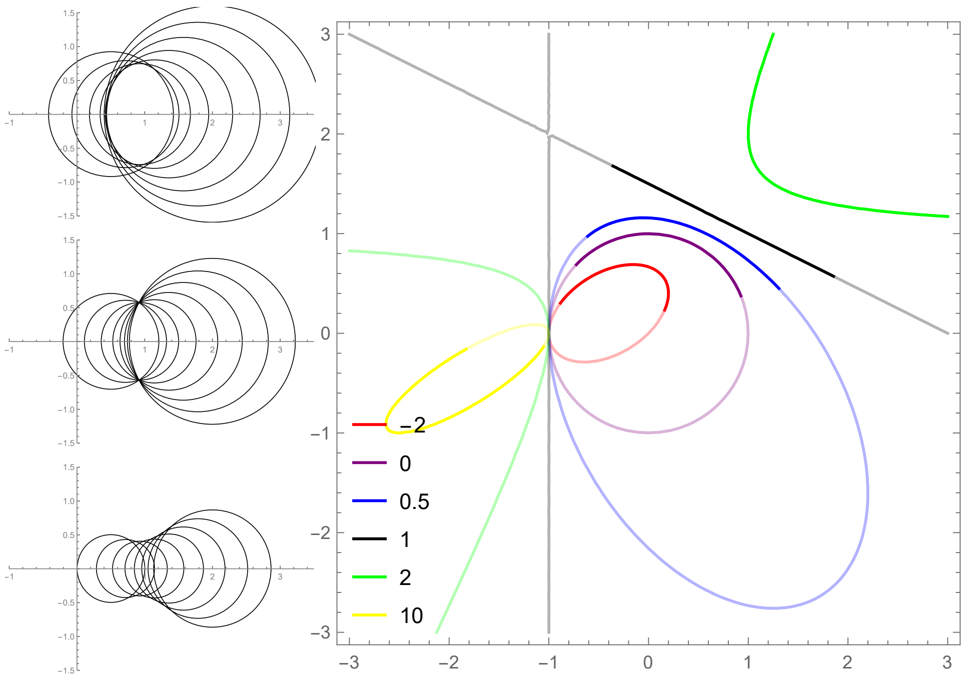

Figure 4.6: $Q^s(\xi)$ (from figure 4.4) for a few values of s, visualized by scaling x_0 to 1, intersected with the projective cone. On the left some examples of the same set of spheres for different s; Each projective quadric represents one.

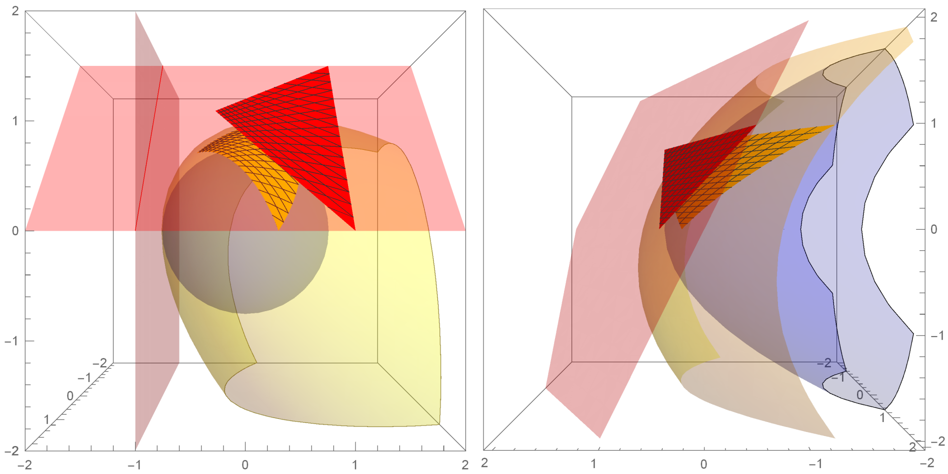

Figure 4.7: A shrunk ($s = \frac{1}{2}$) affine subspace viewed by scaling x_0 to 1 on the left and $x_0 + x_1$ to 1 on the right. The solid red and yellow shapes are a convex hull (of the three corners) and the shrunk convex hull respectively. The blue sphere is the Möbius sphere, which is a paraboloid on the right.

An example of how the convex hull lies in $Q^s(\xi)$ and corresponds to sets of spheres can be found in figure 4.6. Figure 4.7 is a similar case in $\mathbb{P}^3$. A more explicit example, showing the corresponding set of spheres is in figure 4.8. An example a shrunk convex hull of 4 spheres in $\mathbb{R}^2$ (and thus consisting of 2 patches of convex sets) can be found in figure 4.9.

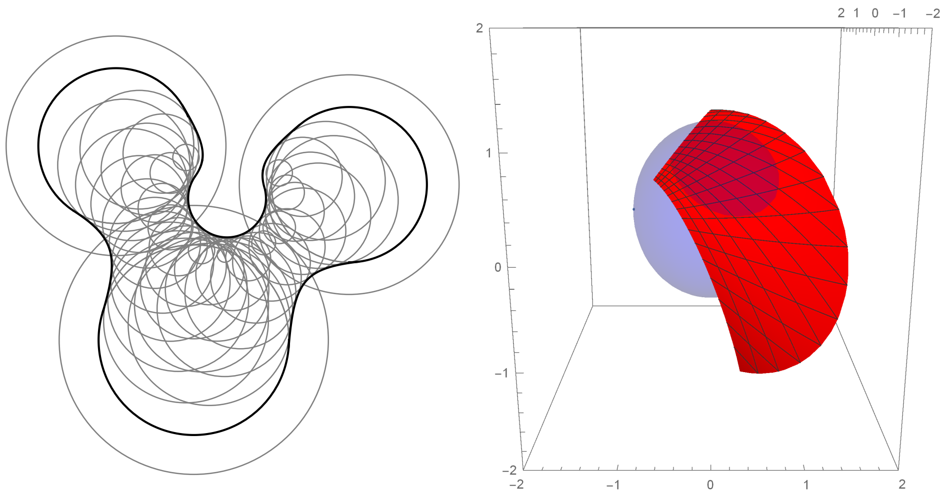

Figure 4.8: A shrunk convex hull of spheres, with its skins curve. On the right the corresponding set in the Möbius space. The red shape lies in a triangular projective cone.

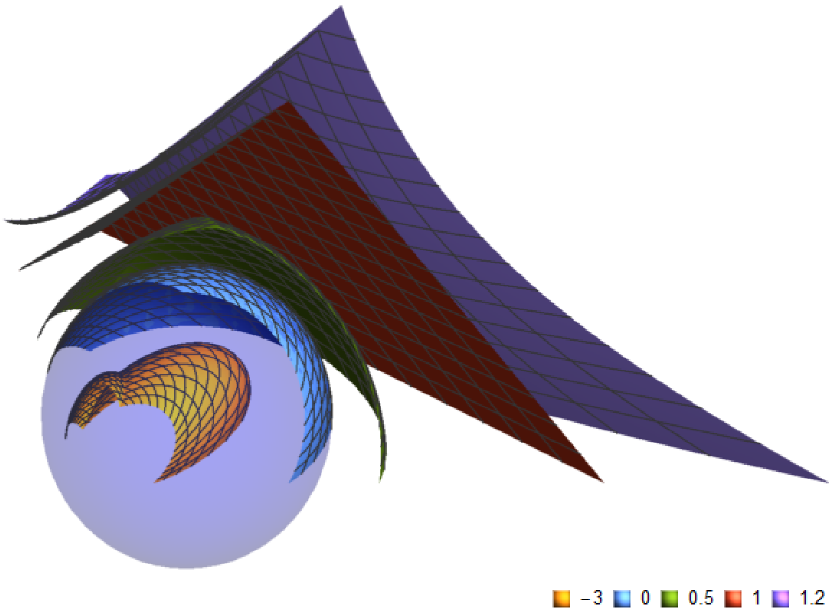

Figure 4.9: Two convex hulls, sharing an edge, shrunk for a few values of s, viewed in the Möbius space $\mathbb{P}^3$ by scaling the first coordinate to 1. The cones $C_{\mathcal{X}}$ visibly clip the shrunk affine hulls to the shrunk convex hulls. The Möbius sphere is shown, opaque in blue, for clarity.

5 The set of all skins: Admissible quadrics

Similar to how, via Möbius geometry, we view spheres in $\mathbb{R}^n$ as points in $\mathbb{P}^{n+1}$, we can also identify quadrics of $\mathbb{P}^{n+1}$ with points in a higher dimensional space. This will allow us to view shrinking affine hulls of $\mathbb{P}^{n+1}$ as moving on a parametrized, projective line in a higher dimensional space. In the previous section, we've described shrunk affine hulls as a specific type of quadric (theorem 7), and given a definition describing shrunk convex hulls.

We will show that the set of quadrics representing shrunk affine hulls (which we will call admissible quadrics) is 'almost' isomorphic to $\mathbb{P}^{n+2}$ in the projective, $\frac{1}{2}n(n+3)$ dimensional space of projective, symmetric matrices PSym. The question of finding skins around a given set of spheres can be done by intersecting the set of admissible quadrics with subspaces. This leads up to a description of all sets of spheres used for defining the skin *and* extended skin surfaces, as subspaces of $\mathbb{P}^{n+2}$.

5.1 Viewing quadrics as matrices

As seen in definition 27 in appendix A, Quadrics in $\mathbb{P}^{n+1}$ can be identified with their determining matrix. A quadric Q is said to be determined by $(n+2) \times (n+2)$-matrix A if:

$$Q = \{[x_0 : \ldots : x_{n+1}] \in \mathbb{P}^{n+1} : x^T A x = 0\}$$

A matrix determines a quadric up to non-zero scalar multiple if it is taken symmetric. Note that the symmetric matrices form a vector space over $\mathbb{R}$, and thus the set of projective, symmetric matrices, PSym form a projective space over $\mathbb{R}$. Recall that we want to describe shrunk flats of spheres, and definition 8 gives an explicit form for these *admissible quadrics*. To view the set of shrunk flats we restrict the identification between quadrics and matrices, to define the map that sends σ and ξ to the matrix defining the quadric representing the shrunk $\xi^\perp$, $Q^s(\xi)$. This map is well defined unless $\sigma = [0:1]$ and $\xi \in \mathcal{H}_P$.

$$\begin{aligned} \delta: \quad (\mathbb{P}^1 \times \mathbb{P}^{n+1}) \setminus (\{0\} \times \mathcal{H}_P) &\to \mathrm{PSym} \\ (\sigma, \xi) &\mapsto A_\sigma(\xi) \end{aligned}$$

The image of δ in PSym is called the *set of admissible quadrics*, Adm, as these are the matrices that can describe a shrunk affine hull. As the space of projective, symmetric matrices is isomorphic to $\mathbb{P}^{(n+2)(n+3)/2-1}$, it quickly becomes unwieldy to check whether a matrix is admissible. Therefore, we will find a much lower dimensional projective subspace of PSym containing this set.

Note: A small recap of the identifications made:

- The points of $\mathrm{Adm} \subset \mathrm{PSym}$ represent admissible quadrics in $\mathbb{P}^{n+1}$.
- The points on these admissible quadrics represent spheres in a shrunk affine hulls of spheres in $\mathbb{R}^n$.
- These shrunk affine hulls can be clipped to become the interior set of spheres for a skin surface.

Using the explicit forms of the matrices A_σ (see appendix B), allows us to make two statements concerning images of certain 'cross sections' of $\mathbb{P}^1 \times \mathbb{P}^{n+1}$. These are given in the following two lemmas.

Lemma 9. For a given $\xi \notin \mathcal{H}_P$, the set $\{Q^s(\xi) : \sigma \in \mathbb{P}^1\}$ is a pencil of quadrics (definition 28). In other words, the image of δ restricted to $\mathbb{P}^1 \times \{\xi\}$ is a projective line.

Proof. Let $\xi \notin \mathcal{H}_P$, and let matrix A_σ determine quadric $Q^\sigma(\xi)$, for $\sigma = (s : t) \in \mathbb{P}^1$. Using linear algebra we know that, $x^T(A+B)x = x^TAx + x^TBx$.

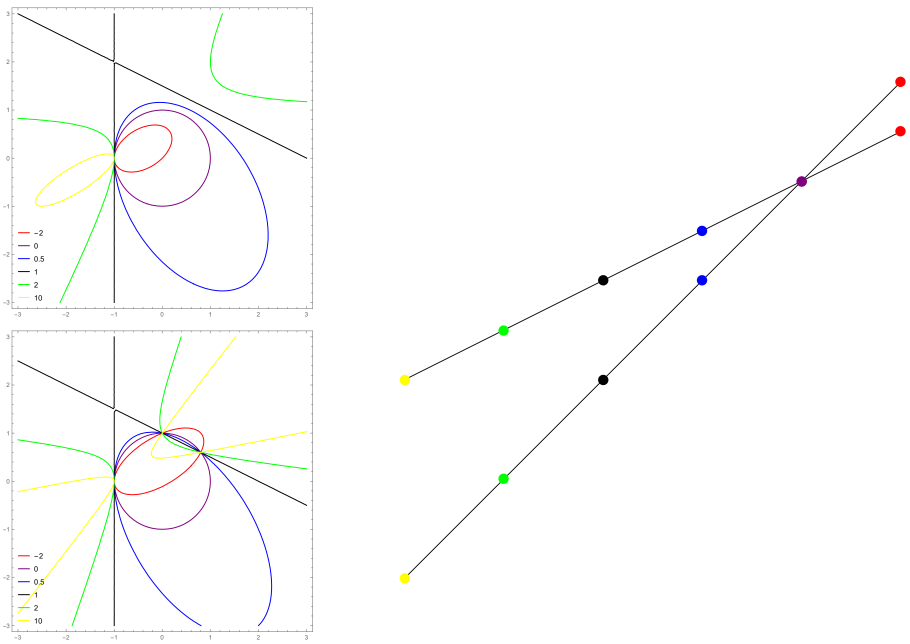

Figure 5.1: Two sets of shrunk pencils of spheres, represented in $\mathbb{P}^2$ (as in figure 4.4, represented (schematically) in PSym as two (ordered) lines, that intersect at M_0.

Therefore addition on quadrics allows us to write:

$$Q^{\sigma}(\xi) = (t-s)\cdot Q^0(\xi) + s\cdot Q^1(\xi) = t\cdot Q^0 + s\cdot Q^{\infty}(\xi)$$

Thus we can write the set as a span of two elements, a pencil. $\square$

For a given set of spheres, we can therefore describe all skin surfaces as a pencil of quadrics. This result is schematically shown in figure 5.1. A consequence of this is that the set of all admissible quadrics can be viewed as a 'bundle of lines' in PSym, where all lines are through the matrix representing the Möbius sphere M_0, and some matrix representing any $Q^s(\xi)$ for fixed $s \neq 0$. This describes a projective cone, hence we need to describe $\{Q^s(\xi) : \xi \in \mathbb{P}^{n+1}\}$ in the space of quadrics, PSym.

Lemma 10. For a given $\sigma \in \mathbb{P}^1 \setminus \{0\}$, the set $\{Q^s(\xi) : \xi \in \mathbb{P}^{n+1}\}$ is represented by an $n+1$ dimensional subspace, called Adm^s, of PSym. In other words the image of δ restricted to $\{\sigma\} \times \mathbb{P}^{n+1}$ is a projective subspace of PSym of dimension $n+1$.

Proof. Using the explicit form of $A_\sigma(\xi)$, for a fixed $\sigma = [s : t]$, with $s \neq 0$, it can be seen that, for $[a : b] \in \mathbb{P}$:

$$A_\sigma(a \cdot \xi + b \cdot \nu) = aA_\sigma(\xi) + bA_\sigma(\nu)$$

This means the restricted δ maps subspaces of $\mathbb{P}^{n+1}$ to subspaces of PSym, and therefore its image, Adm^s is isomorphic to $\mathbb{P}^{n+1}$. $\square$

As a small aside, this allows us to prove the note near theorem 7, that the result of intersecting shrunk quadrics is independent of choice of basis for the orthogonal complement

Corollary 11. Any two quadrics in a pencil have the same intersection and the same holds flats. Therefore, for $\Lambda \subset \mathbb{P}^{n+1}$:

$$\bigcap_{\lambda \in \Lambda} Q^s(\lambda) = \bigcap_{\lambda \in \mathrm{span}\Lambda} Q^s(\lambda)$$

This means that if Λ, Λ' span the same subspace in $\mathbb{P}^{n+1}$ (and thus represent the same affine hull of spheres), then the shrunk affine hull found like 7, is independent on the choice of Λ, Λ'

These previous two lemmas are equivalent to saying that, using $\mathbb{P}^1 \setminus \{0\} \cong \mathbb{R}$, and $\mathbb{P}^{n+1} \setminus \mathcal{H}_P \cong \mathbb{R}^{n+1}$ the (non-projective) map restricted to $\mathbb{R}^1 \times \mathbb{R}^{n+1}$ is bilinear, and therefore maps to a subspace of $\mathrm{PSym} \setminus H$ for some suitable hyperplane H. This can be made more precise.

Lemma 12. The set of admissible quadrics, Adm, is represented in PSym by a subset:

$$\begin{aligned} \mathrm{Adm} &\subset \mathrm{PSym} \\ \mathrm{Adm} &\cong \mathbb{P}^{n+2} \setminus (\mathbb{R}^{n+1} \setminus \{0\}) \\ &\cong \mathbb{R}^{n+2} \cup \mathbb{P}^{n} \cup \{\delta(M_0)\} \end{aligned}$$

Proof. The set of admissible quadrics is represented by a projective cone trough $\delta(M_0)$ and a point and a projective subspace of dimension $n+1$, and therefore can be viewed in a projective space of dimension $n+2$. We have 3 disjoint subsets of this via the restrictions:

$$\begin{array}{rrll} \delta: & \mathbb{P}^1 \setminus \{0\} & \times \mathbb{P}^{n+1} \setminus \mathcal{H}_P & \to \mathbb{R}^{n+2} \\ & \mathbb{P}^1 \setminus \{0\} & \times \mathcal{H}_P & \to \mathbb{P}^{n} \\ & \{0\} & \times \mathbb{P}^{n+1} & \to \delta(M_0) \end{array}$$

A schematic view of this is given in figure 5.2. □

5.2 Skins around given points

Recall that we were looking for all possible skins around a set of given spheres in $\mathbb{R}^n$. A sphere in $\mathbb{R}^n$ is given as a point in $\mathbb{P}^{n+1}$, and any shrunk convex hull in $\mathbb{P}^{n+1}$ containing this point gives a skin around the sphere. Any shrunk convex hull is a subset of shrunk affine hull using corollary 8. This problem thus reduces to finding all admissible quadrics through a set of points in $\mathbb{P}^{n+1}$.

The set of quadrics through a given point $\lambda \in \mathbb{P}^{n+1}$, $H(\lambda)$, is simply a hyperplane (see lemma 29) in PSym. Similarly, for a set of k spheres in $\mathbb{R}^n$, represented by set $\mathcal{X} \subset \mathbb{P}^{n+1}$, the set of quadrics containing all these points is $H(\mathcal{X}) = \bigcap_{\lambda \in \mathcal{X}} H(\lambda)$. This has codimension k if the span of $\mathcal{X}$ is dimension $k-1$. In other words, $\mathcal{X}$ is in general position, and therefore the spheres in $\mathbb{R}^n$ were in general position.

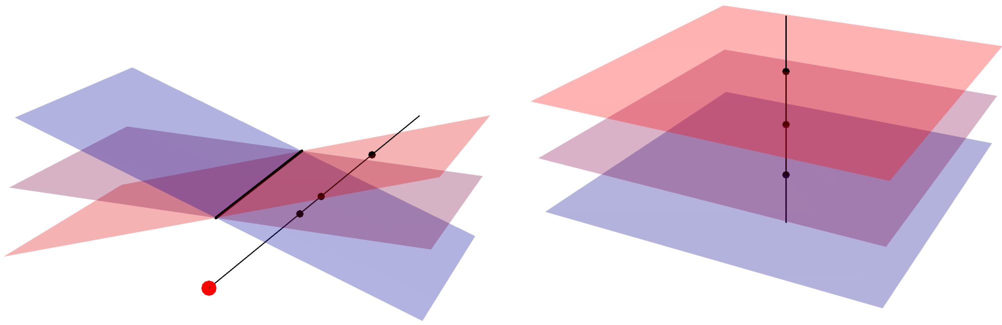

Figure 5.2: A schematic view of the set of admissible quadrics. The planes are the image of Adm^s for fixed $s \neq 0$. These are $n+1$ dimensional projective subspaces, of which 3 are shown. Any two of these planes will intersect in the same subspace (a thick line in the figure), the image of $\{Q^\sigma(\xi) : \xi \in \mathcal{H}_P, \sigma \neq 0\}$. This is an $n-$dimensional projective subspace. As $\delta(M_0)$ is not in either of these spaces, it's drawn as a red point outside of these Adm^s. The line drawn is a pencil of quadrics (as in figure 5.1), the image of $\{Q^\sigma(\xi) : \sigma \in \mathbb{P}^1\}$ for a fixed ξ. On the right, the hyperplane containing the thick line and $\delta(M_0)$ is 'moved to infinity', revealing the decomposition of lemma 12.

Corollary 13. The set of admissible quadrics through a set of spheres represented by $\mathcal{X} \subset \mathbb{P}^{n+1}$ is given by:

$$\mathrm{Adm} \cap H(\mathcal{X})$$

This, in turn, allows us to find all skins around these sphere, by taking convex subsets of these quadrics containing $\mathcal{X}$. In other words, these 4 problems are equivalent:

- Looking for skins around a set of spheres certain spheres.
- Looking for shrunk pencils of quadrics containing these spheres.
- Finding quadrics through their representatives.
- Intersecting $\mathrm{Adm} \subset \mathrm{PSym}$ with hyperplanes.

Using this equivalence we can easily prove a few statements about the uniqueness of the extended skin surface, and prove a bijection between subspaces of PSym and the set of (extended) skin surfaces.

Lemma 14. The extended skin surface (see definition 5) is unique: For each shrink factor $s \neq 0$, and set of $n+1$ spheres in general position in $\mathbb{R}^n$, there is a unique skin around these spheres.

Proof. We know that for fixed $s \neq 0$, the corresponding subset $\mathrm{Adm}^s \subset \mathrm{Adm} \subset \mathrm{PSym}$ is an $n+1$ dimensional subspace (see lemma 10). Let $\mathcal{X} \subset \mathbb{P}^{n+1}$ such that it represents $n+1$ spheres in general position. Note that therefore $H(\mathcal{X})$ is a codimension $n+1$ projective subspace of PSym. The intersection $\mathrm{Adm}^s \cap H(\mathcal{X})$ is not empty (by counting the dimensions), and is therefore a subspace. Assuming the dimension of this subspace is not 0, it contains a $Q^s(\lambda)$, where λ represents a hyperplane in $\mathbb{R}^n$. Thus all spheres in $\mathcal{X}$ are centred on this hyperplane. As the spheres were taken in general position, the dimension of $\mathrm{Adm}^s \cap H(\mathcal{X})$ is 0, allowing for only one admissible quadric containing $\mathcal{X}$, and thus one possible skin. $\square$

Furthermore, using the corollary, the set of all shrunk pencils describing extended skin surfaces through a set of points is 'almost' a projective subspace.

Lemma 15. Similar to the normal skin surface (see lemma 9), the extended skin surface of a set of $n+1$ spheres in general position can also be described by a pencil of quadrics. As the extended skin surface is not defined for $s = 0$, it is viewed as a pencil of quadrics with a point missing.

Proof. Instead of intersecting $H(\mathcal{X})$ with Adm^s (as in the previous proof) we intersect with Adm. The intersection of $H(\mathcal{X})$ with the $n+2$ dimensional subspace containing Adm, which we call $L_{\mathcal{X}}$ is a subspace, and any of the extended skins of the previous lemma are admissible. As for each s, there is only a single quadric of the correct form (the previous lemma), this subspace is 1 dimensional, and contains an element for each $s \neq 0$, hence:

$$H(\mathcal{X}) \cap \mathrm{Adm} \cong L_{\mathcal{X}} \cap \mathrm{Adm} \cong \mathbb{R}$$

This pencil of quadrics, defining an extended skin surface, is shown schematically in figure 5.3. □

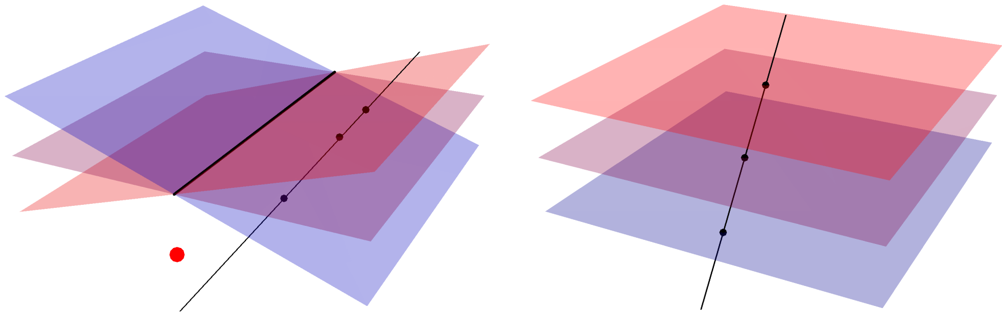

Figure 5.3: The situation of figure 5.2, but instead of the pencil representing the normal skin surface, a pencil is representing an extended skin surface is shown. On the right the line does not intersect $\delta(M_0)$ 'at infinity'.

When finding skin surfaces, in general the set of points is not such that the affine hull is n dimensional. For lower dimensional shrunk affine hulls, corollary 11 allows us to write such a shrunk affine hull as $\cap_\lambda Q^s(\lambda)$ where λ are in a subspace. This set of $Q^s(\lambda)$, is equal to a subspace (lemma 10), which can be identified with the shrunk affine hull. This gives us a natural way to identify these lower dimensional shrunk affine hulls with subspaces of $\mathbb{P}^{n+2} \supset \mathrm{Adm}$.

Corollary 16. The defining sets of spheres, for any skin surface or extended skin surface in $\mathbb{R}^2$, can be viewed as a subspace S of $\mathbb{P}^{n+2} \supset \mathrm{Adm}$, using the identification

$$\{S \subset \mathbb{P}^{n+2} : S \neq \mathrm{Adm}^s \text{ for some } s \in \mathbb{P}^1\} \leftrightarrow \left\{ \bigcap_{Q \in S \cap \mathrm{Adm}^s} Q \subset \mathbb{P}^{n+1} : s \in \mathbb{P}^1 \right\},$$

which identifies, for set of spheres $\mathcal{P} \subset \mathbb{R}^n \times \mathbb{R}$:

$$\begin{array}{ll} \left\{ (\mathrm{aff}\mathcal{P})^s \subset \mathbb{R}^n \times \mathbb{R} : s \in \mathbb{P} \right\} & \text{with } S \text{ such that } \delta(M_0) \in S \\ \left\{ (\mathrm{aff}\mathcal{P}^{\frac{1}{s}})^s \subset \mathbb{R}^n \times \mathbb{R} : s \in \mathbb{P} \setminus \{0\} \right\} & \text{with } S \text{ such that } \delta(M_0) \notin S \end{array}$$

An example of this S, along with some intersections is given in figure 5.4.

5.3 Shrinking in PSym

The last part of this section is devoted to shrinking quadrics in PSym directly. This will be done by parametrizing the pencil L_Q through Q and $\delta(M_0)$ such that this parameter corresponds to the shrink factor. This parametrization will not use the underlying Möbius space, or intuition about shrunk spheres. This allows us to, for example, define $\mathrm{Adm}^s \subset \mathrm{Adm}$ directly in PSym.

Any projective line L can be parametrized with a projective transformation from $\mathbb{P}^1$ to L by fixing 3 points on L. The three points we fix are $\delta(M_0)$ and two unique degenerate quadrics on L_Q. Using the explicit form of a sphere in PSym (example 14), we can see that A_σ is a sphere if and only if it represents $\delta(M_0)$, thus making $\delta(M_0)$ unique. The two points we fix at $\sigma = [1:1]$ and $\sigma = [0:1]$ will be two degenerate quadrics. For this we use the following lemma:

Lemma 17. A quadric of the form $Q^\sigma(\xi)$ for $\xi \notin \mathcal{H}_P$ is degenerate if and only if $\sigma = [1:1]$ or $\sigma = [0:1]$.

Proof. A quadric is degenerate if and only if its matrix is singular. Calcula-

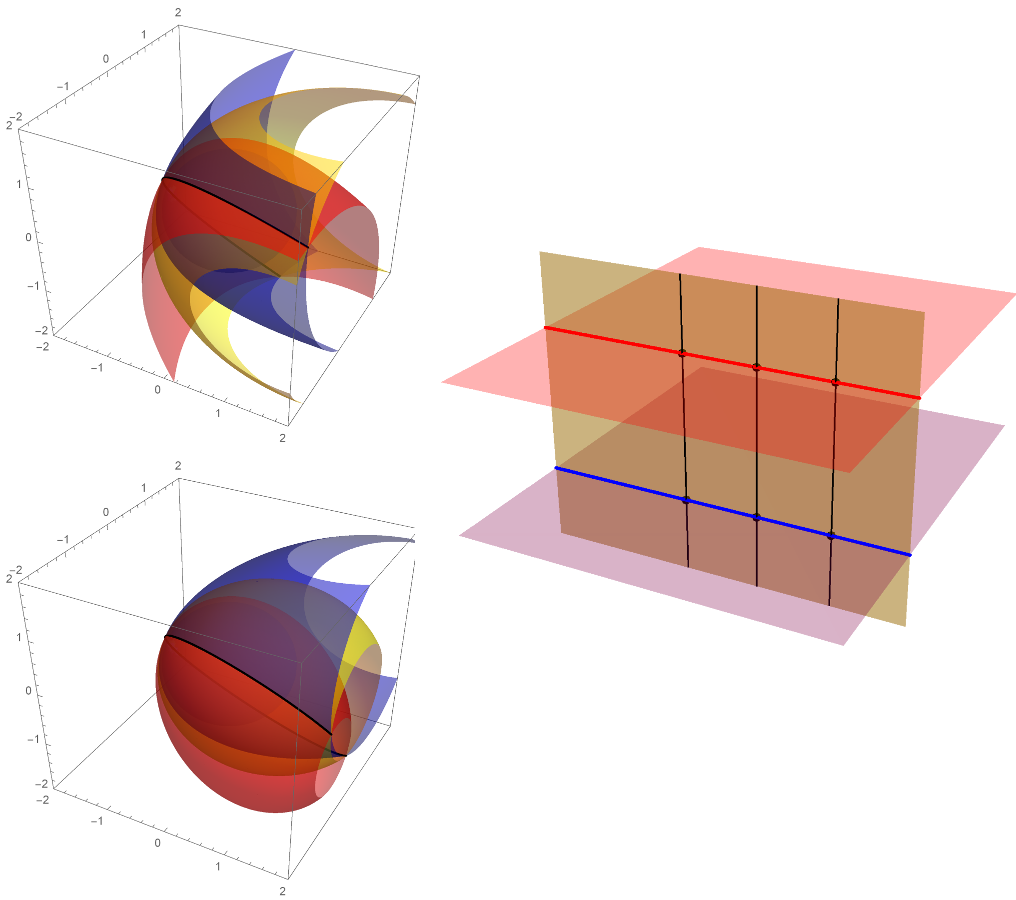

Figure 5.4: On the left a shrunk 1-dimensional affine hull K^s (the black line) shown for two values of s, given as in figure 4.5, as the intersection of a pencil of quadrics. On the right, the identification with a subspace of Adm: The yellow plane represents $K = \{K^s\}$. It intersects each Adm^s in a pencil, which, corresponds to one of the left-side images. The red and blue lines represent the upper- and lower figure respectively.

tion using the explicit form reveals:

$$\det A_{[s:t]}(\xi) = -(a_0 + a_1)^{n+2}(t-s)^n t^2$$

Thus proving the lemma. $\square$

If $n > 2$, we can distinguish the two degenerate quadrics by their degeneracy, thus giving us three identifiable points on the line. This fixes a parametrization that is the same as the one on $\{Q^\sigma(\xi) : \sigma \in \mathbb{P}^1\}$.

Corollary 18. For any $Q \in \mathrm{PSym}$, finding $\sigma \in \mathbb{P}^1$ such that $Q = Q^\sigma(\xi)$ for some ξ can be done by parametrizing L_Q in this way. Finding $\xi \in \mathbb{P}^{n+1}$ can then be done by shrinking to $Q^1(\xi)$ and taking the orthogonal complement of $Q^1(\xi) \setminus \mathcal{H}_P$.

This allows us to define the sets Adm^s in PSym directly, without needing the underlying sets of spheres. In addition to simply shrinking quadrics, this can be used with corollary 16, to find the subspace of $\mathbb{P}^{n+2}$ corresponding to sets of skin surfaces.

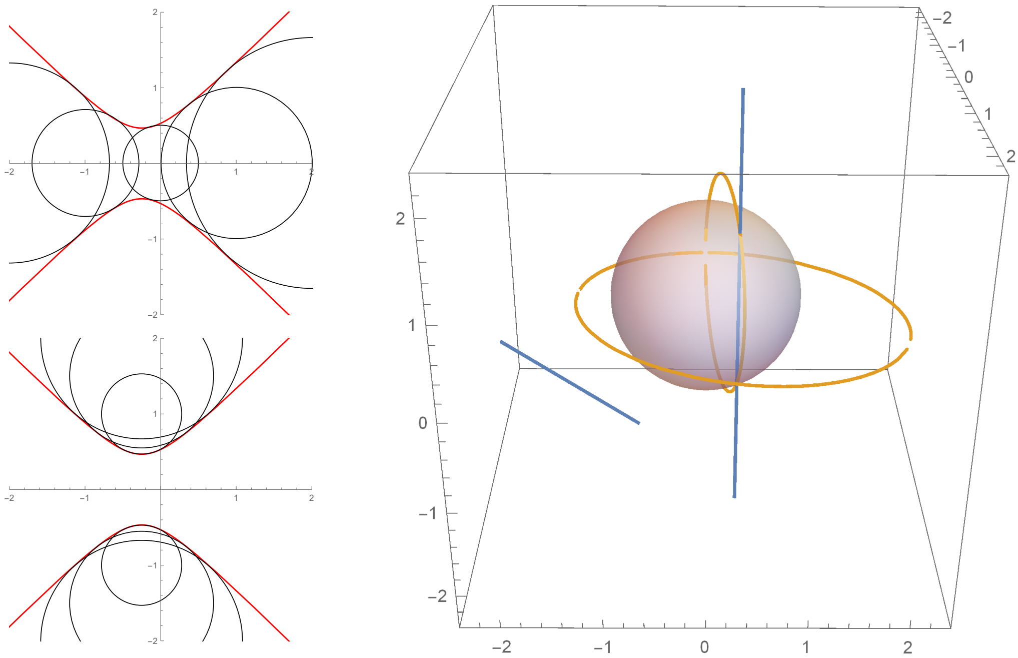

Figure 6.1: The shrunk affine hull of spheres and its shrunk orthogonal complement from figure 2.8, with their representations in the Möbius space. The two blue lines are span$\mathcal{X}$ and $\mathcal{X}^{\perp}$, where the yellow lines are the same sets shrunk.

6 Symmetry and Duality

Using the framework of Möbius geometry allowed us to view a pencil of spheres as a subspace of $\mathbb{P}^{n+1}$. Its orthogonal complement represents the set of all spheres that intersect orthogonally with the pencil. In the Möbius space, lemma 2 states that a subspace shrunk with factor s, defines the same envelope as its orthogonal complement shrunk with factor $1-s$. A simple example is given in figure 6.1. Using the notation, for subset $\mathcal{X} \subset \mathbb{P}^{n+1}$:

$$\text{env}(\text{span}\mathcal{X})^s = \text{env}(\mathcal{X}^{\perp})^{1-s}$$

Finding such an envelope of a shrunk affine hull is useful for finding the skin surface, as lemma 1 gives a decomposition of the skin into such patches using the mixed cells. Furthermore, we have an alternative way of finding the envelope. If two sets define the same envelope from the in- and outside respectively, their intersection is this envelope.

$$\begin{aligned}
\mathrm{skn}^s\mathcal{P} &= \bigcup_{\mathcal{X}\subset\mathcal{P}} \mu^s_{\mathcal{X}} \cap \mathrm{env}(\mathrm{aff}\mathcal{X})^s \\
&= \bigcup_{\mathcal{X}\subset\mathcal{P}} \mu^s_{\mathcal{X}} \cap \mathrm{env}(\mathcal{X}^{\perp})^{1-s} \\
&= \bigcup_{\mathcal{X}\subset\mathcal{P}} \mu^s_{\mathcal{X}} \cap \mathrm{ucl}(\mathrm{aff}\mathcal{X})^s \cap \mathrm{ucl}(\mathcal{X}^{\perp})^{1-s}
\end{aligned}$$

It can be very useful to be able to define the skin from the in- and outside. The sum of dimensions of a subspace and its orthogonal complement in $\mathbb{P}^{n+1}$ is n, and if we are able to describe both sets it is possible to pick the set with the lowest dimension. As the complexity of finding the envelope is heavily dependant on the dimension of the parameter space, this allows us to reduce the complexity of finding the skin surface. This is illustrated in figure 6.2. In particular, for finding skins in $\mathbb{R}^2$ or $\mathbb{R}^3$, this means we only need to find envelopes of (at most) 1 dimensional subspaces.

However, using this alternative decomposition leaves a question: We know each $\mathrm{aff}\mathcal{X}$ contains a section (the convex hulls) such that the union (the convex hull of $\mathcal{P}$) defines the skin immediately:

$$\mathrm{skn}^s\mathcal{P} = \mathrm{env}(\mathrm{conv}\mathcal{P})^s$$

Does the orthogonal complement have an analogue?

It turns out we can find these sections of the orthogonal complements, and as such, the set of spheres defining the skin surface from the outside. Furthermore, these sections are relatively convex in the Möbius space (see definition 9) and can therefore be written as convex hulls of a set of points, $\mathcal{Q}$, that is a dual to $\mathcal{P}$: Their skin surfaces are the same.

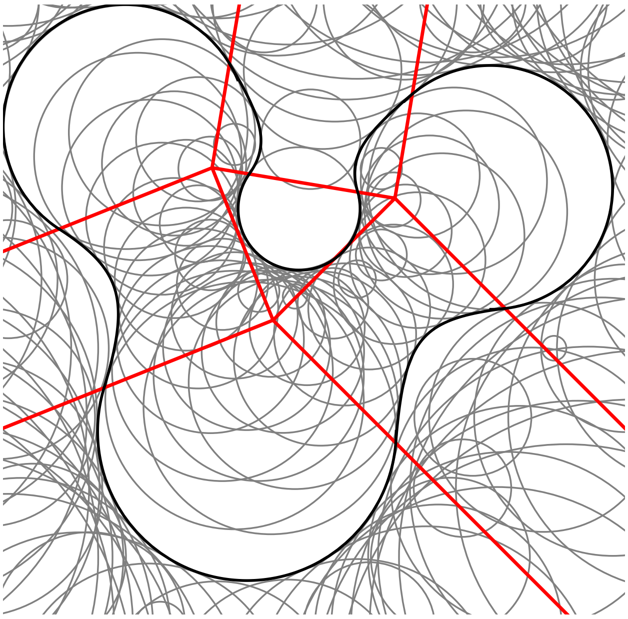

Figure 6.2: The skin surface, shown with the mixed cells, and defined from the in- and outside. Note that if the set on the inside has a lot of complexity, the set on the outside does not, and vice versa.

6.1 Subsets of orthogonal complements

Similar to how the convex hulls of spheres lie in their corresponding affine hulls, we are interested in subsets of the orthogonal complement. In defining the Voronoi cell (definition 4), however, we implicitly already defined such a subset.

Recall the duality between the Delauney and Voronoi complexes; These were both defined as the set of centres of certain spheres. The Delauney cell as the set of centres of a convex hull and the Voronoi cell as the set of centres of the set of spheres orthogonal to $\mathcal{X} \subset \mathcal{P}$, and further than orthogonal to $\mathcal{P} \setminus \mathcal{X}$. The convex hull of spheres is exactly the section of aff$\mathcal{X}$ that locally contributes to the skin, hence suggesting the following theorem:

Lemma 19. The subset of $\mathcal{X}^{\perp}$ that can contribute to the skin is $V'_{\mathcal{X}}$: the set of spheres orthogonal to $\mathcal{X}$, and orthogonal, or further than orthogonal to $\mathcal{P} \setminus \mathcal{X}$. More precisely, for all $\mathcal{X} \subset \mathcal{P}$:

$$\mu_{\mathcal{X}}^s \cap \mathrm{env}(\mathrm{aff}\mathcal{X})^s = \mu_{\mathcal{X}}^s \cap \mathrm{env}(V'_{\mathcal{X}})^{1-s}$$

Note: This immediately implies that:

$$\mathrm{skn}^s\mathcal{P} = \bigcup_{\mathcal{X} \subset \mathcal{P}} \mu_{\mathcal{X}}^s \cap \mathrm{env}(V'_{\mathcal{X}})^{1-s}$$

Proof. We know that $\mathcal{X}^{\perp}$ has this property, hence it suffices to prove that $V'_{\mathcal{X}}$ is a subset with the same property. If a sphere is closer than orthogonal to $\hat{p}$ its upwards closure will still intersect $\hat{p}$ when shrunk with factors s and $1-s$ respectively. Therefore, a sphere closer than orthogonal to $\hat{p} \in \mathcal{P}$ can not define any part of the skin, as the skin does not intersect $\hat{p}^s$. $\square$

Like the convex hull of $\mathcal{P}$, which contains all convex hulls of $\mathcal{X} \subset \mathcal{P}$, we were searching for an analogue to $\mathrm{conv}\mathcal{P}$ for the set of spheres defining the skin surface from the outside. It turns out simply taking the union has this property.

Theorem 20. Define $\mathrm{Vor}\mathcal{P}$ as the union of $V'_{\hat{p}}$ over $\hat{p} \in \mathcal{P}$. Then the set $\mathrm{Vor}\mathcal{P}$ is a set of spheres such that:

$$\begin{aligned} \mathrm{skn}^s\mathcal{P} &= \mathrm{env}(\mathrm{conv}\mathcal{P})^s \\ &= \mathrm{env}(\mathrm{Vor}\mathcal{P})^{1-s} \end{aligned}$$

Proof. First, note that $V'_{\mathcal{X}} \subset V'_{\hat{p}}$ if $\hat{p} \in \mathcal{X}$. Therefore all $V'_{\mathcal{X}} \subset \mathrm{Vor}\mathcal{P}$. If $\hat{p} \in \mathrm{Vor}\mathcal{P} \setminus V'_{\mathcal{X}}$, it is further than orthogonal from $\mathrm{conv}\mathcal{X}$, and hence can not contribute to the envelope in the mixed cell $\mu_{\mathcal{X}}^s$. As the mixed cells cover $\mathbb{R}^n$,

$$\bigcup_{\mathcal{X} \subset \mathcal{P}} \mu_{\mathcal{X}}^s \cap \mathrm{env}(V'_{\mathcal{X}})^{1-s} = \mathrm{env}(\mathrm{Vor}\mathcal{P})^{1-s}$$

Proving the theorem. $\square$

Recall that the Voronoi complex defines a full decomposition of the space $\mathbb{R}^n$. As the spheres in $V'_{\mathcal{X}}$ are centred on the corresponding Voronoi cell, the set Vor$\mathcal{P}$ is the set of spheres where, for each centre $z \in \mathbb{R}^n$ the weight w is the smallest weight such that $\hat{p} = (z, w)$ is orthogonal to at least one element of $\mathcal{P}$.

6.2 Finding $V'_{\mathcal{X}}$ in the Möbius space

This decomposition from the previous section would not be useful if it is very hard to find these subsets $V'_{\mathcal{X}}$ in the Möbius space. It turns out, the condition on $\mathcal{X}^{\perp}$ to get $V'_{\mathcal{X}}$ is quite easy, and can be given by a simple quadratic form. To derive this, we use the use the previously introduced method of shrinking to check whether a point describes a smaller sphere than another.

Lemma 21. Let $\xi = [a_0 : \ldots : a_{n+1}] \notin \mathcal{H}_P$. Define the quadratic form:

$$R_\xi : [x_0 : \ldots : x_{n+1}] \mapsto (a_0 + a_1)(x_0 + x_1)(a, x)$$

The subsets of $\mathbb{P}^{n+1}$ where the form in negative, zero and positive are called $R_\xi^<$, R_ξ^0 and $R_\xi^>$ respectively. Note that these are the light-, time- and space-like points from example 12. Then, $R_\xi^<$ represents all spheres further than orthogonal to the sphere represented by ξ and $R_\xi^>$ represents those less than orthogonal to the sphere represented by ξ. Finally $R_\xi^0 \setminus \mathcal{H}_P$ represents the proper spheres orthogonal to ξ.

Proof. Intuitively, for $\hat{p}, \hat{q} \in \mathbb{R}^n \times \mathbb{R}$ if the sphere $\hat{p}$ is further than orthogonal to $\hat{q}$, $\hat{q}$ can be 'made bigger' to become orthogonal to $\hat{q}$. More formally, there exists a sphere $\hat{p}'$ with the same centre as $\hat{p}$, and weight $w_{p'} > w_p$, such that $\hat{p}'$ is orthogonal to $\hat{q}$. Let ξ represent $\hat{q}$ and let $\mu = [x_0 : ... : x_{n+1}] \in \mathbb{P}^{n+1}$ represent the sphere $\hat{p}$.

Due to symmetry, we only need to prove:

$$\mu \in R_\xi^< \iff \hat{p}, \hat{q} \text{ are further than orthogonal.}$$

If $\hat{p}$ is a point sphere, then $(x_0 + x_1) \neq 0$, and as $\xi \notin \mathcal{H}_P$, we know $\mu \in R_\xi^0$ if and only if $\hat{p}$ lies on the sphere represented by ξ, $\hat{q}$. Due to continuity, the interior points of the sphere $\hat{q}$ correspond either to $R_\xi^<$ or $R_\xi^>$. Simply checking an interior point sphere, the centre, proves the lemma for point spheres.

Now let $\hat{p}$ represent a proper (non-plane, non-point) sphere, then $\mu \in Q^s(\xi)$ for some unique $s \neq 0$, which can be found by intersecting a projective line and a hyperplane (see corollary 13). This means that there is a $\lambda = [y_0 : ... : y_{n+1}]$, with $\lambda \in Q^1(\xi) \setminus \mathcal{H}_P$ (which therefore is orthogonal to ξ), such that λ represents a sphere $\hat{p}'$ with the same centre as $\hat{p}$, and with weight (see corollary 4):

$$w_{\hat{p}'} = \frac{(y, y)}{(y_0 + y_1)^2} = \frac{(x, x)}{s(x_0 + x_1)^2} = \tfrac{1}{s} w_{\hat{p}}$$

For μ to be further than orthogonal to ξ, we need $w_{p'} > w_p$, hence:

$$\tfrac{1}{s}(x, x) > (x, x) \iff \left(\tfrac{1}{s} - 1\right)(x, x) > 0$$

Using the explicit definition of $Q^s(\xi)$ (definition 8), we can rewrite this to:

$$\left(\frac{-2(x_0 + x_1)(a, x)}{(a_0 + a_1)(x, x)}\right)(x, x) > 0 \iff (x_0 + x_1)(a_0 + a_1)(a, x) < 0$$

The fact that $\mathcal{H}_P \subset R_\xi^0$ is immediate, completing the proof. $\square$

This lemma allows us to easily define the set $V'_{\mathcal{X}}$ as represented in the Möbius space $\mathbb{P}^{n+1}$.

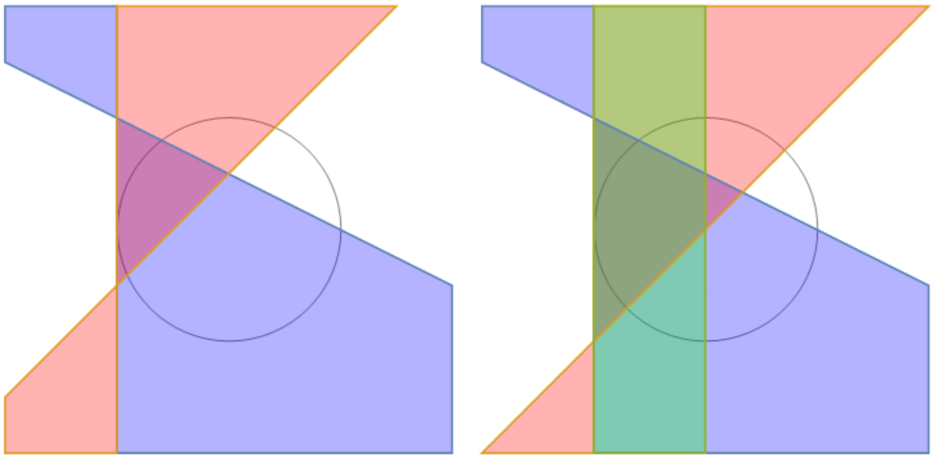

Figure 6.3: An intersection of $R_\lambda^{\leq}$, in $\mathbb{P}^2$, for sets of 2 or 3 elements of $\mathcal{P}$. The $V'_\mathcal{X}$ are all on the boundaries of the polygon formed by the intersection.

Corollary 22. For subset $\mathcal{X} \subset \mathcal{P} \subset \mathbb{P}^{n+1}$ (where $\mathcal{X}$ and $\mathcal{P}$ represent sets of spheres in $\mathbb{R}^n$), $V'_\mathcal{X}$ is the set of all spheres orthogonal to $\mathcal{X}$ and further than orthogonal to $\mathcal{P} \setminus \mathcal{X}$. This is represented in $\mathbb{P}^{n+1}$ by:

$$V'_\mathcal{X} \quad = \mathcal{X}^\perp \cap \bigcap_{\lambda \in \mathcal{P}} R_\lambda^{\leq}$$

Note that in the last equation the intersection is over $\mathcal{P}$, not $\mathcal{P} \setminus \mathcal{X}$. Therefore, finding the representatives of $V'_\mathcal{X}$ in $\mathbb{P}^{n+1}$ is done by intersecting $\mathcal{X}^\perp$ with the same (full dimensional) subset of $\mathbb{P}^{n+1}$. This is shown, in a low dimension, in figure 6.3. The application to skin surfaces is shown in 6.4.

The sets $R_\xi^{\leq}$ are convex in the sense of corollary 8, in fact, they are simply given as half spaces in $\mathbb{P}^{n+1} \setminus \mathcal{H}_P$. This also means that the sets $V'_\mathcal{X}$ are convex. These sets are in general not bounded in $\mathbb{P}^{n+1} \setminus \mathcal{H}_P$, as their sets of centres (the original Voronoi cells) need not be bounded either. Being convex subsets of subspaces of $\mathbb{P}^{n+1}$ allows us to shrink these sets of spheres, as in 4.2. An extensive example is given in figure 6.6.

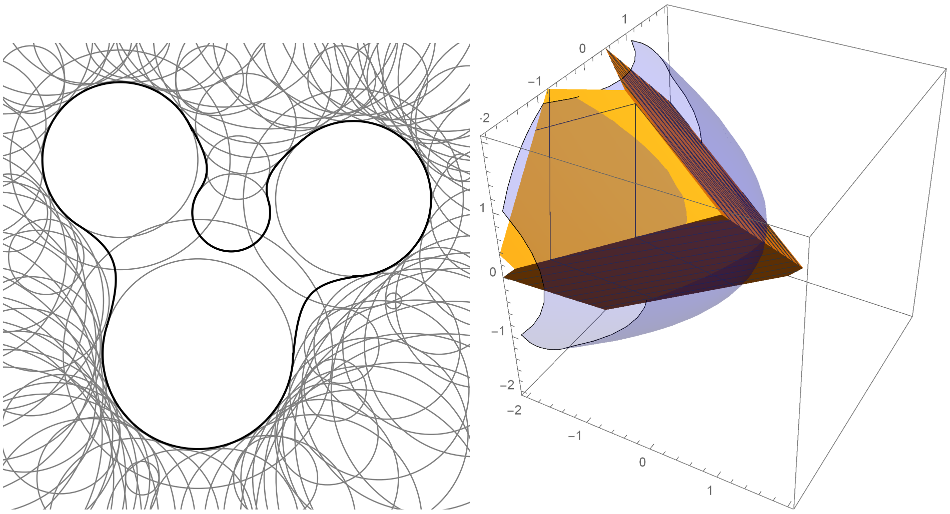

Figure 6.4: The skin curve from figure 4.8, defined from the outside. This set of spheres is shown in the Möbius space (where $x_0 + x_1$ is scaled to 1). The orthogonal planes are clipped to (infinite) triangles. The 'point' represents the sphere in the middle of the skin.

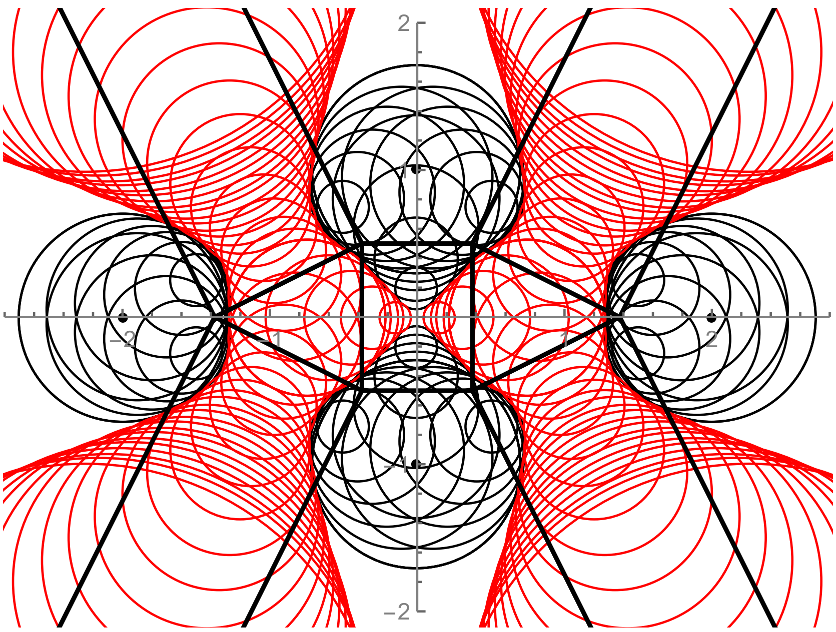

Figure 6.5: The sets of spheres used in figure 6.6, with the mixed cells shown, and the 1- dimensional sets of spheres used for the construction.

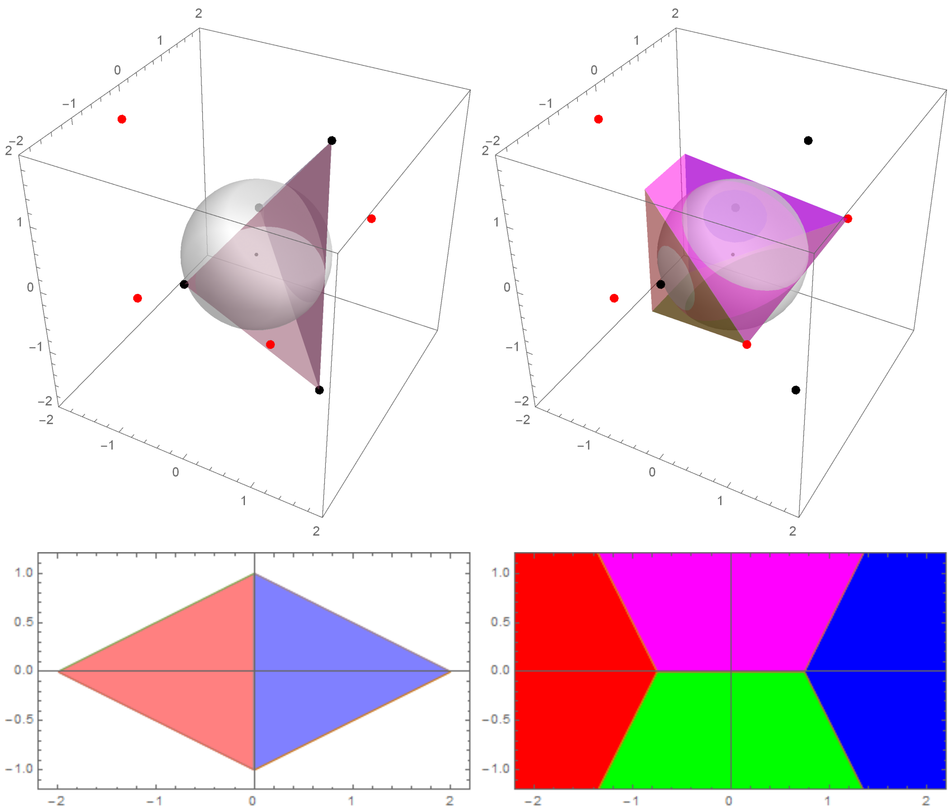

Figure 6.6: The convex hulls in $\mathbb{P}^3$ $(x_0 = 1)$ used for defining a skin surface of the spheres in 6.5. Left from the inside, right from the outside. Above the convex hulls and below this, their the corresponding set of centres (the projection onto the Möbius sphere), colour-coded. The black points are $\mathcal{P}$, and the red points are the basis found by $\mathcal{X}^{\perp}$.

7 Envelopes in the Möbius space

We have been concerned with the sets of spheres defining the skin surfaces, and their properties in the Möbius space. The only step for finding the skin surface that has not been viewed in the Möbius space yet, is taking the envelope. So far this has been glanced over, these sets of spheres and their envelope were viewed as almost synonymous. However, finding the envelope is not entirely trivial. This section will generalize the process of taking envelopes of spheres to work in the Möbius space.

This new process will change the problem of finding an envelope of quadratic shapes (spheres) in $\mathbb{R}^n$ to one of finding the envelope of linear shapes (planes) in the Möbius space $\mathbb{P}^{n+1}$. An intuitive way of viewing this is given in figure 7.1. As we have already found the representatives of all necessary sets of spheres, as patches of admissible quadrics, we will also apply the new process to these quadrics.

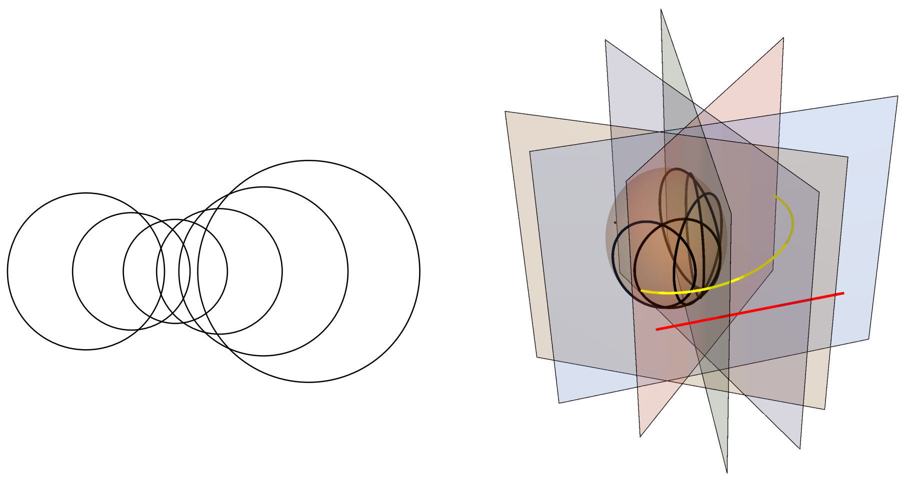

Figure 7.1: Left a shrunk convex hull of spheres in $\mathbb{R}^2$, visualized by 5 elements. On the right the corresponding convex hull in red, the shrunk convex hull in yellow, and some orthogonal planes are shown. The envelope of these planes intersects the Möbius sphere in the projection of the skin.

7.1 Envelopes and orthogonality

Definition 1 introduces the envelope as the boundary of a union of 'interiors'. For the set of $\mathbb{R}^n \to \mathbb{R}$ functions $\{F_\mu : \mu \in C\}$, the envelope was a subset of the discriminant set:

$$D_F = \{x \in \mathbb{R}^n : F(x,\mu) = 0, \nabla_\mu F(x,\mu) = 0 \text{ for some } \mu \in C\}$$

where $F(x,\mu) : (x,\mu) \mapsto F_\mu(x)$. This means that, for a point $x \in \mathbb{R}^n$ to be on the envelope, it needs to be on some sphere in the set ($F(x,\mu) = 0$), and if the parameter is changed infinitesimally, it is still on the sphere ($\nabla_\mu F(x,\mu) = 0$).

In the case of skin surfaces, we are interested in envelopes of spheres. This means that all $F_\mu(x)$ are degree 2 polynomials, and therefore the derivatives with respect to μ are non-trivial. This makes the process of computing the envelope quite hard.

Another way of looking at the envelope of a set of spheres uses orthogonality. The envelope of a set of spheres is the set of points that are orthogonal to (or: on) some sphere in the set and further than orthogonal to (or: outside of) all other spheres. Instead of viewing only point spheres, we can also view this problem in the Möbius space. Let $\mathcal{S} \subset \mathbb{P}^{n+1}$ describe the given set of spheres. Using lemma 21 we can find the subset of $\mathbb{P}^{n+1}$ that is orthogonal to one sphere, and orthogonal, or further than orthogonal to all other elements of $\mathcal{S}$;

$$\mu^\perp \cap \bigcap_{\lambda \in \mathcal{S}} R_\lambda^{\leq} \subset \mathbb{P}^{n+1}$$

Note: When $\mathcal{S}$ is a subset of a pencil, the sets $\mu^\perp$ and $R_\lambda^{\leq}$ are fully determined by its endpoints. This implies that for a finite set of points $\mathcal{X} \subset \mathcal{P}$, the set $V'_\mathcal{X}$ (from lemma 19) can also be viewed as such a set.

Now let $\mathcal{S}$ be shrunk subspace of $\mathbb{P}^{n+1}$, with shrink factor $0 < s < 1$.

If two infinitesimally close spheres are orthogonal to a sphere, this sphere is further than orthogonal from all others, thus if $F(\xi,\mu)$ is viewed as (ξ,μ), the described envelope can be written:

$$\Delta_{\mathcal{S}} = \{\xi \in \mathbb{P}^{n+1} : (\xi,\mu) = 0, \nabla_{\mu}(\xi,\mu) = 0 \text{ for some } \mu \in \mathcal{S}\}$$

This needs a small note, in that this is not a discriminant set according to the previous definition, as the used condition (ξ,μ) is not a well defined function. However, the used zero-sets are well defined. This means that we defined a set $\Delta_{\mathcal{S}}$ that generalises the discriminant set in $\mathbb{R}^n$. Lastly, if $\mathcal{S}$ represents $(\text{aff}\mathcal{P})^s$, and the shrink factor is $0 < s < 1$, any sphere orthogonal to an infinitesimally small neighbourhood in $\mathcal{S}$ is further than orthogonal to all other $\lambda \in \mathcal{S}$. This means that, for $0 < s < 1$, the discriminant set is the envelope.

Corollary 23. The defined set $\Delta_{\mathcal{S}}$ is such that, if $\mathcal{S}$ represents a set of spheres $(\text{aff}\mathcal{P})^s$ in $\mathbb{R}^n$, with $0 < s < 1$ and $|\mathcal{P}| \leq n$, the envelope of $(\text{aff}\mathcal{P})^s$ in $\mathbb{R}^n$ is given as the set of point spheres in $\Delta_{\mathcal{S}}$. In shorthand notation:

$$\text{env}(\text{aff}\mathcal{P})^s \cong \Delta_{\mathcal{S}} \cap M_0$$

Note that this is not an equality, but if we project M_0 back to $\mathbb{R}^n$, using the stereographic projection, these sets are equal.

The envelopes of a shrunk affine hull and its orthogonal complement are equal. Therefore, if $\mathcal{T}$ represents $(\mathcal{P}^{\perp})^{1-s}$, the set $\Delta_{\mathcal{T}} \cap M_0$ is equal to $\Delta_{\mathcal{S}} \cap M_0$. Even more, as both sets define the skin surface from different sides, no proper sphere can be orthogonal to both sets, hence $\Delta_{\mathcal{S}} \cap \Delta_{\mathcal{T}} \subset M_0$. This is shown in figure 7.2. The correlation between the skin and both sets Δ is shown in 7.3.

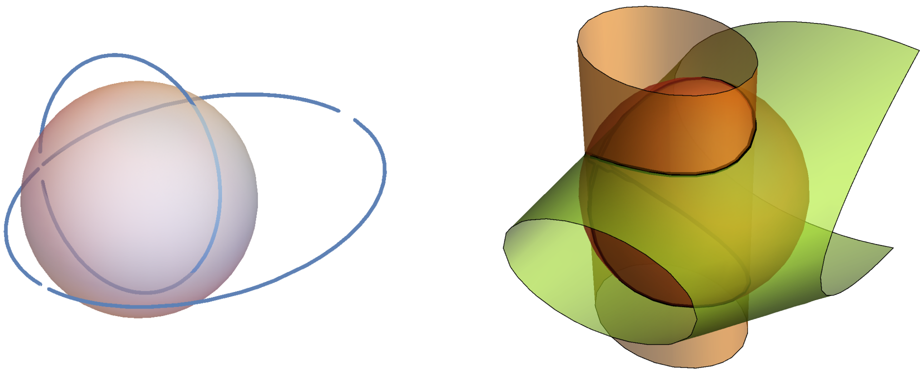

Figure 7.2: On the left a shrunk affine hull and its shrunk orthogonal complement, for the sets of figure 6.1. On the right the two corresponding $\Delta_{\mathcal{S}}$, and the Möbius sphere. Note that any two of the three intersect in the same set of points.

7.2 Explicit envelopes: An example

The previous section gave a method for finding the envelope of shrunk affine hulls using the orthogonal planes. This section will show how easy finding the sets $\Delta_{\mathcal{S}}$ is. We will find its explicit form for a general shrunk affine hull. We will do this in the proof for the following, somewhat surprising, theorem:

Theorem 24. Let the set $\mathcal{S}$ describe a shrunk affine hull of spheres, $(\text{aff}\mathcal{P})^s$ in $\mathbb{P}^{n+1}$. Let $\mathcal{T}$ describe its shrunk orthogonal complement, $(\mathcal{P}^{\perp})^{1-s}$, such that they describe the same envelope. Then the sets $\Delta_{\mathcal{S}}$ and $\Delta_{\mathcal{T}}$ are quadrics in $\mathbb{P}^{n+1}$, and using the space PSym to add and multiply quadrics:

$$(1-s)\Delta_{\mathcal{S}} + s\Delta_{\mathcal{T}} = M_0$$

Proof. If we translate and rotate our original space $\mathbb{R}^n$, we can take the affine hull such that the affine hull is centred on the first m-axes, and the orthogonal complement on the other. This means their respective spaces of centres intersect in the origin. In $\mathbb{P}^{n+1}$, when looking only at proper spheres, we can view $\mathbb{P}^{n+1} \setminus \mathcal{H}_P$ by taking $x_0 = 1 - x_1$. In $\mathbb{P}^{n+1} \setminus \mathcal{H}_P$:

$$\mathcal{S} \quad = (\mathrm{aff}\mathcal{P})^s \quad = \quad \{(S(\vec{x}), \vec{x}, 0 : ..., 0) : \vec{x} \in \mathbb{R}^m\}$$

We know the function $S : \mathbb{R}^m \to \mathbb{R}$ is in general quadratic, but linear for $s = 1$, as the affine hull is a subspace of $\mathbb{P}^{n+1}$. The function S can be written in the form:

$$S(\vec{x}) = \frac{1 - (1-s)(x_1^2 + ... + x_m^2) + s \cdot c}{2}$$

Let $\xi = (a_1, ..., a_{n+1})$ represent a proper sphere in $\mathbb{P}^{n+1} \setminus \mathcal{H}_P$. If $\lambda \in \mathcal{S}$ corresponds to $\vec{x} = (x_1, ..., x_m) \in \mathbb{R}^m$, then:

$$(\xi, \lambda) = -1 + a_1 + S(\vec{x}) + a_2 x_1 + ... + a_{m+1} x_m$$

And therefore:

$$\frac{\partial(\xi, \lambda)}{\partial x_i} = \frac{\partial S}{\partial x_i}(x) + a_{i+1} = -(1-s)x_i + a_{i+1}$$

Equating all conditions to 0, gives explicit formulae for ξ.

$$\xi \in \Delta_{\mathcal{S}} \iff \begin{cases} a_{i+1} & = -\frac{\partial S}{\partial x_i}(x) \quad \forall i \text{ s.t. } 1 \leq i \leq m \\ a_1 & = 1 - S(x) - x_1 a_2 - ... - x_m a_{m+1} \end{cases}$$

Note that these first conditions are linear, and the condition for a_1 quadratic. Therefore, homogenizing the condition for a_1, and using the relations between x_i and a_{i+1}, reveals that $\Delta_{\mathcal{S}}$ is a quadric.

$$\Delta_{\mathcal{S}} = \left\{ \xi \in \mathbb{P}^{n+1} : -a_0^2 + a_1^2 + sc(a_0 + a_1)^2 + \sum_{i=1}^{m} \frac{a_{i+1}^2}{1-s} = 0 \right\}$$

For the orthogonal complement, the function corresponding to S is for $\vec{x}' =$

$(x_{m+1}, ..., x_n)$:

$$T(\vec{x}') = \frac{1 - s(x_{m+1}^2 + ... + x_n^2) + (1-s) \cdot (-c)}{2}$$

Using the same reasoning as for $\Delta_{\mathcal{S}}$, reveals:

$$\Delta_{\mathcal{T}} = \left\{ \xi \in \mathbb{P}^{n+1} : -a_0^2 + a_1^2 - (1-s)c(a_0 + a_1)^2 + \sum_{i=m+1}^{n} \frac{a_{i+1}^2}{s} = 0 \right\}$$

This reveals the wanted equality,

$$(1-s)\Delta_{\mathcal{S}} + s\Delta_{\mathcal{T}} = \left\{ \xi \in \mathbb{P}^{n+1} : -a_0^2 + a_1^2 + ... + a_n^2 \right\}$$

proving the theorem. □

Therefore these two quadrics lie on a pencil of quadrics with M_0.

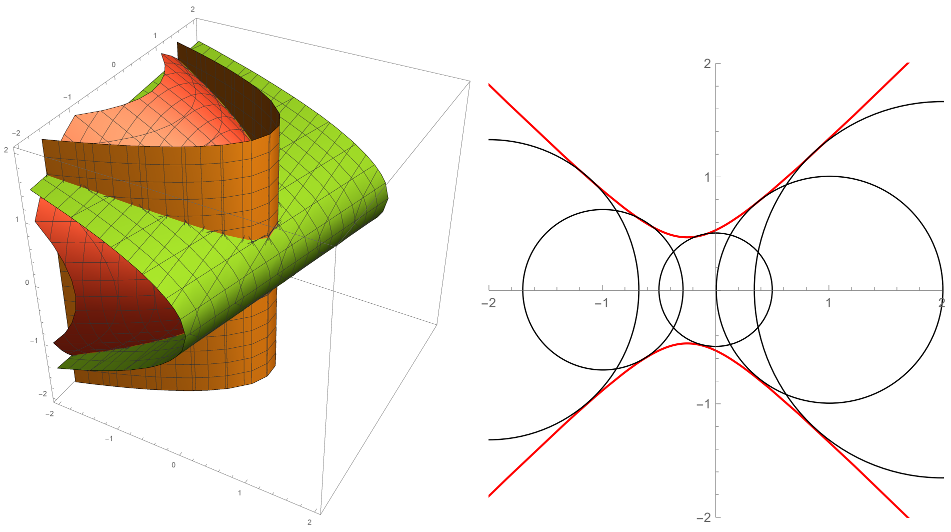

Figure 7.3: The skin on the right is shown in $\mathbb{P}^{n+1} \setminus \mathcal{H}_P$, where the stereographic projection is simply projection onto the plane $x_1 = 0$.

Conclusion

As stated in the introduction, this thesis set out with a single goal:

Describe the skin surface, and its properties using Möbius geometry.

The Möbius geometry turned out to have a few properties useful for, mostly, describing the set of spheres defining the skin surface. Subspaces of the Möbius space correspond to flats of spheres. Furthermore, the orthogonal complement of this subspace, using the natural form on the Möbius space, was closely related to symmetry and decomposability of the skin surface.

The skin surface is given as the envelope of a shrunk convex subset of a flat of spheres. Shrunk flats of spheres were found to correspond to quadrics, on which convexity was defined to correspond to shrunk convex hulls of spheres. Viewing the quadrics as objects of their own a few observations were made. The process of shrinking a quadric could be viewed as moving the object on a pencil of quadrics. The pencil has a natural parametrization, based on the degenerate quadrics it contains. This way of viewing quadrics allows for a natural way of looking at the extended skin surface. Lastly, we defined a bijection between the set of all (not only full dimensional) shrunk affine hulls of spheres determining skin surfaces *and* extended skin surfaces and subspaces of $\mathbb{P}^{n+2}$.

The set of spheres determining a skin surface from the inside is, however, not a shrunk affine hull. Therefore its orthogonal complement is not, generally, the set of spheres determining it from the outside. However, the quadratic form on the Möbius space allowed for an easy way to determine the complex that does determine the skin from the outside. This is easily determined from a set of input spheres, allowing to switch between a k, and an $n-k$ dimensional parameter space, corresponding to a subset of a $k-$flat and a subset of it's orthogonal complement. As the complexity of finding the envelope is heavily reliant on this dimension, this means that the choice can be made to take the envelope of (at most) an $\lfloor \frac{n}{2} \rfloor$ dimensional set of spheres.

Finally (again) the notion of orthogonality was used to find the envelope in the $\mathbb{P}^{n+1}$. This changed taking the envelope of spheres to taking the envelope of planes, and only intersecting the result with a sphere.

However, using the Möbius space is not in every way positive. There is the obvious initial step, doing the stereographic projection, which makes intuitive reasoning on the spheres harder. In addition, the projective space has its own drawbacks, mostly computationally, but also when visualizing the space. Furthermore, the hyperplanes in $\mathbb{R}^n$ are not used for defining the skin. Usually, scaling the space to $x_0 + x_1 = 1$ is enough, resulting in the space of weighted points.

Further research opportunities

In addition to the skin surface, the envelope surfaces, introduced in [1], could be viewed in the Möbius space. This surface is a generalization of the skin, which also uses quadratic interpolation as a the weight function. However, this removes the special role set aside for orthogonality, and therefore the natural link to Möbius geometry.

In addition, having used envelopes of certain quadrics to define surfaces leads to the question whether other weight functions can be used. This thesis already includes a way of using the current skin surfaces to interpolate at most $n+2$ spheres in $\mathbb{R}^n$, but allowing higher degree weight functions will allow for different interpolations. This requires knowing the relation between the shape of the weight function and, for example smoothness or continuity of the resulting envelope.

Other than the Möbius geometry, the more general Lie-sphere geometry might also be a way to view the skin and other interpolation algorithms. This geometry deals with oriented contact of spheres, and therefore has potential to describe envelopes naturally. In a similar way, the set of spheres defining the skin from the outside has a tangent condition to the set of spheres defining the skin from the inside, which might be describable in the Lie-sphere geometry.

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A Definitions in vector- and projective spaces

To be able to introduce a few of the necessary concepts, we will state some facts on vector spaces, and subsets of these. In particular subspaces, affine and convex subsets are defined. After this, the projective space is defined.

A.1 Vector spaces

Definition 10. For a field k, we can take V a *vector space* over k, that is, a $k-$module. In particular, for this thesis, any V will be finite dimensional, and therefore have a finite $k-$basis. As a set, V is isomorphic to k^n, where n is the dimension. A $k-$algebra has the added structure of a ring.

Definition 11. A transformation $f : V \to W$ between $k-$vector spaces V, W is *linear* if $f(v_1 + v_2) = f(v_1) + f(v_2)$ for $v_1, v_2 \in V$ and $f(a \cdot v) = af(v)$ for $a \in k$.

Definition 12. The *group of endomorphisms* on an $n-$dimensional vector space V is written $\mathrm{End}(V)$, and consists precisely of all linear transformations $f : V \to V$. The *automorphism group* $\mathrm{Aut}(V)$ consists of invertible endomorphisms. This group is usually denoted $\mathrm{GL}(V) = \mathrm{Aut}(V)$ for *general linear group*. The further $\mathrm{GL}_n(k) = \mathrm{GL}(k^n)$ is also often used.

Properties:

- As we can define point-wise addition, composition and scalar multiplication with elements of k, $\mathrm{End}(V)$ is a $k-$algebra. In fact it is isomorphic to the algebra $M_n(k)$ of $n \times n$ matrices with entries in k, and therefore of $k-$dimension n^2.
- $\mathrm{GL}(V)$ is therefore isomorphic to the invertible $n \times n$ matrices, $M_n(k)^\times$. For matrices we have the determinant map, det: $M_n(k) \to k$, which is a polynomial in the matrix entries. We know a matrix is invertible if it's determinant is nonzero, therefore $M_n(k)^\times$ is (Zariski) affine open

in $M_n(k)$, and as such also of dimension n^2.

A.2 Subsets of vector spaces

Definition 13. A *(linear) subspace* S of V is a subset of V closed under addition and scalar multiplication. A subspace always contains $0 \in V$ (as 0 is a scalar), and can therefore be viewed as a vector space over k on its own.

Transformations on V may act trivially on the set S, giving rise to the *stabilizer subgroup* of S in a transformation group $T(V)$.

$$\mathrm{Stab}_S(T(V)) = \{f : V \to V \in T(V) : f(s) = s, \forall s \in S\}$$

As transformations on an $m-$dimensional subspace $S \subset V$ can be viewed as transformations on V 'up to' elements in the stabilizer subgroup:

$$\begin{aligned}
\mathrm{End}(S) &= {}^{\mathrm{End}(V)}\!\big/_{\mathrm{Stab}_S(\mathrm{End}(V))} \\
&\cong M_m(k) \\
\mathrm{GL}(S) &= {}^{\mathrm{GL}(V)}\!\big/_{\mathrm{Stab}_S(\mathrm{GL}(V))} \\
&\cong M_m(k)^\times
\end{aligned}$$

Recall that $M_m(k)^\times$ is open in $M_m(k)$ and therefore m^2 dimensional over k.

Definition 14. The *span* of a subset $\mathcal{P} \subset V$, $\mathrm{span}(\mathcal{P})$ is defined as the intersection of all subspaces containing the elements of $\mathcal{P}$.

> **Example 2:** As any subspace is closed under addition and scalar multiplication, the span is precisely all *linear combinations* of elements of $\mathcal{P}$:
>
> $$\mathrm{span}(\mathcal{P}) = \left\{v \in V : v = \sum a_i p_i, \text{ for } p_i \in \mathcal{P}, a_i \in k\right\}$$

Definition 15. A finite set of l points $\mathcal{P} \subset V$ is said to be in *general position* if their span is l-dimensional in V, or equivalently if there is no linear relation between them. $\mathcal{P}$ is in *special position* otherwise.

Properties:

- A single point, not equal to 0, is always in general position.
- If $l > n$, $\mathcal{P}$ is always in special position.

Usually, when considering two points, one would want to be able to draw a straight line between them, giving rise to the following definition:

Definition 16. An *affine subspace* (or a $flat$) $A \subset V$ is a subset of V such that the set of differences $\{x - y : x, y \in A\}$ forms a subspace.

Note: The term 'affine subspace' will be avoided as much as possible due to the confusion with affine subspaces of projective space, however, some of the literature will use this name.

Properties:

- Any flat containing $0 \in V$ is a subspace of V.
- A flat contains the straight line between any two of its points x, y.
- The dimension of an affine subspace is the dimension of the set of differences. An affine subspace of dimension 1 is called a *line*, of dimension 2 a *plane* and of dimension $n - 1$ a *hyperplane*.

This allows us to define the affine hull as, intuitively, the 'smallest affine subspace containing a set'. More precisely:

Definition 17. The *affine hull* of a set $\mathcal{P} \subset V$, $\mathrm{aff}(\mathcal{P})$ is defined as the intersection of all affine subspaces containing $\mathcal{P}$.

Example 3: Taking any element $p \in \mathcal{P}$, and translating the affine hull using $x \mapsto x - p$, results in a subspace, which is the span of $\{q - p : q \in \mathcal{P}\}$. This allows us to write the elements of $\mathrm{aff}(\mathcal{P})$ as:

$$\begin{aligned} \mathrm{aff}(\mathcal{P}) &= \{v \in V : v = p + \textstyle\sum a_i(p_i - p), \text{ for } p_i \in \mathcal{P}, a_i \in k\} \\ &= \{v \in V : v = \textstyle\sum a_i p_i, \text{ for } p_i \in \mathcal{P}, a_i \in k \text{ such that } \textstyle\sum a_i = 1\} \end{aligned}$$

The first notation gives us an intuitive way to view the affine hull, as a translation (by any of the points of $\mathcal{P}$) of a span of differences. The second notation writes all elements as so called *affine combinations*.

In this thesis, we will be looking at ways to interpolate a certain set of points. Therefore it usually makes sense to look at convex sets.

Definition 18. A set $C \subset V$ is convex if it contains the straight line segment bounded by any two points in C.

Similarly to the affine hull, we can intersect convex sets to obtain the convex hull:

Definition 19. The *convex hull* of a set $\mathcal{P} \subset V$, $\operatorname{conv}(\mathcal{P})$ is defined as the intersection of all convex sets containing $\mathcal{P}$.

Example 4: Obviously, the convex hull is a subset of the affine hull, as the affine hull contains entire lines, whereas the convex hull only contains line segments. In keeping with the intuition from example (3) this results in the following:

$$\operatorname{conv}(\mathcal{P}) = \left\{ v \in V : v = \sum a_i p_i \text{ for } p_i \in \mathcal{P}, a_i \in K \text{ such that } \sum a_i = 1, \right\}$$

These kind of elements are called *convex combinations*.

Remark. Following from examples 2, 3 and 4, it is easy to see that:

$$\mathcal{P} \subset \operatorname{conv}(\mathcal{P}) \subset \operatorname{aff}(\mathcal{P}) \subset \operatorname{span}(\mathcal{P}) \subset V$$

A.3 Projective space

On any vector space V over field k, we can take the equivalence relation $\sim$ over $V' = V \setminus \{0\}$ satisfying:

$$v_1 \sim v_2 \iff \exists a \in k^\times \text{ s.t. } a \cdot v_1 = v_2$$

This allows us to define the projective space of V.

Definition 20. The *projective space* of a vectorspace is the quotient space of V' with respect to $\sim$, denoted $\mathbb{P}(V) = {}^{V'}\!/_{\sim}$. We denote the quotient map as $q : V' \to \mathbb{P}(V)$. A projective space is of dimension n if V is dimension $n+1$ as a $k-$vector space.

> **Note:** q is not defined at $0 \in V$, however, if we take $r : V \to V'$ the restriction map, then $q : V' \to \mathbb{P}(V)$ and $q \circ r : V \to \mathbb{P}(V)$ are often used synonymously.

Remark. It is important to keep in mind that $\mathbb{P}(V)$ is almost always **not** a vector space. However, it has the overlying vectorspace V and other useful properties.

The subgroup of non-zero scalar transformations of $k-$algebra $\mathrm{GL}(V)$ is precisely the centre of the multiplication group, and can be denoted $Z(\mathrm{GL}(V))$. Any transformation on V induces an action on $\mathbb{P}(V)$, where $Z(\mathrm{GL}(V))$ acts trivially. This allows us to define the *projective transformations* as:

$$\begin{aligned} \mathrm{PGL}(V) &= {}^{\mathrm{GL}(V)}\!/_{Z(\mathrm{GL}(V))} \\ &\cong \mathbb{P}\left(M_n(k)^\times\right) \end{aligned}$$

Remark. The projective space of $\mathbb{R}^{n+1}$ over $\mathbb{R}$ will be called $\mathbb{P}^n(\mathbb{R})$ or even $\mathbb{P}^n$ if there is no confusion about V and k.

Example 5: $\mathbb{P}(\mathbb{R}^{n+1}) = \mathbb{P}^n$ corresponds to any of the following:

1. The set of $(n+1)$-tuples of elements of $\mathbb{R}$, up to scalar multiplication, denoted $[x_0 : ... : x_n]$, where not all $x_i = 0$. Note that the ':' signifies a ratio. This notation is called *homogeneous coordinates*.

2. $\mathbb{P}^n \cong \mathbb{R}^n \cup \mathbb{P}(\mathbb{R}^n)$ as a set. This can be seen by using homogeneous coordinates and scaling to $[1 : x'_1 : ... : x'_{n-1}]$ if x_0 is nonzero (which is isomorphic to $\mathbb{R}^n$), and 'forgetting' the first coordinate if it is zero.

3. $\mathbb{P}^n \cong \bigcup\limits_{k=0}^{n} \mathbb{R}^k$. This is easily seen by noting that $\mathbb{P}(\mathbb{R})$ is a single point, and applying induction on the previous statement.

4. The set of all straight lines through the origin in $\mathbb{R}^n$, as these lines can be viewed as the normals of planes given by zeroes of degree 1 equations in $\mathbb{R}[X_1, ..., X_n]$, i.e. $\sum a_i X_i = 0$ where not all a_i are zero.

5. The set of points on the 'upper' half of the (n)-sphere: $(S^n)^+ \subset \mathbb{R}^{n+1}$. This can be seen by identifying the previous line with its intersection with the unit sphere.

Definition 21. We can define subspaces of projective space similar to those of vector spaces.

- A subspace S of a projective space $\mathbb{P}(V)$ is the image of a subspace of V under the map $q : V \to \mathbb{P}(V)$.
- Similar to subspaces of vector spaces a subspace of dimension 1 is called a *projective line* and of dimension $n-1$ a *hyperplane*. Using the next lemma, a projective line is the image of either a subspace of dimension 2 or of an affine line of dimension 1 which does not contain $0 \in V$. Note that a vector space of dimension $n+1$ has a projective space of dimension n, hence $\mathbb{P}^n = \mathbb{P}(\mathbb{R}^{n+1})$ is dimension n.
- The span of a subset $\mathcal{X} \subset \mathbb{P}(V)$, span$\mathcal{X}$ is the intersection of all subspaces containing $\mathcal{X}$. Hence it is again the 'smallest' subspace containing $\mathcal{X}$. Note that for $\mathcal{P} \subset V$, span$q(\mathcal{P}) = q($span$\mathcal{P})$

- The transformations on S are simply the projective transformations 'up to' elements of the stabilizer subgroup, which is isomorphic to $\mathbb{P}(M_m(k)^\times)$.

Note: As translations are not well defined on $\mathbb{P}(V)$, there is no concept of affine hulls. Furthermore, in geometry an affine subset of $\mathbb{P}^n$ usually denotes a set that is (as a variety) isomorphic to a subset of k^l for some l. Note that the image under q of a flat (or affine subspace) of dimension greater than 0 in V, is explicitly **not** affine in this sense of the word.

We state the following:

Lemma 25. For $\mathcal{P} \subset V$, $q(\text{aff}\mathcal{P}) = q(\text{span}\mathcal{P})$ in $\mathbb{P}(V)$.

Proof. If $0 \in \mathcal{P}$, the statement is trivial. As $\text{aff}\mathcal{P} \subset \text{span}\mathcal{P}$ it suffices to show that $\text{aff}\mathcal{P} \supset \text{span}\mathcal{P}$.

Take $x \in \text{span}\mathcal{P} \setminus \{0\}$. As in example 2, we can write $x = \sum a_i p_i$, with some finite basis. Without loss of generality, take $\sum a_i \neq 0$ (Note that we can freely scale some nonzero a_i, if we scale p_i as well). As k is a field, it contains $\xi = \left(\sum a_i\right)^{-1}$. Consider $\xi x = \sum (\xi a_i) p_i$. This is an affine combination, as $\sum \xi a_i = \xi \left(\sum a_i\right) = 1$, therefore $\xi x \in \text{aff}(\mathcal{P})$. Because $\xi \in k$, we know $q(\xi x) = q(x)$. $\square$

A.4 Orthogonality

Definition 22. A *symmetric bilinear form* on vector space V over field k is a map $\varphi : V \times V \to k$ such that for all $x, y, z \in V$ and $a \in k$:

$$\begin{aligned} \varphi(x,y) &= \varphi(y,x) \\ \varphi(x,y+z) &= \varphi(x,y) + \varphi(x,z) \\ \varphi(ax,y) &= a\varphi(x,y) \end{aligned}$$

A form is said to be degenerate if there is any $x \in V$ such that $\varphi(x, y) = 0$ for all $y \in V$. The scalar multiplication and (pointwise) sum of forms can be defined in the normal sense, to make the set of symmetric bilinear forms into a vector space over k. If not stated otherwise, we will assume a form to be non-degenerate.

Example 6: If we take a basis for V (which was isomorphic to k^n), we can check what the form does on the basis $\{x_i : i \in I\}$, and write a matrix A:

$$A = \Big(\varphi(x_i, x_j)\Big)_{i,j\in I}$$

Then $\varphi(x, y) = x^T Ay$ for some symmetric matrix A. Therefore, as a $k-$vector space, symmetric, bilinear forms are isomorphic to $\mathrm{Sym}(V) = \mathrm{Sym}_n = \{M \in M_n(k) : M = M^T\}$. A form is degenerate if and only if the corresponding matrix is, i.e. non-degenerate forms correspond precisely to $\mathrm{Sym}(V)^\times$.

Note: The matrix product of two symmetric matrices is not necessarily symmetric, therefore $\mathrm{Sym}(V)$ does not have the structure of a $k-$algebra. However, we also have no concept of multiplication on forms, hence this does not pose a problem.

Example 7: Recall that a matrix A is skew-symmetric if $A = -A^T$. When $\mathrm{char}(K) \neq 2$, any matrix A in $M_n(k)$ can be written as:

$$A = \tfrac{1}{2}\left(A + A^T\right) + \tfrac{1}{2}\left(A - A^T\right)$$

Therefore we have the decomposition $M_n = \mathrm{Sym}_n \oplus \mathrm{Skew}_n$, allowing us to write the following exact sequences of vector spaces:

$$0 \overset{\longleftarrow}{\longrightarrow} \mathrm{Sym}_n \overset{\longleftarrow}{\longrightarrow} M_n \overset{\longleftarrow}{\longrightarrow} \mathrm{Skew}_n \overset{\longleftarrow}{\longrightarrow} 0$$

Notice that the singular matrices are given by the zero set of the determinant, hence the invertible matrices are affine open (same dimension) in these sets.

Definition 23. The *automorphism* group of V with respect to a non-degenerate, symmetric bilinear form φ, $\mathrm{Aut}(\varphi)$ is the subgroup of $\mathrm{GL}(V)$ that preserves the form:

$$\mathrm{Aut}(\varphi) = \{T \in \mathrm{GL}(V) : \varphi(Tx, Ty) = \varphi(x, y) \quad \forall x, y \in V\}$$

Lemma 26. The $k-$dimension of the automorphism group is $n(n-1)/2$, in fact:

$$\mathrm{Aut}(\varphi) \cong \mathrm{Skew}_n^\times$$

Proof. Let A be symmetric, then for any matrix B, $B^T AB$ is symmetric, and therefore:

$$\begin{aligned} \varphi_A(Bx, By) = \varphi_A(x, y) \quad \forall x, y \in V \\ \Leftrightarrow \varphi_{B^TAB} = \varphi_A \\ \Leftrightarrow B^TAB - A = 0 \end{aligned}$$

We can define the following surjective homomorphism:

$$\begin{aligned} \psi_A : M_n &\to \mathrm{Sym}_n \\ B &\mapsto B^TAB - A \end{aligned}$$

Note that by definition $\mathrm{Aut}(\varphi_A) = \ker(\psi_A)^\times$. Using the exact sequence:

$$\begin{tikzcd}
0 \arrow[dr] & & & & 0 \\
& \mathrm{Skew}_n \arrow[dr, "\iota"] & & \mathrm{Sym}_n \arrow[ur] & \\
& & M_n \arrow[ur, "\psi_A"] \arrow[dr, "\tau"'] & & \\
& \ker(\psi_A) \arrow[ur, "\iota"] & & \mathrm{Sym}_n \arrow[dr] & \\
0 \arrow[ur] & & & & 0
\end{tikzcd}$$

Therefore:

$$M_n \big/ \mathrm{Skew}_n \cong M_n \big/ \ker(\psi_A) \Rightarrow \mathrm{Skew}_n \cong \ker(\psi_A)$$

$\square$

Example 8: Over $\mathbb{R}$ this group is usually called the group of *(indefinite) orthogonal transformations.* Using a suitable change of basis, any symmetric bilinear form on $\mathbb{R}^n$ can be represented by a diagonal matrix $(a_{ii})_i$, where $a_{ii} \in \{1, 0, -1\}$. The *signature*, (a, b, c) of the form is the number of positive, zero and negative a_{ii}, respectively. A form is obviously degenerate if and only if $b \neq 0$. If a or c is zero, the form is called *positive-*, respectively *negative definite.* The space $\mathbb{R}^n$ with a form of signature $(p, 0, q)$ will be denoted $\mathbb{R}^{p,q}$, and it's group of (indefinite) orthogonal transformations $\mathcal{O}(p, q)$ (Note that $n = p + q$).

Example 9: As before, we can take the *projective indefinite orthogonal transformations*, $\mathrm{P}\mathcal{O}(p, q)$. Recall that the centre of $\mathrm{GL}(\mathbb{R}^n)$ acted trivially in composition with $q : V \to \mathbb{P}(V)$. The centre Z consisted exactly of all scalar matrices. If T is a scalar matrix then $Tx = tx$ for some nonzero scalar t, and as such, $(Tx, Ty) = t^2(x, y)$. Therefore $Z \cap \mathcal{O}(p, q) = \{\pm I\}$. This can be written as the following commuting diagram.

$$\begin{tikzcd}
\mathrm{GL}(\mathbb{R}^n) \arrow[r, "q_Z"] & \mathrm{GL}(\mathbb{R}^n) \big/ Z \arrow[r, "\sim"] & \mathrm{PGL}(\mathbb{R}^n) \\
\mathcal{O}(p,q) \arrow[r, "q_I"] \arrow[u, "i"] & \mathcal{O}(p,q) \big/ \{\pm I\} \arrow[r, "\sim"] \arrow[u, "i"] & \mathrm{P}\mathcal{O}(p,q) \arrow[u, "i"]
\end{tikzcd}$$

Where all maps are the obvious inclusions (with as inverses restrictions) and quotients.

Definition 24. Two points $v, w \in V$ are called *orthogonal* with respect to form φ if $\varphi(v, w) = 0$. The *orthogonal complement*, $v^\perp$, of $v \in V$ with respect to φ is the set of all points orthogonal to v. The *orthogonal complement of* *a set* $\mathcal{P} \subset V$ with respect to φ, $\mathcal{P}^\perp$, is the set of of all points orthogonal to all $v \in \mathcal{P}$, or equivalently:

$$\mathcal{P}^\perp = \bigcap_{v \in \mathcal{P}} v^\perp = \{w \in V : \varphi(v, w) = 0 \quad \forall v \in \mathcal{P}\}$$

Note: There is usually no confusion about the form φ in question, hence it is not specified in the notation $v^\perp$.

Properties:

- The orthogonal complement is closed under addition and scalar multiplication (as the form is bilinear), and is therefore a subspace.
- The orthogonal complement is inclusion reversing, that is: If $\mathcal{P} \subset \mathcal{Q}$ then $\mathcal{P}^\perp \supset \mathcal{Q}^\perp$.

Lemma 27. For any set $\mathcal{P} \subset V$, $\mathcal{P}^\perp = (\text{span}\mathcal{P})^\perp$.

Proof. We know $\mathcal{P} \subset \text{span}\mathcal{P}$, therefore by the inclusion reversing property, $\mathcal{P}^\perp \supset (\text{span}\mathcal{P})^\perp$. For $v \in \mathcal{P}^\perp$ we know $\varphi(v, p_i) = 0$. We know that any $x \in \text{span}(\mathcal{P})$ can be written as $x = \sum a_i p_i$ for $p_i \in \mathcal{P}$ and $a_i \in k$. As the form φ is bilinear, $\varphi(v, x) = \varphi(v, \sum a_i p_i) = \sum a_i \varphi(v, p_i)$. Hence $\varphi(v, x) = 0$ and therefore $v \in (\text{span}\mathcal{P})^\perp$. $\square$

Corollary 28. Define for $n-$dimensional vector space V:

$$\begin{aligned} \text{Sub}(V) &:= \{H \subset V : H \text{ is a subspace}\} \\ \text{Sub}^k(V) &:= \{H \subset V : H \text{ is a subspace of dimension } k\} \end{aligned}$$

For non-degenerate φ the map $(.)^\perp$ is an automorphism on $\text{Sub}(V)$, such that for any integer $0 \leq k \leq n$ the following induced restriction map is a bijection.

$$\begin{tikzcd} \text{Sub}(V) \arrow[r, "(.)^\perp"] & \text{Sub}(V) \arrow[d, "r"] \\ \text{Sub}^k(V) \arrow[u, "i"] \arrow[r, dashed, leftrightarrow, "1:1"] & \text{Sub}^{n-k}(V) \end{tikzcd}$$

As $\mathcal{P} \subset \mathcal{P}^{\perp\perp}$, we know that on subspaces $(.)^{\perp}$ is it's own inverse A direct consequence is that for subspace $\mathcal{X}$, $\dim\mathcal{X} + \dim\mathcal{X}^{\perp} = \dim V$. This will be illustrated by the following example:

Example 10: Take a line (1 dimensional subspace), $\mathcal{L}$ in V, and any non-zero element $l \in \mathcal{L}$. Then $\mathcal{L}$=span$\{l\}$. Given a basis for V, we can take matrix A representing φ, giving:

$$\begin{aligned} \mathcal{L}^{\perp} &= l^{\perp} \\ &= \{x \in V : l^T A x = 0\} \end{aligned}$$

As φ is non-degenerate $l^T A$ is non-zero, hence $\mathcal{L}^{\perp}$ is the zero set of a single linear polynomial, a hyperplane (of dimension $n-1$). In a k dimensional subspace $\mathcal{K}$ we can take k linearly independent generators, l_i. Let $\mathcal{L}_i$ be the span of l_i.

$$\begin{aligned} & \mathcal{K} &&= (\text{span}\{l_i\}_i) \\ \Rightarrow \forall j : \ & \mathcal{K} &&\supset \mathcal{L}_j \\ \Rightarrow \ & \mathcal{K}^{\perp} &&\subset \cap_{i=1}^{k} \mathcal{L}_i^{\perp} \end{aligned}$$

As φ is non-degenerate, and the l_i are independent, this is the zero set of k independent, linear polynomials, and hence $n-k$ dimensional.

Definition 25. In the projective space $\mathbb{P} = \mathbb{P}(V)$, we define the orthogonal complement, $\xi^{\perp}$, of an element $\xi \in \mathbb{P}$ as the image under $q : V \to \mathbb{P}$ of the orthogonal complement of $(q^{-1}(\xi))^{\perp} \subset V$. Using notation from corollary 28, the orthogonal complement is defined as the induced map:

$$\begin{tikzcd} \text{Sub}(\mathbb{P}) \arrow[r, dashed] \arrow[d, "q^{-1}"] & \text{Sub}(\mathbb{P}) \\ \text{Sub}(V) \arrow[r, "(.)^{\perp}"] & \text{Sub}(V) \arrow[u, "q"'] \end{tikzcd}$$

Example 11: Let $\xi = [x_0 : ... : x_n]$ in homogeneous coordinates on $\mathbb{P}(V)$. Then $q^{-1}(\xi)$ is the line given by $\{\lambda \cdot x : x = (x_0, ..., x_n) \in V, \lambda \in k\}$. This is the span of x, therefore $\xi^\perp = q(x^\perp)$. As V is $n+1$ dimensional, $x^\perp$ has dimension n, hence $\xi^\perp$ has dimension $n-1$ in $\mathbb{P}(V)$.

Properties:

- The orthogonal complement in $\mathbb{P}(V)$ is a projective subspace.
- We know that if $\mathcal{X}$ is a $k-$dimensional (projective) subspace of an $n-$dimensional $\mathbb{P} = \mathbb{P}(V)$, $q^{-1}(\mathcal{X})$ has dimension $k+1$ in V, hence, in terms of corollary 28, the following diagram commutes:

$$\begin{tikzcd}
 & \mathrm{Sub}(V) \arrow[rr, "(.)^\perp"] \arrow[dd] & & \mathrm{Sub}(V) \arrow[dd] \\
\mathrm{Sub}^{k+1}(V) \arrow[ur] \arrow[rr, leftrightarrow, "1:1"] \arrow[dd] & & \mathrm{Sub}^{n-k}(V) \arrow[ur] \arrow[dd] & \\
 & \mathrm{Sub}(\mathbb{P}) \arrow[rr] & & \mathrm{Sub}(\mathbb{P}) \\
\mathrm{Sub}^{k}(\mathbb{P}) \arrow[ur] \arrow[rr, leftrightarrow, dashed, "1:1"] & & \mathrm{Sub}^{n-k-1}(\mathbb{P}) \arrow[ur] &
\end{tikzcd}$$

In particular, in the projective space there is a bijection between points and hyperplanes.

A.5 Quadratic forms and Quadrics

Definition 26. A symmetric bilinear form φ induces a *quadratic form* Q : $V \to k$ by $Q(x) = \varphi(x, x)$. Using a basis for V, this can be viewed as $x^T Ax$ for some symmetric matrix A.

Remark. Q is a homogeneous polynomial, therefore the composition with $q^{-1} : \mathbb{P}(V) \to V$ is well-defined on the zero set, allowing us to define a subset $\{\xi \in \mathbb{P}(V) : Q(q^{-1}(\xi)) = 0\}$.

Note: $Q \circ q^{-1}$ is **not** everywhere well-defined as a function $\mathbb{P}(V) \to k$. Take for example $p \in V$ such that $Q(p) \neq 0$ (note that if Q is nonzero, we can). $q^{-1}(q(p)) = \{ap : a \in k^\times\}$. If $a \in k^\times$ such that $a^2 \neq 1$ then $Q(ap) \neq Q(p)$.

Example 12: As Q is of degree 2, for any $a \in k$: $Q(ax) = a^2 Q(x)$. Therefore if $k = \mathbb{R}$ (where squares are positive), in addition to the zero set, also the sign of the form is preserved. This allows us to subdivide $\mathbb{P}^n$ into:

$$\begin{aligned} Q_{<0} &= \{y \in \mathbb{P}^n : y = q(x), Q(x) < 0\} \\ Q_0 &= \{y \in \mathbb{P}^n : y = q(x), Q(x) = 0\} \\ Q_{>0} &= \{y \in \mathbb{P}^n : y = q(x), Q(x) > 0\} \end{aligned}$$

Classically, based on relativity, these are know as timelike, lightlike and spacelike points respectively. Note that the indefinite orthogonal transformations on $\mathbb{R}^{n+1}$ respect this decomposition.

Definition 27. A *quadric* $\mathcal{C}_Q$ in $\mathbb{P}(V)$ is the image under $q : V' \to \mathbb{P}(V)$ of the zero set of a quadratic form Q. A quadric is degenerate if the underlying bilinear form is degenerate, or equivalently, if the corresponding matrix is singular.

$$\mathcal{C}_Q = \{\xi \in \mathbb{P}(V) : Q(y) = 0 \forall y \in q^{-1}(\xi)\}$$

Remark. As a quadric can also be directly inferred from a bilinear form φ (such that $Q(x) = \varphi(x, x)$) or a representing matrix A (such that $Q(x) = x^T Ax$), we will sometimes write $\mathcal{C}_\varphi$ and $\mathcal{C}_A$ for the same quadric.

Similar to the definition of projective transformations, scalar multiplication results in the same quadric. Recall that the $k-$vector space of $n \times n$, symmetric matrices was called $\mathrm{Sym}(V)$ and was isomorphic to the space of symmetric bilinear forms. Therefore we have an isomorphism ϕ:

$$\begin{tikzcd} \{\text{Symmetric bilinear forms}\} \arrow[r, leftrightarrow] & \mathrm{Sym}(V) \arrow[d, "q_Z"] \\ \{\text{Quadrics in } \mathbb{P}(V)\} \arrow[u] \arrow[r, leftrightarrow, dashed, "\phi"] & \mathrm{PSym}(V) \end{tikzcd}$$

Where $\mathrm{PSym}(V)$ is defined to be:

$$\mathrm{PSym}(V) = {}^{\mathrm{Sym}(V)}\!\big/_{Z} = \mathbb{P}(\mathrm{Sym}(V))$$

The upper diagonal block of a symmetric matrix contains all of its information, therefore $\mathrm{Sym}(V)$ has $k-$dimension $n(n+1)/2$, and therefore (as a vector space) $\mathrm{PSym}(V)$ is isomorphic to the $k-$vector space $\mathbb{P}(k^{n(n+1)/2})$.

Example 13: A conic in $\mathbb{R}^2$ is given as the zero set of a quadratic polynomial in $k[x, y]$. This can be homogenized to a homogeneous polynomial in 3 variables over $\mathbb{P}^3$. Using the definition we can view conics as points in $\mathbb{P}(\mathbb{R}^6)$ by taking:

$$\begin{array}{ccc} \{\text{Quadrics in } \mathbb{P}^3\} & \to & \mathrm{PSym}(\mathbb{R}^3) \\ \begin{array}{l} Ax^2 + By^2 + Cz^2 \\ +2Dxy + 2Exz + 2Fyz = 0 \end{array} & \mapsto & \mathbb{R}^\times \begin{pmatrix} A & D & E \\ D & B & F \\ E & F & C \end{pmatrix} \end{array}$$

A degenerate conic in $\mathbb{P}^3$ is either a line or a pair of lines and is given by:

$$\begin{aligned} 0 &= (ax + by + cz)(a'x + b'y + c'z) \\ &= aa'x^2 + bb'y^2 + cc'z^2 + (ab' + ba')xy + (ac' + ca')xz + (bc' + cb')yz \end{aligned}$$

And indeed:

$$\det \begin{pmatrix} 2aa' & ab' + ba' & ac' + ca' \\ ab' + ba' & 2bb' & bc' + cb' \\ ac' + ca' & bc' + cb' & 2cc' \end{pmatrix} = 0$$

Example 14: A sphere in k^n with radius b and centre $(a_1, ..., a_n)$, can be homogenized to a quadric in $\mathbb{P}(k^{n+1})$, and is represented by the following in $\mathbb{P}(k^{(n+1)(n+2)/2})$

$$\sum_{i=1}^{n}(x_i - a_i x_0)^2 = b^2 x_0^2 \quad \mapsto \quad k^{\times}\begin{pmatrix} b^2+\sum a_i^2 & a_1 & a_2 & \cdots & a_n \\ a_1 & -1 & 0 & \cdots & 0 \\ a_2 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ a_n & 0 & 0 & \cdots & -1 \end{pmatrix}$$

$$\mapsto \quad q\big(b^2+\sum a_i^2, \underbrace{-1,..,-1}_{n}, a_1, ..., a_n, \underbrace{0, ..., 0}_{n(n-1)/2}\,\big)$$

Definition 28. A *pencil of quadrics* is a one-parameter set of quadrics in $\mathbb{P}(V)$ represented by a projective line in $\mathrm{PSym}(V) \cong \mathbb{P}(k^{n(n+1)/2})$.

Pencils of quadrics have a few useful properties:

Properties:

- A pencil is fully determined by 2 quadrics, as a projective line is.
- Take two different quadrics $\mathcal{C}_A, \mathcal{C}_B \subset \mathbb{P}(V)$. Using the fact that the span and affine hull project to the same projective subspace (see lemma 25), we can write the pencil as $\mathcal{C}_{sA+tB}$, where not both $s = 0, t = 0$, . Without loss of generality we can take s, t such that $(s : t) \in P(k^2)$. This is the 'one parameter' from the definition.
- For any quadric on the pencil determined by $\mathcal{C}_A$ and $\mathcal{C}_B$, we know $x^T(sA + tB)x = sx^TAx + tx^TBx$. Therefore all quadrics in the pencil contain $\mathcal{C}_A \cap \mathcal{C}_B$. As a simple result using symmetry, any two different quadrics in a pencil have the same intersection.

Lemma 29. The subset $T_\xi \subset \mathrm{PSym}(V)$ of quadrics containing a certain point $\xi \in \mathbb{P}(V)$ is a hyperplane.

Proof. We know:

$$\begin{aligned} \xi \in \mathcal{C}_B \Leftrightarrow & \quad \forall x \quad \in q^{-1}(\xi): \quad x^T B x = 0 \\ \Leftrightarrow & \quad \exists x \neq 0 \quad \in q^{-1}(\xi): \quad x^T B x = 0 \end{aligned}$$

Therefore the set of all quadrics containing ξ are represented by matrices $M = (m_{ij})$ such that the condition $\sum m_{ij} x_i x_j = 0$ holds. As this is simply a degree 1 polynomial on the elements of matrix M it defines hyperplane. $\square$

Example 15: A well known classical result in geometry is the fact that 5 points 'in general position' fully determine a conic in $\mathbb{P}(\mathbb{R}^3)$. In fact, 4 of these points determine a pencil of conics. This can be seen by simply intersecting codimension 1 subspaces in $\mathbb{P}(\mathbb{R}^6)$. If we denote the line through x, y as L_{xy}, given four points a, b, c, d, we can simply take two of the (degenerate) quadrics $L_{ab}L_{cd}$, $L_{ac}L_{bd}$ or $L_{ad}L_{bc}$ as generators.

B Matrix forms of $Q^s(\xi)$

Let $\xi = [a_0 : \ldots : a_{n+1}]$ and $\sigma = [s : t]$, then the map sending a quadric to it's corresponding projective, symmetric matrix is well defined if $s \neq 0$ or $a_0 + a_1 \neq 0$ (the case that both are zero corresponds to the zero matrix and is not a quadric), and maps:

$$Q^\sigma(\xi) \mapsto$$
$$\begin{pmatrix} s(a_1-a_0)-t(a_1+a_0) & s(a_1-a_0) & sa_2 & \cdots & sa_{n+1} \\ s(a_1-a_0) & s(a_1-a_0)+t(a_1+a_0) & sa_2 & \cdots & sa_{n+1} \\ sa_2 & sa_2 & (t-s)(a_0+a_1) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ sa_{n+1} & sa_{n+1} & 0 & \cdots & (t-s)(a_0+a_1) \end{pmatrix}$$

$$Q^0(\xi) \mapsto$$
$$\begin{pmatrix} -(a_0+a_1) & 0 & 0 & \cdots & 0 \\ 0 & (a_0+a_1) & 0 & \cdots & 0 \\ 0 & 0 & (a_0+a_1) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & (a_0+a_1) \end{pmatrix}$$

$$Q^1(\xi) \mapsto$$
$$\begin{pmatrix} -2a_0 & a_1-a_0 & a_2 & \cdots & a_{n+1} \\ a_1-a_0 & 2a_1 & a_2 & \cdots & a_{n+1} \\ a_2 & a_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n+1} & a_{n+1} & 0 & \cdots & 0 \end{pmatrix}$$

$$Q^\infty(\xi) \mapsto$$
$$\begin{pmatrix} a_1-a_0 & a_1-a_0 & a_2 & \cdots & a_{n+1} \\ a_1-a_0 & a_1-a_0 & a_2 & \cdots & a_{n+1} \\ a_2 & a_2 & (a_0+a_1) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n+1} & a_{n+1} & 0 & \cdots & (a_0+a_1) \end{pmatrix}$$