Melnikov's Method for Homoclinic and Heteroclinic Orbits

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Abstract

A certain class of dynamical systems, which contain fixed points connected by homoclinic or heteroclinic orbits can show chaotic dynamics when subjected to a time periodic perturbation. Melnikov developed a method to predict such behaviour based on the Melnikov function whose zeros correspond to transversal intersections implying chaotic dynamics. The method is described and applied to two cases of the Duffing oscillator.

Contents

1 Introduction

For a certain class of dynamical systems having one or more hyperbolic equilibrium points connected to itself by some homoclinic or heteroclinic orbit we see a splitting of solution regimes under time periodic perturbation, i.e. the stable and unstable manifolds no long coincide. It is shown that this 'breaking up' of the manifolds can give rise to transversal intersections between them. It was shown by Moser [2] that such intersections give rise to chaotic behaviour in the system. For systems having a hyperbolic fixed point connected to itself by the separatix solution Melnikov developed a method to predict the occurrence of such chaotic behaviour.

In this thesis we will describe the derivation of the homoclinic Melnikov function following Wiggins [3]. After the derivation we will be looking at two different cases of the Duffing oscillator: an inverted Duffing oscillator and a soft spring Duffing oscillator respectively. Using the Melnikov function as we have derived it allows us to find zeros of the function which correspond to transversal intersections of the stable and unstable manifold which in turn prove the existence of chaotic behaviour in the system. After computing the threshold curves of the parameters for such behaviour to occur we will check these results numerically.

2 Melnikov's Method

In the following section we will be describing a method for proving the existence of transverse intersections and therefore chaotic dynamics in time periodically perturbed dynamic systems. The discussion of the method will closely follow Wiggins [3].

2.1 Preliminaries

We will be considering systems of the form

$$\begin{aligned} \dot{x} &= \frac{\partial H}{\partial y}(x,y) + \epsilon f_1(x,y,t,\epsilon) \\ \dot{y} &= -\frac{\partial H}{\partial x}(x,y) + \epsilon f_2(x,y,t,\epsilon), \quad (x,y) \in \mathbb{R}^2. \end{aligned} \tag{2.1}$$

Setting $q = (x,y)$, $DH = (\frac{\partial H}{\partial x}, \frac{\partial H}{\partial y})$, $f = (f_1, f_2)$ and $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ we can rewrite this as

$$\dot{q} = JDH(q) + \epsilon f(q,t,\epsilon). \tag{2.2}$$

We assume $C^r, r > 2$ differentiability for H and f and we assume that that f is time periodic with period $T = \frac{2\pi}{\omega}$.

When setting $\epsilon = 0$ the system becomes Hamiltonian. Note that Melnikov's method does not require the system to be Hamiltonian. It does however simplify things and for our purposes it is sufficient to look at the Hamiltonian case. In order for our derivation to work out we do have to make some other assumptions on the system.

Still considering $\epsilon = 0$ we firstly assume the system to have a hyperbolic equilibrium point p_0, connected to itself by a homoclinic orbit. For the orbit we write $q_0(t) = (x_0(t), y_0(t))$.

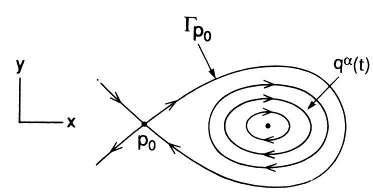


Figure 1: General structure of the system.

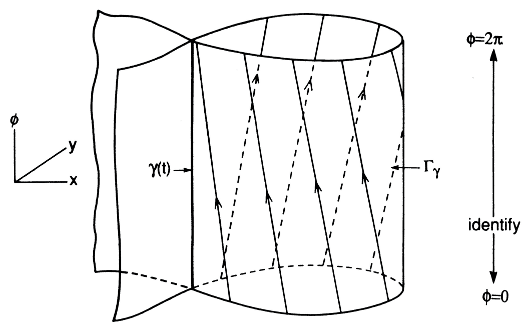


Figure 2: Structure of the autonomous system.

Secondly let the union of the stable and unstable solution set and p_0 be Γ_{p_0}, i.e.

$$\Gamma_{p_0} = \{q \in \mathbb{R}^2 \mid q = q_0(t),\ t \ \in \ \mathbb{R}\} \cup \{p_0\} = W^s(p_0) \cap W^u(p_0) \cup \{p_0\} \quad (2.3)$$

Then the interior of Γ_{p_0} contains a continuous family of peroidic orbits $q^\alpha(t)$ with period $T^\alpha,\ \alpha \in (0, 1)$. We assume that $\lim_{\alpha\to 0} q^\alpha(t) = q_0(t)$, and that $\lim_{\alpha\to 0} T^\alpha = \infty$.

It is clear that (2.1) is time dependent. We want to rewrite it as an autonomous three dimensional system by introducing a new time variable ϕ such that $\dot{\phi} = \omega$, resulting in

$$\begin{aligned} \dot{x} &= \frac{\partial H}{\partial y}(x, y) + \epsilon f_1(x, y, \phi, \epsilon), \\ \dot{y} &= -\frac{\partial H}{\partial x}(x, y) + \epsilon f_2(x, y, \phi, \epsilon), \\ \dot{\phi} &= \omega, \end{aligned} \quad (2.4)$$

or following (2.2)

$$\begin{aligned} \dot{q} &= JDH(q) + \epsilon f(q, \phi, \epsilon), \\ \dot{\phi} &= \omega. \end{aligned} \quad (2.5)$$

As a consequence of this the fixed point p_0 becomes a periodic orbit ($\epsilon = 0$). We denote this orbit as

$$\gamma(t) = (p_0, \phi(t) = \omega t + \phi_0) \quad (2.6)$$

This gives rise to a set of stable and unstable manifolds which can be expressed in term of $\gamma(t)$. We denote these two dimensional manifolds as $W^s(\gamma(t))$ and

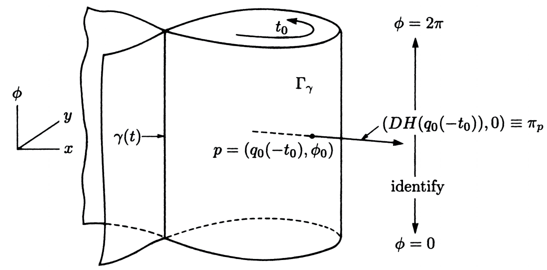


Figure 3: Visualisation of π_p

$W^u(\gamma(t))$ which coincide along a two-dimensional homoclinic manifold. We denote this manifold by Γ_γ.

Having set the field we can continue with our development of the homoclinic Melnikov method. This can roughly be split into three parts, which we will discuss in the following three sections. Firstly we will develop a parametrisation for the homoclinic manifold of the unperturbed system. Then we will find a method for measuring the distance between the split perturbed manifolds which will eventually allow us to derive the Melnikov function.

2.2 Parametrisation

Considering the unperturbed homoclinic orbit q_0 we note that the time of flight to any point on q_0 from p_0 is unique. Therefore any point can be uniquely written as $q_0(-t_0)$ where t_0 is the time of flight from $q_0(t_0)$ to $q_0(0)$ along q_0. Looking at cross-sections $\phi_0 \in (0, 2\pi]$ allows us to represent any point on Γ_γ as $(q_0(-t_0), \phi_0)$. As every point on q_0 has a unique time of flight a map $(t_0, \phi_0) \to (q_0(-t_0), \phi_0)$ is clearly bijective and as a result we can write

$$\Gamma_\gamma = \{(q, \phi) \in \mathbb{R}^2 \times S^1 \mid q = q_0(-t_0),\ t_0 \in \mathbb{R}^1; \phi = \phi_0 \in (0, 2\pi]\}. \tag{2.7}$$

Now our goal is to find a measure for the distance between the stable and unstable manifold. In order to achieve this we first define a vector at each point $p := (q_0(-t_0), \phi_0) \in \Gamma_\gamma$ as

$$\pi_p = \left(\frac{\partial H}{\partial x}(x_0(-t_0), y_0(-t_0)), \frac{\partial H}{\partial y}(x_0(-t_0), y_0(-t_0)), 0 \right). \tag{2.8}$$

Note that the stable and unstable manifold intersect our vector π_p transversely at p.

2.3 Finding a Measure for the Distance Between the Unstable and Stable Manifolds

In order to be able to give a measure for the distance between the manifolds due to the perturbation we first need some preliminary results concerning the behaviour of the system under perturbation. The following proposition is given regarding the persistence of $\gamma(t)$ under perturbation.

Proposition 1. *For sufficiently small ϵ, $\gamma(t)$ will persist as a periodic orbit of the form $\gamma_\epsilon(t) = \gamma(t) + \mathcal{O}(\epsilon)$. $\gamma_\epsilon(t)$ has the same stability type as $\gamma(t)$ with $\gamma_\epsilon(t)$ depending on ϵ in a C^r manner. Also, considering the local (within some $\mathcal{O}(\epsilon)$ region) manifolds, $W^s_{loc}(\gamma_\epsilon(t))$ and $W^u_{loc}(\gamma_\epsilon(t))$ are C^r ϵ-close to $W^s_{loc}(\gamma(t))$ and $W^u_{loc}(\gamma(t))$, respectively.*

The proof of this proposition can be found in [1] and is omitted in this thesis. In order to find the global stable and unstable manifold under perturbation we consider the time evolution of local manifolds. We write $\phi_t(.)$ for the flow generated by the system (2.5). The global stable and unstable manifold can than be defined

$$\begin{aligned} W^s(\gamma_\epsilon(t)) &= \bigcup_{t\leq 0} \phi_t(W^s_{loc}(\gamma_\epsilon(t)), \\ W^u(\gamma_\epsilon(t)) &= \bigcup_{t\geq 0} \phi_t(W^u_{loc}(\gamma_\epsilon(t)). \end{aligned} \tag{2.9}$$

We note that $\phi_t(.)$ is a diffeomorphism which is C^r in ϵ. Now if we restrict ourselves to compact sets in $\mathbb{R}^2 \times S^1$ containing $W^{u,s}(\gamma_\epsilon(t))$, then $W^{u,s}(\gamma_\epsilon(t))$ are C^r functions of ϵ on these compact sets. When studying how the manifolds break up under perturbation we will restrict our self to a $\mathcal{O}(\epsilon)$ neighbourhood of Γ_γ.

Now let us look at Proposition 1 again and give a more quantitative description of its meaning. It states that for some small $\epsilon = \epsilon_0$ we can define a neighbourhood $\mathcal{N}(\epsilon_0)$ that contains $\gamma(t)$ such that $\gamma(t)$ is at a $\mathcal{O}(\epsilon_0)$ distance from the boundary of this neighbourhood. As previously stated $\gamma_\epsilon(t) = \gamma(t) + \mathcal{O}(\epsilon)$ and therefore $\gamma_\epsilon(t)$ is contained in $\mathcal{N}(\epsilon_0)$ for $0 < \epsilon < \epsilon_0$. We also define $W^{u,s}_{loc}(\gamma(t)) := W^{u,s}(\gamma(t)) \cap \mathcal{N}(\epsilon_0)$ and see that this is C^r ϵ-close to $W^{u,s}_{loc}(\gamma_\epsilon(t))$. For our neighbourhood $\mathcal{N}$ we choose a solid torus

$$\mathcal{N}(\epsilon_0) = \{(q,\phi) \in \mathbb{R}^2 |\ |q - p_0| \leq C\epsilon_0, \phi \in (0, 2\pi]\} \tag{2.10}$$

for some positive constant C.

For many of the following arguments it is sufficient to look at a cross-section of

Γ_γ with a surface of section defined as

$$\Sigma^{\phi_0} = \{(q, \phi) \in \mathbb{R}^2 | \phi = \phi_0\}. \tag{2.11}$$

The intersection of Γ_γ with Σ^{ϕ_0} clearly becomes our original unperturbed orbit Γ_{p_0}, independently of a choice of ϕ_0. This is due to the fact that our unperturbed system is independent of ϕ. In the following step we will be looking at projections onto the Σ^{ϕ_0} plane of trajectories of the unperturbed and the perturbed system. We denote these as

$$(q_0(t), \phi_0), \tag{2.12}$$
$$(q_\epsilon(t), \phi_0) \tag{2.13}$$

respectively. Note that as $(q_\epsilon(t), \phi_0)$ is time dependent we do not know the shape of its trajectory. We will now define a measure for the distance between the manifolds as follows. By the definition of π_p at any point $p \in \Gamma_\gamma$ the unperturbed stable and unstable manifold intersect π_p transversely. As we have shown that the perturbed stable and unstable manifolds are C^r in ϵ we know that for sufficiently small ϵ they will intersect π_p. We call the points of intersection p^s_ϵ and p^u_ϵ for the stable and unstable manifold respectively. We can then write

$$d(p, \epsilon) := |p^u_\epsilon - p^s_\epsilon| \tag{2.14}$$

for an unsigned measure of the distance. As a signed measure will be useful in determining the orientations at which the manifolds will *fold* around each other we will rewrite this as

$$d(p, \epsilon) = \frac{(p^u_\epsilon - p^s_\epsilon) \cdot (DH(q_0(-t_0)), 0)}{||DH(q_0(-t_0))||}. \tag{2.15}$$

Now as we do not know the behaviour of the perturbed trajectory it may intersect π_p multiple (infinitely many) times. This raises the question which point we should pick for our calculations. We use the following definition.

Definition 1. *Let $p^s_{\epsilon,i} \in W^s(\gamma(t)) \cap \pi_p$ and $p^u_{\epsilon,i} \in W^u(\gamma(t)) \cap \pi_p$ for the unstable manifold with $i \in \mathcal{I}$ for some indexing set $\mathcal{I}$. Let $q^{s,u}_{\epsilon,i}(t), \phi(t)) \in W^{s,u}(\gamma_\epsilon(t))$ be orbits of the perturbed vector field satisfying $q^{s,u}_{\epsilon,i}(0), \phi(0)) = p^{s,u}_{\epsilon,i}$ respectively. Then*

1. *For some $i = \bar{i} \in \mathcal{I}$ we call $p_\epsilon, \bar{i}^s$ closest to $\gamma_\epsilon(t)$ in terms of positive time of flight along the stable perturbed vector field if for all $t > 0$, $(q^s_{\epsilon,i}, \phi_0) \cap \pi_p = \emptyset$.*
2. *For some $i = \bar{i} \in \mathcal{I}$ we call $p_\epsilon, \bar{i}^u$ closest to $\gamma_\epsilon(t)$ in terms of positive time of flight along the unstable perturbed vector field if for all $t < 0$, $(q^u_{\epsilon,i}, \phi_0) \cap \pi_p = \emptyset$.*

This gives us our desired points to measure from. As $p_\epsilon^{s,u}$ are the intersections of their respective orbits with our vector π_p for some value ϕ_0 we can rewrite (2.15) as

$$d(t, \phi_0, \epsilon) = \frac{DH(q_0(-t_0)) \cdot (q_\epsilon^u - q_\epsilon^s)}{||DH(q_0(-t_0))||}. \tag{2.16}$$

Now we have chosen our points of measurements according to definition 1 but we have not discussed the motivation behind this choice. We explain this by proving the following lemma.

Lemma 1. *Let $p^s_{\epsilon,\bar{i}}$ be a point on the stable manifold intersecting π_p that is not closest to $\gamma_\epsilon(t)$ according to definition 1 and let $\mathcal{N}(\epsilon_0)$ denote the neighbourhood containing $\gamma_{(\epsilon)}(t)$ defined in* (2.10)*. Let $(q_{\epsilon,\bar{i}}(t), \phi(t))$ be a trajectory of the perturbed stable manifold such that the trajectory is at $p^s_{\epsilon,\bar{i}}$ for $t = 0$. Then for sufficiently small ϵ the orbit must pass through $\mathcal{N}(\epsilon_0)$ before it can intersect π_p again (for $t > 0$).*

Note that this lemma considers the stable manifold but it can be proved in an analogous fashion for the unstable manifold. First consider an arbitrary point (q_0^s, ϕ_0) on $W^s(\gamma(t)) \cap \mathcal{N}(\epsilon_0)$ and consider an orbit $(q_0^s(t), \phi(t))$ such that $(q_0^s(0), \phi(0)) = (q_0^s, \phi_0)$. Then for some finite time T^s the orbit will reenter $\mathcal{N}(\epsilon_0)$. We will now consider a point of the perturbed manifold $(q_\epsilon^s, \phi_0) \in W^s_{loc}(y_\epsilon(t)) \cap \mathcal{N}(\epsilon_0)$, with its corresponding trajecotry $(q_\epsilon^s(t), \phi(t)) \in W^s(\gamma_\epsilon(t))$ such that $(q_\epsilon^s(0), \phi(0)) = (q_\epsilon^s, \phi_0)$. Then

$$|(q_\epsilon^s(t), \phi(t) - (q_0^s(t), \phi(t)))| = \mathcal{O}(\epsilon) \tag{2.17}$$

for $0 \leq t \leq \infty$ and according to Gronwall's inequality

$$|(q_\epsilon^s(t), \phi(t) - (q_0^s(t), \phi(t)))| = \mathcal{O}(\epsilon) \tag{2.18}$$

for $T^s \leq t \leq \infty$. So the perturbed orbit must follow the unperturbed orbit $\mathcal{O}(\epsilon)$ close until it reenters $\mathcal{N}(\epsilon_0)$. Now another possibility for multiple intersections with π_p would be the development of *kinks* in the trajectory which could allow for multiple intersections while remaining ϵ-close to the unperturbed orbit. To show that this will not happen we consider the tangent vector of the perturbed and unperturbed trajectories will also stay $\mathcal{O}(\epsilon_0)$ close for $T^s \leq t \leq \infty$. To see this note that (2.18) can be interpreted as

$$(q_\epsilon^s(t), \phi(t)) = (q_0^s(t) + \mathcal{O}(\epsilon), \phi(t)). \tag{2.19}$$

So we get a tangent vector

$$\begin{aligned} \dot{q}_\epsilon^s &= JDH(q_\epsilon^s) + \epsilon f(q_\epsilon^s, \phi(t)), \\ \dot{\phi} &= \omega. \end{aligned} \tag{2.20}$$

If we substitute (2.19) into (2.20) and taking a Taylor expansion over ϵ yields

$$\begin{aligned}\dot{q}_\epsilon^s =&\ JDH(q_0^s) + \mathcal{O}(\epsilon),\\ \dot{\phi} =&\ \omega.\end{aligned} \tag{2.21}$$

Similarly for tangent vectors of the unperturbed are given as

$$\begin{aligned}\dot{q}_0^s =&\ JDH(q_0^s),\\ \phi =&\ \dot{\omega}\end{aligned} \tag{2.22}$$

which is clearly $\mathcal{O}(\epsilon)$ close to (2.21). So multiple intersections due to kinks can not occur for ϵ small enough.

2.4 Derivation of the Melnikov Function

Having defined a function for measuring the distance between the manifolds we are now going to work towards an actually computable Melnikov function. We start by taking a taylor expansion over (2.16) for $\epsilon = 0$:

$$d(t_0, \phi_0, \epsilon) = d(t_0, \phi_0, 0) + \epsilon \frac{\partial d}{\partial \epsilon}(t_0, \phi_0, 0) + \mathcal{O}(\epsilon^2). \tag{2.23}$$

Now $d(t_0, \phi_0, 0)$ is the distance of the manifolds for $\epsilon = 0$, which evidently equals 0, and

$$\frac{\partial d}{\partial \epsilon}(t_0, \phi_0, 0) = \frac{DH(q_0(-t_0)) \cdot (\frac{\partial q_\epsilon^u}{\partial \epsilon}|_{\epsilon=0} - \frac{\partial q_\epsilon^s}{\partial \epsilon}|_{\epsilon=0})}{||DH(q_0(-t_0))||}. \tag{2.24}$$

We can now define the Melnikov functions as follows

$$M(t_0, \phi_0) \equiv DH(q_0(-t_0)) \cdot \left(\frac{\partial q_\epsilon^u}{\partial \epsilon}|_{\epsilon=0} - \frac{\partial q_\epsilon^s}{\partial \epsilon}|_{\epsilon=0} \right). \tag{2.25}$$

The first step towards computing a computable expression for $M(t_0, \phi_0)$ we first define a time dependent version of the Melnikov function

$$M(t; t_0, \phi_0) \equiv DH(q_0(-t_0)) \cdot \left(\frac{\partial q_\epsilon^u(t)}{\partial \epsilon}|_{\epsilon=0} - \frac{\partial q_\epsilon^s(t)}{\partial \epsilon}|_{\epsilon=0} \right) \tag{2.26}$$

where $q_\epsilon^{u,s}(t)$ satisfy $q_\epsilon^{u,s}(0) = q_\epsilon^{u,s}$ respectively and therefore $M(0; t_0, \phi_0) = M(t_0, \phi_0)$. Now in order to be able to keep the manipulations which will follow shortly somewhat readable we want to define some notations first. We set

$$\frac{\partial q_\epsilon^{u,s}(t)}{\partial \epsilon}|_{\epsilon=0} \equiv q_1^{u,s}(t)$$

and

$$\Delta^{u,s}(t) \equiv DH(q_0(t-t_0)) \cdot q_1^{u,s}(t). \tag{2.27}$$

The Melnikov function can then be written as

$$M(t;t_0,\phi_0) = \Delta^u(t) - \Delta^s(t). \tag{2.28}$$

We take the derivative of (2.27) with respect to t

$$\begin{aligned} \frac{d}{dt}(\Delta^{u,s}(t)) = &\left(\frac{d}{dt}(DH(q_0(t-t_0)))\right) \cdot q_1^{u,s}(t) \\ &+ DH(q_0(t-t_0)) \cdot \frac{d}{dt} q_1^{u,s}(t). \end{aligned} \tag{2.29}$$

We will now do some further investigation on the $\frac{d}{dt} q_1^{u,s}(t)$ term. Recall that we have previously defined

$$q_1^{u,s}(t) = \frac{\partial q_\epsilon^{u,s}(t)}{\partial \epsilon}\bigg|_{\epsilon=0}.$$

Now as $q_\epsilon^{u,s}$ solves equation (2.2) we can write

$$\frac{d}{dt}(q_\epsilon^{u,s}(t)) = JDH(q_\epsilon^{u,s}(t)) + \epsilon g(q_\epsilon^{u,s}(t), \phi(t), \epsilon). \tag{2.30}$$

Differentiating this with respect to ϵ and switching the order of differentiation gives us

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial q_\epsilon^{u,s}(t)}{\partial \epsilon}|_{\epsilon=0}\right) =& JD^2H(q_0(t-t_0))\frac{\partial q_\epsilon^{u,s}(t)}{\partial \epsilon}|_{\epsilon=0} \\ &+ g(q_0(t-t_0), \phi(t), 0). \end{aligned} \tag{2.31}$$

Or in terms of q_1 we get

$$\frac{d}{dt} q_1^{u,s} = JD^2H(q_0(t-t_0)) q_1^{u,s}(t) + g(q_0(t-t_0), \phi(t), 0). \tag{2.32}$$

Substituting (2.32) into (2.29) yields

$$\begin{aligned} \frac{d}{dt}(\Delta^{u,s}(t)) = &\left(\frac{d}{dt}(DH(q_0(t-t_0)))\right) \cdot q_1^{u,s}(t) \\ &+ DH(q_0(t-t_0)) \cdot JD^2H(q_0(t-t_0)) q_1^{u,s}(t) \\ &+ DH(q_0(t-t_0)) \cdot g(q_0(t-t_0), \phi(t), 0). \end{aligned} \tag{2.33}$$

Now this could hardly be called a simplification of matters if it were not for the following very convenient result

$$\begin{aligned}&\left(\frac{d}{dt}(DH(q_0(t-t_0)))\right)\cdot q_1^{u,s}(t)\\&+DH(q_0(t-t_0))\cdot JD^2H(q_0(t-t_0))q_1^{u,s}(t)=0.\end{aligned}\tag{2.34}$$

In order to prove (2.34) we first note that

$$\begin{aligned}\frac{d}{dt}(DH(q_0(t-t_0)))&=D^2H(q_0(t-t_0))\dot{q}_0(t-t_0)\\&=(D^2H(q_0(t-t_0)))(JDH(q_0(t-t_0))).\end{aligned}\tag{2.35}$$

Now if we let $q_1^{u,s}(t)=(x_1^{u,s}(t),y_1^{u,s}(t))$. We then get

$$\begin{aligned}(D^2H)(JDH)\cdot q_1^{u,s}&=\begin{pmatrix}\frac{\partial^2H}{\partial x^2}&\frac{\partial^2H}{\partial x\partial y}\\\frac{\partial^2H}{\partial x\partial y}&\frac{\partial^2H}{\partial y^2}\end{pmatrix}\begin{pmatrix}\frac{\partial H}{\partial y}\\-\frac{\partial H}{\partial x}\end{pmatrix}\cdot\begin{pmatrix}x_1^{u,s}\\y_1^{u,s}\end{pmatrix}\\&=x_1^{u,s}\Big[\frac{\partial^2H}{\partial x^2}\frac{\partial H}{\partial y}-\frac{\partial^2H}{\partial x\partial y}\frac{\partial H}{\partial x}\Big]\\&\quad+y_1^{u,s}\Big[\frac{\partial^2H}{\partial x\partial y}\frac{\partial H}{\partial y}-\frac{\partial^2H}{\partial y^2}\frac{\partial H}{\partial x}\Big]\end{aligned}\tag{2.36}$$

and

$$\begin{aligned}DH\cdot(JD^2H)q_1^{u,s}&=\begin{pmatrix}\frac{\partial H}{\partial x}\\\frac{\partial H}{\partial y}\end{pmatrix}\cdot\begin{pmatrix}\frac{\partial^2H}{\partial x\partial y}&\frac{\partial^2H}{\partial y^2}\\-\frac{\partial^2H}{\partial x^2}&\frac{\partial^2H}{\partial x\partial y}\end{pmatrix}\begin{pmatrix}x_1^{u,s}\\y_1^{u,s}\end{pmatrix}\\&=x_1^{u,s}\Big[\frac{\partial^2H}{\partial x\partial y}\frac{\partial H}{\partial x}-\frac{\partial^2H}{\partial x^2}\frac{\partial H}{\partial y}\Big]\\&\quad+y_1^{u,s}\Big[\frac{\partial^2H}{\partial y^2}\frac{\partial H}{\partial x}-\frac{\partial^2H}{\partial x\partial y}\frac{\partial H}{\partial y}\Big]\end{aligned}\tag{2.37}$$

and we get the desired result. Therefore we can write

$$\frac{d}{dt}(\Delta^{u,s}(t))=DH(q_0(t-t_0))\cdot g(q_0(t-t_0),\phi(t),0).\tag{2.38}$$

Now noting that the unstable orbit solves the system for $t\in[-\infty,0]$ and the stable orbit solves the system for $t\in(0,\infty)$ we integrate $\Delta^u(t)$ and $\Delta^s(t)$ from $-\tau$ to 0 and 0 to τ respectively, giving

$$\Delta^u(0)-\Delta^u(-\tau)=\int_{-\tau}^{0}DH(q_0(t-t_0))\cdot g(q_0(t-t_0),\omega t+\phi_0,0)dt\tag{2.39}$$

and

$$\Delta^s(\tau) - \Delta^s(0) = \int_0^\tau DH(q_0(t-t_0)) \cdot g(q_0(t-t_0), \omega t + \phi_0, 0)dt. \tag{2.40}$$

Adding up (2.39) and (2.40) and recalling that

$$M(t_0, \phi_0) = M(0; t_0, \phi_0) = \Delta^u(0) - \Delta^s(0)$$

gives

$$M(t_0, \phi_0) = \int_{-\tau}^\tau DH(q_0(t-t_0)) \cdot g(q_0(t-t_0), \omega t + \phi_0, 0)dt + \Delta^s(\tau) - \Delta^u(-\tau). \tag{2.41}$$

Next we want to consider the limit of (2.41) as $\tau \to \infty$. In order to do this we first want to show that

$$\lim_{\tau\to\infty} \Delta^s(\tau) = \lim_{\tau\to\infty} \Delta^u(-\tau) = 0. \tag{2.42}$$

To prove this we recall that

$$\Delta^{u,s} = DH(q_0(t-t_0)) \cdot q_1^{u,s}(t).$$

Let us consider the unstable manifold first. As $q_0(t-t_0)$ goes to a hyperbolic fixed point $DH(q_0(t-t_0))$ goes to zero. Now as $t \to \infty$ $q_\epsilon^u(t)$ approaches γ_ϵ. So

$$\lim_{t\to\infty} q_1^u(t) = \frac{\partial q_\epsilon^u(t)}{\partial \epsilon}|_{\epsilon=0} \to \frac{\partial \gamma_\epsilon^u(t)}{\partial \epsilon}|_{\epsilon=0}.$$

As $\gamma_\epsilon(t) = \gamma(t) + \mathcal{O}(\epsilon)$ we know that it is bounded. And thus (2.42) holds. Also as we have just reasoned that $DH(q_0(t-t_0))$ approached zero for $t \to \pm\infty$ and as g is a bounded function we know that

$$\int_{-\infty}^\infty DH(q_0(t-t_0)) \cdot g(q_0(t-t_0), \omega t + \phi_0, 0)dt$$

converges absolutely. Transforming $t \to t + t_0$ allows us to write

$$M(t_0, \phi_0) = \int_{-\infty}^\infty DH(q_0(t)) \cdot g(q_0(t), \omega t + \omega t_0 + \phi_0, 0)dt. \tag{2.43}$$

Having found our integral all that remains for us to do is to show that its zeros correspond to transversal intersections of the manifolds. As g is periodic we know $M(t_0, \phi_0)$ is periodic in t_0 and in ϕ_0, with period $2\pi/\omega$ and 2π respectively. Thus varying t_0 and ϕ_0 has the same effect. It follows from this periodicity and (2.43) that

$$\frac{\partial M}{\partial \phi_0}(t_0, \phi_0) = \frac{1}{\omega}\frac{\partial M}{\partial t_0}(t_0, \phi_0); \tag{2.44}$$

and therefore $\frac{\partial M}{\partial t_0} = 0$ iff $\frac{\partial M}{\partial \phi_0} = 0$ and the following theorem can be stated in terms of either one. We will choose to do so in terms of $\frac{\partial M}{\partial t_0}$.

Theorem 1. *Suppose we have a point* $(t_0, \phi_0) = (\bar{t_0}, \bar{\phi_0})$ *such that*

1. $M(\bar{t_0}, \bar{\phi_0}) = 0$ *and*
2. $\frac{\partial M}{\partial t_0}|_{(\bar{t_0}, \bar{\phi_0})} \neq 0$.

Then for ϵ *sufficiently small,* $W^s(\gamma_\epsilon(t))$ *and* $W^u(\gamma_\epsilon(t))$ *intersect transversely at* $q_0(-\bar{t_0}) + \mathcal{O}(\epsilon), \bar{\phi_0})$*. Moreover, if* $M(t_0, \phi_0) \neq 0$ *for all* $(t_0, \phi_0) \in \mathbb{R}^1 \times S^1$*, then* $W^s(\gamma_\epsilon(t)) \cup W^u(\gamma_\epsilon(t)) = \emptyset$*.*

In order to prove this, recall from (2.23), (2.24) and (2.25) that we have

$$d(t_0, \phi_0, \epsilon) = \epsilon \frac{M(t_0, \phi_0)}{||DH(q_0(-t_0))||} + \mathcal{O}(\epsilon^2). \tag{2.45}$$

Defining $\bar{d}(t_0, \phi_0, \epsilon) = \epsilon d(t_0, \phi_0, \epsilon)$ we get

$$\bar{d}(t_0, \phi_0, \epsilon) = \frac{M(t_0, \phi_0)}{||DH(q_0(-t_0))||} + \mathcal{O}(\epsilon) \tag{2.46}$$

and clearly if $\bar{d}(t_0, \phi_0, \epsilon) = 0$ then $d(t_0, \phi_0, \epsilon) = 0$. And we can continue on with $\bar{d}$. At the point $(\bar{t_0}, \bar{\phi_0}, 0)$ the assumption that $M(\bar{t_0}, \bar{\phi_0}) = 0$ clearly implies $\bar{d}(\bar{t_0}, \bar{\phi_0}, 0) = 0$. From the second assumption of the theorem we get

$$\frac{\partial \bar{d}}{\partial t_0}|_{(\bar{t_0}, \bar{\phi_0}, 0)} = \frac{1}{||DH(q_0(-\bar{t_0}))||} \frac{\partial M}{\partial t_0}|_{(\bar{t_0}, \bar{\phi_0})} \neq 0.$$

From the implicit function we now conclude that for $|\phi - \phi_0|$ and ϵ sufficiently small there exists a function

$$t_0 = t_0(\phi_0, \epsilon)$$

s.t.

$$\bar{d}(t_0(\phi_0, \epsilon), \phi_0, \epsilon) = 0.$$

So we have an intersection of $W^s(\gamma_\epsilon(t))$ and $W^u(\gamma_\epsilon(t))$, $\mathcal{O}(\epsilon)$ close to $(q_0(-t_0), \phi_0)$. Now we need to prove the transversality of the intersection in order for us to be able to apply Moser's theorem. Suppose $W^s(\gamma_\epsilon(t))$ and $W^u(\gamma_\epsilon(t))$ intersect at some point p, then the intersection is said to be transversal if

$$T_p W^s(\gamma_\epsilon(t)) + T_p W^u(\gamma_\epsilon(t)) = \mathbb{R}^3. \tag{2.47}$$

Now q_ϵ^u and q_ϵ^s are points on the trajectories $q_\epsilon^u(t)$ and $q_\epsilon^s(t)$ respectively and we have previously show that we can parametrise such points in terms of t_0 and ϕ_0 we can write a basis for $T_p W^s(\gamma_\epsilon(t))$ and $T_p W^u(\gamma_\epsilon(t))$ as

$$\left(\frac{\partial q_\epsilon^u}{\partial t_0}, \frac{\partial q_\epsilon^u}{\partial \phi_0} \right) \tag{2.48}$$

and

$$\left(\frac{\partial q_\epsilon^s}{\partial t_0}, \frac{\partial q_\epsilon^s}{\partial \phi_0}\right) \tag{2.49}$$

respectively. Now intersections will be tangent (thus not transversal) if either

$$\frac{\partial q_\epsilon^u}{\partial t_0} = \frac{\partial q_\epsilon^s}{\partial t_0} \tag{2.50}$$

or

$$\frac{\partial q_\epsilon^u}{\partial \phi_0} = \frac{\partial q_\epsilon^s}{\partial \phi_0}. \tag{2.51}$$

Now we have written zeros of the Melnikov function as $(\bar{t_0} + \mathcal{O}(\epsilon), \bar{\phi_0})$. Looking at the derivative of $d(t_0, \phi_0, \epsilon)$ at these points gives

$$\begin{aligned} \frac{\partial d}{\partial t_0}(\bar{t_0}, \bar{\phi_0}, \epsilon) &= \frac{DH(q_0(-\bar{t_0})) \cdot ((\partial q_\epsilon^u)/(\partial t_0) - (\partial q_\epsilon^s)/(\partial t_0))}{||DH(q_0(-\bar{t_0}))||} \\ &= \epsilon \frac{\partial M/\partial t_0(\bar{t_0}, \bar{\phi_0})}{||DH(q_0(-\bar{t_0}))||} + \mathcal{O}(\epsilon^2) \end{aligned} \tag{2.52}$$

and

$$\begin{aligned} \frac{\partial d}{\partial \phi_0}(\bar{t_0}, \bar{\phi_0}, \epsilon) &= \frac{DH(q_0(-\bar{t_0})) \cdot ((\partial q_\epsilon^u)/(\partial \phi_0) - (\partial q_\epsilon^s)/(\partial \phi_0))}{||DH(q_0(-\bar{t_0}))||} \\ &= \epsilon \frac{\partial M/\partial \phi_0(\bar{t_0}, \bar{\phi_0})}{||DH(q_0(-\bar{t_0}))||} + \mathcal{O}(\epsilon^2). \end{aligned} \tag{2.53}$$

So a sufficient condition for $W^s(\gamma_\epsilon(t))$ and $W^u(\gamma_\epsilon(t))$ not to be tangent at p is

$$\frac{\partial M}{\partial \phi_0}(\bar{t_0}, \bar{\phi_0}) = \omega \frac{\partial M}{\partial t_0}(\bar{t_0}, \bar{\phi_0}) \neq 0.$$

This concludes our proof and gives us all we need to apply Moser's theorem from which follows that transversal intersections between $W^s(p)$ and $W^u(p)$ indicate the system to have chaotic behaviour. The actual implementation of Moser's theorem is beyond the scope of this paper.

3 Application

One of the most common examples of a nonlinear oscillation is the Duffing oscillator. In general the Duffing oscillator can be written as

$$\ddot{x} + \delta\dot{x} + \beta x + \alpha x^3 = F\cos\omega t. \tag{3.1}$$

The Duffing equation describes the oscillations of a mass attached to a nonlinear spring and a linear damper. We get a restoring force from the nonlinear spring which then equals $\beta x + \alpha x^3$. The behaviour changes as α changes sign. When $\alpha < 0$, the restoring force becomes weaker than the linear spring ($\alpha = 0$) as the distance increases and the spring is called soft. In order to demonstrate the Melnikov method we will be considering two cases of the Duffing oscillator. The first one is given by

$$\ddot{x} + x - x^3 = \epsilon(\gamma\cos\omega t - \delta\dot{x}). \tag{3.2}$$

This represents a *soft spring* Duffing oscillator. The second one we will be looking at is given by

$$\ddot{x} - x + x^3 = \epsilon(\gamma\cos\omega t - \delta\dot{x}). \tag{3.3}$$

representing an *inverted* Duffing oscillator, where the parameters are the opposite of the soft spring case. In both cases the forcing term and the damping term are grouped together and represent our periodic perturbation.

3.1 The Soft Spring Duffing Oscillator

For the soft spring Duffing oscillator we can write

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x + x^3 + \epsilon(\gamma\cos\omega t - \delta y). \end{aligned} \tag{3.4}$$

Our unperturbed system ($\epsilon = 0$) can be reduced to

$$\ddot{x} = -x + x^3. \tag{3.5}$$

We see that this system has three equilibrium points, namely $(0,0)$, $(1,0)$ and $(-1,0)$. We are actually only interested in the last two. We check the stability

$$det(Df) = \begin{vmatrix} 0 & 1 \\ 3x^2 - 1 & 0 \end{vmatrix}_{\{(1,0),(-1,0)\}} = -2$$

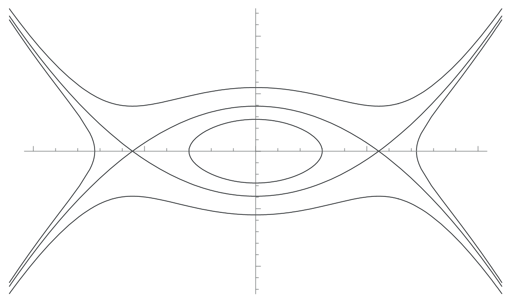


Figure 4: Phase portrait of the soft spring Duffing oscillator.

and the eigenvalues are found as

$$det(Dg - \lambda) = 0 \Rightarrow \lambda = \pm\sqrt{2}.$$

In order to find the Hamiltonian for this function we take (3.5) and multiply both sides by $\dot{x}$. This gives us

$$\ddot{x}\dot{x} + \dot{x}x - \dot{x}x^3 = 0$$

which is equivalent to

$$\frac{d}{dt}\left(\frac{1}{2}\dot{x}^2 + \frac{1}{2}x^2 - \frac{1}{4}x^4\right).$$

So we have found a time invariant function on the state space. This is actually the energy of our system. We write

$$H(x, y) = \frac{1}{2}y^2 - \frac{1}{4}x^4 + \frac{1}{2}x^2. \tag{3.6}$$

We denote values of the function H by h. For the orbit connecting our saddle points we have $H(1, 0) = H(-1, 0) = \frac{1}{4}$. We will now try to find a solution for the heteroclinic orbit connecting these equilibrium points. Substituting $y = \frac{dx}{dt}$ and $H = \frac{1}{4}$ into (3.6) we get

$$\left(\frac{dx}{dt}\right)^2 = \frac{1}{2} - x^2 + \frac{1}{2}x^4$$

Seperating the variables and integrating on both sides yields

$$t = \int dt = \int \frac{dx}{\sqrt{\frac{1}{2} - x^2 + \frac{1}{2}x^4}} = \int \frac{dx}{\sqrt{\frac{1}{2}}\sqrt{1 - 2x^2 + x^4}}$$

which we can reduce to

$$t = \pm\sqrt{2} \int \frac{dx}{(1-x^2)} = \pm\sqrt{2}\tanh^{-1} x + t_0$$

which we can invert to give us our desired function

$$x(t) = \pm\tanh\left(\frac{t-t_0}{\sqrt{2}}\right).$$

Taking the derivative of $x(t)$ gives us $y(t)$. Our heteroclinic orbit then becomes

$$q_0(t)^{\pm} = \left(\pm\tanh(\frac{t-t_0}{\sqrt{2}}), \pm\frac{1}{\sqrt{2}}\operatorname{sech}^2(\frac{t-t_0}{\sqrt{2}})\right) \tag{3.7}$$

This gives us everything we need so compute Melnikovs function. Recall that the Melnikov function can be written

$$M^{\pm}(t_0, \phi_0) = \int_{\infty}^{\infty} DH(q_0^{\pm}(t)) \cdot g(q_0(t), \omega(t) + \omega t_0 + \phi_0, 0)dt. \tag{3.8}$$

From (3.4) we find our perturbation functions

$$g(q_0(t), \omega, \gamma, \delta, t, 0) = (0, \gamma\cos\omega t - \delta y)$$

and from our Hamiltonian we get

$$DH(q_0(t)) = \begin{pmatrix} x - x^3 \\ y \end{pmatrix}.$$

Substituting this into (3.8) gives

$$M^{\pm}(t_0) = \int_{-\infty}^{\infty} y^{\pm}(t)\gamma\cos\omega t - \delta(y^{\pm}(t))^2 dt. \tag{3.9}$$

In order to get a somewhat cleaner notation we split the integral into two parts. Respectively

$$M_1 = \pm\frac{\gamma}{\sqrt{2}} \int_{-\infty}^{\infty} \operatorname{sech}^2\left(\frac{t-t_0}{\sqrt{2}}\right)\cos\omega t dt \tag{3.10}$$

and

$$M_2 = -\frac{\delta}{2} \int_{-\infty}^{\infty} \operatorname{sech}^4\left(\frac{t-t_0}{\sqrt{2}}\right) dt. \tag{3.11}$$

And finally evaluating both parts and adding them up gives us

$$M^{\pm}(\delta, \gamma, \omega) = -\frac{2\sqrt{2}\delta}{3} \pm \sqrt{2}\pi\gamma\omega\operatorname{csch}\left(\frac{\pi\omega}{\sqrt{2}}\right)\cos(\omega t_0 + \phi_0). \tag{3.12}$$

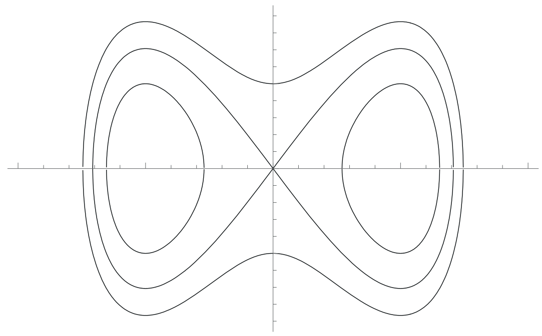


Figure 5: Phase portrait of the inverted Duffing oscillator

3.2 The Inverted Duffing Oscillator

We slightly rewrite our system as

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= x - x^3 + \epsilon(\gamma \cos \omega t - \delta \dot{x}) \end{aligned} \tag{3.13}$$

We first look at the unperturbed system (i.e. $\epsilon = 0$). Our system reduces to

$$\begin{aligned} \dot{x} &= y, \\ \dot{y}\ & x - x^3 \end{aligned} \tag{3.14}$$

which can be reduced to

$$\ddot{x} = \dot{y} = x - x^3. \tag{3.15}$$

The system has a fixed point at $(0, 0)$. Looking at the stability we get

$$Det(Df) = \begin{vmatrix} 0 & 1 \\ 1 - 3x^2 & 0 \end{vmatrix}_{(0,0)} = -1 \neq 0.$$

So our fixed point is a stable fixed point. Moreover, looking at the eigenvalues

$$Deg(Df - \lambda) = 0 \Rightarrow \lambda = \pm 1,$$

tells us we have a hyperbolic fixed point. Analogue to the soft spring case we compute our Hamiltonian

$$H \equiv \frac{1}{2}\dot{x}^2 - \frac{1}{2}x^2 + \frac{1}{4}x^4. \tag{3.16}$$

From which we can compute our homoclinic orbit ($H = 0$)

$$q_0^{\pm}(t) = (\pm\sqrt{2}\,\mathrm{sech}(t), \pm\sqrt{2}\,\mathrm{sech}\,t\tanh t) \tag{3.17}$$

and our integral turns out to be equal to (3.9) with a different orbit. Once again we split it into two parts, respectively

$$M_1 = \pm\sqrt{2}\gamma \int_{-\infty}^{\infty} \mathrm{sech}\,t\tanh t\cos\omega t\ dt$$

and

$$M_2 = -2\delta \int_{-\infty}^{\infty} \mathrm{sech}^2\,t\tanh^2 t\ dt.$$

Which we can compute and combine to get

$$M^{\pm}(t_0, \phi_0) = -\frac{4\delta}{3} \pm \sqrt{2}\gamma\pi\omega\,\mathrm{sech}\left(\frac{\pi\omega}{2}\right)\sin(\omega t_0 + \phi_0). \tag{3.18}$$

3.3 Results

Having computed the Melnikov function for both cases we want to take the results, look at the threshold curves for the parameters and confirm our results. Let us recall once more that for the soft spring and the inverted Duffing oscillator we have

$$\begin{aligned} M_S(\omega,\gamma,\delta) &= -\frac{2\sqrt{2}\delta}{3} \pm \sqrt{2}\pi\gamma\omega\,\mathrm{csch}\,\frac{\pi\omega}{\sqrt{2}}\cos(\omega t_0 + \phi_0) && (3.19)\\ M_I(\omega,\gamma,\delta) &= -\frac{4\delta}{3} \pm \sqrt{2}\gamma\pi\omega\,\mathrm{sech}\,\frac{\pi\omega}{2}\sin(\omega t_0 + \phi_0) && (3.20)\end{aligned}$$

respectively.

In order for us to see chaotic behaviour due to transversal intersections we need simple zeros in the Melnikov function. In order for this to happen we need the amplitude of second part of (3.19) to be greater then the first part. So

$$\begin{aligned} \frac{2\sqrt{2}\delta}{3} &< \sqrt{2}\pi\gamma\omega\,\mathrm{csch}\,\frac{\pi\omega}{\sqrt{2}}, && (3.21)\\ \frac{4\delta}{3} &< \sqrt{2}\pi\gamma\pi\omega\,\mathrm{sech}\,\frac{\pi\omega}{2} && (3.22)\end{aligned}$$

respectively.

3.4 Numerical Verification

It is sufficient to look at ω in terms of the ratio $\frac{\delta}{\gamma}$. This gives us the following threshold condition for the soft spring duffing oscillator:

$$\frac{\delta}{\gamma} < \frac{3\pi\omega \operatorname{csch} \frac{\pi\omega}{\sqrt{2}}}{2} \tag{3.23}$$

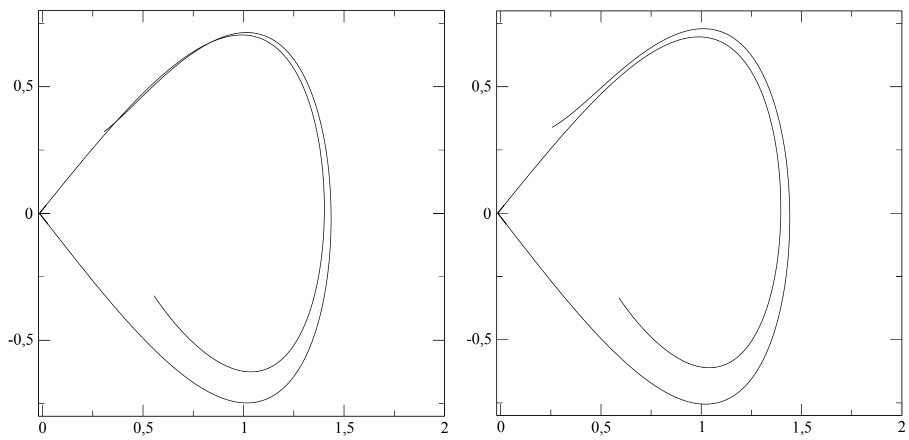
and for the inverted duffing oscillator we find:

$$\frac{\delta}{\gamma} < \frac{3\pi\omega \operatorname{sech} \frac{\pi\omega}{2}}{2\sqrt{2}}. \tag{3.24}$$

Now in order to numerically compute the stable and unstable manifold we first need to find the fixed point after perturbation. We do this using the Newton method of the *C Numerical Recipes* library. Slightly incrementing the perturbation allows us to efficiently set our initial guess. Using a point slightly above and below the fixed point for the unstable and stable manifold respectively allows us to find their respective eigenvectors.

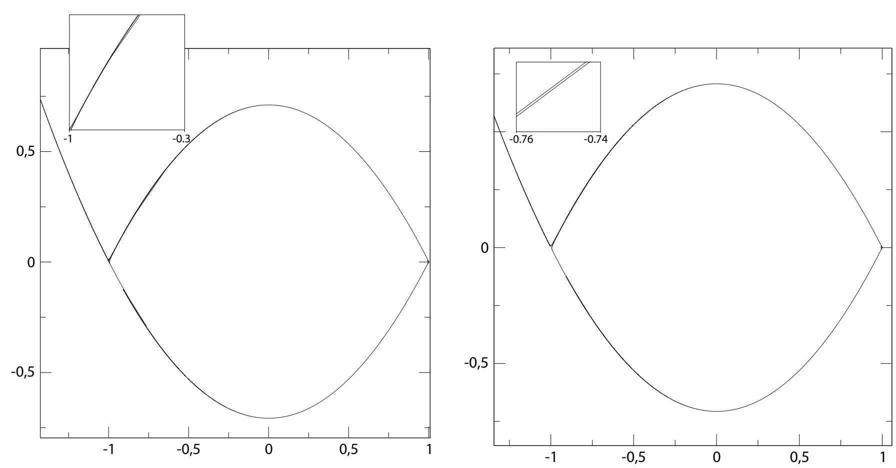
Splitting the eigenvector into a certain amount of equidistant points and integrating each point over one period will now give us intervals of the manifolds. By altering the parameters of this last step (the offset of the vector and the amount of points we split it into) we can now try to get these intervals to overlap. This ultimately gives us the manifolds which we can plot to check our theoretical results.

Setting $\omega = 1$ allow us to get a value for the ratio $\frac{\delta}{\gamma}$ for which we can perform numerical verification. Figures (6) and (7) shows us that setting $\frac{\delta}{\gamma}$ slightly above and below the threshold curve respectively gives us the result we would expect. For the soft spring case the manifolds remain pretty close to each other and even zooming in quite a bit its not easy to see if there are any intersections. However studying the graphs closely enough does confirm our results.



(a) $\frac{\delta}{\gamma} = 1.2$ (b) $\frac{\delta}{\gamma} = 1.6$

Figure 6: The Inverted Duffing Oscillator for different parameters ($\omega = 1$)



(a) $\frac{\delta}{\gamma} = .8$ (b) $\frac{\delta}{\gamma} = 1.2$

Figure 7: The Soft Spring Duffing Oscillator for different parameters ($\omega = 1$)

4 Conclusion

In the first part of this thesis we described the derivation of the Melnikov function. It is stated that zeros of the function will respond to transversal intersections of stable and unstable manifolds which according to Moser's theorem correspond to chaotic behaviour of the system. Having derived the function we were able to apply it to two different versions of the Duffing oscillator and find their respective threshold curves for transversal intersections. Having found these curves we performed numerical experiments to confirm our results.

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