university of groningen
faculty of mathematics and natural sciences

The Kontsevich tetrahedral flow *and* infinitesimal deformations of Poisson structures

Master's project Mathematics
December 2014 – February 2017

Student: A. Bouisaghouane
Supervisor: Dr. A.V. Kiselev
Co-supervisor: Prof. dr. H. Waalkens

Abstract.

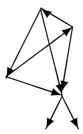
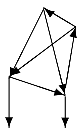
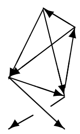
In the paper "Formality conjecture" (1996) Kontsevich designed a universal flow, called the tetrahedral flow as given by the formula $\dot{\mathcal{P}} = \mathcal{Q}_{a:b}(\mathcal{P}) = a\Gamma_1 + b\Gamma_2$, on the spaces of Poisson structures $\mathcal{P}$ on all affine manifolds of dimension $n \geqslant 2$. We investigate several claims made by Kontsevich in *loc. cit.* We reveal, by using several examples of Poisson structures, that, in general, it is only the balance $1:6$ for which the flow preserves the space of Poisson bi-vectors. In collaboration with R. Buring and A.V. Kiselev, we prove that the Kontsevich tetrahedral flow $\dot{\mathcal{P}} = \mathcal{Q}_{a:b}(\mathcal{P})$ infinitesimally preserves the space of Poisson bi-vectors on N^n if and only if the two differential monomials in $\mathcal{Q}_{a:b}(\mathcal{P})$ are balanced by the ratio $a:b = 1:6$. We then investigate the triviality of the flow and prove that for $n = 2$, the flow $\mathcal{Q}_{1:0} = \Gamma_1(\mathcal{P})$ is Poisson-cohomology trivial: $\Gamma_1(\mathcal{P}) = [\![\mathcal{P}, \mathcal{X}]\!]$ for some vector field $\mathcal{X}$; we examine the space of solutions $\mathcal{X}$ and its properties, and represent a specific solution $\mathcal{X}$ by Kontsevich graphs.

Contents

Introduction

In fact, I cheated a little bit.
M. Kontsevich

The master's thesis at hand investigates the existence and properties of the Kontsevich tetrahedral flow, a result of the graph complex as described in [1]. This flow is used to infinitesimally deform Poisson structures on finite dimensional affine manifolds. The formula for the flow is given in terms of the following Kontsevich graphs:

$\mathcal{Q}_{1:6}(\mathcal{P}) = 1 \cdot$  $+ \frac{6}{2} \cdot \Big($  $-$  $\Big).$

This thesis consists of 3 papers, in consecutive order:

1. "Do the Kontsevich tetrahedral flows preserve or destroy the space of Poisson bi-vectors?" [2]
   In his paper "Formality conjecture" [1], Kontsevich introduced a method to deform Poisson structures using graphs. We analyzed the claim that $\mathcal{Q}_{1:0}$ was the formula of a universal flow on the spaces of Poisson bi-vectors. By using existing constructions for Poisson bi-vectors with sufficiently high polynomial degree, we managed to show that the claims for this flow can only hold for the ratio $a : b = 1 : 6$. We end with an attempt at extending the flow to the variational setup.

2. "The Kontsevich tetrahedral flows revisited" [3]
   That same paper, [1], lacked a proof as to why $\mathcal{Q}_{a:b}$ would define an infinitesimal deformation for some ratio of $a : b$. We give an explicit proof, using (Kontsevich) graphs, of the fact that $\mathcal{Q}_{1:6}$ is an infinitesimal deformation of Poisson structures on finite-dimensional affine manifolds. In a sense, this paper fills in the 'gap' in [1]. We also analyze properties of the the solution to the claim.

3. "The Kontsevich tetrahedral flow in 2D: a toy model" [4]
   One more unproven claim in [1], was that the flow $\mathcal{Q}_{1:6}$, when restricted to a 2-dimensional affine manifold, is Poisson-cohomology trivial. We support this claim by a proof. Furthermore, we show that this solution is realizable in terms of Kontsevich graphs, independent of the underlying manifold at hand. Finally, we show that the flow, in general, and the trivializing vector field, in the 2-dimensional case, remain well-defined over a periodic lattice whenever the initial Poisson structure $\mathcal{P}$ is as well.

Preface

A turbulent start

The main goal of this thesis was to extend the concept of the Kontsevich flow, originating from the graph complex, on the space of Poisson bi-vectors. The original flow on finite-dimensional Poisson manifolds was to be translated to the variational setup of spaces of infinite jets of sections of vector bundles equipped with variational Poisson structures. In the original definition of the tetrahedral flow $\mathcal{Q}_{a:b}$, all internal vertices inhibit identical copies of a Poisson bi-vector $\mathcal{P}$ and directed edges, which have a fixed ordering, are decorated with indices. Incoming edges i encode partial derivatives with respect to $\partial/\partial x^i$ and outgoing edges fix an index i in $\mathcal{P}$. Then the graph $(\bullet) \xleftarrow{i} \mathcal{P}^{ij}(\boldsymbol{x}) \xrightarrow{j} (\bullet)$ encodes a bi-differential operator.

In the variational setup, internal vertices inhibit identical copies of a variational Poisson bi-vector $\boldsymbol{\mathcal{P}}$ and edges decorated with label i also carry multi-indices (σ, τ). The head of an edge encodes the partial derivative with respect to the fibre coordinates $\partial/\partial \boldsymbol{u}^i_\tau$, whereas tails encode partial derivatives with respect to parity odd conjugate fibre coordinates $\partial/\partial \boldsymbol{\xi}_{i,\sigma}$. Additionally, the vertices on both ends of a directed edge are differentiated with respect to the total derivatives in the base coordinate $\boldsymbol{x}$, $(\mathrm{d}/\mathrm{d}x)^\sigma$ and $(-\mathrm{d}/\mathrm{d}x)^\tau$. After integration by parts and collecting the total derivatives, one obtains the expression, denoted by $\boldsymbol{\mathcal{Q}_{a:b}}$, for the variational tetrahedral flow. Variational Poisson structures were obtained from Hamiltonian operators with sufficiently high polynomial degree in the fibre coordinates, e.g. the Harry Dym operator. Evaluating the variational Schouten bracket $[\![\boldsymbol{\mathcal{P}}, \boldsymbol{\mathcal{Q}_{a:b}}]\!]$ for such Poisson structures, it turns out that it does not vanish (modulo the image of the horizontal derivative). Therefore, this extension of the tetrahedral flow was stopped by an obstruction.

In [2], the attempt to extend the tetrahedral flow to a variational setup was summarized in no more than a paragraph and a table entry. This result caused us to doubt some of the claims in [1]; these doubts led us to investigate them.

Back to square one

In the 1996 Ascona paper, [1], the following three claims were presented:

i The second graph Γ_2 vanish identically when evaluated on Poisson bi-vectors.

ii The tetrahedral flow preserves Poisson structures infinitesimally.

iii In the 2-dimensional case, the tetrahedral flow of a Poisson bi-vector is the conjugation of that Poisson bi-vector with some vector field.

As no proofs of these claims existed yet, we decided to attempt proving or disproving them.

We first investigated the claim that for the 2-dimensional case the flow was Poisson-cohomology trivial: there exists a vector field $\mathcal{X}$ such that the flow $\dot{\mathcal{P}} = [\![\mathcal{P}, \mathcal{X}]\!]$. This claim turned out to be true, and a step-by-step proof was described in [4] together with some (remarkable) properties of the vector field $\mathcal{X}$. One of these properties is that the trivializing vector field $\mathcal{X}$ is realizable in terms of Kontsevich graphs, which are independent of the affine manifold over which the Poisson structure is defined. This was unexpected, as this claim specifically restricted to manifolds of dimension 2.

> At the time of this research I was working together with Ricardo Buring, and we both presented our interim results at the *Symmetries of Discrete Sytems and Processes III* conference on August 3–7 2015 in Děčín, Czech Republic, organized by the Czech Technical University in Prague.

We then focused on the formula for the flow. This formula consisted of two parts, each originating from a graph. It was claimed that the contribution of the second graph vanishes identically. By testing this claim using examples of Poisson structures with coefficients of sufficiently high polynomial degree, we found it to not hold in general, except for Poisson structures on 2-dimensional affine manifolds. In fact, those examples of Poisson structures of dimension $\geqslant 3$ indicated that for the flow $\mathcal{Q}_{a:b}$ to be a cocycle, the unique ratio $a : b = 1 : 6$ is a necessary condition. These results were collected in [2].

After having shown that the ratio $a : b = 1 : 6$ is a necessary condition for $\mathcal{Q}_{a:b}$ to be a flow, we wanted to prove sufficiency of this claim as we could find no counterexamples where the flow $\mathcal{Q}_{1:6}$ is not a cocycle. There were two approaches, one based on a method of perturbations, the other on an exhaustive search amongst all differential consequences of the Jacobi identity.

> These results and ideas were presented at the *Group Analysis of Differential Equations and Integrable Systems VIII* workshop on June 12–17 2016 in Larnaca, Cyprus, jointly organized by the Department of Mathematics and Statistics of the University of Cyprus and the Department of Applied Research of the Institute of Mathematics of the National Academy of Sciences of Ukraine.

Working in this direction turned out fruitful and the solution to this problem was found, using the software developed by Ricardo Buring, and later described in [3]. There, we also analyzed some properties of the found solution and expanded on the triviality of the flow. This resulted in a, proven, weak statement which says that there does not exist a vector field $\mathcal{X}$, encoded by Kontsevich graphs such that it is independent of the manifold over which the Poisson structure $\mathcal{P}$ is defined, that trivializes the flow $\mathcal{Q}_{1:6}$, i.e. $[\![\mathcal{P}, \mathcal{X}]\!] = \mathcal{Q}_{1:6}$.

The proof of sufficiency for $\mathcal{Q}_{1:6}$ to be a cocycle, through a factorization of the Jacobi identity, was presented at the *Symposium on advances in semi-classical methods in mathematics and physics*, in honor of H.J. Groenewold, on October 19–21 2016 in Groningen, The Netherlands, jointly organized by the Johann Bernoulli Institute for Mathematics and Computer Science (JBI) and the Van Swinderen Institute for Particle Physics and Gravity (VSI).

Uncharted paths

A better understanding of the tetrahedral flow led us to more questions, some of which remain unanswered. For example, when 'experimenting' with examples of Poisson bi-vectors in the tetrahedral flow, we obtained the condition $a : b = 1 : 6$. We do not have an explanation of the origin of this ratio: neither do we know whether or how this ratio is prescribed. And apart from the weak claim in [3] that there does not exist a universal trivializing vector field in realized by Kontsevich graphs, and the stronger claim in [4] that the 2-dimensional flow is always trivial, we do not know much more about the triviality of the tetrahedral flow. For example, we have not been able to find an instance where the flow is a cocycle and not a coboundary. Finally, given the deformation up to first order, $\mathcal{P} + \epsilon\mathcal{Q}_{1:6}(\mathcal{P}) + \bar{o}(\epsilon)$, one can ask oneself whether this deformation can be completed in higher orders using graphs or whether there are more flows for which the leading order term, of order ϵ, is encoded by graphs with $n \geqslant 4$ internal vertices?

Acknowledgements

I am very grateful to my supervisor Arthemy V. Kiselev. Having completed my bachelor's thesis under his supervision, I approached him for a master's thesis subject. He did not disappoint as the subject turned out captivating. During the course of my thesis, Arthemy was available for discussion on a weekly basis. My thesis benefited from this to a great extent; especially during those moments when we had our setbacks. Not only did Arthemy guide me through my thesis, he also motivated me to write our results down in a rigorous fashion with the intent to try and publish the results. He introduced me to the art of writing (and rewriting!) mathematical texts. Arthemy also encouraged me to give presentations to foreign audiences, in Děčín, Czech Republic and Larnaca, Cyprus, the latter of which led to an interesting discussion with professor Vanhaecke, an expert in Poisson geometry.

I am thankful to Ricardo Buring for our collaboration under Arthemy and his help with my subject. I enjoyed working with him from early in the morning to late in the afternoon at our Faculty of Mathematics and Sciences, and I enjoyed our travels to Děčín and Larnaca.

I am also grateful to my parents for encouraging me with finishing my education and my family and friends for their support and thought.

Lastly, a word of thanks to the Graduate School of Science for their financial support of my trips to Děčín and Larnaca.

Hoogezand-Sappemeer, February 2017

Bibliography

[1] M. Kontsevich. Formality conjecture. In Sternheimer D. Rawnsley J. and Gutt S. editors, *Deformation theory and symplectic geometry (Ascona 1996)*, volume 20 of *Math. Phys. Stud.*, pages 139–156, Dordrecht, 1997.

[2] A. Bouisaghouane and A. V. Kiselev. Do the Kontsevich tetrahedral flows preserve or destroy the space of poisson bi-vectors? *Submitted to JPCS*, 2016. Preprint `arXiv:1609.06677` [q-alg] 10 p.

[3] A. Bouisaghouane, R. Buring, and A. V. Kiselev. The Kontsevich tetrahedral flows revisited. *Submitted to JGP*, 2016. Preprint `arXiv:1608.01710` (v3) [q-alg] 26 p.

[4] Anass Bouisaghouane. The Kontsevich tetrahedral flow in 2d: a toy model. 2017. Preprint `arXiv:1702.06044` [q-alg] 6 p.

Do the Kontsevich tetrahedral flows preserve or destroy the space of Poisson bi-vectors?

Anass Bouisaghouane and Arthemy V Kiselev

Johann Bernoulli Institute for Mathematics and Computer Science, University of Groningen, P.O.Box 407, 9700 AK Groningen, The Netherlands

E-mail: A.V.Kiselev@rug.nl

Abstract.

From the paper "Formality Conjecture" (Ascona 1996):

I am aware of only one such a class, it corresponds to simplest good graph, the complete graph with 4 *vertices* (*and* 6 *edges*). *This class gives a remarkable vector field on the space of bi-vector fields on* $\mathbb{R}^d$. *The evolution with respect to the time* t *is described by the following non-linear partial differential equation*: ..., *where* $\alpha = \sum_{i,j} \alpha_{ij}\partial/\partial x_i \wedge \partial/\partial x_j$ *is a bi-vector field on* $\mathbb{R}^d$.

It follows from general properties of cohomology that 1) ***this evolution preserves the class of*** (***real-analytic***) ***Poisson structures***, ...

In fact, I cheated a little bit. In the formula for the vector field on the space of bivector fields which one get from the tetrahedron graph, an additional term is present. ... It is possible to prove formally that ***if*** α ***is a Poisson bracket, i.e. if*** $[\alpha, \alpha] = 0 \in T^2(\mathbb{R}^d)$, ***then the additional term shown above vanishes.***

By using twelve Poisson structures with high-degree polynomial coefficients as explicit counterexamples, we show that both the above claims are false: neither does the first flow preserve the property of bi-vectors to be Poisson nor does the second flow vanish identically at Poisson bi-vectors. The counterexamples at hand suggest a correction to the formula for the "exotic" flow on the space of Poisson bi-vectors; in fact, this flow is encoded by the balanced sum involving both the Kontsevich tetrahedral graphs (that give rise to the flows mentioned above). We reveal that it is only the balance 1 : 6 for which the flow does preserve the space of Poisson bi-vectors.

Introduction. The Kontsevich graph complex is the language of deformation quantisation on finite-dimensional Poisson manifolds [1, 2]. We consider the class of oriented graphs on two sinks and $k \geqslant 1$ internal vertices (of which, each is the tail of two edges and carries a copy of the Poisson bi-vector $\mathcal{P}$). Encoding bi-differential operators, such graphs determine the flows on the space of bi-vectors on a Poisson manifold at hand. The two flows with $k = 4$ internal vertices in the graphs are provided by the two tetrahedra [1], see Fig. 1 on the next page. By producing 12 counterexamples, we prove that the claim [1, 2] of preservation of the Poisson property is false as stated. Simultaneously, we reveal that the flow which is determined by the second graph is not always vanishing by virtue of the skew-symmetry and Jacobi identity for Poisson bi-vectors $\mathcal{P}$.

This paper is structured as follows. First we recall the correspondence between graphs and polydifferential operators [3, 4] and we indicate the mechanism for such an operator to vanish, cf. [5, 6]. In section 2 we recall three constructions of Poisson brackets with polynomial coefficients of arbitrarily high degree (see [7, 8, 9]). In Tables 1–4 on pp. 7–8 we then summarise the properties of all structures from our 12 counterexamples to the claim [1] that

(i) the flow $\dot{\mathcal{P}} = \Gamma_1(\mathcal{P})$ which the first graph in Fig. 1 encodes on the space of bi-vectors $\mathcal{P}$ would preserve their property to be Poisson (in fact, it does not), and that

(ii) the flow $\dot{\mathcal{P}} = \Gamma_2(\mathcal{P})$ would always be trivial whenever the bi-vector $\mathcal{P}$ is Poisson (in fact, this is not true).

In particular, the twelfth counterexample pertains to the infinite-dimensional jet-space geometry of variational Poisson structures [11]. (Quoted from [12], the Hamiltonian differential operator for that variational Poisson bi-vector $\boldsymbol{\mathcal{P}}$ is processed by using the techniques from [13, 14, 15]).

Finally, we examine at which balance the linear combination of the Kontsevich tetrahedral flows preserves the space of Poisson structures on finite-dimensional manifolds. We argue that the ratio 1 : 6 does the job; this claim has been proved in [6].

1. The graphs and operators

Let us formalise a way to encode polydifferential operators using oriented graphs. Consider the space $\mathbb{R}^n$ with Cartesian coordinates $\boldsymbol{x} = (x_1, \ldots, x_n)$, here $2 \leqslant n < \infty$; for typographical reasons we use the lower indices to enumerate the variables, so that $x_1^2 = (x_1)^2$, etc. By definition, the decorated edge $\bullet \xrightarrow{i} \bullet$ denotes at once the derivation $\partial/\partial x_i \equiv \partial_i$ (that acts on the content of the arrowhead vertex) and the summation $\sum_{i=1}^n$ (over the index i in the object which is contained in the arrowtail vertex). For example, the graph $\bullet \xleftarrow{i} \mathcal{P}^{ij}(\boldsymbol{x}) \xrightarrow{j} \bullet$ encodes the bi-differential operator $\sum_{i=1}^n (\cdot) \overleftarrow{\partial_i}\, \mathcal{P}^{ij}(\boldsymbol{x})\, \overrightarrow{\partial_j}(\cdot)$. If its coefficients $\mathcal{P}^{ij}$ are antisymmetric, then the graph $\bullet \xleftarrow{i} \bullet \xrightarrow{j} \bullet$ encodes the bi-vector $\mathcal{P} = \mathcal{P}^{ij}\, \partial_i \wedge \partial_j$, where $\partial_i \wedge \partial_j = \frac{1}{2}(\partial_i \otimes \partial_j - \partial_j \otimes \partial_i)$. It then specifies the Poisson bracket $\{\cdot,\cdot\}_{\mathcal{P}}$ if the $\frac{n(n-1)}{2}$-tuple of coefficients solves the system of equations

$$(\mathcal{P}^{ij})\overleftarrow{\partial_\ell} \cdot \mathcal{P}^{\ell k} + (\mathcal{P}^{jk})\overleftarrow{\partial_\ell} \cdot \mathcal{P}^{\ell i} + (\mathcal{P}^{ki})\overleftarrow{\partial_\ell} \cdot \mathcal{P}^{\ell j} = 0, \qquad (1)$$

hence the bracket $\bullet \underset{L}{\xleftarrow{i}} \mathcal{P}^{ij} \underset{R}{\xrightarrow{j}} \bullet$ satisfies the Jacobi identity. Clearly, $\mathcal{P}^{ij}(\boldsymbol{x}) = \{x_i, x_j\}_{\mathcal{P}}$.

From now on, let us consider only the oriented graphs whose vertices are either sinks, with no issued edges, or tails for an ordered pair of arrows, each decorated with its own index (see Fig. 1). Allowing the only exception in footnote 1, we shall always assume that there are neither tadpoles, nor double oriented edges, nor two-edge loops.

We also postulate that every vertex which is not a sink carries a copy of a given Poisson bi-vector $\mathcal{P} = \mathcal{P}^{ij}(\boldsymbol{x})\, \partial_i \wedge \partial_j$; the ordering of decorated out-going edges coincides with the ordering "first $\prec$ second" of the indexes in the coefficients of $\mathcal{P}$.

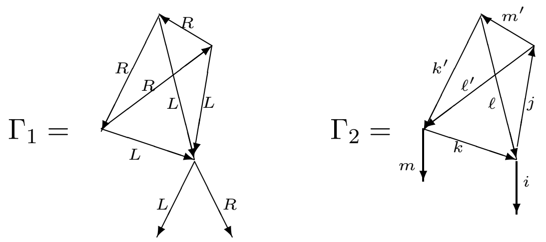


Figure 1. These tetraheral graphs encode flows (2a) and (2b), respectively. Each oriented edge carries a summation index that runs from 1 to the dimension of the Poisson manifold at hand. For each internal vertex (where a copy of the Poisson bi-vector $\mathcal{P}$ is stored), the pair of out-going edges is ordered, L $\prec$ R: the left edge (L) carries the first index and the other edge (R) carries the second index in the bi-vector coefficients. (In retrospect, the ordering and labelling of the indexed oriented edges can be guessed from formulas (2) on p. 3.)

Example 1. Under all these assumptions, the two tetrahedra which are portrayed in Fig. 1 are, up to a symmetry, the only admissible graphs with $k = 4$ internal vertices, $2k = 6 + 2$ edges, and two sinks. The first graph in Fig. 1 encodes the bi-vector

$$\Gamma_1(\mathcal{P}) = \sum_{i,j=1}^{n} \Bigg(\sum_{k,\ell,m,k',\ell',m'=1}^{n} \frac{\partial^3 \mathcal{P}^{ij}}{\partial x_k \partial x_\ell \partial x_m} \frac{\partial \mathcal{P}^{kk'}}{\partial x_{\ell'}} \frac{\partial \mathcal{P}^{\ell\ell'}}{\partial x_{m'}} \frac{\partial \mathcal{P}^{mm'}}{\partial x_{k'}} \Bigg) \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}. \tag{2a}$$

Likewise, the second graph in Fig. 1 yields the bi-vector

$$\Gamma_2(\mathcal{P}) = \sum_{i,m=1}^{n} \Bigg(\sum_{j,k,\ell,k',\ell',m'=1}^{n} \frac{\partial^2 \mathcal{P}^{ij}}{\partial x_k \partial x_\ell} \frac{\partial^2 \mathcal{P}^{km}}{\partial x_{k'} \partial x_{\ell'}} \frac{\partial \mathcal{P}^{k'\ell}}{\partial x_{m'}} \frac{\partial \mathcal{P}^{m'\ell'}}{\partial x_j} \Bigg) \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_m}. \tag{2b}$$

In this paper we examine

(*i*) whether the respective flows $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}(\mathcal{P}) = \Gamma_\alpha(\mathcal{P})$ at $\alpha = 1, 2$ preserve or, in fact, destroy the property of bi-vectors $\mathcal{P}(\varepsilon)$ to be Poisson, provided that the Cauchy datum $\mathcal{P}\big|_{\varepsilon=0}$ is such;

(*ii*) we also inspect whether the second flow is (actually, it is not) vanishing identically at all ε, provided that the Cauchy datum is a Poisson bi-vector.

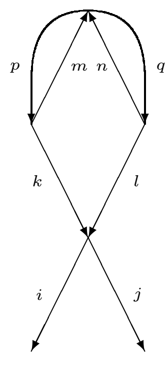
Remark 1. Whenever the bi-vector $\mathcal{P}$ in every internal vertex of a non-empty graph Γ is Poisson, the bi-differential operator which is encoded by Γ can vanish identically. First, this occurs due to the skew-symmetry of coefficients of the bi-vector.[1] Second, the operators encoded using graphs (with a copy of the Poisson bi-vector $\mathcal{P}$ at every internal vertex) can vanish by virtue of the Jacobi identity, see (1), or its differential consequences. This mechanism has been illustrated in [5]; making a part of our present argument (see [6]), it is a key to the proof of the fact that the balanced flow $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}(\mathcal{P}) = \Gamma_1(\mathcal{P}) + 6\,\Gamma_2(\mathcal{P})$ does preserve the property of bi-vectors $\mathcal{P}(\varepsilon)$ to be (infinitesimally) Poisson whenever the Cauchy datum $\mathcal{P}\big|_{\varepsilon=0}$ is such.

So, each of the two claims (*i*–*ii*) is false if it does not hold for at least one Poisson structure (itself already known to have skew-symmetric coefficients and turn the left-hand side of the Jacobi identity into zero for any triple of arguments of the Jacobiator). To examine both claims, we need a store of Poisson structures such that the coefficients $\mathcal{P}^{ij}(\boldsymbol{x})$ are not mapped to zero by the third or second order derivatives in (2a) and (2b), respectively. For that, a regular generator of Poisson structures with polynomial coefficients of arbitrarily high degree would suffice.

2. The generators

Let us recall three regular ways to generate the Poisson brackets or modify a given one, thus obtaining a new such structure. These generators will be used in section 3 to produce the counterexamples to both claims from [1].

[1] For example, consider this oriented graph with ordered pairs of indexed edges ($i \prec j$, $k \prec \ell$, $m \prec n$, $p \prec q$). We claim that due to the antisymmetry of $\mathcal{P}$ which is contained in each of the four internal vertices, the operator (which this graph encodes) vanishes identically. Indeed, it equals minus itself:



$$\begin{aligned}
&\partial_m\partial_n(\mathcal{P}^{pq})\partial_p(\mathcal{P}^{km})\partial_q(\mathcal{P}^{\ell n})\partial_k\partial_\ell(\mathcal{P}^{ij})\,\partial_i \wedge \partial_j \\
&\quad = -\partial_m\partial_n(\mathcal{P}^{qp})\partial_p(\mathcal{P}^{km})\partial_q(\mathcal{P}^{\ell n})\partial_k\partial_\ell(\mathcal{P}^{ij})\,\partial_i \wedge \partial_j \\
&\quad = -\partial_n\partial_m(\mathcal{P}^{pq})\partial_q(\mathcal{P}^{\ell n})\partial_p(\mathcal{P}^{km})\partial_\ell\partial_k(\mathcal{P}^{ij})\,\partial_i \wedge \partial_j = 0.
\end{aligned}$$

To establish the second equality, we interchanged the labelling of indices ($p \rightleftarrows q$, $k \rightleftarrows \ell$, and $m \rightleftarrows n$) and we recalled that the partial derivatives commute.

2.1. The determinant construction

This generator of Poisson bi-vectors is described in [7], cf. [16] and references therein. The construction goes as follows. Let x_1, …, x_n be the Cartesian coordinates on $\mathbb{R}^{n\geqslant 3}$. Let $\vec{g} = (g_1$, …, $g_{n-2})$ be a fixed tuple of smooth functions in these variables. For any a, $b \in \mathcal{C}^\infty(\mathbb{R}^n)$, put

$$\{a, b\}_{\vec{g}} = \det\bigl(\mathbf{J}(g_1, \ldots, g_{n-2}, a, b)\bigr)$$

where $\mathbf{J}(\cdot, \ldots, \cdot)$ is the Jacobian matrix. Clearly, the bracket $\{\cdot, \cdot\}_{\vec{g}}$ is bi-linear and skew-symmetric. Moreover, it is readily seen to be a derivation in each of its arguments: $\{a, b \cdot c\}_{\vec{g}} = \{a, b\}_{\vec{g}} \cdot c + b \cdot \{a, c\}_{\vec{g}}$. For the validity mechanism of the Jacobi identity for this particular instance of the Nambu bracket we refer to [16] again (see also [17]).

Example 2 (see entry 3 in Table 2 on p. 7)**.** Fix the functions $g_1 = x_2^3 x_3^2 x_4$ and $g_2 = x_1 x_3^4 x_4$, and insert them in the determinant generator of Poisson bi-vectors. We thus obtain the bi-vector $\mathcal{P}_0$,

$$\mathcal{P}_0^{ij} = \begin{pmatrix} 0 & -2\,x_1x_2^3x_3^5x_4 & -3\,x_1x_2^2x_3^6x_4 & 12\,x_1x_2^2x_3^5x_4^2 \\ 2\,x_1x_2^3x_3^5x_4 & 0 & -x_2^3x_3^6x_4 & 2\,x_2^3x_3^5x_4^2 \\ 3\,x_1x_2^2x_3^6x_4 & x_2^3x_3^6x_4 & 0 & -3\,x_2^2x_3^6x_4^2 \\ -12\,x_1x_2^2x_3^5x_4^2 & -2\,x_2^3x_3^5x_4^2 & 3\,x_2^2x_3^6x_4^2 & 0 \end{pmatrix}.$$

By construction, the above matrix is skew-symmetric. The validity of Jacobi identity (1) is straightforward: indexed by i, j, k, all the components $[\![\mathcal{P}, \mathcal{P}]\!]^{ijk}$ of the tri-vector vanish.[2] This Poisson bi-vector $\mathcal{P}$ is used in section 3 in the list of counterexamples to the claims under study.

2.2. Pre-multiplication in the 3-dimensional case

Let x, y, z be the Cartesian coordinates on the vector space $\mathbb{R}^3$. For every bi-vector $\mathcal{P} = \mathcal{P}^{ij}\,\partial_i \wedge \partial_j$, introduce the differential one-form $\mathrm{P} = \mathrm{P}_1\,\mathrm{d}x + \mathrm{P}_2\,\mathrm{d}y + \mathrm{P}_3\,\mathrm{d}z$ by setting $\mathrm{P} := -\mathcal{P} \lrcorner \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$, so that $\mathrm{P}_1 = -\mathcal{P}^{23}$, $\mathrm{P}_2 = \mathcal{P}^{13}$, and $\mathrm{P}_3 = -\mathcal{P}^{12}$. It is readily seen [8] that the original Jacobi identity for the bi-vector $\mathcal{P}$ now reads[3] $\mathrm{dP} \wedge \mathrm{P} = 0$ for the respective one-form P. But let us note that the pre-multiplication $\mathrm{P} \mapsto f \cdot \mathrm{P}$ of the form P by a smooth function f preserves this reading of the Jacobi identity: $\mathrm{d}(f\mathrm{P}) \wedge (f\mathrm{P}) = f \cdot \bigl[\mathrm{d}f \wedge \mathrm{P} \wedge \mathrm{P} + f \cdot \mathrm{dP} \wedge \mathrm{P}\bigr] = f^2 \cdot \mathrm{dP} \wedge \mathrm{P} = 0$. This shows that the bi-vector $f\mathcal{P}$ which the form $f\mathrm{P}$ yields on $\mathbb{R}^3$ is also Poisson.

This pre-multiplication trick provides the examples of Poisson structures of arbitrarily high polynomial degree coefficients (in a manifestly non-symplectic three-dimensional set-up).[4]

2.3. The Vanhaecke construction

In [9], Vanhaecke created another construction of high polynomial degree Poisson bi-vectors. Let u be a monic degree d polynomial in λ and v be a polynomial of degree $d-1$ in λ:

$$u(\lambda) = \lambda^d + u_1\lambda^{d-1} + \ldots + u_{d-1}\lambda + u_d, \qquad v(\lambda) = v_1\lambda^{d-1} + \ldots + v_{d-1}\lambda + v_d.$$

[2] Indeed, there are four tuples of distinct values of the indices i, j, and k up to permutations; we let $1 \leqslant i < j < k \leqslant n = 4$ so that the check runs over the set of triples $\{(1,2,3), (1,2,4), (1,3,4), (2,3,4)\}$. For example, $[\![\mathcal{P}, \mathcal{P}]\!]^{123} = 6x_1x_2^5x_3^{11}x_4^2 - 6x_1x_2^5x_3^{11}x_4^2 - 6x_1x_2^5x_3^{11}x_4^2 + 6x_1x_2^5x_3^{11}x_4^2 - 18x_1x_2^5x_3^{11}x_4^2 + 18x_1x_2^5x_3^{11}x_4^2 + 12x_1x_2^5x_3^{11}x_4^2 - 6x_1x_2^5x_3^{11}x_4^2 - 6x_1x_2^5x_3^{11}x_4^2 = 0$. Therefore, $[\![\mathcal{P}, \mathcal{P}]\!] = \sum_{1\leqslant i<j<k\leqslant 4} [\![\mathcal{P}, \mathcal{P}]\!]^{ijk}(\boldsymbol{x})\,\partial_i \wedge \partial_j \wedge \partial_k = 0$.

[3] The exterior differential dP is equal to $\mathrm{dP} = (\partial_x\mathcal{P}^{13} + \partial_y\mathcal{P}^{23})\,\mathrm{d}x \wedge \mathrm{d}y + (-\partial_x\mathcal{P}^{12} + \partial_z\mathcal{P}^{23})\,\mathrm{d}x \wedge \mathrm{d}z + (-\partial_y\mathcal{P}^{11} - \partial_z\mathcal{P}^{13})\,\mathrm{d}y \wedge \mathrm{d}z$. The wedge product is $\mathrm{dP} \wedge \mathrm{P} = \bigl(\partial_x\mathcal{P}^{31}\,\mathcal{P}^{12} + \partial_y\mathcal{P}^{23}\,\mathrm{P}^{21} + \partial_x\mathcal{P}^{12}\,\mathcal{P}^{13} + \partial_z\mathcal{P}^{23}\,\mathcal{P}^{31} + \partial_y\mathcal{P}^{12}\,\mathcal{P}^{23} + \partial_z\mathcal{P}^{31}\,\mathcal{P}^{32}\bigr)\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z = (-[\![\mathcal{P}, \mathcal{P}]\!] \lrcorner \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z)\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$.

[4] In dimension three, this pre-multiplication procedure also provides the examples of Poisson bi-vectors at which the second flow (2b) does not vanish identically.

Consider the space $\Bbbk^{2d}$ (e.g., set $\Bbbk := \mathbb{R}$) with Cartesian coordinates $u_1, \ldots, u_n, v_1, \ldots, v_d$. To define the Poisson bracket, fix a bivariate polynomial $\phi(\cdot,\cdot)$ and for all $1 \leqslant i, j \leqslant d$ set

$$\{u_i, u_j\} = \{v_i, v_j\} = 0, \qquad \{u_i, v_j\} = \text{coeff. of } \lambda^j \text{ in } \phi\big(\lambda, v(\lambda)\big) \cdot \left[\frac{u(\lambda)}{\lambda^{d-i+1}}\right]_+ \mod u(\lambda), \quad (3)$$

where we denote by $[\ldots]_+$ the argument's polynomial part and where the remainder modulo the degree d polynomial $u(\lambda)$ is obtained using the Euclidean division algorithm.

Let us emphasise that these Poisson bi-vector are defined on the even-dimensional spaces. Indeed, the coefficients of Poisson bracket (3) are arranged in the block matrix $\left(\begin{smallmatrix} 0 & U \\ -U & 0 \end{smallmatrix}\right)$, where the components of the matrix U are $U^{ij} = \{u_i, v_j\}$.

2.4. The Hamiltonian differential operators on jet spaces

The variational Poisson brackets $\{\cdot,\cdot\}_{\boldsymbol{\mathcal{P}}}$ for functionals of sections of affine bundles generalise the notion of Poisson brackets $\{\cdot,\cdot\}_{\mathcal{P}}$ for functions on finite-dimensional Poisson manifolds $(N^n, \{\cdot,\cdot\}_{\mathcal{P}})$. Namely, let us consider the space $J^\infty(\pi)$ of infinite jets of sections for a given bundle π over a manifold M^n of positive dimension m. The variational Poisson brackets $\{\cdot,\cdot\}_{\boldsymbol{\mathcal{P}}}$ on $J^\infty(\pi)$ are then specified by using the Hamiltonian differential operators (which we shall denote by A and the order of which is typically positive).[5] The formalism of variational Poisson bi-vectors $\boldsymbol{\mathcal{P}} = \frac{1}{2}\langle \boldsymbol{\xi} \cdot \vec{A}(\boldsymbol{\xi})\rangle$ and the variational Schouten bracket $[\![\cdot,\cdot]\!]$ is standard (see [11, 19]). The geometry of iterated variations is revealed in [13]; the correspondence between the Kontsevich graphs and local variational polydifferential operators is explained in [14].

Example 3. To inspect whether either of the two claims (which we quote from [1] on the title page) would hold in the variational set-up, it is enough to consider a Hamiltionian differential operator with (differential-)polynomial coefficients of degree $\geqslant 3$. Let us take the Hamiltonian operator[6] $A = u^2 \circ \mathrm{d}/\mathrm{d}x \circ u^2$ for the Harry Dym equation (see [12]); here u is the fibre coordinate in the trivial bundle $\pi\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ and x is the base variable. This operator is obviously skew-adjoint, whence the variational Poisson bracket $\{\cdot,\cdot\}_{\boldsymbol{\mathcal{P}}}$ is skew-symmetric. The Jacobi identity for $\{\cdot,\cdot\}_{\boldsymbol{\mathcal{P}}}$ is also easy to check: the variational master-equation $[\![\boldsymbol{\mathcal{P}}, \boldsymbol{\mathcal{P}}]\!] \cong 0$ does hold for the variational bi-vector $\boldsymbol{\mathcal{P}} = \frac{1}{2}\langle \xi \cdot \vec{A}(\xi)\rangle$.

3. The counterexamples

We now examine the properties of both tetrahedral flows (2) whenever each of them is evaluated at a given Poisson bi-vector. (Examples of such bi-vectors are produced by using the techniques from section 2.) To motivate the composition of Tables 1–4 and clarify the meaning of their content, let us consider an example: namely, we first take the Poisson bi-vector which was obtained in section 2.1 (see p. 4).

Example 4 (continued)**.** Rewriting the Poisson bi-vector $\mathcal{P}_0 \in \Gamma\big(\bigwedge^2 TN^4\big)$ in terms of the parity-odd variables $\boldsymbol{\xi}$, we obtain that under the isomorphism $\Gamma\big(\bigwedge^\bullet TN^n\big) \simeq C^\infty(\Pi T^* N^n)$ the bi-vector $\mathcal{P}_0^{ij}(\boldsymbol{x})\,\partial_i \wedge \partial_j$ becomes $\frac{1}{2}\mathcal{P}_0^{ij}(\boldsymbol{x})\,\xi_i\xi_j$; we have that $\mathcal{P}_0 =$

$$-2\,x_1x_2^3x_3^5x_4\xi_1\xi_2 - 3\,x_1x_2^2x_3^6x_4\xi_1\xi_3 + 12\,x_1x_2^2x_3^5x_4^2\xi_1\xi_4 - x_2^3x_3^6x_4\xi_2\xi_3 + 2\,x_2^3x_3^5x_4^2\xi_2\xi_4 - 3\,x_2^2x_3^6x_4^2\xi_3\xi_4.$$

[5] In fact, the Poisson geometry of finite-dimensional affine manifolds $(N^n, \{\cdot,\cdot\}_{\mathcal{P}})$ is a zero differential order sub-theory in the variational Poisson geometry of infinite jet spaces $J^\infty(\pi)$. Indeed, let the fibres in the bundle π be N^n and proclaim that only *constant* sections are allowed.

[6] More examples of variational Poisson structures, which are relevant for our present purpose, can be found in [20] or, e.g., in [21] (see also the references contained therein).

Now, we calculate the right-hand sides $\mathcal{P}_1 := \Gamma_1(\mathcal{P}_0)$ and $\mathcal{P}_2 := \Gamma_2(\mathcal{P}_0)$ of tetrahedral flows (2). The coefficient matrix of the bi-vector $\mathcal{P}_1$ is

$$\mathcal{P}_1^{ij} = \begin{pmatrix} 0 & -24480\,x_1x_2^9x_3^{20}x_4^4 & -51840\,x_1x_2^8x_3^{21}x_4^4 & 12960\,x_1x_2^8x_3^{20}x_4^5 \\ 24480\,x_1x_2^9x_3^{20}x_4^4 & 0 & -15480\,x_2^9x_3^{21}x_4^4 & 2448\,x_2^9x_3^{20}x_4^5 \\ 51840\,x_1x_2^8x_3^{21}x_4^4 & 15480\,x_2^9x_3^{21}x_4^4 & 0 & -18144\,x_2^8x_3^{21}x_4^5 \\ -12960\,x_1x_2^8x_3^{20}x_4^5 & -2448\,x_2^9x_3^{20}x_4^5 & 18144\,x_2^8x_3^{21}x_4^5 & 0 \end{pmatrix}.$$

In a similar way, the polydifferential operator Γ_2 (encoded by the second tetrahedral graph in Fig. 1) yields the matrix

$$\mathcal{P}_2^{ij} = \begin{pmatrix} 16920x_1^2x_2^8x_3^{20}x_4^4 & -12060\,x_1x_2^9x_3^{20}x_4^4 & -16380\,x_1x_2^8x_3^{21}x_4^4 & 42840\,x_1x_2^8x_3^{20}x_4^5 \\ 2700\,x_1x_2^9x_3^{20}x_4^4 & -7200\,x_2^{10}x_3^{20}x_4^4 & 4680\,x_2^9x_3^{21}x_4^4 & -252\,x_2^9x_3^{20}x_4^5 \\ -13140\,x_1x_2^8x_3^{21}x_4^4 & 5040\,x_2^9x_3^{21}x_4^4 & -12060\,x_2^8x_3^{22}x_4^4 & 13716\,x_2^8x_3^{21}x_4^5 \\ -80280\,x_1x_2^8x_3^{20}x_4^5 & -18036\,x_2^9x_3^{20}x_4^5 & 21708\,x_2^8x_3^{21}x_4^5 & -58104\,x_2^8x_3^{20}x_4^6 \end{pmatrix}.$$

Notice that this coefficient matrix is not yet antisymmetric, but its *symmetric* counterpart is skipped out in the construction of the bi-vector $\mathcal{P}_2$ and its transcription by using the anticommuting variables $\boldsymbol{\xi}$. Therefore, we antisymmetrise the above matrix at once, the output to be used in what follows. We obtain that the bi-vector is

$$\begin{aligned}\mathcal{P}_2 = {}&-7380x_1x_2^9x_3^{20}x_4^4\xi_1\xi_2 - 1620x_1x_2^8x_3^{21}x_4^4\xi_1\xi_3 + 61560x_1x_2^8x_3^{20}x_4^5\xi_1\xi_4 \\ &- 180x_2^9x_3^{21}x_4^4\xi_2\xi_3 + 8892x_2^9x_3^{20}x_4^5\xi_2\xi_4 - 3996x_2^8x_3^{21}x_4^5\xi_3\xi_4.\end{aligned}$$

We now see that for the Poisson bi-vector $\mathcal{P}_0$ from Example 2 on p. 4, ***the bi-vector $\mathcal{P}_2$ does not vanish***, thereby disavowing the second claim from [1].

To check the compatibility of the original Poisson bi-vector $\mathcal{P}_0$ with the newly obtained bi-vector $\mathcal{P}_1$, we calculate their Schouten bracket:

$$\begin{aligned}[\![\mathcal{P}_0,\mathcal{P}_1]\!] = {}&46008\,x_1x_2^{11}x_3^{26}x_4^5\xi_1\xi_2\xi_3 + 852768\,x_1x_2^{11}x_3^{25}x_4^6\xi_1\xi_2\xi_4 \\ &+ 1246752\,x_1x_2^{10}x_3^{26}x_4^6\xi_1\xi_3\xi_4 + 340200\,x_2^{11}x_3^{26}x_4^6\xi_2\xi_3\xi_4 \neq 0.\end{aligned}$$

The above expression is not identically zero. Therefore, ***the leading term $\mathcal{P}_1$ in the deformation $\mathcal{P}_0 \mapsto \mathcal{P}(\varepsilon) = \mathcal{P}_0 + \varepsilon\mathcal{P}_1 + \bar{o}(\varepsilon)$ destroys the property of bi-vector $\mathcal{P}(\varepsilon)$ to be Poisson*** at $\varepsilon \neq 0$ on all of $\mathbb{R}^4$.

The same compatibility test for $\mathcal{P}_0$ and its second flow (2b) yields that

$$\begin{aligned}[\![\mathcal{P}_0,\mathcal{P}_2]\!] = {}&-7668\,x_1x_2^{11}x_3^{26}x_4^5\xi_1\xi_2\xi_3 - 142128\,x_1x_2^{11}x_3^{25}x_4^6\xi_1\xi_2\xi_4 \\ &- 207792\,x_1x_2^{10}x_3^{26}x_4^6\xi_1\xi_3\xi_4 - 56700\,x_2^{11}x_3^{26}x_4^6\xi_2\xi_3\xi_4.\end{aligned}$$

Again, this expression does not vanish identically on all of the Poisson manifold $\big(\mathbb{R}^4,\ \{\cdot,\cdot\}_{\mathcal{P}_0}\big)$. We conclude that neither of two flows (2) preserve the property of bi-vector $\mathcal{P}(\varepsilon)$ to stay (infinitesimally) Poisson at $\varepsilon \neq 0$ for this example of Poisson bi-vector.[7]

[7] Let us also inspect whether the Jacobi identity holds for any of the bi-vectors $\mathcal{P}_1$ and $\mathcal{P}_2$. For $\mathcal{P}_1$ we have that the left-hand side of the Jacobi identity is equal to

$$[\![\mathcal{P}_1,\mathcal{P}_1]\!] = -2963589120\cdot\big(x_1x_2^{17}x_3^{41}x_4^8\xi_1\xi_2\xi_3 + 5\,x_1x_2^{17}x_3^{40}x_4^9\xi_1\xi_2\xi_4 - 2\,x_1x_2^{16}x_3^{41}x_4^9\xi_1\xi_3\xi_4\big),$$

which does not vanish. For $\mathcal{P}_2$ the left-hand side of the Jacobi identity equals

$$[\![\mathcal{P}_2,\mathcal{P}_2]\!] = -262517760\cdot\big(x_1x_2^{17}x_3^{41}x_4^8\xi_1\xi_2\xi_3 + 5\,x_1x_2^{17}x_3^{40}x_4^9\xi_1\xi_2\xi_4 - 2\,x_1x_2^{16}x_3^{41}x_4^9\xi_1\xi_3\xi_4\big).$$

This expression also does not vanish, so that neither $\mathcal{P}_1$ nor $\mathcal{P}_2$ are Poisson bi-vectors.

Remark 2. In the above example, the Schouten brackets $[\![\mathcal{P}_0, \mathcal{P}_1]\!]$ and $[\![\mathcal{P}_0, \mathcal{P}_2]\!]$ are determined by the same polynomials in the variables $\boldsymbol{x}$ and $\boldsymbol{\xi}$: we see that $[\![\mathcal{P}_0, \mathcal{P}_1]\!] = -6 \cdot [\![\mathcal{P}_0, \mathcal{P}_2]\!]$. This implies that for this example of Poisson bi-vector $\mathcal{P}_0$, the leading term $\mathcal{Q} := \mathcal{P}_1 + 6\mathcal{P}_2$ does (infinitesimally) preserve the property of $\mathcal{P}(\varepsilon)$ to be Poisson in the course of deformation $\mathcal{P}_0 \mapsto \mathcal{P}_0 + \varepsilon\mathcal{Q} + \bar{o}(\varepsilon)$.

Moreover, it is readily seen that the ratio $1:6$ is the *only* way to balance the two flows, (2a) vs (2b), such that their nontrivial linear combination $\mathcal{Q}$ is compatible with the Poisson bi-vector $\mathcal{P}_0$ from Example 2.[8]

Remark 3. In Example 4 the linear combination $\mathcal{Q} = \mathcal{P}_1 + 6\mathcal{P}_2 \neq 0$ of two flows (2) is not identically equal to zero. (For other examples this may happen incidentally.) The leading term $\mathcal{Q}$ in the infinitesimal deformation $\mathcal{P}_0 \mapsto \mathcal{P}_0 + \varepsilon\mathcal{Q} + \bar{o}(\varepsilon)$ is trivial in the Poisson cohomology with respect to $\boldsymbol{\partial}_{\mathcal{P}_0}$, i. e. $\mathcal{Q} = [\![\mathcal{P}_0, \mathcal{X}]\!]$ for some vector $\mathcal{X}$ on the four-dimensional space.[9] Hence this $\mathcal{Q}$ is trivially compatible with the Poisson bi-vector $\mathcal{P}_0$: namely, $[\![\mathcal{P}_0, \mathcal{Q}]\!] \equiv 0$, see p. 8 below.

In the three tables below we summarise the results about the flows $\mathcal{P}_1$ and $\mathcal{P}_2$, which we evaluate at the examples of Poisson bi-vectors $\mathcal{P}_0$. Special attention is paid to the leading deformation term $\mathcal{Q} = \mathcal{P}_1 + 6\mathcal{P}_2$ in each case: we inspect whether this bi-vector incidentally vanishes and whether it is (indeed, always) compatible with the original Poisson structure $\mathcal{P}_0$.

Table 1. The Poisson bi-vectors $\mathcal{P}_0$ are generated using the determinant method from section 2.1 (the dimension is equal to 3, so we specify the fixed argument g_1); that generator is combined with the pre-multiplication $(f\cdot)$ as explained in section 2.2.

№	dim	Argument & pre-factor	$[\![\mathcal{P}_0, \mathcal{P}_1]\!] = 0$?	$\mathcal{P}_2 \overset{?}{=} 0$	$[\![\mathcal{P}_0, \mathcal{P}_2]\!] = 0$?	$\mathcal{Q} \overset{?}{=} 0$	$[\![\mathcal{P}_0, \mathcal{Q}]\!] = 0$?
1.	3	$[x_1^5 x_2^3 x_3^4 + x_1^2 x_3^5 + x_1 x_2^5 x_3]$ $x_1^3 + x_2^2$	✗	✗	✗	✗	✓
2.	3	$[x_1 x_2 + x_1 x_3 + x_2 x_3]$ $x_1^2 + x_2$	✗	✗	✗	✗	✓

For both examples in Table 1 we have that neither does $\mathcal{P}_1$ preserve the property of $\mathcal{P}_0 + \varepsilon\mathcal{P}_1 + \bar{o}(\varepsilon)$ to be (infinitesimally) Poisson nor does $\mathcal{P}_2$ vanish identically — which is in contrast with both the claims from [1].

Table 2. In dimensions higher than 3, we generate the Poisson bi-vectors $\mathcal{P}_0$ by using the determinant method from section 2.1: the auxiliary arguments g_1, …, g_{n-2} are specified.

№	dim	Arguments	$[\![\mathcal{P}_0, \mathcal{P}_1]\!] = 0$?	$\mathcal{P}_2 \overset{?}{=} 0$	$[\![\mathcal{P}_0, \mathcal{P}_2]\!] = 0$?	$\mathcal{Q} \overset{?}{=} 0$	$[\![\mathcal{P}_0, \mathcal{Q}]\!] = 0$?
3.	4	$[x_2^3 x_3^2 x_4, x_1 x_3^4 x_4]$	✗	✗	✗	✗	✓
4.	4	$[x_1^2 x_2^3 x_3^4 x_4^5, x_1 x_2 x_3 x_4]$	✗	✗	✗	✓	✓
5.	4	$[x_2^2 x_3^2 x_4^2, x_1^2 x_3^2 x_4^2]$	✗	✗	✗	✓	✓
6.	5	$[x_2^3 x_3^2 x_4, x_1 x_3^4 x_4, x_3^3 x_4^2 x_5^4]$	✗	✗	✗	✗	✓

[8] The balance $1 : \frac{4}{3}$ was considered in [22, §5.2] for the linear combination of flows (2a) and (2b), respectively.

[9] In all the two-dimensional Poisson geometries, the first flow $\mathcal{P}_1$ is always cohomologically trivial, i.e. it is of the form $\mathcal{P}_1 = [\![\mathcal{P}_0, \mathcal{X}]\!]$ for some one-vector $\mathcal{X}$, see [1].

In Table 2 we again have that neither is the property to be (infinitesimally) Poisson preserved for $\mathcal{P}_0 + \varepsilon\mathcal{P}_1 + \bar{o}(\varepsilon)$ nor is the bi-vector $\mathcal{P}_2$ vanishing identically.

Table 3. The results for the Vanhaecke method from section 2.3: we here specify the bivariate polynomials ϕ.

№	dim	$\phi(x,y)$	$[\![\mathcal{P}_0,\mathcal{P}_1]\!] \stackrel{?}{=} 0$	$\mathcal{P}_2 \stackrel{?}{=} 0$	$[\![\mathcal{P}_0,\mathcal{P}_2]\!] \stackrel{?}{=} 0$	$\mathcal{Q} \stackrel{?}{=} 0$	$[\![\mathcal{P}_0,\mathcal{Q}]\!] \stackrel{?}{=} 0$
7.	4	$[x^2y^2]$	✗	✗	✗	✗	✓
8.	4	$[x^2y]$	✗	✗	✗	✗	✓
9.	4	$[x^3y^2]$	✗	✗	✗	✗	✓
10.	4	$[x^3y^3]$	✗	✗	✗	✗	✓
11.	6	$[x^2y^2]$	✗	✗	✗	✗	✓

The entries in Table 3 report on the use of the generator from section 2.3: experimentally established, the properties of these Poisson bi-vectors do not match both the claims from [1].

Table 4. The results for the infinite-dimensional case.

№	dim	Operator	$[\![\boldsymbol{\mathcal{P}}_0,\boldsymbol{\mathcal{P}}_1]\!] \stackrel{?}{\cong} 0$	$\boldsymbol{\mathcal{P}}_2 \stackrel{?}{\cong} 0$
12.	∞	$u^2 \circ \mathrm{d}/\mathrm{d}x \circ u^2$	✗	✓

The variational bi-vector $\boldsymbol{\mathcal{P}}_1 = \frac{1}{2}\langle \xi \cdot \vec{A}_1(\xi)\rangle$, which we construct from the variational Poisson bi-vector $\boldsymbol{\mathcal{P}}_0 = \frac{1}{2}\langle \xi \cdot u^2\, \vec{\mathrm{d}}/\mathrm{d}x(u^2\,\xi)\rangle$ by using the geometric technique from [13] (see also [14]), is determined by the (skew-adjoint part of the) first-order differential operator $A_1 = 192\,\bigl(9u^8u_xu_{xx} - u^9u_{xxx}\bigr)\,\mathrm{d}/\mathrm{d}x$ in total derivatives. Again (see Table 4), the two variational bi-vectors are *not* compatible: we check that $[\![\boldsymbol{\mathcal{P}}_0,\boldsymbol{\mathcal{P}}_1]\!] \not\cong 0$ under the variational Schouten bracket. Remarkably, the variational bi-vector $\boldsymbol{\mathcal{P}}_2$ is specified by the second-order total differential operator whose skew-adjoint component vanishes, whence the respective variational bi-vector *is* equal to zero (modulo exact terms within its horizontal cohomology class [11]).

Conclusion

The linear combination $\mathcal{Q} = \mathcal{P}_1 + 6\mathcal{P}_2$ of the Kontsevich tetrahedral flows preserves the space of Poisson bi-vectors $\mathcal{P}_0$ under the infinitesimal deformations $\mathcal{P}_0 \mapsto \mathcal{P}_0 + \varepsilon\mathcal{Q} + \bar{o}(\varepsilon)$. This is manifestly true for all the examples of Poisson bi-vectors on finite-dimensional (vector or affine) spaces $\mathbb{R}^n$ which we have considered so far. We conjectured that the leading deformation term $\mathcal{Q} = \mathcal{Q}(\mathcal{P}_0)$ always has this property, that is, ***the bi-vector $\mathcal{Q}$ marks a $\boldsymbol{\partial}_{\mathcal{P}_0}$-cohomology class for every Poisson bi-vector*** $\mathcal{P}_0$ on a finite-dimensional affine manifold. (Recall that such class can be $\boldsymbol{\partial}_{\mathcal{P}_0}$-trivial; moreover, the bi-vector $\mathcal{Q}$ can vanish identically — yet the above examples confirm the existence of Poisson geometries where neither of the two options is realised.)

Let us conclude that every claim of an object's vanishing by virtue of the skew-symmetry and Jacobi identity for a given Poisson bi-vector, which that object depends on by construction, must be accompanied with an explicit description of that factorisation mechanism (e.g., see [5]) or at least, with a proof of that mechanism's existence. Apart from the trivial case (here, $\mathcal{Q} = 0$ so that $[\![\mathcal{P}_0,\mathcal{Q}]\!] \equiv 0$), such factorisation through the master-equation $[\![\mathcal{P}_0,\mathcal{P}_0]\!] = 0$ can be immediate: here, we have that $[\![\mathcal{P}_0,\mathcal{Q}]\!] = [\![\mathcal{P}_0,[\![\mathcal{P}_0,\mathcal{X}]\!]]\!] = \frac{1}{2}[\![[\![\mathcal{P}_0,\mathcal{P}_0]\!],\mathcal{X}]\!] = \bigl(\frac{1}{2}[\![\cdot,\mathcal{X}]\!]\bigr)\,\bigl([\![\mathcal{P}_0,\mathcal{P}_0]\!]\bigr)$ for all $\boldsymbol{\partial}_{\mathcal{P}_0}$-exact infinitesimal deformations $\mathcal{Q} = \boldsymbol{\partial}_{\mathcal{P}_0}(\mathcal{X})$ of the Poisson bi-vectors $\mathcal{P}_0$. Elaborated in [5], the Poisson cohomology estimate mechanism of the vanishing $[\![\mathcal{P}_0,\mathcal{Q}]\!] \doteq 0$ via $[\![\mathcal{P}_0,\mathcal{P}_0]\!] = 0$

works – for the nontrivial cocycles $\mathcal{Q} \notin \operatorname{im} \boldsymbol{\partial}_{\mathcal{P}_0}$ in the $\boldsymbol{\partial}_{\mathcal{P}_0}$-cohomology – due to much more refined principles. That vanishing mechanism is applied to the factorisation problem at hand in the paper [6] (joint with R. Buring), where we prove the above conjecture.

Acknowledgments
The second author thanks the Organizing committee of XXIV International conference 'Integrable systems & quantum symmetries' (13 – 19 June 2016; CVUT Prague, Czech Republic) for a warm atmosphere during the meeting. The research of A. B. was supported by JBI RUG project 190.135.021; A. K. was supported by NWO grant VENI 639.031.623 (The Netherlands) and JBI RUG project 106552 (Groningen). A. B. thanks R. Buring for fruitful cooperation; A. K. thanks M. Kontsevich for posing the problem and stimulating discussion.

Appendix A. The mechanism of vanishing for $[\![\mathcal{P}, \mathcal{Q}_{1:6}(\mathcal{P})]\!] = 0$: an example

We wish to recognize the differential consequences of the Jacobi identity in the compatibility equation $[\![\mathcal{P}, \mathcal{Q}_{1:6}(\mathcal{P})]\!] = 0$, to understand why it holds. By a straightforward calculation we learn that $[\![\mathcal{P}, \mathcal{Q}_{1:6}(\mathcal{P})]\!] = 0$ for all Poisson bi-vectors on $\mathbb{R}^3$. But as soon as the differential consequences of the Jacobi identity are recognized, they can be translated into graphs. Independent of dimension, the language of graphs then answers the question which we started out with. This answer is found in [6].

Let us now illustrate a more analytic approach to the factorization problem for $[\![\mathcal{P}, \mathcal{Q}_{1:6}]\!] = 0$ via $[\![\mathcal{P}, \mathcal{P}]\!] = 0$ (see [6, App. D] for details). The compatibility equation is a vanishing expression, which is impossible to factorize through the Jacobi identity, which itself is also zero. To make both visible, we perturb a given Poisson bi-vector $\mathcal{P}$ using $\tilde{\mathcal{P}} = \mathcal{P} + \epsilon \cdot \Delta$ for a bi-vector Δ, in such a way that $\tilde{\mathcal{P}}$ is no longer Poisson, thereby $[\![\tilde{\mathcal{P}}, \tilde{\mathcal{P}}]\!] \neq 0$. The goal is to perturb the bi-vector $\mathcal{P}$ such that the left-hand side $[\![\tilde{\mathcal{P}}, \tilde{\mathcal{Q}}_{1:6}]\!]$ becomes non-zero as well. Now the Jacobi identity's non-zero differential consequences becomes recognizable in the non-zero expression $[\![\tilde{\mathcal{P}}, \tilde{\mathcal{Q}}_{1:6}]\!]$.

Example 5. Consider the Poisson bi-vector obtained on $\mathbb{R}^3$ from the determinant construction using two functions $g(z)$ and $f(x)$ as argument and pre-multiplication factor, respectively. Let the perturbation Δ be given component-wise by $\Delta^{12} = f_1(y,z)$, $\Delta^{13} = f_2(y,z)$ and $\Delta^{23} = 0$. The perturbed bi-vector then equals

$$\tilde{\mathcal{P}} = \begin{bmatrix} 0 & f \cdot \mathrm{d}g/\mathrm{d}z & 0 \\ -f \cdot \mathrm{d}g/\mathrm{d}z & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \epsilon \cdot \begin{bmatrix} 0 & f_1 & f_2 \\ -f_1 & 0 & 0 \\ -f_2 & 0 & 0 \end{bmatrix}.$$

The left-hand sides of the Jacobi identity and of the compatibility condition are evaluated to

$$[\![\tilde{\mathcal{P}}, \tilde{\mathcal{P}}]\!]^{123} = \epsilon f_2 \cdot \frac{\mathrm{d}f}{\mathrm{d}x}\frac{\mathrm{d}g}{\mathrm{d}z} + \bar{o}(\epsilon), \qquad [\![\tilde{\mathcal{P}}, \tilde{\mathcal{Q}}]\!]^{123} = -\epsilon \cdot \frac{\partial^3 f_2}{\partial y^3}\left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)^4\left(\frac{\mathrm{d}g}{\mathrm{d}z}\right)^4 + \bar{o}(\epsilon).$$

There is only one way to recognize a differential consequence of the Jacobiator inside $[\![\tilde{\mathcal{P}}, \tilde{\mathcal{Q}}_{1:6}]\!]^{123}$. Namely, the Jacobi identity contains a product of f_2 and derivatives of f and g. The same is true for its non-zero differential consequences. Let us extract this product from $[\![\tilde{P}, \tilde{Q}_{1:6}]\!]^{123}$. The only differential consequences of f_2, $\mathrm{d}f/\mathrm{d}x$, and $\mathrm{d}g/\mathrm{d}y$ in $[\![\mathcal{P}, \mathcal{Q}_{1:6}]\!]^{123}$ are $\partial^3 f_2/\partial y^3$, $\mathrm{d}f/\mathrm{d}x$ and $\mathrm{d}g/\mathrm{d}z$, respectively. This hints that we have the differential consequence $[\![\mathcal{P}, \mathcal{P}]\!]^{123}_{yyy}$. To understand what its coefficient is, we note that the remaining co-factors in $[\![\tilde{\mathcal{P}}, \tilde{\mathcal{Q}}_{1:6}]\!]^{123}$ form $(\mathcal{P}^{12}_x)^3$. We conclude that the left-hand side of the compatibility equation factorizes through the Jacobi identity as follows

$$[\![\mathcal{P}, \mathcal{Q}_{1:6}]\!]^{123} = \mathcal{P}^{12}_x \mathcal{P}^{12}_x \mathcal{P}^{12}_x [\![\mathcal{P}, \mathcal{P}]\!]^{123}_{yyy} + \cdots.$$

Looking at this expression, we construct a list of graphs that can encode it. Such a list fully formed, it is subtracted from $[\![\mathcal{P}, \mathcal{Q}_{1:6}]\!]$ and resolved with respect to the coefficients of every proposed graph. We keep subtracting the already found graphs from any non-zero perturbations of $[\![\mathcal{P}, \mathcal{Q}_{1:6}]\!]$ in the future, once the coefficients are known. The example under study gave us the tripod graph, which is the first entry in [6, Eq. (6)]. Proceeding in the same way, we also recognized the 'elephant' graph, which is the sixth entry in that solution (cf. [6, Remarks 10–11]).

References

[1] Kontsevich M (1997) Formality conjecture *Deformation theory and symplectic geometry (Ascona 1996)* **20** (Dordrecht: Kluwer Acad. Publ.) 139–156

[2] Kontsevich M (2003) Deformation quantization of Poisson manifolds *Lett. Math. Phys.* **66**:3 157–216 (*Preprint* `q-alg/9709040`)

[3] Kontsevich M (1995) Homological algebra of mirror symmetry *Proc. Intern. Congr. Math.* **1** (Basel: Birkhäuser) 120–139

[4] Kontsevich M (1994) Feynman diagrams and low-dimensional topology *First Europ. Congr. of Math.* **2** (Paris, 1992) **120** (Basel: Birkhäuser) 97–121

[5] Buring R and Kiselev A V (2017) On the Kontsevich $\star$-product associativity mechanism *PEPAN Letters* **14**:2 accepted (*Preprint* `arXiv:1602.09036` [q-alg])

[6] Bouisaghouane A, Buring R and Kiselev A V (2016) The Kontsevich tetrahedral flows revisited *Preprint* `arXiv:1608.01710` (v2) [q-alg] 20

[7] Donin J (1998) On the quantization of quadratic Poisson brackets on a polynomial algebra of four variables *Lie Groups and Lie Algebras. Their representations, generalisations and applications* **433** (Dordrecht: Kluwer Acad. Publ.) 17–25

[8] Grabowski J, Marmo G and Perelomov A M (1993) Poisson structures: towards a classification *Mod. Phys. Lett.* **A8**:18 1719–1733

[9] Vanhaecke P (1996) *Integrable systems in the realm of algebraic geometry* **1638** (Berlin: Springer–Verlag)

[10] Gerstenhaber M (1964) On the deformation of rings and algebras *Ann. Math.* **79** 59–103

[11] Olver P J (1993) *Applications of Lie groups to differential equations* **107** (2nd ed.) (NY: Springer–Verlag)

[12] Wang J P (2002) A list of $1+1$ dimensional integrable equations and their properties *J. Nonlin. Math. Phys.* **9** 213–233

[13] Kiselev A V (2013) The geometry of variations in Batalin–Vilkovisky formalism *J. Phys.: Conf. Ser.* **474** Paper 012024 1–51 (*Preprint* `1312.1262 [math-ph]`)

[14] Kiselev A V (2015) Deformation approach to quantisation of field models *Preprint* IHÉS/M/15/13 (Bures-sur-Yvette, France) 37

[15] Kiselev A V (2016) The right-hand side of the Jacobi identity: to be naught or not to be ? *J. Phys.*: *Conf. Ser.* **670** 012030 1–17 (*Preprint* `arXiv:1410.0173` [math-ph])

[16] Vinogradov A and Vinogradov M (1998) On multiple generalizations of Lie algebras and Poisson manifolds *Secondary calculus and cohomological physics* **219** (Providence RI: AMS) 273–287

[17] Nambu Y (1973) Generalized Hamiltonian dynamics *Phys. Rev. D* **7** 2405–2412

[18] Omori H, Maeda Y and Yoshioka A (1993) A construction of a deformation quantization of a Poisson algebra *Geometry and its applications* (River Edge NJ: World Sci. Publ.) 201–218

[19] Kiselev A V (2015) The calculus of multivectors on noncommutative jet spaces *Preprint* IHÉS/M/14/39 (Bures-sur-Yvette, France) `arXiv:1210.0726` (v3) [math.DG] 41

[20] Fokas A S, Olver P J and Rosenau P (1997) A plethora of integrable bi-Hamiltonian equations *Algebraic aspects of integrable systems* **26** (Boston MA: Birkhäuser) 93–101

[21] Vodová J (2013) Low-order Hamiltonian operators having momentum *J. Math. Anal. Appl.* **401**:2 724–732 (*Preprint* `arXiv:1111.6434` [math-ph])

[22] Merkulov S A (2010) Exotic automorphisms of the Schouten algebra of polyvector fields *Preprint* `arXiv:0809.2385` (v6) [q-alg] 37

THE KONTSEVICH TETRAHEDRAL FLOWS REVISITED

A. BOUISAGHOUANE, R. BURING, AND A. KISELEV*

ABSTRACT. We prove that the Kontsevich tetrahedral flow $\dot{\mathcal{P}} = \mathcal{Q}_{a:b}(\mathcal{P})$, the right-hand side of which is a linear combination of two differential monomials of degree four in a bi-vector $\mathcal{P}$ on an affine real Poisson manifold N^n, does infinitesimally preserve the space of Poisson bi-vectors on N^n if and only if the two monomials in $\mathcal{Q}_{a:b}(\mathcal{P})$ are balanced by the ratio $a : b = 1 : 6$. The proof is explicit; it is written in the language of Kontsevich graphs.

Introduction. The main question which we address in this paper is how Poisson structures can be deformed in such a way that they stay Poisson. We reveal one such method that works for all Poisson structures on affine real manifolds; the construction of that flow on the space of bi-vectors was proposed in [14]: the formula is derived from two differently oriented tetrahedral graphs over four vertices. The flow is a linear combination of two terms, each quartic-nonlinear in the Poisson structure. By using several examples of Poisson brackets with high polynomial degree coefficients, we demonstrated in [1] that the ratio 1 : 6 is the only possible balance at which the tetrahedral flow can preserve the property of the Cauchy datum to be Poisson. But does the Kontsevich tetrahedral flow $\dot{\mathcal{P}} = \mathcal{Q}_{1:6}(\mathcal{P})$ with ratio 1 : 6 actually preserve the space of *all* Poisson bi-vectors? In dimension 3 the description of Poisson brackets with smooth coefficients is known from [6]; a brute force calculation then verifies the claim. In this paper we prove the claim in full generality, namely, for all Poisson structures on all affine manifolds of arbitrary finite dimension. The proof is graphical: namely, to prove that equation (1) holds, we find an operator $\Diamond$, encoded by using the Kontsevich graphs, that solves the equation (8). (As soon as solution (9) is obtained, verifying that it does satisfy the determining equation (8) is elementary though tedious.[1]) The first by-product of our proof is that there is no universal mechanism (that would involve the language of Kontsevich graphs) for the tetrahedral flow to be trivial in the respective Poisson cohomology. Secondly, the factorization mechanism, on which the proof of Theorem 3 is based, explains in hindsight why the proven property of tetrahedral flows is false for the variational Poisson brackets. (This was observed empirically in [1]; the geometry of Poisson structures over jet bundles is known from [19].)

Date: 31 October 2016.

2010 *Mathematics Subject Classification.* 53D55, 58E30, 81S10; secondary 53D17, 58Z05, 70S20.

Key words and phrases. Poisson bracket, affine manifold, graph complex, tetrahedral flow, Poisson cohomology.

Address: Johann Bernoulli Institute for Mathematics and Computer Science, University of Groningen, P.O. Box 407, 9700 AK Groningen, The Netherlands. **E-mail*: A.V.Kiselev@rug.nl.

[1]Having a solution $\Diamond$ to equation (8) is analogous to having a rational point on an elliptic curve: finding either is hard, though verifying that it does satisfy the equation at hand is almost immediate.

The text is structured as follows. In section 1 we recall how oriented graphs can be used to encode differential operators acting on the space of multivectors. In particular, differential polynomials in a given Poisson structure are obtained in the frames of this approach as soon as a copy of that Poisson bi-vector is placed in every internal vertex of a graph. Specifically, the right-hand side $\mathcal{Q}_{a:b} = a \cdot \Gamma_1 + b \cdot \Gamma_2$ of the Kontsevich tetrahedral flow $\dot{\mathcal{P}} = \mathcal{Q}_{a:b}(\mathcal{P})$ on the space of bi-vectors on an affine Poisson manifold $\bigl(N^n, \mathcal{P}\bigr)$ is a linear combination of two differential monomials, $\Gamma_1(\mathcal{P})$ and $\Gamma_2(\mathcal{P})$, of degree four in the bi-vector $\mathcal{P}$ that evolves.

In this paper we find out at which balance $a : b$ the Kontsevich tetrahedral flow $\dot{\mathcal{P}} = \mathcal{Q}_{a:b}(\mathcal{P})$ infinitesimally preserves the space of Poisson bi-vectors, that is, the bi-vector $\mathcal{P} + \epsilon\mathcal{Q}_{a:b}(\mathcal{P}) + \bar{o}(\epsilon)$ satisfies the equation

$$[\![\mathcal{P} + \epsilon\mathcal{Q}_{a:b}(\mathcal{P}) + \bar{o}(\epsilon), \mathcal{P} + \epsilon\mathcal{Q}_{a:b}(\mathcal{P}) + \bar{o}(\epsilon)]\!] = \bar{o}(\epsilon) \qquad \text{via } [\![\mathcal{P}, \mathcal{P}]\!] = 0;$$

here we denote by $[\![\cdot, \cdot]\!]$ the Schouten bracket (see formula (11) on page 17; relevant cohomological techniques are reviewed in Appendix A). Expanding, we obtain the cocycle condition,

$$[\![\mathcal{P}, \mathcal{Q}_{1:6}(\mathcal{P})]\!] \doteq 0 \qquad \text{via } [\![\mathcal{P}, \mathcal{P}]\!] = 0, \tag{1}$$

with respect to the Poisson differential $\boldsymbol{\partial}_{\mathcal{P}} = [\![\mathcal{P}, \cdot]\!]$. Viewed as an equation with respect to the ratio $a : b$, condition (1) is the main object of our study.

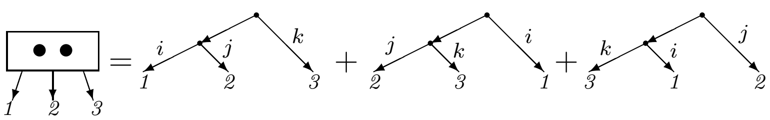
Recent counterexamples [1] show that the bi-vector $\mathcal{P} + \varepsilon\mathcal{Q}_{a:b}(\mathcal{P}) + \bar{o}(\varepsilon)$ stays infinitesimally Poisson *only if* the balance $a : b$ in $\mathcal{Q}_{a:b}$ is equal to $1 : 6$. (Without extra assumptions, the infinitesimal deformation $\mathcal{P} + \varepsilon\mathcal{Q}_{a:b}(\mathcal{P}) + \bar{o}(\varepsilon)$ can be completed to a finite deformation $\mathcal{P}(\varepsilon)$ at $\varepsilon > 0$ if the third Poisson cohomology group $H^3_{\mathcal{P}}(N^n)$ with respect to the differential $\boldsymbol{\partial}_{\mathcal{P}} = [\![\mathcal{P}, \cdot]\!]$ vanishes for the Poisson manifold $\bigl(N^n, \mathcal{P}\bigr)$. Therefore, unlike the Kontsevich formula for the flow $\dot{\mathcal{P}} = \mathcal{Q}_{a:b}(\mathcal{P})$ which is universal for all N^n and $\mathcal{P}$, the integration issue is Poisson model-dependent.)

We now prove that the balance $a : b = 1 : 6$ in the Kontsevich tetrahedral flow is universal in the above sense: for all Poisson bi-vectors $\mathcal{P}$ on every affine manifold N^n, the deformation $\mathcal{P} + \varepsilon\mathcal{Q}_{1:6}(\mathcal{P}) + \bar{o}(\varepsilon)$ is infinitesimally Poisson. The proof is explicit: in section 2 we reveal the mechanism of factorization – via the Jacobi identity – in (1) at $a : b = 1 : 6$. Specifically, we find a linear polydifferential operator $\Diamond(\mathcal{P}, \cdot)$ that acts on the filtered components (see below) of the Jacobiator $\mathrm{Jac}\,(\mathcal{P}) := [\![\mathcal{P}, \mathcal{P}]\!] = 0$ for the bi-vector $\mathcal{P}$; the operator $\Diamond$ provides the factorization $[\![\mathcal{P}, \mathcal{Q}_{1:6}(\mathcal{P})]\!] = \Diamond(\mathcal{P}, \mathrm{Jac}\,(\mathcal{P}))$ of the $\boldsymbol{\partial}_{\mathcal{P}}$-cocycle condition, see (1), through the Jacobiator $\mathrm{Jac}\,(\mathcal{P}) = 0$. On the one side of factorization problem (1) we expand the Poisson differential of the Kontsevich tetrahedral flow at the balance $1 : 6$ into the sum of 39 graphs (see Figure 3 on page 6 and Table 1 in Appendix D). On the other side of that factorization, we take the sum that runs with undetermined coefficients over all those fragments of differential consequences of the Jacobi identity $[\![\mathcal{P}, \mathcal{P}]\!] = 0$ which are known to vanish independently. In our reasoning the differential consequences of the identity $\mathrm{Jac}\,(\mathcal{P}) := [\![\mathcal{P}, \mathcal{P}]\!] = 0$ for Poisson bi-vectors are filtered up to order three according to the differential orders (k, ℓ, m), $k+\ell+m \geqslant 3$, with respect to the arguments of the tri-vector $[\![\mathcal{P}, \mathcal{P}]\!]$. We recall that every differential consequence of order (k, ℓ, m) for the Jacobi identity $\mathrm{Jac}\,(\mathcal{P}) = 0$ then vanishes. To describe the differential operators that produce such consequences of the Jacobi identity, we use the pictorial language of graphs: every internal vertex contains a copy of the bi-vector $\mathcal{P}$ and the operators are reduced by using its skew-symmetry. The resulting sum of graphs is reduced modulo the skew-symmetry of the bi-vector at hand; there remain 7,025 graphs, the coefficients of which are linear in the unknowns. We now solve the arising inhomogeneous linear algebraic system. Its solution yields the polydifferential operator $\Diamond$, encoded using graphs (see p. 11), that provides the sought-for factorization $[\![\mathcal{P}, \mathcal{Q}_{1:6}]\!] = \Diamond(\mathcal{P}, \mathrm{Jac}(\mathcal{P}))$. It is readily seen from formula (9) that the operator $\Diamond$ is completely determined by only 8 nonzero coefficients (out of 1132, see their count in Appendix C). Therefore, although finding the operator $\Diamond$ was hard, verifying that it does solve the factorization problem has become almost immediate. This completes the proof of Theorem 3 and establishes the main result (namely, Corollary 4 on page 4). In section 3 we analyze the properties of the solution at hand. (The maximally detailed description of that solution $\Diamond$ is contained in Appendix D.) The paper concludes with a list of open problems and five appendices.

In Appendix E we outline a different method to tackle the factorization problem, namely, by making the Jacobi identity visible in (1) by perturbing the original structure $\mathcal{P}$ so that it stops being Poisson. Hence it contributes to the right-hand side of (1) such that the respectively perturbed bi-vector $\mathcal{Q}_{1:6}(\mathcal{P})$ stops being compatible (in the sense of (1)) with the perturbed Poisson structure. The first-order balance of both sides of perturbed equation (1) then suggests the coefficients of those differential consequences of the Jacobiatior which are actually involved in the factorization mechanism. The coefficients of operators realized by graphs which were found by following this scheme are reproduced in the full run-through that gave us the solution $\Diamond$ in section 2.

1. The main problem: From graphs to multivectors

1.1. The language of graphs. Let us formalise a way to encode polydifferential operators – in particular multivectors – using oriented graphs. In an affine real manifold N^n (here $2 \leqslant n < \infty$), consider a chart $U_\alpha \hookrightarrow \mathbb{R}^n$ and denote the Cartesian coordinates by $\boldsymbol{x} = (x^1, \ldots, x^n)$. By definition, the decorated edge $\bullet \xrightarrow{i} \bullet$ denotes at once the derivation $\partial/\partial x^i \equiv \partial_i$ (that acts on the content of the arrowhead vertex) and the summation $\sum_{i=1}^n$ (over the index i in the object which is contained within the arrowtail vertex). As it has been explained in [8, 10, 15], the operator which every graph encodes is equal to the sum (running over all the indexes) of products (running over all the vertices) of those vertices content (differentiated by the in-coming arrows, if any). For example, the graph $(1) \xleftarrow[L]{i} \mathcal{P}^{ij}(\boldsymbol{x}) \xrightarrow[R]{j} (2)$ encodes the bi-differential operator $\sum_{i,j=1}^n (1) \overleftarrow{\partial_i} \cdot \mathcal{P}^{ij}(\boldsymbol{x}) \cdot \overrightarrow{\partial_j} (2)$. It then specifies the Poisson bracket on the chart $U_\alpha \subset N^n$. The bracket satisfies the Jacobi identity

$\mathrm{Jac}(\mathcal{P})(1,2,3) =$  $= 0. \quad (2)$

In our notation this encodes a sum over all (i, j, k); instead restricting to fixed (i, j, k) corresponds to taking a coefficient of the differential operator (cf. Lemma 5), which yields the respective component of the Jacobiator. Clearly, the Jacobiator $\mathrm{Jac}(\mathcal{P})$ is totally skew-symmetric with respect to its arguments *1, 2, 3*.

From now on, let us consider only the oriented graphs whose vertices are either sinks (with no issued edges) like the vertices *1, 2, 3* in (2) or tails for an ordered pair of arrows, each decorated with its own index. (We refer to Jacobi identity (2), Figure 2 on p. 5, and to the graphical formulae on pp. 4, 10, 11, and 14). By definition, the arrowtail vertices are called *internal*. For each internal vertex $\bullet$, the pair of out-going edges is ordered L $\prec$ R. Next, every internal vertex $\bullet$ carries a copy of a given Poisson bi-vector $\mathcal{P} = \mathcal{P}^{ij}(\boldsymbol{x})\,\partial_i \wedge \partial_j$ with its own pair of indices. The ordering L $\prec$ R of decorated out-going edges coincides with the ordering "first $\prec$ second" of the indexes in the coefficients of $\mathcal{P}$. Namely, the left edge (L) carries the first index and the other edge (R) carries the second index. Moreover, we let the sinks be ordered (like *1, 2, 3* above), so that every such graph defines a polydifferential operator: its arguments are thrown into the respective sinks.

Remark 1. In principle, we allow the presence of both the tadpoles and cycles over two vertices (or "eyes"); see Figure 1. However, in hindsight there will be neither tadpoles nor eyes in the solution which we shall have found in section 2 below.

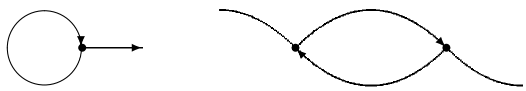
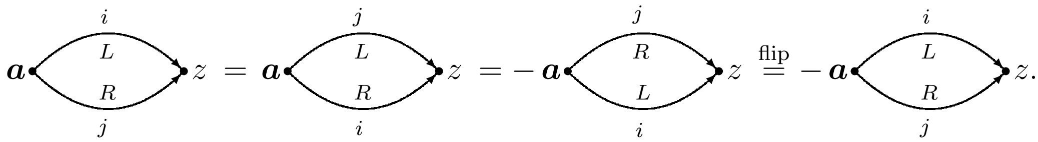
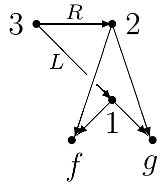


FIGURE 1. A tadpole and an "eye".

Remark 2. Under the above assumptions, there exist inhabited graphs that encode zero differential operators. Namely, consider the graph with a double edge:



By first relabelling the summation indices and then swapping L $\rightleftarrows$ R (and redrawing) we evaluate the operator acting at z to $\sum_{i,j=1}^{n} a^{ij}\partial_i\partial_j(z) = -\sum_{i,j=1}^{n} a^{ij}\partial_i\partial_j(z)$; whence the operator is zero. In the same way, any graph containing a double edge encodes a zero operator. Graphs can also encode zero differential operators in a more subtle way. For example consider the wedge on two wedges:



$$= 0. \qquad (3)$$

Swapping the labels $1 \rightleftarrows 2$ of the lower wedges does not change the operator. On the other hand, doing the same in a different way, namely, by swapping 'left' and 'right' in the top wedge introduces a minus sign. Hence the graph encodes a differential operator equal to minus itself, i.e. zero. Proving that a graph which contains the left-hand side of (3) as a subgraph equals zero is an elementary exercise (cf. Example 2 on p. 22).

Besides the trivial vanishing mechanism in Remark 2, there is Jacobi identity (2) together with its differential consequences, which will play a key role in what follows.

1.2. **The Kontsevich tetrahedral flow.** In the paper [14], Kontsevich proposed the construction of flows $\dot{\mathcal{P}} = \mathcal{Q}(\mathcal{P})$ on the spaces of Poisson structures on affine real manifolds. In particular, he associated one such flow on the space of Poisson bi-vectors $\mathcal{P}$ with the full graph over four vertices, that is, the tetrahedron. Up to symmetry, there are two essentially different ways, resulting in Γ_1 and Γ_2', to orient its edges, provided that every vertex is a source for two arrows and, as an elementary count suggests, there are two arrows leaving the tetrahedron that act on the arguments of the bi-differential operator which the tetrahedral graph encodes. The two oriented tetrahedral graphs are shown in Fig. 2. Unlike the operator encoded by Γ_1, that of Γ_2'

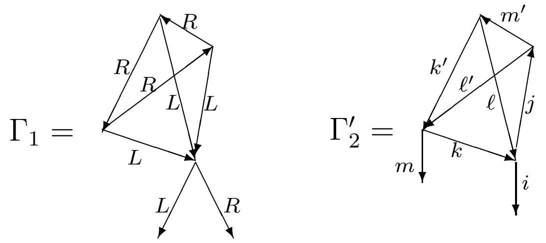


FIGURE 2. The Kontsevich tetrahedral graphs encode two bi-linear bi-differential operators on the product $C^\infty(N^n) \times C^\infty(N^n)$.

is generally speaking not skew-symmetric with respect to its arguments. By definition, put $\Gamma_2 := \frac{1}{2}(\Gamma_2'(1,2) - \Gamma_2'(2,1))$ to extract the antisymmetric part, that is, the bi-vector encoded by Γ_2'. Explicitly, the quartic-nonlinear differential polynomials $\Gamma_1(\mathcal{P})$ and $\Gamma_2(\mathcal{P})$, depending on a Poisson bi-vector $\mathcal{P}$, are given by the formulae

$$\Gamma_1(\mathcal{P}) = \sum_{i,j=1}^{n} \left(\sum_{k,\ell,m,k',\ell',m'=1}^{n} \frac{\partial^3 \mathcal{P}^{ij}}{\partial x^k \partial x^\ell \partial x^m} \frac{\partial \mathcal{P}^{kk'}}{\partial x^{\ell'}} \frac{\partial \mathcal{P}^{\ell\ell'}}{\partial x^{m'}} \frac{\partial \mathcal{P}^{mm'}}{\partial x^{k'}} \right) \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} \tag{4a}$$

and

$$\Gamma_2(\mathcal{P}) = \sum_{i,m=1}^{n} \left(\sum_{j,k,\ell,k',\ell',m'=1}^{n} \frac{\partial^2 \mathcal{P}^{ij}}{\partial x^k \partial x^\ell} \frac{\partial^2 \mathcal{P}^{km}}{\partial x^{k'} \partial x^{\ell'}} \frac{\partial \mathcal{P}^{k'\ell}}{\partial x^{m'}} \frac{\partial \mathcal{P}^{m'\ell'}}{\partial x^j} \right) \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^m}, \tag{4b}$$

respectively. To construct a class of flows on the space of bi-vectors, Kontsevich suggested to consider linear combinations, balanced by using the ratio $a : b$, of the bi-vectors Γ_1 and Γ_2. We recall from section 1.1 that every internal vertex of each graph is inhabited by a copy of a given Poisson bi-vector $\mathcal{P}$, so that the linear combination of two graphs encodes the bi-vector $\mathcal{Q}_{a:b}(\mathcal{P}) = a \cdot \Gamma_1(\mathcal{P}) + b \cdot \Gamma_2(\mathcal{P})$, quartic in $\mathcal{P}$ and balanced using $a : b$. We now inspect at which ratio $a : b$ the bi-vector $\mathcal{P} + \varepsilon \mathcal{Q}_{a:b}(\mathcal{P}) + \bar{o}(\varepsilon)$ stays infinitesimally Poisson for $\varepsilon > 0$, that is (cf. Appendix A),

$$[\![\mathcal{P} + \varepsilon \mathcal{Q}_{a:b}(\mathcal{P}) + \bar{o}(\varepsilon), \mathcal{P} + \varepsilon \mathcal{Q}_{a:b}(\mathcal{P}) + \bar{o}(\varepsilon)]\!] = \bar{o}(\varepsilon). \tag{5}$$

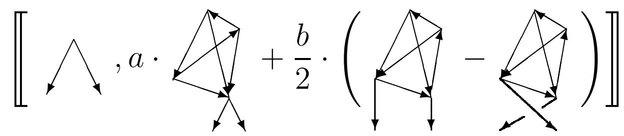
Expanding the left-hand side of equation (5), using the shifted-graded skew-symmetry of the Schouten bracket $[\![\cdot,\cdot]\!]$, and taking into account that $[\![\mathcal{P},\mathcal{P}]\!] = 0$ if and only if

$\mathcal{P}$ is Poisson, we extract the equation

$$[\![\mathcal{P}, \mathcal{Q}_{a:b}(\mathcal{P})]\!] \doteq 0 \qquad \text{via } [\![\mathcal{P},\mathcal{P}]\!] = 0. \tag{1}$$

The left-hand side of equation (1) can be seen in terms of graphs:

$$[\![\mathcal{P}, a\cdot\Gamma_1 + b\cdot\Gamma_2]\!] =$$



(6)

Remark 3. The graphical calculation of the Schouten bracket $[\![\ ,\ ]\!]$ of two arguments amounts to the action –via the Leibniz rule– of every out-going edge in an argument on all the internal vertices in the other argument. For the Schouten bracket of a k-vector with an ℓ-vector, the rule of signs is this. For the sake of definition, enumerate the sinks in the first and second arguments by using $0, \ldots, k-1$ and $0, \ldots, \ell-1$, respectively. Then the arrow into the jth sink in the second argument acts on the internal vertices of the first argument, acquiring the sign factor $(-)^j$; here $0 \leqslant j < \ell$. On the other hand, the arrow to the ith sink in the first argument acts on the second argument's internal vertices with the sign factor $-(-)^{(k-1)-i}$ for $0 \leqslant i \leqslant k-1$. We finally recall that having a totally antisymmetric tri-vector in (6) means that a full skew-symmetrization over the three sinks' content is taken by using $\frac{1}{3!}\sum_{\sigma\in S_3}(-)^\sigma$.

For example, let $a : b = 1 : 6$ (specifically, $a = 1$ and $b = 6$). Then the left-hand side of (1) takes the shape depicted in Figure 3. After the skew-symmetrization and

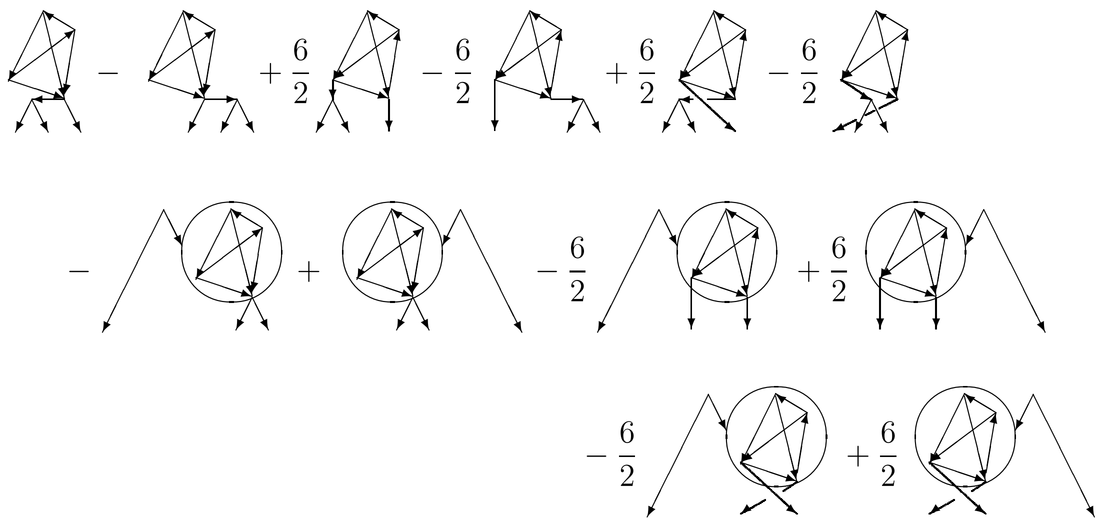


FIGURE 3. Incoming arrows act on the content of boxes via the Leibniz rule; to obtain the tri-vector, the entire picture must be skew-symmetrized over the content of three sinks.

expansion of all Leibniz rules, the sum in Figure 3 simplifies to 39 graphs; they are listed in Table 1 on p. 21 below.

The reason why we are particularly concerned with the ratio $a:b = 1:6$ is that this condition is *necessary* for equation (1) to hold.

Proposition 1 ([1])**.** The tetrahedral flow $\dot{P} = \mathcal{Q}_{a:b}(\mathcal{P})$ preserves the property of $\mathcal{P} + \varepsilon\mathcal{Q}_{a:b}(\mathcal{P}) + \bar{o}(\varepsilon)$ to be infinitesimally Poisson for all Poisson bi-vectors $\mathcal{P}$ on all affine real manifolds N^n *only if* the ratio is $a:b = 1:6$.

Our proof amounts to producing at least one counterexample when any ratio other than $1:6$ violates equation (1) for a given Poisson bi-vector $\mathcal{P}$.

Proof. Let x, y, z be the Cartesian coordinates on $\mathbb{R}^3$. Consider the Poisson bracket $\{u, v\}_{\mathcal{P}} = x \cdot \det\bigl(\partial(xyz + y, u, v)/\partial(x, y, z)\bigr)$ given by the Jacobian, so that the coefficient matrix is

$$\mathcal{P}^{ij} = \begin{pmatrix} 0 & x^2y & -x(xz+1) \\ -x^2y & 0 & xyz \\ -x(xz+1) & -xyz & 0 \end{pmatrix}.$$

The coefficient matrices of both bi-vectors are

$$\Gamma_1(\mathcal{P}) = 6 \cdot \begin{pmatrix} 0 & -x^5y & -x^4(xz+1) \\ x^5y & 0 & -x^3y \\ x^4(xz+1) & x^3y & 0 \end{pmatrix}, \qquad \Gamma_2(\mathcal{P}) = \begin{pmatrix} 0 & x^5y & x^4(xz+2) \\ -x^5y & 0 & -2x^3y \\ -x^4(xz+2) & 2x^3y & 0 \end{pmatrix}.$$

It is readily seen that no non-trivial linear combination $a \cdot \Gamma_1(\mathcal{P}) + b \cdot \Gamma_2(\mathcal{P})$ of the two flows vanishes everywhere on $\mathbb{R}^3 \ni (x, y, z)$ for this example. Acting on the bi-vectors Γ_1 and Γ_2 by the Poisson differential $[\![\mathcal{P}, \cdot]\!]$, we obtain two tri-vectors which are completely determined by one component each. Namely, we have that

$$[\![\mathcal{P}, \Gamma_1(\mathcal{P})]\!]^{123} = 36x^6yz + 48x^5y, \qquad [\![\mathcal{P}, \Gamma_2(\mathcal{P})]\!]^{123} = -6x^6yz - 8x^5y.$$

Clearly, the balance $a : b = 1 : 6$ is the only ratio at which the non-trivial linear combination $\mathcal{Q}_{a:b}(\mathcal{P}) = a \cdot \Gamma_1(\mathcal{P}) + b \cdot \Gamma_2(\mathcal{P})$ solves the equation $[\![\mathcal{P}, \mathcal{Q}_{a:b}(\mathcal{P})]\!] \equiv 0$. $\square$

1.3. **Main result.** In fact, more is known — this time, about the sufficiency of the condition $a : b = 1 : 6$. First, let us recall from [6] that on $\mathbb{R}^3$ with coordinates x, y, and z (almost) all Poisson brackets amount to

$$\{u, v\}_{\mathcal{P}} = f \cdot \det\left(\frac{\partial(g, u, v)}{\partial(x, y, z)}\right) \quad \text{for } u, v \in C^\infty(\mathbb{R}^3), \tag{7}$$

where the free parameter g is a function and the parameter f is a density so that

$$f(x, y, z) \cdot \det\left(\frac{\partial(g, u, v)}{\partial(x, y, z)}\right) \mathrm{d}x\mathrm{d}y\mathrm{d}z = f(x, y, z)\Big|_{\substack{x=x(x',y',z')\\ y=y(x',y',z')\\ z=z(x',y',z')}} \cdot \det\left(\frac{\partial(g, u, v)}{\partial(x', y', z')}\right) \mathrm{d}x'\mathrm{d}y'\mathrm{d}z'.$$

In any given coordinate system the parameter f can be chosen freely; then it is recalculated as shown above.

Proposition 2 ($\mathbb{R}^3$,$\{\cdot, \cdot\}_{\mathcal{P}}$)**.** The tetrahedral flow $\dot{\mathcal{P}} = \mathcal{Q}_{1:6}(\mathcal{P})$ does preserve the property of $\mathcal{P} + \varepsilon\mathcal{Q}_{a:b}(\mathcal{P}) + \bar{o}(\varepsilon)$ to be infinitesimally Poisson for all Poisson structures (7) on $\mathbb{R}^3$.

We use Proposition 2 merely as an heuristic motivation to our main Theorem 3 (see below) in which the claim from Proposition 2 is extended to *all* Poisson structures on all finite-dimensional affine real manifolds. Therefore, in hindsight, Proposition 2 above will have been proven rigorously as soon as Theorem 3 is established by the end of the next section. (In the meantime, a computer-assisted proof by direct calculation is provided for Proposition 2 in Appendix B.)

So, let us no longer restrict the tetrahedral flow $\mathcal{Q}_{1:6}(\mathcal{P})$ to any specific class of Poisson bi-vectors $\mathcal{P}$ but let us work in the full generality. We now examine the mechanism for the tri-vector $[\![\mathcal{P}, \mathcal{Q}_{1:6}(\mathcal{P})]\!]$ in (1) to vanish by virtue of the Jacobi identity $\text{Jac}(\mathcal{P}) := [\![\mathcal{P}, \mathcal{P}]\!] = 0$ for a given Poisson bi-vector $\mathcal{P}$ on N^n of any dimension $n \geqslant 3$. The task is to factorize the content of Figure 3 through the Jacobi identity in (2).

Theorem 3. *There exists a polydifferential operator*

$$\Diamond \in \text{PolyDiff}\,\big(\Gamma(\textstyle\bigwedge^2 TN^n) \times \Gamma(\bigwedge^3 TN^n) \to \Gamma(\bigwedge^3 TN^n)\big)$$

which solves the factorization problem

$$[\![\mathcal{P}, \mathcal{Q}_{1:6}(\mathcal{P})]\!] = \Diamond\,\big(\mathcal{P}, \text{Jac}(\mathcal{P})\big). \tag{8}$$

The polydifferential operator $\Diamond$ is realised using graphs in formula (9)*, see p.* 11 *below.*

Remark 4. Whenever a solution $\Diamond$ of (8) is found – and if it contains a reasonably small number of Leibniz-rule graphs as, e.g., our solution (9), see page 11 below – one can verify the factorization in (8) by a straightforward calculation. Indeed, by expanding the Leibniz rules and collecting similar terms, one obtains 39 graphs from the left-hand side (see Figure 3 and the encoding of those graphs in Table 1 on page 21).

Corollary 4 (Main result)**.** Whenever a bi-vector $\mathcal{P}$ on an affine real manifold N^n is Poisson, the deformation $\mathcal{P} + \varepsilon\mathcal{Q}_{1:6}(\mathcal{P}) + \bar{o}(\varepsilon)$ using the Kontsevich tetrahedral flow is infinitesimally Poisson.

Remark 5. It is readily seen that the Kontsevich tetrahedral flow $\dot{\mathcal{P}} = \mathcal{Q}_{1:6}(\mathcal{P})$ is well defined on the space of Poisson bi-vectors on a given affine manifold N^n. Indeed, it does not depend on a choice of coordinates up to their arbitrary affine reparametrisations. In other words, the velocity $\dot{\mathcal{P}}\big|_{\boldsymbol{u}\in N^n}$ does not depend on the choice of a chart $\mathcal{U} \ni \boldsymbol{u}$ from an atlas in which only *affine* changes of variables are allowed. (Let us remember that affine manifolds can of course be topologically nontrivial.)

Suppose however that a given affine structure on the manifold N^n is extended to a larger atlas on it; for the sake of definition let that atlas be a smooth one. Assume that the smooth structure is now reduced – by discarding a number of charts – to another affine structure on the same manifold. The tetrahedral flow $\dot{\mathcal{P}} = \mathcal{Q}_{1:6}(\mathcal{P})$ which one initially had can be contrasted with the tetrahedral flow $\dot{\tilde{\mathcal{P}}} = \mathcal{Q}_{1:6}(\tilde{\mathcal{P}})$ which one finally obtains for the Poisson bi-vector $\tilde{\mathcal{P}}\big|_{\tilde{\boldsymbol{u}}(\boldsymbol{u})} = \mathcal{P}\big|_{\boldsymbol{u}}$ in the course of a nonlinear change of coordinates on N^n. Indeed, the respective velocities $\dot{\mathcal{P}}$ and $\dot{\tilde{\mathcal{P}}}$ can be different whenever they are expressed by using essentially different parametrisations of a neighbourhood of a point $\boldsymbol{u}$ in N^n. For example, the tetrahedral flow vanishes identically when expressed in the Darboux canonical variables on a chart in a symplectic manifold. But after a nonlinear canonical transformation, the right-hand side $\mathcal{Q}_{1:6}(\tilde{\mathcal{P}})$ can become nonzero at the same points of that Darboux chart.

This shows that an affine structure on the manifold N^n is a necessary part of the input data for construction of the Kontsevich tetrahedral flows $\dot{\mathcal{P}} = \mathcal{Q}_{1:6}(\mathcal{P})$.

2. Solution: From graphs to polydifferential operators

Expanding the Leibniz rules in $[\![\mathcal{P}, \mathcal{Q}_{1:6}(\mathcal{P})]\!]$, we obtain the sum of 39 graphs with 5 internal vertices and 3 sinks (so that from Figure 3 we produce Table 1, see page 21 below). By construction, the Schouten bracket $[\![\mathcal{P}, \mathcal{Q}_{1:6}(\mathcal{P})]\!] \in \Gamma(\bigwedge^3 TN^n)$ is a tri-vector on the underlying manifold N^n, that is, it is a totally antisymmetric tri-linear polyderivation $C^\infty(N^n)\times C^\infty(N^n)\times C^\infty(N^n) \to C^\infty(N^n)$. At the same time, we seek to recognize the tri-vector $[\![\mathcal{P}, \mathcal{Q}_{1:6}(\mathcal{P})]\!]$ as the result of application of the (poly)differential operator $\Diamond$ (see (8) in Theorem 3) to the Jacobiator $\operatorname{Jac}(\mathcal{P})$ (see (2) on p. 3).

We now explain how the operator $\Diamond$ is found by using the method of undetermined coefficients in an expansion of all relevant graphical differential consequences of the Jacobi identity.[2] By construction, the left-hand side of every such differential consequence is a sum of graphs with 5 internal vertices, of which 2 belong to the Jacobiator $\operatorname{Jac}(\mathcal{P})$. We recall that for strictly positive differential order consequences of the Jacobi identity $\operatorname{Jac}(\mathcal{P}) = 0$, the mechanism for operator $\Diamond$ to attain zero value at $\operatorname{Jac}(\mathcal{P}) = 0$ is non-trivial. In fact, it refers to a (possibility of) splitting of every such consequence into the fragments which vanish independently from each other.

Lemma 5 ([2])**.** A tri-differential operator $C = \sum_{|I|,|J|,|K|\geqslant 0} c^{IJK}\ \partial_I \otimes \partial_J \otimes \partial_K$ with coefficients $c^{IJK} \in C^\infty(N^n)$ vanishes identically iff all its homogeneous components $C_{ijk} = \sum_{|I|=i,|J|=j,|K|=k} c^{IJK}\ \partial_I \otimes \partial_J \otimes \partial_K$ vanish for all differential orders (i, j, k) of the respective multi-indices (I, J, K); here $\partial_L = \partial_1^{\alpha_1} \circ \cdots \circ \partial_n^{\alpha_n}$ for a multi-index $L = (\alpha_1, \ldots, \alpha_n)$.

In practice, Lemma 5 states that for every arrow falling on the Jacobiator (for which, in turn, a triple of arguments is specified), the expansion of the Leibniz rule yields four fragments which vanish separately. Namely, there is the fragment such that the derivation acts on the content $\mathcal{P}$ of the Jacobiator's two internal vertices, and there are three fragments such that the arrow falls on the first, second, or third argument of the Jacobiator. It is readily seen that the action of a derivative on an argument of the Jacobiator effectively amounts to an appropriate redefinition of its respective argument. Therefore, a restriction to the order $(1, 1, 1)$ is enough in the run-through over all the graphs which contain Jacobiator (2) and which stand on the three arguments f, g, h of the tri-vector $\Diamond(\mathcal{P}, \operatorname{Jac}(\mathcal{P}))$.

Remark 6. In all the above reasoning, the set $\{\mathit{1}, \mathit{2}, \mathit{3}\}$ of three arguments of the Jacobiator need not coincide with the set $\{f, g, h\}$ of the arguments of the tri-vector $\Diamond(\mathcal{P}, \operatorname{Jac}(\mathcal{P}))$. Of course, the two sets can intersect; this will provide a natural filtration for the components of solution (9). Namely, the number of elements in the intersection runs from three for the first term to zero in the second or third graph.

[2]Another method for solving the factorization problem is outlined in Appendix E.

In fact, Remark 6 reveals a highly nontrivial role of the operator $\Diamond$ in (8). Indeed, some of the three internal vertices of its graphs can be arguments of $\mathrm{Jac}(\mathcal{P})$ whereas some of the other such vertices (if any) can be tails for the arrows falling on $\mathrm{Jac}(\mathcal{P})$. In retrospect, the two subsets of such vertices of $\Diamond$ do not intersect; every vertex in the intersection, if it were nonempty, would produce a two-cycle, but there are no "eyes" in (9).

By ordering the Leibniz-rule graphs in the operator $\Diamond$ according to the number of Jacobiator's arguments which simultaneously are the arguments of (totally skew-symmetric) tri-vector $[\![\mathcal{P}, \mathcal{Q}_{1:6}(\mathcal{P})]\!] = \Diamond(\mathcal{P}, \mathrm{Jac}(\mathcal{P}))$, we count the number of variants in the run-through over all the admissible graphs. (With reference to Fig. 4 below, this is done in Appendix C, see p. 19.) In total, there are 1132 variants.

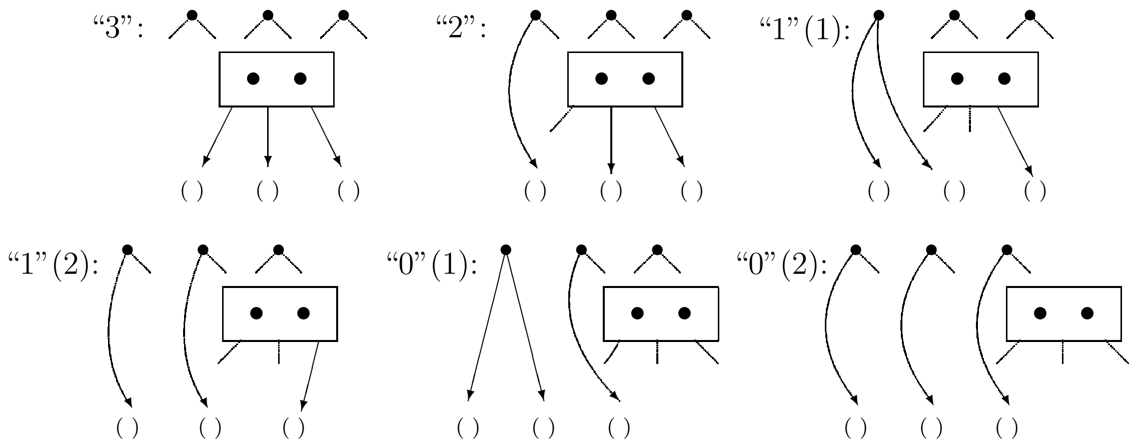


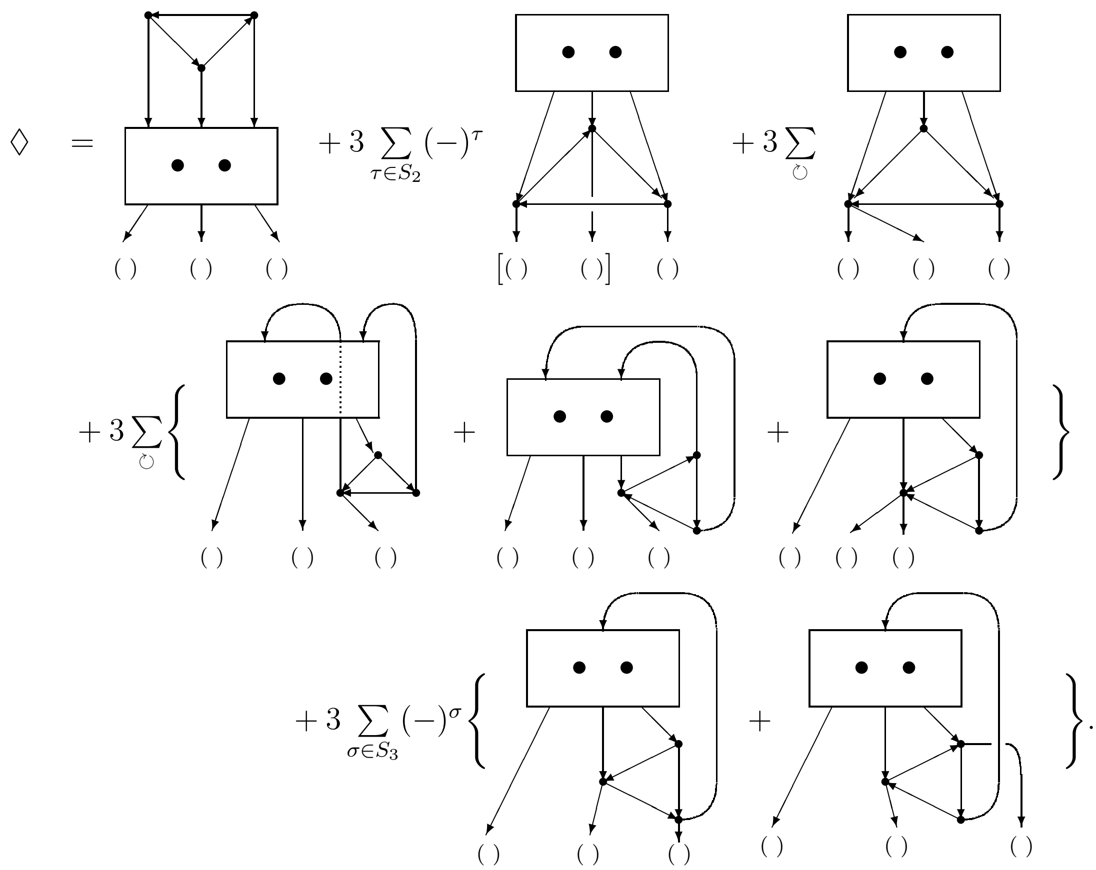
FIGURE 4. This is the list of all different types of differential consequences of the Jacobi identity which are linear in the Jacobiator and which are totally skew-symmetric with respect to the sinks. The list is ordered by the number of ground vertices on which the Jacobiator stands. The number of graphs for each type is deduced in Appendix C: namely, from top-left to bottom-right, there are 216, 432, 108, 288, 24, and 64 Leibniz-rule graphs. The total number of differential consequences is 1132.

We now split all these differential consequences of the Jacobi identity $\mathrm{Jac}(\mathcal{P}) = 0$ by using Lemma 5 (with respect to the *total* differential order (i, j, k) for arguments of $\mathrm{Jac}(\mathcal{P})$ if more than one arrow falls on it), ascribing an undetermined coefficient to every such separately vanishing fragment. That is, we do not restrict only to the differential order (1,1,1) with respect to the arguments of $\mathrm{Jac}(\mathcal{P})$ for every number of derivations acting on the Jacobiator; we agree that this way to introduce the undetermined coefficients is not minimal. However, we always restrict to the order $(1, 1, 1)$ with respect to (f, g, h). We thus have $28,202$ unknowns introduced (counted with possible repetitions of graphs which they refer to).[3] Now we expand all the Leibniz rules that

[3]The relevant algebra of sums of graphs modulo skew-symmetry and the Jacobi identity has been realized in software by the second author. An implementation of those tools in the problem of high-order expansion of the Kontsevich $\star$-product will be explained in a separate paper [3].

run over the internal vertices in every Jacobiator; simultaneously, the object $\mathrm{Jac}(\mathcal{P})$ is expanded using formula (2). As soon as we take into account the order $L \prec R$ and the antisymmetry of graphs under the reversion of that ordering at an internal vertex, the graphs that encode zero differential operators are eliminated. There remain $7,025$ admissible graphs with 5 internal vertices and 3 sinks; the coefficient of every such graph is a linear combination of the undetermined coefficients of the splinters which the Leibniz-rule graphs (see Figure 4) produced from $\mathrm{Jac}(\mathcal{P})$. In conclusion, we view (8) as the system of 7025 linear inhomogeneous equations for the coefficients of graphs in the operator $\Diamond$. Solving this linear system is a way towards a proof of our main results (which is expressed in Corollary 4); The process of finding a solution $\Diamond$ itself does not constitute that proof. Therefore, the justification of the claim in Theorem 3 will be performed separately.

In the meantime, using software tools, we solve this linear algebraic system at hand. The duplications of graph labellings are conveniently eliminated by our request for the program to find a solution with a minimal number of nonzero components. Totally antisymmetric in tri-vector's arguments, the solution consists of 27 Leibniz-rule graphs, which are assimilated into the sum of 8 manifestly skew-symmetric terms as follows:



(9)

To display the $L \prec R$ ordering at every internal vertex and to make possible the arithmetic and algebra of graphs, we use the notation which is explained in Appendix D below.

Proof of Theorem 3. So far, we have constructed operator (9); We emphasize that it is completely determined by as few as only eight integer coefficients. This permits a rigorous proof of our main claim: namely, let us show that operator (9) does satisfy equation (8).

First expand the sums in (9), which gives us 27 Leibniz graphs. Now expand all the Leibniz rules; this yields the sum of 201 Kontsevich graphs with 3 sinks and 5 internal vertices: together with their coefficients, they are listed in Table 3 in Appendix D, see page 20. Clearly, manipulating that number of graphs is still possible for man.

Because we are free to enumerate the five internal vertices in every graph in a way we like, and because the ordering of every pair of outgoing edges is also under our control, at once do we bring all the graphs to their normal form.[4]

It is readily seen that there are many repetitions in Table 3. Now, collect the similar terms. There remain only 39 terms with nonzero coefficients. One verifies that those 39 terms are none other than the entries of Table 1, that is, realizations of the 39 graphs in the left-hand side of (8). This shows that equation (8) holds for the operator $\Diamond$ contained in (9). The proof is complete. $\square$

3. Properties of the found solution

Remark 7. Let us recall that equation (1) yields the linear system of 7,025 inhomogeneous equations for the coefficients of 1132 patterns from Fig. 4. This shows that the algebraic system at hand is extremely overdetermined. Moreover, out of those 1132 admissible totally antisymmetric graphs, solution (9) involves only 8 of them. In this sense, the factorising operator $\Diamond$ in (1) is special; for it expands via (9) over a very low dimensional affine subspace in the affine space of unknowns in that inhomogeneous linear algebraic system.

Property 1. The relevant Leibniz-rule graphs, with respect to which the solution $\Diamond(\mathcal{P},\,\cdot\,)$ expands, do not contain tadpoles nor two-cycles (or "eyes", see Fig. 1 on p. 4).
• None of the arrows that act back on the Jacobiator is issued from any of its arguments.
• In all the graphs the source vertices (if any), on which no arrows fall after all the Leibniz rules are expanded, belong to the Jacobiator (cf. (2) on p. 3).

Property 2. The found solution $\Diamond$ does contain the graphs in which two or three arrows fall on the Jacobiator.[5]

It has been explained in [8, 10] that the existence of two or more such arrows falling on the equation $[\![\mathcal{P},\mathcal{P}]\!]=0$ is an obstruction to an extension of the main claim,

$$[\![\mathcal{P},\mathcal{Q}_{1:6}(\mathcal{P})]\!] \doteq 0 \quad \text{via } [\![\mathcal{P},\mathcal{P}]\!]=0, \tag{1}$$

[4]The normal form of a graph is obtained by running over the group $S_5\times(\mathbb{Z}_2)^5$ of all the relabellings of internal vertices and swaps $L\rightleftarrows R$ of orderings at each vertex. (We recall that every swap negates the coefficient of a graph; the permutations from S_5 are responsible for encoding a given topological profile in seemingly "different" ways.) By definition, the normal form of a graph is the sign (times coefficient) followed by the minimal sequence of five ordered pairs of target vertices viewed as 10-digit base-$(3+5)$ numbers. (By convention, the three ordered sinks are enumerated $0,1,2$ and the internal vertices are the octonary digits $3,\ldots,7$.)

[5]For instance, the first term in $\Diamond$ is the tripod standing on $\mathrm{Jac}(\mathcal{P})$.

to the infinite-dimensional geometry of jet spaces $J^\infty(\pi)$ for affine bundles over a manifold M^m or jet spaces $J^\infty(M^m \to N^n)$ of maps from M^m, and of variational Poisson brackets $\{\,,\,\}_{\boldsymbol{\mathcal{P}}}$ for functionals on such jet spaces (see [19, 7] and [9, 10]). Namely, it can then be that

$$[\![\boldsymbol{\mathcal{P}}, \boldsymbol{\mathcal{Q}}_{1:6}(\boldsymbol{\mathcal{P}})]\!] \ncong 0 \quad \text{through } [\![\boldsymbol{\mathcal{P}}, \boldsymbol{\mathcal{P}}]\!] \cong 0. \tag{10}$$

We denote here by $[\![\,,\,]\!]$ the variational Schouten bracket; the variational bi-vector $\boldsymbol{\mathcal{Q}}_{1:6}$ is constructed from the variational Poisson bi-vector $\boldsymbol{\mathcal{P}}$ by using techniques from the geometry of iterated variations of functionals (see [8, 9, 10]). An explicit counterexample of (10) is known from [1] for the variational Poisson structure of the Harry Dym partial differential equation.

The reason why the obstruction arises is that in the variational setting, the second and higher order variations of a trivial integral functional $\text{Jac}(\boldsymbol{\mathcal{P}}) \cong 0$ in the horizontal cohomology can still be nonzero (although its first variation would of course vanish).[6]

Remark 8. Uniqueness is currently not claimed for the found solution $\Diamond(\mathcal{P}, \cdot)$. The eight graphs in (9) represent a *linear* differential operator with respect to the Jacobiator $\text{Jac}(\mathcal{P})$. However, a quadratic nonlinearity with respect to the two-vertex argument $\text{Jac}(\mathcal{P})$ could be hidden in the five-vertex graphs in formula (9), so that it would in fact encode a bi-differential operator $\Diamond(\mathcal{P}, \cdot, \cdot)$. If this be the case, expansion of one or the other copy of the Jacobiator using (2) in such a polydifferential operator $\Diamond(\mathcal{P}, \cdot, \cdot)$ would produce two seemingly distinct linear differential operators $\Diamond(\mathcal{P}, \cdot)$.

The scenarios to build the bi-linear, bi-differential terms in the operator $\Diamond$ are drawn in Figure 5 below. We consider −in fact, without any loss of generality− only those 8 Leibniz-rule graphs in which

- the three arguments of each copy of Jacobiator (2) are different; in particular,
- neither of the Jacobiators acts on the other copy by two or three arrows (so that only none or one such arrow is possible).

We recall that known solution (9) is the sum of 39 graphs from which a linear dependence on the Jacobiator $\text{Jac}(\mathcal{P})$ is retrieved by using the 27 Leibniz-graphs (see Table 2 on p. 22). Let us inspect whether solution (9) is just linear in $\text{Jac}(\mathcal{P})$ or there is a bi-linear dependence in $\text{Jac}(\mathcal{P})$ hidden in (9).

To this end, we took (with undetermined coefficients) the 27 Leibniz-graphs from (9), which are linear in $\text{Jac}(\mathcal{P})$, and the 8 skew-symmetrized new patters from Fig. 5 (resp., quadratic in $\text{Jac}(\mathcal{P})$). By equating their sum to zero and expanding all the Leibniz rules using the tool [3], we examined the arising system of linear algebraic equations. Due to the presence of homogeneous equations which involve only one unknown, specifically, the coefficient of a new Leibniz-graph from Fig. 5, and by noting that such is the case for every graph from that set, we conclude that the general solution of the homogeneous problem is necessarily linear in the Jacobiator, whence the non-existence of a quadratic part in (9) is manifest. Our computer-assisted reasoning motivates the following claim.

[6]The same effect has been foreseen for a variational lift of deformation quantisation [15]: it has been argued in [10] why the associativity of noncommutative star-product $\star = \times + \hbar\{\,\cdot\,,\,\cdot\,\}_{\boldsymbol{\mathcal{P}}} + \bar{o}(\hbar)$ can leak and it has been shown in [2] that if it actually does at $O(\hbar^k)$, the order k at which this leak of associativity can occur is high: $k \geqslant 4$.

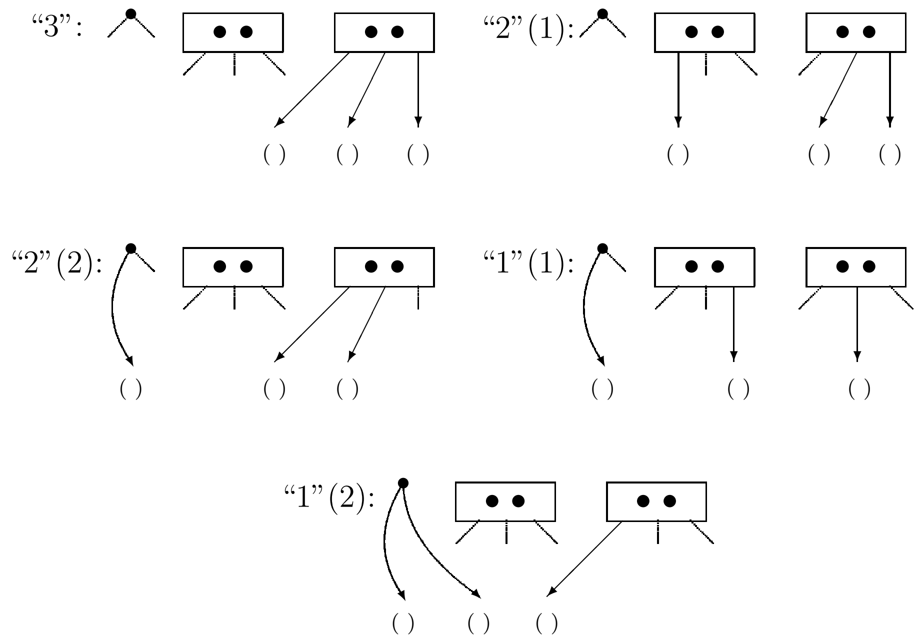


FIGURE 5. The Leibniz-graphs by using which a quadratic – with respect to the Jacobiator – part $\Diamond(\mathcal{P},\cdot,\cdot)$ of the factorizing operator could be sought for in (8); such quadratic part (if any) itself is necessarily totally skew-symmetric with respect to the three sinks ( ).

Conjecture 6. There is no quadratic part in all the solutions of equation

$$[\![\mathcal{P}, \mathcal{Q}_{1:6}(\mathcal{P})]\!] = \Diamond\left(\mathcal{P}, \operatorname{Jac}(\mathcal{P}), \operatorname{Jac}(\mathcal{P})\right) \tag{8'}$$

that expand with respect to the 39 graphs in (9).

Still it could be for equation (8′) that a quadratic dependence of $\Diamond$ on $\operatorname{Jac}(\mathcal{P})$ is established for a solution $\Diamond$ which differs from any operator $\Diamond(\mathcal{P},\cdot)$ that expands only with respect to the graphs contained in (9).

4. DISCUSSION

4.1. For the factorisation $[\![\mathcal{P}, \mathcal{Q}_{1:6}(\mathcal{P})]\!] = \Diamond\big(\mathcal{P}, \operatorname{Jac}(\mathcal{P})\big)$ to guarantee that the equality $\boldsymbol{\partial}_{\mathcal{P}}\big(\mathcal{Q}_{1:6}(\mathcal{P})\big) = 0$ holds if $\operatorname{Jac}(\mathcal{P}) = 0$, its mechanism is nontrivial. Relying on Lemma 5 (see [2]), it tells us how the differential consequences of Jacobi identity are split into separately vanishing expressions. This mechanism works not only in the construction of flows that satisfy (1) but also in the associativity,

$$\operatorname{Assoc}_{\mathcal{P}}(f,g,h) := (f \star g) \star h - f \star (g \star h) \doteq 0 \quad \text{via } [\![\mathcal{P},\mathcal{P}]\!] = 0,$$

of the non-commutative unital star-product $\star = \times + \hbar\{\,\cdot\,,\,\cdot\,\}_{\mathcal{P}} + o(\hbar)$. The formula for $\star$-products was given in [15], establishing the deformation quantisation $\times \mapsto \star$ of the usual product $\times$ in the algebra $C^\infty(N^n) \ni f,g,h$ on a finite-dimentional affine Poisson manifold $(N^n, \mathcal{P})$, see also [2, 10]. In fact, the construction of graph complex and the pictorial language of graphs [14, 15] that encode polydifferential operators is common to all these deformation procedures (cf. [3], also [18]).

Open problem 1. Consider the Kontsevich star-product $\star = \times + \hbar\{\,\cdot\,,\,\cdot\,\}_{\mathcal{P}} + o(\hbar)$ in the algebra $C^\infty(N^n)[[\hbar]]$ on a finite-dimensional affine Poisson manifold $(N^n, \mathcal{P})$. Given by the tetrahedra Γ_1 and Γ_2' (see Fig. 2 on p. 5), the infinitesimal deformation $\mathcal{P} \mapsto \mathcal{P} + \varepsilon\mathcal{Q}_{1:6}(\mathcal{P}) + o(\varepsilon)$ induces the infinitesimal deformation $\star \mapsto \star + \hbar\varepsilon\,[\![[\![\mathcal{Q}_{1:6}(\mathcal{P}),\,\cdot\,]\!],\,\cdot\,]\!] + o(\varepsilon)$ of the star-product. What are the properties of this infinitesimally deformed $\star(\varepsilon)$-product? In particular, is the condition that $\mathcal{Q}_{1:6}(\mathcal{P})$ be $\boldsymbol{\partial}_{\mathcal{P}}$-trivial necessary for the $\star(\varepsilon)$-product to be gauge-equivalent to the unperturbed $\star$-product at $\varepsilon = 0$?

We recall that the theory of (infinitesimal) deformations of associative algebra structures is very well studied in the broadest context (e.g., of the Yang–Baxter equation, Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) equation, Frobenius manifolds and F-structures, etc.), see [17]. We expect that in that theory's part which is specific to the deformation of associative structures on finite-dimensional affine Poisson manifolds N^n, there must be a dictionary between the construction of Kontsevich flows for spaces of Poisson bi-vectors and other instruments to deform the associative product in the algebra $C^\infty(N^n)$.

4.2. The Kontsevich tetrahedral flow $\dot{\mathcal{P}} = \mathcal{Q}_{1:6}(\mathcal{P})$ is a universal procedure to deform a given Poisson bi-vector $\mathcal{P}$ on any finite-dimensional affine real manifold N^n (i. e. not necessarily topologically trivial). The infinitesimal deformation $\mathcal{P} \mapsto \mathcal{P} + \varepsilon\mathcal{Q}_{1:6}(\mathcal{P}) + o(\varepsilon)$ can be completed to the construction of Poisson bi-vector $\mathcal{P}(\varepsilon)$ such that $\mathcal{P}(\varepsilon = 0) = \mathcal{P}$ and $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\big|_{\varepsilon=0}\mathcal{P}(\varepsilon) = \mathcal{Q}_{1:6}(\mathcal{P})$ if the third Poisson cohomology group $H^3_{\mathcal{P}}(N^n)$ with respect to the Poisson differential $\boldsymbol{\partial}_{\mathcal{P}} = [\![\mathcal{P},\,\cdot\,]\!]$ vanishes for the manifold N^n (see Appendix A below). In the symplectic case, i. e. for n even and bracket $\{\,\cdot\,,\,\cdot\,\}_{\mathcal{P}}$ nondegenerate, the Poisson complex is known to be isomorphic to the de Rham complex for N^n (see [16]). We are not yet aware of any way to constrain the Poisson cohomology groups $H^k_{\mathcal{P}}(N^n)$ for *degenerate* Poisson brackets $\{\,\cdot\,,\,\cdot\,\}_{\mathcal{P}}$ on real manifolds N^n of not necessarily even dimension $n < \infty$. (E.g., the algorithm for construction of cubic Poisson brackets on the basis of a class of R-matrices, which is explained in [16], yields a rank-six bracket on $N^9 \subset \mathbb{R}^9$.)

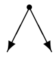
4.3. The second Poisson cohomology group $H^2_{\mathcal{P}}(N^n)$ of the manifold N^n, if nonzero, provides room for the $\boldsymbol{\partial}_{\mathcal{P}}$-nontrivial deformations of $\mathcal{P}$ using $\mathcal{Q}_{1:6}(\mathcal{P})$ such that $\mathcal{Q}_{1:6}(\mathcal{P}) \neq [\![\mathcal{P}, \mathcal{X}]\!]$ for all globally defined 1-vectors $\mathcal{X}$ on N^n. In particular, this implies that there are no $\boldsymbol{\partial}_{\mathcal{P}}$-nontrivial tetrahedral graph flows on even-dimensional star-shaped domains equipped with nondegenerate Poisson brackets.

A possibility for the right-hand side $\mathcal{Q}_{1:6}(\mathcal{P})$ of the tetrahedral flow to be $\boldsymbol{\partial}_{\mathcal{P}}$-trivial is thus a global, topological effect; it cannot always be seen within a single chart in N^n. Moreover, it is not universal with respect to the calculus of graphs.

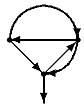
Claim 7. *In contrast with Theorem* 3, *there is no dimension-independent $\boldsymbol{\partial}_{\mathcal{P}}$-triviality mechanism which would be expressed for the tetrahedral flow $\dot{\mathcal{P}} = \mathcal{Q}_{1:6}(\mathcal{P})$ in terms of the Kontsevich graphs* (*see* §1.1 *and* [14, 15]) *and hence, which would be universal*[7] *with respect to all Poisson structures $\mathcal{P}$ on all finite-dimensional affine manifolds N^n.*

[7]Kontsevich notes [14] that if $n = 2$ so that every bi-vector $\mathcal{P}$ on N^2 is Poisson and every flow $\dot{\mathcal{P}} = \mathcal{Q}_{a:b}(\mathcal{P})$ preserves this property, the tetrahedron Γ_1 (or, equivalently, the velocity $\mathcal{Q}_{1:0}(\mathcal{P})$) is

Proof. Indeed, consider the $\boldsymbol{\partial}_{\mathcal{P}}$-coboundary equation,

$$\Gamma_1(\mathcal{P}) + 6\,\Gamma_2(\mathcal{P}) = [\![$$  $$, \ \mathcal{X}]\!],$$

where the graphs Γ_1 and Γ_2', inhabited by a copy of the Poisson bi-vector $\mathcal{P}$ in every internal vertex, are shown in Fig. 2 on p. 5. Because there are four copies of $\mathcal{P}$ in each tetrahedron and the Λ-graph in $\boldsymbol{\partial}_{\mathcal{P}} = [\![\mathcal{P}, \cdot\,]\!]$ contains one copy of the bi-vector $\mathcal{P}$, the number of internal vertices in the 1-vector $\mathcal{X}$ must be equal to 3. Likewise, we recall that neither there are tadpoles in both $\mathcal{P}$ and $\mathcal{Q}(\mathcal{P})$ nor does the Poisson differential $[\![\mathcal{P}, \cdot]\!]$ destroy any tadpoles (cf. Remark 3 on page 6); Therefore, the graph that encodes $\mathcal{X}$ may not contain any tadpoles. The only such Kontsevich graph with three internal vertices but without tadpoles is

$$\mathcal{X} = \text{const} \cdot$$  .

Now it is readily seen that the Schouten bracket of $\mathcal{X}$ with the Poisson bi-vector $\mathcal{P}$ does contain a source vertex (to which no arrows arrive). But there is no such vertex in either of the tetrahedra within the bi-vector $\mathcal{Q}_{1:6}(\mathcal{P})$ in the left-hand side of the $\boldsymbol{\partial}_{\mathcal{P}}$-cocycle equation $\mathcal{Q}_{1:6}(\mathcal{P}) = \boldsymbol{\partial}_{\mathcal{P}}(\mathcal{X})$. This shows that there is no universal solution $\mathcal{X}(\mathcal{P})$ expressed for all $\mathcal{P}$ in terms of graphs. ☐

Remark 9. The same reasoning works for all the Kontsevich graph flows such that none of the graphs besides the bi-vector $\mathcal{P}$ itself contains a source vertex (that is, neither any graph in the flow nor the 1-vector $\mathcal{X}$).

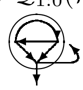
Open problem 2. The formalism developed in [14] suggests that there are, most likely, infinitely many Kontsevich graph flows on the spaces of Poisson bi-vectors on finite-dimensional affine Poisson manifolds. Forming an example $\mathcal{Q}_{1:6}(\mathcal{P})$ of such a cocycle in the graph complex, the tetrahedra Γ_1 and Γ_2' in Fig. 2 are built over four internal vertices. What is or are the next – with respect to the ordering of natural numbers – Kontsevich graph cocycle(s) built over five or more internal vertices?

4.4. The tetrahedral flow $\dot{\mathcal{P}} = \mathcal{Q}_{1:6}(\mathcal{P})$ preserves the space $\{\mathcal{P} \in \Gamma(\bigwedge^2 TN^n) \mid [\![\mathcal{P}, \mathcal{P}]\!] = 0\}$ of Poisson bi-vectors; this is guaranteed by Theorem 3 that asserts $\boldsymbol{\partial}_{\mathcal{P}}(\mathcal{Q}_{1:6}) \doteq 0$ within the (graded-)commutative geometry of finite-dimensional affine real manifolds N^n.

Open problem 3. Does the proven property,

$$[\![\mathcal{P}, \mathcal{Q}_{1:6}(\mathcal{P})]\!] \doteq 0 \quad \text{via } [\![\mathcal{P}, \mathcal{P}]\!] = 0, \tag{1}$$

generalize to the formal noncommutative symplectic supergeometry [11], to the calculus of multivectors performed by using their necklace brackets (see [9] and references

always $\partial_{\mathcal{P}}$-exact. The required 1-vector field $\mathcal{X}(\mathcal{P})$ in the coboundary statement $\mathcal{Q}_{1:0}(\mathcal{P}) = [\![\mathcal{P}, \mathcal{X}]\!]$ can be expressed in terms of the bi-vector $\mathcal{P}$, e.g., by the Leibniz-rule graph $\mathcal{X} =$ . (This is a particular, not general solution.) We recall that after the dimension n is fixed (here $n = 2$), a given differential polynomial in $\mathcal{P}$ can be encoded by the Kontsevich graphs in non-unique way. Details will be discussed elsewhere.

therein), and to Poisson structures on the commutative non-associative unital algebras of cyclic words (e. g., see [20]) ?

Appendix A. The Poisson cohomology

Let us recall several necessary facts from the deformation theory; this material is standard [5]. Denote by ξ_i the parity-odd canonical conjugate of the variable x^i for every $i = 1, \ldots, n$ (see [9] for discussion about the reverse parity symplectic duals). Every bi-vector is then realised in terms of the local coordinates x^i and ξ_i on $\Pi T^* N^n$ by using $\mathcal{P} = \frac{1}{2}\langle \xi_i \, \mathcal{P}^{ij}(\boldsymbol{x}) \, \xi_j \rangle$. We denote by $[\![\cdot,\cdot]\!]$ the Schouten bracket, i.e. the parity-odd Poisson bracket which is locally determined on $\Pi T^*\mathbb{R}^n$ by the canonical symplectic structure $\mathrm{d}\boldsymbol{x} \wedge \mathrm{d}\boldsymbol{\xi}$ (see [8] for details). Our working formula is[8]

$$[\![\mathcal{P}, \mathcal{Q}]\!] = (\mathcal{P})\frac{\overleftarrow{\partial}}{\partial x^i} \cdot \frac{\overrightarrow{\partial}}{\partial \xi_i}(\mathcal{Q}) - (\mathcal{P})\frac{\overleftarrow{\partial}}{\partial \xi_i} \cdot \frac{\overrightarrow{\partial}}{\partial x^i}(\mathcal{Q}). \tag{11}$$

To be Poisson, a bi-vector $\mathcal{P}$ must satisfy the master-equation $[\![\mathcal{P}, \mathcal{P}]\!] = 0$, of which formula (2) is the component expansion with respect to the indices (i, j, k) in the tri-vector $[\![\mathcal{P}, \mathcal{P}]\!](\boldsymbol{x}, \boldsymbol{\xi})$.

Under an infinitesimal deformation $\mathcal{P}(\varepsilon) = \mathcal{P} + \varepsilon\mathcal{Q} + \bar{o}(\varepsilon)$ of the bi-vector $\mathcal{P}$ satisfying $[\![\mathcal{P}, \mathcal{P}]\!] = 0$, the bi-vector $\mathcal{P}(\varepsilon)$ remains Poisson only if $[\![\mathcal{P}(\varepsilon), \mathcal{P}(\varepsilon)]\!] = \bar{o}(\varepsilon)$, whence $[\![\mathcal{P}, \mathcal{Q}]\!] = 0$.

Remark 10. For a Poisson bi-vector $\mathcal{P}$, the operator $\boldsymbol{\partial}_{\mathcal{P}} = [\![\mathcal{P}, \cdot]\!]$ is readily seen to be a differential: by virtue of the Jacobi identity for the Schouten bracket $[\![\cdot,\cdot]\!]$ we have that $\boldsymbol{\partial}_{\mathcal{P}}^2 = 0$. Therefore, the leading order terms $\mathcal{Q}$ in the deformations $\mathcal{P}(\varepsilon) = \mathcal{P} + \varepsilon\mathcal{Q} + \bar{o}(\varepsilon)$ can be trivial in the second $\boldsymbol{\partial}_{\mathcal{P}}$-cohomology, meaning that $\mathcal{Q} = [\![\mathcal{P}, \mathcal{X}]\!]$ for some one-vector $\mathcal{X}$ (whence $[\![\mathcal{P}, [\![\mathcal{P}, \mathcal{X}]\!]]\!] \equiv 0$). Alternatively, for the $\boldsymbol{\partial}_{\mathcal{P}}$-cocycles $\mathcal{Q}$ which are not $\boldsymbol{\partial}_{\mathcal{P}}$-coboundaries, the flows $\mathcal{P}(\varepsilon)$ stay infinitesimally Poisson but leave the $\boldsymbol{\partial}_{\mathcal{P}}$-cohomology class of the Poisson bi-vector $\mathcal{P}$ at $\varepsilon = 0$.

For consistency, let us recall that generally speaking, not every infinitesimal deformation $\mathcal{P} \mapsto \mathcal{P} + \varepsilon\mathcal{Q} + \bar{o}(\varepsilon)$ of a Poisson bi-vector $\mathcal{P}$ can be completed to a Poisson deformation $\mathcal{P} \mapsto \mathcal{P} + \mathcal{Q}(\varepsilon)$ at all orders in ε. The obstructions are contained in the third $\boldsymbol{\partial}_{\mathcal{P}}$-cohomology group $\mathrm{H}^3_{\mathcal{P}} = \{\mathrm{T} \in \Gamma(\bigwedge^3 TN) \mid \boldsymbol{\partial}_{\mathcal{P}}(\mathrm{T}) = 0\} \mathbin{/} \{\mathrm{T} = \boldsymbol{\partial}_{\mathcal{P}}(\mathrm{R}), \mathrm{R} \in \Gamma(\bigwedge^2 TN)\}$. Indeed, cast the master-equation $[\![\mathcal{P} + \mathcal{Q}(\varepsilon), \mathcal{P} + \mathcal{Q}(\varepsilon)]\!] = 0$ for the Poisson deformation to the coboundary statement $[\![\mathcal{Q}(\varepsilon), \mathcal{Q}(\varepsilon)]\!] = \boldsymbol{\partial}_{\mathcal{P}}(-\mathcal{P} - 2\mathcal{Q}(\varepsilon))$, whence $\boldsymbol{\partial}_{\mathcal{P}}([\![\mathcal{Q}(\varepsilon), \mathcal{Q}(\varepsilon)]\!] \equiv 0$ by $\boldsymbol{\partial}_{\mathcal{P}}^2 = 0$. Therefore, the vanishing of the third $\boldsymbol{\partial}_{\mathcal{P}}$-cohomology group guarantees the existence of a power series solution $\mathcal{Q}(\varepsilon)$ to the cocycle-coboundary equation $[\![\mathcal{Q}(\varepsilon), \mathcal{Q}(\varepsilon)]\!] = -2\boldsymbol{\partial}_{\mathcal{P}}(\mathcal{Q}(\varepsilon))$: known to be a cocycle, the left-hand side has been proven to be a coboundary as well.

Remark 11. Nowhere above should one expect that the leading deformation term $\mathcal{Q}$ in $\mathcal{P}(\varepsilon) = \mathcal{P} + \varepsilon\mathcal{Q} + \bar{o}(\varepsilon)$ itself would be a Poisson bi-vector. This may happen for $\mathcal{Q}$ only incidentally.

[8] In the set-up of infinite jet spaces $J^\infty(\pi)$ (see [19] and [8, 9, 10]) the four partial derivatives in the formula for $[\![\cdot,\cdot]\!]$ become the variational derivatives with respect to the same variables, which now parametrise the fibres in the Whitney sum $\pi \times_{M^m} \Pi\widehat{\pi}$ of (super-)bundles over the m-dimensional base M^m.

Appendix B. Computer-assisted proof of Proposition 2

To verify the claim in Proposition 2 by direct calculation, it would take years for man still only a few seconds for a computer.[9] A computer-assisted proof of Proposition 2 is realized through running the script in Maple (see below). (All computations are done with the coefficient matrices of bi-vectors at hand. The bi-vectors are computed by using working formulas (4a) and (4b).) For the balanced flow we have:

```
FlowQ := proc (P, y, a, b)
description "Eval flow Q_a:b of q-dim bi-vector P.";
local i, j, q, A, F, G, B, T, C;
q := op(P)[1];
F := proc (i, j, k, l, m, n, p, r) options operator, arrow;
a*(diff(P[i, j], y[k], y[l], y[m]))*(diff(P[k, n], y[p]))
*(diff(P[l, p], y[r]))*(diff(P[m, r], y[n])) end proc;
G := proc (i, j, k, l, m, n, p, r) options operator, arrow;
b*(diff(P[i, j], y[k], y[l]))*(diff(P[k, m], y[n], y[p]))
*(diff(P[n, l], y[r]))*(diff(P[r, p], y[j])) end proc;
B := Array(1 .. q, 1 .. q);
T := combinat:-cartprod([seq([seq(1 .. q)], i = 1 .. 8)]);
while not T[finished] do
C := op(T[nextvalue]());
B[C[1], C[2]] := B[C[1], C[2]]+F(C);
B[C[1], C[5]] := B[C[1], C[5]]+G(C);
end do;
A := Array(1 .. q, 1 .. q);
for i from 1 to q do
for j from 1 to q do
A[i, j] := simplify((1/2)*B[i, j]-(1/2)*B[j, i]);
end do;
end do;
Matrix(A);
end proc:
```

To implement the Schouten bracket of two bi-vectors A and B, we use a component expansion (cf. [4]):

$$[\![A, B]\!]^{ijk} = \sum_{s=1}^{n} A^{sk}B^{ij}_{s} + B^{sk}A^{ij}_{s} + A^{sj}B^{ki}_{s} + B^{sj}A^{ki}_{s} + A^{si}B^{jk}_{s} + B^{si}A^{jk}_{s},$$

where superscripts and subscripts denote the bi-vector components and partial derivatives with respect to the coordinates y_s, respectively.

```
SchoutenBracket := proc (A, B, y)
description "Evaluate the Schouten-bracket of A and B.";
local T, t, F, n, res, cnt;
n := op(A)[1];
F := proc (i, j, k) options operator, arrow;
```

[9] Running the script below took us approximately 5 seconds.

```
A[s, k]*(diff(B[i, j], y[s]))+B[s, k]*(diff(A[i, j], y[s]))+
A[s, j]*(diff(B[k, i], y[s]))+B[s, j]*(diff(A[k, i], y[s]))+
A[s, i]*(diff(B[j, k], y[s]))+B[s, i]*(diff(A[j, k], y[s])) end proc;
T := combinat:-choose(n, 3);
for t in T do
print([[t[1], t[2], t[3]],simplify(add(F(t[1], t[2], t[3]), s = 1 .. n))]);
end do;
end proc:
```

Finally, the following script provides a computer-assisted proof of Proposision 2.

```
# All 3-dimensional Poisson bi-vectors are of the following form.
> P:=<<0,-f(x,y,z)*(diff(g(x,y,z),z)),f(x,y,z)*(diff(g(x,y,z),y))>|
     <f(x,y,z)*(diff(g(x,y,z),z)),0,-f(x,y,z)*(diff(g(x,y,z),x))>|
     <-f(x,y,z)*(diff(g(x,y,z),y)),f(x,y,z)*(diff(g(x,y,z),x)),0>>:
# We evaluate the balanced flow Q_{1:6} on the above bi-vector.
> Q:=FlowQ(P,{x,y,z},1,6)
                [Length of output exceeds limit of 1000000]
# If so, let us inspect whether the flow Q_{1:6} vanishes.
> LinearAlgebra:-Equal(Q,Matrix(1..3,1..3,0))
                                false
# Still, let us act on this Q_{1:6} by the Poisson differential.
> SchoutenBracket(P,Q,{x,y,z})
                              [[1,2,3], 0]
```

This reasoning hints us that the condition $a : b = 1 : 6$ could be sufficient for equation (1) to hold for all Poisson structures on all finite dimensional affine real manifolds. A rigorous proof of the respective claim in Theorem 3 is provided in section 2.

Appendix C. The count of Leibniz-rule graphs in Fig. 4

We count all possible differential consequences of the Jacobi identity, that is, we consider the differential operators acting on the Jacobiator. We do this by constructing all possible graphs that encode trivector-valued differential consequences (see Lemma 5 on p. 9). The graphs that encode such differential consequences have 3 ground vertices. The Schouten bracket $[\![\mathcal{P}, \mathcal{Q}_{1:6}(\mathcal{P})]\!]$ consists of graphs with 5 internal vertices. Since two of these internal vertices are accounted for by the Jacobi identity, there remain 3 spare internal vertices.

First, let the Jacobiator stand, with all its three edges, on the 3 ground vertices. The only freedom that remains is how the 3 free internal vertices act on each other and on the Jacobiator. With its first edge, every free internal vertex can act on itself, on its 2 neighbouring free vertices, or on the Jacobiator; there are 4 possible targets. No second edge can meet the first edge at the same target (as this would yield no contribution due to the anti-symmetry, which is explained in Remark 2). Hence there are only 3 possible targets for this second edge. Finally, again due to anti-symmetry, every possibility is constructed exactly twice this way. Swapping the targets of the first and second edge only contributes to the sign of the graph. The total number of this type of differential consequence is therefore $\left(\frac{4\cdot3}{2}\right)^3 = 216$ graphs. This type of graph is drawn first from the top-left in Figure 4.

Now let the Jacobiator stand on only 2 of the ground vertices. The remaining edge of the Jacobiator has only 3 possible targets, as the third edge cannot fall back onto the Jacobiator itself. One of the free internal vertices acts with an edge on the remaining ground vertex. The other edge has 4 candidates as its target, namely the vertex itself, the neighbouring 2 free internal vertices, and the Jacobiator. The 2 internal vertices not falling on a ground vertex have each $\frac{4\cdot3}{2}$ possible targets. The total number of graphs is therefore equal to $3\cdot4\cdot\left(\frac{4\cdot3}{2}\right)^2 = 432$. This type of graph is the second from the top-left in Figure 4.

Next, let the Jacobiator stand on only 1 ground vertex. We distinguish between two cases: namely, the case where 1 free internal vertex stands on both the remaining ground vertices and the case where two different internal vertices act by one edge each on the remaining two ground vertices. These are the third and fourth graphs from the top-left in Figure 4, respectively.

- In the first case, the remaining 2 internal vertices each have $\frac{4\cdot3}{2}$ possible targets. The Jacobiator must act with its two remaining free edges on two different targets out of the 3 available, yielding 3 possibilities. The number of graphs in the first case is $3\cdot\left(\frac{4\cdot3}{2}\right)^2 = 108$.
- For the second case, two internal vertices can each act on themselves, on the neighbouring 2 internal vertices, or on the Jacobiator. With two of its edges, the Jacobiator can act in 3 different ways on the 3 internal vertices. The third internal vertex has $\frac{4\cdot3}{2}$ possible targets. This brings the total number of graphs for the second case to $4\cdot4\cdot\frac{4\cdot3}{2}\cdot3 = 288$.

The last case to consider is where the Jacobiator does not act on any of the ground vertices. Again, since the outgoing edges of the Jacobiator must have different targets, it is clear that the Jacobiator acts in a unique way on all 3 internal vertices. We now distinguish two cases: namely, the case where 1 free internal vertex stands on 2 ground vertices, 1 free internal vertex acts on 1 ground vertex, and 1 free internal vertex falling on no ground vertex, and the second case where each internal vertex acts with one edge on one ground vertex. These two cases are represented by the last 2 graphs in Figure 4, respectively.

- In the first case, there is a free internal vertex with one free edge, which has 4 possible targets. The remaining free internal vertex with two free edges has $\frac{4\cdot3}{2}$ possible targets. The total number of graphs for this case is $4\cdot\frac{4\cdot3}{2} = 24$.
- In the second case, each internal vertex can act on itself, on its 2 neighbouring internal vertices, and on the Jacobiator. This results in a total of $4^3 = 64$ graphs.

Summarizing, the total number of all trivector-valued Leibniz-rule graphs, linear in the Jacobiator and containing five internal vertices, is 1132.

Appendix D. Encoding of the solution

Let Γ be a labelled Kontsevich graph with n internal and m external vertices. We assume the ground vertices of Γ are labelled $[0, \ldots, m-1]$ and the internal vertices are labelled $[m, \ldots, m+n-1]$. We define the *encoding* of Γ to be the *prefix* (n, m), followed by a list of *targets*. The list of targets consists of ordered pairs where the kth pair ($k \geqslant 0$) contains the two targets of the internal vertex number $m + k$.

The expansion of the Schouten bracket $[\![\mathcal{P}, \mathcal{Q}_{a:b}]\!]$ for the ratio $a : b = 1 : 6$ depicted in Figure 3 simplifies to a sum of 39 graphs with coefficients $\pm 1, \pm 3$. The encodings of these graphs, followed by their respective coefficients, are listed in Table 1. The graphs

TABLE 1. Machine-readable encoding of Figure 3 on p. 6.

1.1	3	5	4 2 0 1 4 6 4 7 4 5	1	7.1	3	5	6 2 7 0 1 4 4 5 5 6	3
1.2	3	5	4 0 1 2 4 6 4 7 4 5	1	7.2	3	5	6 0 7 1 2 4 4 5 5 6	3
1.3	3	5	4 1 2 0 4 6 4 7 4 5	1	7.3	3	5	6 1 7 2 0 4 4 5 5 6	3
2.1	3	5	7 0 3 5 3 6 3 4 1 2	1	8.1	3	5	7 2 7 0 1 4 4 5 5 6	3
2.2	3	5	7 1 3 5 3 6 3 4 2 0	1	8.2	3	5	7 0 7 1 2 4 4 5 5 6	3
2.3	3	5	7 2 3 5 3 6 3 4 0 1	1	8.3	3	5	7 1 7 2 0 4 4 5 5 6	3
3.1	3	5	5 2 0 1 4 6 4 7 4 5	3	9.1	3	5	4 2 7 1 0 4 4 5 5 6	−3
3.2	3	5	5 0 1 2 4 6 4 7 4 5	3	9.2	3	5	4 0 7 2 1 4 4 5 5 6	−3
3.3	3	5	5 1 2 0 4 6 4 7 4 5	3	9.3	3	5	4 1 7 0 2 4 4 5 5 6	−3
4.1	3	5	6 7 0 3 3 4 4 5 1 2	3	10.1	3	5	5 2 7 1 0 4 4 5 5 6	−3
4.2	3	5	6 7 1 3 3 4 4 5 2 0	3	10.2	3	5	5 0 7 2 1 4 4 5 5 6	−3
4.3	3	5	6 7 2 3 3 4 4 5 0 1	3	10.3	3	5	5 1 7 0 2 4 4 5 5 6	−3
5.1	3	5	4 2 7 0 1 4 4 5 5 6	3	11.1	3	5	6 2 7 1 0 4 4 5 5 6	−3
5.2	3	5	4 0 7 1 2 4 4 5 5 6	3	11.2	3	5	6 0 7 2 1 4 4 5 5 6	−3
5.3	3	5	4 1 7 2 0 4 4 5 5 6	3	11.3	3	5	6 1 7 0 2 4 4 5 5 6	−3
6.1	3	5	5 2 7 0 1 4 4 5 5 6	3	12.1	3	5	7 2 7 1 0 4 4 5 5 6	−3
6.2	3	5	5 0 7 1 2 4 4 5 5 6	3	12.2	3	5	7 0 7 2 1 4 4 5 5 6	−3
6.3	3	5	5 1 7 2 0 4 4 5 5 6	3	12.3	3	5	7 1 7 0 2 4 4 5 5 6	−3
					13.1	3	5	6 0 7 3 3 4 4 5 1 2	−3
					13.2	3	5	6 1 7 3 3 4 4 5 2 0	−3
					13.3	3	5	6 2 7 3 3 4 4 5 0 1	−3

are collected into groups of three, consisting of the skew-symmetrization – by a sum over cyclic permutations – of a single graph. Within the encodings in the groups of three, the lists of targets only differ by a cyclic permutation of the target vertices $0, 1, 2$.

Consisting of 8 skew-symmetric terms, the solution (see (9) on p. 11) is encoded in Table 2: the sought-for values of coefficients are written after the encoding of the respective 27 Leibniz-rule graphs. Here the sums over permutations of the ground vertices are expanded (thus making the 27 Leibniz-rule graphs out of the 8 skew-symmetric groups). In every entry of Table 2, the sum of three graphs in Jacobiator (2) is represented by its first term. For all the in-coming arrows, the vertex 6 is the placeholder for

Table 2. Machine-readable encoding of solution (9) on p. 11.

1.1	3	5	4 6 5 6 3 6 0 1 6 2	−1
2.1	3	5	0 4 1 5 2 3 3 4 6 5	−3
2.2	3	5	0 4 2 5 1 3 3 4 6 5	3
3.1	3	5	0 4 1 2 3 4 3 4 6 5	−3
3.2	3	5	0 1 2 3 3 4 3 4 6 5	−3
3.3	3	5	0 2 1 3 3 4 3 4 6 5	3
4.1	3	5	4 5 1 6 4 6 0 2 6 3	−3
4.2	3	5	4 5 0 6 4 6 1 2 6 3	3
4.3	3	5	5 6 3 5 2 6 0 1 6 4	−3
5.1	3	5	1 4 5 6 3 6 0 2 6 3	3
5.2	3	5	0 4 5 6 3 6 1 2 6 3	−3
5.3	3	5	5 6 2 3 4 6 0 1 6 4	−3
6.1	3	5	1 2 3 5 3 6 0 3 6 4	3
6.2	3	5	0 2 3 5 3 6 1 3 6 4	−3
6.3	3	5	4 6 0 1 3 4 2 4 6 5	−3
7.1	3	5	1 5 3 5 2 6 0 3 6 4	−3
7.2	3	5	1 5 3 5 0 6 2 3 6 4	3
7.3	3	5	0 5 3 5 2 6 1 3 6 4	3
7.4	3	5	2 5 3 5 1 6 0 3 6 4	3
7.5	3	5	2 5 3 5 0 6 1 3 6 4	−3
7.6	3	5	0 5 3 5 1 6 2 3 6 4	−3
8.1	3	5	1 4 2 5 3 6 0 3 6 4	−3
8.2	3	5	1 5 2 3 4 6 0 3 6 4	−3
8.3	3	5	0 4 2 5 3 6 1 3 6 4	3
8.4	3	5	0 5 2 3 4 6 1 3 6 4	3
8.5	3	5	4 6 0 5 1 3 2 4 6 5	−3
8.6	3	5	4 6 1 5 0 3 2 4 6 5	3

the Jacobiator (again, see (2) on p. 3); in earnest, the Jacobiator contains the internal vertices 6 and 7. This convention is helpful: for every set of derivations acting on the Jacobiator with internal vertices 6 and 7, only the first term is listed, namely the one where each edge lands on 6.

Example 1. The first entry of Table 2 encodes a graph containing a three-cycle over internal vertices $3, 4, 5$. Issued from each of these three, the other edge lands on the vertex 6: the placeholder for the Jacobiator. This entry is the first term in (9) on p. 11.

Example 2. The entry 3.1 is one of three terms produced by the third graph in solution (9); the Jacobiator in this entry is expanded using formula (2), resulting in three terms (by definition). It is easy to see that the first term contains picture (3) from Remark 2 as a subgraph. Hence the polydifferential operator encoded by this graph vanishes due to skew-symmetry. However, the other two terms produced in the entry 3.1 by formula (2) do not vanish by skew-symmetry. Likewise, there is one term vanishing by the same mechanism in the entry 3.2 and in 3.3.

The proof of Theorem 3 amounts to expanding the Leibniz rules on Jacobiators in Table 2 according to the rules above (resulting in Table 3 on pp. 23–24, where the prefix "3 5" of each graph has been omitted for brevity), simplifying by collecting terms, and seeing that one obtains Table 1.

Appendix E. Perturbation method

In section 2 above, the run-through method gave all the terms at once in the operator $\Diamond$ that establishes the factorization $[\![\mathcal{P}, \mathcal{Q}_{1:6}]\!] = \Diamond(\mathcal{P}, \mathrm{Jac}(\mathcal{P}))$. At the same time, there is another method to find $\Diamond$; the operator $\Diamond$ is then constructed gradually, term after

Table 3. Expansion of Leibniz rules on Jacobiators in Table 2.

0 1 2 3 3 6 3 7 3 5	−1	0 4 2 5 6 7 1 3 3 6	−3	0 4 2 5 3 6 3 4 1 6	−3
0 1 2 3 3 6 3 7 4 5	−1	0 4 5 6 1 3 3 5 2 3	3	0 4 2 5 3 6 1 7 3 4	−3
0 1 2 3 3 6 3 7 4 5	−1	0 4 5 6 1 3 5 7 2 3	−3	0 4 2 5 3 6 1 4 3 6	3
0 1 2 3 3 6 4 7 4 5	−1	0 4 5 6 1 7 3 5 2 3	3	0 4 2 5 3 6 3 7 1 4	3
0 1 2 3 3 6 3 7 4 5	−1	0 4 5 6 1 7 5 7 2 3	−3	0 4 5 6 1 3 2 3 5 6	−3
0 1 2 3 3 6 4 7 4 5	−1	0 4 1 2 3 4 3 7 4 5	−3	0 4 5 6 2 3 5 7 1 3	−3
0 1 2 3 3 6 4 7 4 5	−1	0 4 1 2 3 6 4 7 4 5	3	0 4 5 6 2 3 3 5 1 6	−3
0 1 2 3 4 6 4 7 4 5	−1	0 4 5 6 1 2 5 7 4 5	3	0 4 5 6 1 7 2 3 3 6	3
0 4 1 2 4 6 4 7 4 5	−1	0 4 5 6 1 2 5 7 3 5	3	0 4 5 6 1 6 2 3 3 5	−3
0 4 1 2 3 6 4 7 4 5	−1	0 4 5 6 1 2 3 5 3 5	3	0 4 5 6 2 3 3 7 1 5	3
0 4 1 2 3 6 4 7 4 5	−1	0 4 5 6 1 2 5 7 3 5	−3	0 4 2 5 1 3 3 4 5 6	3
0 4 1 2 3 6 3 7 4 5	−1	0 4 1 5 2 6 3 4 3 5	3	0 4 2 5 6 7 1 3 3 4	3
0 4 1 2 3 6 4 7 4 5	−1	0 4 1 5 2 6 4 7 3 5	−3	0 4 2 5 3 6 3 4 1 5	−3
0 4 1 2 3 6 3 7 4 5	−1	0 4 5 6 1 6 2 7 4 5	−3	0 4 2 5 1 6 3 7 3 4	−3
0 4 1 2 3 6 3 7 4 5	−1	0 4 5 6 1 6 2 7 3 5	−3	0 4 2 5 1 6 3 4 3 5	3
0 4 1 2 3 6 3 7 3 5	−1	0 4 2 5 3 6 1 4 3 6	−3	0 4 2 5 3 6 1 7 3 4	3
0 2 1 3 3 6 3 7 3 5	1	0 4 2 5 6 7 1 4 3 6	3	0 4 1 5 2 3 3 5 4 6	3
0 2 1 3 3 6 3 7 4 5	1	0 4 1 5 3 6 2 4 3 6	3	0 4 5 6 1 7 3 7 2 3	3
0 2 1 3 3 6 3 7 4 5	1	0 4 1 5 6 7 2 4 3 6	−3	0 4 5 6 2 3 3 5 1 4	−3
0 2 1 3 3 6 4 7 4 5	1	0 4 5 6 1 7 2 5 4 6	−3	0 4 1 5 6 7 2 3 3 6	−3
0 2 1 3 3 6 3 7 4 5	1	0 4 5 6 1 7 2 5 3 6	−3	0 4 1 5 3 6 2 3 4 6	−3
0 2 1 3 3 6 4 7 4 5	1	0 4 2 5 1 6 3 4 3 5	−3	0 4 5 6 1 7 2 3 3 6	−3
0 2 1 3 3 6 4 7 4 5	1	0 4 2 5 1 6 4 7 3 5	3	0 4 1 5 2 3 3 4 5 6	−3
0 2 1 3 4 6 4 7 4 5	1	0 4 1 5 2 3 3 7 4 5	−3	0 4 1 5 6 7 2 3 3 4	−3
0 1 2 5 3 6 3 4 3 4	−3	0 4 1 5 2 6 3 7 4 5	−3	0 4 1 5 3 6 3 4 2 5	3
0 1 2 5 3 6 4 7 3 4	3	0 4 5 6 1 7 5 7 2 4	3	0 4 1 5 2 6 3 7 3 4	3
0 1 2 5 6 7 3 4 3 4	−3	0 4 5 6 1 7 5 7 2 3	3	0 4 1 5 2 6 3 4 3 5	−3
0 1 2 5 6 7 3 4 4 6	3	0 4 5 6 1 6 2 3 3 5	3	0 4 1 5 3 6 2 7 3 4	−3
0 4 1 5 2 6 4 7 4 5	3	0 4 5 6 1 6 2 7 3 5	3	0 4 2 5 1 3 3 5 4 6	−3
0 4 1 5 2 6 4 7 3 5	3	0 4 2 5 1 3 3 7 4 5	3	0 4 5 6 2 7 3 7 1 3	−3
0 4 1 5 2 6 3 7 4 5	3	0 4 2 5 1 6 3 7 4 5	3	0 4 5 6 1 3 3 5 2 4	3
0 4 1 5 2 6 3 7 3 5	3	0 4 5 6 2 7 5 7 1 4	−3	0 4 2 5 6 7 1 3 3 6	3
0 4 2 5 3 6 3 4 1 3	−3	0 4 5 6 2 7 5 7 1 3	−3	0 4 2 5 3 6 1 3 4 6	3
0 4 2 5 3 6 4 7 1 3	3	0 4 5 6 1 3 2 5 3 6	3	0 4 5 6 1 3 2 7 3 5	−3
0 4 2 5 6 7 1 3 3 4	−3	0 4 5 6 1 7 2 5 3 6	3	0 1 2 3 3 4 3 5 4 6	3
0 4 2 5 6 7 1 3 4 6	3	0 4 5 6 1 2 3 5 3 5	−3	0 1 2 3 3 6 3 7 4 5	3
0 1 2 3 3 4 3 7 4 5	−3	0 4 5 6 1 2 3 7 3 5	−3	0 1 2 5 3 6 3 7 3 5	−3
0 1 2 5 3 6 4 7 3 4	−3	0 4 5 6 3 7 3 7 1 2	−3	0 1 2 5 3 6 3 7 3 4	−3
0 1 2 3 3 6 4 7 4 5	3	0 4 5 6 3 6 3 7 1 2	3	0 1 2 5 3 6 3 4 3 4	3
0 1 2 5 3 6 4 7 4 5	3	0 4 5 6 2 3 3 5 1 5	−3	0 1 2 5 3 6 3 7 3 4	3
0 4 1 5 6 7 2 4 4 6	3	0 4 5 6 2 3 3 7 1 5	−3	0 4 1 5 3 6 2 3 4 6	3
0 4 1 5 6 7 2 3 4 6	3	0 4 5 6 1 7 3 7 2 3	−3	0 4 1 5 3 6 4 7 2 3	3
0 4 1 5 6 7 2 4 3 6	3	0 4 5 6 1 7 3 5 2 3	−3	0 4 1 5 3 6 3 4 2 6	3
0 4 1 5 6 7 2 3 3 6	3	0 4 5 6 1 3 3 5 2 5	3	0 4 1 5 3 6 2 7 3 4	3
0 4 5 6 2 3 3 5 1 3	−3	0 4 5 6 1 3 3 7 2 5	3	0 4 1 5 3 6 2 4 3 6	−3
0 4 5 6 2 7 3 5 1 3	−3	0 4 5 6 2 7 3 7 1 3	3	0 4 1 5 3 6 3 7 2 4	−3
0 4 5 6 2 3 5 7 1 3	3	0 4 5 6 2 7 3 5 1 3	3	0 4 5 6 1 3 2 5 3 6	−3
0 4 5 6 2 7 5 7 1 3	3	0 4 1 2 3 4 3 5 4 6	3	0 4 5 6 1 3 3 7 2 5	−3
0 2 1 5 3 6 3 4 3 4	3	0 4 1 2 3 6 3 7 4 5	3	0 4 5 6 1 3 3 5 2 6	3
0 2 1 5 3 6 4 7 3 4	−3	0 4 5 6 1 2 3 7 3 5	3	0 4 5 6 1 3 2 7 3 5	3
0 2 1 5 6 7 3 4 3 4	3	0 4 5 6 1 2 3 7 3 4	3	0 4 5 6 1 3 2 3 5 6	3
0 2 1 5 6 7 3 4 4 6	−3	0 4 2 5 3 6 3 4 1 4	−3	0 4 5 6 1 3 5 7 2 3	3
0 4 2 5 1 6 4 7 4 5	−3	0 4 2 5 3 6 3 7 1 4	−3	0 1 2 3 3 4 3 4 5 6	−3
0 4 2 5 1 6 4 7 3 5	−3	0 4 1 5 2 6 3 7 3 5	−3	0 1 2 3 3 4 3 7 4 5	3
0 4 2 5 1 6 3 7 4 5	−3	0 4 1 5 2 6 3 7 3 4	−3	0 1 2 3 3 4 3 5 4 6	−3
0 4 2 5 1 6 3 7 3 5	−3	0 4 1 5 3 6 3 4 2 4	3	0 2 1 3 3 4 3 4 5 6	3
0 4 1 5 3 6 3 4 2 3	3	0 4 1 5 3 6 3 7 2 4	3	0 2 1 3 3 4 3 7 4 5	−3
0 4 1 5 3 6 4 7 2 3	−3	0 4 2 5 1 6 3 7 3 5	3	0 2 1 3 3 4 3 5 4 6	3
0 4 1 5 6 7 2 3 3 4	3	0 4 2 5 1 6 3 7 3 4	3	0 4 1 2 3 4 3 4 5 6	−3
0 4 1 5 6 7 2 3 4 6	−3	0 2 1 3 3 4 3 5 4 6	−3	0 4 1 2 3 4 3 7 4 5	3

Table 3 (continued).

0 2 1 3 3 4 3 7 4 5	3	0 2 1 3 3 6 3 7 4 5	−3	0 4 1 2 3 4 3 5 4 6	−3
0 2 1 3 3 6 4 7 4 5	−3	0 2 1 5 3 6 3 7 3 5	3	0 4 1 5 2 3 3 4 5 6	3
0 2 1 5 3 6 4 7 3 4	3	0 2 1 5 3 6 3 7 3 4	3	0 4 1 5 2 3 3 7 4 5	3
0 2 1 5 3 6 4 7 4 5	−3	0 2 1 5 3 6 3 4 3 4	−3	0 4 1 5 2 3 3 5 4 6	−3
0 4 2 5 6 7 1 4 4 6	−3	0 2 1 5 3 6 3 7 3 4	−3	0 4 2 5 1 3 3 4 5 6	−3
0 4 2 5 6 7 1 4 3 6	−3	0 4 2 5 3 6 1 3 4 6	−3	0 4 2 5 1 3 3 7 4 5	−3
0 4 2 5 6 7 1 3 4 6	−3	0 4 2 5 3 6 4 7 1 3	−3	0 4 2 5 1 3 3 5 4 6	3

term in (9), by starting with a zero initial approximation for $\Diamond$. This is the perturbation scheme which we now outline.

In fact, the perturbation method was tried first, revealing the typical graph patterns and their topological complexity. From Proposition 2 we already know that $[\![\mathcal{P}, \mathcal{Q}_{1:6}]\!] = 0$ for Poisson brackets on $\mathbb{R}^3$. The difficulty is that because the condition $[\![\mathcal{P}, \mathcal{Q}_{1:6}]\!] = 0$ and the Jacobi identity $[\![\mathcal{P}, \mathcal{P}]\!] = 0$ are valid, it is impossible to factorize one through the other; both are invisible. So, we first make both expressions visible by perturbing the Poisson bi-vector $\mathcal{P} \mapsto \mathcal{P}_\epsilon = \mathcal{P} + \epsilon\Delta$ in such a way that the tri-vector $[\![\mathcal{P}_\epsilon, \mathcal{Q}_{1:6}(\mathcal{P}_\epsilon)]\!]$ and the Jacobiator $[\![\mathcal{P}_\epsilon, \mathcal{P}_\epsilon]\!]$ stop vanishing identically:

$$[\![\mathcal{P}_\epsilon, \mathcal{Q}_{1:6}(\mathcal{P}_\epsilon)]\!] \neq 0 \quad \text{and} \quad [\![\mathcal{P}_\epsilon, \mathcal{P}_\epsilon]\!] \neq 0.$$

To begin with, put $\Diamond := 0$. Now consider the description [6] of Poisson brackets on $\mathbb{R}^3$ by using the pre-factor $f(x, y, z)$ and arbitrary function $g(x, y, z)$ in the formula

$$\{u, v\}_\mathcal{P} = f \cdot \det\left(\frac{\partial(g, u, v)}{\partial(x, y, z)}\right);$$

it is helpful to start with some very degenerate dependencies of f and g of their arguments (see [1] and [21]). The next step is to perturb the coefficients of the Poisson bracket $\{\cdot,\cdot\}_\mathcal{P}$ at hand; in a similar way, one starts with degenerate dependency of the perturbation Δ. The idea is to take perturbations which destroy the validity of Jacobi identity for $\mathcal{P}_\epsilon$ in the linear approximation in the deformation parameter ϵ. It is readily seen that the expansion of (8) in ϵ yields the equality

$$[\![\mathcal{P}_\epsilon, \mathcal{Q}_{1:6}]\!](\epsilon) = (\Diamond + \bar{o}(1))\,([\![\mathcal{P}_\epsilon, \mathcal{P}_\epsilon]\!]) = 2\epsilon \cdot (\Diamond + \bar{o}(1))\,([\![\mathcal{P}, \Delta]\!]) + (\Diamond + \bar{o}(1))\,([\![\mathcal{P}, \mathcal{P}]\!]) + \bar{o}(\epsilon).$$

Knowing the left-hand side at first order in ϵ and taking into account that $[\![\mathcal{P}, \mathcal{P}]\!] \equiv 0$ for the Poisson bi-vector $\mathcal{P}$ which we perturb by Δ, we reconstruct the operator $\Diamond$ that now acts on the known tri-vector $2[\![\mathcal{P}, \Delta]\!]$. In this sense, the Jacobiator $[\![\mathcal{P}, \mathcal{P}]\!]$ shows up through the term $[\![\mathcal{P}, \Delta]\!]$.

For each pair $(\mathcal{P}, \Delta)$, the above balance at ϵ^1 contains sums over indexes that mark the derivatives falling on the Jacobiator. By taking those formulae, we guess the candidates for graphs that form the next, yet unknown, part of the operator $\Diamond$. Specifically, we inspect which differential operator(s), acting on the Jacobi identity, become visible and we list the graphs that provide such differential operators via the Leibniz rule(s). For a while we keep every such candidate with an undetermined coefficient. By repeating the iteration, now for a different Poisson bi-vector $\mathcal{P}$ or its new, less degenerate perturbation Δ, we obtain linear constraints for the already introduced undetermined coefficients. Simultaneously, we continue listing the new candidates and introducing new coefficients for them.

Remark 12. By translating formulae into graphs, we convert the dimension-dependent expressions into the dimension-independent operators which are encoded by the graphs. An obvious drawback of the method which is outlined here is that, presumably, some parts of the operator $\Diamond$ could always stay invisible for all Poisson structures over $\mathbb{R}^3$ if they show up only in the higher dimensions. Secondly, the number of variants to consider and in practice, the number of irrelevant terms, each having its own undetermined coefficient, grows exponentially at the initial stage of the reasoning.

By following the loops of iterations of this algorithm, we managed to find two non-zero coefficients and five zero coefficients in solution (9). Namely, we identified the coefficient ± 1 for the tripod, which is the first term in (9), and we also recognized the coefficient ± 3 of the sum of 'elephant' graphs, which is the second to last term in (9).

Remark 13. Because of the known skew-symmetry of the tri-vector $[\![\mathcal{P}, \mathcal{Q}_{1:6}]\!]$ with respect to its arguments f, g, h, finding one term in a sum within formula (9) for $\Diamond$ means that the entire such sum is reconstructed. Indeed, one then takes the sum over a subgroup of S_3 acting on f, g, h, depending on the actual skew-symmetry of the term which has been found.

For instance, the first term in (9), itself making a sum running over $\{\text{id}\} \prec S_3$, is obviously totally antisymmetric with respect to its arguments. The other graph which we found by using the perturbation method (see the last graph in the second line of formula (9) on p. 11) is skew-symmetric with respect to its second and third arguments but it is not yet totally skew-symmetric with respect to the full set of its arguments. This shows that is suffices to take the sum over the group $\circlearrowleft = A_3 \prec S_3$ of cyclic permutations of f, g, h, thus reconstructing the sixth term in solution (9).

Acknowledgements. A. K. thanks M. Kontsevich for posing the problem; the authors are grateful to P. Vanhaecke and A. G. Sergeev for stimulating discussions.

This research was supported in part by JBI RUG project 106552 (Groningen). A. B. and R. B. thank the organizers of the 8^{th} international workshop GADEIS VIII on Group Analysis of Differential Equations and Integrable Systems (12–16 June 2016, Larnaca, Cyprus) for partial financial support and warm hospitality. A. B. and R. B. are also grateful to the Graduate School of Science (Faculty of Mathematics and Natural Sciences, University of Groningen) for financial support. We thank the Center for Information Technology of the University of Groningen for providing access to the Peregrine high performance computing cluster.

References

[1] *Bouisaghouane A., Kiselev A. V.* (2016) Do the Kontsevich tetrahedral flows preserve or destroy the space of Poisson bi-vectors? *Preprint* IHÉS/M/16/12 (Bures-sur-Yvette, France), `arXiv:1602.09036` [q-alg], 10 p.

[2] *Buring R., Kiselev A. V.* (2017) On the Kontsevich $\star$-product associativity mechanism, *PEPAN Letters* **14**:2 (accepted), 4 p. (*Preprint* `arXiv:1602.09036` [q-alg])

[3] *Buring R., Kiselev A. V.* (2016) Software modules and computer-assisted proof schemes in the Kontsevich deformation quantization (in preparation), see link: `https://github.com/rburing/kontsevich_graph_series-cpp`

[4] *Dubrovin B.* (2005) Bihamiltonian structures of PDE's and Frobenius manifolds, Summer School ICTP, `http://indico.ictp.it/event/a04198/session/47/contribution/26/material/0/0.pdf`.

[5] *Gerstenhaber M.* (1964) On the deformation of rings and algebras, *Ann. Math.* **79**, 59–103.

[6] *Grabowski J., Marmo G., Perelomov A. M.* (1993) Poisson structures: towards a classification, *Mod. Phys. Lett.* **A8**:18, 1719–1733.

[7] *Kiselev A. V.* (2012) The twelve lectures in the (non)commutative geometry of differential equations, *Preprint* IHÉS/M/12/13 (Bures-sur-Yvette, France), 140 pp.

[8] *Kiselev A. V.* (2013) The geometry of variations in Batalin–Vilkovisky formalism, *J. Phys.: Conf. Ser.* **474**, Paper 012024, 1–51. (*Preprint* `1312.1262 [math-ph]`)

[9] *Kiselev A. V.* (2015) The calculus of multivectors on noncommutative jet spaces. *Preprint* IHÉS/M/14/39 (Bures-sur-Yvette, France), `arXiv:1210.0726` (v3) [math.DG], 41 p.

[10] *Kiselev A. V.* (2015) Deformation approach to quantisation of field models, *Preprint* IHÉS/M/15/13 (Bures-sur-Yvette, France), 37 p.

[11] *Kontsevich M.* (1993) Formal (non)commutative symplectic geometry, The Gel'fand Mathematical Seminars, 1990-1992 (L. Corwin, I. Gelfand, and J. Lepowsky, eds), Birkhäuser, Boston MA, 173–187.

[12] *Kontsevich M.* (1994) Feynman diagrams and low-dimensional topology. First Europ. Congr. of Math. **2** (Paris, 1992), Progr. Math. **120**, Birkhäuser, Basel, 97–121.

[13] *Kontsevich M.* (1995) Homological algebra of mirror symmetry. Proc. Intern. Congr. Math. **1** (Zürich, 1994), Birkhäuser, Basel, 120–139.

[14] *Kontsevich M.* (1997) Formality conjecture. Deformation theory and symplectic geometry (Ascona 1996, D. Sternheimer, J. Rawnsley and S. Gutt, eds), Math. Phys. Stud. **20**, Kluwer Acad. Publ., Dordrecht, 139–156.

[15] *Kontsevich M.* (2003) Deformation quantization of Poisson manifolds, *Lett. Math. Phys.* **66**:3, 157–216. (*Preprint* `q-alg/9709040`)

[16] *Laurent–Gengoux C., Picherau A., Vanhaecke P.* (2013) Poisson structures. Gründlehren der mathematischen Wissenschaften **347**, Springer–Verlag, Berlin.

[17] *Manin Yu. I.* (1999) Frobenius manifolds, quantum cohomology, and moduli spaces. AMS Colloquium publications **47**, Providence RI.

[18] *Merkulov S. A.* (2010) Exotic automorphisms of the Schouten algebra of polyvector fields. *Preprint* `arXiv:0809.2385` (v6) [q-alg], 37 p.

[19] *Olver P. J.* (1993) Applications of Lie groups to differential equations, Grad. Texts in Math. **107** (2nd ed.), Springer–Verlag, NY.

[20] *Olver P. J., Sokolov V. V.* (1998) Integrable evolution equations on associative algebras, *Comm. Math. Phys.* **193**:2, 245–268;
Olver P. J., Sokolov V. V. (1998) Non-abelian integrable systems of the derivative nonlinear Schrödinger type, *Inverse Prob.* **14**:6, L5–L8.

[21] *Vanhaecke P.* (1996) Integrable systems in the realm of algebraic geometry, Lect. Notes Math. **1638**, Springer–Verlag, Berlin.

THE KONTSEVICH TETRAHEDRAL FLOW IN 2D: A TOY MODEL

ANASS BOUISAGHOUANE

ABSTRACT. In the paper "Formality conjecture" (1996) Kontsevich designed a universal flow $\dot{\mathcal{P}} = \mathcal{Q}_{a:b}(\mathcal{P}) = a\Gamma_1 + b\Gamma_2$ on the spaces of Poisson structures $\mathcal{P}$ on all affine manifolds of dimension $n \geqslant 2$. We prove a claim from *loc. cit.* stating that if $n = 2$, the flow $\mathcal{Q}_{1:0} = \Gamma_1(\mathcal{P})$ is Poisson-cohomology trivial: $\Gamma_1(\mathcal{P}) = [\![\mathcal{P}, \mathcal{X}]\!]$ for some vector field $\mathcal{X}$; we examine the structure of the space of solutions $\mathcal{X}$. Both the construction of differential polynomials $\Gamma_1(\mathcal{P})$ and $\Gamma_2(\mathcal{P})$ and the technique to study them remain valid in higher dimensions $n \geqslant 3$, but neither the trivializing vector field $\mathcal{X}$ nor the setting $b := 0$ survive at $n \geqslant 3$, where the balance is $a : b = 1 : 6$.

Introduction. Let $\mathcal{P} = (\mathcal{P}^{ij})$ be a Poisson bi-vector (whose coefficient matrix is skew-symmetric and satisfies the Jacobi identity[1] $[\![\mathcal{P}, \mathcal{P}]\!] = 0$) on a real affine n-dimensional manifold N^n and denote by $\mathbf{x} = (x^1, \ldots, x^n)$ local coordinates. In [1], Kontsevich introduced two differential polynomials[2] in the coefficients $\mathcal{P}^{\alpha\beta}$ and their derivatives $\mathcal{P}^{\alpha\beta}_{\sigma} \stackrel{\text{def}}{=} \partial^{|\sigma|}/\partial x^{\sigma}(\mathcal{P}^{\alpha\beta})$, where $\sigma = (\sigma_1 \cdots \sigma_k)$, $k \in \mathbb{N}$, denotes the indices of the coordinates with respect to which we differentiate, e.g. $\partial^{|(122)|}/\partial x^{(122)} = \partial^3/\partial x^1 x^2 x^2$:

$$\Gamma_1^{ij}(\mathcal{P}) = \sum_{k,\ell,m,k',\ell',m'=1}^{n} \mathcal{P}^{ij}_{klm} \mathcal{P}^{kk'}_{\ell'} \mathcal{P}^{\ell\ell'}_{m'} \mathcal{P}^{mm'}_{k'}, \quad \Gamma_2^{im}(\mathcal{P}) = \sum_{j,k,\ell,k',\ell',m'=1}^{n} \mathcal{P}^{ij}_{k\ell} \mathcal{P}^{km}_{k'\ell'} \mathcal{P}^{k'\ell}_{m'} \mathcal{P}^{m'\ell'}_{j}.$$

From any initial bi-vector $\mathcal{P}$, the coefficients of a new bi-vector are constructed using the differential polynomial Γ_1: $\dot{\mathcal{P}}_1^{ij}(\omega, \eta) = \Gamma_1^{ij}(\mathcal{P}) \partial_{x^i}(\omega) \wedge \partial_{x^j}(\eta)$ where the functions ω, $\eta \in C^\infty(N^n)$. We thereby obtain a 'flow' on the space of bi-vectors with infinitesimal deformation $\mathcal{P} \mapsto \mathcal{P} + \epsilon\Gamma_1(\mathcal{P}) + \bar{o}(\epsilon)$, $\epsilon \in \mathbb{R}$. From a Poisson bi-vector $\mathcal{P}$, we construct the classical Poisson differential $\partial_{\mathcal{P}} = [\![\mathcal{P}, \cdot]\!]$ and obtain the Poisson complex:

$$0 \to \Bbbk \hookrightarrow C^\infty(N^n) \to \mathcal{X}^1(N^n) \to \mathcal{X}^2(N^n) \to \cdots \to \mathcal{X}^n(N^n) \to 0.$$

Date: February 21, 2017.
2010 *Mathematics Subject Classification.* 35R01, 53D17, 70G45 .
Key words and phrases. Poisson bracket, deformation, tetrahedral flow, cohomology.
Address: Johann Bernoulli Institute for Mathematics and Computer Science, University of Groningen, P.O. Box 407, 9700 AK Groningen, The Netherlands. **E-mail*: a.bouisaghouane.1@student.rug.nl.

[1]The Schouten bracket $[\![\cdot, \cdot]\!]$ is a unique extension of the commutator $[\cdot, \cdot]$ on the space of vector fields $\mathcal{X}^1(N^n)$ to the space $\mathcal{X}^k(N^n)$ of polyvector fields. By definition, the Schouten bracket coincides with the Lie bracket when evaluated on 1-vectors. When evaluated on p-vector $\mathcal{X}$, q-vector $\mathcal{Y}$ and r-vector $\mathcal{Z}$, the Schouten bracket satisfies the equations $[\![\mathcal{X}, \mathcal{Y}]\!] = -(-1)^{(p-1)(q-1)}[\![\mathcal{Y}, \mathcal{X}]\!]$ and $[\![\mathcal{X}, \mathcal{Y} \wedge \mathcal{Z}]\!] = [\![\mathcal{X}, \mathcal{Y}]\!] \wedge \mathcal{Z} + (-1)^{(p-1)q}\mathcal{Y} \wedge [\![\mathcal{X}, \mathcal{Z}]\!]$.

[2]The second differential polynomial Γ_2^{im} is not skew-symmetric in (i, m) so that one must skew-symmetrize in (i, m): $\dot{\mathcal{P}}_2^{im}(\omega, \eta) = \frac{1}{2}\left(\Gamma_2^{im}(\mathcal{P}) - \Gamma_2^{mi}(\mathcal{P})\right) \partial_{x^i}(\omega) \wedge \partial_{x^m}(\eta)$, in order to construct a similar flow as the one obtained from Γ_1.

Does either of the flows mark a $\partial_{\mathcal{P}}$-trivial class in the Poisson cohomology? If $n=2$, every flow is $\partial_{\mathcal{P}}$-closed because the Schouten bracket of the bi-vector $\mathcal{P}$ with the bi-vector $\Gamma_1(\mathcal{P})$ or $\Gamma_2(\mathcal{P})$ is a tri-vector that sure vanishes on a two-dimensional affine manifold. The property we explore in this paper is the exactness, with respect to the Poisson differential, of the Kontsevich tetrahedral flows over 2-dimensional affine manifolds.

We first expand both the bi-vectors obtained from Γ_1 and skew-symmetrized Γ_2 with respect to the local coordinates $x^1 = x, x^2 = y$ of our 2-dimensional manifold. This means that we can expand the differential polynomials with their indices $i, j, k, \ell, m, k', \ell', m' \in \{1, 2\}$. In dimension $n = 2$, every bi-vector has only one unique component, $\mathcal{P}^{12}$, which is denoted by u.

Proposition 1. The only non-zero component of the bi-vector flow $\dot{\mathcal{P}}_1$ is expressed in terms of u, x and y via

$$\dot{\mathcal{P}}_1 = u_{xxx}u_y^3 - u_{yyy}u_x^3 - 3u_{xxy}u_xu_y^2 + 3u_{xyy}u_x^2u_y. \tag{1}$$

The bi-vector flow $\dot{\mathcal{P}}_2$ vanishes identically (in dimension $n = 2$ under study).

Proof. We expand the differential formula for $\Gamma_1^{12}(\mathcal{P})$. Taking into account that the bi-vector coefficient $\mathcal{P}^{ij}$ can be non-zero only when $i = 1, j = 2$ or $i = 2, j = 1$, the sum on the right-hand side of the formula for $\Gamma_1^{12}(\mathcal{P})$ yields:

$$\begin{aligned}\Gamma_1^{12}(\mathcal{P}) &= \sum_{k,\ell,m,k',\ell',m'=1}^{n} \mathcal{P}^{12}_{klm}\mathcal{P}^{kk'}_{\ell'}\mathcal{P}^{\ell\ell'}_{m'}\mathcal{P}^{mm'}_{k'} = \sum_{\ell,m,\ell',m'=1}^{n} \mathcal{P}^{12}_{1lm}\mathcal{P}^{12}_{\ell'}\mathcal{P}^{\ell\ell'}_{m'}\mathcal{P}^{mm'}_{2} + \mathcal{P}^{12}_{2lm}\mathcal{P}^{21}_{\ell'}\mathcal{P}^{\ell\ell'}_{m'}\mathcal{P}^{mm'}_{1} \\ &= \sum_{m,m'=1}^{n} \mathcal{P}^{12}_{11m}\mathcal{P}^{12}_{2}\mathcal{P}^{12}_{m'}\mathcal{P}^{mm'}_{2} + \mathcal{P}^{12}_{12m}\mathcal{P}^{12}_{1}\mathcal{P}^{21}_{m'}\mathcal{P}^{mm'}_{2} + \mathcal{P}^{12}_{21m}\mathcal{P}^{21}_{2}\mathcal{P}^{12}_{m'}\mathcal{P}^{mm'}_{1} + \mathcal{P}^{12}_{22m}\mathcal{P}^{21}_{1}\mathcal{P}^{21}_{m'}\mathcal{P}^{mm'}_{1} \\ &= \mathcal{P}^{12}_{111}\mathcal{P}^{12}_{2}\mathcal{P}^{12}_{2}\mathcal{P}^{12}_{2} + \mathcal{P}^{12}_{222}\mathcal{P}^{21}_{1}\mathcal{P}^{21}_{1}\mathcal{P}^{21}_{1} + 3\mathcal{P}^{12}_{112}\mathcal{P}^{12}_{2}\mathcal{P}^{12}_{2}\mathcal{P}^{21}_{2} + 3\mathcal{P}^{12}_{122}\mathcal{P}^{12}_{2}\mathcal{P}^{21}_{1}\mathcal{P}^{21}_{2}\end{aligned}$$

A similar computation for $\dot{\mathcal{P}}_2$ is given in Appendix A. □

From now on, we denote the bi-vector $\dot{\mathcal{P}}_1$ obtained from Γ_1 unambiguously by $\dot{\mathcal{P}}$.

The definition of a flow $\dot{\mathcal{P}}$ to be $\partial_{\mathcal{P}}$-exact is that there exists a vector field $\mathcal{X}$, defined globally on N^n, whose coefficients are differential polynomials in $\mathcal{P}$, such that the flow is the Schouten bracket of this vector field $\mathcal{X}$ with the Poisson bi-vector: $\dot{\mathcal{P}} = \partial_{\mathcal{P}}\mathcal{X} = [\![\mathcal{P}, \mathcal{X}]\!]$. It is readily seen that for a two-dimensional vector field $\mathcal{X} = F\,\partial/\partial x + G\,\partial/\partial y$, we have that the only non-zero component of $[\![\mathcal{P}, \mathcal{X}]\!]$ is $[\![\mathcal{P}, \mathcal{X}]\!]^{12} = u_xF + u_yG - u(F_x + G_y)$. The conjugation equation $\dot{\mathcal{P}} = [\![\mathcal{P}, \mathcal{X}]\!]$ is now expressed via:

$$u_{xxx}u_y^3 - u_{yyy}u_x^3 - 3u_{xxy}u_xu_y^2 + 3u_{xyy}u_x^2u_y = u(F_x + G_y) - u_xF - u_yG. \tag{2}$$

Let us examine using Jets [5] whether equation (2) has any globally defined solutions $\mathcal{X}$.

Example 1. A solution $\mathcal{X} = F\,\partial/\partial x + G\,\partial/\partial y$ to equation (2), depending on the differential monomials u_σ not exceeding order $|\sigma| = 3$, is:

$$\begin{aligned} F &= u_{yyy}u_x^2 - 2u_{xyy}u_xu_y + u_{xxy}u_y^2 + 2u_{xx}u_{yy}u_y - 2u_{xy}^2u_y + cu_y, \\ G &= -u_{xxx}u_y^2 + 2u_{xxy}u_xu_y - u_{xyy}u_x^2 - 2u_{xx}u_{yy}u_x + 2u_{xy}^2u_x - cu_x,\ c \in \mathbb{R}. \end{aligned} \tag{3}$$

It is easily verified that this vector field is divergence-free[3]: $F_x + G_y = 0$.

One can obtain a more general solution by allowing higher differential orders $|\sigma|$ of the monomials u_σ that make up the vector field components F and G. The following method was hinted to us by P. Safronov and I. Khavkine.[4] Consider the weight homomorphism $\mathrm{wt}_x(u_\sigma) = (\#x \in \sigma)$, such that $\mathrm{wt}_x(u_\sigma u_\tau) = \mathrm{wt}_x(u_\sigma) + \mathrm{wt}_x(u_\tau)$. Similarly, we let $\mathrm{wt}_y(u_\sigma) = (\#y \in \sigma)$. We now let the differential polynomials that make up either side of equation (2) be quartic in u and homogeneous in wt_x and wt_y. Under these assumptions, it follows from comparing the left-hand and righ-hand sides of equation (2) that the polynomial terms of F and G are cubic in u_σ, that the differential polynomials of F have $\mathrm{wt}_x = 2$ and $\mathrm{wt}_y = 3$, and those of G have $\mathrm{wt}_x = 3$ and $\mathrm{wt}_y = 2$. Specifically, $F = \sum_{i=1}^N c_i u_{\sigma_1^i} u_{\sigma_2^i} u_{\sigma_3^i}$ such that $\mathrm{wt}_x(u_{\sigma_1^i} u_{\sigma_2^i} u_{\sigma_3^i}) = 2$ and $\mathrm{wt}_y(u_{\sigma_1^i} u_{\sigma_2^i} u_{\sigma_3^i}) = 3$ for all $1 \leqslant i \leqslant N$, where $N \in \mathbb{N}$ and $c_i \in \mathbb{R}$. Similarly, $G = \sum_{j=1}^M d_j u_{\tau_1^j} u_{\tau_2^j} u_{\tau_3^j}$ such that $\mathrm{wt}_x(u_{\tau_1^j} u_{\tau_2^j} u_{\tau_3^j}) = 3$ and $\mathrm{wt}_y(u_{\tau_1^j} u_{\tau_2^j} u_{\tau_3^j}) = 2$ for all $1 \leqslant j \leqslant M$, where $M \in \mathbb{N}$ and $d_j \in \mathbb{R}$. In fact, $M = N = 12$. We now substitute these polynomial expressions with undetermined coefficients for F and G into equation (2) and solve for the coefficients.

Proposition 2. Under all the above assumptions, the space of solutions to equation (2) is given by the vector fields $\mathcal{X} = F\,\partial/\partial x + G\,\partial/\partial y$, where

$$\begin{aligned}
F &= (a+b)u_{xxyyy}u^2 + b u_{xxyy}u_y u + c u_{xyyy}u_x u + u_{xxy}u_y^2 + d u_{xxy}u_{yy}u + e u_{xyy}u_{xy}u \\
&\quad + (a+d)u_{yyy}u_{xx}u - 2u_{xyy}u_x u_y + u_{yyy}u_x^2 + 2u_{xx}u_{yy}u_y - 2u_{xy}^2 u_y, \\
G &= -(a+b)u_{xxxyy}u^2 + a u_{xxxy}u_y u - (a+b+c)u_{xxyy}u_x u - (a+d)u_{xxx}u_{yy}u \\
&\quad + (a+c-e)u_{xxy}u_{xy}u - (a+c+d)u_{xyy}u_{xx}u - u_{xxx}u_y^2 + 2u_{xxy}u_x u_y - u_{xyy}u_x^2 \\
&\quad - 2u_{xx}u_{yy}u_x + 2u_{xy}^2 u_x,
\end{aligned} \tag{4}$$

and a, b, c, d, e are real constants. These vector fields are not divergence-free except for the case when all coefficients vanish (without the entire solution vanishing, because the solution then becomes equal to (3) at $c = 0$).

Now that we have a vector field $\mathcal{X}$ such that $\partial_{\mathcal{P}}\mathcal{X} = \dot{\mathcal{P}}$ over a 2-dimensional affine manifold, we inspect whether its construction can be repeated to trivialize $\dot{\mathcal{P}}$ on all finite-dimensional affine manifolds. For this, we pass from the dimension dependent differential polynomials to dimension independent representations of said differential polynomials, by using Kontsevich graphs.

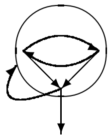
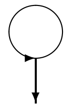
The graphs consist of ground vertices, drawn at the bottom of the figure, and internal vertices. All internal vertices in the graph represent a copy of $\mathcal{P}^{ij}$. Every vertex is the tail for an ordered pair of outgoing edges. These edges are labelled by the summation indices i, j in $\mathcal{P}^{ij}$ inside the vertex. In the 2-dimensional case, every vertex with $u = \mathcal{P}^{12}$

[3]If the vector field $\mathcal{X}$ is divergence-free, then equation (2) with respect to F and G splits into an algebraic equation and an additional restriction on the vector field components. The solution to the divergence-free equation is of the form $F = \varphi(x-y) - \psi(y)$ and $G = \varphi(x-y) - \chi(x)$, for functions φ ,ψ and χ to be determined.

[4]See http://mathoverflow.net/questions/209376. A solution reported there is obtained by setting the constant $c = 0$ in equation (3). Choosing constants a, b, c, d, e equal to zero in solution (4) from Proposition 2 yields again that solution.

has outgoing edges labelled 1 and 2, respectively. Incoming edges with value 1 or 2 imply differentiation of the target vertex with respect to x^1 or x^2, respectively. Finally, the internal vertices connected by edges form a product of bi-vector components $u = \mathcal{P}^{12}$ and derivatives thereof. The result is then multiplied by the arguments of the ground vertices that are differentiated as specified by the incoming edges.

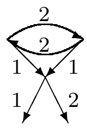
Lemma 3. Solving equation (2), the divergence-free vector field (3) in Example 1 is realizable in terms of the Kontsevich graphs:

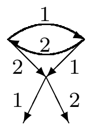
$\mathcal{X} =$  $+ c \cdot$  . (5)

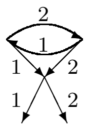
Proof. Whenever $F_x + G_y = 0$ on $\mathbb{R}^2 \ni (x, y)$, the vector field components F and G can be potentiated such that $F = H_y, G = -H_x$; we let

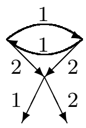
$$H = u_{xx}u_y^2 - 2u_{xy}u_xu_y + u_{yy}u_x^2 + cu,$$

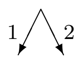
where H can be viewed as a scalar function. These differential monomials can be written as Kontsevich graphs (see above):

$\mathcal{H} =$ 

$-$ 

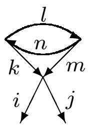
$-$ 

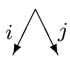
$+$ 

$+ c \cdot$ 

.

The five graphs with fixed values either 1 or 2 can be obtained from the two labelled graphs below, by letting the labels in (i, j, k, l, m, n) and (i, j) all run over the values 1 and 2 in a sum.

$\mathcal{H} = \frac{1}{2} \Bigg($ 

$+ c \cdot$ 

$\Bigg)$.

As a graph, $\mathcal{H}$ encodes a bi-vector. In order to obtain F and G, respectively, one must differentiate with respect to x and y. Observe that there are two edges, labelled i and j, falling on ground vertices. Since a one-vector is encoded by a graph with only 1 ground vertex, we let one of these edges, edge j, fall, by the Leibniz rule, on the above three internal vertices. Summation over the index j guarantees differentiation with respect to both x and y.The same goes for the second graph. We have represented the vector field $\mathcal{X}$ by two Kontsevich graphs, see (5). □

Remark 1. The generic vector field described in Proposition 2 cannot be realized in terms of Kontsevich graphs, due to the presence of terms like $u_{xxyyy}u^2$ when $a + b \neq 0$. This term would have to be represented by a graph with 5 edges falling on a single vertex. Since there are only 3 vertices in total, one vertex would have to send both its edges to the vertex encoding u_{xxyyy}. We know however, that graphs containing such 'double edges' vanish identically.

Remark 2. The vector field described by the two graphs in (5) exists in higher dimensions but no longer solves the conjugation equation $[\![\mathcal{P}, \mathcal{X}]\!] = \dot{\mathcal{P}}$. This is verified by evaluating the Poisson differential acting on the graphs encoding $\mathcal{X}$. Comparing the result with the graph realization of the first tetrahedral flow, as described in [4, Figure

2], one observes that the graphs in the Schouten bracket $[\![\mathcal{P}, \mathcal{X}]\!]$ do not equal those in the flow $\mathcal{Q}_{1:6}$. Therefore, the graphs in equation (5) cannot be expected to trivialize the flows of all Poisson structures on all manifolds of dimension $n \geqslant 3$.

Given any lattice L in $\mathbb{R}^n$, as a finitely generated abelian group, there exist a basis for this lattice, denoted by $b_1, \ldots, b_m$ with $m \leqslant n$. By the Gram-Schmidt orthogonalization process, there exists an invertible map between the this basis and an orthogonal basis $o_1, \ldots, o_m$. This orthogonal basis is not necessarily a basis for the lattice L, but rather one for a lattice $\widehat{L}$ that is isomorphic to L.

Proposition 4. Consider a lattice L in $\mathbb{R}^n$ and the associated projection under taking the quotient $\pi : \mathbb{R}^n \to \mathbb{R}^n/L$. Let $\mathcal{P}$ be a Poisson bi-vector on the n-dimensional affine manifold $\mathbb{R}^n/L$, whose coefficients are L-periodic Fourier series. Then

(1) the image of the flow $\dot{\mathcal{P}}$ under projection is a well-defined bi-vector on $M^n = \mathbb{R}^n/L$, and
(2) for $n = 2$, the image under projection of the trivializing vector field $\mathcal{X}$ is well defined on $M^n = \mathbb{R}^n/L$.

Proof. The coefficients $\mathcal{P}^{ij}$ of $\mathcal{P}$ are L-periodic Fourier series. Products of derivatives of these Fourier series yield new products of trigonometric functions with certain wavenumbers. The expressions for $\dot{\mathcal{P}}$, $\mathcal{Q}_{1:6}$ and $\mathcal{X}$ are all of this form. These products are reduced by trigonometric substitutions, resulting in Fourier series of higher wavenumbers; The wavenumbers of these new expression are no smaller than the original wavenumbers. Therefore, the coefficients of $\dot{\mathcal{P}}$, $\mathcal{Q}_{1:6}$ and $\mathcal{X}$ are also L-periodic Fourier series. We conclude that the image under the projection π of the flows $\dot{\mathcal{P}}$ and $\mathcal{Q}_{1:6}$ and the vector field $\mathcal{X}$ are well-defined on $M^n = \mathbb{R}^n/L$. □

Example 2. Let us consider a flow defined on the 2-torus $\mathbb{T}^2$ with periods $\{1, 1\}$. We inspect the following 2-dimensional bi-vector $\mathcal{P}$, well-defined, on that torus:

$$\mathcal{P}^{12}(x, y) = u(x, y) = \alpha_{1,1} \sin(2\pi x) \cos(2\pi y), \ \alpha_{1,1} \in \mathbb{R}.$$

We can compute the corresponding flow using formula (1):

$$\begin{aligned} \mathcal{Q}^{12}_{1:0} = &-128\alpha^4_{1,1}\pi^6 \cos(2\pi x) \cos(2\pi y) \sin^3(2\pi x) \sin^3(2\pi y) \\ &+ 128\alpha^4_{1,1}\pi^6 \sin(2\pi x) \sin(2\pi y) \cos^3(2\pi x) \cos^3(2\pi y). \end{aligned}$$

This expression contains products of sines and cosines, which we reduce using trigonometric substitutions to obtain:

$$\mathcal{Q}^{12}_{1:0} = 16\alpha^4_{1,1}\pi^6 \left(\sin(8\pi x) \sin(4\pi y) + \sin(4\pi x) \sin(8\pi y)\right),$$

Which is again well-defined on our torus. From a direct computation and trigonometric substitutions, it follows that:

$$\begin{aligned} H = -16\alpha^3_{1,1}\pi^4 \sin(2\pi x) \cos(2\pi y) \big(&\sin^2(2\pi x) \sin^2(2\pi y) \\ &+2 \cos^2(2\pi x) \sin^2(2\pi y) + \cos^2(2\pi y) \cos^2(2\pi x) \ \big). \end{aligned}$$

The vector field $\mathcal{X}$ is obtained through $F = H_y$ and $G = -H_x$. The obtained expressions for F and G are again reduced, by trigonometric substitutions, yielding:

$$F = \quad 8\alpha^3_{1,1}\pi^5 \left(-2 \cos(2\pi x) \cos(2\pi y) - 3 \cos(6\pi x) \cos(2\pi y) + \cos(2\pi x) \cos(6\pi y)\right),$$

$$G = -8\alpha_{1,1}^3\pi^5 \left(\; 2\sin(2\pi x)\sin(2\pi y) - 3\sin(2\pi x)\sin(6\pi y) + \sin(6\pi x)\sin(2\pi y)\right).$$

We now verify that this vector field $\mathcal{X}$ on $\mathbb{R}^2$ obtained from the first graph in the above Lemma is well-defined on $\mathbb{T}^2$.

Conclusion. When restricting to 2-dimensional affine manifolds, only one of the two graphs in the Kontsevich tetrahedral flow has a non-zero contribution. The cocycle condition holds trivially and we proved that the flow is a coboundary by constructing a trivializing vector field. We also showed how the divergence free part of the trivializing vector field is realizable in terms of Kontsevich graphs. Finally, we explained why both the flow $\dot{\mathcal{P}}$ and vector field $\mathcal{X}$ remain well-defined when taking a quotient over a lattice with respect to which the original Poisson bi-vector was periodic.

Acknowledgements. The author thanks R. Buring for cooperation and A. V. Kiselev for guidance and constructive criticism. The author is grateful to the organizers of SDSP III conference (3–7 August 2015, CVUT Děčín, Czech Republic) for stimulating discussions. This research was supported in part by the Graduate School of Science (RuG).

References

[1] *Kontsevich M.* (1997) Formality conjecture. Deformation theory and symplectic geometry (Ascona 1996, D. Sternheimer, J. Rawnsley and S. Gutt, eds), Math. Phys. Stud. **20**, Kluwer Acad. Publ., Dordrecht, 139–156.

[2] *Kontsevich M.* (2003) Deformation quantization of Poisson manifolds, *Lett. Math. Phys.* **66**:3, 157–216. (*Preprint* `q-alg/9709040`)

[3] *Bouisaghouane A., A. V. Kiselev* (2016) Do the Kontsevich tetrahedral flows preserve or destroy the space of Poisson bi-vectors? *Preprint* `arXiv:1609.06677` [q-alg], 10 p.

[4] *Bouisaghouane A., Buring R. and A. V. Kiselev* (2016) The Kontsevich tetrahedral flows revisited, *Preprint* `arXiv:1608.01710` (v3) [q-alg], 26 p.

[5] *Baran H. and Marvan M.* Jets. A software for differential calculus on jet spaces and diffieties, ver. 5.82 (May 2015) for Maple 15.

Appendix A. Vanishing of Γ_2 in 2D

To show that $\frac{1}{2}\left(\Gamma_2^{12}(\mathcal{P}) - \Gamma_2^{21}(\mathcal{P})\right) = 0$ holds in the 2-dimensional case, we expand the sum in the formula of Γ_2^{ij} for $i, j, k, \ell, m, k', \ell', m' \in \{1, 2\}$:

$$\begin{aligned}
\Gamma_2^{12}(\mathcal{P}) &= \sum_{j,k,\ell,k',\ell',m'=1}^{2} \mathcal{P}_{k\ell}^{1j}\mathcal{P}_{k'\ell'}^{k2}\mathcal{P}_{m'}^{k'\ell}\mathcal{P}_{j}^{m'\ell'} \\
&= \sum_{\ell,k',\ell',m'=1}^{2} \mathcal{P}_{1\ell}^{12}\mathcal{P}_{k'\ell'}^{12}\mathcal{P}_{m'}^{k'\ell}\mathcal{P}_{2}^{m'\ell'} \\
&= \sum_{k',\ell',m'=1}^{2} \mathcal{P}_{11}^{12}\mathcal{P}_{k'\ell'}^{12}\mathcal{P}_{m'}^{k'1}\mathcal{P}_{2}^{m'\ell'} + \mathcal{P}_{12}^{12}\mathcal{P}_{k'\ell'}^{12}\mathcal{P}_{m'}^{k'2}\mathcal{P}_{2}^{m'\ell'} \\
&= \sum_{\ell',m'=1}^{2} \mathcal{P}_{11}^{12}\mathcal{P}_{2\ell'}^{12}\mathcal{P}_{m'}^{21}\mathcal{P}_{2}^{m'\ell'} + \mathcal{P}_{12}^{12}\mathcal{P}_{1\ell'}^{12}\mathcal{P}_{m'}^{12}\mathcal{P}_{2}^{m'\ell'} \\
&= \mathcal{P}_{11}^{12}\mathcal{P}_{22}^{12}\mathcal{P}_{1}^{21}\mathcal{P}_{2}^{12} + \mathcal{P}_{11}^{12}\mathcal{P}_{21}^{12}\mathcal{P}_{2}^{21}\mathcal{P}_{2}^{21} \\
&\quad + \mathcal{P}_{12}^{12}\mathcal{P}_{12}^{12}\mathcal{P}_{1}^{12}\mathcal{P}_{2}^{12} + \mathcal{P}_{12}^{12}\mathcal{P}_{11}^{12}\mathcal{P}_{2}^{12}\mathcal{P}_{2}^{21} \\
&= -u_{xx}u_{yy}u_xu_y + u_{xx}u_{xy}u_y^2 + u_{xy}^2u_xu_y - u_{xy}u_{xx}u_y^2 \\
&= -u_{xx}u_{yy}u_xu_y + u_{xy}^2u_xu_y,
\end{aligned}$$

$$\begin{aligned}
\Gamma_2^{21}(\mathcal{P}) &= \sum_{j,k,\ell,k',\ell',m'=1}^{2} \mathcal{P}_{k\ell}^{2j}\mathcal{P}_{k'\ell'}^{k1}\mathcal{P}_{m'}^{k'\ell}\mathcal{P}_{j}^{m'\ell'} \\
&= \sum_{\ell,k',\ell',m'=1}^{2} \mathcal{P}_{2\ell}^{21}\mathcal{P}_{k'\ell'}^{21}\mathcal{P}_{m'}^{k'\ell}\mathcal{P}_{1}^{m'\ell'} \\
&= \sum_{k',\ell',m'=1}^{2} \mathcal{P}_{21}^{21}\mathcal{P}_{k'\ell'}^{21}\mathcal{P}_{m'}^{k'1}\mathcal{P}_{1}^{m'\ell'} + \mathcal{P}_{22}^{21}\mathcal{P}_{k'\ell'}^{21}\mathcal{P}_{m'}^{k'2}\mathcal{P}_{1}^{m'\ell'} \\
&= \sum_{\ell',m'=1}^{2} \mathcal{P}_{21}^{21}\mathcal{P}_{2\ell'}^{21}\mathcal{P}_{m'}^{21}\mathcal{P}_{1}^{m'\ell'} + \mathcal{P}_{22}^{21}\mathcal{P}_{1\ell'}^{21}\mathcal{P}_{m'}^{12}\mathcal{P}_{1}^{m'\ell'} \\
&= \mathcal{P}_{21}^{21}\mathcal{P}_{22}^{21}\mathcal{P}_{1}^{21}\mathcal{P}_{1}^{12} + \mathcal{P}_{21}^{21}\mathcal{P}_{21}^{21}\mathcal{P}_{2}^{21}\mathcal{P}_{1}^{21} \\
&\quad + \mathcal{P}_{22}^{21}\mathcal{P}_{12}^{21}\mathcal{P}_{1}^{12}\mathcal{P}_{1}^{12} + \mathcal{P}_{22}^{21}\mathcal{P}_{11}^{21}\mathcal{P}_{2}^{12}\mathcal{P}_{1}^{21} \\
&= -u_{xy}u_{yy}u_x^2 + u_{xy}^2u_xu_y + u_{xy}u_{yy}u_x^2 - u_{xx}u_{yy}u_xu_y \\
&= -u_{xx}u_{yy}u_xu_y + u_{xy}^2u_xu_y.
\end{aligned}$$