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Unifying Conformal Gravity and the Standard Model of particle physics

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By convention there is color,
By convention sweetness,
By convention bitterness,
But in reality there are atoms and space

Democritus

RIJKSUNIVERSITEIT GRONINGEN

Abstract

Faculty of Mathematics and Natural Sciences
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Unifying Conformal Gravity and the Standard Model of particle physics

by Karin Dirksen

This thesis is a literature study centered around the question 'How can we use conformal invariance to unify the Standard Model with Gravity?' After refreshing the Standard Model and renormalization and regularization techniques, as well as Einstein's theory of General Relativity, it is explained why conformal invariance is a useful tool in the unification program. The differences between the related concepts of 'conformal', 'scale' and 'Weyl' invariance are explained and the two types of Conformal Gravity theories are introduced: Conformal Weyl gravity based on the squared Weyl tensor and Conformal Dilaton Gravity which uses a Stückelberg trick to turn the Einstein-Hilbert action into a conformally invariant theory. Pursuing the latter due to unitarity concerns in the former, the Conformal Standard Model in the presence of gravity is developed. We distinguish between two toy models, one with an unphysical scalar dilaton field χ and one with a physical dilaton φ. As conformal invariant theories do not allow the explicit presence of scales, conformal symmetry breaking is necessary to generate the scale needed for electroweak symmetry breaking (EWSB). The Weyl invariance of the theory with the unphysical dilaton can be extended to the quantum theory and the additional gauge freedom allows gauge fixing of the dilaton to a constant, thus ensuring EWSB. The theory with the physical dilaton suffers from a conformal anomaly. Boldly assuming that this anomaly is cancelled at some scale, a Gildener-Weinberg analysis of the theory shows the possibility of radiative breaking of the conformal symmetry. The two theories differ in one minus sign, but have vastly different results. Experiments and astronomical observations could help in understanding which, if any, of these theories could be a toy model for a Theory of Everything.

Preface

Context

It was not until the early 19th century when Hans Christian Ørsted noted the deflection of a magnetic compass needle caused by an electric current and demonstrated that the effect is reciprocal. With contributions from Michael Faraday this lead to Maxwell's equations forming a unified theory of electricity, magnetism, and light: electromagnetism.

Michael Faraday believed that the forces of nature were mutually dependent and more or less convertible into one and another. Furthermore, the electrical and gravitational forces share fundamental characteristics: they both diminish with the inverse square of the distance; they are both proportional to the product of the interacting masses or charges; and both forces act along the line between them. This let him to search for a connection between gravity and electric or magnetic action, which his experiments were unable to ascertain [1].

After the publication of Albert Einstein's theory of gravity, the search for a theory which would relate gravity and electromagnetism to a unified field[1] began with a renewed interest. "If the special theory of relativity had unified electricity and magnetism and if the general theory had geometrized gravitation, should not one try next to unify and geometrize electromagnetism and gravity?" [2].

Ultimately unsuccessful in his quest[2], Einstein considered a variety of approaches which were by and large, a reaction to proposals advanced by others like Hermann Weyl, Theodor Kaluza and Arthur Edington. These approaches proved troublesome for various reasons and were to some extent discarded by Einstein. The first original approach put forward by Einstein himself was published in a paper of 1925 in which also the term 'unified field theory' appeared for the first time in a title. The last approach of Einstein's work along his unified field theory program was based on a local Riemannian metric but on an asymmetric one. Einstein spent the rest of his life elaborating the asymmetric theory and his very last considerations were presented by his last assistant, Bruria Kaufmann, a few weeks after Einstein's death.

Though unification of electromagnetism with gravity proved unsuccessful, the unlikely unification of electromagnetism with the weak force was established 35 years after the first theory on the weak interaction by Enrico Fermi. The road to electroweak unification (adapted from Kibble [4]) started already when Paul Dirac attempted to quantize the electromagnetic field in the 1920s. This eventually resulted in Quantum Electrodynamics (QED), a quantum field theory that successfully describes processes where the number of particles changes like the emission of a photon by an electron dropping into a quantum state of lower energy. The theory was plagued

[1]Hence the name 'Unified Field Theory'. In the past it solely was used in the context of a unified field theory in which electromagnetism and gravity would emerge as different aspects of a single fundamental field. Nowadays this term is used for a type of field theory that allows *all* fundamental forces and elementary particles to be written in terms of a single field. The 'Theory of Everything' and 'Grand Unified Theory' are closely related to unified field theory, but differ by not requiring the basis of nature to be fields.

[2]See Sauer [3] for a further, more detailed discussion on Einstein's unified field theory program from a conceptual, representational, biographical, and philosophical perspective.

by infinite, divergent contributions from for example the self-energy of the electron. These difficulties were overcome by the development of renormalization theory by Julian Swinger, Sin-Itiro Tomonaga, Freeman Dyson and Richard Feynman which rapidly promoted QED to the most accurately verified theory in the history of physics. The next goal was to find similar elegant theories describing the other forces of nature, hoping of course to find a unified theory.

In 1958 Richard Feynman and Murray Gell-Mann published the $V-A$ theory which showed that the weak interactions could be seen as proceeding via the exchange of spin-1 $W^{\pm}$ bosons, just as the electromagnetic interactions are mediated by the photon. This hinted at a possible unification, but was complicated by the fact that the W-bosons needed to be massive, while the photon is massless. Furthermore, weak interactions do not conserve parity whereas the electromagnetic interactions are parity-conserving. The latter problem was resolved by Sheldon Glashow, who proposed an extended model with a larger symmetry group, $SU(2) \times U(1)$, and a fourth gauge boson Z_0.

Any mass term appearing in the Lagrangian will spoil the gauge-invariance property because gauge symmetry prohibit the generation of a mass for the vector field. Therefore, the nonzero W and Z masses require incorporating spontaneous symmetry breaking into the theory. This breaking mechanism was described by the Goldstone theorem which stated that the appearance of massless spin-zero Nambu–Goldstone bosons is a consequence of spontaneous symmetry breaking in a relativistic theory. For a gauge theory it was later shown that spontaneous breaking let to massive bosons. This is known as the Higgs mechanism.

Then in 1968 Steven Weinberg and Abdus Salam independently combined these ideas into a unified gauge theory of weak and electromagnetic interactions of leptons. Meaningful calculations were the made possible when Gerardus 't Hooft and Martinus Veltman showed the theory was renormalizable. One immediate prediction of this now proven viable theory was the "neutral current" which has to exist to assure its renormalizability. In 1973 the neutral current was discovered in the Gargamelle bubble chamber at CERN. This verification of what nowadays is known as the Glashow-Weinberg-Salam model (GWS model) lead to the Nobel prize for its namesakes. The later discovery of the W and Z particles was even further evidence of the validity of the GWS model.

Meanwhile, during the 1970s and 1980s there had been a separate, parallel development of a gauge theory of strong interactions, quantum chromodynamics (QCD). This lead to the development of the Standard Model (SM). The model started with the proposal of quarks by Murray Gell-Mann and George Zweig, followed by significant experimental evidence on its validity like the discovery of the charmed quark, the tau neutrino and the Higgs particle. However, the SM contained a degree of arbitrariness as well as too many unresolved questions to be considered the final theory. These are the problems which grand unified theories (GUT[3]) which unify the strong force with the electroweak force, hope to address.

In 1974 Abdus Salam and Jogesh Pati proposed the first GUT, known as the Pati-Salam model[4]. It addresses the intriguing similarity between quarks and leptons, namely the fact that each generation of fermions in the Standard Model has two quarks and two leptons. The Pati-Salam model identifies the 'lepton-ness' (non-quark-ness) of leptons as the fourth color, lilac, of a larger $SU(4)_c$ gauge group which then needs to be broken by means of some Higgs scalar. The model also considers the difference between left- and right-handed fermions: where the former are in a nontrivial representation of $SU(2)$, the latter are part of a trivial representation and thus non-participating in the weak force. The model instead treats them on equal footing by assuming the existence of a 'right-handed' weak isospin $SU(2)_R$ gauge group. Assuming discrete parity symmetry $\mathbb{Z}_2$, the

[3] The acronym GUT was first coined in 1978 by CERN researchers John Ellis, Andrzej Buras, Mary K. Gaillard, and Dimitri Nanopoulos.

[4] Technically, it does not give a unified description. Even at high energies, the gauge group is the product of three groups and not a single group as is the case in the more conventional GUTs such as $SU(5)$ or $SO(10)$. Still, due to the unification of quarks and leptons and the fact that left-handed and right-handed fields are treated on the same footing, it is referred to as a GUT.

total symmetry group of the Pati-Salam model becomes $SU(4)_c \times SU(2)_R \times SU(2)_L \times \mathbb{Z}_2$.

Although quarks and leptons are now unified[5], there are nevertheless two[6] independent gauge couplings resulting in two arbitrary parameters. This difficulty is resolved by embedding the Standard Model gauge group into the simple unified gauge group $SU(5)$, with one universal gauge coupling α_G defined at the grand unification scale M_G which is expected to be of the order 10^{16} GeV. Quarks and leptons sit in two irreducible representations, as before, but the low energy gauge couplings are now determined in terms of the two independent parameters α_G and M_G. Hence there is one prediction. This is known as the Georgi-Glashow model and elegantly explains the fractional charges of quarks. Unfortunately, this theory has since been ruled out by experiment; it predicts that protons will decay faster than the current lower bound on proton lifetime. Furthermore, LEP data showed that gauge coupling unification is not achieved in non-supersymmetric $SU(5)$ GUTs, further promoting the development of supersymmetric models.

As the Pati-Salam model fully unifies fermions but not the gauge fields and the Georgi-Glashow model fully unifies gauge fields but not fermions, a bigger/other model that unifies them all is sought. Besides $SO(10)$ models, unification through string theory, supersymmetry or (compact) extra dimensions is pursued as well. These GUT models become highly complex because they try to reproduce e.g. the observed fermion masses and mixing angles. Due to this difficulty, and due to the lack of any observed effect of grand unification so far, there is no generally accepted GUT model.

Though the existence of a GUT is far from a proven fact, the idea of unification can be extended all the way to the Planck scale, the energy scale at which the gravitational attraction of elementary particles becomes comparable to their strong, weak and electromagnetic interactions. Already at energies of the order 10^{18} GeV the gravitational attraction becomes comparable to the gauge force due to the vector bosons of a GUT. Though this scale is slightly larger than the scale at which the SM couplings meet, a link between grand unification and unification of gravity is is not unreasonable.

A Theory of Everything (ToE) which deals with this unification of gravity and matter faces a multitude of challenges. One of the biggest is that the existence of quantum matter and the fact that this matter acts on spacetime seems to make it unavoidable to assign quantum nature also to spacetime itself under the name of Quantum Gravity. At present, cosmological observations give us no unambiguous clue as to how such a theory would have to look. Though experiments have put constraints on parameters of indirect relevance to quantum gravity, they still allow a wide variety of theories.

One of the most important and well understood aspects of Quantum Gravity is the so-called semiclassical approach, where quantized matter fields are treated using a classical curved metric.However, even this situation is plagued by severe technical and conceptual problems, since crucial tools of QFT in flat spacetime such as energy- momentum conservation, Fourier transformation and analyticity, Wick rotation, particle interpretation of asymptotic scattering states, are no longer available due to the lack of spacetime symmetries.

Even at the classical level there are numerous attempts at unifying gravity with the three other fundamental forces. As Einstein's theory of General Relativity is such a success on solar system distance scales, the most straightforward generalization of gravity is to augment the Einstein-Hilbert action with additional general coordinate invariant pure metric terms which due to the smallness of theory coefficients or structure have negligible effects in the solar system. Other options are introducing additional gravitational fields (often scalars) besides the metric tensor itself as proposed e.g. by Carl H. Brans and Robert H. Dicke in 1961, or increasing the number of spacetime dimensions as original put forward by Theodor Kaluza and Oskar Klein (1919).

[5] Note that unification of quarks and leptons leads to proton-decay since B and L numbers are violated. Some $\beta B + \alpha L$- number is preserved: $B+3L$ number for the Pati-Salam model and $B-L$ number for the Georgi-Glashow model.

[6] Without assuming parity symmetry ($L \leftrightarrow R$ symmetry) there are three independent gauge couplings.

Less orthodox is the possibility to modify the nature of the geometry itself, for example by introducing torsion (e.g. Elie Cartan, 1922), discarding the symmetry of the indices of the metric (e.g. Behram Kurşunoğlu, 1952), or, more radical, replace the Riemann geometry with a new type of geometry as was proposed by Hermann Weyl (1918).

Focus of the thesis

In this thesis we cannot hope to give a comprehensive overview of the current research on Standard Model unification with (quantum) gravity, much less come up with a new theory. Therefore, we will focus on one particular avenue which has proved beneficial in the development of the Standard Model, namely we will require a additional symmetry to be present at the Planck scale which will subsequently be broken. This additional symmetry is invariance under the conformal group.

The Standard Model contains a number of freely adjustable coupling constants and mass parameters. There seems to be no physical principle to determine these parameters as long as they stay within certain domains dictated by the renormalization group. However, Gerard ‘t Hooft [e.g. 5] argues that when gravity is coupled to the system, local conformal invariance should be a spontaneously broken exact symmetry. This condition fixes all parameters, including masses and the cosmological constant. Before this result can be grasped, the connection between conformal symmetry and gravity in what is known as Conformal Gravity should be understood.

We will look at a toy model Lagrangian which includes the Standard Model and Conformal Gravity contributions and assume it to be valid for high energy scales, e.g. the Planck scale. When lowering the energy one encounters a breaking scale of conformal symmetry where the theory breaks down in gravity and for example some Grand Unified Theory. This breaking is necessary as a conformally invariant theory is also scale invariant whereas our daily reality certainly includes scales. We will consider the different mechanisms in the case of our toy model.

Outline of this thesis

This thesis starts in Chapter 1 with explaining the Standard Model, specifically the elements of the Lagrangian. This Lagrangian has terms that in perturbation theory receive infinite contributions from diverging Feynman diagrams. In the second part of the first chapter regularization and renormalization are used to deal with these infinities. These procedures render the coupling constants energy dependent. This is known as the ‘running of the coupling constants’. The exact dependence on the energy scale is encoded in the beta function. Having then established a firm understanding of the Standard Model, the chapter ends by concisely explaining the reasons for looking Beyond the Standard Model.

As pointed out already, one of the major reasons is that we want to include Gravity in our description. Chapter 2 starts with an introduction of Einstein's theory of General Relativity. We show why Einstein Gravity is unsatisfying and give arguments why a conformal invariant theory of gravity would give more desirable results. The intricate relationship between conformal invariance and scale and Weyl invariance is explained with great care after which the conformal algebra and its restrictions on the theory are introduced. We end the chapter by applying the idea of conformal invariance on a theory of gravity. There we will introduce two important theories, namely Conformal Weyl Gravity as advanced by P. D. Mannheim and Conformal Dilaton Gravity as advanced by Gerard ‘t Hooft. These theories are intimately linked with each other, as will be shown. We will also include matter fields into our theory, and for that the tetrad formalism will be developed. This formalism tells us how to minimally couple the Standard Model to Gravity. The constraint of conformal invariance then requires us to further include a non-minimal term, finally giving us our toy model.

Chapter 3 proceeds by exploring how scales are generated through symmetry breaking. First the different symmetry breaking mechanisms are explained, including an extensive review of the Coleman-Weinberg and Gildener-Weinberg formalisms for radiative symmetry breaking. This allows us to investigate the generation of scales in the toy model with the physical and unphysical dilaton and to compare them.

In the last chapter some of the advantages and drawbacks of the developed theory are presented. We will try to give suggestions for further research.

Appendices include derivations that are too long to include in the main body of the text.

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Thank you.

Notation and conventions

Indices

$i, j, \ldots$	$SU(2)$ gauge group indices running from 1 to 3
$a, b, \ldots$	$SU(3)$ gauge group indices running from 1 to 8
C	Color index associated with the quarks in the fundamental representation of $SU(3)$: $C = 1, 2, 3$ corresponds to red (R), green (G) and blue (B).
M, N	Generation or family index. The STandard Model has 3 generations, each divided into two types of leptons (one electron-like and one neutrino-like) and two types of quarks (up- and down-like): $M, N = 1, 2, 3$.
$\mu, \nu, \ldots$	General coordinate indices in X_D
$m, n, \ldots$	Local Lorentz indices in X_D, used e.g. in tetrad formulations of gravitational theories

The Einstein summation convention is used. Furthermore, i is imaginary constant (upright in mathmode, as compared to i which is used as an index).

Fields and gauge group objects

Most of the following notations and conventions will be introduced more detailed in Chapter 1.

Φ	Generic field
ϕ	Scalar field
$\psi, \bar{\psi}$	Fermion field, $\bar{\psi} = \psi^\dagger \gamma^0$.
X_μ^a	General gauge field. The gauge potentials associated with the $SU(2)_L, U(1)_Y, SU(3)_c$ gauge groups are W_μ^i, B_μ, G_μ^a.
χ, ω, φ	Dilaton field as a unphysical St´uckelberg field, unphysical metascalar or physical scalar field, respectively.
T^a	Generators of the group
f^{abc}	Structure coefficients $[T^a, T^b] = \mathrm{i} f^{abc} T^c$
D_μ	Gauge covariant derivative acting on matter fields Φ like $D_\mu \Phi = \left(\partial_\mu + \mathrm{i}\eta g A_\mu^a T_a\right)\Phi$ where g is coupling strength and $\eta = \pm 1$ to signal the different conventions used in the literature (see below).
$F_{\mu\nu}^a$	General field strength tensor: $F_{\mu\nu}^a = \partial_\mu F_\nu^a - \partial_\nu F_\mu^a - \eta g f^{abc} F_\mu^b F_\nu^c$. The field strength tensors associated with the $SU(2)_L, U(1)_Y, SU(3)_c$ gauge groups are $W_{\mu\nu}^i, B_{\mu\nu}, G_{\mu\nu}^a$.

The fundamental representation of the $SU(2)$ gauge group is given by $T^i = \frac{\mathrm{i}}{2}\sigma^i$ with σ^i the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

which are Hermitian and unitary.

The fundamental representation of the $SU(3)$ gauge group is given by $T^a = \frac{1}{2}\lambda^a$ with λ^a the Gell-Mann matrices

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\lambda_5 = \begin{pmatrix} 0 & 0 & -\mathrm{i} \\ 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 \end{pmatrix}, \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

These matrices are traceless, Hermitian, and obey the extra trace orthonormality relation $\mathrm{Tr}(\lambda_i \lambda_j) = 2\delta_{ij}$.

The Dirac matrices γ^μ in the Dirac representation are

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & \mathrm{i} & 0 \\ 0 & \mathrm{i} & 0 & 0 \\ -\mathrm{i} & 0 & 0 & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

They obey the anti-commutating relation $\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$. The gamma matrices can be raised and lowered with the metric $g_{\mu\nu}$. The product of four gamma matrices is $\gamma^5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3$. Furthermore, the Feynman slash notation is used: $\not{v} = v_\mu \gamma^\mu$.

From [6], we denote the different sign conventions $\eta = \pm 1$ for the Standard Model used by some well known texts which are used in this thesis:

Ref.	η	η'	η_Z	η_θ	η_Y	η_e	Y
Peskin and Schroeder [7]	-	-	+	+	+	-	
Zee [8]	-	-	+	+	+	-	*
Srednicki [9]	-	-	+	+	+	-	

where η, η' is associated with the electroweak coupling constant g, g' from the $SU(2)_L \times U(1)_Y$ model, η_s with the strong coupling constant g_s from $SU(3)_c$, η_Y with the sign of the hypercharge Y, and η_Z, η_θ are associated with the signs used for the Z_μ gauge boson and Weinberg mixing angle. We have set $\eta_s = \eta$ and $\eta_e e = \eta\eta_\theta g \sin\theta_W = \eta' g' \cos\theta_W$. An asterisk on the last column means that such authors have $Q = (T^3 + \eta_Y Y)/2$ instead of our definition $Q = T^3 + \eta_Y Y$.

Geometric spaces

Most of the following notations and conventions will be introduced more detailed in Chapter 2.

L_D	D-dimensional differentiable manifold
U_D	D-dimensional Riemann-Cartan space
V_D	D-dimensional Riemann space
M_D	D-dimensional Minkowski space with metric $\eta_{ab} = (+, -, -, -)$
E_D	D-dimensional Euclidean space

Geometric objects

∂_μ, ∂_a	Coordinate and local frame base of the tangent space related by $\partial_\mu = e_\mu{}^a \partial_a$, $\partial_a = e^\mu{}_a \partial_\mu$ where $e_\mu{}^a$ are the D-beins (specifically vierbeins or tetrads) and $e^\mu{}_a$ their inverses
$\mathrm{d}x^\mu, \mathrm{d}\xi^a$	Coordinate and local frame base of the cotangent space
g	metric tensor with components $g_{\mu\nu} = g(\partial_\mu, \partial_\nu)$ in a general coordinate frame and $\eta_{ab} = g(\partial_a, \partial_b)$ in a local (i.e. flat) frame of reference
$\Gamma^\lambda{}_{\mu\nu}$	Affine connection
$T^\lambda{}_{\mu\nu}(\Gamma)$	Torsion of the connection Γ: $T^\lambda{}_{\mu\nu}(\Gamma) = \Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu}$
$R^\mu{}_{\nu\rho\sigma}(\Gamma)$	Curvature of the connection Γ: $R^\mu_{\nu\rho\sigma}(\Gamma) = \partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\rho} + \Gamma^\lambda_{\nu\sigma}\Gamma^\mu_{\lambda\rho} - \Gamma^\lambda_{\nu\rho}\Gamma^\mu_{\lambda\sigma}$
$\nabla_\mu \Gamma$	Covariant derivative with respect to an affine connection
$\{^{\mu}_{\nu\lambda}\}$	Levi-Civita connection which is a linear, metric compatible and torsion-free connection associated with V_4 and is given in terms of the Christoffel symbols: $\{^{\mu}_{\nu\lambda}\} = \frac{1}{2} g^{\mu\rho} \left(\partial_\nu g_{\lambda\rho} + \partial_\lambda g_{\nu\rho} - \partial_\rho g_{\lambda\nu}\right)$
$\nabla_\mu(\{\})$	Covariant derivative with respect to the Levi-Civita connection. Often shortened to $\nabla_\mu(\{\})F = F_{;\mu}$
$\omega_\mu{}^a{}_b$	Spin connection, which is related to the affine connection as $\omega_\mu{}^a{}_b = e_\nu{}^a e^\lambda{}_b \Gamma^\nu_{\mu\lambda} - e^\lambda{}_b \partial_\mu e_\lambda{}^a$
$\nabla_\mu()$	Covariant derivative with respect to the spin connection

Further conventions

Units are chosen such that $c = \hbar = 1$ where c is the velocity of light and $\hbar = h/(2\pi)$ in terms of Planck's constant h. Next to that we denote the different sign conventions $\epsilon = \pm 1$ for the the following tensors as used by some well known texts which are used in this thesis:

$\epsilon_g\, g = -(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2$	Sign of the metric
$\epsilon_R R^\mu{}_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\rho} + \Gamma^\lambda_{\nu\sigma}\Gamma^\mu_{\lambda\rho} - \Gamma^\lambda_{\nu\rho}\Gamma^\mu_{\lambda\sigma}$	Sign of Riemann tensor
$\epsilon \quad R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu}$	Sign of the Ricci tensor
$\epsilon_T 8\pi T_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$	Sign of the Einstein equation

Ref.	ϵ_g	ϵ_R	ϵ	ϵ_T
Alvarez et al. [10]	-	+	+	+
‘t Hooft [11]	+	+	+	+
Mannheim [12]	+	-	+	-
Misner, Thorne and Wheeler [13]	+	+	+	+
Sundermeyer [14]	-	+	+	-
Wald [15]	+	+	+	+
Yepez [16]	-	+	+	-
This thesis	-	+	+	+

We also use $\partial_\mu = \frac{\partial}{\partial x^\mu}$, $\Box = \partial_\mu \partial^\mu$ and $x^2 = x_\mu x^\mu$.

Abbreviations and acronyms

BEH	Brout-Englert-Higgs
CDG	Conformal Dilaton Gravity
CKM	Cabibbo-Kobayashi-Maskawa
CW	Coleman-Weinberg
CWG	Conformal Weyl gravity
dS	de Sitter
EAdS	Euclidean Anti-de Sitter
EEP	Einstein's Equivalence Principle
EFT	Effective Field Theory
EWSB	Electroweak symmetry breaking
GR	General Relativity
GUT	Grand Unified Theory
GW	Gildener-Weinberg
GWS model	Glashow-Weinberg-Salam model
NEP	Newton's Equivalence Principle
NJL	Nambu and Jona-Lasinio
PGB	Pseudo-Goldstone boson
PMNS	Pontecorvo-Maki-Nakagawa-Sakata
QCD	Quantum Chromodynamics
QED	Quantum Electrodynamics
QFT	Quantum Field Theory
SEP	Strong Equivalence Principle
SM	Standard Model
SR	Special Relativity
SSB	Spontaneous Symmetry Breaking
ToE	Theory of Everything, a quantum gravity theory that is also a grand unification of all known interactions
WEP	Weak Equivalence Principle

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Chapter 1

The Standard Model and beyond

The Standard Model is constructed by first postulating a set of symmetries of the system, and then by writing down the most general renormalizable Lagrangian from its field content that respects these symmetries. In Lie algebra jargon the Standard Model is known as a non-abelian gauge theory[7]. The global Poincaré symmetry[8] is postulated for all relativistic quantum field theories. Additionally there is the local $U(1)_Y \times SU(2)_L \times SU(3)_c \rightarrow U(1)_{\text{EM}} \times SU(3)_c$ gauge symmetry which is based on the electroweak gauge group of the GWS model $SU(2)_L \times U(1)_Y$ and the $SU(3)_c$ color gauge group of QCD. The electroweak symmetry group is spontaneously broken to the electromagnetic symmetry $U(1)_{\text{EM}}$ by the Brout-Englert-Higgs mechanism.

We start this chapter by identifying the particle content of the Standard Model and discuss the group structure in more detail. In the next section the different components of the Lagrangian are recounted culminating in the full Standard Model Lagrangian $\mathcal{L}_{\text{SM}}$ [6]. To deal with the infinities that arise in the calculations, we introduce regularization and renormalization in section 1.3. Despite its enormous success, the Standard Model is not the final Theory of Everything as we will argue in the last section.

1.1 Field content and structure

1.1.1 Field content

The Standard Model including including neutrino masses and mixing angles (also known as the minimal[9] extended Standard Model) depends on 25 free parameters. Namely, 3 lepton masses, 6 quark masses, the coupling constants g, g_s, g', the Higgs VEV, 3 quark flavor mixing angles, 3 neutrino mixing angles, and 2 CP violating phases (or 4 if massive neutrino's are Majorana fermions). Note that the set of parameters is not unique, e.g. fermion masses can be replaced by Yukawa couplings. Most of their numerical values have been established by experiment (see Appendix A). The ESM is able to calculate any experimental observable in terms of its input parameter set and has done so successfully. The Standard Model precisely predicted a wide variety of phenomena, e.g. the existence of the Higgs boson and the Z and $W^{\pm}$ masses.

[7] The term 'gauge' refers to local nature of the symmetry transformations. The gauge group of the theory is a Lie group of gauge transformations. For each group generator of the Lie group there arises a corresponding vector field called the gauge field. Gauge fields are included in the Lagrangian to ensure gauge invariance. When such a theory is quantized, the quanta of the gauge fields are called gauge bosons. If the symmetry group is non-commutative, the gauge theory is referred to as non-Abelian.

[8] The Poincaré symmetry comprises symmetry under translations, rotations and boosts, which are transformations connecting two uniformly moving bodies.

[9] It is still called 'minimal' because we only assume the number of flavors that are experimentally verified.

The field content of the Standard Model consists of 12 flavors or matter fields, 12 gauge fields and the Higgs boson defined as follows:

- Matter fields (spin-$\frac{1}{2}$ fermions), namely:
  - 6 leptons e_M and ν_M where $M = 1, 2, 3$ is the generation index such that e_M is the electron, the muon or the tau and ν_M is the corresponding neutrino.
  - 6 quarks u_M^C and d_M^C with $C = 1, 2, 3$ corresponding to the three types of $SU(3)$ color R, G, B and $M = 1, 2, 3$ is the generation index such that u_M^C is either the up, charm or top quark and d_M^C is the down, strange or bottom quark.
- Gauge fields (spin-1 bosons), namely:
  - Photon A_μ which mediates the electromagnetic interaction
  - 3 weak bosons $W_\mu^\pm$ and Z_μ^0 which mediate the charged and neutral current weak interactions
  - 8 gluons G_μ^a where $a = 1, 2, \ldots 8$ which mediate the strong interactions
- Higgs boson (spin-0) H which is the result of the complex Higgs fields ϕ^+ and ϕ^0 that spontaneously break the electroweak $SU(2)_L \times U(1)_Y$ symmetry

1.1.2 Gauge group $SU(3)_c$

The strong interactions between quarks and gluons are described by Quantum Chromodynamics (QCD), a non-Abelian gauge theory with $SU(3)_c$ color symmetry. The $SU(3)_c$ symmetry group has 8 generators $T^a, a = 1, 2, \ldots 8$ satisfying $[T^a, T^b] = \mathrm{i} f_{abc} T^c$ where f_{abc} are the antisymmetric structure constants of the group[10]. The fundamental representation is given by the Gell-Mann matrices λ^a according to $T^a = \frac{1}{2}\lambda^a$. The other important representation is the adjoint representation: $(T^a_{adj})^{bc} = -\mathrm{i} f^{abc}$. In QCD the gluons transform under the adjoint representation whereas quarks transform under the fundamental representation and are given by a triplet:

$$q = \left(q^R, q^G, q^B\right) \tag{1.1}$$

The dynamics of the quarks are given by the field strength tensor, which in general is defined as $F_{\mu\nu} = \frac{\mathrm{i}}{g}[D_\mu, D_\nu]$ with g the coupling constant and D_μ the appropriate gauge covariant derivative.

$$D_\mu q = \left(\partial_\mu + \mathrm{i}\eta_s g_s T^a G_\mu^a\right) q \tag{1.2}$$

with g_s is the gauge coupling constant related to $SU(3)_c$ and $G_\mu^a (a = 1, 2, \ldots 8)$ the gauge vector fields known as gluons. Using $\eta_s = \pm 1$ to reflect the two sign conventions used in the literature (see 'Notations and conventions', page x), the field strength tensor of $SU(3)_c$ is then given by:

$$G_{\mu\nu}^a = \partial_\mu G_\nu^a - \partial_\nu G_\mu^a - \eta_s g_s f^{abc} G_\mu^b G_\nu^c \tag{1.3}$$

The combination D_μ provides the coupling between the fields and ensures that the equation is invariant under the local $SU(3)_c$ gauge transformation of the quark fields and fields strength tensor, which is given by the matrix

$$U(\beta) = e^{\mathrm{i}\eta_s g_s T^a \beta^a} \tag{1.4}$$

such that the fields transform as

[10] The structure constants are given by: $f_{123} = +1$, $f_{458} = f_{678} = \frac{1}{2}\sqrt{3}$, $f_{147} = f_{165} = f_{246} = f_{257} = f_{345} = f_{376} = \frac{1}{2}$ and all others that are not related by permutations are zero. Note that we don't have any structure constants that have both a 3 and an 8 since λ_3 and λ_8 commute.

$$q' = q + \delta q, \qquad \delta q = \mathrm{i}\eta_s g_s T^a \beta^a q \tag{1.5}$$

$$G_\mu^{a\prime} = G_\mu^a + \delta G_\mu^a, \quad \delta G_\mu^a = -\partial_\mu \beta^a - \eta_s g_s f^{abc} \beta^b G_\mu^c \tag{1.6}$$

where $\beta^a (a = 1, 2, \ldots 8)$ is the group parameter and the second equality holds for infinitesimal transformations. With these definitions one can verify that the covariant derivative transforms like the field itself, ensuring the gauge invariance of the Lagrangian.

1.1.3 Gauge group $SU(2)_L \times U(1)_Y$

Electroweak interactions are described by the GWS model which is based on chiral $SU(2)_L$ gauge invariance: parity violation is built into the model by assigning the left- and right handed fermions to different representations of $SU(2)_L$ where the subscript signals the fact that the right-handed fermions do not transform under $SU(2)_L$ (they are singlets). The $SU(2)_L$ symmetry group has 3 generators $T^i, i = 1, 2, 3$ satisfying $[T^i, T^j] = \mathrm{i}\varepsilon^{ijk} T^k$ where ε^{ijk} is the Levi-Civita symbol. The fundamental representation of the group generators is given by the Pauli matrices: $T^i = \frac{\mathrm{i}}{2}\sigma^i$ and the adjoint representation is $(T^i_{adj})^{jk} = -\mathrm{i}\varepsilon^{ijk}$.

The other group of the GWS model is $U(1)_Y$ which is the Abelian group of phase transformations. Its generator is the hypercharge Y and it is related to the charge of the fermion Q and the T_3 generator of $SU(2)$ by $Q = T_3 + \eta_Y Y$. Both the left-handed and right-handed fermions transform non-trivially under this group.

Left- and right-handed fermions are defined as follows:

$$\psi_L \equiv \tfrac{1}{2}(1 - \gamma_5)\psi, \quad \psi_R \equiv \tfrac{1}{2}(1 + \gamma_5)\psi \tag{1.7}$$

In the Standard Model, the left-handed leptons and neutrino's are grouped together in the fundamental representation of $SU_L(2)$, as are the left-handed quarks. Right-handed fermions[11] are invariant, i.e. singlet states. Suppressing the generation and color indices, we have

$$\psi_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}, \begin{pmatrix} u_L \\ d_L \end{pmatrix}, \quad \psi_R = \nu_R, e_R, u_R, d_R \tag{1.8}$$

Also for this symmetry group a covariant derivative needs to be introduced to uphold gauge invariance:

$$D_\mu = \partial_\mu + \mathrm{i}\eta g W_\mu^i T^i + \mathrm{i}\eta' g' \eta_Y Y B_\mu \tag{1.9}$$

where $W_\mu^i (i = 1, 2, 3)$ and B_μ are the gauge boson fields that correspond to the $SU(2)_L$ and $U(1)_Y$ symmetry group, respectively. The covariant field strength tensors become

$$W_{\mu\nu}^i = \partial_\mu W_\nu^i - \partial_\nu W_\mu^i - \eta g \varepsilon^{ijk} W_\mu^j W_\nu^k \tag{1.10}$$

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \tag{1.11}$$

The local transformations of the fields under $SU(2)_L \times U(1)_Y$ are given by the matrices

$$U(\alpha) = e^{\mathrm{i}\eta g T^i \alpha^i(x)}, \quad U(\theta) = e^{\mathrm{i}\eta' g' \eta_Y Y \theta(x)} \tag{1.12}$$

such that

[11] There is no evidence that there are right-handed neutrino's, yet they are needed for the generation of neutrino masses.

$$SU(2)_L: \begin{cases} \psi_L' = \psi_L + \delta\psi_L, & \delta\psi_L = \mathrm{i}\eta g T^i \alpha^i \psi_L \\ \psi_R' = \psi_R \\ W_\mu^{i\,\prime} = W_\mu^i + \delta W_\mu^i, & \delta W_\mu^i = -\partial_\mu \alpha^i - \eta g \varepsilon^{ijk} \alpha^j W_\mu^k \end{cases} \tag{1.13}$$

$$U(1)_Y: \begin{cases} \psi_L' = \psi_L + \delta\psi_L, & \delta\psi_L = \mathrm{i}\eta' g' \eta_Y Y \theta \psi_L \\ \psi_R' = \psi_R + \delta\psi_R, & \delta\psi_R = \mathrm{i}\eta' g' \eta_Y Y \theta \psi_R \\ B_\mu' = B_\mu + \delta B_\mu, & \delta B_\mu = -\partial_\mu \theta \end{cases} \tag{1.14}$$

where $\alpha^i (i = 1, 2, 3)$ and θ are the group parameters for the weak isospin and weak hypercharge operators, respectively. Again $\eta, \eta', \eta_Y = \pm 1$ to reflect the different sign conventions used in the literature. Note that the gauge fields only transform under their associated subgroup.

For later computations it is convenient to write the covariant derivative in terms of the mass eigenstates $W_\mu^\pm, Z_\mu$ (weak gauge bosons) and A_μ (photon):

$$\begin{gathered} \eta_Z Z_\mu = \cos(\theta_W) W_\mu^3 - \eta_\theta \sin(\theta_W) B_\mu, \quad A_\mu = \eta_\theta \sin(\theta_W) W_\mu^3 + \cos(\theta_W) B_\mu \\ W_\mu^\pm = \frac{W_\mu^1 \mp i W_\mu^2}{\sqrt{2}}, \quad \tan(\theta_W) = \frac{\eta' g'}{\eta\, \eta_\theta\, g} \end{gathered} \tag{1.15}$$

Where $\eta_\theta = \pm 1$ and θ_W the Weinberg angle that relates the former W_μ^i and B_μ fields with the physical gauge bosons.

$$D_\mu = \partial_\mu + \frac{\mathrm{i}\eta g}{2}\left[\sigma_+ W_\mu^+ + \sigma_- W_\mu^-\right] + \mathrm{i}\eta_e e Q A_\mu + \frac{\mathrm{i}\eta g}{\cos(\theta_W)}\left[\frac{\sigma_3}{2} - Q\sin^2(\theta_W)\right]\eta_z Z_\mu \tag{1.16}$$

where

$$\sigma_\pm = \frac{\sigma_1 \pm \mathrm{i}\sigma_2}{\sqrt{2}}, \quad \eta_e e = \eta\, \eta_\theta\, g \sin(\theta_W) = \eta' g' \cos(\theta_W) \tag{1.17}$$

1.2 Standard Model Lagrangian

1.2.1 Fermions and the gauge sector

The fermion Lagrangian becomes

$$\begin{aligned} \mathcal{L}_{\text{Fermion}} = &\sum_{\text{leptons}} (\bar{\nu}_L, \bar{e}_L)\, \mathrm{i}\not{D} \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} + \bar{\nu}_R \mathrm{i}\not{D} \nu_R + \bar{e}_R \mathrm{i}\not{D} e_R \\ &+ \sum_{\text{quarks}} (\bar{u}_L, \bar{d}_L)\, \mathrm{i}\not{D} \begin{pmatrix} u_L \\ d_L \end{pmatrix} + \bar{u}_R \mathrm{i}\not{D} u_R + \bar{d}_R \mathrm{i}\not{D} d_R \end{aligned} \tag{1.18}$$

where the full gauge covariant derivative for fermions in the fundamental representation is given by

$$\begin{aligned} D_\mu = \partial_\mu &+ \frac{\mathrm{i}\eta g}{2}\left[\sigma_+ W_\mu^+ + \sigma_- W_\mu^-\right] + \mathrm{i}\eta_e e Q A_\mu \\ &+ \frac{\mathrm{i}\eta g}{\cos(\theta_W)}\left[\frac{\sigma_3}{2} - Q\sin^2(\theta_W)\right]\eta_z Z_\mu + \frac{\mathrm{i}\eta_s g_s}{2}\lambda_a G_\mu^a \end{aligned} \tag{1.19}$$

with σ^i $(i = 1, 2, 3)$ the Pauli matrices, λ^a $(a = 1, \ldots 8)$ the Gell-Mann matrices and the quantum numbers of the fields are given in table 1.1.

Field	$\frac{\sigma_3}{2}$	$\eta_Y Y$	Q	Field	$\frac{\sigma_3}{2}$	$\eta_Y Y$	Q
ν_{mL}	$+\frac{1}{2}$	$-\frac{1}{2}$	0	ν_{mR}	0	0	0
e_{mL}	$-\frac{1}{2}$	$-\frac{1}{2}$	-1	e_{mR}	0	-1	-1
u_{mL}	$+\frac{1}{2}$	$+\frac{1}{6}$	$+\frac{2}{3}$	u_{mR}	0	$+\frac{2}{3}$	$+\frac{2}{3}$
d_{mL}	$-\frac{1}{2}$	$+\frac{1}{6}$	$-\frac{1}{3}$	d_{mR}	0	$-\frac{1}{3}$	$-\frac{1}{3}$

Table 1.1 – *The quantum numbers of the different fields of the GWS model.*

The gauge invariant dynamical terms of the gauge bosons is built from the asymmetric, covariant field strength tensor for each group, namely equation (1.3), (1.10) and (1.11). For the the Standard Model the Yang-Mills Lagrangian thus becomes:

$$\mathcal{L}_{\mathrm{YM}} = -\tfrac{1}{4}B_{\mu\nu}B^{\mu\nu} - \tfrac{1}{4}W^i_{\mu\nu}W^{i\mu\nu} - \tfrac{1}{4}G^a_{\mu\nu}G^{a\mu\nu} \tag{1.20}$$

Note that due to the nonlinearity of the $W^i_{\mu\nu}$ and $G^a_{\mu\nu}$, this Lagrangian contains trilinear and quartic self-interaction terms of the non-Abelian gauge fields W^i_μ and G^a_μ.

1.2.2 Brout-Englert-Higgs mechanism

At this point in the review of the Standard Model both the gauge bosons and fermions are still massless as there is no $SU(2) \times U(1)$ invariant mass term possible. In 1964 another mechanism of mass generation was investigated by three separate groups: (i) Robert Brout and François Englert; (ii) Peter Higgs; and later by (iii) Gerald Guralnik, Carl Hagen and Tom Kibble. Their results are known as the Brout-Englert-Higgs-Guralnik-Hagen-Kibble mechanism, often just called the Brout-Englert-Higgs (BEH) or Higgs mechanism, which is a prescription for breaking the gauge symmetry spontaneously (SSB).

The Higgs multiplet $\mathcal{H}$ is introduced in the theory. It needs to have four degrees of freedom as there are four weak gauge bosons W^i_μ $(i = 12, 3)$ and B_μ. The simplest model for the Higgs field is that of a $SU(2)$ doublet with 2 complex scalar fields ϕ^+ and ϕ^0:

$$\mathcal{H} = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}, \text{ with } \begin{array}{ll} \phi^+ : & T^3 = +\frac{1}{2}, \quad \eta_Y Y = +\frac{1}{2} \text{ and } \quad Q = +1 \\ \phi^0 : & T^3 = -\frac{1}{2}, \quad \eta_Y Y = +\frac{1}{2} \text{ and } \quad Q = 0 \end{array} \tag{1.21}$$

The CP-even neutral component of the complex doublet scalar field $\mathcal{H}$ acquires a nontrivial vacuum expectation value (VEV) $v \approx 246$ GeV which sets the scale of electroweak symmetry breaking (EWSB):

$$\langle \mathcal{H} \rangle \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}, \quad \text{such that} \quad \sigma^i \begin{pmatrix} 0 \\ v \end{pmatrix} \neq 0, \quad Y \begin{pmatrix} 0 \\ v \end{pmatrix} \neq 0, \quad Q \begin{pmatrix} 0 \\ v \end{pmatrix} = 0 \tag{1.22}$$

which means that the vacuum is not invariant under $SU(2)_L$ transformations and that $U(1)_Y$ is also broken. However, the electromagnetic symmetry $U(1)_{em}$ remains exact. The Higgs field thus 'spontaneously breaks' the local gauge symmetry $SU(2)_L \times U(1)_Y$ to $U(1)_{em}$.

The above mentioned vacuum expectation value (VEV) is realized by a 'Mexican hat potential' containing a tachyonic mass term and quartic self-interaction:

$$V(\mathcal{H}^\dagger \mathcal{H}) = -\mu^2 \mathcal{H}^\dagger \mathcal{H} + \lambda (\mathcal{H}^\dagger \mathcal{H})^2 \tag{1.23}$$

where $\mu^2 > 0$ and $\lambda > 0$ are real, constant parameters.

The minimum of the potential is at:

$$\tfrac{1}{2}\upsilon^2 = \mathcal{H}_0^\dagger \mathcal{H}_0 = \frac{\mu^2}{2\lambda} \Rightarrow \upsilon = \sqrt{\frac{\mu^2}{\lambda}} \tag{1.24}$$

The gauge invariant Higgs Lagrangian is then established by replacing the normal derivative with the covariant derivative (1.19) using the appropriate quantum numbers as mentioned in equation (1.21).

$$\mathcal{L}_{\text{Higgs}} = (D_\mu \mathcal{H})^\dagger (D_\mu \mathcal{H}) + \mu^2 \mathcal{H}^\dagger \mathcal{H} - \lambda(\mathcal{H}^\dagger \mathcal{H})^2 \tag{1.25}$$

Perturbation theory requires smallness of the terms in the expansion, so the fields must have average value zero in the ground state, but $\langle 0|\mathcal{H}|0\rangle = \upsilon \neq 0$. Therefore, we need to redefine $\mathcal{H}$. Here we treat two of these redefinitions: the unitary gauge and the R_ξ gauges.

Unitary gauge

This parametrization is based on the introduction of four new fields $H(x)$ and $\zeta^i(x)$ with $i = 1, 2, 3$:

$$\mathcal{H} = \frac{1}{\sqrt{2}} U^{-1}(\zeta) \begin{pmatrix} 0 \\ \upsilon + H(x) \end{pmatrix}, \quad \text{with } U^{-1}(x) = e^{\frac{-i\zeta_i(x)T^i}{\upsilon}} \tag{1.26}$$

where $H(x)$ is the Higgs boson. Exploiting the gauge invariance, the fields $\zeta^i(x)$ can be transformed away via a gauge transformation (equation (1.13) with $\vec{\alpha} = \frac{\vec{\zeta}}{\upsilon}$):

$$\mathcal{H}' = U(\zeta)\mathcal{H} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \upsilon + H(x) \end{pmatrix} \tag{1.27}$$

and replacing $\mathcal{H}$ with $\mathcal{H}'$ everywhere. The Higgs Lagrangian $\mathcal{L}_{\text{Higgs}}$ in the unitary gauge becomes (up to a constant):

$$\begin{aligned} \mathcal{L}_{Higgs} = {} & \tfrac{1}{2}\partial_\mu H \partial^\mu H - \tfrac{1}{2} M_H^2 H^2 - \frac{M_H^2}{2\upsilon} H^3 - \frac{M_H^2}{8\upsilon^2} H^4 \\ & + M_W^2 W_\mu^+ W^{\mu-} \left(1 + \frac{2}{\upsilon} H + \frac{1}{\upsilon^2} H^2\right) + \tfrac{1}{2} M_Z^2 Z_\mu Z^\mu \left(1 + \frac{2}{\upsilon} H + \frac{1}{\upsilon^2} H^2\right) \end{aligned} \tag{1.28}$$

We see that expanding the potential around the minimum such that terms linear in $H(x)$ drop out, gives physical masses to the Higgs and weak gauge bosons:

$$M_W = \frac{g\upsilon}{2}, \quad M_Z = \frac{M_W}{\cos(\theta_W)}, \quad M_H = \sqrt{2\mu^2} \tag{1.29}$$

Before SSB there were 4 massless gauge fields and 4 degrees of freedom form the Higgs field of which 3 would-be Goldstone bosons ($\vec{\xi}$). After SSB, the Goldstone bosons are absorbed to give masses to the $W_\mu^\pm$ and Z_μ gauge bosons, leaving the photon massless. The remaining component of the complex doublet becomes the Higgs boson, a new fundamental scalar particle. The number of degrees of freedom before and after spontaneous symmetry breaking is thus equal.

R_ξ gauge

The disadvantage of the unitary gauge is that the propagators of the gauge fields behave as k^0 when the momentum $k \to \infty$ seemingly indicating that the theory is non-renormalizable. However, since physical quantities are gauge invariant, any physical quantity can be calculated in a gauge where renormalizability is manifest. While the particle content is manifest in the unitary gauge, we could also choose to work in the more conventional class of R_ξ gauges.

$$\mathcal{H} = \begin{pmatrix} \phi^+(x) \\ \frac{\upsilon+H(x)+\mathrm{i}\phi_Z(x)}{\sqrt{2}} \end{pmatrix} \tag{1.30}$$

Plugging this into equation (1.25) gives:

$$\begin{aligned}\mathcal{L}_{\text{Higgs}} = {} & \tfrac{1}{2}\partial_\mu H \partial^\mu H - \tfrac{1}{2}M_H^2 H^2 - \frac{M_H^2}{2\upsilon}H^3 - \frac{M_H^2}{8\upsilon^2}H^4 \\ & + M_W^2 W_\mu^+ W^{\mu-}\left(1+\frac{2}{\upsilon}H+\frac{1}{\upsilon^2}H^2\right) + \tfrac{1}{2}M_Z^2 Z_\mu Z^\mu \left(1+\frac{2}{\upsilon}H+\frac{1}{\upsilon^2}H^2\right) \\ & + \partial_\mu\phi^+\partial^\mu\phi^- + \tfrac{1}{2}\partial_\mu\phi_Z\partial^\mu\phi_Z + \mathrm{i}\eta M_W\left(W_\mu^-\partial^\mu\phi^+ - W_\mu^+\partial^\mu\phi^-\right) - \eta\eta_Z M_Z Z_\mu \partial^\mu \phi_Z \\ & + \text{trilinear interactions } + \text{quadrilinear interactions}\end{aligned} \tag{1.31}$$

where the masses are defined as in equation (1.29). As is now evident, spontaneous symmetry breaking in the R_ξ gauge introduces mixing terms between the gauge bosons and the would-be Goldstone fields. As we will see in section 1.2.4, these terms can be cancelled but the unphysical $\phi_Z, \phi_\pm$ do not disappear and will contribute to the propagator. It means we have the desirable behavior of the propagator at the cost of increasing the number of particles and Feynman diagrams. These R_ξ gauges are therefore only used when calculating higher-order corrections to transition amplitudes.

1.2.3 Yukawa interactions

The Yang-Mills, fermion and Higgs Lagrangian of the Standard Model have now been discussed. Next, we note that the introduced Higgs field $\mathcal{H}$ also couples to the fermions in the so-called Yukawa interaction terms. These interaction terms have to be $SU(2)_L$ singlets with the property $\Sigma Y = 0$ in order for them to be $SU(2)_L \times U(1)_Y$ gauge invariant. Because of this, the charge conjugated Higgs doublet is needed.

$$\tilde{\mathcal{H}} = \mathrm{i}\sigma_2\mathcal{H}^\dagger = \begin{pmatrix} {\phi^0}^\dagger \\ -\phi^- \end{pmatrix}, \text{ with } \begin{matrix} {\phi^0}^\dagger: & T^3 = +\frac{1}{2}, & Y = -\frac{1}{2} & \text{and} & Q = 0 \\ \phi^-: & T^3 = -\frac{1}{2}, & Y = -\frac{1}{2} & \text{and} & Q = -1 \end{matrix} \tag{1.32}$$

Summing over the generations and allowing for mixing of generations, the gauge invariant Yukawa Lagrangian becomes:

$$\begin{aligned}\mathcal{L}_{\text{Yukawa}} = & -\sum_{\text{quarks}} \left(\bar{u}_L \;\; \bar{d}_L\right)\Gamma^u\tilde{\mathcal{H}}u_R + \bar{u}_R\Gamma^{u*}\tilde{\mathcal{H}}^\dagger\begin{pmatrix}u_L\\d_L\end{pmatrix} + \left(\bar{u}_L \;\; \bar{d}_L\right)\Gamma^d\mathcal{H}d_R + \bar{d}_R{\Gamma^d}^*\mathcal{H}^\dagger\begin{pmatrix}u_L\\d_L\end{pmatrix} \\ & -\sum_{\text{leptons}} \left(\bar{\nu}_L \;\; \bar{e}_L\right)\Gamma^\nu\tilde{\mathcal{H}}\nu_R + \bar{\nu}_R\Gamma^{\nu*}\tilde{\mathcal{H}}^\dagger\begin{pmatrix}\nu_L\\e_L\end{pmatrix} + \left(\bar{\nu}_L \;\; \bar{e}_L\right)\Gamma^e\mathcal{H}e_R + \bar{e}_R\Gamma^{e*}\mathcal{H}^\dagger\begin{pmatrix}\nu_L\\e_L\end{pmatrix}\end{aligned} \tag{1.33}$$

where $\Gamma^{u,d,\nu,e}$ are the Yukawa couplings represented by 3×3 complex matrices. In general, the matrices are not diagonal because the fermion fields in the Lagrangian are not mass eigenstates. The "true fermions" (primed fields) with well-defined masses are linear combinations of those in $\mathcal{L}_{\text{Yukawa}}$ which are flavor eigenstates.

$$\bar{\psi}_L\Gamma^\psi\psi_R = \left(\bar{\psi}_L\mathbf{U}_L^\psi\right)\left({\mathbf{U}_L^\psi}^\dagger\,\Gamma^\psi\,\mathbf{U}_R^\psi\right)\left({\mathbf{U}_R^\psi}^\dagger\psi_R\right) = \bar{\psi}'_L M^\psi \psi'_R \tag{1.34}$$

Here, $\psi = u, d, \nu, e$ and M^f is the mass matrix with the masses of the three generations of ψ on the diagonal. The unitary matrices $\mathbf{U}$ can be derived from the Cabibbo-Kobayashi-Maskawa

(CKM) mixing matrix $\mathbf{V}^q = \mathbf{U}_L^u \mathbf{U}_L^{d\,\dagger}$ and Pontecorvo-Maki-Nakagawa-Sakata (PMNS) mixing matrix $\mathbf{V}^\ell = \mathbf{U}_L^\nu \mathbf{U}_L^{e\,\dagger}$ with $\mathbf{V}$ in both cases parametrized as

$$\mathbf{V} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-\mathrm{i}\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{\mathrm{i}\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.35}$$

where $s_{ij} = \sin\theta_{ij}, c_{ij} = \cos\theta_{ij}$, and δ is the phase responsible for all CP-violating phenomena in flavor-changing processes in the SM. The known values are listed in Appendix A.

Once the Higgs acquires a VEV (i.e. plug either (1.26) or (1.30) in $\mathcal{L}_{\text{Yukawa}}$), and after rotation to the fermion mass eigenstate basis that also diagonalizes the Higgs-fermion interactions, all fermions acquire a mass given by $m_\psi = \lambda_\psi \frac{v}{\sqrt{2}}$. The fermion masses, accounting for a large number of the free parameters of the SM are also listed in Appendix A. It should be noted that the electroweak symmetry breaking mechanism provides no additional insight on possible underlying reasons for the large variety of masses of the fermions, often referred to as the flavor hierarchy.

Before continuing we point out that in the above we assumed neutrino's to be Dirac particles. The idea of the Dirac neutrino works in the sense that we can generate neutrino masses via the Higgs mechanism. Yet a strong point of critique is that this suggests that neutrinos should have similar masses to the other particles in the Standard Model, something that can be avoided if we also assume that interaction strength of the neutrino with the Higgs boson is at least 10^{12} times weaker than that of the top quark. Few physicists accept such a tiny number as a fundamental constant of nature. The other option is to consider neutrino's as Majorana particles, which is a fermion that is its own antiparticle, contrary to Dirac fermions. They were hypothesized by Ettore Majorana in 1937. In this scheme, it is possible for right-handed neutrinos to have a mass of their own without relying on the Higgs boson, but two additional CP violating phases must be added to the PMNS mixing matrix via the matrix $\mathrm{diag}(1, e^{\mathrm{i}\alpha_{21}/2}, e^{\mathrm{i}\alpha_{31}/2})$.

Currently, one uses a mixture of the two possibilities to explain the smallness of the neutrino masses. This (type I) see-saw mechanism of neutrino mass generation assumes Yukawa interactions with the SM leptons and Higgs doublet as in (1.33) as well as a Majorana mass term for right-handed neutrino's:

$$\mathcal{L}_{\text{see-saw}} = \mathcal{L}_{\text{Yukawa}} - \frac{1}{2} M \bar{\nu}_R \nu_R^c \tag{1.36}$$

where M is a diagonal Majorana mass matrix when using the mass eigenstates and ν_R^c is the CP conjugate of a right-handed neutrino field, in other words a left-handed antineutrino field. In the case when the elements of Dirac mass matrices m_ν are much smaller than those of the Majorana mass matrix M, the complete neutrino mass matrix may be diagonalized yielding a small mass term for the left-handed flavor neutrinos [17]: $M_\nu = -m_\nu M^{-1} m_\nu^T$. Adding the Majorana mass term thus yields the desired result of small neutrino masses. It is important to note that this term could also be generated through the vacuum expectation value of some extra scalar singlet.

1.2.4 Faddeev-Popov gauge-fixing procedure

Similarly to an Abelian theory like QED, the gauge invariance of the theory leads to too many degrees of freedom, ruining the naive way of calculating the propagator. In the unitary gauge it is possible to determine the propagators without these difficulties. However, if we want to apply the arguments of power counting renormalizability, the boson propagators have to behave as k^{-2} for large momenta k^2. Therefore, one uses the class of R_ξ gauges, but also fix the gauge to reduce the degrees of degree of freedom.

To address the problem of the degrees of freedom gauge-fixing terms are added to the gauge invariant Lagrangian in perturbation theory.

$$\mathcal{L}_{GF} = -\frac{1}{2\xi_G}\mathcal{C}_G^2 - \frac{1}{2\xi_A}\mathcal{C}_A^2 - \frac{1}{2\xi_Z}\mathcal{C}_Z^2 - \frac{1}{\xi_W}\mathcal{C}_-\mathcal{C}_+ \tag{1.37}$$

where

$$\begin{aligned} \mathcal{C}_G^a &= \partial^\mu G_\mu^a & \delta\mathcal{C}_G^a &= \partial^\mu(\delta G_\mu^a) \\ \mathcal{C}_A &= \partial^\mu A_\mu & \delta\mathcal{C}_A &= \partial^\mu(\delta A_\mu) \\ \mathcal{C}_Z &= \partial^\mu Z_\mu + \eta\eta_Z\zeta_Z M_Z\phi_Z & \delta\mathcal{C}_Z &= \partial^\mu(\delta Z_\mu) + \eta\eta_Z\zeta_Z M_Z\delta\phi_Z \\ \mathcal{C}_\pm &= \partial^\mu W_\mu^\pm \pm \mathrm{i}\eta\zeta_W M_W\phi^\pm & \delta\mathcal{C}_\pm &= \partial^\mu(\delta W_\mu^\pm) \pm \mathrm{i}\eta\zeta_W M_W\delta\phi^\pm \end{aligned} \tag{1.38}$$

The class of R_ξ gauges refers to the different values we can give to ξ_a and ζ_a in the above equation to simplify calculations. Familiar combinations are the 't Hooft gauge where $\zeta_Z = \xi_z$ and $\zeta_W = \xi_W$ and the 't Hooft-Feynman gauge has in addition $\xi_w = \xi_Z = \xi_A = 1$. Both of which would also cancel the cross terms that arise in equation (1.31). Another gauge is the Landau gauge $\xi_W = \xi_Z = \xi_A = 0$ and $\zeta_w = \zeta_Z = 0$, whereas working in the unitary gauge is equivalent to $\zeta_W = \xi_W^2 \to \infty$ and $\zeta_Z = \xi_Z^2 \to \infty$. All these choices are allowed as the dependence on the gauge should cancel in physical quantities.

Next recall the gauge transformations under $SU(3)$, $SU_L(2)$ and $U_Y(1)$, (1.5), (1.13) and (1.14) respectively. Applying these gauge transformations on the gauge-fixing terms, to find the missing terms in the second column of (1.38), shows that gauge fixing breaks the local gauge symmetry non-linearly. As a consequence of the broken gauge Ward identity the unphysical part of the vector bosons interacts and contributes to the physical scattering matrix in the tree approximation. In order to cancel these interactions additional fields, the Faddeev-Popov ghosts, are needed[12]. For each gauge field there is such a ghost field, but as they are unphysical particles they only occur inside loops. Where the gauge field acquires a mass via the Brout-Englert-Higgs mechanism, the associated ghost field acquires the same mass but only in the Feynman-'t Hooft gauge. Furthermore, they obey Fermi statistics just like fermions do, giving rise to an extra minus sign in Feynman diagrams as compared to the same diagrams made up of gauge fields.

$$\mathcal{L}_{\mathrm{FP}} = \eta_G \sum_{i=1}^{4}\left[\bar{c}_+\frac{\partial(\delta\mathcal{C}_+)}{\partial\alpha^i} + \bar{c}_-\frac{\partial(\delta\mathcal{C}_-)}{\partial\alpha^i} + \bar{c}_Z\frac{\partial(\delta\mathcal{C}_Z)}{\partial\alpha^i} + \bar{c}_A\frac{\partial(\delta\mathcal{C}_A)}{\partial\alpha^i}\right]c_i + \eta_G\sum_{a,b=1}^{8}\bar{\omega}^a\frac{\partial(\delta\mathcal{C}_G^a)}{\partial\beta^b}\omega^b \tag{1.39}$$

In the above equation, $c_\pm, c_A, c_Z$ are the electroweak ghosts associated with $U(\alpha^i)$ with $i = 1,\ldots 4$ and ω^a with $a = 1,\ldots 8$ the ghosts associated with the $SU(3)$ color transformations. The various δC are given in equation (1.38). Gauge invariance of the full Lagrangian of the Standard Model can be explicitly checked [see e.g. 6].

Choosing the R_ξ means that the Goldstone boson is still present and has acquired a mass from the gauge fixing term. It has interactions with the gauge boson, with the Higgs scalar and with itself. Furthermore, Faddeev- Popov ghosts needed to be introduced, which interact with the gauge bosons, the Higgs scalar and the Goldstone bosons.

[12] Note that the conventional way for introducing Faddeev-Popov fields into gauge theories does not start from unitarity arguments but from the path integral formulation of quantum field theory: To implement the gauge fixing program in path integrals one needs a compensating determinant. This determinant can be rewritten in the form a path integral over a set of anti-commuting scalar fields. Since these scalar fields have the wrong statistics (they should have been bosons instead of fermions) they are not physical and therefore called ghosts. A third way of introducing Faddeev-Popov ghosts in the theory is provided by the algebraic method of BRS quantization [18].

1.2.5 Complete Lagrangian

The full Standard Model Lagrangian consists of the contributions from the Yang-Mills sector (1.20), the fermions (1.18), the Higgs sector (1.25), the yukawa interactions (1.33), the gauge-fixing terms (1.37), and the Faddeev-Popov ghosts (1.39).

$$\mathcal{L}_{SM} = \mathcal{L}_{\text{YM}} + \mathcal{L}_{\text{Fermion}} + \mathcal{L}_{\text{Higgs}} + \mathcal{L}_{\text{Yukawa}} + \mathcal{L}_{\text{GF}} + \mathcal{L}_{\text{ghosts}} \tag{1.40}$$

Including neutrino masses via the see-saw mechanism requires (1.36) instead of (1.33) and add an additional Majorana mass term for right-handed neutrino's to the above lagrangian. Before spontaneous symmetry breaking of the electroweak interactions, this is Lagrangian takes the following form:

$$\begin{aligned}
\mathcal{L}_{\text{SM}} = & -\tfrac{1}{4}B_{\mu\nu}B^{\mu\nu} - \tfrac{1}{4}W^i_{\mu\nu}W^{i\mu\nu} - \tfrac{1}{4}G^a_{\mu\nu}G^{a\mu\nu} && \text{(U(1), SU(2), SU(3) gauge terms)} \\
& + \begin{pmatrix}\bar{\nu}_L & \bar{e}_L\end{pmatrix} \mathrm{i}\not{D} \begin{pmatrix}\nu_L \\ e_L\end{pmatrix} + \bar{\nu}_R \mathrm{i}\not{D}\nu_R + \bar{e}_R \mathrm{i}\not{D} e_R && \text{(lepton dynamical terms)} \\
& - \begin{pmatrix}\bar{\nu}_L & \bar{e}_L\end{pmatrix} \Gamma^e \mathcal{H} e_R - \bar{e}_R \Gamma^{e*} \mathcal{H}^\dagger \begin{pmatrix}\nu_L \\ e_L\end{pmatrix} && \text{(e, } \mu\text{, } \tau \text{ mass terms)} \\
& - \begin{pmatrix}\bar{\nu}_L & \bar{e}_L\end{pmatrix} \tilde{\mathcal{H}} \Gamma^\nu \nu_R - \bar{\nu}_R \Gamma^{\nu*} \tilde{\mathcal{H}}^\dagger \begin{pmatrix}\nu_L \\ e_L\end{pmatrix} - \frac{1}{2} M \bar{\nu}_R \nu_R^c && \text{(neutrino mass term)} \\
& + \begin{pmatrix}\bar{u}_L & \bar{d}_L\end{pmatrix} \mathrm{i}\not{D} \begin{pmatrix}u_L \\ d_L\end{pmatrix} + \bar{u}_R \mathrm{i}\not{D} u_R + \bar{d}_R \mathrm{i}\not{D} d_R && \text{(quark dynamical terms)} \\
& - \begin{pmatrix}\bar{u}_L & \bar{d}_L\end{pmatrix} \Gamma^d \mathcal{H} d_R - \bar{d}_R \Gamma^{d*} \mathcal{H}^\dagger \begin{pmatrix}u_L \\ d_L\end{pmatrix} && \text{(d, s, b mass term)} \\
& - \begin{pmatrix}\bar{u}_L & \bar{d}_L\end{pmatrix} \Gamma^u \tilde{\mathcal{H}} u_R - \bar{u}_R \Gamma^{u*} \tilde{\mathcal{H}}^\dagger \begin{pmatrix}u_L \\ d_L\end{pmatrix} && \text{(u, c, t mass term)} \\
& + (D_\mu \mathcal{H})^\dagger (D_\mu \mathcal{H}) + \mu^2 \mathcal{H}^\dagger \mathcal{H} - \lambda(\mathcal{H}^\dagger \mathcal{H})^2 && \text{(Higgs dynamical and mass term)} \\
& - \frac{1}{2\xi_G}\mathcal{C}_G^2 - \frac{1}{2\xi_A}\mathcal{C}_A^2 - \frac{1}{2\xi_Z}\mathcal{C}_Z^2 - \frac{1}{\xi_W}\mathcal{C}_-\mathcal{C}_+ && \text{(Gauge-fixing term)} \\
& + \eta_G \sum_{a,b=1}^{8} \bar{\omega}^a \frac{\partial(\delta C_G^a)}{\partial \beta^b} \omega^b && \text{(Faddeev- Popov ghosts)} \\
& + \eta_G \sum_{i=1}^{4} \left[\bar{c}_+ \frac{\partial(\delta\mathcal{C}_+)}{\partial\alpha^i} + \bar{c}_- \frac{\partial(\delta\mathcal{C}_-)}{\partial\alpha^i} + \bar{c}_Z \frac{\partial(\delta\mathcal{C}_Z)}{\partial\alpha^i} + \bar{c}_A \frac{\partial(\delta\mathcal{C}_A)}{\partial\alpha^i} \right] c_i
\end{aligned} \tag{1.41}$$

and the full gauge covariant derivative is given in (1.19)

$$\begin{aligned}
D_\mu &= \partial_\mu - \frac{\mathrm{i}g}{2}\sigma_i A^i_\mu - \mathrm{i}g' Y B_\mu - \frac{\mathrm{i}g_s}{2}\lambda_a G^a_\mu \\
&= \partial_\mu + \frac{\mathrm{i}\eta g}{2}\left[\sigma_+ W^+_\mu + \sigma_- W^-_\mu\right] + \mathrm{i}\eta_e e Q A_\mu + \frac{\mathrm{i}\eta g}{\cos(\theta_W)}\left[\frac{\sigma_3}{2} - Q\sin^2(\theta_W)\right]\eta_z Z_\mu + \frac{\mathrm{i}\eta_s g_s}{2}\lambda_a G^a_\mu
\end{aligned}$$

Summation over the generation, color indices and contracted indices (Einstein convention) is implied in all above formulae.

1.3 Dealing with infinities

We start by noting that the Standard Model Lagrangian of the previous chapter is still classical. In order to correctly calculate physical observables like cross sections and decay rates, we'll have to transform it to a quantum theory. To change the classical Lagrangian to a quantum theory we'll write the particle fields in terms of creation and annihilation operators and then impose the canonical (anti)commutation relations. This procedure is called canonical quantization.

Instead, we can also use the path integral formalism as shown by means of the LSZ reduction formula in Section 1.3.1. Added benefits of that method are that it uses $\mathcal{L}_{\mathrm{SM}}$ of the previous section directly, and explicitly preserves all symmetries of the theory. The LSZ formula allows us to calculate transition amplitudes via the n-point correlation functions. To calculate those we need the procedures of regularization and renormalization as explained in Section 1.3. This introduces an energy dependence in the coupling constant and the function describing this dependence is explained in the next-to-last section and is applied to the Standard Model in Section 1.3.4.

1.3.1 LSZ reduction formula and generating functionals

We start by writing down the transition amplitude $\mathcal{T}_{i\to f}$ with an initial state $|i\rangle$ of n incoming particles at $t = -\infty$ and a final state $|f\rangle$ of n' of outgoing particles at time $t = +\infty$. The LSZ reduction formula, named after its inventors Harry Lehmann, Kurt Symanzik en Wolfhart Zimmermann, relates the amplitude to an expression involving the particle fields Φ: the n-point correlation function or Green's function.

$$\begin{aligned}\mathcal{T}_{i\to f} = \langle f|S|i\rangle = i^{n+n'} \int d^4x_1 e^{\mathrm{i}k_1x_1}(-\partial_1^2 + m^2)\dots d^4x_{1'} e^{\mathrm{i}k_{1'}x_{1'}}(-\partial_{1'}^2 + m^2)\dots \\ \times \langle\Omega|T\{\Phi(x_1)\dots\Phi(x_n)\Phi(x_{1'})\dots\Phi(x_{n'})\}|\Omega\rangle\end{aligned} \tag{1.42}$$

where S is the S-matrix, $|\Omega\rangle$ the vacuum of the theory and T is a time-ordering symbol that orders operators such that operators at later times are left of operators at earlier times. The n-point correlation function can be expressed in terms of functional integrals. Actually, using Wick's theorem, the correlation functions can be evaluated in a perturbative expansion in powers of the coupling g based on all connected Feynman diagrams. The result is

$$\langle\Phi(x_1)\dots\Phi(x_n)\rangle = \lim_{T\to\infty(1-\mathrm{i}\epsilon)} \frac{\int \mathcal{D}\Phi\Phi(x_1)\dots\Phi(x_n)e^{\mathrm{i}\int d^4x\mathcal{L}}}{\int \mathcal{D}\Phi e^{\mathrm{i}\int d^4x\mathcal{L}}} = \begin{pmatrix}\text{sum of Feynman} \\ \text{diagrams with } x_i\end{pmatrix}$$

Defining the generating functional $Z[J]$ of all general correlation functions, where $J(x)$ an external current that acts as a source for every field $\Phi(x)$, we get

$$\langle\Phi(x_1)\dots\Phi(x_n)\rangle = \frac{(-\mathrm{i})^n}{Z[J]} \left.\frac{\delta^n Z[J]}{\delta J(x_1)\dots\delta J(x_n)}\right|_{J=0} \quad \text{with} \quad Z[J] = \int \mathcal{D}\Phi\, e^{\mathrm{i}\int d^4x(\mathcal{L}(x)+J(x)\Phi(x))}$$

Next, we can define $W[J]$ as the generating functional of all *connected* correlation functions:

$$\langle\Phi(x_1)\dots\Phi(x_n)\rangle_{conn.} = -\mathrm{i}^{n+1} \left.\frac{\delta^n W[J]}{\delta J(x_1)\dots\delta J(x_n)}\right|_{J=0} \quad \text{with} \quad W[J] = -\mathrm{i}\log Z[J]$$

Lastly, we define $\Gamma[\Phi_{cl}]$ as the generating functional for the 1-particle irreducible[13] (1PI) correlation functions, most often referred to as the (1PI) effective action:

$$\Gamma[\Phi_{cl}] = W[J] - \int d^4x J(x)\Phi_{cl}(x) \quad \text{and} \quad \Phi_{cl} = \frac{\delta W[J]}{\delta J(x)} \tag{1.43}$$

[13] A 1-particle irreducible diagram is an amputated Feynman diagram which does not fall into two pieces if you cut one internal line.

such that

$$\langle\Phi(x_1)\ldots\Phi(x_n)\rangle_{1\text{PI}} = \mathrm{i}\,\frac{\delta^n\Gamma[\Phi_{cl}]}{\delta\Phi_{cl}(x_1)\ldots\delta\Phi_{cl}(x_n)}\bigg|_{\Phi_{cl}=\langle\Phi\rangle} \quad \text{if} \quad \langle\Phi\rangle = \frac{\delta W[J]}{\delta J(x)}\bigg|_{J=0}$$

and if the connected 2-point function takes a nonzero value which is identified with the exact propagator D. The effective action $\Gamma[\Phi_{cl}]$ contains the full information about the quantum dynamics of a theory in a classical language, in the sense that the full loop-corrected value of any S-matrix element can be obtained by using the effective action and computing only tree-level diagrams.

Now taking into account that we usually have an interacting, multiparticle theory, we find that additional conditions are needed for the LSZ formula to work. This is because the ground state is in general $\langle 0|\Phi(0)|0\rangle \equiv \upsilon \neq 0$. In a free theory the annihilation and creation operators annihilate the vacuum to the right or left respectively. Thus, we require

$$\langle 0|\Phi(x)|0\rangle = \langle 0|\Phi(0)|0\rangle = 0 \tag{1.44}$$

to hold also in the interacting theory, which can be achieved by performing a linear shift, as was also done in Section 1.2.2: $\Phi \to \Phi + \upsilon$. With this linear shift, the ground state is again $|0\rangle$, normalized via $\langle 0|0\rangle = 1$. Given that only tadpole Feynman diagrams (diagrams with only one external line) contribute, setting $\langle 0|\Phi(x)|0\rangle = 0$ is equivalent to omitting from the generating functional all the tadpole graphs [9].

Furthermore, we need to make sure the asymptotic states at $t = -\infty$ and $t = +\infty$ are one-particle states. In the free theory $\Phi(0)\,|0\rangle$ creates a single particle state $|k\rangle$ with an appropriate normalization[14], because

$$\langle k|\Phi(x)|0\rangle = e^{-\mathrm{i}k^\mu x_\mu}\langle k|\Phi(0)|0\rangle = e^{-\mathrm{i}k^\mu x_\mu} \tag{1.45}$$

For this relation to hold in a multiparticle, interacting theory we define

$$\Phi_0(x) = \xi\sqrt{Z}\Phi_r(x), \quad \text{with} \quad Z = |\langle k|\Phi(0)|0\rangle|^2 \tag{1.46}$$

The subscript 0 signals that this is the bare field whereas Φ_r is the renormalized field, which has a propagator with a pole at the physical mass m with residue ξ. The renormalized field strength Z_Φ is the probability that $\Phi(0)$ creates a given state from the vacuum and ξ is the field-strength normalization factor.

We can write the bare Lagrangian as the sum of the renormalized Lagrangian which has the same form as the original but with field-strength normalized to 1 and physical mass m and coupling g, and residual counter terms. These counterterms are finally determined order by order in perturbation theory to enforce the renormalization conditions defining the physical mass m and couplings g.

Consider for example the ϕ^4 bare Lagrangian with only scalar fields

$$\mathcal{L}(\phi_0,\lambda_0) = \tfrac{1}{2}\partial_\mu\phi_0\partial^\mu\phi_0 - \tfrac{1}{2}m_0^2\phi_0^2 - \tfrac{1}{4!}\lambda_0\phi_0^4$$

Next we shift and apply the reparametrization such that the Lagrangian satisfies the normalization requirements (1.44) and (1.45). The Lagrangian can now be re-written as

$$\begin{aligned}\mathcal{L}(\phi,\lambda) + \mathcal{L}_{\text{counter}} = \mathcal{L}(\phi_0,\lambda_0) &= \tfrac{1}{2}Z\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{2}Zm_0^2\phi^2 - \tfrac{1}{4!}Z^2\lambda_0\phi^4 \\ &= \left[\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{2}m^2\phi^2 - \tfrac{1}{4!}\lambda\phi^4\right] + \left[\tfrac{1}{2}\Delta_\phi\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{2}\Delta_m\phi^2 - \tfrac{1}{4!}\Delta_\lambda\phi^4\right]\end{aligned}$$

[14]The one-particle state is normalized via $\langle k'|k\rangle = (2\pi)^3 2k^0\delta^3(\mathbf{k}' - \mathbf{k})$

where $\xi = 1$ and $\Delta_Z = Z - 1, \Delta_m = m_0^2 Z - m^2, \Delta_\lambda = \lambda_0 Z^2 - \lambda$. Note that equation (1.45) fixes only one of the three normalization parameters introduced. The remaining two parameters are fixed by equating m^2 to the location of the pole of the propagator (exact two-point function) and λ to the magnitude of the scattering amplitude at zero momentum.

To see how the effective action is affected by writing the Lagrangian $\mathcal{L}_0[\Phi_r] = \mathcal{L}_r[\Phi_r] + \Delta\mathcal{L}[\Phi_r]$, we similarly split the external current

$$J(x) = J_r(x) + \Delta J(x) \quad \text{where} \quad \left.\frac{\delta S_r[\Phi_r]}{\delta \Phi_r}\right|_{\Phi_r = \Phi_{cl}} = -J_r(x)$$

where J_r and the counterterm ΔJ are defined to enforce the definition of $\Phi_{cl}(x)$ at the lowest order and order by order in perturbation theory, respectively. We can then evaluate the exact integral as a saddle-point expansion, corresponding to a loop expansion in powers of $\hbar$, by writing $\Phi_r(x) = \Phi_{cl}(x) + \eta(x)$. Then we can write

$$\Gamma[\Phi_{cl}] = S_r[\Phi_{cl}] \pm \frac{\mathrm{i}}{2}\,\mathrm{Tr}\log\left(-\frac{\delta^2 S_r[\Phi_{cl}]}{\delta \Phi_r^2}\right) - \mathrm{i}\begin{pmatrix}\text{sum of}\\ \text{conn. diag.}\end{pmatrix} + \Delta S[\Phi_{cl}] \tag{1.47}$$

where the sign depends on whether the fields are bosonic or fermionic. Applying it for example to our ϕ^4-model:

$$\Gamma^0[\Phi_{cl}] = \int \mathrm{d}^4 x \left(\tfrac{1}{2}\partial_\mu \Phi_{cl}\partial^\mu \Phi_{cl} - \tfrac{1}{2}m^2\Phi_{cl}^2 - \tfrac{1}{4!}\lambda\Phi_{cl}^4\right) \tag{1.48a}$$

$$\begin{aligned}\Gamma^1[\Phi_{cl}] &= \frac{\mathrm{i}}{2}\,\mathrm{Tr}\log\left(\Box + m^2 + \frac{\lambda}{2}\Phi_{cl}^2\right) + \Delta^1 S \\ &= \frac{\mathrm{i}}{2}\,\mathrm{Tr}\log\left(\Box + m^2\right) + \frac{\mathrm{i}}{2}\,\mathrm{Tr}\log\left(1 + \frac{\lambda}{2}D\Phi_{cl}^2\right) + \Delta^1 S \\ &= \text{const.} - \sum_n \frac{\mathrm{i}}{2n}\,\mathrm{Tr}\left(-\frac{\lambda}{2}D\Phi_{cl}^2\right)^n + \tfrac{1}{2}\Delta_\phi \partial_\mu\Phi_{cl}\partial^\mu\Phi_{cl} - \tfrac{1}{2}\Delta_m m^2\Phi_{cl}^2 - \tfrac{1}{4!}\Delta_\lambda\lambda\Phi_{cl}^4\end{aligned} \tag{1.48b}$$

where we used $\log(1-x) = -\sum_n \frac{x^n}{n}$ in the last line. Diagrammatically, this corresponds to the counter-terms and summing up all the 1-loop diagrams with an arbitrary number n of vertices between the fluctuation field η and the background field Φ_{cl}. They indeed reproduce the correct factor $1/(2n)$ because rotations $(1/n)$ and reflections $(1/2)$ of the loop do not change the diagram. The factor $1/2$ in the numerator is an extra bose factor, since exchanging two external lines at the same vertex does not give a new diagram.

1.3.2 Renormalization and regularization

As explained in the previous section, we know that the LSZ-formula allows us to calculate transition amplitudes via the n-point correlation function. This function can be expressed as a sum of fully connected and amputated Feynman diagrams. Feynman diagrams and rules for external lines, vertices, propagators and counterterms offer a very transparent way of constructing process amplitudes, order-by-order in perturbation theory. This section explains the procedure for calculating the correlation function, see also e.g. any textbook on QFT like [7, 9] which contain explicit examples.

First, use the Feynman rules to write down the amplitude of each diagram. We then change to an integral representation and use the identities (valid for any positive definite $A, B, C, \dots$)

$$\frac{1}{AB} = \int_0^1 \mathrm{d}x\,\frac{1}{\left[Ax + B(1-x)\right]^2}, \qquad \frac{1}{A^2 B} = \int_0^1 \mathrm{d}x\,\frac{2x}{\left[Ax + B(1-x)\right]^3},$$

$$\frac{1}{ABC} = 2!\int_0^1 \mathrm{d}y\,y\int_0^1 \mathrm{d}x\,\frac{1}{\left[A(1-y) + y\left(Bx + C(1-x)\right)\right]^3} \quad \text{etc.}$$

where x, y are Feynman parameters. This method of Feynman parameters allows us to combine the denominators of the propagators to a single expression. Next we complete the square in the new dominator by introducing a new loop momentum variable ℓ and express the numerator in terms of this variable. The integral over terms linear in ℓ vanishes.

Using another trick called Wick rotation to Euclidean momentum($\ell^0 \equiv i\ell_E^0, \boldsymbol{\ell} = \boldsymbol{\ell}_E$) the momentum integral can be evaluated using four-dimensional spherical coordinates. Standard integrals can be used to simplify the expression further and they can be derived from only one integral:

$$I_0(\alpha) \equiv i\frac{(-1)^\alpha}{(2\pi)^d}\int \frac{d^dk}{(k^2+A^2-i\epsilon)^\alpha} = i\frac{(-1)^\alpha}{(4\pi)^{d/2}}\frac{\Gamma(\alpha-d/2)}{\Gamma(\alpha)}A^{d-2\alpha} \tag{1.49}$$

If $\alpha > d/2$, the integral is finite, but in relativistic QFTs we often encounter $\alpha \leq d/2$ and the integral becomes UV divergent. To deal with this divergence the integral must be 'regularized' using a regularization scheme. Here we use UV dimensional regularization, developed by Gerard 't Hooft, where we change the dimension of the integral to $d = 4 - 2\epsilon$. Then the loop integrals can be split in a divergent part in the form of inverse powers of ϵ and an arbitrary finite part. We can then choose the renormalization constants Z_i such that the counterterms exactly cancel these divergent Feynman diagrams, e.g. 'renormalize' these parameters to maintain the renormalization conditions. Finally, we take the limit $\epsilon \to 0$ to get finite physical observables. This procedure is known as renormalized perturbation theory.

Dimensional regularization introduces a mass scale μ into the theory. Because we extend from dimension $d = 4$ to dimension $d = 4-2\epsilon$, a previously dimensionless g must now have dimensions 2ϵ. So we let μ be some scale with mass dimension one and then $g \to g\tilde{\mu}^{2\epsilon}$ such that g itself is dimensionless and the Lagrangian will still have dimension four. Here, $\mu^2 = 4\pi e^{-\gamma_E}\tilde{\mu}^2$ and γ_E the Euler-Mascheroni constant[15]. Note that bare parameters are per definition independent of μ.

Notice that there is a freedom in splitting the integral into two pieces. The amount of the finite piece that we absorb into the divergent piece depends on what regularization scheme we use. Specifically, the modified minimal subtraction scheme[16] ($\overline{\text{MS}}$) is used. This prescription absorbs only the infinities that arise in perturbative calculations beyond leading order into the counterterms.

QFTs are called renormalizable if there are a finite number of counter-terms needed to render perturbation theory useful. The superficial degree of divergence D can give a tentative indication whether or not a particular diagram diverges.

$$D = d + \left(\frac{d-4}{2}\right)V - \left(\frac{d-2}{2}\right)N_\gamma - \left(\frac{d-1}{2}\right)N_e \tag{1.50}$$

with d the dimension, V the number of vertices, N_γ the number of external photon lines, and N_e the number of external electron lines. Naively, a diagram will show divergence $\propto \Lambda^D$ for $D > 0$ with Λ the momentum cut-off, the divergence will be logarithmic for $D = 0$, and there will be no divergence when $D < 0$. There are of course exceptions.

The coefficient in front of V is the mass dimension of the coupling constant of the theory. Therefore, we can say that renormalizability is related to the mass dimensions of the theory.

- Super-renormalizable theory: Only a finite number of Feynman diagrams superficially diverge. This is the case when the coupling constant has positive mass dimension.
- Renormalizable theory: When the coupling constant is dimensionless, only a finite number of amplitudes superficially diverge. However, divergencies occur at all orders in perturbation theory.

[15] It appears in $\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma_E + \mathcal{O}(\epsilon)$ and is $\gamma_E \approx 0.5772$. It will cancel in observable quantities.
[16] Had we set $\tilde{\mu} = \mu$ in $g\tilde{\mu}^\epsilon$ then the scheme would be just the minimal subtraction or MS scheme.

- Non-renormalizable theory: All amplitudes are divergent at a sufficiently high order in perturbation theory. This happens when the coupling constant has negative mass dimension.

The spin of the matter fields also has to be taken into account: in $d = 4$ dimensions spin 0 and spin $\frac{1}{2}$ fields are renormalizable, spin 1 fields only when associated with a gauge symmetry, and fields with higher spin are never renormalizable for $d \geq 4$ dimensions.

The 'old' view was that only renormalizable Lagrangians can function as models for physical phenomena. In nonrenormalizable theories, physical observables do not decouple from the UV modes and we cannot make predictions in the high energy limit. However, Kenneth Wilson realized that nonrenormalizable theories can still make valid predictions below a certain ultraviolet cutoff Λ [19] because then the UV modes are negligible. These theories are so called Effective Field Theories (EFT) and allow us to make predictions at present energies without making unwarranted assumptions about what is going on at higher energies.

The central ingredient of EFT is the Wilsonian effective action (not to be mistaken with the 1PI effective action). Suppose we have a theory at a high energy scale, the bare scale Λ_0, which is described by the bare action $S_{\Lambda_0} = \int \mathrm{d}^d x \mathcal{L}_0$. Now, we integrate out degrees of freedom between the bare scale and a lower, effective scale, Λ and rescale the theory. These continuously generated transformations of Lagrangians are referred to as the ERG. The resulting action, the Wilsonian effective action S_Λ is in general different from the original action. This effective action now describes the theory at the effective scale [20]. Unlike the 1PI effective action, one still has to perform the path integral in the Wilsonian case.

On the other hand, if we do not know the high energy theory, we adopt the following general procedure:

1. Identify the low energy degrees of freedom and symmetries.
2. Using only these fields write down the most general Lagrangian consistent with these symmetries. This Lagrangian can be ordered in an energy expansion in terms of increasing dimensions of the operators involved, where the operators of the lowest dimension(s) form the leading order interactions.
3. Quantize the fields and identify the propagators in the usual way.
4. Observables like the cross section can be calculated in the usual way, using regularization and renormalization to deal with the infinities. However, an effective theory requires an infinite amount of terms in order to absorb all divergences. The coefficients of various terms in the effective Lagrangian cannot be predicted and are thus determined by matching them to experiments. This only needs to be done once per term and once fixed each coefficient can be reliably used to compute further observables.
5. Use the theory to make predictions.

As can be seen, this procedure differs little from conventional renormalizable theories. The only difference being in the number of terms to absorb the infinities. However, only a finite number of terms are required to make predictions to within a required accuracy.

1.3.3 Callan-Symanzik equation

In this section we want to focus on one particular important aspect of the above explained renormalization and regularization procedure, namely the fact that a scale dependence is introduced. Specifically, the renormalization conditions that we have imposed involve an arbitrary renormalization scale μ and field strength normalization ξ. However, we could have equally well used μ', ξ'. In order for it to be the same theory, there needs to exist a continuous relationship between the regular and primed expressions. This again means that the dependence of the Green's function on μ, ξ, g is constrained.

Recall the connected n-point function, which we call $G^{(n)}(x_1,\ldots,x_n;\mu,g)$

$$G^{(n)}(x_1,\ldots,x_n;g_0) = \langle\Phi(x_1)\ldots\Phi(x_n)\rangle_{\text{conn.}} \tag{1.51}$$

The renormalized Green's function is

$$G_r^{(n)}(x_1,\ldots,x_n;\mu,g) = \langle\Phi_r(x_1)\ldots\Phi_r(x_n)\rangle_{\text{conn.}} \tag{1.52}$$

where the only renormalization scale dependence is hidden in Z such that

$$G^{(n)}(x_1,\ldots,x_n;g_0) = Z_\Phi^{n/2}G_r^{(n)}(x_1,\ldots,x_n;\mu,g) \tag{1.53}$$

Then the trajectories of the parameters (μ,g) defining the same theory through different renormalization conditions are determined by requiring that the bare Green's function should be invariant under infinitesimal transformations $\mu\to\mu+\delta\mu, g\to g+\delta g$ combined with $Z_\Phi\to Z_\Phi+\delta Z_\Phi$, done at fixed values for the bare coupling g_0.

$$\begin{aligned}
0 &= \left(\delta\mu\frac{\partial}{\partial\mu}+\delta g\frac{\partial}{\partial g}+\delta Z\frac{\partial}{\partial Z}\right)Z^{n/2}G_r^{(n)}(x_1,\ldots,x_n;\mu,g)\\
&= \left(\delta\mu\frac{\partial}{\partial\mu}+\delta g\frac{\partial}{\partial g}+\frac{n}{2}\frac{\delta Z}{Z}\right)G_r^{(n)}(x_1,\ldots,x_n;\mu,g)\\
&= \left(\mu\frac{\partial}{\partial\mu}+\beta\frac{\partial}{\partial g}+n\gamma\right)G_r^{(n)}(x_1,\ldots,x_n;\mu,g)
\end{aligned}$$

In going from the first to the second line we did the derivative with respect to Z and multiplied everything by $Z^{-n/2}$. Going from the second to third line, which is known as the Callan-Symanzik equation, we multiplied by $\mu/\delta\mu$ and defined

$$\beta = \mu\frac{\partial g}{\partial\mu} = \frac{\partial g}{\partial\log\mu}, \quad \gamma = \frac{\mu}{2Z}\frac{\partial Z}{\partial\mu} = \frac{\partial\log\sqrt{Z}}{\partial\log\mu} \tag{1.54}$$

Allowing also for mass terms by rewriting the dimensional coupling m^2 in terms of a dimensionless coupling $g_m = m^2/\mu^2$

$$0 = \left(\mu\frac{\partial}{\partial\mu}+\beta\frac{\partial}{\partial g}+\beta_m\frac{\partial}{\partial g_m}+n\gamma\right)G_r^{(n)}(x_1,\ldots,x_n;\mu,g,g_m), \quad \text{with} \quad \beta_m = \mu\frac{\partial g_m}{\partial\mu} = \frac{\partial g_m}{\partial\log\mu}$$

The general Callan-Symanzik equation for multiple interactions with dimensionless couplings $\{g_i\}$ and fields $\{\Phi_f\}$ of which some contain mass terms, is

$$\left[\mu\frac{\partial}{\partial\mu}+\sum_i\beta_i\frac{\partial}{\partial g_i}+\sum_j(\beta_m)_j\frac{\partial}{\partial(g_m)_j}+\sum_f n_f\gamma_f\right]G_r^{(\{n_f\})}\left(x_1,\ldots x_n;\mu,\{g_i\},\{g_{mi}\}\right)=0 \tag{1.55}$$

The function $\beta(g)$ is the beta function, which gives the behavior of the coupling constant as a function of μ ('the running of the coupling'), and γ_x is the scaling of the field Φ_x and named the anomalous dimension. The general solution to the Callan-Symanzik equation in the presence of mass parameters will involve not only a running coupling and a running field-strength normalization, but also a running mass parameter.

From now on, we are only interested in the beta functions, since they determine the strength of the interaction and the conditions under which perturbation theory is still valid. They provide insights in the energy dependence of cross sections, hints to phase transitions and can provide evidence to the energy range in which a particular theory is valid. Consider, for example, what we can learn from the three possible signs of the beta function for small g.

1. $\beta(g)>0$: the g goes to zero in the infrared, leading to definite predictions about the small-momentum behavior of the theory. However, the coupling $g(\mu)$ increases with an increase

of the mass scale. Thus the short-distance behavior of the theory cannot be computed using Feynman diagram perturbation theory. Theories with this particular property, everywhere or only in a particular region, are called infrared free. The theory of QED is an example of such a theory.

2. $\beta(g) = 0$: by the definition of the beta function, the coupling constant is independent of the energy scale μ. These so-called finite quantum field theories are scale invariant.

3. $\beta(g) < 0$: the coupling decreases with an increase of μ, so the theory is strongly coupled at small energies and weakly coupled at large energies. This means that we can only use perturbation theory at large energies. Theories with this property, like QCD, are called asymptotic free.

The zeros of the beta function, so called fixed points, dictate the ultraviolet and infrared properties of the theory. Both QCD and QED have a trivial, also called Gaussian, fixed point at $g = 0$, but their behavior in that point differs greatly. The difference lies in the energy regime at which the coupling constant approaches its critical value. Non-trivial fixed points are those for which $\beta(g^*) = 0$ with $g^* \neq 0$ and they come in two flavors: an ultraviolet fixed point (UVFP) when for $g < g^*$ the beta function is positive and for $g > g^*$ $\beta(g) < 0$, and an infrared fixed point (IRFP) when the opposite is true. Theories with the latter are scale invariant at large distances, since the coupling constant renormalization flow stops in the infrared. This scale invariance is part of a larger conformal symmetry which we will encounter in the next chapter.

1.3.4 Standard Model beta functions

Now we turn our attention back to the Standard Model. Specifically, we are interested in the one-loop beta function of the different gauge coupling constants, though we will mention the current precision as well.

Computing the beta functions is rather involved. For later convenience we write $g_i = \sqrt{4\pi\alpha_i}$, such that the beta function is given by:

$$\mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2} \frac{\alpha_i}{\pi} = \beta_i(\{\alpha_j\}, \epsilon) \tag{1.56}$$

where the dependence on the regulator ϵ explicitly indicates that we work with Dimensional Regularization. For gauge couplings, the beta functions are scalar functions, while for the Yukawa and scalar couplings, the beta functions have the same dimension as the Yukawa-coupling tensor and the scalar-coupling tensor.

First consider the simpler theory of QED. We write the theory in terms of the renormalized parameters. Comparison with the bare lagrangian shows that the QED coupling constant can be written as $\alpha_0 = Z_1^2 Z_2^{-2} Z_3^{-1} \tilde{\mu}^{2\epsilon}\alpha$. The different renormalization factors Z_i have to be determined, which can be done by calculating the relevant Feynman diagrams for the QED vertex (Z_1), the electron propagator (Z_2) and the photon propagator (Z_3). Using the fact α_0 does not depend on the renormalization scale, the beta function of α can be successfully determined via (1.56).

To calculate the gauge coupling beta functions of the Standard Model, we rewrite equation (1.56) using

$$\alpha_i^0 = \mu^{2\epsilon} Z_{\alpha_i} \alpha_i \quad \text{with} \quad Z_{\alpha_i} = \frac{Z_{\text{vrtx}}^2}{\Pi_k Z_k} \tag{1.57}$$

Noting that α_i^0 is independent of the mass scale μ, we find [21]:

$$
\begin{aligned}
0 &= \mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2}\alpha_i^0 = \mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2}\left(\mu^{2\epsilon} Z_{\alpha_i}\alpha_i\right) \\
&= \epsilon + \mu^2 \frac{1}{Z_{\alpha_i}}\frac{\mathrm{d}Z_{\alpha_i}}{\mathrm{d}\mu^2} + \mu^2 \frac{1}{\alpha_i}\frac{\mathrm{d}\alpha_i}{\mathrm{d}\mu^2} \\
&= \alpha_i\epsilon + \mu^2 \frac{\alpha_i}{Z_{\alpha_i}} \sum_{j=1}^{n} \frac{\partial Z_{\alpha_i}}{\partial \alpha_j}\frac{\mathrm{d}\alpha_j}{\mathrm{d}\mu^2} + \mu^2 \frac{\mathrm{d}\alpha_i}{\mathrm{d}\mu^2} \\
&= \alpha_i\epsilon + \frac{\alpha_i}{Z_{\alpha_i}} \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{\partial Z_{\alpha_i}}{\partial \alpha_j}\pi\beta_j + \left(1 + \frac{\alpha_i}{Z_{\alpha_i}}\frac{\partial Z_{\alpha_i}}{\partial \alpha_i}\right)\pi\beta_i \\
\Rightarrow \beta_i &= -\left[\epsilon\frac{\alpha_i}{\pi} + \mu^2 \frac{\alpha_i}{Z_{\alpha_i}} \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{\partial Z_{\alpha_i}}{\partial \alpha_j}\beta_j\right]\left(1 + \frac{\alpha_i}{Z_{\alpha_i}}\frac{\partial Z_{\alpha_i}}{\partial \alpha_i}\right)^{-1}
\end{aligned}
\tag{1.58}
$$

In the third line we used that Z_{α_i} depends on μ^2 implicitly via α_j. In the fourth line we used the definition of the beta function (1.56). The number of couplings n is equal to 7 in the Standard Model. Namely $\alpha_{1,2,3}$ are the gauge couplings, $\alpha_{4,5,6}$ refer to Yukawa couplings for the three heaviest leptons $x = t, b$ and τ respectively, and α_7 is the Higgs self-coupling λ

$$
\begin{gathered}
\alpha_1 = \frac{5}{3}\frac{\alpha}{\cos^2(\theta_W)} = \frac{5}{3}\frac{g_1^2}{4\pi}, \quad \alpha_2 = \frac{\alpha}{\sin^2(\theta_W)} = \frac{g_2^2}{4\pi}, \quad \alpha_3 = \alpha_s = \frac{g_3^2}{4\pi}, \\
\alpha_x = \frac{\alpha m_x^2}{2\sin^2(\theta_W) M_W^2}, \quad \alpha_7 = \hat{\lambda} = \frac{\lambda}{4\pi} = \frac{1}{4\pi}\frac{M_H^2}{2v^2}
\end{gathered}
\tag{1.59}
$$

where $\alpha = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{\hbar c}$ is the fine structure constant, θ_W the weak mixing angle, α_s the strong coupling constant and m_x and M_W are the fermion and W boson mass, respectively. The factor $\frac{5}{3}$ in α_1 is a normalization constant and arises from embedding the Standard Model in the larger gauge group $SU(5)$.

The first term in the first factor of equation (1.58) vanishes in four space-time dimensions. The second term in the first factor contains the beta functions of the remaining six couplings of the SM. That means that if the coupling renormalization Z_{α_i} depends on α_j we also need β_j. Suppose you need one of the gauge coupling beta functions β_i at loop order ℓ, then you'll need Z_i also at loop order ℓ. All other couplings except α_7 are present from two loops onward, meaning we need to know β_j for $j = 1, \ldots 6, j \neq i$ to loop order $\ell - 2$ and β_7 to order $\ell - 3$. Zeroth order is just the ϵ-dependent term. If we are just interested in the one-loop gauge coupling beta functions, we will see no contributions from the other sectors in the result. However, $\beta_j^{(1)}$ for the Yukawa couplings and scalar self-interaction are affected by all sectors.

Using this equation in combination with equation (1.57) for Z_{α_i} and the renormalization procedure to calculate[17] the field and vertex renormalization constants, the beta functions for the gauge couplings $\alpha_1, \alpha_2, \alpha_3$ can be determined.

The one loop contribution to the gauge β-function for a non-abelian gauge theory was first presented in light of asymptotic freedom by Gross and Wilczek [22], and Politzer [23]. The beta functions for a general theory involving fermions and scalars with Yukawa and quartic scalar interactions like the SM, were presented in [24]. The subsequent two loop corrections for such a general theory were presented in a series of papers by Marie Machacek and Michael Vaughn [25]. The most precise and complete set of gauge beta functions is established by Mihaila et al. [21],

[17]Note that the number of diagrams grows exponentially with each loop order. Nowadays, advanced computer programs are used to calculate these diagrams. The workings of these programs are outside the scope of these thesis.

which correspond to the results of Bednyakov et al. [26] though there is a factor 4 difference due to a different definition of the beta function than (1.56). They have computed the beta functions for the three gauge couplings of the Standard Model in the minimal subtraction scheme to three loops, taking into account contributions from all sectors of the Standard Model. Furthermore, in [27] the four-loop beta-function of the strong coupling in the Standard Model was calculated, though only the top-Yukawa and self-Higgs interactions were taken into account. Higgs [28, 29] and Yukawa couplings [30] taking into account contributions from all sectors of the Standard Model are known to three loop order. Additionally, leading top-Yukawa and QCD contribution to the β_7 function are known to fourth order [31].

As said earlier, we are merely interested in the 1-loop results which we will provide below: It goes beyond the scope of this thesis explain the calculations and results for the higher loop beta functions, so we refer to the given references for more information.

To understand the 1-loop results from Mihaila et al. [21] we need to introduce group theoretical invariants. Recall that the generators T^a of the gauge group can be represented by matrices, e.g. the Pauli matrices and the Gell-Mann matrices in case of the fundamental representation of $SU_L(2)$ and $SU_c(3)$. We use this to define an inner product: $\mathrm{Tr}\big(T^aT^b\big) = S(R)\delta^{ab}$. Here $S(R)$ can be any number and depends on the representation R of the generator. This number is called the Dynkin index. The other invariant we are interested in, is the quadratic Casimir operator $C_2(R) = T^aT^b$, which is a $d(R)\times d(R)$ matrix with $d(R)$ the dimension of the representation. A useful formula for determining the Casimir operator is $T(R)d(A) = C_2(R)d(R)$. Here $d(A)$ is the dimension of the adjoint representation of the group and $T(R)$ is the index of the representation. This gives $(N^2-1)/2N$ for the fundamental representation of $SU(N)$ and N for the adjoint representation of the same group.

The generic 1-loop result for α for a theory with 1 gauge group is:

$$(4\pi)^2\beta_\alpha^{(1)} = 4\big(-\tfrac{11}{3}C_2(G) + \tfrac{4}{3}\kappa S_2(F) + \tfrac{1}{6}S_2(S)\big)\alpha^2 \tag{1.60}$$

where S, F, G are the representations of the scalar, fermion and gauge field, respectively. The constant $\kappa = 1, \frac{1}{2}$ appears in terms originating from fermion loops and takes the first or second value depending on whether the fermion representation is Dirac or Weyl, respectively. Similar generic function for the Yukawa and Higgs contribution exist as well, but are omitted for readability considerations.

For the Standard Model the result needs to be extended to the three gauge groups $SU_c(3)\times SU_L(2)\times U_Y(1)$. Using Table 1.2, the Standard Model results for the 1-loop gauge coupling beta functions $\beta_i^{(1)}$ in terms of α_i follow:

$$(4\pi)^2\beta_1^{(1)} = \frac{82}{5}\alpha_1^2, \quad (4\pi)^2\beta_2^{(1)} = -\frac{38}{3}\alpha_2^2, \quad (4\pi)^2\beta_3^{(1)} = -28\alpha_3^2 \tag{1.61}$$

The three Yukawa coupling beta functions at 1-loop are:

$$(4\pi)^2\beta_4^{(1)} = \Big(6\alpha_5 + 18\alpha_4 + 4\alpha_6 - \tfrac{17}{5}\alpha_1 - 9\alpha_2 - 32\alpha_3\Big)\alpha_4 \tag{1.62}$$

$$(4\pi)^2\beta_5^{(1)} = \Big(18\alpha_5 + 6\alpha_4 + 4\alpha_6 - \alpha_1 - 9\alpha_2 - 32\alpha_3\Big)\alpha_5 \tag{1.63}$$

$$(4\pi)^2\beta_6^{(1)} = \Big(10\alpha_6 + 12\alpha_5 + 12\alpha_4 - 9\alpha_1 - 9\alpha_2\Big)\alpha_6 \tag{1.64}$$

Lastly, the beta function of the self-interaction of the Higgs field at 1-loop order [28]:

$$\begin{aligned}(4\pi)^2\beta_7^{(1)} = {} & 48\alpha_7^2 - \tfrac{36}{10}\alpha_1\alpha_7 - 18\alpha_2\alpha_7 + \tfrac{27}{100}\alpha_1^2 + \tfrac{9}{10}\alpha_1\alpha_2 + \tfrac{9}{4}\alpha_2^2 \\ & - 12\alpha_4^2 - 12\alpha_5^2 - 4\alpha_6^2 + 24\alpha_4\alpha_7 + 24\alpha_5\alpha_7 + 8\alpha_6\alpha_7\end{aligned} \tag{1.65}$$

where we accounted for the factor 4 difference between our convention and those of the source.

	$C_2(G)$	$C_2(S)$	$C_2(F)$	$S_2(S)$	$S_2(F)$
$SU(3)$	3	0	$\frac{4}{3}R^3$	0	6
$SU(2)$	2	$\frac{3}{4}\mathbb{1}_4$	$\frac{3}{4}R^2$	1	6
$U(1)$	0	$\frac{1}{4}\mathbb{1}_4$	$(R^1)^2$	1	10

Table 1.2 – *Dynkin indices and Casimir operators for the Standard Model.*
For $SU(3)$ we note that S_2 measures the amount of color: scalars have no color and of the fermions only the 6 quarks have color. Fermions are in the fundamental representation and the gauge fields in the adjoint representation, from this we can determine C_2. R^3 is a block diagonal matrix with ones and zeros depending on the Gell-Mann matrices. For $SU(2)$ S_2 counts the number of left-handed components, i.e. 3 lepton doublets and 3 quark doublets. The Casimir depends on the pauli matrices and R^2 is diagonal matrix with ones and zero depending on the fermions considered. In $U(1)$ the total hypercharge is measured by S_2. C_2 can be determined by noting that a scalar has hypercharge $+\frac{1}{2}$. Furthermore R^1 is a diagonal matrix with the hypercharges of the respective spinors on the diagonal. Based on Appendix D from [21].

1.4 Beyond the Standard Model

The Standard Model including including neutrino masses and mixing angles (also known as the minimal extended Standard Model) depends on 25 free parameters. Most of their numerical values have been established by experiment (see Appendix A). The ESM is able to calculate any experimental observable in terms of its input parameter set and has done so successfully. The Standard Model precisely predicted a wide variety of phenomena, e.g. the existence of the Higgs boson and the Z and W masses. However, the SM is not able to calculate these 25 parameters and in this sense the SM is not a predictive theory, nor expected to be the final theory. And there are more reasons for expecting physics beyond the Standard Model. In no particular order a few examples are [32]:

i. The Hierarchy problem: the free parameters of the Standard Model need to be extremely fine-tuned in order to get the observed Higgs mass.

ii. Why do the fermion masses and mixing exhibit structure when in the SM they are just free parameters?

iii. Why is the number of fermion families equal to 3?

iv. Why is the number of spacetime dimensions 3+1?

v. What is the nature of neutrinos?

vi. The strong CP problem: there is another invariant term which could be added to the SM Lagrangian:

$$\mathcal{L}_\theta = \theta \frac{g_s^2}{64\pi^2} \epsilon^{\mu\nu\rho\sigma} F^a_{\mu\nu} F^a_{\rho\sigma}$$

where $\epsilon^{\mu\nu\rho\sigma}$ a totally antisymmetric tensor. However, the parameter θ must be exceedingly small in order not to give rise to strong interaction contributions to CP violating quantities such as the electric dipole moment of the neutron. The current experimental limits on this dipole moment tell us that $\theta < 10^{-10}$. Why this term is absent (or so small)?

vii. How does color confinement emerge from QCD?

viii. What gave rise to the matter-antimatter asymmetry?

ix. What is the nature of dark matter and dark energy?

x. Can the four forces of nature be unified in one Theory of Everything and if yes, what are its properties? Should Gravity be quantized?

Chapter 2

Conformal Gravity

Our current understanding of the physics of our Universe is based on the quantum field theory (QFT) description of the Standard Model (SM) and Einstein's theory of gravity, General Relativity (GR). As such, GR is relevant in regions of both large-scale and high-mass whereas the SM is the theoretical framework in which the other three forces of nature are explained, applicable in regions of both small scale and low mass. Moreover, Einstein drastically changed the concepts of time and space in GR when he promoted the metric of space-time to a dynamical object. QFT, however, takes for granted some fixed background spacetime determining the causal structure, as one of its very foundations. It furthermore requires that any dynamical field is quantized and is governed by probabilistic laws. Combining the two theories irrevocably leads to fundamental conceptual difficulties regarding the nature of time and spacetime.

Nonetheless, GR and QFT should be unified in a theory of Quantum Gravity[18] not only to be able to unify gravity with the Standard Model, but also to explain for example black holes and address the aforementioned problem of (space-)time.

With the lack of any direct experimental hints, most focus on constructing a mathematically and conceptually consistent (and appealing) Quantum Gravity framework. We distinguish 'primary' and 'secondary' theories of quantum gravity. In the former one starts with a given classical theory and applies heuristic quantization rules. A further division in canonical and covariant approaches can be made: the first uses a Hamiltonian formalism whereas the latter employs four-dimensional covariance at some stage. The main advantages of primary theories is that the starting point is known unlike for the secondary theories. There one starts with a fundamental quantum framework of all interactions and tries to derive (quantum) GR in certain limiting situations, for example, through an energy expansion [33].

Here we will use the covariant approach and thus start this chapter by looking at Einstein gravity as a (classical) field theory, following [14, 16] in the conventions of this thesis (see 'Notations and conventions', page xi). Despite the fact that GR can be made an effective field theory, it is not suited to unify with the Standard model interactions. In Section 2.2 we argue why we want to impose conformal invariance on our theory. In Section 2.3 we develop a conformal field theory of gravity, called Conformal Gravity and show that it can indeed describe the results from GR. In the last section Section 2.4 we will construct a Conformal Standard Model and use it to find a conformal toy model.

[18] Strictly speaking, the aim of Quantum Gravity is only to describe the quantum behavior of the gravitational field. However, some quantum gravity theories also try to unify gravity with the other fundamental forces. We refer to such a theory as a 'Theory of Everything' and its existence is the most prominent unresolved question in modern day physics.

2.1 Einstein Gravity

In Einstein's theory of General Relativity, spacetime is represented as a connected real, 4-dimensional linear differentiable manifold L (a collection of smoothly connected points). This manifold can only be determined in connection with a solution to the field equation and has by itself no physical meaning; it gets meaning only through fields defined on it. The field equations follow from the principle of least action using an action that respects background independence[19].

$$S = S_g + S_{\text{mat}} = \int \mathrm{d}^d x \sqrt{g}(\mathcal{L}_g + \mathcal{L}_{\text{mat}}) \tag{2.1}$$

where $g = |\det(g_{\mu\nu})|$ and $\sqrt{g}$ ensures that $\sqrt{g}\mathcal{L}$ transforms as a scalar density under coordinate transformations. Note that $g_{\mu\nu}$ is a nondegenerate metric with signature[20] (1,3) and is said to be dynamical. This Lorentizian metric guarantees that Special Relativity (SR) with its non-gravitational laws remains approximately valid locally even if gravitational fields are taken into account.

The gravitational Lagrangian $\mathcal{L}_g$ needs to be a scalar, but also needs to depend on the derivatives of the metric[21] to get dynamics in the vacuum (i.e. the Einstein equations). The notion of differentiation on a smooth manifold requires the introduction of the affine connection $\nabla(\Gamma)$, which can be identified as a covariant derivative. Consider a vector field $X^\alpha(x)$ at a point x^α on a manifold. The curvature of a manifold will cause a distortion: the parallel transported vector field to $x^\alpha + \delta x^\alpha$ is different from $X^\alpha(x + \delta x)$. This is depicted in figure 2.1.

The change of the coordinate field due to the manifold is defined as

$$X^\alpha(x + \delta x) = X^\alpha(x) + \delta x^\beta \partial_\beta X^\alpha = X^\alpha(x) + \delta X^\alpha(x)$$

On the other hand, the parallel transported vector field is $X^\alpha(x) + \tilde{\delta} X^\alpha(x)$, where the second term must vanish if either δx^α vanishes or X^α vanishes. Therefore, we write

$$\tilde{\delta} X^\alpha(x) = -\Gamma^\alpha_{\beta\gamma}(x) X^\beta(x) \delta x^\gamma$$

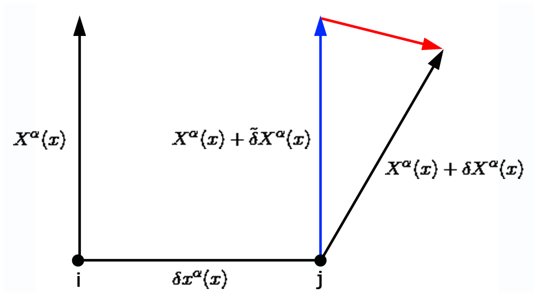

Figure 2.1 – *Two spacetime points x^α and $x^\alpha + \delta x^\alpha$ are labeled as i and j, respectively. The vector field at i is $X^\alpha(x)$, and at j it is $X^\alpha(x+\delta x) = X^\alpha(x)+\delta X^\alpha(x)$. However, the parallel transported vector (the blue vector) at j is $X^\alpha(x) + \tilde{\delta} X^\alpha(x)$. The difference between the two vectors at j is colored red. (The figure is based on figure 1 from Yepez [16]).*

[19] Other terms used are 'general coordinate invariance', 'general covariance' or 'diffeomorphism invariance'. There have been extensive discussions on the precise definition especially as myriad terminology exist to describe these concepts, see e.g. section 4 of Pooley [34] for the details. Superficially this just means that one can use whatever coordinates or frames of reference one likes to formulate the laws that govern the joint evolution of matter and space-time. However, Erich Kretschmann (1917) already noted that any physical law can be rewritten in an equivalent but generally covariant form. Hence general covariance alone cannot rule out any physical law. Therefore we distinguish on the one side passive diffeomorphism invariance (i.e. general covariance or coordinate invariance) and on the other side active diffeomorphism invariance (i.e. background independence). In this thesis we assume the latter, which we understand as the fact that the laws of GR, in contrast to those of e.g. Newtonian physics and SR, do not presuppose the existence of an absolute spacetime structure which is specified categorically prior to dynamical laws and not influenced by physical processes.

[20] The signature (p, q, r) of a metric tensor g is the number of positive, negative and zero eigenvalues of the real symmetric matrix $g_{\mu\nu}$ of the metric tensor with respect to a basis. In this thesis the signature is either denoted by a pair of integers (p, q) (because we work in spaces where $r = 0$) or as an explicit list of signs of eigenvalues such as $(+, -, -, -)$ or $(-, +, +, +)$ for the signature (1,3) respectively (3,1).

[21] From the link between GR and Newtonian gravity, it follows that the metric has the meaning of a gravitational potential, and that derivatives of the metric take on the role of gravitational forces.

The covariant derivative can then be written as

$$\nabla_\gamma(\Gamma)X^\alpha = \lim_{\delta x^\gamma \to 0} \tfrac{X^\alpha(x+\delta x)-(X^\alpha(x)+\tilde{\delta}X^\alpha(x))}{\delta x^\gamma}$$

where the numerator is the difference between the two vectors at j (depicted red in the figure). Using the definitions, the above formula gives after dropping the explicit dependence on x

$$\nabla_\gamma(\Gamma)X^\alpha = \partial_\gamma X^\alpha + \Gamma^\alpha_{\beta\gamma}X^\beta \tag{2.2}$$

We call $\Gamma^\alpha_{\beta\gamma}$ the affine connection and it transforms under a coordinate transformation as

$$\Gamma^{\mu}_{\nu\rho}{}'(x') = \frac{\partial x^\beta}{\partial x^{\nu'}}\frac{\partial x^\gamma}{\partial x^{\rho'}}\Gamma^\alpha_{\beta\gamma}\frac{\partial x^{\mu'}}{\partial x^\alpha} + \frac{\partial x^{\mu'}}{\partial x^\sigma}\frac{\partial^2 x^\sigma}{\partial x^{\nu'}\partial x^{\rho'}} \tag{2.3}$$

to ensure that the covariant derivative of a tensor is again a tensor, i.e. it ensures the coordinate independence.

The covariant derivative with respect to a general affine connection $\nabla_\gamma(\Gamma)$ of a tensor is thus given by its partial derivative and correction terms, one for each index, involving the connection coefficients contracted with the tensor. Explicitly, the covariant derivative of a generic tensor $T^{\mu_1,\ldots,\mu_m}_{\nu_1,\ldots\nu_n}$

$$\begin{aligned}\nabla_\gamma(\Gamma)T^{\mu_1,\ldots,\mu_m}_{\nu_1,\ldots\nu_n} &= \partial_\gamma T^{\mu_1,\ldots,\mu_m}_{\nu_1,\ldots\nu_n} \\ &+ \Gamma^{\mu_1}_{\alpha\gamma}T^{\alpha,\mu_2,\ldots,\mu_m}_{\nu_1,\ldots\nu_n} + \cdots + \Gamma^{\mu_m}_{\alpha\gamma}T^{\mu_1,\ldots,\mu_{m-1},\alpha}_{\nu_1,\ldots\nu_n} \\ &- \Gamma^\beta_{\gamma\nu_1}T^{\mu_1,\ldots,\mu_m}_{\beta,\nu_2,\ldots\nu_n} - \cdots - \Gamma^\beta_{\gamma\nu_n}T^{\mu_1,\ldots,\mu_m}_{\nu_1,\ldots\nu_{n-1},\beta}\end{aligned} \tag{2.4}$$

From this definition one immediately sees that when tensor has no indices, i.e. if it is a scalar, the covariant derivative simplifies to an ordinary one. At this point, it also becomes clear that the definition of $\Gamma^\mu_{\nu\rho}$ contains some ambiguity, because (2.4) remains a tensor if one adds to $\Gamma^\mu_{\nu\rho}$ any tensor $\tilde{\Gamma}^\mu_{\nu\rho}$.

From the affine connection, two tensors can be built. Firstly, there is the torsion tensor which is related to the fact that infinitesimal parallelograms on the manifold do not close in general, the closure failure being proportional to the torsion tensor:

$$T^\lambda{}_{\mu\nu}(\Gamma) = \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu} = 2\Gamma^\lambda_{[\mu\nu]} \tag{2.5}$$

The other tensor built form the connection is the (Riemann-Cartan) curvature tensor

$$R^\mu_{\nu\rho\sigma}(\Gamma) = \partial_\rho\Gamma^\mu_{\nu\sigma} - \partial_\sigma\Gamma^\mu_{\nu\rho} + \Gamma^\lambda_{\nu\sigma}\Gamma^\mu_{\lambda\rho} - \Gamma^\lambda_{\nu\rho}\Gamma^\mu_{\lambda\sigma} \tag{2.6}$$

The name 'curvature tensor' stems from the fact that a given spacetime will be flat if and only if every component of the curvature tensor is zero. In that case, the geodesic equation describes Newton's second law of motion for a free particle in the absence of gravity in an arbitrary accelerating coordinate frame. If the curvature tensor is nonzero, the geodesic equation in combination with the Schwarzschild metric will give Newtonian gravity.

The curvature and torsion tensors are related as can be seen by considering the non-commutativity of the covariant derivatives. For a scalar field ϕ and vector field X^α we have

$$\begin{aligned}[\nabla_\mu, \nabla_\nu]\phi &= -T^\sigma_{\mu\nu}(\Gamma)\partial_\sigma\phi \\ [\nabla_\mu, \nabla_\nu]X^\alpha &= R^\alpha_{\beta\mu\nu}(\Gamma)X^\beta - T^\sigma_{\mu\nu}(\Gamma)\nabla_\sigma(\Gamma)X^\alpha\end{aligned}$$

Note that spacetime strictly has neither curvature nor torsion as a manifold can have different connections and both torsion and curvature are properties of the connection.

As stated at the start of this section, for General Relativity we assume we work on a manifold which is also a metric space, i.e. it is equipped with a non-degenerate metric tensor $g_{\mu\nu}$. These manifolds are known as Riemannian and pseudo-Riemannian manifolds[22]. With the metric we can use the torsion and curvature tensor to define related tensors:

$$
\begin{aligned}
R_{\mu\nu} &= R^{\alpha}_{\mu\alpha\nu} && \text{Ricci tensor} && (2.7)\\
R &= g^{\mu\nu}R_{\mu\nu} && \text{Ricci scalar} && (2.8)\\
Q_{\mu\nu\lambda} &= -\nabla_{\mu}(\Gamma)g_{\nu\lambda} && \text{Nonmetricity tensor} && (2.9)\\
K^{\lambda}{}_{\mu\nu} &= \tfrac{1}{2}(T^{\lambda}{}_{\mu\nu} - T_{\nu}{}^{\lambda}{}_{\mu} - {}_{\mu}{}^{\lambda}{}_{\nu}) && \text{Contortion tensor} && (2.10)
\end{aligned}
$$

Note that the contortion tensor is antisymmetric in the first two indices ($K_{\lambda\mu\nu} = -K_{\mu\lambda\nu}$), while torsion itself is antisymmetric in the last two indices ($T^{\lambda}{}_{\mu\nu} = -T^{\lambda}{}_{\nu\mu}$).

The above definitions allow us to decompose a general affine connection as

$$\Gamma^{\mu}{}_{\nu\lambda} = \{{}^{\mu}_{\nu\lambda}\} + K^{\mu}{}_{\nu\lambda} + (\Gamma^{\mu}{}_{\nu\lambda})_S \tag{2.11}$$

where $(\Gamma^{\mu}{}_{\nu\lambda})_S$ is the segmental connection

$$(\Gamma^{\mu}{}_{\nu\lambda})_S = \tfrac{1}{2}g^{\mu\sigma}\left(Q_{\lambda\nu\sigma} + Q_{\nu\lambda\sigma} - Q_{\sigma\lambda\nu}\right)$$

and $\{{}^{\mu}_{\nu\lambda}\}$ are the Christoffel symbols

$$\{{}^{\mu}_{\nu\lambda}\} = \tfrac{1}{2}g^{\mu\rho}\left(\partial_{\nu}g_{\lambda\rho} + \partial_{\lambda}g_{\nu\rho} - \partial_{\rho}g_{\lambda\nu}\right) \tag{2.12}$$

The splitting of a connection according to (2.11) allows for the distinction of various spaces, each with its own geometry, see item 2.1. One particular geometry is the Weyl geometry, which we will encounter later. For Einstein Gravity, we use the fact that we can define a unique connection as given by the 'Fundamental Theorem of Riemannian Geometry'. This theorem states that for a Riemannian or pseudo-Riemannian manifold there is a unique connection $\Gamma^{\mu}{}_{\nu\lambda}$ which satisfies the following conditions:

i. It preserves the metric, i.e. $Q_{\mu\nu\lambda} = 0$. This condition is sometimes referred to as the metric compatibility condition and ensures that lengths and angles are preserved under parallel transport.

ii. The torsion tensor T vanishes.

These two properties uniquely define the so-called the Levi-Civita connection given by the Christoffel symbols $\Gamma^{\mu}{}_{\nu\lambda} = \{{}^{\mu}_{\nu\lambda}\}$. If we only had the first condition, the resulting space would be a Riemann-Cartan space where the metric and the connection are still independent objects, unlike the connection in Riemann geometry, which can be calculated completely from the derivatives of the metric (the geometry is completely described by the metric). Due to the symmetry properties of the Christoffel symbols, the Riemann curvature tensor with respect to the Levi-Civita connection $R^{\mu}_{\nu\rho\sigma}(\{\})$ then obeys the following relations

$$R_{\mu\alpha\nu\beta} = R_{\nu\beta\mu\alpha} \tag{2.13}$$

$$R_{\mu\alpha\nu\beta} = -R_{\alpha\mu\nu\beta} = -R_{\mu\alpha\beta\nu} \tag{2.14}$$

$$R_{\mu\alpha\nu\beta} + R_{\mu\beta\alpha\nu} + R_{\mu\nu\beta\alpha} = \tfrac{1}{6}R_{\mu(\alpha\nu\beta)} = 0 \tag{2.15}$$

$$\nabla_{\beta}R^{\lambda}{}_{\mu\alpha\nu} + \nabla_{\nu}R^{\lambda}{}_{\mu\beta\alpha} + \nabla_{\alpha}R^{\lambda}{}_{\mu\nu\beta} = \tfrac{1}{6}R^{\lambda}{}_{\mu(\alpha\nu;\beta)} = 0 \tag{2.16}$$

where equation (2.15) and (2.16) are known as the first and second Bianchi identity, respectively.

[22] A Riemannian manifold is a smooth manifold M equipped with a positive-degenerate metric tensor $g_{\mu\nu}$. A pseudo-Riemannian manifold has a metric tensor which is only required to be nondegenerate rather than the stronger positive-definite requirement of a Riemannian manifold.

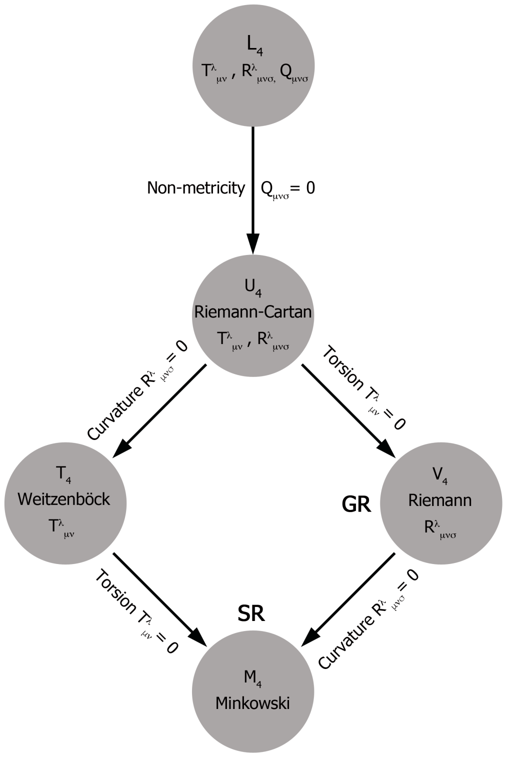

Figure 2.2 – *In order to preserve lengths and angles under parallel transport one imposes the metric compatibility condition (nonmetricity equals zero) on the linear differentiable manifold L_4. The space with the most general metric-compatible linear connection is called a Riemann-Cartan space U_4. If the torsion (and thus also the contortion) vanishes we have a Riemann space V_4, and if, alternatively, the curvature vanishes, we have the teleparallel space T_4 of Weitzenböck. If both curvature and torsion vanish identically, we arrive at Minkowski space M_4, the stage for Special Relativity. (The figure is based on figure 7.2 from Sundermeyer [14]).*

This means the Ricci tensor is symmetric. The Ricci scalar, which is the object of our interest in constructing the gravity lagrangian, becomes

$$R = g^{\mu\nu} R_{\mu\nu} = g^{\mu\nu}(\{{}^{\sigma}_{\mu\nu}\}_{,\sigma} - \{{}^{\sigma}_{\mu\sigma}\}_{,\nu} + \{{}^{\rho}_{\mu\nu}\}\{{}^{\sigma}_{\sigma\rho}\} - \{{}^{\rho}_{\mu\sigma}\}\{{}^{\sigma}_{\nu\rho}\})$$

where the covariant derivative with respect to the Levi-Civita connection is traditionally abbreviated by a semicolon $\nabla_\mu(\Gamma)T = T_{;\mu}$ and a partial derivative is abbreviated by a comma $\partial_\mu T = T_{,\mu}$.

We can now construct a general coordinate invariant action known as the Einstein-Hilbert action $S_g = S_{\mathrm{EH}}$:

$$S_{\mathrm{total}} = S_{\mathrm{EH}} + S_{\mathrm{mat}} = \frac{1}{2\kappa^2}\int \mathrm{d}^d x\sqrt{g}\,(R - 2\Lambda) + S_{\mathrm{mat}}, \quad \text{where} \quad g = |\det g_{\mu\nu}| \tag{2.17}$$

The Einstein-Hilbert action is the simplest form[23] that adheres the coordinate invariance: the square root of the determinant is the simplest possible volume element and R is the simplest possible scalar that can be formed from the covariant derivatives of the metric. The second term in equation (2.17) is related to the cosmological constant. It should in principle be included, but cosmology bounds give $|\Lambda| < 10^{-56}\mathrm{cm}^{-2}$ meaning that this constant is unimportant at ordinary energies. Furthermore, κ has been chosen such that the non-relativistic limit yields the usual form of Newton's gravity law.

$$2\kappa^2 = M_p^{-2} = 16\pi G_N \quad \text{with} \quad G_N = 6.674 \times 10^{-11}\mathrm{Nm}^2\mathrm{kg}^{-2} \tag{2.18}$$

M_p is the 4-dimensional Planck mass. From the Einstein-Hilbert action Einstein's gravitational field equations can de derived via the principle of least action (see Appendix B for the full calculation):

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi G_N T_{\mu\nu} \tag{2.19}$$

[23]Here, we justify the choice of the Einstein-Hilbert action on grounds of simplicity, but we do note that simplicity itself is not a law of nature.

where $T_{\mu\nu}$ is Hilbert's energy-momentum tensor corresponding to the matter Lagrangian $\mathcal{L}_{\text{mat}}$

$$T^{\mu\nu} \equiv -\frac{2}{\sqrt{g}}\frac{\delta}{\delta g_{\mu\nu}}\left(\sqrt{g}\mathcal{L}_{\text{mat}}\right) = -2\frac{\delta \mathcal{L}_{\text{mat}}}{\delta g_{\mu\nu}} + g_{\mu\nu}\mathcal{L}_{\text{mat}} \tag{2.20}$$

which is a symmetric, conserved quantity containing the energy density (T_{00}), the energy flux in the i-direction (T_{0i}), the 3-momentum density (T_{i0}), and the 3-momentum flux (T_{ij}).

Einstein's equations work well as a classical theory with more and more of its predictions being verified by experiments (including gravitational waves). The problems arise at the next step, namely the quantization of the theory (as a quantum field theory of the spin-two graviton). We refrain from entering in too much detail at this point. Simply stated, it is known that some quantization of gravity is inevitable because part of the metric is determined by the energy-momentum tensor which depends upon the other particle fields whose quantum nature has been well established. Divergences arise due to these other fields. The quadratic and quartic divergences can be absorbed into renormalizations of Newton's constant and the cosmological constant, but there is no parameter in general relativity in which to absorb the logarithmic divergence. One can only absorb it if new, fourth derivative terms of the metric are added to the gravitational field equations [35]. Kellogg Stelle [36] showed that if the equations of motion be changed to include terms with up to four derivatives (the theory is then different from GR), the resulting quantum theory is perturbatively renormalizable. However, the theory also becomes unstable as cross sections blow up thus violating the unitarity of the theory. This is of course inconsistent with the observed reality of a universe which is 13.8 billion years old.

In conclusion, the quantization of Einstein gravity gives a theory that is unitary, but nonrenormalizable[24]. On the other hand, higher order derivative theories turn out to be renormalizable at the one-loop quantum level, but at the price of losing the unitarity of the S-matrix. As of yet, it is not known how to build a higher-derivative theory that is renormalizable and unitary at the same time.

Nonetheless, within the framework of Effective Field Theory, quantization of nonrenormalizable theories can make perfect sense provided they are applied to low energy (i.e. well below some ultraviolet (UV) cutoff of the EFT) predictions. Thanks to the work of John F. Donoghue [e.g. 38, and references therein] it appears that the Einstein-Hilbert Lagrangian is just the least suppressed term in the Lagrangian of an effective field theory containing every possible generally covariant function of the metric and its derivatives.

$$S = \int \mathrm{d}^4x\sqrt{g}\left(\lambda + \tfrac{1}{2\kappa^2}R + c_1R^2 + c_2C^2 + c_3E + c_4\Box R + \cdots + \mathcal{L}_{\text{mat}}\right) \tag{2.21}$$

where $C^2 = R^2_{\mu\nu\rho\sigma} - 2R^2_{\mu\nu} + \frac{1}{3}R^2$ is the square of the Weyl tensor which is conformal invariant in 4 dimensions (see Section 2.3), and E is the so called Euler term:

$$E = \left(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2\right) \tag{2.22}$$

which in four space-time dimensions, also known as a Lanczos term or Gauss-Bonnet topological invariant, vanishes for space-times topologically equivalent to flat space [39].

As with the Standard Model this requires the introduction of gauge-fixing terms as well as Faddeev-Popov ghost fields. The background field method, which relies on the expansion of the metric about a smooth background field $\bar{g}_{\mu\nu}(x)$ as $g_{\mu\nu}(x) = \bar{g}_{\mu\nu}(x) + \kappa h_{\mu\nu}$ offers a small relief as it ensures that no gauge-fixing terms are needed for terms of order $\mathcal{O}(\partial^2)$. This is because

[24] 't Hooft and Veltman [37] calculated that in a theory of gravity alone the counterterms can be absorbed by a renormalization of the metric tensor $g_{\mu\nu}$. They showed that this is no longer the case and a real infinity remains for the simplest case of one massless Klein-Gordon field ϕ interacting with gravity. If other fields or other interactions are added in $\mathcal{L}$, the number of possible terms in the counterterm Lagrangian increases rapidly and only miraculous cancellations could restore renormalizability.

the total set of terms linear on $h_{\mu\nu}$ (including those from the matter Lagrangian) will vanish if $\bar{g}_{\mu\nu}$ satisfies Einstein's equations. The remaining Lagrangian is of higher order and still requires gauge fixing and the introduction of the associated Faddeev-Popov ghost fields.

2.2 Using conformal symmetry

So we can make a quantum field theory out of General Relativity, but only an effective field theory. Furthermore, it uses the Einstein equation (2.19) as given while any extrapolation from its weak classical gravity solar system origin runs into trouble (e.g. the dark matter problem, dark energy problem, singularity problem, etc.)[12]. To describe the physics at the Planck scale an alternative theory has to be developed, preferably one that 1) does not encounter these problems and 2) would allow unification with what we already know of the strong, weak and electromagnetic interactions.

Despite intense efforts over the last years it is far from clear at this time what a consistent theory of Quantum Gravity will look like and what its main features will be. The most straightforward way to alter the physics of a field theory is to change the action principle for that theory. Specifically, to extend the theory of General Relativity by changing the symmetry of the action because using symmetry to relate and unify physical theories has been very successful in the past, the Standard Model being one of those successes.

The use of conformal invariance is one of the possibilities and numerous reasons support its use. First of all, the classical SM Lagrangian is devoid of any intrinsic mass or length scales, if the Higgs mass term is dropped. As it is associated with the energy-momentum tensor, it is thus quite natural that gravity should be devoid of any intrinsic mass or length scales too. Furthermore, we know from the ultrarelativistic limit of special relativity that rest masses of particles have negligible effects. This is another argument why one might expect a high-energy theory of physics to lack any explicit mass scales. A conformally invariant theory would fit this expectation beautifully.

Besides, conformal invariance is highly restrictive, more so than simple scale invariance: the energy-momentum tensor of the theory should be traceless. Lacking experimental constraints in the quantum gravity domain, these conformal constraints would be very welcome in our search for a Theory of Everything. Furthermore, it allows us to address several issues in SM, most notably it can naturally accommodate a see-saw mechanism to give mass to neutrino's and addresses the hierarchy problem due to missing tachyonic mass term.

Last but not least, conformal field theories are fundamentally linked to quantum field theories. Firstly because they are a very usefull tool in describing a system close to its critical point, i.e. the the point at the end of a phase equilibrium curve where a continuous phase transition occurs. It turns out there are quantum critical points at T = 0 where transitions are driven by quantum fluctuations. These phase transitions also exhibit infinite correlation lengths and thus are also describable via CFTs. Secondly, we can think of any quantum field theory as a perturbation of a conformal field theory by relevant operators (which push the theory away from the fixed point). In other words, any point in our parameter space can be considered as a renormalization group flow perturbed away from some fixed point CFT.

Having now clear why we are interested in a conformal invariant theory, we continue this section by explaining the terms 'scale invariance', 'Weyl invariance' and 'conformal invariance' and their relation to each other as they are often used interchangeably in the literature. Following Chapter 4 of Di Francesco et al. [40], unless otherwise specified, we proceed by discussing the conformal group. This allows us to show that conformal invariance is indeed highly restrictive. In the next section we will use conformal invariance to develop a theory of Conformal Gravity.

2.2.1 Scale, Weyl and conformal invariance

We define a general coordinate transformation as

$$x \to x', \quad g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x^{\mu\prime}} \frac{\partial x^\beta}{\partial x^{\nu\prime}} g_{\alpha\beta}(x) \tag{2.23}$$

Following Forger and Romer [41], a rigid *scale transformation* (also known as a dilation or dilatation) in flat spacetime is then

$$x^a \to \tilde{\Omega} x^a, \quad \Phi(x) \to \tilde{\Omega}^{-\Delta_\Phi} \Phi(\lambda^{-1} x) \tag{2.24}$$

where λ is a real, arbitrary constant and Δ_Φ the scaling dimension of the field Φ. The flat metric is understood to be invariant under dilatations and so can be viewed as having scaling dimension 0 and only pick up factors of λ due to the scaling of the coordinates. From (2.24) it can also be deduced that if a field Φ has scaling dimension Δ_Φ, then all its partial derivatives $\partial_\mu \Phi$ will have scaling dimension $\Delta_\Phi + 1$, meaning that the partial derivative has scaling dimension $+1$. To ensure that the covariant derivative has the same dimension, also the gauge fields will need to carry a scaling dimension equal to $+1$. However, considering the scale invariance of the Yang-Mills sector shows that the field strength tensor has scaling dimension $\frac{d}{2}$ and the gauge field thus has dimension $\frac{d-2}{2}$, i.e. scale invariance of the Yang-Mills sector thus occurs only in $d = 4$ spacetime dimensions. The scaling dimension of specific fields can be found by inspection of their kinetic terms and gives a scaling dimension of $\frac{1}{2}(d-2)$ for scalar fields and $\frac{1}{2}(d-1)$ for fermions.

Rigid transformations like translations, Lorentz transformations and dilatations do not have a direct analogue on arbitrary spacetime manifolds. However, the equivalence principle and the principle of general covariance suggest that when considering generally covariant classical field theories on a arbitrary spacetime manifold, rigid spacetime symmetries should be replaced by flexible spacetime symmetries. This can be resolved by switching from the active to the passive point of view: In a general background we should consider scale transformations not as active transformations that move points in spacetime, but rather as passive transformations that change the scale of the metric by which we measure distances between points in spacetime. Scale transformations in this latter sense are called *global Weyl rescalings*. The link between them is that in flat spacetime both should lead to the same rescaling for the distance between points, i.e. the metric and more general a field should transform under a global Weyl rescaling as:

$$g_{\mu\nu} \to \tilde{\Omega}^2 g_{\mu\nu}, \quad \Phi \to \tilde{\Omega}^{-\tilde{\Delta}_\Phi} \Phi \tag{2.25}$$

The Weyl weight $\tilde{\Delta}_\Phi$ differs from the scaling dimension: we see that the metric and its inverse have weight 2 and -2, respectively. Furthermore, the partial derivative and with it the covariant derivative, the gauge fields and the connection, now have weight 0. The Weyl weight of scalar and fermion fields is again found by inspection of the kinetic terms, which means that the Weyl weight in those cases coincides with the scaling dimension.

As the list of theories that are scale-invariant coincides exactly with the list of global Weyl invariant theories, we can indeed say that the generalization of scale invariance in flat spacetime is the invariance under global Weyl rescaling in general spacetime (though this is not a coordinate transformation but a simultaneous pointwise transformations of both the metric and the fields) [41, 42].

Next we need to interpret what we mean by a *local Weyl rescalings*, which is a rescaling by an arbitrary function on spacetime:

$$g_{\mu\nu} \to \tilde{\Omega}(x)^2 g_{\mu\nu}, \quad \Phi \to \tilde{\Omega}(x)^{-\tilde{\Delta}_\Phi} \Phi \tag{2.26}$$

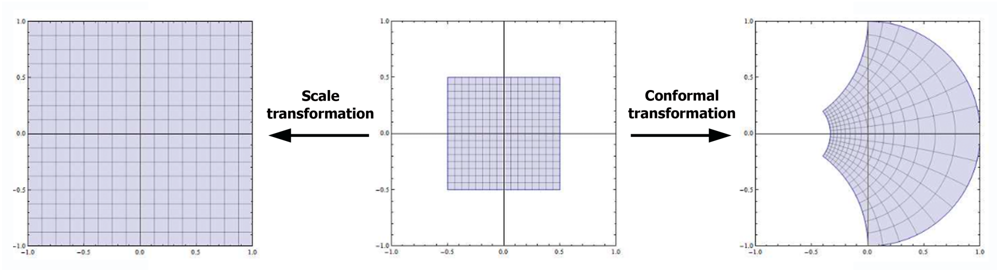

Figure 2.3 – *Difference between a scale and a conformal transformation (based on figure 1 from Nakayama [44]).*

Fields that transform like this are called Weyl covariant. We note that partial derivatives of Weyl covariant fields are no longer Weyl covariant. This is cured by "gauging" the global symmetry under Weyl rescaling so that it becomes local, i.e. replacing the partial derivative with an appropriate gauge covariant derivative, e.g. for scalars this means

$$\partial_\mu \Phi \rightarrow \mathcal{D}_\mu \Phi = (\nabla_\mu - \tilde{\Delta}_\Phi W_\mu)\Phi \tag{2.27}$$

which is called the Weyl covariant derivative in order to distinguish it from the gauge covariant derivative D_μ and the space-time covariant derivative ∇_μ.

However, Iorio et al. [42] show that a necessary and sufficient condition for a scale invariant action to become local Weyl invariant without the need for introducing the Weyl gauge field is that the flat space limit of the ungauged action is *conformally invariant.* The conformal symmetry in a d-dimensional spacetime is defined as the subgroup of coordinate transformations that leaves the metric invariant up to a conformal scaling factor, i.e.

$$x^\mu \rightarrow x^{\mu\prime}, \quad g_{\mu\nu}(x) \rightarrow g'_{\mu\nu}(x') = g_{\alpha\beta}(x)\frac{\partial x^\alpha}{\partial x^{\mu\prime}}\frac{\partial x^\beta}{\partial x^{\nu\prime}} = \Omega(x)^2 g_{\mu\nu}(x) \tag{2.28}$$

where Ω form a subgroup of the group of local Weyl transformations that is induced by conformal transformations. So local Weyl invariance does imply conformal invariance, but conformal invariance does not necessary imply Weyl invariance, as shown in [43] where explicit examples are given.

The name 'conformal' comes from a Latin word 'conformalis' which means 'having the same shape'. It is the group that contains all the coordinate transformations that preserve the angle $\frac{v \cdot w}{\sqrt{v^2 w^2}}$ between any two vectors v, w with $v \cdot w = g_{\mu\nu}v^\mu w^\nu$, which is more general than scale invariance (see figure 2.3). It is important to note that, unlike the Weyl transformation (2.26), a conformal transformation does not act on the metric, but changes the integration variables and the derivatives.

Last but not least, we note that upon considering conformal transformations (2.28), isometries correspond to $\Omega^2(x) = 1$, meaning they are a subset of conformal transformations. In flat spacetime this group is simply the Poincaré group. Another subset of conformal transformations are the scale transformations (dilations) which correspond to $\Omega^2(x) =$ constant. Conformal invariance thus demands scale invariance, but the converse is not necessarily true. The question regarding the precise condition under which a scale-invariant field theory is also invariant under the conformal group was already posed in 1971 by Coleman and Jackiw. In two space-time dimensions Zomolodchikov (1986) showed that a unitary and scale-invariant action is necessarily conformally invariant. In four space-time dimensions this issue is more subtle since unitary, scale-invariant but not conformally invariant Lagrangians are known. Nakayama [44] spoke of a consensus on this issue in d = 4 dimension under the assumptions of 1) unitarity, 2) Poincarè invariance, 3) discrete spectrum in scaling dimension, 4) existence of scale current and 5) unbroken scale invariance in the vacuum. Sachs [45] showed that requiring diffeomorphism invariance additional to unitarity ensures scale invariance is enlarged to conformal invariance in $d = 4$.

To summarize, we have shown that global Weyl invariance generalizes scale invariance to general spacetimes. A global Weyl invariant theory can be made local Weyl invariant, without introducing the scale covariant derivative if the flat spacetime limit of the theory is conformally invariant. So if a theory is locally Weyl invariant is is also conformally invariant, but a conformally invariant theory is not necessarily local Weyl invariant. This means that even though many texts on Conformal Gravity actually use the stricter Weyl invariance, conformal invariance is also a property of the theory.

Furthermore, we have shown that conformal invariance implies scale invariance, but that the converse is not necessarily true. We have given some conditions that, if met, allow us to promote scale invariance to conformal invariance. These conditions are so reasonable that many texts that discuss scale-invariant theories can actually be considered conformal invariant as well.

2.2.2 The Conformal Group

Consider the metric for the infinitesimal transformation $x^{\mu\prime} = x^\mu + \varepsilon^\mu$

$$g_{\mu\nu} \to g_{\mu\nu} + \nabla_\mu f_\nu + \nabla_\nu f_\mu$$

The requirement that the transformation be conformal (2.28)

$$g_{\mu\nu} \to \Omega(x)^2 g_{\mu\nu} \approx (1 + 2w(x))g_{\mu\nu}$$

gives

$$\nabla_\mu f_\nu + \nabla_\nu f_\mu = 2w(x)g_{\mu\nu}, \qquad w(x) = \tfrac{1}{d}\nabla_\mu f^\mu \tag{2.29}$$

where $\nabla_\mu(\{\})$ is the Levi-Civita covariant derivative and $w(x)$ is determined by taking the trace on both sides. Note that for a local Weyl transformation $w(x)$ would be unconstrained, again showing that local Weyl invariance implies conformal invariance though not the converse.

Equation (2.29) is known as the conformal Killing equation and f^μ as a Killing field, which is a vector field that preserves the metric on a Riemannian manifold. In flat spacetime, it becomes

$$\partial_\mu\varepsilon_\nu + \partial_\nu\varepsilon_\mu = \eta_{\mu\nu}\tfrac{2}{d}\partial_\mu\varepsilon^\mu \tag{2.30}$$

with $\eta_{\mu\nu}$ the Minkowski metric with signature $(d-1,1)$, and ε^μ the flat space-time analog of f^μ. Using this result, we find that for $d > 2$ the most general conformal Killing vector has the form

$$\varepsilon^\mu(x) = a^\mu + bx^\mu + \lambda^\mu_\nu x^\nu + c_\nu\left(2x^\mu x^\nu - \eta^{\mu\nu}x^2\right) \tag{2.31}$$

with $(a_\mu, \omega^{\mu\nu}, b, c_\nu)$ a total of $\frac{(d+1)(d+2)}{2}$ parameters and $x^2 = x_\nu x^\nu$. We find that the constant term a_μ can be seen as an infinitesimal translation (d in total). The term bx_μ represents a dilation (infinitesimal scale transformation) and because $\omega_{\mu\nu} = -\omega_{\nu\mu}$ that term can be identified as a rigid rotation ($d(d-1)/2$ Lorentz transformations). The quadratic term corresponds to the d special conformal transformations. The results are summarized in table 2.1.

If the dimension of the set of conformal Killing vectors is 15, then the space is conformally flat. However, if we allowed $(a_\mu, \omega^{\mu\nu}, b, c_\nu)$ to become spacetime dependent, we anticipate that we have only 11 local symmetries instead of 15 (in $d = 4$). This is because the global special conformal transformation of $x^2 \to \frac{x^2}{1+2c_\mu x^\mu + x^2}$ can be considered a particular transformation under the local dilations $x^2 \to \lambda(x)x^2$.

Type	Infinitesimal form	Finite form
Translation	$x^\mu \to x^\mu + a^\mu$	$x_\mu \to x^\mu + \alpha^\mu$
Dilatation	$x^\mu \to x^\mu + bx^\mu$	$x_\mu \to \lambda x^\mu$
Lorentz transformation	$x^\mu \to x^\mu + \lambda^\mu_\nu x^\nu$	$x_\mu \to \Lambda^\mu_\nu x^\nu$
Special conformal transformation	$x^\mu \to x^\mu + 2(x^\nu c_\nu)x^\mu - c^\mu x^2$	$x^\mu \to \frac{x^\mu - \beta^\mu x^2}{1-2\beta_\mu x^\mu + \beta^2 x^2}$

Table 2.1 – *Overview of the infinitesimal and finite transformations of the conformal group.*

With the conformal transformations as given in table 2.1, we can now turn to the generators of the conformal group. The general definition of a generator G_a of a symmetry transformation $x \to x'$ and $\Phi(x) \to \Phi'(x') = \mathcal{F}(\Phi(x))$ where Φ is the collection of fields, is given as

$$\left.\begin{aligned} x^{\mu\prime} &= x^\mu + \varepsilon_a \frac{\delta x^\mu}{\delta \varepsilon_a} \\ \Phi'(x') &= \Phi(x) + \varepsilon_a \frac{\delta \mathcal{F}}{\delta \varepsilon_a} \\ -\mathrm{i}\varepsilon_a G_a \Phi(x) &\equiv \Phi'(x) - \Phi(x) \end{aligned}\right\} \Rightarrow \mathrm{i}G_a\Phi = \frac{\delta x^\mu}{\delta\varepsilon_a}\partial_\mu\Phi - \frac{\delta\mathcal{F}}{\delta\varepsilon_a} \tag{2.32}$$

Assuming that the fields are unaffected by the transformation (i.e. $\mathcal{F}(\Phi) = \Phi$), the conformal group enlarges the flat space Poincaré group[25] of the P_μ (translation) and $L_{\mu\nu} = -L_{\nu\mu}$ (Lorentz transformation) generators to include a dilatation operator D and special conformal transformation generators K_μ.

$$\begin{aligned} P_\mu &= -\mathrm{i}\partial_\mu & L_{\mu\nu} &= \mathrm{i}(x_\mu\partial_\nu - x_\nu\partial_\mu) \\ D &= -\mathrm{i}x^\mu\partial_\mu & K_\mu &= -\mathrm{i}(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu) \end{aligned} \tag{2.33}$$

The generators obey

$$\begin{aligned} [L_{\mu\nu}, L_{\rho\sigma}] &= \mathrm{i}\left(\eta_{\nu\rho}L_{\mu\sigma} + \eta_{\sigma\mu}L_{\nu\rho} - \eta_{\mu\rho}L_{\nu\sigma} - \eta_{\nu\sigma}L_{\mu\rho}\right) & [P_\mu, P_\nu] &= 0, \\ [L_{\mu\nu}, P_\rho] &= -\mathrm{i}\left(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu\right), & [D, D] &= 0, \\ [L_{\mu\nu}, D] &= 0, & [D, P_\mu] &= \mathrm{i}P_\mu, \\ [L_{\mu\nu}, K_\rho] &= -\mathrm{i}\left(\eta_{\mu\rho}K_\nu - \eta_{\nu\rho}K_\mu\right) & [D, K_\mu] &= -\mathrm{i}K_\mu, \\ [K_\mu, P_\nu] &= 2\mathrm{i}\left(\eta_{\mu\nu}D - L_{\mu\nu}\right), & [K_\mu, K_\nu] &= 0 \end{aligned} \tag{2.34}$$

where the first three commutation relations indicate that the conformal algebra has indeed a Poincaré subalgebra. The first three lines together show that generators $L_{\mu\nu}, P_\mu$ and D span a subalgebra, sometimes called the Weyl algebra. This illustrates our previous point that mathematically full conformal symmetry is not necessarily implied by scale symmetry plus Poincaré invariance.

Because the conformal group is isomorphic to $SO(4,2)$, we can define

$$J_{\mu\nu} = L_{\mu\nu} \quad J_{4\mu} = \tfrac{1}{2}(P_\mu - K_\mu), \quad J_{45} = D, \quad J_{5\mu} = \tfrac{1}{2}(P_\mu + K_\mu)$$

where $J_{ab} = -J_{ba}$ with $a, b \in 0, 1, \ldots 5$ and $\mu, \nu \in 0, 1, 2, 3$. The conformal algebra is then given by the commutation relations of the $SO(4,2)$ group of four dimensional spacetime:

$$[J_{ab}, J_{cd}] = \mathrm{i}\left(\eta_{ad}J_{bc} + \eta_{bc}J_{ad} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac}\right) \tag{2.35}$$

where the diagonal metric η_{ab} has signature $(-++++-)$.

[25] Poincaré symmetry applies to flat Minkowski space-time. It would be in principle replaced by either de Sitter (dS) $SO(4,1)$ or anti-de Sitter (AdS) $SO(3,2)$ symmetries in a positively or negatively curved spacetime.

Having established the space-time part of the generators of the conformal group, we need to take into account how a field transforms under a conformal transformation (2.28)

$$x^\mu \to x^{\mu\prime}, \quad g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \Omega(x)^2 g_{\mu\nu}(x)$$

For fields Φ with spin the result is the following:

$$\Phi(x) \to \Phi'(x') = \left|\frac{\partial x'}{\partial x}\right|^{\Delta_\Phi/d} \Phi(x) = \Omega(x)^{-\Delta_\Phi} S_\mu^{\mu'} \Phi(x) \tag{2.36}$$

where Δ_Φ is the scaling dimension of the field and $S_\mu^{\mu'}$ the matrix representation of the Lorentz generator acting on the indices of Φ (S is the spin operator associated with Φ and usually constructed from the gamma matrices). Fields that transform in this way are called 'quasi-primary' fields. Now, using equation (2.36) for the $\mathcal{F}(\Phi)$ term in the definition for the generator (2.32), the generators of the conformal algebra become:

$$\begin{aligned} P_\mu &= -\mathrm{i}\partial_\mu & L_{\mu\nu} &= S_{\mu\nu} + \mathrm{i}(x_\mu\partial_\nu - x_\nu\partial_\mu) \\ D &= -\mathrm{i}(x^\mu\partial_\mu + \Delta) & K_\mu &= \mathrm{i}(-2x_\mu x^\nu\partial_\nu + x^2\partial_\mu) - 2\mathrm{i}x_\mu\Delta - S_{\mu\nu}x^\nu \end{aligned} \tag{2.37}$$

A theory is called conformal invariant at the classical level if its action is invariant under conformal transformations of table 2.1. This is local symmetry if the metric $g_{\mu\nu}$ is dynamical as in gravity theories, but global if the metric is fixed, which is usually assumed in QFTs. Conformal invariance at the quantum level is a whole different story as the renormalization procedure, as outlined in section 1.3, introduces a renormalization scale in the theory, thus breaking scale (and conformal) invariance. We will come back to this in a later chapter. Classical or quantum theory, spontaneously broken conformal symmetry gives only 1 Goldstone bosons, namely the one related to the scale operator.

2.2.3 Restrictions due to conformal invariance

In the presence of fields, there is a nonzero stress-energy tensor $T_{\mu\nu}$. Next we will be interested in how conformal transformations affect $T_{\mu\nu}$. For that we consider a classical field theory on flat $d > 2$-dimensional space-time containing matter and gauge fields Φ_i and a metric tensor g. The Lagrangian is furthermore assumed to be the sum of two terms, a purely gravitational part $\mathcal{L}_g$ depending only on the metric tensor g and its first and second order partial derivatives but not on the matter fields or their derivatives, and a matter field part $\mathcal{L}_m$ depending on the matter fields and their first order partial derivatives as well as on the metric tensor g and its first and second order partial derivatives [41]:

$$\mathcal{L}(g, \partial g, \partial^2 g, \Phi, \partial\Phi) = \mathcal{L}_g(g, \partial g, \partial^2 g) + \mathcal{L}_m(g, \partial g, \partial^2 g, \Phi, \partial\Phi)$$

The equations of motion for the fields are then derived from the variational principle $\delta S = 0$ with respect to the fields. This leads to the Euler-Lagrange formula:

$$\frac{\delta\mathcal{L}_m}{\delta\Phi^i} = \frac{1}{\sqrt{|g|}}\left(\frac{\partial(\sqrt{|g|}\mathcal{L})}{\partial\Phi^i} - \partial_\mu\frac{\partial(\sqrt{|g|}\mathcal{L})}{\partial\partial_\mu\Phi^i}\right) = 0 \tag{2.38}$$

Recall the Hilbert energy-momentum tensor[26] from equation (2.20)

[26] Different definitions of the energy-momentum tensor exist, namely the so-called canonical energy-momentum tensor. This is described via the Noether prescription and improved by Belinfante to be symmetrical. In a similar way, the Belinfante tensor can be further improved as was done by C. Callan, S. Coleman and R. Jackiw (1970) who gave a recipe for constructing an energy-momentum tensor which is divergence-free, symmetric and traceless within conformal invariant theories. Iorio et al. [42] provide a more structured, differential-geometry based approach to the rather ad hoc recipe followed by the previously mentioned authors. Taking into account that the processes of varying with respect to $g_{\mu\nu}$ and taking the flat limit do not commute, we note that on shell the canonical energy-momentum tensor and the Hilbert energy-momentum tensor coincide.[46].

$$T_{\mu\nu} = -\frac{2}{\sqrt{g}} \frac{\delta}{\delta g_{\mu\nu}} \left(\sqrt{g}\mathcal{L}_{\text{mat}}\right)$$

For an infinitesimal local Weyl transformation (2.26)

$$g_{\mu\nu} \to \tilde{\Omega}(x)^2 g_{\mu\nu} \approx (1 + 2\tilde{w}(x)) g_{\mu\nu}$$

we have

$$\begin{aligned} \delta S_m &= \int \mathrm{d}^d x \left(\frac{\delta \sqrt{g}\mathcal{L}_m}{\delta \Phi^i} \delta\Phi^i + \frac{\delta(\sqrt{g}\mathcal{L}_m)}{\delta g_{\mu\nu}} \delta g_{\mu\nu} \right) = -\tfrac{1}{2} \int \mathrm{d}^d x T^{\mu\nu} \delta g_{\mu\nu} \\ &= -\int \mathrm{d}^d x T^{\mu\nu} \tilde{w}(x) g_{\mu\nu} \qquad\qquad = -\int \mathrm{d}^d x T^\mu_\mu \tilde{w}(x) = 0 \end{aligned}$$

where we assumed the fields are on shell, i.e. satisfy the equation of motions, in the first line. We conclude that invariance of the action under Weyl transformations requires tracelessness of the energy-momentum tensor $T^\mu_\mu = 0$ because $\tilde{w}(x)$ is an arbitrary function. The inverse is not necessarily true, because tracelessness can also be achieved if under infinitesimal local Weyl rescaling, $\mathcal{L}_m$ picks up a total divergence or terms that vanish upon insertion of the equations of motion for the matter fields.

For a scale transformation (i.e. global Weyl rescaling) where $\tilde{w}(x) = \tilde{w}$ =constant, the above result is modified to

$$\delta S = -\tilde{w} \int \mathrm{d}^d x T^\mu_\mu = 0$$

Meaning that the trace of the energy-momentum tensor for a scale-invariant theory vanishes up to a total derivative in flat spacetime: $T^\mu_\mu = -\partial_\mu D^\mu$. As conformal invariance implies scale invariance, this is also a property of conformal invariant theories. Actually, invariance under special conformal transformation is equivalent to T^μ_μ being a double divergence ($T^\mu_\mu = \partial_{\mu\nu} D^{\mu\nu}$), which is a much more restrictive condition [47]. Following the standard procedure, we can improve the energy-momentum tensor by

$$\Theta^{\mu\nu} = T^{\mu\nu} + \partial_\alpha \partial_\beta \Upsilon^{\mu\alpha\nu\beta}$$

where

$$\Upsilon^{\mu\nu} = \frac{1}{d-2}\left(\eta^{\alpha\nu} C^{\beta\mu} + \eta^{\beta\mu} C^{\alpha\nu} - \eta^{\mu\nu} C^{\alpha\beta} - \eta^{\alpha\beta} C^{\mu\nu}\right) + \frac{1}{(d-1)(d-2)}\left(\eta^{\mu\nu}\eta^{\alpha\beta} - \eta^{\alpha\nu}\eta^{\beta\mu}\right) C^\rho_\rho$$

Unitarity demands that the only allowed improvement term in unitary quantum field theories in $d > 2$ is from $C^{\mu\nu} = \eta^{\mu\nu} C$ with a dimension $d-2$ scalar operator C if we demand the energy-momentum tensor has the canonical scaling dimension Δ [44]. Following this improvement procedure, we find that the improved tensor $\Theta^{\mu\nu}$ for a conformal invariant theory also exhibits tracelessness [44, 46].

A theory with conformal invariance thus satisfies the following properties [48, p.9-12]:

1. There is a set of fields $\tilde{\Phi}_i(x)$, which in general is infinite and contains in particular the derivatives of all the fields.
2. There is a subset of quasi-primary fields $\Phi_j \in \tilde{\Phi}_i$ that transform according to (2.36). The theory is then covariant under this transformation, in the sense that the correlation functions satisfy

$$\langle \Phi_1(x_1) \dots \Phi_n(x_n) \rangle = \left| \frac{\partial x'}{\partial x} \right|_{x=x_1}^{\Delta_1/d} \cdots \left| \frac{\partial x'}{\partial x} \right|_{x=x_n}^{\Delta_n/d} \langle \Phi_1(x_1') \dots \Phi_n(x_n') \rangle \tag{2.39}$$

where Δ_j is the scaling dimension of Φ_j.

3. The rest of the $\tilde{\Phi}_i$'s can be expressed as linear combinations of the quasi-primary fields and their derivatives.
4. There is a vacuum $|0\rangle$ invariant under the global conformal group.
5. The energy-momentum tensor is traceless.

Equation (2.39) in combination with the transformations of table 2.1 is highly restrictive. Consider for example a theory with several spinless fields ϕ_i, the two-point function is then:

$$\langle \phi_i(x_1)\phi_j(x_2)\rangle = \left|\frac{\partial x'}{\partial x}\right|_{x=x_1}^{\Delta_1/d} \left|\frac{\partial x'}{\partial x}\right|_{x=x_2}^{\Delta_2/d} \langle \phi_i(x_1')\phi_j(x_2')\rangle$$

Translation invariance implies that the left-hand side does not depend on the independent coordinates but rather their differences. Rotational invariance limits this further to $r_{12} = |x_1 - x_2|$. Scale invariance allows only the dependence on ratios

$$\langle \phi_i(x_1)\phi_j(x_2)\rangle = \frac{M_{ij}}{r_{12}^{\Delta_i+\Delta_j}}$$

where we note that the matrix M_{ij} will be positive definite in a unitary theory. This implies that there exists a field basis such that $M_{ij} = M\delta_{ij}$. Lastly, the special conformal transformation of r_{ij} in combination with its jacobian, further restrict the form of the two-point function to:

$$\langle \phi_i(x_1)\phi_j(x_2)\rangle = \begin{cases} \frac{M\delta_{ij}}{r_{12}^{2\Delta}} & \text{for} \quad i = j,\ \Delta_i = \Delta_j = \Delta \\ 0 \quad \text{for} \quad i \neq j \end{cases} \tag{2.40}$$

Similarly, the 3-point function is uniquely determined up to a constant C_{123}, which can be determined using e.g. the Operator Product Expansion (OPE):

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle = \frac{C_{123}}{r_{12}^{\Delta-2\Delta_3} r_{23}^{\Delta-2\Delta_1} r_{13}^{\Delta-2\Delta_1}}, \quad \text{with} \quad \Delta = \sum_i \Delta_i \tag{2.41}$$

However, $n \geq 4$-point functions are not fully determined. In general, they have an arbitrary (i.e. not fixed by conformal symmetry) dependence on the $n(n-3)/2$ ratios $\frac{r_{ij}r_{kl}}{r_{ik}r_{jl}}$:

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\rangle = f(u,v) \prod_{i<j}^{4} r_{ij}^{2\Delta_l+2\Delta_k-\Delta} \tag{2.42}$$

where

$$u = \frac{r_{12}^2 r_{34}^2}{r_{13}^2 r_{24}^2},\ v = \frac{r_{14}^2 r_{23}^2}{r_{13}^2 r_{24}^2}, \quad \text{and} \quad \left(\frac{v}{u}\right)^{\Delta} f(u,v) = f(v,u)$$

2.3 A conformal invariant theory of Gravity

Having explained the conformal group and shown how restrictive conformal invariance is, it is time to turn our attention to a conformal theory of gravity. For that we note that already in 1918 did Hermann Weyl attempt at unification of electromagnetism with gravitation. His ideas were discarded because all mass parameters would be identically zero, which was unacceptable in the time prior to the development of spontaneous symmetry breaking. However, Weyl also discovered a tensor with a remarkable geometric property, the so-called conformal or Weyl tensor $C_{\mu\nu\rho\sigma}$. Renewed interest in Weyl's work and subsequent work by Rudolf Bach, was sparked by the work of Philip Mannheim ([12]), who used the conformal tensor as the basis for a conformal invariant theory of gravity dubbed *Conformal Weyl Gravity* (CWG) (Section 2.3.1).

Another popular approach to obtain a conformally invariant theory of quantum gravity is *Conformal Dilaton Gravity* (CDG), named like this by Alvarez et al. [10]. One particular theory has already been proposed by Paul A.M. Dirac in a paper from 1973, where he used the procedure of group averaging, that is, perform a conformal transformation on the Einstein-Hilbert lagrangian and promote the Weyl rescaling factor to the status of a new field. This received new interest after the work of e.g. Gerard 't Hooft ([11]) and Alessandro Codello [49] and will be explained in the second section. In Section 2.3.3 we will compare the two theories.

2.3.1 Conformal Weyl Gravity

In this section we study the 4 dimensional conformal invariant theory of gravity based on the conformal (actually local Weyl) invariant quadratic curvature terms built from the contractions of the Riemann tensor, i.e. the Weyl tensor squared. The Weyl tensor $C_{\mu\nu\rho\sigma}$ in d dimensions is given as [10]

$$\begin{aligned} C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} &+ \frac{1}{(d-2)(d-1)}\left(g_{\mu\rho}g_{\nu\sigma} - g_{\nu\rho}g_{\mu\sigma}\right)R \\ &- \frac{1}{d-2}\left(g_{\mu\rho}R_{\nu\sigma} - g_{\mu\sigma}R_{\nu\rho} - g_{\nu\rho}R_{\nu\sigma} + g_{\nu\sigma}R_{\mu\rho}\right) \end{aligned} \tag{2.43}$$

For dimensions $d \geq 3$ it is defined as the trace-free part of the Riemann curvature tensor. The Weyl tensor furthermore has all the algebraic properties of $R_{\mu\alpha\nu\beta}$ i.e.

$$C_{\mu\nu\rho\sigma} = C_{\rho\sigma\mu\nu} \tag{2.44}$$
$$C_{\mu\nu\rho\sigma} = -C_{\nu\mu\rho\sigma} = -C_{\mu\nu\sigma\rho} \tag{2.45}$$
$$C_{\mu\nu\rho\sigma} + C_{\mu\sigma\nu\rho} + C_{\mu\rho\sigma\nu} = 0 \tag{2.46}$$

and in addition is traceless. It transforms under the conformal transformation (2.28) as $C^{\mu}_{\nu\rho\sigma} \to C^{\mu}_{\nu\rho\sigma}$ (see Appendix C). Therefore, the Weyl tensor can be used to construct an action which is invariant under the local conformal transformation of equation (2.28), namely the Conformal Weyl Gravity action in d dimensions [10]

$$\begin{aligned} S_{\text{CWG}} &= \alpha_g \int \mathrm{d}^d x \sqrt{g} C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} \\ &= \alpha_g \int \mathrm{d}^d x \sqrt{g}\left(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - \frac{4}{d-2}R_{\mu\nu}R^{\mu\nu} + \frac{2}{(d-1)(d-2)}R^2\right) \end{aligned} \tag{2.47}$$

where α_g is a dimensionless gravitational coupling constant. In $d = 4$, the CWG action is the unique conformally invariant action constructed solely from the Weyl tensor. In 4 spacetime dimensions we can use the Euler term (2.22) to write the CWG action as:

$$S_{\text{CWG}} = 2\alpha_g \int \mathrm{d}^4 x \sqrt{-g}\left(R_{\mu\nu}R^{\mu\nu} - \tfrac{1}{3}R^2\right) \tag{2.48}$$

Conformal invariance of the gravitational action forbids any fundamental (Planck) scale, meaning Newton's constant as well as the Einstein-Hilbert action are not allowed. Luckily it is not necessary to achieve the Einstein equations exactly as long as the dynamical equations of Conformal Gravity pass the three classical tests[27] of General Relativity, namely the gravitational red shift, gravitational bending of light, and the precession of planetary orbits. Specifically, what is needed for the three tests is knowledge of the metric exterior to a static, spherically symmetric source, namely the Schwarzschild metric which obeys $R_{\mu\nu} = 0$ in the source-free region. But

[27] Mannheim notes that besides these three tests, there is one other phenomena which needs to be explained on solar sized distance scales. This is the decay of the orbit of binary pulsar. This decay will happen in any covariant metric theory since in all of them gravitational information cannot be communicated faster than the speed of light. The problem lies in the calculating of the specific amount of decay which has actually been observed, with calculations having so far only been carried through (and with stunning success) is the second order Einstein theory itself.

the exterior Schwarzschild solution is an exterior solution to any pure metric theory of gravity which replaces the EH action with any Ricci tensor based higher order action. So having a Ricci tensor based action will suffice to recover the standard solar system phenomenology[28] [12].

Rudolf Bach derived the fourth order dynamical gravitational field equations for Conformal Weyl Gravity (2.47) in 1921. We present the results for the vacuum case here (the derivation can be found in Appendix D):

$$B_{\mu\nu} = \nabla^{\alpha}(\Gamma)\nabla^{\beta}(\Gamma)C_{\mu\alpha\nu\beta} - \tfrac{1}{2}R^{\alpha\beta}C_{\mu\alpha\nu\beta} = 0 \tag{2.49}$$

where we call $B_{\mu\nu}$ the Bach tensor, which is symmetric, divergence-free $\nabla_\nu B^{\mu\nu} = 0$, traceless $g_{\mu\nu}B^{\mu\nu} = 0$ and behaves under a conformal transformation as $B_{\mu\nu} \to \Omega^{-2}B_{\mu\nu}$. In the literature one encounters the above expression also in terms of the Schouten tensor $S_{\mu\nu}$ or in terms of the Ricci tensor and scalar. These equivalent expressions are also given in Appendix D.

The Bach equation admits trivial solutions, i.e. all conformal Einstein spaces[29]. These are topological spaces that are conformally related to an Einstein space for which $R_{\mu\nu} = \alpha g_{\mu\nu}$. This is equivalent to saying that the metric is a solution to the vacuum Einstein field equation with arbitrary value of the constant α. Specifically, this includes solutions with vanishing Ricci tensor. Solutions with vanishing Ricci tensor include all vacuum solutions to Einstein gravity such as the Schwarzschild solution exterior to a static, spherically symmetric source.

To be specific, start by assuming a static, spherically symmetric metric in the vacuum. The line element is then given by

$$ds^2 = -A(r)\,\mathrm{d}t^2 + B(r)dr^2 + C(r)\,\mathrm{d}\Omega^2$$

where $\mathrm{d}\Omega^2 = \mathrm{d}\psi^2 + \sin^2(\psi)\,\mathrm{d}\phi^2$ is the metric of the standard 2-sphere. The functions A, B and C have to be positive. This line element is conformally equivalent to the Mannheim-Kazanas solution [50, 51]

$$ds^2 = -A(r)\,\mathrm{d}t^2 + \tfrac{\mathrm{d}r^2}{A(r)} + r^2\,\mathrm{d}\Omega^2\,, \quad \text{where} \quad A(r) = 1 - 3\beta\gamma - \tfrac{(2-3\beta\gamma)\beta}{r} + \gamma r - kr^2 \tag{2.50}$$

Now we see that $\gamma = k = 0$ and $\beta = m$ exactly returns the Schwarzschild metric from the ordinary flat space Einstein-Hilbert theory. The other two parameters (k, γ) stem from the fact that the theory is fourth order and thus the vacuum solution possesses two extra constants of integration as compared to Einstein Gravity. For $k \neq 0$, the background is a de Sitter (or anti-de Sitter) geometry. Note that this could only occur in the presence of a cosmological cosntant in Einstein gravity, which is not needed in the Weyl theory. The second integration constant (γ) effectively measures all departures of the above metric from that in the Einstein case.

Several authors have used this solution to explain galactic rotation curves as well as deflection of light and time delay in the exterior of a static spherically symmetric source without the need for copious amounts of dark matter. For this they relayed on the γr term, though its validity was doubted at the time. Hans-Jürgen Schmidt[30] [52] resolved the discussion by showing that the value of γ can be made to vanish by a conformal transformation. Even though Dark Matter is not needed to explain rotation curves, we do point out that other astronomical observations still point to its existence.

[28] This would not necessarily be true for any higher order Riemann tensor based action since the Schwarzschild metric is not Riemann flat, only Ricci flat. The CWG Lagrangian does contain the Riemann tensor but in 4 dimensions it vanishes due to the Euler term.

[29] To see that this is indeed a solution to the Bach equation, plug in $R_{\mu\nu} = \alpha g_{\mu\nu}$. Then we find that R is a constant and that all derivative terms in $B_{\mu\nu}$ vanish automatically. The remaining terms can only be proportional to $g_{\mu\nu}$, but the tracelessness of the Bach tensor ensures that these terms also vanish, leading to the conclusion that for an Einstein space indeed $B_{\mu\nu} = 0$.

[30] Schmidt also showed conclusively the existence of solutions to Conformal Weyl Gravity which are not conformally equivalent to Einstein spaces. These non-trivial solutions include all the homogeneous and isotropic spacetimes (i.e., described by the Robertson-Walker line element) have zero Weyl tensor. How our Universe could result from such an empty high symmetric spacetime was investigated in [82, and references therein].

Finally, we need to consider the fact that the Bach field equations are fourth-order and special attention for the unitarity of the theory is warranted. Recall that in quantum field theories we want conservation of probabilities, i.e. the Hamiltonian should be hermitian and the S-matrix unitary. Unitarity is also related to evolution of time and is characterized by the absence of negative norm states in the theory. On the other hand, the Bach equation is a fourth-order equation meaning that its correlation functions, upon quantization, could lead to ghost excitations and/or tachyon behavior (negative norm states).

Based on the work of Carl Bender, Mannheim [53, and references therein] provided a solution for this problem by showing that PT symmetry, i.e. symmetry under the combined operation of parity P and time reversal T, is a necessary and sufficient condition for unitary time evolution whereas hermiticity is only a sufficient condition: Though hermiticity of the Hamiltonian implies unitarity, one cannot conclude that lack of Hermiticity implies lack of unitarity. To be specific, for Conformal Weyl Gravity, Mannheim argues that the presence of a negative Dirac norm (ghost state) could signal that the theory is not Hermitian rather than not unitary. Moreover, by establishing that the fourth-order derivative conformal gravity theory is indeed a PT theory[31], meaning that the gravitational field $g_{\mu\nu}$ is anti-Hermitian and a PT eigenstate, Conformal Weyl Gravity is now able to emerge as a fully renormalizable and unitary theory of quantum gravity in four spacetime dimensions.

Dropping the hermiticity requirement in favor of PT symmetry is not generally accepted and other solutions have been proposed. Another approach to the problem of unitarity, for example, is by James T. Wheeler [54]. Specifically, he proposes to vary all of the connection fields of conformal gravity independently instead of only the metric. The torsion-free solutions of the resulting field equations differ from those of GR only in the fact that they show local dilatational covariance. This way the whole discussion regarding the unitarity of the theory is irrelevant.

Another way to deal with the fourth-order aspect of CG was proposed by Juan Maldacena [55]. He started with the observations that 1) the on shell action for four dimensional Einstein gravity in an Einstein space that is locally asymptotically Euclidean Anti-de Sitter (EAdS) can be computed in terms of the action of Weyl gravity, and 2) any space that is conformal to an Einstein space is a solution to the equations of motion of conformal gravity. Knowing this, he wanted to select the solutions of Einstein gravity from the solutions of conformal gravity so that the Einstein action can be replaced by the Conformal Weyl Gravity action and at the same time only second-order dynamical equations in the pure gravity sector will be retrieved. Imposing a Neumann boundary condition on the metric at the boundary results in the semiclassical (or tree level) wave function of the universe of four dimensional asymptotically de-Sitter (dS) or Euclidean anti-de Sitter spacetimes.

This can be illustrated, without entering in too much detail, as follows. Given that conformal gravity has fourth order equations, we expect that it has four solutions for a given spatial momentum. Requiring a EAdS or dS spacetime, kills already two of the solutions. The Neumann boundary condition $\partial_z g_{ij} = 0$ kills another solution. Since an Einstein space is conformal to a solution obeying all the boundary conditions, we conclude that the remaining solution is a (conformal) Einstein space and that classically there is complete equivalence between ordinary gravity with a cosmological constant and conformal gravity. However, keep in mind that this equivalence has not been proved at the quantum level nor for other than EAdS and dS spacetimes.

At last we stress that in order for the quantum system to be unitary, its classical counterpart has to have stable evolution for arbitrary initial conditions. Otherwise quantum tunneling will connect stable and unstable regions of the phase space and will inevitably lead to violation of unitarity. Examples are known, for example the Pais-Uhlenbeck oscillator, which are unitary even though they are in possession of a ghost mode [56].

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[31] Unlike Hermiticity, PT symmetry does not need to be postulated as it is derivable from Poincaré invariance. Recall from section 2.2 that the conformal group is an extension of the Poincaré group: conformal invariant theories include Poincaré invariance and thus, according to Mannheim, PT symmetry.

2.3.2 Conformal Dilaton Gravity

While the objective of Wheeler and Maldacena in the previous section was to generate Einstein's theory of gravity starting from the conformal theory, the framework of Weyl gauging as used by e.g. ‘t Hooft [11] and Codello et al. [49] allows us to do the converse. We will use the latter to introduce this general framework and comment on ’t Hooft alternative interpretation afterwards.

Any theory can be made local Weyl invariant by Weyl gauging, that is, we treat Weyl invariance as local Abelian gauge symmetry, promote one mass parameter to a field called the dilaton (Stückelberg trick) and use it to construct the Weyl gauge field. Then we find the Weyl covariant derivative and construct the Weyl connection. Next we construct the Weyl-invariant curvature tensor using Weyl covariant derivative, which upon twice contracting the indices gives Ricci scalar curvature for Weyl geometry. Now we express every dimensional parameter of our theory in terms of the dilaton and replace all spacetime covariant derivatives by Weyl covariant derivatives and all Ricci scalar curvatures R by the Weyl covariant curvatures $\tilde{R}$. The resulting theory is then locally Weyl invariant by construction. (Recall that this symmetry implies conformal invariance, but is more restraining.)

Next we use the above described procedure on the Einstein-Hilbert action (2.17)

$$S_{\text{total}} = S_{\text{EH}} + S_{\text{mat}} = \frac{1}{19\pi G_N}\int \mathrm{d}^4x\sqrt{g}\,(R-2\Lambda) + S_{\text{mat}}, \quad g = |\det g_{\mu\nu}|,\; M_p^{-2} = 16\pi G_N \tag{2.51}$$

and demand it should be invariant under a Weyl transformations (2.26)

$$g_{\mu\nu} \to g'_{\mu\nu} = \Omega(x)^2 g_{\mu\nu}, \quad \Phi \to \Phi' = \Omega(x)^{-\frac{(d-2)}{2}}\Phi$$

To do that we treat the Planck mass M_p as a Stückelberg field, which we call the dilaton:

$$\chi(x) = \sqrt{\frac{8(d-1)}{16(d-2)\pi G_N}}e^{\sigma(x)} \tag{2.52}$$

which transforms as $\chi \to \Omega^{-1}\chi$ under a Weyl transformation. We then use it to define the Weyl gauge vector W_μ as

$$W_\mu = -\chi^{-1}\partial_\mu\chi$$

This means we have extended the electroweak gauge symmetry to to include Weyl invariance $SU(2)_L \times U(1)_Y \times U(1)_W$ where the local Abelian $U(1)_W$ introduces the above defined scalar field $\chi(x)$ and gauge boson W_μ. To eliminate the additional terms that arise under these gauge transformations the partial derivative operator should replaced with a gauge covariant derivative D_μ:

$$D_\mu\Phi = \partial_\mu\Phi - \tilde{\Delta}_\Phi W_\mu\Phi$$

with $\tilde{\Delta}$ the Weyl weight of the field.

Now we can find the Weyl connection, which is constructed to ensure that gauge covariant derivative terms are conformal invariant on a general manifold:

$$\tilde{\Gamma}^\mu_{\nu\rho} = \tfrac{1}{2}g^{\mu\rho}\left(\mathcal{D}_\nu g_{\lambda\rho} + \mathcal{D}_\lambda g_{\nu\rho} - \mathcal{D}_\rho g_{\lambda\nu}\right) = \left\{{}^{\;\mu}_{\nu\rho}\right\} - \tfrac{1}{2}\tilde{\Delta}\big(\delta^\mu_\nu W_\rho + \delta^\mu_\rho W_\nu - g^{\mu\lambda}g_{\nu\rho}W_\lambda\big)$$

where $\left\{{}^{\;\mu}_{\nu\rho}\right\}$ is the Levi-Civita connection given by the Christoffel symbols (2.12). The partial derivative operator in D_μ should now be replaced with a spacetime covariant derivative based on the Weyl connection $\nabla_\mu(\tilde{\Gamma})$, such that we find the Weyl covariant derivative (2.27):

$$\mathcal{D}_\mu \Phi = \nabla_\mu(\tilde{\Gamma})\Phi - \tilde{\Delta}_\Phi W_\mu \Phi$$

Equation (2.6) then allows us to construct the curvature tensor for the Weyl connection. Twice contracting gives the Ricci scalar for Weyl geometry:

$$\tilde{R} = R + 2(d-1)\nabla_\mu(\Gamma)W^\mu - (d-1)(d-2)W_\mu W^\mu$$

Plugging $\tilde{R}$ into the Einstein-Hilbert action (2.51) and writing all dimensional parameters in terms of the dilaton field $\chi(x)$, gives

$$S_{\text{total}} = S_{\text{CDG}} + S_{\text{mat}} = \int \mathrm{d}^d x\sqrt{g}\left(\frac{d-2}{8(d-1)}\chi^2 R - \tfrac{1}{8}(d-2)^2\partial_\mu\chi\partial^\mu\chi + \lambda\chi^4\right) + S_{\text{mat}} \quad (2.53)$$

or more precisely,

$$S_{\text{total}} = S_{\text{CDG}} + S_{\text{mat}} = \int \mathrm{d}^d x\sqrt{g}\left(\frac{d-2}{8(d-1)}\phi^2 R + \tfrac{1}{4}(d-2)\chi\partial^\mu\partial_\mu\chi + \lambda\chi^{\frac{2d}{d-2}}\right) + S_{\text{mat}} \quad (2.54)$$

which differs from the previous one by a covariant total divergence of the form $\frac{d-2}{4}\nabla^\mu(\Gamma)(\chi\partial_\mu\chi)$, and hence has the same equations of motion and the same energy-momentum tensor.

In 4 spacetime dimensions, we have

$$S_{\text{CDG}} = \int \mathrm{d}^4 x\sqrt{g}\left(\frac{1}{12}\chi^2 R - \tfrac{1}{2}\partial_\mu\chi\partial^\mu\chi + \lambda\chi^4\right) \quad (2.55)$$

This action is therefore referred to as the Conformal Dilaton Gravity action (name is due to Alvarez et al. [10]). For the dilaton we can identify a kinetic term as well as a 'mass' term proportional to the background scalar curvature R. It is the same action as that obeyed by a conformally coupled scalar field. Observe that CDG lagrangian does not contain a kinetic term for the $\hat{g}_{\mu\nu}$ field. Any such kinetic terms should arise solely from higher order effects due to the interactions with the matter fields.

We immediately see that the kinetic term of the dilaton has the wrong sign considering we used the mostly minus convention for the metric. This means that the dilaton is a unphysical scalar field. However, the the additional gauge freedom can be used to gauge fix the dilaton to a constant such that we can retrieve Einstein gravity: $\sigma(x) = 0$, or

$$\chi(x) = \sqrt{\frac{12}{16\pi G_N}}, \quad \lambda = \frac{2}{9}\pi G_N \Lambda$$

The above demonstrated equivalence between CDG action and the EH action need not survive at the quantum level, due to quantum fluctuations and in particular to Faddeev-Popov terms associated with the Weyl symmetry.

Actually, the above results were also obtained by 't Hooft [11] by splitting the metric in such a way that all scale dependences are contained in the dilaton field[32] $\omega(x)$. He extends the classical theory to the quantum theory. Key in his approach is treating the dilaton in the metric as an independent dynamical degree of freedom instead of evaluating the path integral as a perturbative series in the metric components (which would render the Einstein-Hilbert action nonrenormalizable). He then finds that the renormalization procedure leads to a quantum counter-term in the total action which is proportional to the Conformal Weyl Gravity action, which we discussed before.

[32] To distinguish between the dilaton field introduced in the geometric approach from 't Hooft's approach, we use $\chi(x)$ for the former and $\omega(x)$ for the latter. For an overview of the used notation, see the 'Notations and conventions' section at the start of this thesis.

To be specific, the metric tensor is split similar to a Weyl rescaling

$$g_{\mu\nu} = \hat{\omega}^2(x)\hat{g}_{\mu\nu} \tag{2.56}$$

where the quantity $\hat{g}_{\mu\nu}$ does not transform as an ordinary tensor but as a 'metatensor', meaning that it transforms as a tensor, but with prefactors containing unconventional powers of the Jacobian of the coordinate transformation. In the same sense, $\omega(x)$ is then called a metascalar, encoding all scale dependencies of $g_{\mu\nu}$.

The functional integration procedures thus becomes

$$\int \mathcal{D}g_{\mu\nu} e^{S_{\text{total}}}[\dots] = \int \mathcal{D}\hat{\omega}(x) \int \mathcal{D}\hat{g}_{\mu\nu}(x) e^{S_{\text{total}}}[\dots]$$

where additional constraints are needed to deal with the coordinate reparametrization ambiguity (i.e. a gauge-fixing condition that only depends on $\hat{g}_{\mu\nu}$) and right-hand side additionally requires a constraint for the conformal gauge ambiguity. One choice could be $\partial_\mu \hat{g}^{\mu\nu} = 0$ and $\det(\hat{g}_{\mu\nu}) = -1$.

't Hooft then continues by proposing to *first* integrate over $\hat{\omega}(x)$ together with the matter fields $\Phi^{\text{mat}}(x)$, and then over $\hat{g}_{\mu\nu}(x)$. (Recall that in standard perturbation theory the integration order does not matter.) Anticipating dimensional regularization and renormalization procedures, we switch to d dimensions by replacing $\hat{\omega} \to \hat{\omega}^{2/(d-2)}$ in equation (2.56), which then becomes

$$g_{\mu\nu} = \hat{\omega}^{\frac{4}{d-2}}(x)\hat{g}_{\mu\nu} \tag{2.57}$$

Plugging this into the Einstein-Hilbert action (2.17) and rescaling the $\hat{\omega}$ field according to

$$\hat{\omega} = \sqrt{\xi\kappa^2}\omega, \quad \text{where} \quad \xi = \frac{(d-2)}{4(d-1)}$$

such that Newton's constant (recall $\kappa^2 = 8\pi G_N$) completely disappears[33] in the gravity part of the action and becomes (see Appendix E for the derivation)

$$S_{\text{split}} = \int \mathrm{d}^d x \sqrt{\hat{g}} \left(\tfrac{1}{2} \tfrac{d-2}{4(d-1)} \hat{R}\omega^2 - \tfrac{1}{2}\hat{g}^{\mu\nu}\partial_\mu\omega\partial_\nu\omega \right) \tag{2.58}$$

where $\hat{g} = |\det\{\hat{g}_{\mu\nu}\}|$, ω is a scalar field called the dilaton and everything in S is now associated with $\hat{g}_{\mu\nu}$. In 4 dimensions this equals the CDG action found earlier, but there is one important difference. The geometric method modifies the connection while leaving the metric as it is whereas 't Hooft modifies the metric which in turn also modifies the connection. The introduction of Weyl invariance is enforced in the geometric theory by extending the gauge symmetry whereas it emerges from the peculiar treatment of the metric as done here. The methods would thus lead to different results for the matter part of the action.

However, it is important to observe that the split action for the ω field has an overall sign opposite to that of ordinary scalar fields, just as the dilaton found in CDG. Regardless of which action we consider, 't Hooft claims that since it is an overall sign, it has no net effect on the Feynman rules. This can be exploited by rotating the field in the complex plane[34][35]: $\omega = \mathrm{i}\eta$ (or $\chi = \mathrm{i}\eta$). Then, the dilaton integration is technically identical to the integration over a conventional, renormalizable scalar field. From the work of 't Hooft and Veltman [37], we then find that this action leads to an effective action for $\hat{g}_{\mu\nu}$ that largely coincides with the familiar

[33] Interesting question is whether a constant that can be scaled away can be considered fundamental. Recall that it is not possible to scale away the coupling constant in a Yang-Mills theory for example.

[34] If the dilaton had been chosen real, then the Wick-rotated functional integral would diverge exponentially, so that ω or χ would no longer function properly as a Lagrange multiplier.

[35] The validity of this step is questioned by Dietz et al. [83].

conformally invariant action from Conformal Weyl Gravity (2.47), but with an infinite numerical coefficient, which would have to be renormalized (we come back to this in the next chapter). So even if a squared Weyl tensor term was not included in the initial action as a kinetic term for the metric, one would be generated as a renormalization counterterm.

2.3.3 CWG versus CDG

In this chapter we motivated our search for a conformally invariant theory of gravity. The usual procedures to obtain a conformally invariant gravity theory are either to adopt the Conformal Weyl Gravity action (2.47) (or for $d = 4$ (2.48)) as for example proposed by Mannheim, or the conformal Einstein-Hilbert action called Conformal Dilaton Gravity (2.55) as proposed by Codello by Weyl gauging the EH action or 't Hooft via the splitting of the metric. Both theories have their merits and drawbacks. So lets compare Conformal Dilaton Gravity and Conformal Weyl Gravity at this stage.

Conformal Weyl Gravity is renormalizable and its fourth-order dynamical equations can reproduce the Schwarzschild metric of our Solar System. However, it leads to ghost excitations which spoil the unitarity of the theory. Mannheim proposes to deal with this by relaxing the hermiticity constraint to PT symmetry, which amounts to making the gravitational field anti-Hermitian. However, the full proof of the unitarity of the PT theory is not complete since interactions are not yet considered. Other ideas for handling this problems were for example proposed by Wheeler and Maldacena. Conformal Weyl Gravity may explain gravitational rotation curves without the need of Dark Matter. However, the theory is formulated only in the negative gravitational constant, so the gravitational interaction is not attractive but repulsive. It seems to be difficult for this exotic feature to be reconciled with the observation of the cosmic microwave background [57].

Conformal Dilaton Gravity is related to Einstein Gravity, i.e. gauge-fixing the CDG action returns the Einstein-Hilbert action. However, it has inherited the nonrenormalizability of the theory. Explicit calculations have shown that if we can rotate the dilaton in the complex plane, S_{CDG} results in a one-loop effective action which is proportional to the CWG action S_{CWG} with a divergent factor in front, which needs to be dealt with. Adding the same term with opposite sign as a counterterm may resolve the divergence, but spoils again the unitarity. This theory was proposed by e.g. Faria [58] under the name 'Massive Conformal Gravity'

$$S_{\mathrm{MCG}} = -\tfrac{1}{12}\int \mathrm{d}^4x\sqrt{g}\Big[(\phi^2 R + 6\partial_\mu\phi\partial^\mu\phi) - \frac{1}{m^2}C^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma}\Big] \tag{2.59}$$

and plays the role of a proper quantization action when quantizing the dilaton ϕ in the fixed curved background $g_{\mu\nu}$. Faria argued that MCG is a renormalizable quantum theory of gravity which has two massive ghost states, but Myung [59] refuted the renormalizability based on the work from Stelle [36].

Currently, there is no obvious way to attain the renormalizability without violating the unitarity in quantizing the gravity. So we can only conclude that CDG is either renormalizable but non-unitary due to the adding of the Weyl tensor squared counterterm or unitary but nonrenormalizable due to the infinities present in the CDG action.

Assuming that our final Theory of Everything is a 4 spacetime dimensional, renormalizable, unitary, conformally invariant theory, we are thus faced with a problem. Insisting on the 4 spacetime dimensions and conformal symmetry, we can either pursue the ideas of Mannheim further and hope to address the issue of its unitarity at a later stage or we work with Conformal Dilaton Gravity and address the renormalization aspect later. Here we choose the latter path for a multiple of reasons.

Firstly, the principle of unitarity is heavily embedded in various domains of physics. It will be difficult to judge which techniques and results are applicable to a situation where we are working with a theory that is not unitary. Secondly, accepting nonrenormalizable field theories as effective field theories has become commonplace. Predictions from CDG as viewed as an EFT are not without meaning. Experimental and astronomical evidence should then provide more insights into its validity, not forgetting that the assumptions of four spacetime dimensions and conformal invariance are no more than that, namely assumptions. Lastly, quantization of gravity is far from understood and so ideas and procedures regarding the quantization and renormalizablity of a theory of Gravity are still under development. Dismissing or accepting a theory purely on this aspect is hardly a sufficient reason at this stage.

2.4 Adding matter: the Conformal Standard Model

With the gravity sector discussed in the preceding sections and the Standard Model (SM) explained in Chapter 1, the remaining question of this chapter and main question of this last section is, assuming Conformal Gravity, what is the Lagrangian of the complete theory?

First, we need to look at how the SM Lagrangian is modified in the presence of gravitational field. This can be achieved by means of Einstein's Equivalence Principle. As a small reminder from any course on gravity, we list the different equivalence principles again[60]:

i. Newton's equivalence principle (NEP): In the Newtonian limit, the inertial and gravitational masses of a body are equal.

ii. Weak equivalence principle (WEP) is the empiric law of the universality of the free-fall. It indicates that the worldlines of test particles in a gravitational field do not depend on the particle properties, but only on their gravitational environment.

iii. Einstein's equivalence principle (EEP): Fundamental non-gravitational test physics is not affected, locally and at any point of spacetime, by the presence of a gravitational field. In other words, a non-gravitational test experiment in a locally non-rotating, freely-falling frame in a gravitational field gives the same results as performed in an inertial frame in the absence of gravity.

iv. Strong equivalence principle (SEP): All test fundamental physics (including gravitational physics) is not affected, locally, by the presence of a gravitational field.

The EEP is often also stated as the joint requirement of the WEP, local position invariance, and the local Lorentz invariance for non-gravitational test experiments. Local position invariance means that the outcome of any local non-gravitational experiment is independent of where and when in the universe it is performed. Furthermore, with local Lorentz invariance we mean that the outcome of any local non-gravitational experiment is independent of the velocity of the free-falling reference frame in which it is performed.

Regardless of the specific formulation, the general idea at the core of EEP is the correspondence between local reference frames in a gravitational field and reference frames in the absence of gravity. But EEP says more; namely, that fundamental non-gravitational physics in a curved spacetime is locally Minkowskian. The procedure one should follow in order to introduce the gravitational interaction in any field theory built in flat space-time starts by taking the Lorentz invariant action of the theory and identify the coordinates appearing in it with that of the locally inertial system. Gravitational interaction will then appear once a coordinate change to an arbitrary system is made. For this, Einstein's vierbein or tetrad formalism is needed. After its introduction in Section 2.4.1, we will investigate how it affects the Standard Model Lagrangian. In the last section, Section 2.4.3, we include conformal invariance and establish the Conformal Standard Model.

2.4.1 Tetrad formalism

Recall from Section 2.1 that the affine connection $\Gamma^{\mu}_{\nu\rho}$ was introduced to implement translation invariance on a general manifold. To also implement Lorentz invariance a set of vierbeins[36] e^a_μ is introduced, where the coordinate a refers to a fixed, special-relativistic reference coordinate system with metric η_{ab} and the coordinate μ refers to general coordinate system with metric $g_{\mu\nu}$.

The vierbein field theory approach was proposed in 1928 by Einstein in his pursuit of a unified field theory of gravity and electricity. Full understanding of the principle is based on the mathematics of fiber bundles. However, here we limit ourselves to a mathematically less involved introduction of only the elements that concern our discussion, based on Yepez [16] (supported by Sundermeyer [14]).

The conventional coordinate-based approach to GR uses a 'natural' differential basis for the tangent space T_P at a point P given by the partial derivatives of the coordinates at P (the holonomic basis), i.e.

$$\hat{\boldsymbol{e}}_\mu = \partial_\mu, \qquad \hat{\boldsymbol{e}}^\mu = \mathrm{d}x^\mu\,, \qquad \hat{\boldsymbol{e}}^\mu \otimes \hat{\boldsymbol{e}}_\nu = \mathbf{1}^\mu{}_\nu \tag{2.60}$$

where we use a bold face symbol to denote a basis vector and apply a caret symbol to denote a *unit* basis vector. Also, the use of a Greek indices denotes a component in a (general) coordinate system representation. They are called holonomic (or world, or coordinate space) indices and are raised and lowered with the metric tensor $g_{\mu\nu}$. With this notation a contravariant vector $A \in T_P$ is $A = A^\mu \hat{\boldsymbol{e}}_\mu = (A_0, A_1, A_2, A_3)$ and a covariant vector $B = B_\mu \hat{\boldsymbol{e}}^\mu = g_{\mu\nu} B^\nu \hat{\boldsymbol{e}}^\mu = (B^0, B^1, B^2, B^3)$.

We are free to choose any orthonormal basis we like to span T_P, so long as it has the appropriate signature of the manifold on which we are working. Therefore, we use that we can find at any point P a local inertial frame for which the physical laws become those known from Minkowski space. In other words, we choose a basis such that the metric $g_{\mu\nu}$ becomes locally flat het point P. The vierbein field e^a_μ is then the 4×4 transformation matrix between these locally free-falling basis vectors $\hat{\boldsymbol{e}}_a$ (Riemann normal coordinates) and the conventional coordinate vectors.

$$\hat{\boldsymbol{e}}_\mu = e_\mu{}^a \hat{\boldsymbol{e}}_a = e_\mu{}^a \partial_a, \qquad \hat{\boldsymbol{e}}^\mu = e^\mu{}_a \hat{\boldsymbol{e}}^a = e^\mu{}_a\, \mathrm{d}\xi^a \tag{2.61}$$

where inertial coordinates ξ^a are labeled by latin letters, which are called the Minkowski, or tangent frame indices and can be raised and lowered with the Minkowski metric η_{ab}.

Because the vierbein field satisfies the orthonormality condition

$$e^\mu{}_a(x) e_\nu{}^a(x) = \delta^\mu_\nu, \qquad e_\mu{}^a(x) e^\mu{}_b(x) = \delta^a_b$$

the above equation can be reversed and we write the tetrad basis in terms of the coordinate basis

$$\hat{\boldsymbol{e}}_a = e^\mu{}_a \hat{\boldsymbol{e}}_\mu, \qquad \hat{\boldsymbol{e}}^a = e_\mu{}^a \hat{\boldsymbol{e}}^\mu, \qquad \hat{\boldsymbol{e}}^a \otimes \hat{\boldsymbol{e}}_n = \mathbf{1}^m{}_n \tag{2.62}$$

This way a line element can be written as

$$\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = \eta_{ab}\,\mathrm{d}\xi^a\,\mathrm{d}\xi^b \quad \Rightarrow \quad g_{\mu\nu} = \eta_{ab} e_\mu{}^a(x) e_\nu{}^b(x) \tag{2.63}$$

with $\eta_{mn} = \mathrm{diag}(+,-,-,-)$ (the mostly minus convention). Note that taking the determinant of this last expression, we see that the volume element is $\mathrm{d}^4\xi = \sqrt{|g|}\,\mathrm{d}^4x$, explaining the form

[36] In 4 dimensions we speak of vierbein or tetrad field, where the former is German for 'four legs'. The term 'local frame fields' is also used. There are generalizations to different dimensions, for example in three dimensions there is the triad or dreibein. In general, vielbein (German for 'many legs') is used to refer to a collection in arbitrary dimensions.

of (2.1) and subsequent equations.

Equation (2.63) imposes 10 constraints on the 16 components of the tetrad, leaving 6 components arbitrary. These 6 components are determined by local Lorentz transformations

$$\hat{\boldsymbol{e}}_a = e^\mu{}_a \hat{\boldsymbol{e}}_\mu \rightarrow \hat{\boldsymbol{e}}_{a'} = e^\mu{}_{a'} \hat{\boldsymbol{e}}_\mu = \Lambda^a{}_{a'}(x) e^\mu{}_a \hat{\boldsymbol{e}}_\mu \tag{2.64a}$$

$$\hat{\boldsymbol{e}}^a = e_\mu{}^a \hat{\boldsymbol{e}}^\mu \rightarrow \hat{\boldsymbol{e}}^{a'} = e_\mu{}^{a'} \hat{\boldsymbol{e}}^\mu = \Lambda^{a'}{}_a(x) e_\mu{}^a \hat{\boldsymbol{e}}_\mu \tag{2.64b}$$

where $\Lambda^a{}_{a'}(x)$ is a position-dependent transformation which, due to (2.63), satisfies the orthogonality condition $\Lambda^a{}_{a'}\Lambda^b{}_{b'}\eta_{ab} = \eta_{a'b'}$, i.e. transformed tetrads leave the metric invariant. Consequently, the Lorentz group can be regarded as the group of tetrad rotations in GR.

Any vector at a spacetime point has components in the coordinate and non-coordinate orthonormal basis related to each other by the vierbein field:

$$\boldsymbol{V} = V^\mu \hat{\boldsymbol{e}}_\mu = V^a \hat{\boldsymbol{e}}_a, \qquad V^a = e_\mu{}^a V^\mu, \qquad V^\mu = e^\mu{}_a V^a$$

A multi-index tensor can then have both Latin and Greek indices in its components:

$$T^{a\mu}{}_{b\nu} = e^\mu{}_m e_\nu{}^n T^{am}{}_{bn} = e_\alpha{}^a a^\beta{}_b T^{\alpha\mu}{}_{\beta\nu}$$

and a general coordinate transformation then becomes

$$T^{a\mu}{}_{b\nu} \rightarrow T^{a'\mu'}{}_{b'\nu'} = \Lambda^{a'}{}_a \frac{\partial x^{\mu'}}{\partial x^\mu} \Lambda^b{}_{b'} \frac{\partial x^\nu}{\partial x^{\nu'}} T^{a\mu}{}_{b\nu}$$

With the introduction of the tetrad basis we need to review again the covariant derivative as we explicitly used the coordinate basis in the derivation in section 2.1. In non-coordinate-based differential geometry, the derivation is the same except that the ordinary affine connection coefficients $\Gamma^\lambda_{\mu\nu}$ are replaced by the Lorentz or vector connection coefficients $\omega_\mu{}^a{}_b$, i.e. parallel transport of the vector $X^a(x)$ is given by

$$X^a(x) + \tilde{\delta} X^a(x) = X^a(x) - \omega_\gamma{}^a{}_b X^b(x) \delta x^\gamma$$

The non-coordinate based equivalent of equation (2.2) then becomes

$$\nabla_\gamma(\omega) X^a = \partial_\gamma X^a + \omega_\gamma{}^a{}_b X^b \tag{2.65}$$

where the dependence on x is to be understood. Comparing the above equation with (2.2), note that we use $\nabla_\mu(\Gamma)$ for the spacetime covariant derivative with respect to the linear connection Γ (e.g. $\nabla_\mu(\{\})$ for the Levi-Civita connection) and $\nabla_\mu(\omega)$ for the spacetime covariant derivative with respect to the Lorentz connection ω.

The covariant derivative for a generic (non-coordinate based) tensor $T^{a_1,\ldots,a_k}_{b_1,\ldots b_l}$ is then

$$\begin{aligned} \nabla_\gamma(\omega) T^{a_1,\ldots,a_k}_{b_1,\ldots b_l} &= \partial_\gamma T^{a_1,\ldots,a_k}_{b_1,\ldots b_l} \\ &+ \omega^{a_1}{}_{\alpha\gamma} T^{\alpha,a_2,\ldots,a_k}_{b_1,\ldots b_l} + \cdots + \omega^{a_k}{}_{\alpha\gamma} T^{a_1,\ldots,a_{k-1},\alpha}_{b_1,\ldots b_l} \\ &- \omega^\beta{}_{\gamma b_1} T^{a_1,\ldots,a_k}_{\beta,b_2,\ldots b_l} - \cdots - \omega^\beta{}_{\gamma b_l} T^{a_1,\ldots,a_k}_{b_1,\ldots b_{l-1},\beta} \end{aligned} \tag{2.66}$$

Similar to the affine connection coefficients which transform as (2.3) to ensure coordinate invariance of the covariant derivative, the Lorentz connections coefficients have to transform in a particular way as to ensure Lorentz invariance of the covariant derivative. Namely,

$$\omega_\mu{}^{a'}{}_{b'} = \Lambda^{a'}{}_a \omega_\mu{}^a{}_b \Lambda^b{}_{b'} - \Lambda^b{}_{b'} \partial_\mu \Lambda^{a'}{}_a$$

Because tensor equations are valid regardless of the bases, the two formalisms have to agree. Therefore, we express (2.65) in the coordinate basis and compare with (2.2) to establish the affine connection in terms of the Lorentz connection and vice versa

$$\Gamma^{\nu}_{\mu\lambda} = e^{\nu}{}_{a}\partial_{\mu}e_{\lambda}{}^{a} + e^{\nu}{}_{a}e_{\lambda}{}^{b}\omega_{\mu}{}^{a}{}_{b} \tag{2.67}$$

$$\omega_{\mu}{}^{a}{}_{b} = e_{\nu}{}^{a}e^{\lambda}{}_{b}\Gamma^{\nu}_{\mu\lambda} - e^{\lambda}{}_{b}\partial_{\mu}e_{\lambda}{}^{a} = -e^{\nu}{}_{b}\nabla_{\mu}(\Gamma)e_{\nu}{}^{a} \tag{2.68}$$

Left multiplying the last expression and rearranging terms gives:

$$\nabla_{\mu}(\Gamma,\omega)e_{\sigma}{}^{a} \equiv \partial_{\mu}e_{\sigma}{}^{a} - \Gamma^{\nu}_{\mu\sigma}e_{\nu}{}^{a} + \omega_{\mu}{}^{a}{}_{b}e_{\sigma}{}^{b} = 0 \tag{2.69}$$

where $\nabla(\Gamma,\omega)$ is used to signal a covariant derivative that 'sees' the holonomic indices with the affine connection, and all anholonomic indices with the Lorentz connection. (In some texts this derivative is abbreviated by a vertical bar $\nabla_{\mu}(\Gamma,\omega)T = T_{|\mu}$.) The above statement is known as the 'tetrad postulate' and states that the vierbein field is invariant under parallel transport.

Using the expression for the affine connection (2.67) in the definition of the torsion and curvature tensor, (2.5) and (2.6) respectively, we find

$$T^{\lambda}{}_{\mu\nu}(\Gamma) = e^{\lambda}{}_{a}T^{a}{}_{\mu\nu}(\omega) = \tfrac{1}{2}e^{\lambda}{}_{a}\left(\nabla_{\mu}(\omega)e_{\nu}{}^{a} - \nabla_{\nu}(\omega)e_{\mu}{}^{a}\right) \tag{2.70}$$

$$R^{\lambda}{}_{\sigma\mu\nu}(\Gamma) = e^{\lambda}{}_{a}e_{\sigma}{}^{b}R^{a}_{b\mu\nu}(\omega) = e^{\lambda}{}_{a}e_{\sigma}{}^{b}(\partial_{\mu}\omega_{\nu}{}^{a}{}_{b} - \partial_{\nu}\omega_{\mu}{}^{a}{}_{b} + \omega_{\mu}{}^{a}{}_{c}\omega_{\nu}{}^{c}{}_{b} - \omega_{\nu}{}^{a}{}_{c}\omega_{\mu}{}^{c}{}_{b}) \tag{2.71}$$

Similarly, the Ricci tensor and Ricci scalar definitions, (2.7) and (2.8), become

$$R^{a}_{\mu}(\omega) = R^{ab}{}_{\mu\nu}(\omega)e^{\nu}{}_{b} \qquad R = R^{a}{}_{\mu}e^{\mu}{}_{a} \tag{2.72}$$

2.4.2 The Standard Model in the presence of gravity

The next step is to look at the Standard Model in the presence of a gravitational field. The effect of gravity is encoded in the properties of spacetime itself as deviations from e.g. the Minkowski or Euclidean metric are the graviton field itself: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ where $h_{\mu\nu}$ is the graviton with spin 2 (two-index tensor).

In order to construct the actions of matter fields in an external gravitational field, the principles of locality and general covariance are imposed. It is also natural to forbid the introduction of new parameters with the dimension of inverse mass, besides renormalizability and simplicity. Following these three principles (locality, covariance and restricted dimension), the form of the action is fixed except the values of some new parameters which remain arbitrary (non-minimal scheme). We could furthermore follow the Einstein Equivalence Principle, i.e. we demand the symmetries of the original flat-space theory also hold for the general curved-space theory. This is referred to as 'minimal coupling'[37], which is the idea that the basic equations of physics in the presence of gravity differ from their counterparts in flat spacetime only by replacing partial derivatives for spacetime covariant derivatives, the flat metric for the generic metric $g_{\mu\nu}$ and the volume element $\mathrm{d}^4x \to \sqrt{g}\mathrm{d}^4x$.

Following the EEP to establish the minimal coupling of the Standard Model with gravity, we start with the Dirac action and identify the coordinates appearing in it with that of the locally inertial system. Specifically, we will treat local Lorentz invariance in a similar manner to gauge symmetries to establish the curved spacetime theory.

[37]Strictly speaking, the EEP is not identical to the minimal coupling scheme. The EEP requires that under given conditions physical phenomena in a sufficiently small region of spacetime unfold in the same way when there is, or is not, a gravitational field. This is a condition on solutions, rather than on equations unlike the idea of minimal coupling. The term 'minimal coupling' refers to the fact that the fields are coupled to the background metric only via the covariant derivative, which is obviously not the case for the non-minimal coupling scheme where explicit coupling terms are introduced.

Consider the Dirac equation for fermions in locally free-falling coordinates ξ^a:

$$(\mathrm{i}\gamma^a \partial_a - m)\psi = 0$$

where γ^a are the coordinate-invariant Dirac matrices which satisfy $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$. The spacetime-dependent Dirac matrices, $\gamma^\mu = e^\mu{}_a \gamma^a$, thus obey

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$

Following Yepez [16], we distinguish between the external Lorentz transformations that act on 4-vectors, $\Lambda^\mu{}_\nu$ of the Lorentz group $SO(3,1)$, and the internal Lorentz transformations that act on spinor wave functions, $U(\Lambda)$ of the spinor representation of the $SU(4)$ group. These commute: $[\Lambda^\mu{}_\nu, U(\Lambda)] = 0$. Explicitly, the internal Lorentz transformation of a quantum spinor field in unitary form is

$$U(\Lambda) = e^{\frac{1}{2}\lambda_{ab}(x)G^{ab}} \approx 1 + \tfrac{1}{2}\lambda_{ab}(x)G^{ab}$$

where $\lambda_{ab} = -\lambda_{ba}$ is the group parameter and G^{ab} is the tensor generator of the transformation, which appears in the representation corresponding to the object that is transformed (vector, tensor, scalar, etc.). The fundamental representation of SU(4) are the $4^2 - 1 = 15$ Dirac matrices, which include four vectors γ^a, six tensors $\sigma^{ab} = \frac{1}{4}[\gamma^a, \gamma^b]$, one pseudo scalar $\gamma^5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3$, and four axial vectors $\gamma^5\gamma^a$. The generator associated with the internal Lorentz transformation of a fermion (spinor) is $G^{ab} = \sigma^{ab}$. With these formulae, the invariance of the Dirac equation under a Lorentz transformation can be explicitly checked.

Next we make the transformation local for a general spacetime, and we replace the partial derivative with the spinor covariant derivative

$$\boldsymbol{\nabla}_\mu = \partial_\mu + \boldsymbol{\Gamma}_\mu \tag{2.73}$$

we get, in the spirit of a local gauge theory for the special Lorentz group, the generally covariant expression

$$(\mathrm{i}\gamma^a e^\mu{}_a \boldsymbol{\nabla}_\mu - m)\psi = 0 \tag{2.74}$$

where the Lorentz transformations of a Lorentz 4-vector x^a, 4-spinor ψ and γ^μ are

$$x^{a\prime} = \Lambda^a{}_b x^b, \quad \psi' = U(\Lambda)\psi, \quad \bar{\psi}' = \bar{\psi}U(\Lambda)^{-1}, \quad \gamma^{\mu'} = U(\Lambda)\gamma^\mu U(\Lambda)^{-1} \tag{2.75}$$

and the spinor connection $\boldsymbol{\Gamma}_\mu$ transforms according to

$$\boldsymbol{\Gamma}'_\mu = U(\Lambda)\boldsymbol{\Gamma}_\mu U(\Lambda)^{-1} + (\partial_\mu U(\Lambda))U(\Lambda)^{-1}$$

To derive the form of the spinor connection, we look at the parallel transport of the Lorentz transformation matrix. From the Lorentz transformation of the vierbein (2.64) we have

$$e_\mu{}^{a'}(x^\alpha) e^\mu{}_b(x^\alpha) = \Lambda^{a'}{}_b(x^\alpha)$$

Because the left-hand side has two upper indices of a different kind, the Latin non-coordinate index a' and the Greek coordinate index μ, we need to take extra care in treating them with respect to parallel transport. Specifically, a Taylor expansion can be used to connect a quantity in its Latin non-coordinate index at one point to a neighboring point, but the affine connection must be used for the Greek coordinate index:

$$\begin{aligned}\Lambda^{a'}{}_b(x^\alpha + \delta x^\alpha) &= e_\mu{}^{a'}(x^\alpha + \delta x^\alpha) e^\mu{}_b(x^\alpha + \delta x^\alpha) \\ &= \left(e_\mu{}^{a'}(x^\alpha) + \frac{\partial e_\mu{}^{a'}}{\partial x^\alpha}\delta x^\alpha \right) \left(e^\mu{}_b(x^\alpha) - \Gamma^\mu{}_{\beta\alpha}(x^\alpha) e^\beta{}_b(x^\alpha)\delta x^\alpha \right)\end{aligned}$$

$$
\begin{aligned}
&= \delta_b^{a'} + \left(e^{\mu}{}_b \partial_\alpha e_\mu{}^{a'} - \Gamma^\mu{}_{\beta\alpha} e_\mu{}^{a'} e^\beta{}_b \right) \delta x^\alpha \\
&= \delta_b^{a'} - \omega_\alpha{}^{a'}{}_b \delta x^\alpha
\end{aligned}
$$

where equation (2.4) was used in the second line, the explicit dependence on x^α was dropped in line 3, and the Lorentz connection in the last line is given by (2.68). Writing the Lorentz transformation matrix in its infinitesimal form, gives the group parameter λ_{ab}:

$$
\Lambda^a{}_b = \delta^a{}_b + \lambda^a{}_b = \delta^a{}_b - \omega_\alpha{}^a{}_b \delta x^\alpha = \delta^a{}_b + e^\nu{}_b (\nabla_\alpha(\Gamma) e_\nu{}^a) \delta x^\alpha
$$

Next, recall the Lorentz transformation of a spinor (2.75)

$$
U(\Lambda)\psi = \left(1 + \tfrac{1}{2}\lambda_{ab}(x) G^{ab}\right)\psi = \psi + \delta\psi \quad \text{where} \quad \delta\psi = -\tfrac{1}{2}(\omega_\alpha)_{ab}\delta x^\alpha G^{ab}\psi = \mathbf{\Gamma}_\alpha \psi \delta x^\alpha
$$

The generalized derivative that is needed to correctly differentiate the Dirac 4-spinor field in curved space time is thus the spinor covariant derivative (2.73) with the spinor connection

$$
\mathbf{\Gamma}_\mu = -\tfrac{1}{2}(\omega_\mu)_{ab} G^{ab} = \tfrac{1}{2} e^\nu{}_b (\nabla_\alpha(\Gamma) e_{\nu a}) G^{ab} = \tfrac{1}{2} e^\nu{}_b (\partial_\alpha e_{\nu a}) G^{ab}, \quad G^{ab} = \tfrac{\mathrm{i}}{2}[\gamma^a, \gamma^b] \tag{2.76}
$$

Because the spinor covariant derivative of the γ matrices is zero, the Dirac equation in curved spacetime with the above definition of the spinor covariant derivative is obtained by variation with respect to $\bar{\psi}$ of the Lagrangian[38]

$$
\mathcal{L}_{\text{Dirac}} = \tfrac{\mathrm{i}}{2}\left(\bar{\psi}\gamma^\mu (\mathbf{\nabla}_\mu \psi) - \bar{\mathbf{\nabla}}_\mu \bar{\psi}\gamma^\mu \psi \right) - m\bar{\psi}\psi = \tfrac{\mathrm{i}}{2}\bar{\psi}\gamma^\mu \overset{\leftrightarrow}{\mathbf{\nabla}}_\mu \psi - m\bar{\psi}\psi \tag{2.77}
$$

Following the above arguments we can write the complete Standard Model Lagrangian, which was developed in Section 1.2 in flat spacetime, now in curved spacetime. First the fermion Lagrangian (1.18), which becomes

$$
\begin{aligned}
S_{\text{Fermion}} = \int \mathrm{d}^4x \sqrt{g} \Big[& \left(\bar{\nu}_L, \bar{e}_L\right)_L \mathrm{i}\not{D} \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} + \bar{\nu}_R \mathrm{i}\not{D} \nu_R + \bar{e}_R \mathrm{i}\not{D} e_R \\
& + \left(\bar{u}_L, \bar{d}_L\right) \mathrm{i}\not{D} \begin{pmatrix} u_L \\ d_L \end{pmatrix} + \bar{u}_R \mathrm{i}\not{D} u_R + \bar{d}_R \mathrm{i}\not{D} d_R \Big]
\end{aligned} \tag{2.78}
$$

where summation over all leptons and quarks is assumed. The gauge covariant derivative (1.19) is extended to include the spinor covariant derivative as understood in (2.77):

$$
\begin{aligned}
\mathcal{D}_\mu = \overset{\leftrightarrow}{\mathbf{\nabla}}_\mu + \tfrac{\mathrm{i}\eta g}{2}\left[\sigma_+ W_\mu^+ + \sigma_- W_\mu^-\right] + \mathrm{i}\eta_e e Q A_\mu \\
+ \tfrac{\mathrm{i}\eta g}{\cos(\theta_W)}\left[\tfrac{\sigma_3}{2} - Q\sin^2(\theta_W)\right]\eta_z Z_\mu + \tfrac{\mathrm{i}\eta_s g_s}{2}\lambda_a G_\mu^a
\end{aligned} \tag{2.79}
$$

[38] If we include torsion, but still work with a metric compatible metric, the Dirac equation is written as

$$
\mathcal{L}_{\text{Dirac}} = \mathrm{i}\bar{\psi}\left(\gamma^\mu\partial_\mu - \tfrac{\mathrm{i}}{2}\gamma^\mu\{\mu ab\}G^{ab} - \tfrac{\mathrm{i}}{8}\gamma^5\gamma^\mu S_\mu\right)\psi - m\bar{\psi}\psi
$$

which is the same result as e.g. equation 8.15 from Dobado et al. [61]. It can be established by inserting the general affine connection (2.11) in the expression for the spin connection (2.68), but keeping the nonmetricity equal to zero:

$$
\omega_\mu{}^a{}_b = e_\nu{}^a e^\lambda{}_b \Gamma^\nu_{\mu\lambda} - e^\lambda{}_b \partial_\mu e_\lambda{}^a, \quad \Gamma^\mu{}_{\nu\lambda} = \{{}^{\mu}_{\nu\lambda}\} + K^\mu{}_{\nu\lambda}
$$

and S_μ is defined as the axial part of the torsion tensor $S_\rho = \epsilon_{\mu\nu\lambda\rho}T^{\mu\nu\rho}$.

There is no consensus on the status of torsion within a (quantum) theory of gravity. Arguments for keeping it include that quantum effects or modification of Einstein gravity to include higher derivative terms could produce torsion. More important, Friedrich W. Hehl [84] already advocated in 1979 that where mass is coupled to the metric via the energy-momentum tensor, the other elementary notion needed to describe an elementary particle, spin, should also be coupled, namely to a geometrical quantity related to rotational degrees of freedom in space-time which is the contortion tensor $K^\mu{}_{\nu\lambda}$. More recently, Luca Fabbri argues for a conformal theory including torsional degrees of freedom, see e.g. [85]. Some other interesting papers on the subject are e.g. [86–89]. Because the inclusion of torsion, especially in a nonminimal scheme, leads to an increasing number of possible terms that can be included in the Lagrangians, we will not consider it further.

Gravity enters only via the above covariant derivative (minimal coupling) as there are no non-minimal terms algebraically possible.

In contrast, scalar contributions are not modified in the minimal scheme because they do not transform under Lorentz transformations: $G^{ab} = 0$ and the spacetime covariant derivative is just an ordinary partial derivative. However, a nonminimal generalization of the scalar action is

$$S_{\text{Scalar}} = \tfrac{1}{2} \int \mathrm{d}^4x \sqrt{g} \left(g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - m^2 \phi^2 - \xi \phi^2 R \right) \tag{2.80}$$

where ξ is a new dimensionless quantity called the nonminimal parameter with $\xi = 0$ corresponding to the minimal coupling. The reason for considering a nonminimal term of the form $R\phi^2$ and not some other extension is twofold. Firstly, we are interested in the simplest generalization from a special relativistic field theory to a general space-time and are only inclined to consider one of the unlimited number of other possible additions when there is other evidence suggesting we should do so. For this term there is such evidence, because the renormalization of an interacting field in curved spacetime necessarily involves a counterterm proportional to $R\phi^2$.The second reason for considering this term in the nonminimal scheme is its relevance to conformal invariance, as we will see in the next section.

This nonminimal extension of the scalar sector only influences the Higgs sector. So the Higgs and Yukawa sectors of the Standard Model in the presence of a gravitational field become:

$$S_{\text{Higgs + Yukawa}} = \int \mathrm{d}^4x \sqrt{g} \left(g^{\mu\nu} (D_\mu \mathcal{H})^\dagger (D_\nu \mathcal{H}) - V(\mathcal{H}^\dagger \mathcal{H}) - \xi \mathcal{H}^\dagger \mathcal{H} R + \mathcal{L}_{\text{Yukawa}} \right) \tag{2.81}$$

where D_μ and $\mathcal{L}_{\text{Yukawa}}$ are given by (1.19) and (1.33), respectively.

The last sector we need to consider is the gauge sector in flat spacetime which is given by the Yang-Mills sector (1.20). Gauge fields transform under Lorentz transformations as ${G_{ab}}^i{}_j = \delta^i_a \eta_{bj} - \delta^i_b \eta_{aj}$. The spinor covariant derivative of a gauge field thus becomes:

$$\begin{aligned}
\partial_a A_j \to \nabla_a A_j &= e^\mu_a \big(\partial_\mu - \tfrac{1}{2} {\omega_\mu}^{ab} G_{ab} \big) A_i \\
&= e^\mu_a \partial_\mu \big(e^\nu_j A_\nu \big) - \tfrac{1}{2} e^\mu_a ({\omega_\mu}^i{}_j - {\omega_{\mu j}}^i) e^\nu_i A_\nu \\
&= e^\mu_a e^\nu_j \partial_\mu A_\nu + e^\mu_a (\partial_\mu e^\nu_j - {\omega_\mu}^i{}_j e^\nu_i) A_\nu
\end{aligned}$$

In combination with the tetrad postulate (2.69), this gives $F_{aj} = e^\mu_a e^\nu_j (\partial_\mu A_\nu - \partial_\nu A_\mu)$, meaning that the product $F^a_{mn} F^{mn}_a$ is invariant under Lorentz transformations. The Yang-Mills sector the requires thus only replacing the flat metric with the general metric:

$$S_{\text{YM}} = -\tfrac{1}{4} \int \mathrm{d}^4x \sqrt{g} g^{\mu\rho} g^{\nu\sigma} \left(B_{\mu\nu} B_{\rho\sigma} + W^i_{\mu\nu} W^i_{\rho\sigma} + G^a_{\mu\nu} G^a_{\rho\sigma} \right) \tag{2.82}$$

where field strength tensors of $SU(3)_c, SU(2)_L, U(1)_Y$ are given by equations (1.3), (1.10) and (1.11).

2.4.3 A conformal toy model

Having established the form of the Standard Model in the presence of a nontrivial metric $g_{\mu\nu}$, it is now time to impose the conformal restrictions and include the gravity sector. Conformal invariance includes scale invariance meaning mass terms or other scales explicitly break the symmetry. Therefore no mass scales - no gravitational constant, no mass for the Higgs field, no cosmological constant and no mass parameters for the quarks, leptons or gauge bosons - can be included in the theory. This requirement means that the tachyonic Higgs mass term in the Higgs sector and the Majorana mass term in the Yukawa sector cannot be included in the Conformal Standard Model.

Next we need to investigate the behavior of the different sectors under a conformal transformation (2.28):

$$\Phi(x) \to \Omega^{\tilde{\Delta}_\Phi}\Phi(x), \quad e_\mu{}^a \to \Omega(x) e_\mu{}^a, \quad \text{such that} \quad g_{\mu\nu} = \eta_{ab} e_\mu{}^a e_\nu{}^b \to \Omega(x)^2 g_{\mu\nu} \tag{2.83}$$

where we included explicitly the behavior of the vierbein field under a conformal transformation. Recalling the Weyl weight of the fields from Section 2.2.1, we can readily see that the Yang-Mills action transforms in a conformally invariant way:

$$S'_{\mathrm{YM}} = \int \mathrm{d}^4x \sqrt{\Omega^8 g}(\Omega^{-2} g^{\mu\rho})(\Omega^{-2} g^{\nu\sigma})(\Omega^0 F_{\mu\nu})(\Omega^0 F_{\rho\sigma}) = S_{\mathrm{YM}}$$

We will not demonstrate conformal invariance of the fermion sector explicitly, but instead refer to [41, Section 5].

That leaves the Higgs sector of the Standard Model. A conformal invariant scalar action requires a nonminimal coupling term where $\xi = \frac{d-2}{4(d-1)}$ in d spacetime dimensions, i.e. in four spacetime dimensions

$$S_{\mathrm{Higgs}} = \tfrac{1}{2} \int \mathrm{d}^4x \sqrt{g} \left(g^{\mu\nu} (D_\mu \mathcal{H})^\dagger D_\nu \mathcal{H} + \tfrac{1}{6} \mathcal{H}^\dagger \mathcal{H} R \right) \tag{2.84}$$

which can be derived upon considering the equation of motion of the nonminimal, massless scalar theory (2.80) (see Appendix F). Curiously, this value can also be argued on the basis of the EEP and thus without reference to any conformal argument [62].

Now we have all the ingredients for the construction of a Conformal Standard Model. There remains one aspect to be discussed and that is the coupling with Gravity. In the above equations we have minimal coupling in the Yukawa, fermion and Yang-Mills sectors and nonminimal coupling with the Ricci scalar curvature in the scalar sector. We could furthermore include the CDG action (2.55) of the previous section either by 1) identifying the Higgs boson with the (unphysical) dilaton or 2) include the (unphysical) dilaton as an additional field. Another option is to 3) include the dilaton as an additional physical scalar field. The remainder of this section is devoted to investigating these different options.

Suppose we identify the Higgs doublet with the dilaton[39]:

$$\mathcal{H} = \frac{1}{\sqrt{2}} U^{-1}(\zeta) \begin{pmatrix} 0 \\ \phi(x) \end{pmatrix} \quad \text{with} \quad U^{-1}(\zeta) = e^{\frac{-\mathrm{i}\zeta_i(x)T^i}{v}}$$

where the 3 Goldstone bosons that appear in the exponent can be eliminated by a local $SU(2)$ transformation fixing the unitary gauge for the Higgs field. In the broken phase of the theory the scalar field in (2.84) is gauge fixed, following the procedure of Appendix E, to a fixed value to retrieve the familiar Einstein-Hilbert action. We need to choose $\phi = v$:

$$\frac{1}{2\kappa^2} = \frac{\xi v^2}{2} \quad \Rightarrow \quad G_N = \frac{1}{8\pi}\frac{1}{\xi v^2} \tag{2.85}$$

Because Newton's constant $G_N = M_p^{-2}$ (we work in $\hbar = c = 1$), we get that v is on the order of the Planck mass ($\sim 10^{19}$ GeV), which is inconsistent with the experimental value of the electroweak VEV $v \approx 246$ GeV. Therefore, we cannot identify this conformally coupled scalar with the Higgs field, but need to allow it as an extra field to the Standard Model and Gravity.

[39]This model was actually proposed by e.g. Pawlowski and Raczka [63] because the Higgs boson at that time was still not experimentally verified leading scientists to doubt its existence altogether. In the model of Pawlowksi one gets the mass generation without the mechanism of spontaneous symmetry breaking and without the remaining real dynamical Higgs field.

A two scalar theory is:

$$
\begin{aligned}
S^{\pm}_{\text{CSMG}} = \int \mathrm{d}^4x\sqrt{g}\Big[&-\tfrac{1}{4}g^{\mu\rho}g^{\nu\sigma}\left(B_{\mu\nu}B_{\rho\sigma} + W^i_{\mu\nu}W^i_{\rho\sigma} + G^a_{\mu\nu}G^a_{\rho\sigma}\right) && \text{(Yang-Mills sector)} \\
&+ \begin{pmatrix}\bar{\nu}_L & \bar{e}_L\end{pmatrix} \mathrm{i}\hat{\not{\mathcal{D}}} \begin{pmatrix}\nu_L \\ e_L\end{pmatrix} + \bar{\nu}_R \mathrm{i}\hat{\not{\mathcal{D}}}\nu_R + \bar{e}_R\mathrm{i}\hat{\not{\mathcal{D}}}e_R && \text{(lepton dynamical terms)} \\
&+ \begin{pmatrix}\bar{u}_L & \bar{d}_L\end{pmatrix} \mathrm{i}\hat{\not{\mathcal{D}}} \begin{pmatrix}u_L \\ d_L\end{pmatrix} + \bar{u}_R \mathrm{i}\hat{\not{\mathcal{D}}}u_R + \bar{d}_R\mathrm{i}\hat{\not{\mathcal{D}}}d_R && \text{(quark dynamical terms)} \\
&- \begin{pmatrix}\bar{\nu}_L & \bar{e}_L\end{pmatrix}\mathcal{H}\Gamma^e e_R - \begin{pmatrix}\bar{\nu}_L & \bar{e}_L\end{pmatrix}\mathrm{i}\sigma_2\mathcal{H}^*\Gamma^\nu\nu_R + \text{h.c.} && \text{(lepton mass terms)} \\
&- \begin{pmatrix}\bar{u}_L & \bar{d}_L\end{pmatrix}\mathcal{H}\Gamma^d d_R - \begin{pmatrix}\bar{u}_L & \bar{d}_L\end{pmatrix}\mathrm{i}\sigma_2\mathcal{H}^*\Gamma^u u_R + \text{h.c.} && \text{(quark mass terms)} \\
&\pm \tfrac{1}{2}g^{\mu\nu}\partial_\mu\chi\partial_\nu\chi + g^{\mu\nu}(D_\mu\mathcal{H})^\dagger D_\nu\mathcal{H} && \text{(scalar dynamical terms)} \\
&+ \tfrac{1}{12}(\pm\chi^2 + 2\mathcal{H}^\dagger\mathcal{H})R - V(\mathcal{H},\chi)\Big] && \text{(scalar interactions)}
\end{aligned}
\tag{2.86}
$$

where the different covariant derivatives are[40]:

$$
\begin{aligned}
D_\mu &= \partial_\mu + \mathrm{i}\eta g W^i_\mu T^i + \mathrm{i}\eta' g'\eta_Y Y B_\mu \\
\mathcal{D}_\mu &= D_\mu - \tfrac{\mathrm{i}}{4}(\omega_\mu)_{ab}[\gamma^a,\gamma^b]
\end{aligned}
$$

The above toy model is invariant under the gauge symmetry of $SU_c(3)\times SU_L(2)\times U_Y(1)$ and local Weyl rescalings in 4 spacetime dimensions:

$$
g_{\mu\nu}\to\Omega^2(x)g_{\mu\nu},\quad \chi\to\Omega^{-1}\chi,\quad \mathcal{H}\to\Omega^{-1}\mathcal{H},\quad \psi\to\Omega^{-\frac{3}{2}}\psi,\quad A_\mu\to\Omega^0 A_\mu \tag{2.87}
$$

where ψ is a Dirac fermion field (either a lepton or a quark) and A_μ is a general gauge boson (either the photon, gluon, $W^\pm$ or Z boson).

In case we choose the minus sign S^-_{CSMG} we have one scalar field which comes from the Conformal Standard Model and the other is the unphysical dilaton from Conformal Dilaton Gravity was for example considered by Bars [66]. The term $\frac{1}{12}R(\chi^2 - 2\mathcal{H}^\dagger\mathcal{H})$ leads to Newton's constant and allows us to find General Relativity back. To allow for the possibility for it to be positive, we must indeed have one scalar, namely the dilaton ϕ, to have the wrong sign kinetic term, otherwise we would end up with a purely negative gravitational parameter. Furthermore, the relative minus sign between the two scalars enables us to cover all patches of field space that are required for geodesic completeness[41] of all cosmological solutions for all times and any set of initial conditions [66].

A proposal for the specific form of the potential $V(\mathcal{H},\chi)$ will follow in the next chapter. Possible gauge-fixing terms as well as the compensating Faddeev-Popov ghosts will not be explicitly considered. We also, again, note that this toy model does not include a kinetic term for the gravitational field. Recall from the discussion in Section 2.3.2 that the only possibility is the Weyl tensor squared from the Conformal Gravity action (2.47). See e.g. Oda [57] for the toy model including this term.

[40] It is possible that these gauge covariant derivatives also include the Weyl gauge field as e.g. proposed by [64, 65].

[41] Geodesic completeness refers to two notions. Firstly, it means geodesic continuation through all singularities separating patches of spacetime. Secondly it also means avoiding unnatural initial conditions by requiring infinite action for geodesics that reach arbitrarily far in the past. Though inequivalent, both notions are satisfied for the model which we are developing in this section [67].

The dilaton in the above model S^{-}_{CSMG} (2.86) is an unphysical but otherwise regular scalar field if we follow the interpretation of [49, 67]. On the other hand, we could also follow 't Hooft [11]. Then the metric displayed is actually a metatensor from which all scale dependences are contained in the unphysical, 'metascalar' field ω, also called the dilaton. The metric's behavior under coordinate transformations thus differs depending on the method we consider. Taking into account these differences, both methods lead to the same final toy model as long as the potential $V(\hat{\omega}, \mathcal{H})$ remains unspecified. However, to distinguish between the different interpretations, we use the superscript 'split' ($S^{\mathrm{split}}_{\mathrm{CSMG}}$) to signal that we talk about the model in the interpretation by 't Hooft.

Last but not least, we could also go for the plus sign in (2.86). The dilaton is then an additional real scalar field, which in its simplest form would be a gauge singlet.

Chapter 3

Scales in a scaleless theory

In the previous chapter we established different toy models where the Standard Model, Gravity and conformal invariance are combined. The two models differ in the meaning of the dilaton: in CSMG$^-$ the dilaton is a ghost particle, whereas it is a real scalar field in CSMG$^+$. This chapter is devoted to investigating how scales can be generated in a theory which is classically without scales. Some symmetry breaking mechanism needs to be responsible and the possibilities are discussed in Section 3.1. In Section 3.2 and Section 3.3 symmetry breaking in the two different toy models is discussed, where the former includes special attention to the interpretation of the dilaton by 't Hooft. In the last section we will compare the two models.

3.1 Origin of mass

The Nambu and Jona-Lasinio (NJL) model describes how the mass of the nucleon, then assumed to be an elementary particle, is generated dynamically from the vacuum through the nucleon-antinucleon pair condensate. This idea is now used in the mass generation mechanism of composite particles via the quark-antiquark condensate in QCD. The model thus accounts for 99% of the mass of the visible world. The remaining 1% of the mass, the masses of the elementary particles themselves, are generated by the Higgs boson via the spontaneous symmetry breaking of the $SU(2)_L \times U(1)_Y$ gauge symmetry as induced by the explicit mass term of the Higgs boson. However, saying that the Higgs boson gives masses to *all* elementary particles in the SM is incorrect because the source of its own mass as well as the (possibly different) source of the neutrino masses in the Standard Model are still unclear.

The Higgs boson mass found at the Large Hadron Collider as well as subsequent experiments are in agreement with the Brout-Englert-Higgs-mechanism of the Standard Model. This means we could accept the Higgs field with is tachyonic mass in $\mathcal{L}_{\text{SM}}$ as a fundamental field. However, while the Standard Model is not contradicting with the current data at the LHC within the error, some extended Higgs sectors can also reproduce the data. Besides, if the standard model is correct, the measured values e.g. the mass of the Higgs boson, imply the Universe is metastable [68]. This again would imply that the Standard Model cannot be valid all the way to the Planck scale, and that new particles and interactions must contribute to the scalar potential.

However, here we work under the assumption that both the Standard Model and Einstein Gravity are not the final theories but rather effective theories arising from breaking of conformal invariance of toy model. Such conformal symmetry breaking can arise in several ways. For example it can be collateral damage of the spontaneous breaking of the GUT and/or electroweak gauge symmetries, as the fermions and gauge bosons in the conformal theory acquire masses and thereby spoil the conformal symmetry. The breaking mechanism could, on the contrary, be preceding or even completely unrelated to the electroweak breaking.

There are two types of breaking of a symmetry in general, namely

i. Explicit breaking at the classical level occurs when a term in the Lagrangian is not invariant. When the theory possess a symmetry when its dynamics is analyzed in terms of unquantized, commuting variables, the symmetry may disappear when the dynamics is quantized and analysis is performed in terms of non-commuting quantum variables. In other words, quantization of the symmetric classical theory can lead to anomalies that explicitly break the symmetry. This is called to anomalous symmetry breaking.

ii. Spontaneous symmetry breaking occurs when the classical Lagrangian is invariant but the ground state is not. Specifically, we have to be in a situation in which the ground state is such that the symmetry is still a property of the system when the fields are in an unstable stationary point (saddle point) of the potential; because of the instability, the configuration of fields will spontaneously tend to the stable stationary point of the potential, which is, however, not invariant. The mechanism of spontaneous symmetry breaking is described by the Goldstone mechanism and the Brout-Englert-Higgs mechanism in case of a global and local symmetry breaking, respectively. It is also possible to have a symmetric classical Lagrangian but that quantization leads to radiative corrections that spontaneously break the symmetry. This is for example achieved via the Coleman-Weinberg breaking mechanism.

Explicit breaking at the classical level would invalidate the conclusions based on the restrictions from conformal invariance and will therefore not be considered. However, as is well known (and further explained in the **??**), quantization of a classically conformal lagrangian gives rise to a trace anomaly which anomalously breaks the conformal symmetry. As there are ways of restoring the conformal symmetry at quantum level, we are left with investigating how scales are generated in a quantum conformal theory. The remainder of this section is therefore devoted to explaining the Coleman-Weinberg mechanism in Section 3.1.1 and its extension to the multiple scalar case, the Gildener-Weinberg formalism, in Section 3.1.2.

3.1.1 The Coleman-Weinberg mechanism

Sidney Coleman and Erick Weinberg [69] (for a review see also [70]) investigated symmetry breaking in classical massless theories as to replace the BEH mechanism based on the rather ugly tachyonic Higgs mass term. In their paper, they showed that for QED, even though the minimum of the interaction potential is zero at the tree-level, the radiative corrections at the one-loop level to the (effective) potential change the potential to a mexican-hat-type hence inducing spontaneous symmetry breaking. Here we will demonstrate the general features of the Coleman-Weinberg (CW) mechanism using the trivial ϕ^4-theory from Section 1.3:

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu\phi)^2 - \tfrac{1}{2}m^2\phi^2 - \tfrac{1}{4!}\lambda\phi^4 + \tfrac{1}{2}\Delta_\phi(\partial_\mu\phi)^2 - \tfrac{1}{2}\Delta_m\phi^2 - \tfrac{1}{4!}\Delta_{\phi^4}\phi^4 \tag{3.1}$$

where $\Delta_\phi, \Delta_m, \Delta_\lambda$ are the counter-terms of the wave-function, mass and the coupling constant. This Lagrangian is symmetric under the transformation $\phi \to -\phi$ and the vacuum shares that symmetry for $m^2 > 0$, but the symmetry is spontaneously broken for $m^2 < 0$, as can be recalled from our discussion of the Higgs mechanism. Here, we focus on the borderline case of a classically massless theory, i.e. $m^2 = 0$.

Classically, the positivity of the quartic term would be sufficient to guarantee a symmetric vacuum. However, in a quantum field theory the vacuum energy includes the zero-point energies of the various fields that enter the theory and these zero-point energies can potentially change this situation. Thus, to investigate spontaneous symmetry breaking in flat spacetime we require the quantum corrected potential, also known as the effective potential V_{eff}, of the theory. The minima of the effective potential give, without any approximation, the true vacuum states of the theory.

Recall the effective action (1.43):

$$\Gamma[\Phi_{cl}] = W[J] - \int \mathrm{d}^4x J(x)\Phi_{cl}(x) \quad \text{where} \quad \Phi_{cl} = \frac{\delta W[J]}{\delta J(x)}$$

which is the generating functional for all the 1-particle irreducible (1PI) correlation functions $\Gamma^{(r)}(x_1, \ldots, x_r)$ and consequently encodes the full quantum dynamics of the theory in a classical language.

One of the leading procedures to determine the structure of the effective potential is the loop expansion method in which one finds the contributions coming from each separate part of the diagrams order by order (i.e., starting from tree-level to r-loops) and then sum them to find the result with the desired precision.

$$\Gamma[\Phi_{cl}] = \mathrm{i}\sum_r \frac{1}{r!}\int \mathrm{d}^4x_1 \ldots \mathrm{d}^4x_r \Gamma^{(r)}(x_1, \ldots, x_r)\Phi_{cl}(x_1)\ldots\Phi_{cl}(x_r) \tag{3.2}$$

Alternatively, one can also express the effective action as a momentum (or derivative) expansion. The general form of the effective action then consists of the standard two-derivative kinetic term multiplied by some non-trivial wave-function factor, an effective potential without derivatives, and in general also an infinite series of higher-derivative corrections:

$$\Gamma[\Phi_{cl}] = \int \mathrm{d}^4x \Big(Z_{\text{eff}}[\Phi_{cl}]\partial_\mu\Phi_{cl}\partial^\mu\Phi_{cl} - V_{\text{eff}}[\Phi_{cl}] + \ldots \Big) \tag{3.3}$$

In particular, if we study $\Phi_{cl}(x) = \Phi_{cl}$ (constant), then only the first term contributes to the effective action, which is called the effective potential and in the tree approximation it is just the classical potential. By comparing the two expansions, it can be seen that the nth derivative of V_{eff} is the sum of all 1PIs diagrams with vanishing momentums of external legs. In other words, the effective action and the effective potential contain all the effective vertices among the fields Φ_{cl} which are induced by quantum fluctuations through loops, respectively at any momentum and at zero-momentum.

The usual renormalization conditions of perturbation theory can be expressed in terms of the functions that occur in the above equation. For example, we define the squared mass of the meson as the value of the inverse propagator at zero momentum[42], and the coupling constant equals the four-point function at zero external momentum. However, massless theories suffer from IR divergencies in the Green's functions when $\Phi_{cl} \to 0$, thus requiring us to enforce the renormalization conditions at an arbitrary scale μ:

$$m^2 = 0 = \left.\frac{\mathrm{d}^2V_{\text{eff}}}{\mathrm{d}\Phi_{cl}^2}\right|_{\Phi_{cl}=\mu}, \quad \lambda = \left.\frac{\mathrm{d}^4V_{\text{eff}}}{\mathrm{d}\Phi_{cl}^4}\right|_{\Phi_{cl}=\mu}, \quad Z_{\text{eff}}[\Phi_{cl} = \mu] = 1 \tag{3.4}$$

With the general theory outlined, we turn back to our scalar theory. Specifically, the effective action is just the classical action. Furthermore, we had determined the 1-loop effective action (1.48) in Section 1.3.1:

$$\Gamma^1[\Phi_{cl}] = \frac{\mathrm{i}}{2}\operatorname{Tr}\log\left(\Box + \frac{\lambda}{2}\Phi_{cl}^2\right) + \tfrac{1}{2}\Delta_\phi\partial_\mu\Phi_{cl}\partial^\mu\Phi_{cl} - \tfrac{1}{2}\Delta_m^1\Phi_{cl}^2 - \tfrac{1}{4!}\Delta_\lambda^1\Phi_{cl}^4 \tag{3.5}$$

where we need to keep the mass renormalization term, even though we are have a massless theory, because the theory possesses no symmetry that would guarantee vanishing bare mass in the limit of vanishing renormalized mass.

[42]If we were interested in higher order corrections this in general gauge-dependent definitions needs to be replaced by the gauge-invariant definition of particle masses as the locations of poles in propagators. At the order of our current interest, the two definitions coincide.

The tree level effective potential is then the classical potential and the 1-loop effective potential is computed by considering the 1-loop effective action for constant configurations for Φ_{cl}.

$$V^0_{\text{eff}} = \tfrac{1}{4!}\lambda\Phi^4_{cl} \quad V^1_{\text{eff}} = -\tfrac{\mathrm{i}}{2}\,\mathrm{Tr}'\log\!\left(\Box + \tfrac{1}{2}\lambda\Phi^2_{cl}\right) + \tfrac{1}{2}\Delta^1_m\Phi^2_{cl} + \tfrac{1}{4!}\Delta^1_\lambda\Phi^4_{cl} \tag{3.6}$$

where Tr' denotes the trace over non-zero modes and can be evaluated by performing a Wick rotation to Euclidean space ($k^0 = \mathrm{i}k^0_E$) and using the standard integral (1.49)

$$-\tfrac{\mathrm{i}}{2}\,\mathrm{Tr}'\log\!\left(\Box + \tfrac{1}{2}\lambda\Phi^2_{cl}\right) = \tfrac{1}{2}\int\frac{\mathrm{d}^d k_E}{(2\pi)^d}\log\!\left(k^2_E + \tfrac{1}{2}\lambda\Phi^2_{cl}\right)$$

Now observe that the integral has a logarithmic singularity at $\Phi_{cl} = 0$. The IR divergence can be avoided by staying away from vanishing Φ_{cl}. Besides, the integral is also UV divergent, which we handle by introducing a momentum cut-off (following [69]). The 1-loop effective action becomes:

$$V^1_{\text{eff}} = \left(\frac{\lambda\Lambda}{64\pi^2}\right)\Phi^2_{cl} + \frac{\lambda^2\Phi^4_{cl}}{256\pi^2}\left(\log\!\left(\frac{\lambda\Phi^2_{cl}}{2\Lambda^2}\right) - \tfrac{1}{2}\right) + \tfrac{1}{2}\Delta^1_m\Phi^2_{cl} + \tfrac{1}{4!}\Delta^1_\lambda\Phi^4_{cl}$$

With this potential and the renormalization conditions (3.4), we can now determine the counterterms:

$$\begin{aligned} 0 &= \left.\frac{\mathrm{d}^2 V_{\text{eff}}}{\mathrm{d}\Phi^2_{cl}}\right|_{\Phi_{cl}=0} &&\Rightarrow\quad \Delta_m = -\frac{\lambda\Lambda^2}{32\pi^2} \\ \lambda &= \left.\frac{\mathrm{d}^4 V_{\text{eff}}}{\mathrm{d}\Phi^4_{cl}}\right|_{\Phi_{cl}=\mu} &&\Rightarrow\quad \Delta_\lambda = -\frac{3\lambda^2}{32\pi^2}\left(\log\!\left(\frac{\lambda\mu^2}{2\Lambda^2}\right) + \frac{11}{3}\right) \end{aligned}$$

We could evaluate the first renormalization condition at $\Phi_{cl} = 0$ whereas the second condition should be evaluated at an arbitrary scale μ because the fourth derivative of V_{eff} at the origin does not exist.

Combining it all, gives the final expression for effective potential in the 1-loop approximation for the massless scalar theory:

$$V_{\text{eff}} = \frac{1}{4!}\lambda\Phi^4_{cl} + \frac{\lambda^2\Phi^4_{cl}}{256\pi^2}\left(\log\!\left(\frac{\Phi^2_{cl}}{\mu^2}\right) - \frac{25}{6}\right) \tag{3.7}$$

The theory is renormalizable and, as expected, all dependence on the momentum cut-off Λ have disappeared in the above expression. Finally, we ask ourselves the important question with which we started this section: does this potential give rise to spontaneous symmetry breaking? Spontaneous symmetry breaking occurs if the quantum field develops a nonzero vacuum expectation value, even when the source $J(x)$ vanishes. This occurs if

$$\frac{\delta\Gamma}{\delta\Phi_{cl}} = 0 \quad\Rightarrow\quad \frac{\mathrm{d}V_{\text{eff}}}{\mathrm{d}\Phi_{cl}} = 0$$

where we assumed that the vacuum expectation value is translationally invariant. Using (3.7) seems to give a non-trivial VEV:

$$\left.\frac{\mathrm{d}V_{\text{eff}}}{\mathrm{d}\Phi_{cl}}\right|_{\langle\Phi\rangle} = \frac{\lambda\langle\Phi\rangle^3}{6} - \frac{22\lambda^2\langle\Phi\rangle^3}{384\pi^2} + \frac{\lambda^2\langle\Phi\rangle^3}{64\pi^2}\log\!\left(\frac{\langle\Phi\rangle^2}{\mu^2}\right) = 0 \quad\Rightarrow\quad \lambda\log\!\left(\frac{\langle\Phi\rangle}{\mu}\right) = -\frac{16\pi^2}{3} + \mathcal{O}(\lambda)$$

However, choosing the arbitrary renormalization scale as $\mu^2 = \frac{\langle\Phi\rangle}{2}$, we see that $\lambda(\langle\Phi\rangle) \sim 50$ which is far outside the expected range of validity of the one-loop approximation. In other words, the minimum arises from balancing a term of $\mathcal{O}(\lambda)$ against a term of $\mathcal{O}\big(\lambda^2\log(\Phi/M)\big)$ which inevitably means that for small λ the minimum lies outside the domain of validity.

The range of validity may be improved using the Renormalization Group. As we can see in (3.7), the arbitrary renormalization mass is still there. Its only function is to define the coupling constant and the field renormalization strength through (3.4) and that a small change in μ can be compensated by appropriate changes in λ and Z. Recall from our discussion in Section 1.3.3 that this continuous relationship is constrained by the Callan-Symanzik equation (1.55). We apply the Callan-Symanzik equation to the effective action (3.2) to resum all the 1-loop 1PI diagrams in order to obtain the renormalization group improved Coleman-Weinberg potential in which the independence of the renormalization mass is exact to every order. In the process, we also obtain the β-function for λ without computing Feynman diagrams.

From [7, Chapter 13] we know that the irreducible n-point function is related to the corresponding function of bare fields:

$$\Gamma^{(n)} = Z(\mu)^{n/2}\Gamma_0^{(n)}$$

This relation is identical to the relation between renormalized and bare Green's function (1.53) except for the change of sign in the exponent. Following the same argumentation as Section 1.3.3 gives then the Callan-Symanzik equation n-point function

$$\left(\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial\lambda} - n\gamma(\lambda)\right)\Gamma^{(n)}(\{x_i\};\mu,\lambda) = 0$$

This equation, integrated with n powers of Φ_{cl} and summed over n, is equivalent to

$$\left(\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial\lambda} - \gamma(\lambda)\int \mathrm{d}x\,\frac{\delta}{\delta\Phi_{cl}}\right)\Gamma([\Phi_{cl}];\mu,\lambda) = 0$$

Assuming constant Φ_{cl} then gives the Callan-Symanzik equation for the effective potential:

$$\left(\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial\lambda} + \delta\frac{\partial}{\partial\alpha} - \gamma(\lambda)\Phi_{cl}\frac{\partial}{\partial\Phi_{cl}}\right)V_{\text{eff}}(\Phi_{cl},\mu,\lambda) = 0, \quad \beta = \mu\frac{\partial g}{\partial\mu}, \quad \gamma = \frac{\mu}{2Z}\frac{\partial Z}{\partial\mu} \tag{3.8}$$

Using the Landau gauge, $\delta = 0$ in the one-loop approximation such that the term for the gauge parameter α disappears. Each term of the equation then is

$$\begin{aligned}
\mu\frac{\partial V_{\text{eff}}}{\partial\mu} &= -\frac{2\lambda^2}{256\pi^2}\Phi_{cl}^4 \\
\beta_\lambda\frac{\partial V_{\text{eff}}}{\partial\lambda} &= \beta_\lambda\left(\frac{1}{4!}\Phi_{cl}^4 + \text{higher orders}\right) \\
\gamma\Phi_{cl}\frac{\partial V_{\text{eff}}}{\partial\Phi_{cl}} &= \gamma\left(\frac{1}{3!}\lambda\Phi_{cl}^4 + \text{higher orders}\right)
\end{aligned}$$

To determine γ we note that we have not calculated the one-loop corrections to Z. However, the one-loop propagator correction is completely cancelled by the mass counterterm and the first non-trivial corrections is at $\mathcal{O}(\lambda^2)$. This means that at 1 loop we have $Z(\mu) = 1 + \mathcal{O}(\lambda^2) \to \gamma = 0$. Plugging this all into the Callan-Symanzik equation gives the one-loop beta function for the coupling constant:

$$\beta_\lambda = \frac{3}{16\pi^2}\lambda^2 + \mathcal{O}(\lambda^3) \tag{3.9}$$

The next step is to find the solution to the Callan-Symanzik equation (which is a differential equation). Introducing the dimensionless quantities $V^{(4)}$ and t, and a redefinition of β and γ according to

$$V_4 = \frac{\partial^4 V_{\text{eff}}}{\partial\Phi_{cl}^4}, \quad t = \log\left(\frac{\Phi_{cl}}{\mu}\right), \quad \bar{\beta} = \frac{\beta}{1-\gamma}, \quad \bar{\gamma} = \frac{\gamma}{1-\gamma}$$

changes (3.4) and (3.8) to

$$\left(-\frac{\partial}{\partial t} + \bar{\beta}\frac{\partial}{\partial \lambda} + 4\bar{\gamma}\right)V_4(t,\lambda) = 0, \quad \text{where} \quad \lambda = V_4(0,\lambda) \quad \text{and} \quad Z(0,\lambda) = 1$$

Assuming we know $\bar{\beta}$ and $\bar{\gamma}$ exactly, we can construct the general solution to this equation (see e.g. [69] for details)

$$V_4(t,\lambda) = \bar{\lambda}(t,\lambda)e^{2\int_0^t \mathrm{d}t\bar{\gamma}(\bar{\lambda}(t,\lambda))} \tag{3.10}$$

where $\bar{\lambda}(t,\lambda)$ gives the running of the coupling and is defined as the solution of the ordinary differential equation

$$\frac{\mathrm{d}\bar{\lambda}}{\mathrm{d}t} = \bar{\beta}(\lambda'), \quad \lambda'(0,\lambda) = \lambda \tag{3.11}$$

Upon generalizing to a general massless theory, the above equation becomes a system of coupled ordinary differential equations, one for each coupling constant. Unfortunately, there is one problem with the above picture: we do not know $\bar{\beta}$ and $\bar{\gamma}$ exactly. Suppose we can construct an approximation to $\bar{\lambda}$ by using the one-loop result of $\bar{\beta}$ in the above equation. We then expect that the approximation is reliable in the range of t for which $\bar{\lambda}(t,\lambda)$ remains small and if we are lucky, this is the case for a large range of t. It is exactly in this way that the renormalization group gives us an 'improved' result because the domain of validity of the one-loop approximation is determined not only by small coupling constant but also by the condition that the logarithmic factor t not be too large.

Using the fact that $\gamma = 0$ and thus $\beta_\lambda = \bar{\beta}_\lambda$, we get the following approximate differential equation (3.11) and solution:

$$\frac{\mathrm{d}\bar{\lambda}}{\mathrm{d}t} = \frac{3}{16\pi^2}\lambda^2, \quad \Rightarrow \quad \bar{\lambda} = \frac{\lambda}{1-\frac{3\lambda t}{16\pi^2}} = \lambda + \sum_{n=1}^{\infty} 3^n \frac{\lambda^{n+1}}{(4\pi)^{2n}} \log\left(\frac{\Phi_{cl}}{\mu}\right)^n \tag{3.12}$$

The first term in this expansion is the scale-independent vertex, whereas the first simple logarithm gives the momentum dependence coming from the 1-loop correction and so forth. Evidently, the 1-loop approximation to the running coupling resums the leading logarithmic behavior of all the loop diagrams. This idea can be extended such that when we use the k-loop expression for the β function it resums not only the leading logarithm $\lambda^{n+1}\log(\Phi/\mu)^n$ arising at each loop order n, but also the first $k-1$ subleading logarithms $\lambda^{n+1}\log(\Phi/\mu)^{n-k+1}$. However, here we are satisfied with the one-loop improved results.

Because $V_4(t,\lambda) = \bar{\lambda}(t,\lambda)$ we obtain the improved approximation for the effective potential:

$$V_{\text{eff}} = \frac{\lambda\Phi_{cl}^4}{4!\left(1-\frac{3\lambda t}{16\pi^2}\right)} \tag{3.13}$$

Plugging in the sum of (3.12), we see that this result agrees with the earlier found (3.7), but in addition is also valid for negative t. We now see that the minimum of the classical potential at the origin is indeed turned into a maximum due to the radiative corrections, a conclusion which we now can make because of the extended range of validity of our approximation. Our conclusion that the minimum we found earlier may be false is also verified by the above potential where instead of a minimum we find a pole $\Lambda_{UV} = \mu\exp\left(\frac{16\pi^2}{3\lambda}\right)$, equally outside the range of our approximation as earlier.

Elizalde and Odintsov [71] reported on the extension of the Coleman-Weinberg mechanism to curved spacetimes. Using the RG equation for the effective potential in curved spacetime

$$\left(\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial\lambda} + \delta\frac{\partial}{\partial\alpha} + \beta_\xi\frac{\partial}{\partial\xi} - \gamma(\lambda)\Phi_{cl}\frac{\partial}{\partial\Phi_{cl}}\right)V_{\text{eff}}(\Phi_{cl},\mu,\lambda,\xi) = 0$$

for ϕ^4-theory in curved spacetime

$$V^{(0)} = a\lambda\Phi_{cl}^4 - b\xi R\Phi_{cl}^2, \quad \text{where} \quad a, b > 0, \quad \xi = \frac{d-2}{4(d-1)}$$

He found the following RG improved effective action up to one loop using the Landau gauge

$$V_{\text{eff}} = \frac{\lambda\Phi_{cl}^4}{4!\left(1 - \frac{3\lambda t}{16\pi^2}\right)} - \frac{1}{2}R\left[\frac{1}{6} + \left(\xi - \frac{1}{6}\right)\left(1 - \frac{3\lambda t}{16\pi^2}\right)^{-1/3}\right] \tag{3.14}$$

which in flat spacetime reduces to equation (3.13).

Although we need to reject the nontrivial minimum as an artifact of our approximation, there are physical theories for which the above described Coleman-Weinberg mechanism does result in spontaneous symmetry breaking. One such example is massless QED (see [69] for further details). However, two comments are in place. Firstly, including higher-loop corrections will change the above sketched picture only that they might turn the maximum at the origin back into a minimum, but they cannot turn it into an absolute minimum, and the asymmetric vacuum we have found remains a local minimum definitely lower than that at the origin. Secondly, using the CW mechanism on massless QED, one encounters a phenomenon called dimensional transmutation: initially there are 2 free parameters (e, λ), but after the CW procedure we have traded the dimensionless λ for the dimensional $\langle\Phi\rangle$. This is a general feature of spontaneous symmetry breaking in fully massless field theories.

Being able to explain spontaneous symmetry breaking without introducing (artificial) mass terms in the theory, lead many to wonder whether the Coleman-Weinberg mechanism could be responsible for the spontaneous symmetry breaking in the Standard Model. In an influential paper, Sher [70] provided the one-loop effective potential for the Standard Model, which meant extending the mechanism to include non-Abelian gauge fields and include the Higgs mass term in the Renormalization Group equations. He established a lower bound on the Higgs mass:

$$m_H^2 \geq m_{CW}^2 \approx (10.4 \pm 0.3\text{GeV})^2[1 - 0.009(m_{\text{top}}/25\text{GeV})]$$

Note that if the top quark is heavier than about 83 GeV, which it is according to current precision measurements, the mass becomes negative. Furthermore, the large top mass gives a large negative contribution to effective potential, destabilizing the potential. Therefore, the Coleman-Weinberg symmetry breaking is ruled out for the Standard Model with small Higgs coupling λ.

For example Steele and Wang [72] investigated whether CW breaking is possible if we allow a much larger Higgs self-coupling λ. Already a Higgs mass upper bound of 165 GeV from the five-loop effective potential was found. The authors of [72] used Padé approximation methods in combination with subsequent averaging to estimate the contributions of higher-loop contributions to the potential, thus finding a nine-loop Higgs mass upper bound of 141 GeV. Therefore, they concluded that the CW mechanism with higher Higgs coupling ($\lambda = 0.23$) could still be consistent with the ATLAS and CMS $m_H = 125$ GeV Higgs boson. However, a larger Higgs self-coupling gives rise to concerns regarding possible nearby Landau poles and the validity of the one-loop approximation (or even perturbation theory in general). Besides enhancing of $HH \to HH$ scattering, they found that the Higgs decays to gauge bosons are unaltered but that the scattering processes $W^+W^+ \to HH, ZZ \to HH$ are also enhanced, providing signals to distinguish conventional and radiative electroweak symmetry breaking mechanisms.

The other remaining possibility for the Coleman-Weinberg mechanism to be responsible for the spontaneous breakdown of the electroweak symmetry is by extending the Standard Model. This could either be done by increasing the scalar content of the model by adding e.g. a scalar singlet, or by extending the gauge symmetry with e.g. an extra $U(1)$ group. In our discussion we have encountered examples of both extensions, but our toy model falls in the first category.

3.1.2 Gildener-Weinberg formalism

The Coleman-Weinberg mechanism of spontaneous symmetry breaking by radiative corrections was extended to theories with arbitrary numbers of scalar fields by Eldad Gildener and Steven Weinberg [73] (for a review see also [70]). In order to discuss spontaneous symmetry breaking in that case, the potential has to be minimalized. However, even for the minimal singlet extension of the Standard Model

$$V(\mathcal{H},\omega) = \frac{\lambda_1}{4}(\mathcal{H}^\dagger\mathcal{H})^2 + \frac{\lambda_2}{2}\omega^2(\mathcal{H}^\dagger\mathcal{H}) + \frac{\lambda_3}{4}\omega^4 \tag{3.15}$$

the one-loop effective potential (in the $\overline{\text{MS}}$ scheme) is quite cumbersome[43]

$$\begin{aligned} V^{(1)}_{\text{eff}} = &\frac{1}{64\pi^2}\left[\frac{3}{4}\left(\lambda_1\mathcal{H}^\dagger\mathcal{H}+\lambda_2\omega^2\right)^2\log\left(\frac{\lambda_1\mathcal{H}^\dagger\mathcal{H}+\lambda_2\omega^2}{\mu^2}\right) + F_+^2\log\left(\frac{F_+^2}{\mu^2}\right) + F_-^2\log\left(\frac{F_-^2}{\mu^2}\right)\right] \\ &- \frac{6}{32\pi^2}g_t^4(\mathcal{H}^\dagger\mathcal{H})^2\log\left(\frac{\mathcal{H}^\dagger\mathcal{H}}{\mu^2}\right) - \frac{1}{32\pi^2}(\operatorname{Tr}\Gamma^\omega)^4\omega^4\log\left(\frac{\omega^2}{\mu^2}\right) \end{aligned}$$

with

$$4F_\pm = (3\lambda_1+\lambda_2)\mathcal{H}^\dagger\mathcal{H} + (3\lambda_3+\lambda_2)\omega^2 \pm \sqrt{\left[(3\lambda_1-\lambda_2)\mathcal{H}^\dagger\mathcal{H} - (3\lambda_3-\lambda_2)\omega^2\right]^2 + 16\lambda_2^2\omega^2\mathcal{H}^\dagger\mathcal{H}}$$

and cannot be minimalized analytically [74]. Rather than using numerical models, Gildener and Weinberg use a analytical approximation method based on the Renormalization Group Equations (RGEs). Their formalism amounts to finding the minimum of classical potential on a unit sphere and then use the RGEs to set potential to zero at this point. They found that then there should exist a number of heavy Higgs bosons, with masses comparable to the intermediate vector bosons, plus one light Higgs boson, which they called a "scalon", with mass of order $\alpha G_F^{-1/2}$. This scalon is considered the pseudo-Goldstone boson (PGB) associated with scale invariance and therefore also called the dilaton.

We will demonstrate the main results from the Gildener-Weinberg (GW) formalism using a renormalizable field theory as before with a real, color-neutral scalar multiplet Φ containing all scalar degrees of freedom of a given theory. The general tree-level scalar potential with a weak but otherwise arbitrary quartic coupling constant $f_{ijkl} \sim g^4$ (where $g \ll 1$ a typical gauge coupling constant) is

$$V^{(0)}(\Phi) = \frac{1}{4!}f_{ijkl}\Phi_i\Phi_j\Phi_k\Phi_l$$

We choose the renormalization scale Λ_{GW} such that the potential develops a nontrivial minimum, which lies on a ray through the origin of the multi-dimensional field space called the flat direction. This imposes certain conditions on the scalar coupling at $\mu = \Lambda_{GW}$. To be specific, we parametrize $\Phi_i = N_i\phi$ where N_i is a unit vector and ϕ is the distance from the origin of field space. Let the minimum value of V^0 on the unit sphere occur for $N_i = n_i$ with $\sum_i n_i^2 = 1$ which normalizes the VEVs such that the sum of squares lies on a unit sphere. The Gildener-Weinberg

[43] This result is calculated for the Standard Model in the presence of right-chiral neutrino's and a minimally enlarged scalar sector with $O(4)\times O(1)$ invariant quartic interactions (4 for the complex doublet and 1 due to the real dilaton singlet). The computation of the fermionic contribution is done using the approximation that the quark sector is dominated by the top quark. For the leptonic sector all terms involving Γ^e have been ignored and the neutrino-Higgs Yukawa coupling Γ^ν is assumed much smaller then the neutrino-dilaton yukawa coupling Γ^ω. Also, the terms from $SU(2)_L\times U(1)_Y$ gauge fields are not included because the respective gauge couplings are small, nor from $SU(3)_c$ gauge fields because it is a two-loop effect. Lastly, note that the kinetic terms of the scalars all have the right sign as opposed to our model where the additional scalar reflects the coupling between the Standard Model and Einstein Gravity and has a wrong sign kinetic term.

condition then reads that at the renormalization scale Λ_{GW} the potential along the tree level flat direction $V(\mathbf{N}\phi)$ should vanish:

$$\min_{N_iN_i=1} V(\mathbf{N}\phi)\Big|_{\mathbf{N}=\mathbf{n}} = \min_{N_iN_i=1} f_{ijkl}(\Lambda_{GW})N_iN_jN_kN_l\phi^4\Big|_{\mathbf{N}=\mathbf{n}} = 0 \tag{3.16}$$

This is simply the statement that the potential restricted to the single degree of freedom ϕ, is of the form $\frac{1}{4}\lambda_\phi\phi^4$ with the corresponding coupling constant vanishing at Λ_{GW}. Furthermore, the GW condition (3.16) defines a hypersurface in the space of couplings. It is essential to study the RG flow (i.e. the beta functions) to see whether this hypersurface is reached and, if there are multiple, which one is reached first.

Next, requiring that the flat direction is a stationary line gives

$$\frac{\partial V^{(0)}(\Phi)}{\partial \Phi_i}\Big|_{\mathbf{N}=\mathbf{n}} = 0 \quad \Rightarrow \quad f_{ijkl}n_jn_kn_l = 0 \tag{3.17}$$

Lastly we require that $V^{(0)}(\mathbf{N}\phi)$ not only vanishes and is stationary at $\mathbf{N} = \mathbf{n}$; we are also demanding that it is a minimum there. This means that the matrix

$$P_{ij} = \frac{\partial^2 V^{(0)}(N)}{\partial N_i \partial N_j}\Big|_{\mathbf{N}=\mathbf{n}} = \tfrac{1}{2} f_{ijkl}n_kn_l \tag{3.18}$$

needs to have either positive or zero eigenvalues. Actually, all eigenvalues of P are positive definite, except for the zero eigenvalues associated with eigen vectors n and $n\Theta$, where Θ is a generator of a continuous symmetry that the theory possesses.

The GW conditions (3.16)-(3.18) ensure that the tree-level potential achieves a minimum along the flat direction independent of the value ϕ. Including higher-order terms (loop corrections) in the potential gives a small curvature to the potential along the flat direction, such that a particular value $\langle\phi\rangle$ is singled out as the true minimum. We can establish this minimum by writing the effective potential in a form similar to the one in Coleman-Weinberg case along the flat direction if the scalar couplings in all other directions in field space are sufficiently large.

$$V_{\text{eff}}(n\phi) = \phi^4 F(t,g) = A(g)\phi^4 + B(g)\phi^4 t + C(g)\phi^4 t^2, + \ldots, \quad t = \log\left(\frac{\phi^2}{\Lambda_{GW}^2}\right) \tag{3.19}$$

where perturbation theory requires $|g| \ll 1$ and $|gt| \ll 1$ and $A, B, \ldots$ are functions of the dimensionless coupling constants g. We are only interested in the one-loop effective potential and thus focus on $A(g)$ and $B(g)$

$$A(g) = \frac{1}{64\pi^2\langle\phi\rangle^4}\sum_i (-1)^{2s_i} d_i\, M_i^4(n\langle\phi\rangle)\left(\log\left(\frac{M_i^2(n\langle\phi\rangle)}{\langle\phi\rangle^2}\right) - c_i\right) \tag{3.20a}$$

$$B(g) = \frac{1}{64\pi^2\langle\phi\rangle^4}\sum_i (-1)^{2s_i} d_i\, M_i^4(n\langle\phi\rangle) \tag{3.20b}$$

where the index i in the above sums runs over all particles in the given theory with d_i the number of the particle's real degrees of freedom and s_i its spin. Then, $M_i(n\langle\phi\rangle)$ is the field-dependent tree-level mass evaluated along the flat direction for each particle. Lastly, the constant c_i depends on the renormalization scheme, i.e. for the $\overline{\text{MS}}$ scheme we have $c_i = \frac{5}{6}$ for gauge bosons and $c_i = \frac{3}{2}$ for scalars and fermions.

Writing the scalar, fermion and vector contributions out explicitly, gives

$$A = \frac{1}{64\pi^2}\left[\sum_s \frac{M_s^4}{\langle\phi\rangle^4}\log\left(\frac{M_s^2}{\langle\phi\rangle^2}\right) - 4\sum_f \zeta_f \frac{m_f^4}{\langle\phi\rangle^4}\log\left(\frac{M_f^2}{\langle\phi\rangle^2}\right) + 3\sum_v \frac{M_v^4}{\langle\phi\rangle^4}\log\left(\frac{m_v^2}{\langle\phi\rangle^2}\right)\right] \tag{3.21a}$$

$$B = \frac{1}{64\pi^2}\left[\sum_s \frac{M_s^4}{\langle\phi\rangle^4} - 4\sum_f \zeta_f \frac{M_f^4}{\langle\phi\rangle^4} + 3\sum_v \frac{M_v^4}{\langle\phi\rangle^4}\right] \tag{3.21b}$$

where $\zeta_f = 1, \frac{1}{2}$ for Dirac and Majorana (or Weyl) fermions, respectively. These sums are dominated by largest tree-level masses in the theory and the dependence on the coupling constant is hidden in M_i, but will be shown explicitly below.

Through a straightforward calculation we can show that the GW potential (3.19) has an extremum along the flat direction at

$$\log\left(\frac{\langle\phi\rangle}{\Lambda_{GW}}\right) = -\frac{1}{4} - \frac{A}{2B} \tag{3.22}$$

Note that if A is of the same order as B, $\langle\phi\rangle$ is of the same order as Λ_{GW} and perturbation theory remains valid. Plugging this back into the GW potential gives

$$V_{\text{eff}}^{(1)}(n\phi) = B\phi^4\left(\log\left(\frac{\phi^2}{\langle\phi\rangle^2}\right) - \frac{1}{2}\right) \tag{3.23}$$

and we can evaluate the potential at the minimum to be $V_{\text{eff}}^{(1)}(\langle\phi\rangle) = -\frac{1}{2}B\langle\phi\rangle^4$. This is a lower minimum than the one at the origin if and only if B is positive. From (3.21b) we know that this is definitely the case if there are no fermions in the theory. In the presence of fermions B can be positive as long as the Yukawa couplings (which determine the fermion mass) are not too large.

Assuming $B > 0$, the squared masses of the scalar bosons in zeroth order are given by the eigenvalues of the matrix

$$(M_s^2)_{ij} = \left.\frac{\partial^2 V^{(0)}(\Phi)}{\partial\Phi_i\partial\Phi_j}\right|_{\Phi=n\langle\phi\rangle} = P_{ij}\langle\phi\rangle^2$$

As explained, this matrix has a set of positive-definite eigenvalues leading to masses of a set of Higgs bosons (of the order of typical intermediate vector boson masses), plus a set of zero eigenvalues with eigenvectors $(\Theta n)_j$ corresponding to massless Goldstone bosons (which in a realistic model need to be cancelled by the BEH mechanism). There is also one zero eigenvalue with eigenvector n which leads to one light pseudo-goldstone boson also with vanishing zeroth-order mass. Assuming that the only symmetry that is broken by the higher order corrections is the scale invariance corresponding to the dilaton, the Goldstone bosons remain massless, but the dilaton develops a mass from the loop corrections in the flat direction.

$$M_{\text{PGB}}^2 = n_i n_j \left.\frac{\partial^2 V_{\text{eff}}^{(1)}(\Phi)}{\partial\Phi_i\partial\Phi_j}\right|_{n\langle\phi\rangle} = \left.\frac{\mathrm{d}^2 V_{\text{eff}}^{(1)}(n\phi)}{\mathrm{d}\phi^2}\right|_{\langle\phi\rangle} = 8B\langle\phi\rangle^2 \tag{3.24}$$

where B is given by (3.21b) and the sum over the scalars is the sum over the heavy Higgs bosons of the theory. Gildener and Weinberg then assumed a model based on the gauge group $SU(2)\times U(1)$ (like the Standard Model, but with an arbitrary number of scalars) such that $\langle\phi\rangle^2 = v = 2^{-1/2}G_F^{-1} = 247$ GeV. They furthermore assumed that the fermion masses were much lighter than the intermediate vector bosons and proceed to find a lower bound by dropping the Higgs masses and using the weak mixing angle, which at that time was estimated at 35° (compare to the current value $\theta_W \approx 28.57°$ from Table A.1) such that $M_{\text{PGB}} \geq 7\,\text{GeV}$.

With the discovery of the top quark quark which is heavier that the intermediate vector bosons of the Standard Model, we should include it explicitly in an estimation of the PGB mass (3.24). Denoting (3.21b) for the Standard Model as B_{SM}, we have

$$B_{\text{SM}} = \frac{1}{64\pi^2 v^4}\left[-12M_t^4 + 3M_Z^4 + 6M_W^4\right] = \frac{3}{64\pi^2 v^4}\left[\frac{3g_1^4 + 2g_1^2g_2^2 + g_2^4}{16} - g_t^4\right] \tag{3.25}$$

$$B(g) = B_{\mathrm{SM}} + \frac{1}{64\pi^2 v^4}\left[\sum_s M_s^4 - 4\sum_{f'} \zeta_{f'} M_{f'}^4 + 3\sum_{v'} M_{v'}^4\right] = B_{\mathrm{SM}} + B_{\mathrm{add}} \tag{3.26}$$

where in the second line we plugged in the expressions for the gauge boson and Higgs masses (1.29) and $M_t = \frac{g_t v}{\sqrt{2}}$. The sum over f' runs over additional fermions (for example right-handed neutrino's) and the sum over v' accounts for additional vector bosons (for example due to including an additional $U(1)$ gauge symmetry). We immediately see that $B_{\mathrm{SM}} < 0$ and as there are no additional scalars, we verify our earlier conclusion that spontaneous symmetry breaking of the electroweak symmetry in the Standard Model due to radiative corrections is not possible.

Equation (3.24) then becomes

$$M_{\mathrm{PGB}}^2 = 8B_{\mathrm{add}}\,\langle\phi\rangle^2 - K, \quad \text{where} \quad K = -8B_{\mathrm{SM}}\,\langle\phi\rangle^2 > 0 \tag{3.27}$$

These formulae will help analyzing the possibility of Coleman-Weinberg breaking in extended Standard Models. As with the Coleman-Weinberg mechanism of the previous section, the results then also need to be checked for the presence of Landau poles or instabilities (negative coupling constants) below the Planck scale, and to confirm the validity of the one-loop approximation. Ideally this requires calculating the full resummed and renormalization group invariant effective action, but the usually one is content with investigating the 1PI based beta functions.

3.2 CSMG toy model: an unphysical dilaton

Having now established the relevant theory, we continue to investigate the breaking of conformal symmetry in our Conformal Standard Model with Gravity toy model (2.86) with an unphysical dilaton. In Section 3.2.1, we first consider the general CSMG model in curved spacetime with $g_{\mu\nu}$ the metric in the usual sense and χ is a dilaton multiplet. There we establish that we have insufficient information to investigate symmetry breaking classically in the flat spacetime scenario. Therefore we will quantize the theory in Section 3.2.2 and discuss the appearance of the conformal anomaly and how to handle it. The last subsection is devoted to 't Hooft's interpretation of our toy model.

3.2.1 Symmetry breaking in curved CSMG

Recall the non-kinetic part of the scalar sector of our conformal toy model:

$$S_{\mathrm{CSMG}} = \int \mathrm{d}^4x\sqrt{g}\Big[\ldots - \tfrac{1}{12}(\chi^2 - 2\mathcal{H}^\dagger\mathcal{H})R - V(\mathcal{H},\chi) + \ldots\Big]$$

From (**??**) we know that the potential will contain quartic self-interactions of both the Higgs field and the dilaton field, and possibly a mixing term. Therefore, we propose the following purely quartic renormalizable potential involving one scalar multiplet additional to the complex Higgs doublet [67]

$$S_{\mathrm{CSMG}} = \int \mathrm{d}^4x\sqrt{g}\Big[\ldots - \tfrac{1}{12}(\chi^2 - 2\mathcal{H}^\dagger\mathcal{H})R - \lambda_1(\mathcal{H}^\dagger\mathcal{H})^2 - \lambda_2\chi^2(\mathcal{H}^\dagger\mathcal{H}) - \lambda_3\chi^4 \tag{3.28}$$

This situation is analogous to the Higgs sector of the Standard Model, but now $\mathcal{H}^\dagger\mathcal{H}$ is proportional to the scalar curvature R, which is neither the Higgs mass nor a constant. However, if we confine ourselves to the case of positive quartic scalar self-coupling, $\lambda_1 > 0$, and constant curvature $R = bd(d-1)$, where d is the dimension and b a constant signalling the type of space: when $b = 0$ the spacetime is Minkowski space, $b < 0$ the spacetime is de-Sitter space, and for $b > 0$ the spacetime is anti-de-Sitter (AdS) space (these identification depend on the convention of the metric signature).

To investigate the possibility for spontaneous symmetry breaking in this model, we calculate the Hessian which allows us to see whether the vacuum is a saddle point or not. We use the unitary gauge for the Higgs doublet, but do not expand around the minimum, and perform a $SU(2)$ gauge transformation to remove the Goldstone bosons such that $\frac{1}{2}\phi^2 = \mathcal{H}^\dagger\mathcal{H}$ and $\phi_0 = \upsilon$. If the dilaton is a singlet, it is not affected by this transformation. Denoting *all* non-kinetic terms of the scalar sector as depicted in (3.28) with $-V$, we have:

$$\frac{\partial V}{\partial \phi} = -\xi R\phi + 4\lambda_1\phi^3 + 2\lambda_2\chi^2\phi, \quad \frac{\partial^2 V}{\partial \phi^2} = -\xi R + 12\lambda_1\phi^2 + 2\lambda_2\chi^2$$
$$\frac{\partial V}{\partial \chi} = \xi R\chi + 4\lambda_3\chi^3 + 2\lambda_2\chi\phi^2, \quad \frac{\partial^2 V}{\partial \chi^2} = \xi R - 12\lambda_3\chi^2 + 2\lambda_2\phi^2,$$
$$\frac{\partial^2 V}{\partial \chi \partial \phi} = 4\lambda_2\chi\phi$$

such that the Hessian at the origin becomes:

$$\mathrm{H} = \frac{\partial^2 V}{\partial \phi^2}\frac{\partial^2 V}{\partial \chi^2} - \left(\frac{\partial^2 V}{\partial \chi \partial \phi}\right)^2\Bigg|_{\phi,\chi=0} = -\xi^2 R = -\frac{d(d-2)^2 b}{4(d-1)} \quad \begin{cases} <0 & \text{if} \quad b>0, \\ >0 & \text{if} \quad b<0, \\ =0 & \text{if} \quad b=0 \end{cases} \tag{3.29}$$

From this we see that the ground state is a saddle point for $b > 0$, a local minimum for $b < 0$ and that we don't have enough information yet to handle the flat space case. Having a saddle point at the origin, implies there is a nontrivial minimum elsewhere, i.e. in AdS spacetimes we have spontaneous symmetry breaking à la Higgs. During electroweak symmetry breaking the Higgs acquires a VEV thus creating a massive theory which is also non-conformal invariant. It is possible that the dilaton acquires a VEV at the same time or before the Higgs does.

Classically, CSMG reduces in the *broken* phase to Einstein Gravity and the Standard Model by gauge-fixing the dilaton to a constant ω_0 for all spacetime.

$$\tfrac{1}{2}\kappa^{-2} = M_p^2 = \tfrac{1}{12}(\chi_0^2 - \upsilon^2), \; \upsilon \ll M_p \quad \Rightarrow \quad \chi_0 \approx \sqrt{12}M_p$$

Using this so-called Einstein gauge, the usual Standard Model including a conformally coupled Higgs boson is retrieved. The Higgs potential ensures the subsequent spontaneous symmetry breaking of the electroweak gauge symmetry.

In flat spacetime and at low energies (e.g. at the LHC) there is no discernible difference between our toy model and the usual Standard Model. In a cosmological context the difference between the regular Standard Model versus the conformal invariant model does become apparent as the conformally coupled $\mathcal{H}$ plays an important role in the cosmological evolution of the Universe. For this it is important to understand that cosmological inflation is thought to arise from symmetry breaking due to a scalar field called the inflaton. The inflaton could be identified with the dilaton meaning inflation arises from conformal (or scale) invariance breaking. On the other hand, Bezrukov and Shaposhnikov have proposed a model in which the Higgs boson assumes this role, meaning that inflation is a result of electroweak spontaneous symmetry breaking, which would have observational difference compared to the first option [67]. However, we refrain from investigating the curved spacetime scenario any further as quantum field theory is far from established there.

3.2.2 Going quantum: anomalous breaking

In flat spacetime ($R = 0$), there is no term that can substitute the role of the tachyonic mass term in the Higgs potential. Therefore, we want to use the techniques of the previous section, i.e. the Gildener-Weinberg formalism, to see whether spontaneous symmetry breaking occurs at the quantum level. However, there are two concerns. Firstly, the equivalence between the Conformal Dilaton Gravity and Einstein's General Relativity might not survive at the quantum level due to quantum fluctuations and in particular to the FP ghosts associated with the Weyl symmetry. Secondly, a theory with an additional symmetry is only physical if we can extend the classical theory to the quantum level, i.e. no anomalies should arise. However, we will see that quantizing our Conformal Standard Model with Gravity toy model leads to an anomaly: conformal symmetry is anomalously broken.

Simply stated, the theory at the classical level is conformal invariant ($S(\Omega^2 g_{\mu\nu}, \Omega^{\tilde{\Delta}}\Phi) = S(g_{\mu\nu}, \Phi)$, with $\tilde{\Delta}$ the Weyl dimension of the field) and as such contains no dimensionfull parameters. When quantizing this theory some mass scale is always introduced during the regularization of the integrals, e.g. a cutoff scale (Pauli-Villars regularization) or a mass to preserve the proper canonical dimensions (dimensional regularization). Then, the effective action of theory carries some trace of this mass scale, thus breaking the conformal invariance[44]. This violation of classical scale invariance by quantum corrections can be described as a current conservation or trace anomaly and was first put forward by Capper and Duff [75]. They also stress that simply avoiding the mass from entering the effective action via some Weyl invariant regularization prescription is not enough as any and all identities used in the calculation must also hold in all dimensions (and not e.g. only in 4 spacetime dimensions).

To see this, recall from Section 2.2, that scale invariance (i.e. global Weyl invariance in flat spacetime) requires the trace of the energy-momentum tensor to be a total divergence: $T^\mu_\mu = \partial_\mu D^\mu$, where D^μ is the dilatation current and the EMT is given by equation (2.20)

$$T^{(0)}_{\mu\nu} = -\frac{2}{\sqrt{g}} \frac{\delta(\sqrt{g}\mathcal{L}_{\text{mat}})}{\delta g_{\mu\nu}}\bigg|_{g_{\mu\nu}=\eta_{\mu\nu}}$$

with $\mathcal{L}_{\text{mat}}$ given by the CSMG model $S^{\pm}_{\text{CSMG}}$ excluding the dilatonic kinetic terms. Conformal invariance (i.e. local Weyl invariance in flat spacetime) requires the trace of the energy-momentum tensor to be a double divergence: $T^\mu_\mu = \partial_\mu\partial_\nu D^{\mu\nu}$. In both cases, current conservation implies tracelessness of the EMT.

On a curved space-time manifold this result is supplemented by terms quadratic in the Riemann tensor

$$g^{\alpha\beta}T_{\alpha\beta} = b(F + \tfrac{2}{3}\Box R) + b'G + cH$$

where

$$\begin{aligned}
F &= R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu} + \tfrac{1}{3}R^2 \overset{\text{n=4}}{=} C^2 \\
G &= R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2 \overset{\text{n=4}}{=} E^2 \\
H &= F^a_{\mu\nu}F^{\mu\nu a} \\
b &= \frac{1}{16\pi^2}\frac{1}{120}\big((1+N_0) + 6N_{1/2} + 12N_1\big) \\
b' &= -\frac{1}{16\pi^2}\frac{1}{360}\big((1+N_0) + 11N_{1/2} + 62N_1\big)
\end{aligned}$$

with C^2 the Weyl tensor squared, E^2 the Euler density squared and$H$ is included to account for possible gauge fields and is called the internal gauge anomaly.

The trace anomaly of free matter vanishes in the limit of flat space, but this is not true for interacting fields: the trace is then proportional to the beta functions.

$$\left.\begin{matrix} g \to g + w\beta(g) \\ \mathcal{L} \to w\beta(g)\mathcal{L} \end{matrix}\right\} \quad \Rightarrow \quad \int \mathrm{d}^4x\, wT^\mu_\mu = -\frac{\delta\Gamma[\Phi_{cl}, w]}{\delta w}\bigg|_{w=0,\, g_{\mu\nu}=\eta_{\mu\nu}} = \int \mathrm{d}^4x\, w\beta(g)\mathcal{L}_{\text{int}}$$

[44] Capper et al. [**?**] show that the non-invariance of the gauge-fixing and ghost terms compensate each other in the effective action, so that the breaking of Weyl invariance is indeed only due to the presence of the scale μ.

where the functional derivative is calculated with respect to infinitesimal local Weyl transformations and we put $g_{\mu\nu}(x) = \eta_{\mu\nu}$ at the end (recall that varying with respect to the metric and taking the flat spacetime limit do not commute!). So for nonvanishing $\beta(g)$ the EMT is no longer traceless, hence the name 'trace anomaly'.

In contrast to the work of Capper and Duff, Codello et al. [49] show that local Weyl symmetry remains a valid symmetry at the quantum level as already anticipated by Englert et al. [76]. The contradiction between these conclusions can be traced back to the role of the dilaton. The theory considered in [75] contains only dimensionless parameters. On the other hand, the authors of [49], start with an arbitrary theory and promote a mass parameter to the status of a field (the dilaton) such that the dimensionfull parameters can be written as dimensionless quantities.

Due to the presence of the dilaton there exist two quantization procedures, which can be understood as different functional measures. The standard measure breaks Weyl invariance, but replacing the fixed regularization mass μ with the dilaton ω one obtains a measure that maintains it. This second measure can be obtained from the standard by applying the Stückelberg trick after quantization (though the two commute). Then we find that the Weyl invariant effective action is the ordinary effective action to which a Wess-Zumino term has been added, with the effect of canceling the Weyl anomaly. Furthermore, the trace of the energy-momentum tensor derived from the two actions are the same. We refrain from entering into detail and confine ourselves to saying that the authors of [49] continue by generalizing these ideas for interacting theories with a dynamical metric and dilaton, allowing them to find Weyl-invariant effective action with a nontrivial potential for the dilaton and

$$\left.\frac{\delta\Gamma[\Phi_{cl}, w]}{\delta w}\right|_{w=0,\, g_{\mu\nu}=\eta_{\mu\nu}} = 0$$

Thus they conclude that the conformally invariant scalar-tensor gravity coupled to various matter fields is free of the Weyl anomaly when the Weyl symmetry is spontaneously broken by radiative corrections.

It is important to stress that the above result should not be interpreted as vanishing trace of the energy-momentum tensor because the EMT is independent of the chosen measure. This has lead to great confusion in the past and can be clarified by noting that the trace of the matter stress tensor is distinct from the generator of Weyl transformations that includes the additional dilaton field. Put differently, the dilaton is part of the gravitational sector and thus not part of $\mathcal{L}_{\text{mat}}$ which is used in the definition of the EMT.

Whether or not the conformal anomaly is cancelled, boils down to your point of view. Most particle physicists view that true conformal invariance is only achieved at a fixed point of the theory ($\beta = 0$) because even if the theory is classically conformal invariant, this is spoiled in the quantum theory by the introduction of the renormalization/cut-off scale. On the other hand, according to most relativists Weyl transformations can be seen as relating different local choices of units. Since the choice of units is arbitrary and cannot affect the physics, it follows that essentially any physical theory can be formulated in a Weyl-invariant way. Codello et al. then showed that it is possible to also have a quantum Weyl-invariant theory.

3.2.3 't Hooft's interpretation

Lastly, we revisit the 't Hooft's interpretation of our CSMG toy model again. We will show how scales are generated in the CSMG model with a ghostlike dilaton metascalar field. We start by considering again our toy model (2.86) where the potential is specified as the quartic self-coupling of the Higgs field. First consider the Higgs and Yukawa sector in the unitary gauge, i.e. (1.26) from Section 1.2.2:

$$S_{\text{Higgs}} = \int \mathrm{d}^4x\sqrt{g}\left[\tfrac{1}{2}g^{\mu\nu}D_\mu H D_\nu H + \tfrac{3}{2}\lambda v^2 H^2 + \lambda v H^3 + \tfrac{1}{4}\lambda H^4 + \frac{1}{12}H^2 R\right] \tag{3.30}$$

$$S_{\text{Yukawa}} = -\int \mathrm{d}^4x\sqrt{g}\Big[\bar{e}_L M_e e_R + \bar{u}_L M_u u_R + \bar{d}_L M_d d_R + \bar{e}_L H\Gamma_e e_R + \bar{u}_L H\Gamma_u u_R + \bar{d}_L H\Gamma_d d_R + \text{h.c.}\Big] \tag{3.31}$$

where $\mathrm{M}_f = \frac{v}{\sqrt{2}}\Gamma_f$ a 3×3 Yukawa coupling matrix in the fermion generation indices that can be converted to the diagonal mass matrix using (1.34). The mass of the Higgs boson is thus $M_H^2 = 3\lambda v^2$.

Next, we proceed with the splitting of the metric $g_{\mu\nu}(x) = \omega^2(x)\hat{g}_{\mu\nu}$, i.e. all dimensionfull parameters receive appropriate powers of the dilaton depending on their scaling behavior

$$S_{\text{CSMG}}^{\text{split}} = \int \mathrm{d}^4x\sqrt{\hat{g}}\Big[\mathcal{L}_g + \mathcal{L}_{\text{mat}}^{\text{kin}} + \mathcal{L}_{\text{mat}}^{\text{mass}} + \mathcal{L}_{\text{mat}}^{\text{int}}\Big]$$

where the gravity sector is the EH action after splitting of the metric (see Appendix E for details)

$$S_g = \frac{1}{2\kappa^2}\int \mathrm{d}^4x\sqrt{\hat{g}}\,\left(\hat{R}\omega^2 + 6\hat{g}^{\mu\nu}\partial_\mu\omega\partial_\nu\omega\right) \tag{3.32}$$

The kinetic part consists of the Yang-Mills sector (2.82), and the kinetic terms of the fermions (2.78) as well as the Higgs field. The masses of the Higgs boson and fermions (no neutrino mass term) are part of mass part of the action. The interactions then include dilaton-Higgs interactions, the quartic self-interaction of both the Higgs boson and the dilaton (which comes from the cosmological constant term), and the Yukawa sector.

$$\begin{aligned}
\mathcal{L}_{\text{mat}}^{\text{kin}} &= -\tfrac{1}{4}\hat{g}^{\mu\rho}\hat{g}^{\nu\sigma}\left(B_{\mu\nu}B_{\rho\sigma} + W^i_{\mu\nu}W^i_{\rho\sigma} - G^a_{\mu\nu}G^a_{\rho\sigma}\right) + \begin{pmatrix}\bar{\nu}_L & \bar{e}_L\end{pmatrix}\mathrm{i}\hat{\not{D}}\begin{pmatrix}\nu_L\\ e_L\end{pmatrix} + \bar{\nu}_R\mathrm{i}\hat{\not{D}}\nu_R + \bar{e}_R\mathrm{i}\hat{\not{D}}e_R\\
&\quad + \begin{pmatrix}\bar{u}_L & \bar{d}_L\end{pmatrix}\mathrm{i}\hat{\not{D}}\begin{pmatrix}u_L\\ d_L\end{pmatrix} + \bar{u}_R\mathrm{i}\hat{\not{D}}u_R + \bar{d}_R\mathrm{i}\hat{\not{D}}d_R + \tfrac{1}{2}\hat{g}^{\mu\nu}D_\mu H D_\nu H - \tfrac{1}{12}H^2 R \qquad (3.33)\\
\mathcal{L}_{\text{mat}}^{\text{mass}} &= -\tfrac{1}{2}M_H^2\omega^2 H^2 - \bar{e}_L\omega M_e e_R - \bar{u}_L\omega M_u u_R - \bar{d}_L\omega M_d d_R + \text{h.c.} \qquad (3.34)\\
\mathcal{L}_{\text{mat}}^{\text{int}} &= -\frac{\Lambda}{\kappa^2}\omega^4 - \lambda v\omega H^3 - \frac{\lambda}{4}H^4 - \bar{e}_L H(x)\Gamma^e e_R - \bar{u}_L H(x)\Gamma^u u_R - \bar{d}_L H(x)\Gamma^d d_R + \text{h.c.} \qquad (3.35)
\end{aligned}$$

where the covariant derivative $\hat{\mathcal{D}}$ (2.79) is now associated with $\hat{g}_{\mu\nu}$.(Recall that fermions get a nontrivial spinor covariant derivative besides the gauge covariant derivative.)

The next step is to rescale the dilaton field: $\omega = \frac{\kappa}{\sqrt{6}}\hat{\omega} = \hat{\kappa}\hat{\omega}$. This is equivalent to picking a mass parameter of the theory (κ), and promoting it to a function[45]. The gravity part of the action S_g thus becomes the Conformal Dilaton Gravity action in 4 dimensions

$$S_{\text{CDG}} = \int \mathrm{d}^dx\sqrt{\hat{g}}\,\left(\tfrac{1}{12}\hat{R}\hat{\omega}^2 + \tfrac{1}{2}\hat{g}^{\mu\nu}\partial_\mu\hat{\omega}\partial_\nu\hat{\omega}\right) \tag{3.36}$$

Remarkably, the mass terms then turn into conformally invariant quartic coupling terms between matter fields and the $\hat{\omega}$ field, because $\hat{\kappa}$ has the dimension of an inverse mass and exactly cancels the mass dimension of M_f and M_H. The same applies for the scalar 3-field interaction where the mass dimension of v is cancelled, and for the cosmological constant term:

[45] Instead of $\hat{\omega}(x)$ one often encounters also $e^{\sigma(x)}$, as used in i.e. [49].

$$\lambda \upsilon \omega H^3 \to \lambda_3 \hat{\omega} H^3 \quad \text{with} \quad \lambda_3 = \frac{\kappa}{\sqrt{6}} \lambda \upsilon$$
$$\kappa^{-2} \Lambda \omega^4 \to \lambda_4 \hat{\omega}^4 \quad \text{with} \quad \lambda_4 = \frac{\kappa^2}{36} \Lambda$$

where λ_3, λ_4 dimensionless coupling constants. The full toy model in flat spacetime according to 't Hooft thus becomes:

$$\begin{aligned}
S^{\text{split}}_{\text{CSMG}} = \int \mathrm{d}^4x \sqrt{\hat{g}} \Big[&- \tfrac{1}{4} \hat{g}^{\mu\rho} \hat{g}^{\nu\sigma} \left(B_{\mu\nu} B_{\rho\sigma} + W^i_{\mu\nu} W^i_{\rho\sigma} - G^a_{\mu\nu} G^a_{\rho\sigma} \right) && \text{Yang-Mills sector} \\
&+ \begin{pmatrix} \bar{\nu}_L & \bar{e}_L \end{pmatrix} \mathrm{i} \hat{\not{D}} \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} + \bar{\nu}_R \mathrm{i} \hat{\not{D}} \nu_R + \bar{e}_R \mathrm{i} \hat{\not{D}} e_R && \text{Lepton kinetic terms} \\
&+ \begin{pmatrix} \bar{u}_L & \bar{d}_L \end{pmatrix} \mathrm{i} \hat{\not{D}} \begin{pmatrix} u_L \\ d_L \end{pmatrix} + \bar{u}_R \mathrm{i} \hat{\not{D}} u_R + \bar{d}_R \mathrm{i} \hat{\not{D}} d_R && \text{Quark kinetic terms} \\
&+ \tfrac{1}{12} \hat{R} (\hat{\omega}^2 - H^2) - \tfrac{1}{2} \hat{g}^{\mu\nu} (\partial_\mu \hat{\omega} \partial_\nu \hat{\omega} + D_\mu H D_\nu H) && \text{Scalar kinetic terms} \quad (3.37) \\
&- \bar{e}_L \hat{\kappa} \hat{\omega} M_e e_R - \bar{u}_L \hat{\kappa} \hat{\omega} M_u u_R - \bar{d}_L \hat{\kappa} \hat{\omega} M_d d_R + \text{h.c.} && \text{Fermion mass terms} \\
&- \tfrac{1}{2} M_H^2 \hat{\kappa} \hat{\omega}^2 H^2 - \lambda_4 \hat{\omega}^4 - \lambda_3 \hat{\omega} H^3 - \frac{1}{4} \lambda H^4 - \frac{1}{12} \hat{R} (H^2 - \hat{\omega}^2) && \text{Scalar mass and interactions} \\
&- \bar{e}_L H(x) \Gamma^e e_R - \bar{u}_L H(x) \Gamma^u u_R - \bar{d}_L H(x) \Gamma^d d_R + \text{h.c.} \Big] && \text{Yukawa interactions}
\end{aligned}$$

So, even though the gauge symmetry is spontaneously broken because we used the unitary gauge for the Higgs doublet, conformal symmetry is still a property of the theory! This way we can turn any theory into a conformal invariant theory.

Subsequently, 't Hooft proposed to rotate the dilaton to the complex plane $\tilde{\omega}(x) = \mathrm{i}\eta(x)$ to deal with the fact that the dilaton field $\tilde{\omega}$ has an overall sign opposite to that of ordinary scalar fields ϕ. This then leads to unconventional factors of -1 and i in the mass and interaction terms.The path integral over the dilaton field can now be done.

To be explicit, 't Hooft and Veltman [37] (strongly based on the work of [37]) considered a scalar particle in an external gravitational field, i.e. a Lagrangian which is invariant under general coordinate transformations

$$\mathcal{L} = \sqrt{-g} \left(-\tfrac{1}{2} \partial_\mu \varphi g^{\mu\nu} \partial_\nu \varphi + \tfrac{1}{2} \varphi M \varphi \right)$$

where M is symmetric function of external fields and does not depend on the bosons φ. The counterterm lagrangian that eliminates all one-loop divergencies is then of the form[46]

$$\Delta \mathcal{L} = - \frac{\sqrt{-g}}{8\pi^2 (4-d)} \left(\tfrac{1}{120} \left(R_{\mu\nu} R^{\mu\nu} - \tfrac{1}{3} R^2 \right) + \tfrac{1}{4} \left(M + \tfrac{1}{6} R \right)^2 \right)$$

Note that this is equivalent to saying that the lagrangian $\mathcal{L}$ generates an effective action whose divergent part is of the form $\Gamma^{\text{div}} = - \int \mathrm{d}^d x \Delta \mathcal{L}$.

Comparing this with the rotated CSMG action, we see that $M = -\frac{d-2}{4(d-1)} \hat{R} \to -\frac{1}{6} \hat{R}$ in 4 dimensions. The ω path integral (in 4 dimensions) thus leads to an effective action which includes only one divergent term, namely:

$$\Gamma^{\text{div}}_{\text{CDG}} = C \int \mathrm{d}^d x \sqrt{-g} \left(R_{\mu\nu} R^{\mu\nu} - \tfrac{1}{3} R^2 \right), \quad \text{with} \quad C = \frac{1}{8\pi^2 (4-d)} \frac{1}{120} \qquad (3.38)$$

[46] Note that we have a sign switch in the definition of the Ricci curvature as compared to [37].

which we recognize as the Weyl Gravity Lagrangian (2.48).

Adding other matter fields and ignoring interactions between them, one finds that these fields contribute to the divergence in the counterterm action (3.38) as well. In fact, all these divergences take the same form and they each just add to the overall coefficient:

$$C(d) = \frac{1}{16\pi^2(4-d)}\frac{1}{120}\left((1+N_0)+6N_{1/2}+12N_1-\frac{7}{2}N_{3/2}+\frac{424}{3}N_2\right) \tag{3.39}$$

in which the 1 is the effect of the metascalar component ω of gravity, N_0 is the number of real scalar fields, $N_{1/2}$ complex Dirac fields (or $\frac{1}{2}N_{1/2}$ Majorana spinors[47]), N_1 real vector fields, $N_{3/2}$ gravitinos and N_2 spin-2 fields. The last two terms are not considered as they refer to non-renormalizable fields.

't Hooft shows that adding non-conformal matter does not affect the formal conformal invariance of the effective action nor do the non-conformal parts, such as the mass term, have any effect on the divergence of C. Having established that, he is hesitant to follow the 'standard' procedure where, in some cases, the divergencies cancel out exactly by introducing a local counter term of the same form as (3.38), but with opposite sign,

$$-\Delta\mathcal{L} = C(d)\omega^{\frac{2(d-4)}{d-2}}\sqrt{-\hat{g}}\hat{C}_{\alpha\beta\mu\nu}\hat{C}^{\alpha\beta\mu\nu}$$

The d dependence in ω cannot be removed as it would break conformal invariance. However, if we keep it, the theory has an essential singularity at $\omega = 0$, which is not normally present in field theories. On the other hand, it is the only conformal invariant possible kinetic term for the $\hat{g}_{\mu\nu}$ field. Nonetheless, it violates unitarity. As mentioned earlier, unitarity can be restored by modifying the hermiticity condition in favor of PT symmetry, which effectively means that the integration contours need to be rotated in the complex plane such that $\hat{g}_{\mu\nu}$ becomes anti-Hermitian. Instead of pursuing this counterterm strategy further, 't Hooft leaves the question on the functional integration of $\hat{g}_{\mu\nu}$ at that.

Without entering into details, which are actually provided by Codello et al., 't Hooft does point out that during the above regularization and renormalization procedure the conformal anomaly must somehow be avoided. This is the case when the beta functions of the theory in the presence of the dilaton equal zero. For a renormalizable theory there are just as many beta functions as adjustable parameters, thus allowing to completely determine the theory. It is not immediately clear how the conformal anomaly was handled, it would be better to follow the procedure by Codello [49] such that we ensure Weyl invariance at the quantum level.

3.3 CSMG toy model: a physical dilaton

This section is devoted to explaining the Conformal Standard Model with Gravity toy model where the dilaton is a physical scalar field φ. That means we cannot reproduce Einstein's theory of General Relativity in the broken phase, but only have a coupling to Gravity via the nonminimal coupling term $\varphi^2 R$. As a simplification, we consider the theory in flat spacetime, though it would be more rigorous to take full model on curved spacetime, quantize it and then take flat spacetime limit.

As was already discussed in the case of an unphysical dilaton, our toy model is plagued by the conformal anomaly: when we quantize the theory, we find terms that explicitly break conformal invariance. However, demanding that conformal symmetry is a fundamental symmetry of Nature, there should be a scale at which the trace anomaly vanishes. The anomaly is given by

[47] This is misprinted in main body of the original paper [11] and thus in subsequent papers of 't Hooft, though it was correctly stated in its appendix.

the weighted sum of the beta functions and from the Standard Model we know that the hypercharge gauge coupling will increase with energy. We can either imbed the Standard Model in a non-Abelian gauge group or assume that Gravity somehow cancels the hypercharge contribution to the trace anomaly. Pursuing this second possibility, means there shouldn't be any intermediate scale between the EW and Planck scale. Furthermore, the running couplings should exhibit neither Landau poles nor instabilities over this whole range of energies.

Following Helmboldt et al. [77][48], we want to find the effective potential of our Conformal Standard Model with Gravity toy model (2.86) with the following scalar sector ($R = 0$)

$$S^{+}_{\text{CSMG}} = \int \mathrm{d}^4x\sqrt{g}\Big[\ldots+\tfrac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi+g^{\mu\nu}D_\mu\mathcal{H}D_\nu\mathcal{H}-\big(\lambda_1(\mathcal{H}^\dagger\mathcal{H})^2+\lambda_2\varphi^2(\mathcal{H}^\dagger\mathcal{H})+\lambda_3\varphi^4\big)+\ldots\Big] \tag{3.40}$$

where the dilaton is a colorless scalar $SU(3)\times SU(2)\times U(1)$ multiplet with a given hypercharge. For now we will focus on the real singlet case, though some other possibilities are briefly discussed at the end. Choosing a suitable parametrization, we minimize the above potential using the Gildener-Weinberg conditions for the scalar couplings. If radiative corrections dynamically generate a mass scale, this obviously breaks the conformal symmetry spontaneously. Accordingly, we expect the theory's low-energy phase to contain one pseudo-Goldstone boson (PGB) which obtains its finite mass only at loop level. Having analyzed the masses of the fields in the minimum, we need to test the stability of the system and see whether the minimum is attainable by considering the beta functions.

In the following we use the unitary gauge for the Higgs doublet. Then the potential becomes

$$V_{\text{CSMG+}} = \frac{\lambda_1}{4}H^4 + \frac{\lambda_2}{2}\xi^2H^2 + \lambda_3\xi^4 \tag{3.41}$$

Following the Gildener-Weinberg formalism, we parametrize the scalar fields in fields space in terms of the angle α:

$$H = \phi\sin\alpha, \quad \xi = \phi\cos\alpha$$

where ϕ is the distance from the origin in field space and $\alpha \in (-\frac{\pi}{4}, \frac{\pi}{2})$. The Gildener-Weinberg mechanism tells us that a non-zero vacuum expectation value will be generated in the flat direction in scalar field space $\alpha = \bar{\alpha}$,

$$\langle H\rangle = v_H = \langle\phi\rangle\sin\bar{\alpha}, \quad \langle\xi\rangle = v_\xi = \langle\phi\rangle\cos\bar{\alpha}$$

which we can find from the Gildener-Weinberg conditions (3.16) and (3.17):

$$\left.\begin{aligned} 0 &= \left.\frac{\partial V^{(0)}}{\partial\sin\alpha}\right|_{\alpha=\bar{\alpha},\phi=\langle\phi\rangle} \\ 0 &= \left.\frac{\partial V^{(0)}}{\partial\cos\alpha}\right|_{\alpha=\bar{\alpha},\phi=\langle\phi\rangle} \\ V(0)(\phi,\alpha)&\Big|_{\alpha=\bar{\alpha},\phi=\langle\phi\rangle} = 0 \end{aligned}\right\} \quad\Rightarrow\quad \begin{aligned} &4\lambda_1\lambda_3 - \lambda_2^2 = 0 \\ &\tan^2\bar{\alpha} = -\frac{\lambda_2}{\lambda_1} > 0 \end{aligned} \tag{3.42}$$

where the coupling constants are evaluated at the Gildener-Weinberg scale Λ_{GW}. Assuming $\lambda_1(\Lambda_{GW}) > 0$, we thus require $\lambda_2(\Lambda_{GW}) < 0$.

Next we find the tree-level masses by expanding the potential (3.41) around the vacuum:

[48] We follow the procedure and ideas of [77] (similar to [78]), but after carefully recalculating some results we found that [77] mistakenly used $\tan^2\alpha = -\frac{\lambda_2}{2\lambda_1}$ instead of $\tan^2\alpha = -\frac{\lambda_2}{\lambda_1}$ which we use in this thesis. Therefore some formulae used here do not match with Helmboldt et al.

	PGB	$\mathrm{M}_{\text{Higgs}}$	υ_H/υ_ξ
$\alpha > 0$	$= \tilde{\rho}_\xi$	m_+	< 1
$\alpha < 0$	$= H_{\text{LHC}}$	m_{PGB}	> 1

Table 3.1 – *The results of the Gildener-Weinberg analysis on the minimally extended Conformal Standard Model.*

$$V_{\text{CSM}} = \frac{\lambda_1}{4}(\upsilon_H + \rho_H)^4 + \frac{\lambda_2}{2}(\upsilon_\xi + \rho_\xi)^2(\upsilon_H + \rho_H)^2 + \lambda_3(\upsilon_\xi + \rho_\xi)^4$$

where fields ρ_i are the fluctuations around the vacua υ_i and (3.42) ensures that the terms linear in fluctuations vanish. Then we find

$$V_{\text{quadratic}} = \frac{1}{2}(\rho_H\,\rho_\xi)M^2\begin{pmatrix}\rho_H\\ \rho_\xi\end{pmatrix} \quad \text{where} \quad M^2 = \begin{pmatrix}3\lambda_1\upsilon_H^2 + \lambda_2\upsilon_\xi^2 & 2\lambda_2\upsilon_h\upsilon_\xi\\ 2\lambda_2\upsilon_h\upsilon_\xi & 12\lambda_3\upsilon_\xi^2 + \lambda_2\upsilon_H^2\end{pmatrix} \tag{3.43}$$

At tree level and at the Gildener-Weinberg scale, the potential is minimized and a massive and massless scalar field can be distinguished. To identify the Goldstone boson of broken conformal invariance we use the orthogonal matrix U to diagonalize the mass matrix and define the mass eigenstates:

$$\begin{pmatrix}\bar{\rho}_H\\ \bar{\rho}_\xi\end{pmatrix} = U\begin{pmatrix}\rho_H\\ \rho_\xi\end{pmatrix} = \begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}\rho_H\\ \rho_\xi\end{pmatrix}$$

where $\bar{\rho}_H$ is to be identified with the Higgs boson found at the LHC H_{LHC}as we want it to mainly consist out of the Standard Model doublet field. The masses of $\bar{\rho}_i$ are given by the eigenvalues of the mass matrix:

$$2m_\pm^2 = \mathrm{Tr}\big(M^2\big) \pm \sqrt{[\mathrm{Tr}(M^2)]^2 - 4\det(M^2)} \quad \Rightarrow \quad m_+^2 = 2(\lambda_1 - \lambda_2)\upsilon_H^2, \quad m_-^2 = 0 \tag{3.44}$$

where we used (3.42) to write $\lambda_3 = \lambda_2^2/4\lambda_1$ and $\upsilon_\xi^2 = -\frac{\lambda_1\upsilon_H^2}{\lambda_2}$. As expected, the mass matrix is only diagonalized if $\tan(2\alpha) = \tan(2\bar{\alpha})$ and the spectrum in the broken phase contains one scalar degree of freedom with vanishing tree-level mass.

To see which state corresponds to the PGB, we distinguish between $\alpha > 0$ and $\alpha < 0$. Starting with the former, we see that the diagonalized mass matrix is $\mathrm{diag}(m_+^2, m_-^2)$, meaning that $\bar{\rho}_H = H_{\text{LHC}}$ has mass m_+ and $\bar{\rho}_\xi$ is to be identified with the PGB for broken conformal invariance. Also

$$\tan(2\alpha) > 0 \to \tan(2\bar{\alpha}) > 0 \to \tan^2(\bar{\alpha}) < 1 \to \upsilon_H < \upsilon_\xi$$

For $\alpha < 0$ the opposite is true and the pseudo-Goldstone boson corresponds to the Higgs boson found at the LHC. These results are summarized in Table 3.1.

The next step is take the radiative corrections into account, allowing us to establish the PGB mass at 1-loop. In the flat direction the 1-loop effective potential is given by (3.19)

$$V_{\text{eff}}^{(1)}(n\phi) = \phi^4 F(t,g) = A(g)\phi^4 + B(g)\phi^4\log\left(\frac{\phi^2}{\Lambda_{GW}^2}\right)$$

where the coefficients are given by equation (3.20). This potential develops a nontrivial minimum when $B > 0$ which we analyze in terms of the SM contributions and additional scalar contributions as in (3.26). If $\bar{\rho}_\xi$ was the PGB, then its mass would be determined by B_{SM} and the Higgs mass. However, m_{Higgs} is not large enough to overcome the large negative contribution of the top quark and we would still end up with $B < 0$. Therefore, $\alpha > 0$ is ruled out. Accordingly, we should identify the Higgs boson as the PGB of broken conformal invariance, which as such has vanishing tree-level mass at Λ_{GW} and instead has the 1-loop mass (3.27)

$$M^2_{\text{Higgs}} = 8B_{\text{add}} \langle\phi\rangle^2 - K(\Lambda_{GW}), \quad \text{where} \quad K(\Lambda_{GW}) = -8B_{\text{SM}}(\Lambda_{GW}) \langle\phi\rangle^2 > 0$$

The theory's spectrum contains the additional scalar with mass m_+:

$$B_{\text{add}} = \frac{m_+^4}{64\pi^2 \langle\phi\rangle^4} = \frac{(\lambda_1(\Lambda_{GW}) - \lambda_2(\Lambda_{GW}))^2}{16\pi^2} \sin^4 \bar{\alpha}$$

Using the above and that we can generically write the electroweak VEV as $\langle\phi\rangle = v = v_H + cv_\xi$ with $c \leq 0$, we find

$$(\lambda_1(\Lambda_{GW}) - \lambda_2(\Lambda_{GW}))^2 \sin^2 \bar{\alpha} = \frac{2\pi^2(M^2_{\text{Higgs}} + K)}{v_H^2} > \frac{2\pi^2(M^2_{\text{Higgs}} + K)}{v^2} \equiv r^2 \tag{3.45}$$

Recalling that $\lambda_2 < 0$, we solve for λ_1

$$0 < \lambda_1(\lambda_2) < \frac{\lambda_2^2 - r^2}{\lambda_2} \quad \Rightarrow \quad |\lambda_2| < r \tag{3.46}$$

Furthermore, from (3.42) we know that negative scalar mixing angle α implies $\lambda_1 \leq -\lambda_2$. This in turn means $|\lambda_2| > \frac{r}{\sqrt{2}}$. We can now make a best-case approximation for the constraint on the scalar mixing λ_2 if we ignore the Standard Model contributions to the Higgs mass as given by $K(\Lambda_{GW})$ and plug in the numerical values for r:

$$3.6 \leq |\lambda_2| \leq 5.1 \tag{3.47}$$

This is a best-case approximation because K is positive and the exact value for $\lambda_2(\Lambda_{GW})$ will thus always be larger than the above constraints.

To complete the analysis we can use the Callan-Symanzik equation on the effective potential and determine the beta functions. Solving the coupled set of differential equation would allow to check for the emergence of Landau poles below the Planck scale. One can then furthermore check whether there is a scale at which the Gildener-Weinberg conditions are met. Because of the limited relevance to our toy model, we refrain from carrying put this analysis and only give the conclusions.

The authors in [77] concluded that Landau pole is found below the Planck scale for the real singlet case. Carrying out the same analysis for other multiplets[49], they finally conclude that for the theory to develop no Landau poles up till the Planck scale the conformal invariant Standard Model needs to be extended with 2 real gauge singlet fields of which one needs to develop a nonzero VEV. For the addition of only one multiplet they found in each instance that the scalar mixing is too large, making the RG running highly unstable as gauge boson contributions cannot decelerate the running enough. With two additional fields, the authors speculate that one could be a viable Dark Matter candidate.

Before continuing, there is one important remark left to be made regarding the validity of the

[49]Experimental evidence for the scalar sector of the Standard Model is much weaker than that for the other sectors. One of the experimental parameters for the scalar sector is ρ, which is a measure of the ratio of the neutral current to charged current strength in the effective low-energy Lagrangian. In the standard GWS model, at tree level $\rho = 1$. If one introduces N scalar multiplets Φ_i with vacuum expectation values σ_i which have isospin I_i and hypercharges Y_i then ρ for a general (charge-conserving) Higgs structure becomes [70]

$$\rho = \frac{\sum_{i=1}^N [I_i(I_i+1) - \frac{1}{4}Y_i^2]\sigma_i}{\sum_{i=1}^N \frac{1}{2}Y_i^2\sigma_i}$$

Experimentally, we have $\rho = 1.00037 \pm 0.00023$ [17]. The simplest method of satisfying the constraint is to choose only representations such that $I(I+1) = \frac{3}{4}Y^2$: $SU(2) \times U(1)$ singlets obey this restriction, as do SU(2) doublets with $Y = \pm 1$. Although other representations exist, they are very large and will not be considered here. We furthermore restrict ourselves to one additional multiplet to the Standard Model.

results of Helmboldt et al. [77]. We know from Section 3.1.2 that this is the expected one-loop form using $\overline{\text{MS}}$ regularization. However, in the previous section we already pointed out that regularization breaks the conformal invariance on a quantum level. Even under the assumption that the anomaly is cancelled, it not necessarily mean that we retrieve the above effective potential. Using a regularization prescription that is conformal invariant Ghilencea [79] found a (finite) correction term to the one-loop scalar potential for ϕ and φ, beyond the Coleman-Weinberg term. This would thus change the above picture quantitatively, but does not invalidate the procedure itself.

3.4 Conclusion

The difference between the particle physicist's and relativist's approaches lies in the way the additional scalar is introduced and how scales are generated. We started our discussion regarding a conformal toy model with pointing out that no Higgs mass term is allowed and that a scalar field requires a nonminimal extension in order for it to be conformally invariant. We also showed that a conformal model actually requires an additional scalar besides the Higgs doublet. Due to the gauge symmetry of $\mathcal{L}_{\text{mat}} = \mathcal{L}_{\text{SM}}$, the only allowed Yukawa couplings of the dilaton are then to the right handed neutrino's for which it can become a source of mass, or potential dark matter candidates. Because it cannot interact substantially with visible matter, the dilaton itself could potentially be a dark matter particle candidate, though we will refrain from investigating this line of thought further.

On the other hand, according to 't Hooft's the additional scalar field follows from the unconventional splitting of the metric. If we gauge fix the dilaton field we retrieve the known theories and the wrong sign of the kinetic term for the dilaton does not spoil the unitarity of theory as feared. However, some authors find this a questionable approach as we not only gauge fix the field, but rob the scalar field of all its dynamics: the wrong sign problem is only solved in the conformally broken phase. Furthermore, by setting the value of the scalar field equal to a constant of dimension proportional to mass, any hope for renormalizability of the theory is lost. For example Ohanian [65] therefore argues for an approach where an additional Weyl gauge vector is introduced to deal with the wrong sign and provide an explicit dynamic mechanism for conformal symmetry breaking. This mechanism is similar to the BEH mechanism, but should also explain the transition from a Weyl to Riemannian geometry. Moreover, in Ohanian's model the dilaton is complexified which means introducing 2 additional real scalar fields into the Standard Model as opposed to just one. The latter was also required for Renormalization group stability, analyzed by Helmboldt et al. [77].

Yet, Codello et al. started with promoting a mass parameter to a field with no intent that it should be a dynamical field. So gauge-fixing doesn't seem so objectionable in that respect. Furthermore, non-renormalizability is not a problem in the framework of [49]. 't Hooft goes even one step further by saying that rotating the dilaton to the complex plane results in a renormalizable theory. Its β functions would then allow us to completely fix the adjustable parameters of the theory and thus address the problem of arbitrariness.

Chapter 4

Strengths, challenges and outlook

We started with refreshing our knowledge of the Standard Model. Having dealt with the problem of infinities, in the end we still had to point out several unresolved issues, specifically the unification with gravity. As a way towards a Theory of Everything, we proposed to use conformal invariance as additional symmetry. Let us briefly point out some of the arguments:

1. Conformal symmetry is a highly restraining symmetry allowing us to control for example the number of counterterms that should be included in a quantum theory.
2. Besides the tachyonic mass term, the Standard Model is already conformally invariant.
3. The classical conformal action does not allow a cosmological constant though it will emerge after gauge-fixing the theory.
4. The dilaton, the Goldstone boson of spontaneously broken conformal invariance, can couple to right-handed neutrino's and as such may provide a source for neutrino masses.
5. Similar to the Higgs giving mass to the elementary particles, the dilaton gives mass to the Higgs boson via e.g. the Coleman-Weinberg (and Gildener-Weinberg) procedure.
6. Astronomical and accelerator evidence
    a. 'Another striking hint of scale symmetry occurs on cosmic scales: the (nearly) scale invariant spectrum of primordial fluctuations, as measured by WMAP and the Planck satellite.' [67].
    b. 'Another motivation of the present study comes from a recent important observation that the Standard Model seems to be valid all the way up to the Planck scale and there is no new physics between the electroweak scale and the Planck one.' [57].
    c. In the particular case that the Coleman-Weinberg mechanism is indeed responsible for the breaking of conformal invariance: 'It has been shown that models with extended Higgs sectors can avoid washing out any previous baryon number, as well as generate a sufficient baryon asymmetry. In fact, generation of a baryon asymmetry also requires a first order electroweak phase transition, and CW models always have a first order transition. This leads to greater incentive for considering CW symmetry breaking in extended Higgs models.' [70].

If one requires an exact Weyl symmetry to be realized in gravitational theories at the classical level, only two candidate theories are viable though flawed. The first theory we constructed was based on the work of Mannheim and is called Conformal Weyl Gravity, for which the action is described in terms of the square term of the conformal Weyl tensor. Despite the advantage of renormalizability, the fact that it is not unitary swayed us to consider the other plausible model that is also conformally invariant, namely Conformal Dilaton Gravity.

In this theory, a (ghost-like) scalar field is introduced in such a way that it couples to the scalar curvature in a conformally invariant manner. Even if this theory is a unitary theory owing to the presence of only second-order derivative terms, it is nonrenormalizable. With the great progress made in working with effective field theories, we do choose to develop a toy model in this sense.

In developing our toy model we made various assumptions, each of which needs to be validated. We have assumed (in no particular order)

1. 4 spacetime dimensions. Maybe there are more? Or should we work in less dimensions to deal with the encountered problems?
2. flat spacetime (though not always explicitly). Maybe we should consider curved spacetime?
3. the minimal number of fermion generations. Maybe there are more?
4. the Higgs mechanism is responsible for breaking the electroweak sector and subsequent generation of mass. We assumed the breaking of conformal invariance leads to the electroweak breaking.
5. a torsionfree connection. Maybe it should be included as well?
6. a conformal toy model that required one additional scalar. This is a real $SU_c(3) \times SU_L(2) \times U_Y(1)$ singlet. Maybe we need more scalars and/or maybe they are in a doublet representation? Or a whole other representation in case the Standard Model is embedded in e.g. $SU(5)$ as suggested in various GUT models?
7. our theory should be unitary. However, maybe PT symmetry is sufficient as Mannheim suggests, making Conformal Weyl Gravity more appealing yet at the same time opening up a whole can of possibilities.
8. our theory should be locally Weyl invariant (i.e. conformally invariant) such that Einstein Gravity and the Standard Model emerge as effective theories. On the other hand, maybe global Weyl invariance (scale invariance) or some other global symmetry is sufficient? Or Einstein Gravity in the form of the EInstein-Hilbert action need not necessarily emerge as an effective field theory?
9. our action should contain only local terms.

A wealth of different theories have emerged in the literature, each slightly different from the other in one or more of the above aspects. Because the terms ‘conformal’ and ‘scale’ symmetry are often abused, especially a lot has been developed in the field of global Weyl invariance.

Even though we are not able to fundamentally support each of the above assumptions, before anything else we should check whether our toy model could reproduce the known phenomenology. More importantly, we had to explain how our toy model could still account for scales. In that discussion we distinguished between a physical and an unphysical dilaton field.

According to the first the conformal anomaly arises upon quantization of a conformal invariant model. We demonstrated that under the bold assumption that there exist a scale above which the conformal anomaly is cancelled, Coleman-Weinberg radiative symmetry breaking will bring back a dimensionfull parameter in our theory which ensures subsequent electroweak symmetry breaking [77]. For such an analysis to be carried out for our full Conformal Standard Model with Gravity toy model we need to take the nonminimal coupling to the Ricci curvature into account and further investigate whether the used effective action indeed preserves local Weyl invariance (i.e. research the topic of scale-invariant regularization schemes).

The theory with an unphysical dilaton resulted from a more geometrical approach, namely Weyl gauging as explained by Codello et al. [49]. Because in that framework a conformal transformation switches units, and the physics should not depend on the choice of unit, the conformal anomaly should thus not arise in this geometric setting. Using one mass parameter to define the dilaton, all dimensionfull parameters can be expressed in terms of the dilaton. Codello et al. showed that there is a renormalization method that ensures the conformal invariance is transferred to the quantum theory. Scales exist after gauge fixing the dilaton back to a constant.

Having established that conformal invariance can be transferred to a quantum theory and scales can be generated, there is still an abundance of issues and questions that need to be addressed, besides assessing the validity of the above assumptions. To name a few:

1. Besides the conformal anomaly, are there any other anomalies in our toy model?
2. Is our model truly conformally invariant? According to [91] theories whose action is Weyl gauge invariant but which are associated with Riemannian spacetime manifolds cannot be actually conformal invariant since the affine properties of the Riemann space are modified by the conformal transformations.
3. How do we properly extend the framework of quantum field theory to curved spacetime and nonzero temperature? Theories hoping to describe the very early universe ought to be treated as quantum field theories in curved spacetime with finite temperature. Unfortunately, at present we do not have a clear prescription how to combine quantum field theory at non-zero temperature and quantum field theory in curved spacetime. In the separate fields there also some problems. For example because generic curved spacetimes are not time-translation invariant, no meaningful notion of a Hamilton operator or 'energy' exists, and thus we have no means to select or define a vacuum state. Actually, the whole concept of what a particle is, is rather meaningless.
4. In our discussion we have ignored surface terms. However, from the derivation of the Einstein equation from the Einstein-Hilbert tensor that is not always trivial nor correct.

Finally, we also would like to point to [92] for a nice review on the foundational questions any theory of Quantum Gravity should be able to address.

Appendices

Appendix A

Standard Model parameters

The Standard Model including including neutrino masses and mixing angles depends on 25 independent parameters. Namely, for example, 6 lepton masses, 6 quark masses, the coupling constants $g_1 = g', g_2 = g, g_s$, 3 quark flavour mixing angles and 1 complex CP violating phase, 3 neutrino mixing angles and 1 CP violating phase (assuming neutrino's are Dirac), the Higgs mass M_H and quartic coupling constant λ. Note that this set is not unique as we could have equally well used e.g. the $W^\pm, Z$ gauge boson masses, the Weinberg mixing angle and the Higgs VEV υ instead of g_1, g_2, M_H and λ because they are related to each other (see Section 1.2.2).

$$M_H = \sqrt{2\mu^2}, \quad \upsilon = \sqrt{\frac{\mu^2}{\lambda}}, \quad M_W = \frac{g_1 \upsilon}{2}, \quad \tan(\theta_W) = \frac{g_1}{g_2}, \quad M_Z = \frac{M_W}{\cos(\theta_W)}$$

The value of υ is fixed by the Fermi coupling $G_F = 1.1663787(6) \cdot 10^{-5}\,\mathrm{GeV}^{-2}$ according to $\upsilon = (\sqrt{2G_F})^{-1/2}$ which is determined from muon decay measurements.

The values below are taken rom the Particle Data Group [17]. The figures in parentheses after the values give the 1-standard-deviation uncertainties in the last digits, i.e. $m_H = 125.09(24)\mathrm{GeV}/c^2 = 25.09 \pm 0.24\mathrm{GeV}/c^2$. For the top quark mass $173.21(51)(71)\mathrm{GeV} = 173.21 \pm 0.51 \pm 0.71\mathrm{GeV}$. However, before giving the data, we need to make two comments.

Firstly, the mass of the neutrino's is an ongoing subject of research. The mass eigenstates ν_i, $i = 1, 2, 3$ are distinct from the flavour eigenstates ν_M, $M = e, \mu, \tau$. Currently we are only able to give the mass differences

$$\Delta m_{ij}^2 = m_i^2 - m_j^2$$

where we know that $m_2 > m_1$. The normal convention is $m_3 > m_{1,2}$ as opposed to the inverted convention $m_3 < m_{1,2}$. Here we give the results for the former. For the mxing angle we will use the same convention as Section 1.2.3: $s_{ij} = \sin\theta_{ij}$.

The second comment concerns the quark mixing. Usually, the Cabibbo-Kobayashi-Maskawa (CKM) matrix is given in terms of the Wolfenstein parametrization. Rather than specefying the mixing angles and CP-violating phase, $\tilde{\lambda}, \bar{\eta}, A, \bar{\rho}$ are used[50]:

$$s_{12} = \tilde{\lambda}, \quad s_{23} = A\tilde{\lambda}^2, \quad s_{13} e^{\mathrm{i}\delta} = A\tilde{\lambda}^3(\rho + \mathrm{i}\eta)$$

These defintions ensures that $\bar{\rho} + \mathrm{i}\bar{\eta}$ is phase convention independent.

[50] Actually, the PDG uses λ but to avoid confusion with the Higgs quartic coupling, we use $\tilde{\lambda}$ for the CKM parameter.

Parameters of the Standard Model			
Symbol	Description	Renormalization point	Value
m_e	Electron mass		$0.510\,998\,9461(31)$ MeV/c^2
m_μ	Muon mass		$105.658\,3745(24)$ MeV/c^2
m_τ	Tau mass		$1776.86(12)$ MeV/c^2
Δm_{21}^2	Mass difference between ν_1 and ν_2		$7.53(18)\cdot 10^{-5}$eV^2
$\left\|\Delta m_{32}^2\right\|$	Mass difference between ν_2 and ν_3		$2.44(6)\cdot 10^{-3}$eV^2
m_u	Up quark mass	$\mu_{\overline{MS}} = 2$ GeV	$2.2^{+0.6}_{-0.4}$ MeV/c^2
m_d	Down quark mass	$\mu_{\overline{MS}} = 2$ GeV	$4.7^{+0.5}_{-0.4}$ MeV/c^2
m_s	Strange quark mass	$\mu_{\overline{MS}} = 2$ GeV	96^{+8}_{-4} MeV
m_c	Charm quark mass	$\mu_{\overline{MS}} = m_c$	$1.27(3)$ GeV
m_b	Bottom quark mass	$\mu_{\overline{MS}} = m_b$	$4.18^{+0.04}_{-0.03}$ GeV
m_t	Top quark mass		$173.21(51)(71)$ GeV
$(s_{12})^2$	PMNS 12-mixing angle		$0.304(14)$
$(s_{23})^2$	PMNS 23-mixing angle		$0.51(5)$
$(s_{13})^2$	PMNS 13-mixing angle		$0.0219(12)$
δ^ν	PMNS CP-violating phase		1.35π
$\tilde{\lambda}$	CKM parameter		$0.22496(48)$
A	CKM parameter		$0.823(13)$
$\bar{\rho}$	CKM parameter		$0.141(19)$
$\bar{\eta}$	CKM parameter		$0.349(12)$
$\alpha_s = \frac{g_s^2}{4\pi}$	Strong coupling constant	$\mu_{\overline{MS}} = m_Z$	$0.1182(12)$
m_H	Higgs mass		$125.09(24)$ GeV/c^2
m_W	$W^\pm$ mass		$80.385(15)$ GeV/c^2
m_Z	Z^0 mass		$91.1876(21)$ GeV/c^2
m_γ	Photon mass		$< 1\cdot 10^{-18}$ eV
$\sin^2(\theta_W)$	Weak mixing angle	$\mu_{\overline{MS}} = m_Z$	$0.231\,29(5)$

Table A.1 – *The table contains the experimental values of the parameters that determine the Standard Model of particle physics retrieved from the Particle Data Group [17].*

Appendix B

Derivation of the Einstein equations

In this appendix we will present the derivation of the Einstein equation via the principle of least action. In order to do that we make use of the following results from [80, Chapter 16]:

$$
\begin{aligned}
\delta g^{\mu\nu} &= -g^{\alpha\mu} g^{\beta\nu} \delta g_{\alpha\beta} && \text{(B.1a)}\\
\delta\sqrt{-g} &= \tfrac{1}{2}\sqrt{-g} g^{\mu\nu} \delta g_{\mu\nu} && \text{(B.1b)}\\
\delta\Gamma^{\alpha}_{\sigma\nu} &= \tfrac{1}{2} g^{\alpha\beta}\left[(\delta g_{\beta\sigma})_{;\nu} + (\delta g_{\beta\nu})_{;\sigma} - (\delta g_{\sigma\nu})_{;\beta}\right] && \text{(B.1c)}\\
\delta R^{\alpha}{}_{\nu\lambda\sigma} &= (\delta\Gamma^{\alpha}_{\sigma\nu})_{;\lambda} - (\delta\Gamma^{\alpha}_{\lambda\nu})_{;\sigma} && \text{(B.1d)}\\
\delta R_{\mu\nu\lambda\sigma} &= \delta g_{\mu\alpha} R^{\alpha}{}_{\nu\lambda\sigma} + g_{\mu\alpha}\delta R^{\alpha}{}_{\nu\lambda\sigma} && \text{(B.1e)}\\
\delta R_{\mu\nu} &= \delta R^{\alpha}{}_{\mu\alpha\nu} = (\delta\Gamma^{\alpha}_{\mu\nu})_{;\alpha} - (\delta\Gamma^{\alpha}_{\alpha\nu})_{;\mu} \\
&= \tfrac{1}{2} g^{\sigma\tau}\left((\delta g_{\mu\nu})_{;\sigma;\tau} + (\delta g_{\sigma\tau})_{;\mu;\nu} - (\delta g_{\mu\sigma})_{;\nu;\tau} - (\delta g_{\nu\tau})_{;\mu;\sigma}\right) && \text{(B.1f)}\\
\delta R &= R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu} = g_{\mu\nu}\delta R^{\mu\nu} - R^{\mu\nu}\delta g_{\mu\nu} \\
&= g^{\mu\nu} g^{\sigma\tau}\left((\delta g_{\mu\nu})_{;\sigma;\tau} - (\delta g_{\mu\sigma})_{;\nu;\tau}\right) - R^{\mu\nu}\delta g_{\mu\nu} && \text{(B.1g)}
\end{aligned}
$$

Start by recalling the Einstein-Hilbert action (2.17)

$$
S_{\text{EH}} = \frac{1}{2\kappa^2}\int \mathrm{d}^d x \sqrt{g}\,(R - 2\Lambda)\,, \quad \text{with} \quad \kappa^2 = 8\pi G_N
$$

Writing $R = g^{\mu\nu} R_{\mu\nu}$, we vary the action to get

$$
\begin{aligned}
\delta S_{\text{EH}} &= \frac{1}{2\kappa^2}\int \mathrm{d}^d x \left[\delta\sqrt{g}\,(g^{\mu\nu}R_{\mu\nu} - 2\Lambda) + \sqrt{g}(\delta g^{\mu\nu})R_{\mu\nu} + \sqrt{g} g^{\mu\nu}\delta R_{\mu\nu}\right] \\
&= \frac{1}{2\kappa^2}\int \mathrm{d}^d x \left[-\tfrac{1}{2}\sqrt{-g} g_{\mu\nu}\delta g^{\mu\nu}\,(g^{\mu\nu}R_{\mu\nu} - 2\Lambda) + \sqrt{g} R_{\mu\nu}\delta g^{\mu\nu}\right] + \frac{1}{2\kappa^2}\int \mathrm{d}^d x \sqrt{g} g^{\mu\nu}\delta R_{\mu\nu} \\
&= \frac{\sqrt{g}}{2\kappa^2}\int \mathrm{d}^d x \left[-\tfrac{1}{2} g_{\mu\nu} R + g_{\mu\nu}\Lambda + R_{\mu\nu}\right]\delta g^{\mu\nu} + \frac{1}{2\kappa^2}\int \mathrm{d}^d x \sqrt{g} g^{\mu\nu}\delta R_{\mu\nu}
\end{aligned}
\tag{B.2}
$$

In the first term we already recognize the vacuum Einstein equation:

$$
G_{\mu\nu} = -\tfrac{1}{2} g_{\mu\nu} R + g_{\mu\nu}\Lambda + R_{\mu\nu} = 0
$$

Now we just need to show that the second term in (B.2) is a boundary term and indeed vanishes. Using (D.6f), we write

$$\begin{aligned} g^{\mu\nu}\delta R_{\mu\nu} &= g^{\mu\nu}\Big[(\delta\Gamma^{\lambda}_{\mu\nu})_{;\lambda} - (\delta\Gamma^{\lambda}_{\lambda\mu})_{;\nu}\Big] = g^{\mu\nu}\Big[\delta\Gamma^{\lambda}_{\mu\nu} - \delta^{\lambda}_{\nu}\delta\Gamma^{\beta}_{\beta\mu}\Big]_{;\lambda} \\ &= \Big[g^{\mu\nu}\delta\Gamma^{\lambda}_{\mu\nu} - g^{\lambda\mu}\delta\Gamma^{\nu}_{\nu\mu}\Big]_{;\lambda} \quad = \nabla_\lambda(\{\})\delta w^\lambda \end{aligned}$$

where in the second line we used the fact that we are using the Levi-Cevita connection, meaning that the nonmetricity $g_{\mu\nu;\alpha} = 0$ vanishes. To proof that this is indeed a divergence, we follow [16] and start by recalling the Christoffel symbols (2.12), which, upon contracting λ and μ, become

$$\begin{aligned} \{^{\mu}_{\mu\nu}\} &= \tfrac{1}{2}g^{\mu\rho}\left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\right) \\ &= \tfrac{1}{2}g^{\mu\rho}\partial_\nu g_{\rho\mu} + \tfrac{1}{2}g^{\mu\rho}(\partial_\mu g_{\rho\nu} - \partial_\rho g_{\mu\nu}) \\ &= \tfrac{1}{2}g^{\mu\rho}\partial_\nu g_{\rho\mu} \\ &= \tfrac{1}{2}\,\mathrm{Tr}\big(g^{\lambda\rho}\partial_\nu g_{\rho\mu}\big) \\ &= \frac{1}{\sqrt{g}}\partial_\nu\sqrt{g} \end{aligned} \tag{B.3}$$

where we used $Tr[M^{-1}\partial_\nu M] = \partial_\nu \ln\det M$ in going from the almost last to the last line. With the above result, we can write the divergence of a vector as

$$\nabla_\mu X^\mu = \partial_\mu X^\mu + \Gamma^{\mu}_{\mu\nu}X^\nu = \frac{1}{\sqrt{g}}\partial_\nu(\sqrt{g}X^\mu)$$

and thus

$$\begin{aligned} \sqrt{g}g^{\mu\nu}\delta R_{\mu\nu} &= \nabla_\lambda\Big(g^{\mu\nu}\delta\Gamma^{\lambda}_{\nu\mu} - g^{\mu\lambda}\delta\Gamma^{\nu}_{\nu\mu}\Big) \\ &= \nabla_\lambda\Big((g^{\mu\lambda}g^{\nu\sigma} - g^{\mu\nu}g^{\lambda\sigma})\nabla_\sigma\delta g_{\mu\nu}\Big) \\ &= \partial_\lambda(\sqrt{g}\delta w^\lambda) \end{aligned}$$

where we plugged in the variation for the Chrstoffel symbols (D.6c) in line 2. Now, equation (B.2) becomes

$$\delta S_{\mathrm{EH}} = \frac{\sqrt{g}}{2\kappa^2}\int \mathrm{d}^d x G_{\mu\nu}\delta g^{\mu\nu} + \frac{1}{2\kappa^2}\int \mathrm{d}^d x\partial_\alpha(\sqrt{g}\delta w^\alpha) = \frac{\sqrt{g}}{2\kappa^2}\int \mathrm{d}^d x G_{\mu\nu}\delta g^{\mu\nu} + B_{\mathrm{EH}}$$

However, the boundary term does not drop out when integrated over all space because, as we will show, it depends both on the variation of the metric on the boundary and on its normal derivative, and it is not consistent to require both to be zero (i.e. to impose both Dirichlet and Neumann boundary conditions).

Performing the integral over a space-time region $\mathcal{V}$ bounded by the hypersurface $\partial\mathcal{V} = \Sigma$, the boundary terms becomes, upon using thethe Gauss integral formula

$$B_{\mathrm{EH}} = \frac{1}{2\kappa^2}\int_{\mathcal{V}} \mathrm{d}^d x\sqrt{g}\nabla_\lambda\delta w^\lambda) = \epsilon\oint_\Sigma \mathrm{d}^n y\sqrt{h}N_\lambda\delta w^\lambda$$

where N_α is the normal vector to the boundary Σ in $\mathcal{V}$, y the coordinates on the boundary, and $h_{\mu\nu}$ is the induced metric on the boundary: $g_{\mu\nu} = h_{\mu\nu} + \epsilon N_\mu N_\nu$ with $\epsilon = N_\mu N^\mu = \pm 1$ (+ when the boundary is timelike and - when the boundary is spacelike). Then,

$$N_\lambda\delta w^\lambda = N_\lambda(g^{\mu\lambda}g^{\nu\sigma} - g^{\mu\nu}g^{\lambda\sigma})\nabla_\sigma\delta g_{\mu\nu} = (N^\mu h^{\nu\sigma} - N^\sigma h^{\mu\nu})\nabla_\sigma\delta g_{\mu\nu}$$

The first term is zero under Dirichlet boundary conditions $\delta g_{\mu\nu}\big|_\Sigma = 0$ on the metric at the boundary Σ. However, the second term is normal derivative and nonzero at the boundary:

$$N_\lambda\delta w^\lambda\Big|_\Sigma = -N^\sigma h^{\mu\nu}\nabla_\sigma\delta g_{\mu\nu}\Big|_\Sigma = -h^{\mu\nu}N^\sigma\partial_\sigma\delta g_{\mu\nu}\Big|_\Sigma$$

Thus

$$B_{\text{EH}} = \frac{1}{2\kappa^2}\epsilon \oint_\Sigma \mathrm{d}^n y \sqrt{h}(-h^{\mu\nu} N^\sigma \partial_\sigma \delta g_{\mu\nu})$$

Therefore, we need to look for an action $S_{\text{Gravity}} = S_{\text{EH}} + S_{\text{Boundary}}$ with S_{Boundary} a surface term such that

$$B_{\text{EH}} + \delta S_{\text{Boundary}}\Big|_{\delta g=0} = 0$$

which suggest we could add $\delta S_{\text{Boundary}} = -B_{\text{EH}}$ to ensure proper behaviour of the Einstein-Hilbert action under Dirichlet boundary conditions. However, the above condition does not fix the surface term uniquely because $B_{\text{EH}} + \delta S_{\text{Boundary}}$ may differ from zero away from Σ. We want to make use of this freedom becausetThe current form is not very attractive, in particular as it is non-covariant not only with respect to bulk coordinate transformations but also with respect to boundary coordinate transformations. Accordingly, J. W. York, and by G. Gibbons and S. Hawking, constructed the currently preferred, geometrically transparent, boundary term

$$S_{\text{GHY}} = \frac{1}{2\kappa^2} 2\epsilon \oint_\Sigma \mathrm{d}^n y \sqrt{h} K, \quad \text{with} \quad K = \nabla_\alpha N^\alpha$$

K is the trace of the extrinsic curvature of the boundary Σ. We determine its variation to show that it cancels B_{EH}:

$$\begin{aligned}
K &= \nabla_\alpha N^\alpha = h^{\alpha\beta}\nabla_\alpha N_\beta = h^{\alpha\beta}(\partial_\alpha N_\beta - \Gamma^\sigma_{\alpha\beta} N_\sigma) \\
\delta K &= -\tfrac{1}{2} h^{\alpha\beta} g^{\sigma\rho} N_\sigma \left[(\delta g_{\rho\beta})_{;\alpha} + (\delta g_{\alpha\rho})_{;\beta} - (\delta g_{\alpha\beta})_{;\rho}\right] \\
&= +\tfrac{1}{2} h^{\alpha\beta} N^\rho \partial_\rho \delta g_{\alpha\beta}
\end{aligned}$$

where we have use the fact that the tangential derivatives of $\delta g_{\alpha\beta}$ vanish on Σ. We immediately see that the variation of the Gibbs-Hawking-York boundary term indeed cancels the boundary term which we get from the variation of the Einstein-Hilbert action (B_{EH}). Thus, for Dirichlet boundary conditions the variation of S_{Gravity} gives the gravitational part of the Einstein equations:

$$\frac{\delta S_{\text{Gravity}}}{\delta g^{\mu\nu}} = \frac{\delta S_{\text{EH}}}{\delta g^{\mu\nu}} + \frac{\delta S_{\text{GHY}}}{\delta g^{\mu\nu}} = \frac{\sqrt{g}}{2\kappa^2}\left[-\tfrac{1}{2} g^{\mu\nu} R + g_{\mu\nu}\Lambda + R_{\mu\nu}\right] = 0$$

Including matter to the above gives

$$\frac{\delta S_{\text{GR}}}{\delta g^{\mu\nu}} = \frac{\delta S_{\text{EH}}}{\delta g^{\mu\nu}} + \frac{\delta S_{\text{GHY}}}{\delta g^{\mu\nu}} + \frac{\delta S_{\text{mat}}}{\delta g^{\mu\nu}} = 0$$

But we defined the Hilbert energy-momentum tensor (2.20) exactly as

$$T^{\mu\nu} \equiv -\frac{2}{\sqrt{g}} \frac{\delta(\sqrt{g}\mathcal{L}_{\text{mat}})}{\delta g_{\mu\nu}}$$

meaning we indeed retrieve the Einstein equations (2.19):

$$R_{\mu\nu} - \tfrac{1}{2} g^{\mu\nu} R + g_{\mu\nu}\Lambda = 8\pi G_N T_{\mu\nu}$$

Appendix C

Conformal invariance of Weyl tensor

Here we show explicitly the invariance of the Weyl tensor (2.43):

$$C^{\mu}{}_{\nu\rho\sigma} = R^{\mu}{}_{\nu\rho\sigma} + \tfrac{2}{(n-2)(n-1)} g^{\mu}{}_{[\rho} g_{\sigma]\nu} R + \tfrac{2}{n-2} \left(g^{\mu}{}_{[\sigma} R_{\rho]\nu} + g_{\nu[\rho} R_{\sigma]}{}^{\mu} \right)$$

under the conformal transformation (2.28)

$$\tilde{g}_{\mu\nu} = \Omega^2(x) g_{\mu\nu} \qquad \tilde{g}^{\mu\nu} = \Omega^{-2}(x) g^{\mu\nu}$$

where the second equation follows from the orthogonality of the metric. For completeness we note that the determinant transforms as

$$\tilde{g} = \det\{\tilde{g}_{\mu\nu}\} = \epsilon^{\mu\ldots\sigma} \tilde{g}_{0\mu} \ldots \tilde{g}_{n\sigma} = \epsilon^{\mu\ldots\sigma} \Omega^2(x) g_{0\mu} \ldots \Omega^2(x) g_{n\sigma} = \Omega^{2n}(x) g$$

and that $\tilde{g}_{\mu\nu}$ is used to raise and lower tilded expressions, whereas $g_{\mu\nu}$ is used for the regular expressions, e.g.

$$\tilde{g}_{\nu\sigma} \tilde{R}^{\mu}_{\rho} = \tilde{g}_{\nu\sigma} \tilde{g}^{\mu\lambda} \tilde{R}_{\lambda\rho} = g_{\nu\sigma} g^{\mu\lambda} \tilde{R}_{\lambda\rho}$$

The following derivation is based on [15, Appendix D] and uses the finite expressions for the behaviour of tensors under a conformal transformation. Equivalently we could also have chosen to do this derivation in terms of infinitesimal expressions. Note that the explicit x dependence is dropped to minimize cluttering of the notation.

First we look at the metric-compatible and torsion-free Levi-Cevita connection

$$\Gamma^{\mu}_{\nu\lambda} = \left\{{\mu \atop \nu\lambda}\right\} = \tfrac{1}{2} g^{\mu\rho} \left(\partial_\nu g_{\lambda\rho} + \partial_\lambda g_{\nu\rho} - \partial_\rho g_{\lambda\nu} \right)$$

which transforms under a conformal transformation according to

$$\begin{aligned} \tilde{\Gamma}^{\mu}_{\nu\lambda} &= \tfrac{1}{2} \tilde{g}^{\mu\rho} \left(\partial_\nu \tilde{g}_{\lambda\rho} + \partial_\lambda \tilde{g}_{\nu\rho} - \partial_\rho \tilde{g}_{\lambda\nu} \right) \\ &= \tfrac{1}{2} g^{\mu\rho} \left(\partial_\nu g_{\lambda\rho} + \partial_\lambda g_{\nu\rho} - \partial_\rho g_{\lambda\nu} \right) + \Omega^{-1} g^{\mu\rho} \left(g_{\lambda\rho} \partial_\nu \Omega + g_{\nu\rho} \partial_\lambda \Omega - g_{\lambda\nu} \partial_\rho \Omega \right) \\ &= \Gamma^{\mu}_{\nu\lambda} + 2\delta^{\mu}{}_{(\nu} \partial_{\lambda)} \ln \Omega - g_{\nu\lambda} g^{\mu\sigma} \partial_\sigma \ln \Omega \end{aligned} \tag{C.1}$$

where we denote the last two terms of the last line with $H^{\mu}_{\nu\lambda}$. The covariant derivative with respect to the affine connection thus transforms as

$$\tilde{\nabla}_\nu X_\lambda = \nabla_\nu X_\lambda - H^{\mu}_{\nu\lambda} X_\mu \tag{C.2}$$

The Riemann curvature tensor (2.6) can then be written as[51]

$$
\begin{aligned}
\tilde{R}^{\mu}{}_{\nu\rho\sigma} &= \partial_\rho \tilde{\Gamma}^{\mu}_{\nu\sigma} - \partial_\sigma \tilde{\Gamma}^{\mu}_{\nu\rho} + \tilde{\Gamma}^{\lambda}_{\nu\sigma}\tilde{\Gamma}^{\mu}_{\lambda\rho} - \tilde{\Gamma}^{\lambda}_{\nu\rho}\tilde{\Gamma}^{\mu}_{\lambda\sigma} \\
&= R^{\mu}{}_{\nu\rho\sigma} + \partial_\rho H^{\mu}{}_{\sigma\nu} - \partial_\sigma H^{\mu}{}_{\rho\nu} + H^{\lambda}_{\nu\sigma} H^{\mu}_{\lambda\rho} - H^{\lambda}_{\nu\rho} H^{\mu}_{\lambda\sigma} \\
&= R^{\mu}{}_{\nu\rho\sigma} + 2\delta^{\mu}{}_{[\sigma}\partial_{\rho]}\partial_\nu \ln\Omega - 2g_{\nu[\sigma}\partial_{\rho]} g^{\mu\lambda}\partial_\lambda \ln\Omega + 2(\partial_{[\sigma}\ln\Omega)\delta^{\mu}{}_{\rho]}\partial_\nu \ln\Omega \\
&\quad - 2(\partial_{[\sigma}\ln\Omega) g_{\rho]\nu} g^{\mu\lambda}\partial_\lambda \ln\Omega - 2g_{\nu[\sigma}\delta^{\mu}{}_{\rho]} g^{\lambda\kappa}(\partial_\lambda \ln\Omega)\partial_\kappa \ln\Omega \\
&= R^{\mu}{}_{\nu\rho\sigma} + 2\delta^{\mu}{}_{[\sigma}A_{\rho]\nu} - 2g^{\mu\lambda}g_{\nu[\sigma}A_{\rho]\lambda} + 2(\delta^{\mu}{}_{[\rho}B_{\sigma]\nu} - 2g^{\mu\lambda}g_{\nu[\rho}B_{\sigma]\lambda} - 2g_{\nu[\sigma}\delta^{\mu}{}_{\rho]}g^{\lambda\kappa}B_{\lambda\kappa}
\end{aligned}
\tag{C.3}
$$

It is understood that this is the curvature tensor is with respect to the Levi-Cevita connection (the explicit reference to that fact is dropped). Furthermore,

$$
A_{\mu\nu} = \partial_\mu \partial_\nu \ln\Omega, \qquad B_{\mu\nu} = \partial_\mu \ln\Omega \partial_\nu \ln\Omega
$$

Contracting over μ and ρ gives the transformed Ricci tensor:

$$
\begin{aligned}
\tilde{R}_{\nu\sigma} = \tilde{R}^{\mu}{}_{\nu\mu\sigma} &= R_{\nu\sigma} - (n-2)\partial_\nu\partial_\sigma \ln\Omega - g_{\nu\sigma}g^{\mu\lambda}\partial_\mu\partial_\lambda \ln\Omega \\
&\quad + (n-2)(\partial_\nu \ln\Omega)\partial_\sigma \ln\Omega - (n-2)g_{\nu\sigma}g^{\mu\lambda}(\partial_\mu \ln\Omega)\partial_\lambda \ln\Omega \\
&= R_{\nu\sigma} - (n-2)A_{\nu\sigma} - g_{\nu\sigma}g^{\mu\lambda}A_{\mu\lambda} + (n-2)B_{\nu\sigma} - (n-2)g_{\nu\sigma}g^{\mu\lambda}B_{\mu\lambda}
\end{aligned}
\tag{C.4}
$$

The Ricci scalar is then retrieved after contracting the above expression with $\tilde{g}^{\sigma\nu} = \Omega^{-2}g^{\sigma\nu}$:

$$
\begin{aligned}
\tilde{R} = \tilde{g}^{\nu\sigma}\tilde{R}_{\nu\sigma} &= \Omega^{-2}\Big[R - 2(n-1)g^{\nu\sigma}\partial_\nu\partial_\sigma \ln\Omega - (n-2)(n-1)g^{\nu\sigma}(\partial_\nu \ln\Omega)\partial_\sigma \ln\Omega\Big] \\
&= \Omega^{-2}\Big[R - 2(n-1)g^{\nu\sigma}A_{\nu\sigma} - (n-2)(n-1)g^{\nu\sigma}B_{\nu\sigma}\Big]
\end{aligned}
\tag{C.5}
$$

From this result it is immediately obvious that the Einstein-Hilbert action (2.17) is not invariant under conformal transformations.

Next we plug the above formulae for the transformations of the Riemann and Ricci tensor and Ricci scalar in the defintion of the Weyl tensor

$$
\begin{aligned}
\tilde{C}^{\mu}{}_{\nu\rho\sigma} &= \tilde{R}^{\mu}{}_{\nu\rho\sigma} + \tfrac{2}{(n-2)(n-1)}\delta^{\mu}{}_{[\rho}\tilde{g}_{\sigma]\nu}\tilde{R} + \tfrac{2}{n-2}\left(\delta^{\mu}{}_{[\sigma}\tilde{R}_{\rho]\nu} + g^{\mu\lambda}g_{\nu[\rho}\tilde{R}_{\sigma]\lambda}\right) \\
&= R^{\mu}{}_{\nu\rho\sigma} + 2\delta^{\mu}{}_{[\sigma}A_{\rho]\nu} - 2g^{\mu\lambda}g_{\nu[\sigma}A_{\rho]\lambda} + 2(\delta^{\mu}{}_{[\rho}B_{\sigma]\nu} - 2g^{\mu\lambda}g_{\nu[\rho}B_{\sigma]\lambda} - 2g_{\nu[\sigma}\delta^{\mu}{}_{\rho]}g^{\lambda\kappa}B_{\lambda\kappa} \\
&\quad + \tfrac{2}{(n-2)(n-1)}\delta^{\mu}{}_{[\rho}g_{\sigma]\nu}\Big[R - 2(n-1)g^{\nu\sigma}A_{\nu\sigma} - (n-2)(n-1)g^{\nu\sigma}B_{\nu\sigma}\Big] \\
&\quad + \tfrac{2}{n-2}\left[\delta^{\mu}{}_{[\sigma}R_{\rho]\nu} - (n-2)\delta^{\mu}{}_{[\sigma}A_{\rho]\nu} - \delta^{\mu}{}_{[\sigma}g_{\rho]\nu}g^{\kappa\lambda}A_{\kappa\lambda} + (n-2)\delta^{\mu}{}_{[\sigma}B_{\rho]\nu}\right. \\
&\quad - (n-2)\delta^{\mu}{}_{[\sigma}g_{\rho]\nu}g^{\kappa\lambda}B_{\kappa\lambda} + g^{\mu\lambda}g_{\nu[\rho}R_{\sigma]\lambda} - (n-2)g^{\mu\lambda}g_{\nu[\rho}A_{\sigma\lambda} - g^{\mu\lambda}g_{\nu[\rho}g_{\sigma]\lambda}g^{\mu\kappa}A_{\mu\kappa} \\
&\quad \left. +(n-2)g^{\mu\lambda}g_{\nu[\rho}B_{\sigma]\lambda} - (n-2)g^{\mu\lambda}g_{\nu[\rho}g_{\sigma]\lambda}g^{\mu\kappa}B_{\mu\kappa}\right] \\
&= R^{\mu}{}_{\nu\rho\sigma} + \tfrac{2}{(n-2)(n-1)}\delta^{\mu}{}_{[\rho}g_{\sigma]\nu}R + \tfrac{2}{n-2}\left(\delta^{\mu}{}_{[\sigma}R_{\rho]\nu} + g_{\nu[\rho}R_{\sigma]}{}^{\mu}\right) \\
&= C^{\mu}{}_{\nu\rho\sigma}
\end{aligned}
\tag{C.6}
$$

One index needs to be raised for the conformal invariance to hold, because

[51] To compare equation D.7 from Wald [15] to $R^{\mu}{}_{\nu\rho\sigma}$, we first note that we have defined the covariant derivative with respect to the partial derivative: $\nabla_\mu X_\nu = \partial_\mu \Gamma^{\lambda}_{\mu\nu} X_\lambda$ unlike Wald, meaning that we need to substititute ∂_μ for ∇_a in equation D.7. Furthermore, we note that $R_{abc}{}^{d} = R_{c}{}^{d}{}_{ab} = R^{d}{}_{cba}$. Relabeling indices as $d \to \mu, c \to \nu, b \to \rho, a \to \sigma$ then gives $R^{\mu}{}_{\nu\rho\sigma}$.

$$\tilde{C}_{\lambda\nu\rho\sigma} = \tilde{g}_{\lambda\mu}\tilde{C}^{\mu}{}_{\nu\rho\sigma} = \Omega^2 g_{\lambda\mu} C^{\mu}{}_{\nu\rho\sigma} = \Omega^2 C_{\lambda\nu\rho\sigma}$$

The Conformal Weyl action (2.47) behaves under a conformal transformation like

$$\begin{aligned}
S_{\mathrm{CWG}} &= -\alpha_g \int \mathrm{d}^n x \sqrt{-\tilde{g}}\, \tilde{C}_{\mu\nu\rho\sigma}\tilde{C}^{\mu\nu\rho\sigma} \\
&= -\alpha_g \int \mathrm{d}^4 x \sqrt{-\tilde{g}} \tilde{C}^{\alpha}{}_{\beta\mu\nu}\tilde{C}^{\beta}{}_{\alpha\rho\sigma}\tilde{g}^{\mu\rho}\tilde{g}^{\nu\sigma} \\
&== -\alpha_g \int \mathrm{d}^4 x \\
sqrt{-}\Omega^{2n} g\, C^{\alpha}{}_{\beta\mu\nu} C^{\beta}{}_{\alpha\rho\sigma}\, \Omega^{-2} g^{\mu\rho}\, \Omega^{-2} g^{\nu\sigma} & \\
&= -\alpha_g \int \mathrm{d}^d x \sqrt{-g}\, \Omega^{n-4}\, C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}
\end{aligned}$$

which shows that the Conformal Weyl action is only invariant under conformal transformations in $n = 4$ dimensional spacetime. As we have actually nowhere in the above used the constraint from (2.29), we have actually established Weyl invariance (i.e. local scale invariance) of the Weyl tensor, which is a stronger symmetry than conformal symmetry.

Some further notes on the Weyl tensor:

i. From the definition of the Weyl tensor it is obvious that it must be zero for $n \leq 2$.

ii. By comparing the number of indepent components between the Riemann curvature tensor ($\frac{n^2(n^2-1)}{12}$) with those of the Weyl curvature tensor ($\frac{n(n+1)}{2}$) we can show that for $n = 3$ the Weyl tensor also vanishes identically.

iii. In dimensions $n \geq 4$, the $C_{\mu\nu\rho\sigma}$ is generally nonzero. If, however, the Weyl tensor vanishes, the metric is locally conformally flat: there exists a local coordinate system in which the metric tensor is proportional to a constant tensor.

iv. In GR, if matter is absent, the Ricci tensor vanishes. However, spacetime is not necessarily flat in this case since the Weyl tensor contributes curvature to the Riemann curvature tensor and so the gravitational field is not zero in the vaccum. The Weyl tensor allows gravity to propagate in regions where there is no matter/energy source.

A tensor field f has conformal weight α if under a conformal transformation of (2.26) there is a real number α such that $f \to f' = e^{\alpha\sigma(x)} f$. The metric thus has conformal weight 2, its inverse -2 and $\det(g)$ has weight d. If $\omega = 0$ then the tensor field is conformally invariant. Conformal weights tell us how objects transform under rotations and scalings and are related to the spin and scaling dimension of the object.

Appendix D

Derivation of the Bach equation of motion

In this appendix we will derive the Bach equation starting from (??). Then we will give equivalent expressions of (2.49) in terms of the Schouten tensor and in terms of the Ricci tensor and scalar.

Recall the Weyl tensor (2.43) in 4 dimensions, i.e.

$$C_{\mu\alpha\nu\beta} = R_{\mu\alpha\nu\beta} + \tfrac{1}{2}\left(-g_{\mu\nu}R_{\alpha\beta} + g_{\mu\beta}R_{\alpha\nu} + g_{\alpha\nu}R_{\alpha\beta} - g_{\alpha\beta}R_{\mu\nu}\right) + \tfrac{1}{6}\left(g_{\mu\nu}g_{\alpha\beta} - g_{\alpha\nu}g_{\mu\beta}\right)R \tag{D.1}$$

which has the same symmetry properties as the Riemann curvature tensor $R_{\mu\alpha\nu\beta}$:

$$R_{\mu\alpha\nu\beta} = R_{\nu\beta\mu\alpha} \tag{D.2}$$

$$R_{\mu\alpha\nu\beta} = -R_{\alpha\mu\nu\beta} = -R_{\mu\alpha\beta\nu} \tag{D.3}$$

$$R_{\mu\alpha\nu\beta} + R_{\mu\beta\alpha\nu} + R_{\mu\nu\beta\alpha} = R_{\mu[\alpha\nu\beta]} = 0 \tag{D.4}$$

$$\nabla_\beta R^\lambda{}_{\mu\alpha\nu} + \nabla_\nu R^\lambda{}_{\mu\beta\alpha} + \nabla_\alpha R^\lambda{}_{\mu\nu\beta} = R^\lambda{}_{\mu[\alpha\nu;\beta]} = 0 \tag{D.5}$$

where equation (D.4) and (D.5) are known as the first and second Bianchi identity, respectively. The covariant derivatives are written using the semicolon convention $\nabla_\mu v = v_{;\mu}$ and $\nabla_\mu \nabla_\nu v = v_{;\nu;\mu}$.

For the derivation of the Bach equation we furthermore make use of the following results from [80, Chapter 16]:

$$\begin{aligned}
\delta g^{\mu\nu} &= -g^{\alpha\mu}g^{\beta\nu}\delta g_{\alpha\beta} && \text{(D.6a)}\\
\delta\sqrt{-g} &= \tfrac{1}{2}\sqrt{-g}g^{\mu\nu}\delta g_{\mu\nu} && \text{(D.6b)}\\
\delta\Gamma^\alpha_{\sigma\nu} &= \tfrac{1}{2}g^{\alpha\beta}\left[(\delta g_{\beta\sigma})_{;\nu} + (\delta g_{\beta\nu})_{;\sigma} - (\delta g_{\sigma\nu})_{;\beta}\right] && \text{(D.6c)}\\
\delta R^\alpha{}_{\nu\lambda\sigma} &= (\delta\Gamma^\alpha_{\sigma\nu})_{;\lambda} - (\delta\Gamma^\alpha_{\lambda\nu})_{;\sigma} && \text{(D.6d)}\\
\delta R_{\mu\nu\lambda\sigma} &= \delta g_{\mu\alpha}R^\alpha{}_{\nu\lambda\sigma} + g_{\mu\alpha}\delta R^\alpha{}_{\nu\lambda\sigma} && \text{(D.6e)}\\
\delta R_{\mu\nu} &= \delta R^\alpha{}_{\mu\alpha\nu} \\
&= \tfrac{1}{2}g^{\sigma\tau}\left((\delta g_{\mu\nu})_{;\sigma;\tau} + (\delta g_{\sigma\tau})_{;\mu;\nu} - (\delta g_{\mu\sigma})_{;\nu;\tau} - (\delta g_{\nu\tau})_{;\mu;\sigma}\right) && \text{(D.6f)}\\
\delta R &= g^{\mu\nu}\delta R_{\mu\nu} + R_{\mu\nu}\delta g^{\mu\nu} = g_{\mu\nu}\delta R^{\mu\nu} - R^{\mu\nu}\delta g_{\mu\nu} \\
&= g^{\mu\nu}g^{\sigma\tau}\left((\delta g_{\mu\nu})_{;\sigma;\tau} - (\delta g_{\mu\sigma})_{;\nu;\tau}\right) - R^{\mu\nu}\delta g_{\mu\nu} && \text{(D.6g)}
\end{aligned}$$

To derive the Bach equation, we rewrite the Conformal Weyl Gravity action (??) as

$$S_{\mathrm{CWG}} = -\alpha_g \int \mathrm{d}^4x\sqrt{-g}C^\alpha{}_{\beta\mu\nu}C^\beta{}_{\alpha\rho\sigma}g^{\mu\rho}g^{\nu\sigma} \tag{D.7}$$

And vary to get

$$
\begin{aligned}
0 &= \delta S_{CG} \\
&= -\alpha_g \int \mathrm{d}^4x \left[2C^\alpha{}_{\beta\mu\nu}\delta C^\beta{}_{\alpha\rho\sigma} g^{\mu\rho} g^{\nu\sigma}\sqrt{-g} + C^\alpha{}_{\beta\mu\nu} C^\beta{}_{\alpha\rho\sigma}\delta\left(g^{\mu\rho}g^{\nu\sigma}\sqrt{-g}\right)\right] \\
&= -\alpha_g \int \mathrm{d}^4x\sqrt{-g} \left[2C^\alpha{}_{\beta\mu\nu}\delta C^\beta{}_{\alpha\rho\sigma} g^{\mu\rho} g^{\nu\sigma} + C^\alpha{}_{\beta\mu\nu} C^\beta{}_{\alpha\rho\sigma}\left(2g^{\mu\rho}\delta^\nu_\lambda\delta^\sigma_\tau - \tfrac{1}{2}g^{\mu\rho}g^{\nu\sigma}g_{\lambda\tau}\right)\delta g^{\lambda\tau}\right]
\end{aligned}
$$

In going from the second to the third line we have used (D.6a) and (D.6b).

Next we use equation 10a from [81] (which is only valid in 4 spacetime dimensions) and the symmetries of the Weyl tensor to see that the second term of the last line equals zero:

$$
\begin{aligned}
C^\alpha{}_{\beta\mu\nu} C^\beta{}_{\alpha\rho\sigma}\left(2g^{\mu\rho}\delta^\nu_\lambda\delta^\sigma_\tau \tfrac{1}{2}g^{\mu\rho}g^{\nu\sigma}g_{\lambda\tau}\right) &= 2C^\alpha{}_{\beta\lambda\nu}C^\beta{}_{\alpha\tau}{}^\nu - \tfrac{1}{2}C^\alpha{}_\beta{}^{\rho\sigma}C^\beta{}_{\alpha\rho\sigma}g_{\lambda\tau} \\
&= 2C^{\alpha\beta}{}_\lambda{}^\nu C_{\beta\alpha\tau\nu} + \tfrac{1}{2}C^{\alpha\beta\rho\sigma}C^{\beta\alpha\rho\sigma}g_{\lambda\tau} \\
&= -2\left(C^{\alpha\beta\nu}{}_\lambda C_{\alpha\beta\nu\tau} - \tfrac{1}{4}C^{\alpha\beta\rho\sigma}C_{\beta\alpha\rho\sigma}g_{\lambda\tau}\right) \\
&= 0
\end{aligned}
$$

For the remaining term in δS, we use the definition of the Weyl curvature to write:

$$
\begin{aligned}
\delta C^\beta{}_{\alpha\rho\sigma} = \delta R^\beta{}_{\alpha\rho\sigma} &+ \tfrac{1}{2}\left[\delta g_{\alpha\rho}R^\beta{}_\sigma - \delta g_{\alpha\sigma}R^\beta{}_\rho\right] + \tfrac{1}{2}\left[g_{\alpha\rho}\delta R^\beta{}_\sigma - g_{\alpha\sigma}\delta R^\beta{}_\rho\right] \\
&- \tfrac{1}{2}\left[\delta^\beta_\rho\delta R_{\alpha\sigma} - \delta^\beta_\sigma\delta R_{\alpha\rho}\right] - \tfrac{1}{6}\left[\delta R(g_{\alpha\rho}\delta^\beta_\sigma - g_{\alpha\sigma}\delta^\beta_\rho) + \delta^\beta_\sigma R\delta g_{\alpha\rho} - \delta^\beta_\rho R\delta g_{\alpha\sigma}\right]
\end{aligned}
$$

Next we substitute (D.6f) and (D.6g) in the above equation. Contracting indices shows that the last term on the righthandside vanishes. Using integration by parts and dropping surface terms means that the third and fourth term on the righthandside also vanishe[52]. This means we are left with:

$$
0 = \delta S_{CG} = -2\alpha_g \int \mathrm{d}^4x C^\alpha{}_{\beta\mu\nu}\left(\delta R^\beta{}_{\alpha\rho\sigma} + \tfrac{1}{2}\left[\delta g_{\alpha\rho}R^\beta{}_\sigma - \delta g_{\alpha\sigma}R^\beta{}_\rho\right]\right) g^{\mu\rho}g^{\nu\sigma}\sqrt{-g}
$$

Now we insert (D.6d) and integrate by parts (first line), use the symmetries of Weyl tensor (second line) and substitute the variation of the connection (D.6c) (third line) and integrate by parts (fourth line):

$$
\begin{aligned}
0 &= -2\alpha_g \int \mathrm{d}^4x\sqrt{-g}C^\alpha{}_\beta{}^{\rho\sigma}\left(\left[\nabla_\sigma\delta\Gamma^\beta_{\alpha\rho} - \nabla_\rho\delta\Gamma^\beta_{\alpha\sigma}\right] + \tfrac{1}{2}\left[\delta g_{\alpha\rho}R^\beta{}_\sigma - \delta g_{\alpha\sigma}R^\beta{}_\rho\right]\right) \\
&= 2\alpha_g \int \mathrm{d}^4x\sqrt{-g}\left(2\nabla_\sigma C^\alpha{}_\beta{}^{\rho\sigma}\delta\Gamma^\beta_{\alpha\rho} - \delta g_{\alpha\rho}R^\beta{}_\sigma C^\alpha{}_\beta{}^{\rho\sigma}\right) \\
&= 2\alpha_g \int \mathrm{d}^4x\sqrt{-g}\left(\nabla_\sigma C^\alpha{}_\beta{}^{\rho\sigma}g^{\beta\mu}\left[(\delta g_{\mu\alpha})_{;\rho} + (\delta g_{\mu\rho})_{;\alpha} - (\delta g_{\alpha\rho})_{;\mu}\right] - \delta g_{\alpha\rho}R^\beta{}_\sigma C^\alpha{}_\beta{}^{\rho\sigma}\right) \\
&= 2\alpha_g \int \mathrm{d}^4x\sqrt{-g}\left(\left[-\nabla_\rho\nabla_\sigma C^{\alpha\mu\rho\sigma}\delta g_{\mu\alpha} - \nabla_\alpha\nabla_\sigma C^{\alpha\mu\rho\sigma}\delta g_{\mu\rho} + \nabla_\mu\nabla_\sigma C^{\alpha\mu\rho\sigma}\delta g_{\alpha\rho}\right] - \delta g_{\alpha\rho}R^\beta{}_\sigma C^\alpha{}_\beta{}^{\rho\sigma}\right)
\end{aligned}
$$

Because $\nabla_\rho\nabla_\sigma C^{\alpha\mu\rho\sigma} = 0$, this expression reduces to the Bach equation (2.49) (where we have renamed the indices):

$$
B_{\mu\nu} = \nabla^\alpha\nabla^\beta C_{\mu\alpha\nu\beta} - \tfrac{1}{2}R^{\alpha\beta}C_{\mu\alpha\nu\beta} = 0 \tag{D.8}
$$

We can express this in terms of the Ricci tensor and scalar if we know the covariant derivative of the Weyl tensor in terms of R and $R_{\mu\nu}$. Therefore, we write the second Bianci identity (D.5) in terms of the conformal Weyl tensor

$$
0 = R_{\mu\alpha[\nu\beta;\lambda]} = C_{\mu\alpha[\nu\beta;\lambda]} + \tfrac{1}{d-3}g_{\mu[\nu}C_{\beta\lambda]\alpha}{}^\sigma{}_{;\sigma} + \tfrac{1}{d-3}g_{\alpha[\nu}C_{\lambda\beta]\mu}{}^\sigma{}_{;\sigma}
$$

[52] Recall that the covariant derivative of the metric is identically zero because that is how we have chosen to define the connection $\Gamma^\alpha_{\mu\nu}$

Next we contract the second Bianchi identity twice[53]:

$$\nabla_\beta C_{\mu\alpha\nu}{}^\beta = \tfrac{d-3}{d-2}\left(-2\nabla_{[\mu}R_{\alpha]\nu} + \tfrac{1}{n-1}g_{\nu[\alpha}\nabla_{\mu]}R\right)$$

Taking the divergence of this expression[54]:

$$\begin{aligned} C_{\mu\alpha\nu\beta}{}^{;\beta;\alpha} &= \tfrac{d-3}{d-2}R_{\mu\nu}{}^{;\alpha}{}_{;\alpha} - \tfrac{d-3}{2(d-1)}R_{;\mu;\nu} + \tfrac{d(d-3)}{(d-2)^2}R_{\mu\alpha}R^\alpha{}_\nu \\ &- \tfrac{d-3}{d-2}R^{\alpha\beta}C_{\mu\alpha\nu\beta} - \tfrac{d-3}{(d-2)^2}g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} - \tfrac{d(d-3)}{(d-1)(d-2)^2}RR_{\mu\nu} \\ &- \tfrac{d-3}{2(d-1)(d-2)}g_{\mu\nu}R^{;\alpha}{}_{;\alpha} + \tfrac{d-3}{(d-1)(d-2)^2}g_{\mu\nu}R^2 \end{aligned}$$

Plugging this into the Bach equation (2.49)/(D.8) gives:

$$\begin{aligned} B_{\mu\nu} &= \nabla^\alpha\nabla^\beta C^{\mu\alpha\nu\beta} - \tfrac{1}{2}R^{\alpha\beta}C_{\mu\alpha\nu\beta} \\ &= \tfrac{1}{2}R_{\mu\nu}{}^{;\alpha}{}_{;\alpha} - \tfrac{1}{6}R_{;\mu;\nu} + R_{\mu\alpha}R^\alpha{}_\nu - \tfrac{1}{4}g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} \\ &- \tfrac{1}{3}RR_{\mu\nu} - \tfrac{1}{12}g_{\mu\nu}R^{;\alpha}{}_{;\alpha} + \tfrac{1}{12}g_{\mu\nu}R^2 - R^{\alpha\beta}C_{\mu\alpha\nu\beta} \end{aligned}$$

Next we note that

$$\begin{aligned} R^{\alpha\beta}C_{\mu\alpha\nu\beta} &= R_{\alpha\beta}R_{\mu\alpha\nu\beta} + \tfrac{1}{6}R^{\alpha\beta}R\left(g_{\mu\nu}g_{\alpha\beta} - g_{\mu\beta}g_{\nu\alpha}\right) \\ &\quad + \tfrac{1}{2}R^{\alpha\beta}\left(R_{\mu\beta}g_{\alpha\nu} + R_{\alpha\nu}g_{\mu\beta} - R_{\mu\nu}g_{\alpha\beta} - R_{\alpha\beta}g_{\mu\nu}\right) \\ &= \left(-\tfrac{1}{2}R_{;\mu;\nu} + R_\mu{}^\beta R_{\nu\beta} + R_{\mu\beta;\nu}{}^{;\beta}\right) + \tfrac{1}{6}\left(g_{\mu\nu}R^2 - R_{\mu\nu}R\right) \\ &\quad + \tfrac{1}{2}\left(R_\nu{}^\beta R_{\mu\beta} + R^\alpha{}_\mu R_{\alpha_\nu} - RR_{\mu\nu} - g_{\mu\nu}R^{\alpha\beta}R_{\alpha\beta}\right) \end{aligned}$$

where we have plugged in the definition of the Weyl tensor and determined $R^{\alpha\beta}R_{\mu\alpha\nu\beta}$ via the Ricci identity:

$$\begin{aligned} (\nabla_\alpha\nabla_\mu - \nabla_\mu\nabla_\alpha)T^{\alpha_1\dots\alpha_r}{}_{\beta_1\dots\beta_s} &= -R^{\alpha_1}_{\nu\mu\alpha}T^{\nu\alpha_2\dots\alpha_r}{}_{\beta_1\dots\beta_s}\cdots - R^{\alpha_r}_{\nu\mu\alpha}T^{\nu\alpha_1\dots\alpha_{r-1}}{}_{\beta_1\dots\beta_s} \\ &+ R^\beta_{\beta_1\mu\alpha}T^{\alpha_1\dots\alpha_r}{}_{\beta\beta_2\dots\beta_s}\cdots + R^\beta_{\beta_s\mu\alpha}T^{\alpha_1\dots\alpha_r}{}_{\beta_1\dots\beta_{s-1}\beta} \end{aligned} \tag{D.9}$$

Thus, the Bach equation in terms of the Ricci tensor and scalar is

$$\begin{aligned} B_{\mu\nu} = &\left(\tfrac{1}{2}g_{\mu\nu}R^{;\beta}{}_{;\beta} + R_{\mu\nu}{}^{;\beta}{}_{;\beta} - R_\mu{}^\beta{}_{;\nu;\beta} - R_\nu{}^\beta{}_{;\mu;\beta} - 2R_{\mu\beta}R_\nu{}^\beta + \tfrac{1}{2}g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta}\right) \\ &- \tfrac{1}{3}\left(2g_{\mu\nu}R^{;\beta}{}_{;\beta} - 2R_{;\mu;\nu} - 2RR_{\mu\nu} + \tfrac{1}{2}g_{\mu\nu}R^2\right) = 0 \end{aligned} \tag{D.10}$$

where the structure of the Weyl Lagrangian (**??**) is made evident.

Another equivalent way of giving the Bach tensor is in terms of the Schouten tensor $S_{\mu\nu}$:

$$B_{\mu\nu} = \nabla^\alpha\nabla_\mu S_{\nu\alpha} - \nabla^2 S_{\mu\nu} + \tfrac{1}{2}S^{\alpha\beta}C_{\mu\alpha\nu\beta} = 0 \tag{D.11}$$

with

$$S_{\mu\nu} = \tfrac{-1}{d-2}R_{\mu\nu} + \tfrac{1}{2(d-1)(d-2)}Rg_{\mu\nu}$$

[53]The idea is similar to that for the regular expression of the second Bianchi identity (D.5). You define a tensor like $T_{\mu\alpha\nu\beta\lambda} = R_{\mu\alpha[\nu\beta;\lambda]} = 0$. Then contract T on the first and fourth indices: $U_{\alpha\nu\lambda} = T_{\mu\alpha\nu}{}^\mu{}_\lambda = R_{\alpha\nu;\lambda} - R_{\alpha\lambda;\nu} + R_{\mu\alpha\lambda\nu}{}^{;\mu} = 0$. Then you contract U on the first two indices, which yields the familiar result: $R_{\alpha\lambda}{}^{;\alpha} - \frac{1}{2}R_{;\lambda} = 0$.

[54]Note that the gradient of the Ricci curvature is $R^\mu{}_{\nu;\mu} = \frac{1}{2}R_{;\nu}$. Thus it follows from this expression that indeed $\nabla_\mu\nabla_\nu C^{\alpha\beta\mu\nu} = 0$

To check the equivalence between (D.8) and (D.11), simply plug in the 4 dimensional Schouten tensor

$$\begin{aligned}B_{\mu\nu} &= \nabla^\alpha \nabla_\mu S_{\nu\alpha} - \nabla^2 S_{\mu\nu} + \tfrac{1}{2} S^{\alpha\beta} C_{\mu\alpha\nu\beta} \\ &= \tfrac{-1}{2} R_{\alpha\nu;\mu}{}^{;\alpha} + \tfrac{1}{12} R_{;\mu}{}^{;\alpha} g_{\alpha\nu} + \tfrac{1}{2} R_{\mu\nu}{}^{;\alpha}{}_{;\alpha} - \tfrac{1}{12} R^{;\alpha}{}_{;\alpha} g_{\mu\nu} \\ &- \tfrac{1}{4} R^{\alpha\beta} C_{\mu\alpha\nu\beta} + \tfrac{1}{24} R g^{\alpha\beta} C_{\mu\alpha\nu\beta}\end{aligned} \tag{D.12}$$

Noting that $g^{\alpha\beta} C_{\mu\alpha\nu\beta} = -C^{\beta}{}_{\mu\nu\beta} = 0$ (the Weyl tensor is tracefree) and using the expression for $R^{\alpha\beta} C_{\mu\alpha\nu\beta}$ from above, we will indeed retrieve equation (D.8).

Appendix E

CDG Lagrangian derivation

We use the results from Appendix C to derive the conformal invariant CDG Lagrangian (2.55) starting from the Einstein-Hilbert Lagrangian (2.17)

$$S_{\rm EH} = \frac{1}{2\kappa^2}\int \mathrm{d}^n x\sqrt{g}\,(R-2\Lambda)\,, \quad \text{with} \quad \frac{1}{\kappa^2} = M_p^{d-2} = \frac{1}{16\pi G_N}$$

Next, following 't Hooft [11], we split the metric according to

$$g_{\mu\nu} = \omega^{\frac{4}{n-2}}(x)\hat{g}_{\mu\nu}, \quad g^{\mu\nu} = \omega^{\frac{-4}{n-2}}(x)\hat{g}^{\mu\nu}, \quad g = \omega^{\frac{4n}{n-2}}\hat{g}$$

which is similar to doing a conformal transformation (2.28) with $\Omega = \omega^{\frac{2}{n-2}}$. Hence, we can replace the Ricci scalar with the conformally transformed Ricci scalar (C.5)

$$R = g^{\nu\sigma}R_{\nu\sigma} = \omega^{-2}\Big[\hat{R} - 2(n-1)\hat{g}^{\nu\sigma}\partial_\nu\partial_\sigma\ln\Omega - (n-2)(n-1)\hat{g}^{\nu\sigma}(\partial_\nu\ln\Omega)\partial_\sigma\ln\Omega\Big]$$

where we have matched the different (tilde, hat or regular) expressions according to the notation used in this section.

The integrand of the Einstein-Hilbert action becomes

$$\begin{aligned}
\sqrt{g}\,(R-2\Lambda) &= \sqrt{\hat{g}}\,\Big(\Omega^{n-2}\big[\hat{R} - 2(n-1)\hat{g}^{\nu\sigma}\partial_\nu\partial_\sigma\ln\Omega - (n-2)(n-1)\hat{g}^{\nu\sigma}(\partial_\nu\ln\Omega)\partial_\sigma\ln\Omega\big] - 2\sqrt{\hat{g}}\,\Omega^n\Lambda\Big)\\
&= \sqrt{\hat{g}}\,\Big(\big[\Omega^{n-2}\hat{R} - 2(n-1)\Omega^{n-3}\hat{g}^{\nu\sigma}\partial_\nu\partial_\sigma\Omega - (n-4)(n-1)\Omega^{n-4}\hat{g}^{\nu\sigma}(\partial_\nu\Omega)\partial_\sigma\Omega\big] - 2\Omega^n\Lambda\Big)\\
&= \sqrt{\hat{g}}\,\Big(\big[\Omega^{n-2}\hat{R} - [(n-4)(n-1) - 2(n-1)(n-3)]\Omega^{n-4}\hat{g}^{\nu\sigma}(\partial_\nu\Omega)\partial_\sigma\Omega\big] - 2\Omega^n\Lambda\Big)\\
&= \sqrt{\hat{g}}\,\Big(\big[\omega^2\hat{R} + (n-1)(n-2)\Omega^{\frac{2(n-4)}{n-2}}\hat{g}^{\nu\sigma}(\partial_\nu\omega^{\frac{2}{n-2}})\partial_\sigma\omega^{\frac{2}{n-2}}\big] - 2\omega^{\frac{2n}{n-2}}\Lambda\Big)\\
&= \sqrt{\hat{g}}\,\left(\big[\omega^2\hat{R} + 4\frac{(n-1)}{(n-2)}\hat{g}^{\nu\sigma}(\partial_\nu\omega)\partial_\sigma\omega\big] - 2\omega^{\frac{2n}{n-2}}\Lambda\right)
\end{aligned} \tag{E.1}$$

where we performed a partial integration in the third line and inserted $\Omega = \omega^{\frac{2}{n-2}}$ in the fourth line. Now we recognize the action:

$$S = \frac{1}{2\kappa^2}\int \mathrm{d}^n x\sqrt{\hat{g}}\,\left(\hat{R}\omega^2 + \frac{1}{2}\frac{8(n-1)}{(n-2)}\hat{g}^{\mu\nu}\partial_\mu\omega\partial_\nu\omega - 2\Lambda\omega^{\frac{2n}{n-2}}\right) \tag{E.2}$$

Rescaling the field as

$$\omega = \sqrt{\kappa^2 \xi}\hat{\omega}, \quad \text{with} \quad \xi = \frac{(n-2)}{4(n-1)} \tag{E.3}$$

gives the Conformal Dilaton Gravity action (2.55)

$$S_{\text{CDG}} = \int \mathrm{d}^d x \sqrt{\hat{g}} \left(\tfrac{1}{2}\, \xi \hat{R} \hat{\omega}^2 + \tfrac{1}{2} \hat{g}^{\mu\nu} \partial_\mu \hat{\omega} \partial_\nu \hat{\omega} + V(\hat{\omega}) \right) \tag{E.4}$$

where $V(\hat{\omega})$ is such that it is minimized when $\hat{\omega} = \upsilon$. We demand that upon gauge-fixing this action according to $\hat{\omega} = \upsilon$, we indeed receive back the EH action:

$$\xi \upsilon^2 = \frac{1}{\kappa^2}, \qquad V(\upsilon) = \frac{\Lambda}{\kappa^2} = \lambda \upsilon^{\frac{2n}{n-2}}$$

where λ is some self-interaction coupling constant.

Appendix F

Conformal covariance of the non-minimal scalar action

A conformal invariant scalar action requires a nonminimal coupling term where $\xi = \frac{d-2}{4(d-1)}$ in d spacetime dimensions, i.e. in four spacetime dimensions

$$S_{\text{Scalar}} = \tfrac{1}{2} \int \mathrm{d}^4x \sqrt{g} \left(g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi + \tfrac{1}{6} \phi^2 R \right) \tag{F.1}$$

To see this consider the equation of motion of the nonminimal, massless scalar theory (2.80) as given by the Euler-Lagrange equation (2.38)

$$\begin{aligned}
0 &= \frac{1}{\sqrt{g}} \left(\frac{\partial(\sqrt{g}\mathcal{L})}{\partial \phi} - \partial_\mu \frac{\partial(\sqrt{g}\mathcal{L})}{\partial \partial_\mu \phi} \right) \\
&= \frac{1}{\sqrt{g}} \left(-\sqrt{g} \xi R \phi - \partial_\mu \left(\sqrt{g} g^{\mu\nu} \nabla_\nu \phi \right) \right) \\
&= -\xi R \phi - \frac{1}{\sqrt{g}} \Big(\sqrt{g} \left(-\{{}^{\mu}_{\alpha\mu}\} g^{\alpha\nu} - \{{}^{\nu}_{\alpha\mu}\} g^{\mu\alpha} \right) \nabla_\nu \phi + \sqrt{g} g^{\mu\nu} \partial_\mu \nabla_\nu \phi + g^{\mu\nu} \nabla_\nu \phi (\partial_\mu \sqrt{g}) \Big) \\
&= -\xi R \phi - \left(g^{\mu\nu} \partial_\mu \nabla_\nu \phi - \{{}^{\nu}_{\alpha\mu}\} g^{\mu\alpha} \nabla_\nu \phi \right) \\
&= \xi R \phi - g^{\mu\nu} \nabla_\mu \nabla_\nu \phi
\end{aligned}$$

where $\nabla_\mu = \nabla_\mu(\{\})$ is the Levi-Cevita covariant derivative. We furthermore used $\nabla_\mu g^{\alpha\beta} = 0$ in the second line and the identity (B.3) as derived in the appendix in the fourth line. Using the conformal transformation of the covariant derivative (C.2)

$$\tilde{\nabla}_\nu X_\lambda = \nabla_\nu X_\lambda - H^{\mu}_{\nu\lambda} X_\mu$$

and Ricci scalar (C.5),

$$\begin{aligned}
\tilde{R} = \tilde{g}^{\nu\sigma} \tilde{R}_{\nu\sigma} &= \Omega^{-2} \Big[R - 2(n-1) g^{\nu\sigma} \partial_\nu \partial_\sigma \ln \Omega - (n-2)(n-1) g^{\nu\sigma} (\partial_\nu \ln \Omega) \partial_\sigma \ln \Omega \Big] \\
&= \Omega^{-2} \Big[R - 2(n-1) g^{\nu\sigma} A_{\nu\sigma} - (n-2)(n-1) g^{\nu\sigma} B_{\nu\sigma} \Big]
\end{aligned}$$

it can be easily verified that the above Klein-Gordon equation transforms under a conformal transformation $g_{\mu\nu} \to \tilde{g}_{\mu\nu} = \Omega^2(x) g_{\mu\nu}$ as

$$
\begin{aligned}
\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\tilde{\nabla}_\nu\tilde{\phi} - \xi\tilde{R}\tilde{\phi} = g^{\mu\nu}\Big[&\Omega^{\tilde{\Delta}-3}\nabla_\mu\Omega\nabla_\nu\phi\big(n+2\tilde{\Delta}-2\big) + \Omega^{\tilde{\Delta}-3}\phi\nabla_\mu\nabla_\nu\Omega\tilde{\Delta} \\
&+ \Omega^{\tilde{\Delta}-4}\phi\nabla_\mu\Omega\nabla_\nu\Omega\big(\tilde{\Delta}(\tilde{\Delta}-3+n)\big) + \Omega^{\tilde{\Delta}-2}\nabla_\mu\nabla_\nu\phi\Big] \\
&- \xi\Big[\Omega^{\tilde{\Delta}-2}R - g^{\mu\nu}\Omega^{\tilde{\Delta}-3}\partial_\mu\partial_\nu\Omega\big(2n-2\big) - g^{\mu\nu}\Omega^{\tilde{\Delta}-4}(\partial_\mu\Omega)\partial_\nu\Omega\big(n^2-5n+4\big)\Big]\phi \\
= g^{\mu\nu}\Omega^{\tilde{\Delta}-4}\phi\nabla_\mu\Omega\nabla_\nu\Omega&\left(\frac{3n}{2} - \frac{n^2}{4} - 2 + (n^2-5n+4)\xi\right) \\
&+ g^{\mu\nu}\Omega^{\tilde{\Delta}-3}\phi\nabla_\mu\nabla_\nu\Omega\left(\frac{2-n}{2} + 2(n-1)\xi\right) + g^{\mu\nu}\Omega^{\tilde{\Delta}-2}\nabla_\mu\nabla_\nu\phi - \xi\Omega^{\tilde{\Delta}-2}R\phi
\end{aligned}
$$

where we used the fact that the Weyl weight of a scalar field is $\tilde{\Delta} = -\frac{n-2}{2}$ to eliminate the first term on the righthandside in the first line. Defining $\xi = \frac{n-2}{4(n-1)}$ removes the extra terms and we see that in $n > 1$ dimensions the equation of motion is conformally covariant.

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