# An investigation of a nonlinear susy theory

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**Abstract**

Supersymmetry is one of the most promising ideas for physics beyond the standard model. If it is indeed a symmetry of nature, it is realized nonlinearly. In this project, a global nonlinear realization of susy (Volkov-Akulov) is examined. First, a general understanding of symmetries in theoretical physics is developed, after which susy is treated on a basic level. The spontaneous breaking of symmetries is discussed, focusing on susy breaking. VA theory is introduced, and the equation of motion is given and analyzed. No negative-energy degrees of freedom or 'ghosts' appear to be present. Finally, another Lagrangian is treated (Komargodski-Seiberg), equivalent to that of Volkov and Akulov, emphasizing the universality of VA theory.

# Contents

# 1 Introduction

In this bachelor research project, we will take a look at a nonlinear realization of supersymmetry (or susy). Before starting the discussion leading up to this theory, let us quickly voice a motivation for the project and explain the outline of this text.

The interest in supersymmetric theories does not come out of thin air. As physicists know, symmetries in general are a topic of great interest in the pursuit of new and better theories to describe reality. As will be explained in the next section, supersymmetry is no exception to this rule. However, the motivation for susy is not that it is obviously present in nature, but rather that its presence in a theory solves a lot of problems of theoretical physics, perhaps most notably the hierarchy problem; the enormously larger size of the Planck mass as compared to the mass of the Higgs boson.

The reason a nonlinear realization is interesting to look at, is that if susy is present in nature, it must be a 'broken symmetry', which is described by a nonlinear theory. While the theory treated, Volkov-Akulov theory, is a theory with broken susy, it is not regarded as a description of nature itself, but rather as a tool to gain insight into the behaviour of such theories. Of course, this remark would hold for every effective field theory (theories that aim to describe the dynamics of a system in a certain range of the total energy spectrum in a relatively simple manner), but even more so for VA theory. After all, it was proposed by the name givers as a way to describe the massless neutrino [1], while we know today that at least two of the three known kinds of neutrinos have a nonzero mass!

The question this research is trying to answer is as follows: What is the significance of Volkov-Akulov theory in theoretical physics? 'Significance' can be interpreted in a broad sense, meaning both 'physical meaning' and how it relates to other parts of theoretical physics. To answer this question, it is of course necessary to first cover the basics of supersymmetry and spontaneous symmetry breaking.

In the next section, we will first acquaint ourselves with the role of symmetries in physics and try to distinguish between different types of symmetries. After that, we will focus on supersymmetry specifically, aiming to describe a supersymmetric action, treating important concepts such as superspace and chiral superfields along the way. Having hopefully understood susy up to a basic level, we find out what happens when symmetries spontaneously break, again focusing on susy specifically. Then we have all the tools needed to derive and analyze the Volkov-Akulov action and equations of motion. After this, we will look at a different way to find an action with the same symmetry as the VA-action: the Komargodski-Seiberg action, which is related to the VA-action in a fundamental way.

## 2 Symmetries and their importance

Why are physicists so obsessed with symmetries? To answer that question, let us turn to Nobel laureate David Gross [2]. In a lecture in 1996 he explained that for a long time, although regarded as fascinating and helpful notions, symmetries were not seen as the cause of the conservation laws that they gave rise to. These conservation laws were thought to result from the dynamical laws of nature themselves.

According to Gross, this changed around the year 1905, when Einstein, in constructing his theory of special relativity, put the symmetry first, and regarded the symmetry as the primary feature of nature, constraining the dynamical laws. After this, Emmy Noether published her seminal article 'Invariante Variationsprobleme'[3] in 1918, which proves the existence of a conserved current for every continuous symmetry. Of course, earlier theorists such as Lagrange and especially Hamilton (1733 and around 1830 resp. [4]) did admire the beauty and usefulness symmetries in their formalisms, but they did not see the symmetry as the source of the conservation laws.

In describing any physical system, it helps enormously to be able to point out regularities in its behaviour. For, using the regularities, it is possible to bring structure and coherence to the laws of nature, just like the laws bring coherence and structure into the set of events in the system. These regularities are summarized in the symmetry principles. Without them, we would not be able to make sense of events; there would be no pattern to recognize and therefore no law to be discovered. Nowadays, it is realized that symmetries are even more powerful; they dictate the form of the laws of nature.

This is why, to quote Gross, 'Today [symmetry] serves as a guiding principle in the search for further unification and progress.'; symmetries imply understanding of the laws of nature, which is after all what fundamental physics is all about. The unification Gross is referring to, is the unification of the forces of nature. This is where new symmetries come in.

Among those new symmetries, is supersymmetry. Supersymmetry is a very promising symmetry, as it has the ability to relate bosonic and fermionic states to each other (unification of bosons and fermions, forces and matter); to describe them in a single pattern. In this way, it can help to solve a few of the most pressing problems of modern physics, such as the extreme smallness of cosmological constant, the hierarchy problem, and the issue of renormalization of quantum gravity [5]. However, it is clear that supersymmetry is not fully realized in nature, since the predicted 'superpartners' of known particles are not present. This means that, if susy is to be a symmetry in a successful theory, it needs to be 'broken'.

To explore further the idea of symmetry, we will have a short look at the different natures symmetries can have (spacetime versus internal symmetries), their implications (conserved currents; partner particles), and different ways they can be realized (linear versus nonlinear).

## 2.1 Internal and spacetime symmetries

As is probably known by the reader, a symmetry of a system in the more technical sense of the word, is an invariance of the action describing it under the operation of certain transformations. These transformations then together for a group, called the symmetry group.

The symmetries in physical systems can be classified into two classes: spacetime symmetries and internal symmetries. While the first is generated by transformations of both the fields and the coordinates, the second leaves the coordinates invariant and therefore acts the same at every point in spacetime. An example of spacetime symmetries is easy enough: Think of invariance under rotation in space. The Poincaré group is the group formed by translation transformations and the Lorentz transformations, which generate rotations and boosts. It is obviously a spacetime symmetry group and guarantees (via Lorentz invariance) that special relativity is obeyed.

An example of an internal symmetry is a bit less tentative. Let us consider an action made out of a spinor field $\psi$ and its conjugate $\psi^*$. Then the action could have an invariance under phase transformation $\psi \to e^{i\alpha}\psi$.

In an article published in 1967 [6], Coleman and Mandula proof a 'no go' theorem, stating that there is no way to combine internal and spacetime symmetries, unless by taking the simple direct product of the poincaré group and an internal symmetry group. Or as in their abstract: 'We prove a new theorem on the impossibility of combining space-time and internal symmetries in any but a trivial way.'.

What the theorem does leave room for in the massless case (and disregarding infrared problems and symmetry breaking [7]), is an extension of the poincaré group, giving what is called the conformal group. The conformal group adds to the poincaré group dilatation (or scale transformation) and special conformal transformations. But this is where the expanding of the spacetime group must end, argue Coleman and Mandula in their theorem. Add to this in a direct product the internal symmetry group and one has all symmetries of the scattering matrix, or S matrix, describing interactions in the physical system.

This seemed to be more or less the final word on the topic, until in 1974 Haag, Łopuzański, and Sohnius generalized the theorem [7], by noting that, when one allows for spinorial, anti-commuting generators, it is possible to extend the spacetime symmetry group even further. Not only succeeded the spinorial generators in extending the spacetimegroup further, they also created a non-trivial interaction between the spacetime and internal symmetries of the system; something that the Coleman-Mandula theorem seemed to forbid! This exciting new extension, as we will see in the next section, leads to supersymmetry.

Another way to classify symmetries is by the label of global and local. Global symmetries hold at every point in spacetime, as the word suggests, while local symmetries have different symmetry transformations at different points; the transformations depend on the coordinates of spacetime. In this text, only global susy is treated. That is not to say that local supersymmetry is not interesting. On the contrary: Imposing local susy on a system results in general relativity being automatically included, thereby giving a supergravity theory.

## 2.2 Implications of symmetries

The most profound consequence of a (continuous and global) symmetry in a physical system is the conservation of a four current $j^\mu$. This is the statement made by Noether's theorem [3], as the reader should know. This implication is of grave importance for two reasons. Firstly, a conserved four current gives rise to a conserved charge, which is perhaps the most simple and important notion one can give about any physical system. Secondly, and this is more a fortification of the first reason than a separate reason entirely, a conserved current implies that the charge is conserved *locally*[8]. Or in mathematical language, for any volume $V$ with area $A$, the charge $Q_V$ in the volume and the three current $\vec{j}$ are related via

$$\frac{Q_V}{dt} = -\int_A \vec{j} \cdot d\vec{S}.$$

Every decrease in charge must be accompanied by a current flow out of the volume. This introduces a kind of continuity in our physical theory that greatly structures the behaviour of the system.

As mentioned in the introduction of this section, a consequence of supersymmetry is the required existence of superpartners; particles with the same mass (in the unbroken case) as known particles but spin $S = j + \frac{1}{2}$ if the spin of the known particle is $S = j$. This last property is a consequence of the fermionic nature of the susy generators, which transform particles to other particles, with a spin differing by one half. The existence of superpartners can however also be seen as a requirement for the symmetry to be present, rather than as a consequence of the symmetry.

## 2.3 Realization of symmetries

Symmetries can be realized either linearly or nonlinearly. This refers to whether the transformation of a field in the Lagrangian realizing the symmetry is linear in itself or not. This difference turns out to be of importance, since nonlinear theories appear to describe theories with spontaneously broken symmetries at low energies. That is, the transformations of nonlinear realizations do not leave the vacuum state invariant, such that the symmetry can be considered to be 'broken' by the vacuum; the symmetry is still present in the theory, the vacuum state itself just does not display the same symmetry.

The theory of spontaneously broken symmetries and thus of nonlinear realizations of symmetries is important in physics, since many symmetries encountered in nature appear to be spontaneously broken. To get a general idea of this phenomenon, let us briefly look at a probably familiar situation. Take for example ferromagnetism in iron. Above a temperature of about 1000 Kelvin (Curie temperature, $T_c$), the iron is paramagnetic. All the spins of the electrons in the material, which are the largest cause of magnetism, will point in different directions, as long as no magnetic field is applied, see 1. There is a spherical symmetry in the system and in the Lagrangian describing it. Below 1000 K, iron becomes a ferromagnet, because the spins in a domain now all point in the same direction, for that is the energetically favourable way to arrange themselves, even when no magnetic field is present. In a bulk of iron, there will not be a net magnetization, because different domains have different directions of magnetization. Within a domain, however, the symmetry is clearly broken. This process is unsurprisingly called 'spontaneous magnetization'.

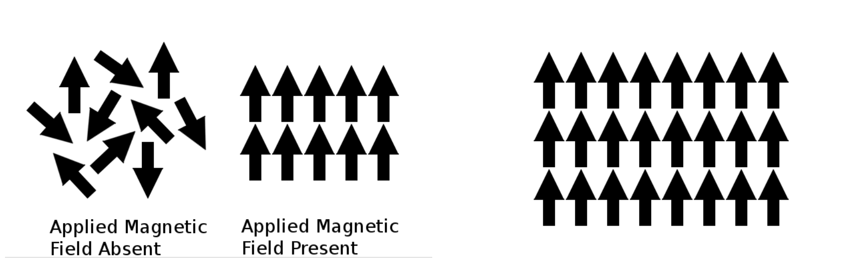


Figure 1: In the first picture, the domain is shown at a temperature above $T_c$, with spins all oriented randomly in the absence of a magnetic field. In the second picture, the same domain is seen, now with spins aligned, due to spontaneous magnetization [9].

As pointed out earlier, if susy is present in nature, it must be a broken symmetry as well. To briefly come back to the last subsection, nonlinearly realized theories still have the conserved currents, as guaranteed by Noether's theorem. The superpartners predicted by susy are also present, with spin differing by a half, but the masses are not necessarily equal. They may even be factors of a thousand larger[10].

The prediction of superpartners gives a very clear check for susy, so as long as experiments (such as LHC) of higher and higher energies do not find them, we know we are either still in the broken range, or susy is not a symmetry of nature at all. Because of that, after exploring the necessary basics of supersymmetry, it is logical to continue right away with nonlinear theories in the following sections.

# 3 Understanding supersymmetry

## 3.1 Supersymmetry algebra

Supersymmetry is, as mentioned, a new symmetry, added to the already known ones, such as translational invariance and Lorentz symmetry. These are given by the so called Poincaré group, obeying the Poincaré algebra [5]:

$$\begin{aligned} [P_\mu, P_\nu] &= 0, \\ [M_{\mu\nu}, M_{\rho\sigma}] &= i\eta_{\nu\rho}M_{\mu\sigma} - i\eta_{\mu\rho}M_{\nu\sigma} - i\eta_{\nu\sigma}M_{\mu\rho} + i\eta_{\mu\sigma}M_{\nu\rho}, \\ [M_{\mu\nu}, P_\rho] &= -i\eta_{\rho\mu}P_\nu + i\eta_{\rho\nu}P_\mu, \end{aligned}$$

where $M_{\mu\nu} = -M_{\nu\mu}$, defined as $M_{0i} = K_i$ and $M_{ij} = \epsilon_{ijk}J_k$ are the Lorentz generators (boosts and rotations respectively) and $P_\mu$ the translation generators. The $\eta_{\mu\nu}$ tensors are the Minkowski metrics, with the $(+1, -1, -1, -1)$ convention. The generators can also be given as differential operators:

$$P_\mu = -i\partial_\mu, \qquad M_{\mu\nu} = i(x_\nu\partial_\mu - x_\mu\partial_\nu) = x_\mu P_\nu - x_\nu P_\mu.$$

Here especially the first operator is easy to recognize.

To add another symmetry to the already known symmetry group, we extend the algebra with (anti-)commutation relations involving the transformations under which we want our supersymmetric system to be invariant.

What we want from the transformations is to be fermionic generators, so that our total set of symmetries indeed relates bosons to fermions. This can be done by imposing that our additional generators transform as fermions under the Lorentz group.

The matrices from which the Lorentz generators for fermions can be composed are the Dirac matrices, given in Weyl representation as

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix}, \quad \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix},$$

following Bilal. These matrices work on four-component Dirac spinors, which can be 'cut up' into so called undotted spinors $\psi_\alpha$ and dotted spinors $\overline{\chi}^{\dot{\alpha}}$, having two components each.

The Lorentz generators for fermions in Weyl representation are

$$\Sigma^{\mu\nu} = \frac{i}{2}\gamma^{\mu\nu}, \quad \gamma^{\mu\nu} = \gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu = \begin{pmatrix} \sigma^\mu\overline{\sigma}^\nu - \sigma^\nu\overline{\sigma}^\mu & 0 \\ 0 & \overline{\sigma}^\mu\sigma^\nu - \overline{\sigma}^\nu\sigma^\mu \end{pmatrix},$$

from which we immediately see that the separate transformations of the undotted and dotted spinors are

$$\begin{aligned} i(\sigma^{\mu\nu})_\alpha^{\ \beta} &= \frac{i}{4}(\sigma^\mu_{\alpha\dot{\gamma}}\overline{\sigma}^{\nu\dot{\gamma}\beta} - \sigma^\nu_{\alpha\dot{\gamma}}\overline{\sigma}^{\mu\dot{\gamma}\beta}) \\ i(\overline{\sigma}^{\mu\nu})^{\dot{\alpha}}_{\ \dot{\beta}} &= \frac{i}{4}(\overline{\sigma}^{\mu\dot{\alpha}\gamma}\sigma^\nu_{\gamma\dot{\beta}} - \overline{\sigma}^{\nu\dot{\alpha}\gamma}\sigma^\mu_{\gamma\dot{\beta}}). \end{aligned}$$

Now we know how we want the new generators to transform under Lorentz transformations: as undotted spinors $Q^I_\alpha$ and dotted spinors $\overline{Q}^I_{\dot{\alpha}}$. Here $I = 1, 2, ..., N$ labels the new generator pairs, in case we want to include multiple. We can then form the relation with the other generators by noting we require commutation of the susy generators with translations, and the transformations under the Lorentz generators mentioned above:

$$\begin{aligned}
[P_\mu, Q^I_\alpha] &= \quad 0, \\
[P_\mu, \overline{Q}^I_{\dot\alpha}] &= \quad 0, \\
[M_{\mu\nu}, Q^I_\alpha] &= \quad i(\sigma^{\mu\nu})_\alpha^{\ \beta} Q^I_\beta, \\
[M_{\mu\nu}, \overline{Q}^{I\dot\alpha}] &= \quad i(\overline{\sigma}^{\mu\nu})^{\dot\alpha}_{\ \dot\beta} \overline{Q}^{I\dot\beta}.
\end{aligned}$$

The operators $Q^I_\alpha$ and $\overline{Q}^J_{\dot\beta}$ are fermionic, so they must anti-commute. Furthermore, we know their anti-commutator itself must transform as a four vector, which can in this group only be $P_\mu$. Also, the product must be Lorentz invariant, so including a $\sigma^\mu$ is necessary. Using this, we arrive at [5]

$$\{Q^I_\alpha, \overline{Q}^J_{\dot\beta}\} = 2\sigma^\mu_{\alpha\dot\beta} P_\mu \delta^{IJ},$$

while the anti-commutators of $Q^I_\alpha$ and $\overline{Q}^I_\beta$ with themselves are:

$$\{Q^I_\alpha, Q^J_\beta\} = \epsilon_{\alpha\beta} Z^{IJ}, \quad \{\overline{Q}^I_{\dot\alpha}, \overline{Q}^J_{\dot\beta}\} = \epsilon_{\dot\alpha\dot\beta} (Z^{IJ})^*,$$

with $Z^{IJ} = -Z^{JI}$ as central charges.

Since every irreducible representation (irrep) of the Poincaré algebra corresponds to a particle, an irrep of the susy algebra, being an extension of this, corresponds (in general) to multiple particles. These particles are related to each other by the generators $Q^I_\alpha$ and $\overline{Q}^I_{\dot\beta}$, meaning their spins differ from each other by one half. In this way, the particles form a supermultiplet, containing both fermions and bosons. These particles all have the same mass, because the mass squared generator $P^2$ commutes with $Q^I_\alpha$. This does however not hold for nonlinear realizations of susy, which have mass differences in a multiplet.

## 3.2 Constructing a susy action

Now we understand how supersymmetry is defined and how it relates bosons and fermions, the next question is of course: How do we incorporate this idea into our theory? Or equivalently, how do we construct a susy action? In the following, we will restrict ourselves to unextended susy, where $N = 1$. Then there is only one set of susy operators.

### 3.2.1 Superspace

Firstly, we will identify a very natural environment to define the fields we will be using. This environment is superspace. Superspace has the big advantage over ordinary spacetime that it makes it easier to identify susy invariants and to write susy Lagrangians [11]. Coordinates in superspace are the regular space time coordinate $x^\mu$, extended with two Grassmann coordinates (or Grassman spinors) $\theta^\alpha$ and $\overline{\theta}^{\dot{\alpha}}$. The $x^\mu$ as always has four components, while the Grassmann spinors both have two.

Defining a function in superspace, a superfield, it is useful to note that the Grassmann coordinates are spinors, so they anticommute amongst themselves. Using this we have for example $\theta^1\theta^1\theta^2 = -\theta^1\theta^1\theta^2 = 0$, from which it is easy to see that every product involving more than two $\theta$'s or $\overline{\theta}$'s vanishes, since there is always more than one of the same component present in such a product. This means we can write down an exact Taylor expansion for all superfields:

$$\begin{aligned} F(x,\theta,\overline{\theta}) &= f(x) + \theta\psi(x) + \overline{\theta}\overline{\chi}(x) + \theta\theta m(x) + \overline{\theta\theta}n(x) \\ &\quad + \theta\sigma^\mu\overline{\theta}v_\mu(x) + \theta\theta\overline{\theta}\overline{\lambda}(x) + \overline{\theta\theta}\theta\rho(x) + \theta\theta\overline{\theta\theta}d(x). \end{aligned} \tag{1}$$

In calculating things with two component Grassmann coordinates, it is useful to define the anti-symmetric, Lorentz-invariant matrices

$$\epsilon_{\alpha\beta} = \epsilon_{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \epsilon^{\alpha\beta} = \epsilon^{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

to be able to raise and lower indices. Then $\theta\theta = \theta^\alpha\theta_\alpha = -\epsilon_{\alpha\beta}\theta^\alpha\theta^\beta = -\epsilon_{12}\theta^1\theta^2 - \epsilon_{21}\theta^2\theta^1 = -\theta^1\theta^2 + \theta^2\theta^1 = -2\theta^1\theta^2 = 2\theta^2\theta^1$ and similarly $\overline{\theta\theta} = 2\overline{\theta}^1\overline{\theta}^2$.

Integrations in superspace are defined for a single component $\theta^1$ as $\int d\theta^1(a + b\theta^1) = b$. And since $\theta\theta = 2\theta^2\theta^1$ and $\overline{\theta\theta} = 2\overline{\theta}^1\overline{\theta}^2$, we define $d^2\theta = d\theta^1 d\theta^2$ and $d^2\overline{\theta} = d\overline{\theta}^2 d\overline{\theta}^1$, so that

$$\int d^2\theta\ \theta\theta = \int d^2\overline{\theta}\ \overline{\theta\theta} = 1, \quad \int d^2\theta d^2\overline{\theta}\ \theta\theta\overline{\theta\theta} = 1.$$

### 3.2.2 Constructing susy transformations in superspace

The next step is to actually find the $Q_\alpha$'s and their hermitian conjugate $\overline{Q}_{\dot{\alpha}}$'s as differential operators on superspace. We want the infinitesimal transformation $i\epsilon^\alpha Q_\alpha$ to generate a translation in $\theta^\alpha$ with constant infinitesimal spinor $\epsilon^\alpha$ and some translation in $x^\mu$, since the anti-commutator with its conjugate is a translation in spacetime. So in $iQ_\alpha$, a term $\frac{\partial}{\partial\theta^\alpha}$ must be present, as well as a term with $P_\mu$ for the spacetime translation, $\sigma^\mu$ to make it Lorentz invariant (and satisfy the algebra), and $\overline{\theta}_{\dot{\alpha}}$ to get picked out by the conjugate derivative. Taking this together, we need something of the form

$$iQ_\alpha = \frac{\partial}{\partial\theta^\alpha} + c(\sigma^\mu\overline{\theta})_\alpha P_\mu = \frac{\partial}{\partial\theta^\alpha} - ic(\sigma^\mu\overline{\theta})_\alpha\partial_\mu,$$

and similarly the conjugate

$$i\overline{Q}_{\dot{\alpha}} = -\left(\frac{\partial}{\partial\overline{\theta}^{\dot{\alpha}}} - ic^*(\theta\sigma^\mu)_{\dot{\alpha}}\partial_\mu\right),$$

where $c$ (complex conjugate $c^*$) is an arbitrary constant.

Then we find

$$
\begin{aligned}
\{Q_\alpha, \overline{Q}_{\dot\beta}\} &= \left(\frac{\partial}{\partial\theta^\alpha} + c(\sigma^\mu\overline{\theta})_\alpha P_\mu\right)\left(\frac{\partial}{\partial\overline{\theta}^{\dot\beta}} + c^*(\theta\sigma^\mu)_{\dot\beta}P_\mu\right) \\
&+ \left(\frac{\partial}{\partial\overline{\theta}^{\dot\beta}} + c^*(\theta\sigma^\mu)_{\dot\beta}P_\mu\right)\left(\frac{\partial}{\partial\theta^\alpha} + c(\sigma^\mu\overline{\theta})_\alpha P_\mu\right) \\
&= \left(c^*\sigma^\mu_{\alpha\dot\beta} - c\sigma^\mu_{\dot\beta\alpha}\right)P_\mu = (c^* + c)\sigma^\mu_{\alpha\dot\beta}P_\mu = 2\,\mathrm{Re}(c)\sigma^\mu_{\alpha\dot\beta}P_\mu,
\end{aligned}
$$

which reduces to the already shown anti-commutation relation when $\mathrm{Re}(c) = 1$ (from now on we therefore set $c = 1$). It is easy to see that for more than one pair of operators (so $N > 1$), it suffices to add a factor $\delta^{IJ}$, since only with $I = J$ the partial derivatives will have something to act on.

The transformations are therefore given by

$$
\begin{aligned}
Q_\alpha &= -i\frac{\partial}{\partial\theta^\alpha} - \sigma^\mu_{\alpha\dot\beta}\overline{\theta}^{\dot\beta}\partial_\mu, \\
\overline{Q}_{\dot\alpha} &= i\frac{\partial}{\partial\overline{\theta}^{\dot\alpha}} + \theta^\beta\sigma_{\beta\dot\alpha}\partial_\mu,
\end{aligned} \tag{2}
$$

which on a general superfield $F(x^\mu, \theta^\alpha, \overline{\theta}^{\dot\alpha})$ have the following effect:

$$
\begin{aligned}
&(1 + i\epsilon Q + i\overline{\epsilon}\overline{Q})F(x^\mu, \theta^\alpha, \overline{\theta}^{\dot\alpha}) = F(x^\mu, \theta^\alpha, \overline{\theta}^{\dot\alpha}) \\
&+ \left(\Delta^\mu\partial_\mu + \epsilon^\alpha\frac{\partial}{\partial\theta^\alpha} - \overline{\epsilon}^{\dot\alpha}\frac{\partial}{\partial\overline{\theta}^{\dot\alpha}}\right)F(x^\mu, \theta^\alpha, \overline{\theta}^{\dot\alpha}) = F(x^\mu + \Delta^\mu, \theta^\alpha + \epsilon^\alpha, \overline{\theta}^{\dot\alpha} + \overline{\epsilon}^{\dot\alpha}),
\end{aligned} \tag{3}
$$

where $\Delta^\mu = -i\epsilon\sigma^\mu\overline{\theta} + i\theta\sigma^\mu\overline{\epsilon}$.

### 3.2.3 Chiral superfields

A generic superfield $F$ has a lot of components. More than necessary even, since it is a reducible representation[11]. It is therefore useful to find an irrep of $N = 1$ susy. We can find such an irrep by imposing a constraint on the superfields. This will be done by finding covariant derivatives, $D_\alpha$ and $\overline{D}_{\dot\alpha}$, that anticommute with the susy generators $Q$ and $\overline{Q}$. Then since $D_\alpha(\delta_{\epsilon,\overline{\epsilon}}F) = \delta_{\epsilon,\overline{\epsilon}}(D_\alpha F)$ and for susy invariance we need $\delta_{\epsilon,\overline{\epsilon}}F = (i\epsilon Q + i\overline{\epsilon}\overline{Q})F = 0$, we can conclude $D_\alpha F = 0$ is a susy restriction. Similarly we find $\overline{D}_{\dot\alpha}F = 0$.

For the covariant derivatives we have [5]:

$$
\begin{aligned}
D_\alpha &= \frac{\partial}{\partial\theta^\alpha} + i\sigma^\mu_{\alpha\dot\beta}\overline{\theta}^{\dot\beta}\partial_\mu, \\
\overline{D}_{\dot\alpha} &= \frac{\partial}{\partial\overline{\theta}^{\dot\alpha}} + i\theta^\beta\sigma^\mu_{\beta\dot\alpha}\partial_\mu,
\end{aligned}
$$

so that

$$\{D_\alpha, \overline{D}_{\dot\beta}\} = 2i\sigma^\mu_{\alpha\dot\beta}\partial_\mu,$$

and all other anti-commutation relations are zero.

A field $\phi$ satisfying the condition $\overline{D}_{\dot\alpha}\phi = 0$ is called a chiral superfield, while a field $\overline{\phi}$ obeying $D_\alpha\overline{\phi} = 0$ is anti-chiral. Of course, the name is no surprise, as the procedure done here is analogous to the way chiral fields are found in the step from Dirac to Weyl spinors. In that case, one has a reducible representation, Dirac spinors, and one can define projection operators $P_\pm$ to find the irreducible Weyl spinors (and check other spinors if they already are). This gives us instead of 4 complex components, only 2 for every spinor.

In the context of (anti-)chiral superfields, the condition makes it possible to switch from three coordinates $(x^\mu, \theta^\alpha, \overline{\theta}^{\dot\alpha})$ to two, by noting that

$$\begin{aligned} &D_\alpha\overline{\theta} = \overline{D}_{\dot\alpha} = D_\alpha\overline{y}^\mu = \overline{D}_{\dot\alpha}y^\mu = 0, \\ &y^\mu = x^\mu + i\theta\sigma^\mu\overline{\theta}, \quad \overline{y}^\mu = x^\mu - i\theta\sigma^\mu\overline{\theta}. \end{aligned}$$

The component expansions are now way simpler:

$$\begin{aligned} \phi(y,\theta) &= z(y) + \sqrt{2}\theta\psi(y) - \theta\theta f(y), \\ \overline{\phi}(\overline{y},\overline{\theta}) &= \overline{z}(\overline{y}) + \sqrt{2}\overline{\theta\psi}(\overline{y}) - \overline{\theta\theta f}(\overline{y}), \end{aligned}$$

while the operators become

$$Q_\alpha = -i\frac{\partial}{\partial\theta^\alpha}, \quad \overline{Q}_{\dot\alpha} = i\frac{\partial}{\partial\overline{\theta}^{\dot\alpha}} + 2\theta^\beta\sigma^\mu_{\beta\dot\alpha}\frac{\partial}{\partial y^\mu}.$$

In the expansion, the field $z(y)$ describes a complex scalar, $\psi(y)$ a Weyl fermion and $f(y)$ is only an auxiliary field [5]. Using the expansions and new operators, we can find the variation of the chiral field to be

$$\begin{aligned} \delta\phi(y,\theta) &= (i\epsilon Q + i\overline{\epsilon}\overline{Q})\phi(y,\theta) = \left(\epsilon^\alpha\frac{\partial}{\partial\theta^\alpha} + 2i\theta^\beta\sigma^\mu_{\beta\dot\alpha}\overline{\epsilon}\frac{\partial}{\partial y^\mu}\right)\phi(y,\theta) \\ &= \sqrt{2}\epsilon\psi - 2\epsilon\theta f + 2i\theta\sigma^\mu\overline{\epsilon}(\partial_\mu z + \sqrt{2}\theta\partial_\mu\psi) \\ &= \sqrt{2}\epsilon\psi + \sqrt{2}\theta(\sqrt{2}i\sigma^\mu\overline{\epsilon}\partial_\mu z - \sqrt{2}\epsilon f) - \theta\theta(-i2\sqrt{2}\sigma^\mu\overline{\epsilon}\partial_\mu\psi). \end{aligned}$$

#### 3.2.4 Susy invariant actions

Finally, we can actually construct a susy invariant action. The most generic susy Lagrangian will look like this:

$$\mathcal{L} = \int d^2\theta d^2\overline{\theta}\ F(x^\mu, \theta, \overline{\theta}) + \int d^2\theta\ W(\phi) + \int d^2\overline{\theta}\ [W(\phi)]^\dagger.$$

Here the $F$ is any old superfield and $W(\phi)$ ($[W(\phi)]^\dagger = \overline{W}(\overline{\phi})$) is a chiral (anti-chiral) field, which can in general be composed of other chiral (anti-chiral) fields.

To see that this Lagrangian will lead to an invariant action, we first consider the susy variation of a superfield (3):

$$\begin{aligned}\delta F &= \left(\Delta^\mu \partial_\mu + \epsilon^\alpha \frac{\partial}{\partial\theta^\alpha} - \bar{\epsilon}^{\dot{\alpha}} \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}\right) F(x^\mu, \theta^\alpha, \bar{\theta}^{\dot{\alpha}}) \\ &= \epsilon^\alpha \frac{\partial F}{\theta^\alpha} - \bar{\epsilon}^{\dot{\alpha}} \frac{\partial F}{\partial\bar{\theta}^{\dot{\alpha}}} + \partial_\mu[i(\theta\sigma^\mu\bar{\epsilon} - \epsilon\sigma^\mu\bar{\theta})F],\end{aligned}$$

of which only the last term remains after the integration $\int d^2\theta d^2\bar{\theta}$. This last term is a total space time derivative, so this part is indeed invariant.

Secondly, for the chiral field one finds the variation:

$$\delta\phi = \frac{\partial}{\partial\theta^\alpha}(-\epsilon^\alpha\phi(y,\theta)) + \frac{\partial}{\partial y^\mu}[i(\theta\sigma^\mu\bar{\epsilon} - \epsilon\sigma^\mu\bar{\theta})\phi(y,\theta)].$$

Here the last term becomes a space-time derivative in the integration $\int d^2\theta$ and the other terms vanish. Of course a similar results holds for its anti-chiral counterpart. This proves that the Lagrangian given above after space-time integration gives a supersymmetric action.

### 3.3 An example of linear susy

To get a more thorough understanding of linear supersymmetric theories before moving on to nonlinear realizations, we will now treat one of the most simple susy field theories: The Wess-Zumino model. Julius Wess and Bruno Zumino in 1973 first wrote down what they called 'supergauge transformations' in four dimensions [12] and Lagrangians transforming with a total derivative under them, so that the actions were invariant. The invariance under this kind of transformation was later renamed 'supersymmetry'.

In the WZ model, the rather easy choice of $F = \phi^\dagger\phi$, where $\phi$ is a chiral field, is made. $F$ itself is then neither chiral nor anti-chiral. Of this part of the action, only the terms $\theta\theta\bar{\theta}\bar{\theta}$, also called the D-term, will remain after integration. It is easy to find these terms once we expand $\phi$ and $\phi^\dagger$ in terms of $x^\mu$:

$$\begin{aligned}&\phi(y,\theta) = z(y) + \sqrt{2}\theta\psi(y) - \theta\theta f(y) = z(x) + \sqrt{2}\theta\psi(x) - \theta\theta f(x) \\ &+ \partial_\mu z(x)(y^\mu - x^\mu) + \sqrt{2}\theta\partial_\mu\psi(x)(y^\mu - x^\mu) + \frac{\partial_\nu\partial_\mu z(x)}{2!}(y^\nu - x^\nu)(y^\mu - x^\mu),\end{aligned}$$

which using $y^\mu - x^\mu = i\theta\sigma^\mu\bar{\theta}$ and the identities

$$\begin{aligned}\theta\psi\theta\chi &= -\frac{1}{2}\theta\theta\psi\chi \\ \theta\sigma^\mu\bar{\theta}\theta\sigma^\nu\bar{\theta} &= \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}\eta^{\mu\nu},\end{aligned}$$

gives

$$\phi(y,\theta) = z(x) + \sqrt{2}\theta\psi(x) - \theta\theta f(x) + i\theta\sigma^\mu\overline{\theta}\partial_\mu z(x)$$
$$- \frac{i}{\sqrt{2}}\theta\theta\partial_\mu\psi\sigma^\mu\overline{\theta} - \frac{1}{4}\partial_\nu\partial_\mu z(x)\eta^{\nu\mu}\theta\theta\overline{\theta\theta}.$$

Similarly, for $\overline{\phi}$ we have

$$\overline{\phi}(\overline{y},\overline{\theta}) = \overline{z}(\overline{y}) + \sqrt{2}\overline{\theta\psi}(\overline{y}) - \overline{\theta\theta}\overline{f}(\overline{y}) = \overline{z}(x) + \sqrt{2}\overline{\theta\psi}(x) - \overline{\theta\theta}\overline{f}(x)$$
$$- i\theta\sigma^\mu\overline{\theta}\partial_\mu\overline{z}(x) + \frac{i}{\sqrt{2}}\overline{\theta\theta}\theta\sigma^\mu\partial_\mu\overline{\psi}(x) - \frac{1}{4}\theta\theta\overline{\theta\theta}\partial_\nu\partial_\mu\overline{z}(x)\eta^{\nu\mu}.$$

The D-tem, notation $|_{\theta\theta\overline{\theta\theta}}$, then simply is all terms of the product containing the right $\theta$'s

$$\phi^\dagger\phi = ... - \frac{1}{4}z^\dagger\partial^2 z\theta\theta\overline{\theta\theta} - i\overline{\theta\psi}\theta\theta\partial_\mu\psi\sigma^\mu\overline{\theta} + \theta\sigma^\mu\overline{\theta}\partial_\mu z^\dagger\theta\sigma^\nu\overline{\theta}\partial_\nu z + f^\dagger f\theta\theta\overline{\theta\theta}$$
$$+ i\overline{\theta\theta}\theta\sigma^\mu\partial_\mu\overline{\psi}\theta\psi - \frac{1}{4}\partial^2 z^\dagger z\theta\theta\overline{\theta\theta}$$

Since using the Fierz identities (see appendix 8.1) we have

$$-i(\overline{\theta\psi})(\theta\theta)(\partial_\mu\psi\sigma^\mu\overline{\theta}) = -i(\overline{\theta\psi})(\overline{\theta}\partial_\mu\psi\sigma^\mu)(\theta\theta) = \frac{i}{2}\overline{\psi}\partial_\mu\psi\sigma^\mu\theta\theta\overline{\theta\theta},$$
$$(\theta\sigma^\mu\overline{\theta})\partial_\mu z^\dagger(\theta\sigma^\nu\overline{\theta})\partial_\nu z = \frac{1}{2}\partial_\mu z^\dagger\partial_\nu z\theta\theta\overline{\theta\theta}\eta^{\mu\nu} = \frac{1}{2}\partial_\mu z^\dagger\partial^\mu z\theta\theta\overline{\theta\theta},$$
$$i(\overline{\theta\theta})(\theta\sigma^\mu\partial_\mu\overline{\psi})(\theta\psi) = i(\theta\sigma^\mu\partial_\mu\overline{\psi})(\theta\psi)(\overline{\theta\theta}) = -\frac{i}{2}\psi\sigma^\mu\partial_\mu\overline{\psi}\theta\theta\overline{\theta\theta},$$

giving

$$\phi^\dagger\phi|_{\theta\theta\overline{\theta\theta}} = -\frac{1}{4}z^\dagger\partial^2 z - \frac{1}{4}\partial^2 z^\dagger z + \frac{1}{2}\partial_\mu z^\dagger\partial^\mu z + f^\dagger f + \frac{i}{2}\partial_\mu\psi\sigma^\mu\overline{\psi} - \frac{i}{2}\psi\sigma^\mu\partial_\mu\overline{\psi}$$
$$= \partial_\mu z^\dagger\partial^\mu z + \frac{i}{2}(\partial_\mu\psi\sigma^\mu\overline{\psi} - \psi\sigma^\mu\partial_\mu\overline{\psi}) + f^\dagger f + TD,$$

where $TD$ stands for total derivative. Then to write out the action

$$S = \int d^4x d^2\theta d^2\overline{\theta}\ \phi^\dagger\phi + \int d^4x d^2\theta\ W(\phi) + \int d^4x d^2\overline{\theta}\ [W(\phi)]^\dagger,$$

we need a taylor expansion for the superpotential to find the terms $\theta\theta$ and $\overline{\theta\theta}$. This expansion is done around $z(y)$ and all $\frac{\partial W}{\partial z}$ and $\frac{\partial^2 W}{\partial z^2}$ are evaluated at $z(y)$:

$$W(\phi) = W(z(y)) + \frac{\partial W}{\partial z}(\phi - z(y)) + \frac{1}{2}\frac{\partial^2 W}{\partial z^2}(\phi - z(y))^2 + ...$$
$$= W(z(y)) + \sqrt{2}\frac{\partial W}{\partial z}\theta\psi(y) - \theta\theta\frac{\partial W}{\partial z}f(y) + \frac{1}{2}\frac{\partial^2 W}{\partial z^2}(\sqrt{2}\theta\psi(y))(\sqrt{(2)}\theta\psi(y))$$
$$= W(z(y)) + \sqrt{2}\frac{\partial W}{\partial z}\theta\psi(y) - \theta\theta\left(\frac{\partial W}{\partial z}f(y) + \frac{1}{2}\frac{\partial^2 W}{\partial z^2}\psi(y)\psi(y)\right),$$

where in the last step again a Fierz identity was used. The action becomes

$$S = \int d^4x \; [|\partial_\mu z|^2 - i\psi\sigma^\mu\partial_\mu\overline{\psi} + f^\dagger f - \frac{\partial W}{\partial z} f + h.c - \frac{1}{2}\frac{\partial^2 W}{\partial z^2}\psi\psi + h.c.].$$

Here we still have the auxiliary field $f^\dagger$. We can get rid of that by noting that from the action

$$f^\dagger = \frac{\partial W}{\partial z}$$

results. This leads to

$$S = \int d^4x \; [|\partial_\mu z|^2 - i\psi\sigma^\mu\partial_\mu\overline{\psi} - \left|\frac{\partial W}{\partial z}\right|^2 - \frac{1}{2}\frac{\partial^2 W}{\partial z^2}\psi\psi + h.c.],$$

and substituting in a superpotential $W = \frac{m}{2}\phi^2 + \frac{g}{3}\phi^3$ as in [5], we find

$$\begin{aligned} S = \int d^4x \; [&|\partial_\mu z|^2 - i\psi\sigma^\mu\partial_\mu\overline{\psi} - m^2|z|^2 - \frac{m}{2}(\psi\psi + \overline{\psi\psi}) \\ &- mg(z^\dagger z^2 + (z^\dagger)^2 z) - g^2|z|^4 + g(z\psi\psi + z^\dagger\overline{\psi\psi})]. \end{aligned}$$

Here we see that the terms with $mg$, $g^2$, and $g$ are interaction terms, since they have more than a square of a field. The mentioned constants therefore are coupling constants. The size of these constants obviously determines in large part how important certain interactions are in the system. Coupling of the scalar $z$ to the spinor $\psi$ happens in the term with constant $g$, which therefore couples bosons to fermions.

It is worth noting that in this example we chose to have only one field $\phi$ in the action, while taking multiple different chiral $\phi^i$ would have been possible as well.

# 4 Spontaneous breaking of supersymmetry

Spontaneous breaking of susy happens when the vacuum state of the system is not invariant under the susy transformations. This is the case when there exists a nonzero energy of the vacuum state $|0\rangle$, for then using the susy algebra and $\sigma^0 = I_2$,

$$\begin{aligned} 0 \neq 2\sigma^0_{\alpha\dot{\alpha}} \langle 0| P_0 |0\rangle &= \langle 0| \{Q_\alpha, \overline{Q}_{\dot{\alpha}}\} |0\rangle = \langle 0| (Q_\alpha \overline{Q}_{\dot{\alpha}} + \overline{Q}_{\dot{\alpha}} Q_\alpha) |0\rangle \\ &= \left\|\overline{Q}_{\dot{\alpha}} |0\rangle\right\|^2 + \|Q_\alpha |0\rangle\|^2, \end{aligned}$$

indicating that not all $Q_\alpha$ and $\overline{Q}_{\dot{\alpha}}$ leave the vacuum invariant. We know that, for a vacuum state, Lorentz invariance must be realized, as well as stability. This means all non-scalar fields and spacial derivatives must vanish. The condition for a vacuum then simply becomes minimality of the scalar potential we have left. This scalar potential for supersymmetry is given by [5]

$$V(z, z^\dagger) = f_i^\dagger f^i + \frac{1}{2} D^a D^a, \tag{4}$$

where $f_i^\dagger = \frac{\partial W(z)}{\partial z^i}$ and $D^a = g^a \left( z_i^\dagger (T^a)^i{}_j z^j + \xi^a \right)$. Here $W(z)$ is the superpotential, $g^a$ is a scaling constant, $T^a$ are the generators of the gauge group, and $\xi^a$ are the Fayet-Iliopoulos terms. The condition for unbroken symmetry becomes

$$f_i^\dagger = D^a = 0. \tag{5}$$

When either $f^i \neq 0$ or $D^a \neq 0$, scalar potential V is not zero, but strictly positive. This means there is a strictly positive energy for the vacuum, breaking susy.

## 4.1 Goldstone's Theorem

When a continuous, global symmetry is broken, we can invoke Goldstone's theorem to conclude there exists a massless mode in the spectrum, i.e. there exists a massless particle [5]. The quantum numbers of such a particle are related to the broken symmetry.

In 1960, Nambu was the first to come across a massless particle, while investigating broken symmetries [13]. He even found it in two separate ways. When Goldstone in 1961 also found a massless particle in a model with a broken symmetry, he conjectured that every time a continuous symmetry group leaves the Lagrangian, but not the vacuum invariant, such a particle arises. Subsequently, Goldstone, Salam, and Weinberg in 1962 published an article giving three proofs of the conjecture [13].

A complete proof of Goldstone's theorem will not be given in this text, as it is deemed too involving for its purposes by the author. However, the general idea is as follows. When a Lagrangian is invariant under certain continuous symmetry transformations, but the vacuum state is not, this indicates the existence of many equivalent but distinct vacuum states. They are equivalent in the sense that their energy is identical, yet distinct because the transformation does not map them to zero, but to another vacuum. The possibility of transitioning between the different vacuum states without needing to supply energy to the system, can be seen as a massless mode in the spectrum, i.e. a massless particle. Perhaps the easiest way to show this line of reasoning is by introducing the example of the Sombrero potential (see 2). If we consider a point particle living in this potential, clearly the ground (or vacuum) state would be reached when the particle is in the valley of the hat. However, contrary to the potential itself, the ground state of the system is not invariant under rotation around the central axis. After all, when the particle is in the valley at an angle $\theta$ with respect to an axis pointing out of the page, the picture is clearly different from the case where the particle is at an angle $\phi \neq \theta$. Obviously, there is a degree of freedom in the arbitrary choice of the angle. This degree of freedom is parametrized by, or interpreted as, a massless mode.

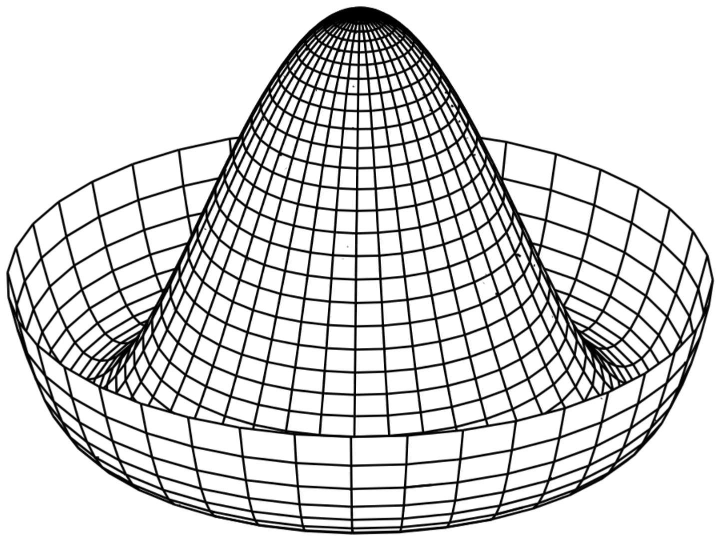

Figure 2: The Sombrero potential, having a continuous rotational symmetry around the axis pointing up through the top of the hat [14].

### 4.2 The Goldstino

Although strictly speaking, the Goldstone theorem can only be applied to internal symmetries, there is an equivalent theorem for supersymmetry as shown in for example [15]. While the massless particles encountered in the early 60's by Nambu and Goldstone were particles with zero spin, for supersymmetry we can expect the presence of a massless spin one-half particle. After all, the broken symmetry is generated by spin one-half operators. This particle is generally called a Goldstino.

To show the emergence of a fermionic massless mode, we use what we know about the susy breaking vacuum. Since it is a vacuum,

$$\frac{\partial V(\langle z^i\rangle, \langle z_i^\dagger\rangle)}{\partial z^i} = \langle f^j\rangle \frac{\partial W}{\partial z^i \partial z^j} - g^a \langle D^a\rangle z_j^\dagger (T^a)^j{}_i = 0,$$

and $\langle f^i\rangle = f^i(\langle z^i\rangle) \neq 0$ or $\langle D^a\rangle = D^a(\langle z^i\rangle, \langle z_i^\dagger\rangle) \neq 0$. This can be combined with the requirement of gauge invariance of the superpotential $W$,

$$0 = \delta^a_{gauge} W = \frac{\partial W}{\partial z^i} \delta^a_{gauge} z^i = f_i^\dagger (T^a)^i{}_j z^j,$$

giving a set of equations, that can be written in terms of a matrix equation, where the matrix has a zero eigenvalue:

$$M = \begin{pmatrix} \langle \frac{\partial W}{\partial z^i \partial z^j}\rangle & -g^a \langle z_l^\dagger\rangle (T^a)^l{}_i \\ -g^b \langle z_l^\dagger\rangle (T^b)^l{}_j & 0 \end{pmatrix}, \quad M \begin{pmatrix} \langle f^i\rangle \\ \langle D^a\rangle \end{pmatrix} = 0.$$

The matrix $M$ constructed here turns out to be exactly the fermion mass matrix [5]. That this matrix has a zero eigenvalue, means that there is a massless fermion: the Goldstino.

## 4.3 Mechanisms for spontaneous susy breaking

Before we saw that when the scalar potential (4) is nonzero for all $z$, the vacuum necessarily breaks supersymmetry. This lead us to the condition of unbroken susy (5). We will now explore the line of thought (as shown in [5]) behind two mechanisms by which susy can be broken. It will be clear that whether or not susy is broken depends on the choice for the superpotential $W$ and whether the Fayet-Iliopoulos parameters $\xi^\alpha$ are zero or not.

### 4.3.1 O'Raifeartaigh mechanism

For this mechanism, we must first assume that the $\xi^a$ (Fayet-Iliopoulos parameters) vanish or no U(1) factors are present. Then susy will be broken if there is no $z$ satisfying both $\frac{\partial W}{\partial z^i} = 0$ and $z_i^\dagger (T^a)^i{}_j z^j = 0$. Obviously, if there is no linear term in superpotential $W$, $\langle z^i \rangle = 0$ is always a solution. So let us assume there is a linear term, meaning $W = a_i z^i + ....$ Gauge invariance then requires that the representation carried by $z^i$ contains at least one singlet. This inevitably leads to a set of equations via $\frac{\partial W}{\partial z^i} = 0$ that cannot simultaneously be satisfied. Take for example the superpotential

$$W = Y(a - X^2) + bZX + w(X, z^i),$$

where X, Y, and Z are singlets. Then $f_Y^\dagger = a - X^2$ and $f_Z^\dagger = bX$ can clearly not both be zero at the same time. This means that there is no vacuum that preserves susy.

### 4.3.2 Fayet-Iliopoulos mechanism

Assume now that $\xi$ does not vanish and there exists at least one U(1) factor. Then the relevant part of $D = 0$ is

$$\sum_i q_i |z^i|^2 + \xi,$$

$q_i$ being the U(1) charges of $z^i$. Now since there are no chiral anomalies, i.e. the chiral symmetry U(1) is preserved, $\Sigma_i q_i^3 = 0$. This means charges of both signs must exist.

If we now also introduce $f^i = 0$, susy must be broken. Again, let u look at a simple example. Take $W = m\phi^1\phi^2$ with $\phi^1$ and $\phi^2$ chiral multiplets with charges $q_1 = -q_2 = 1$. Then $D = 0$ gives $|z^1|^2 - |z^2|^2 + \xi = 0$. Clearly $z^1$ and $z^2$ cannot both be zero, so that $f^1 = mz_2^\dagger$ or $f^2 = mz_1^\dagger$ or both are nonzero when $m \neq 0$. This means susy is broken.

# 5 The Volkov-Akulov action

In this section, we finally encounter the Volkov-Akulov action. This model was first introduced in 1973, when proposing that the neutrino was in fact a Goldstone particle [1]; a massless particle found due to the breaking of a symmetry. The symmetry in question was supersymmetry.

## 5.1 Deriving the Lagrangian

Here we will try to derive the Lagrangian in a natural way, starting at the susy transformation:

$$\begin{aligned} x'^{\mu} &= x^{\mu} + i(\theta\sigma^{\mu}\overline{\epsilon} - \epsilon\sigma^{\mu}\overline{\theta}) \\ \theta'^{\alpha} &= \theta^{\alpha} + \epsilon^{\alpha} \\ \overline{\theta}'_{\dot{\alpha}} &= \overline{\theta}_{\dot{\alpha}} + \overline{\epsilon}_{\dot{\alpha}}. \end{aligned}$$

For an arbitrary spinor field $\psi(x)$, we can simply take $\theta = \kappa\psi(x)$, so that:

$$\begin{aligned} \psi'(x') &= \psi(x) + \frac{\epsilon}{\kappa} \\ \overline{\psi}'(x') &= \overline{\psi}(x) + \frac{\overline{\epsilon}}{\kappa}. \end{aligned}$$

Noting that by Taylor expanding $\psi'(x')$ we get to first order

$$\psi'(x') = \psi'(x + i\kappa(\psi\sigma\overline{\epsilon} - \epsilon\sigma\overline{\psi})) = \psi'(x) + i\kappa(\psi\sigma\overline{\epsilon} - \epsilon\sigma\overline{\psi})\partial_{\mu}\psi'(x),$$

so that since $\partial_{\mu}\psi'(x) = \partial_{\mu}\psi(x)$

$$\psi'(x') = \psi'(x) + i\kappa(\psi\sigma^{\mu}\overline{\epsilon} - \epsilon\sigma^{\mu}\overline{\psi})\partial_{\mu}\psi(x).$$

This helps us find the transformations:

$$\begin{aligned} \delta^{Q}(\epsilon,\overline{\epsilon})\psi^{\alpha} &= \psi'^{\alpha}(x) - \psi^{\alpha}(x) = \frac{\epsilon^{\alpha}}{\kappa} - i\kappa(\psi\sigma^{\mu}\overline{\epsilon} - \epsilon\sigma^{\mu}\overline{\psi})\partial_{\mu}\psi^{\alpha} \\ \delta^{Q}(\epsilon,\overline{\epsilon})\overline{\psi}_{\dot{\alpha}} &= \overline{\psi}'_{\dot{\alpha}}(x) - \overline{\psi}'_{\dot{\alpha}}(x) = \frac{\overline{\epsilon}_{\dot{\alpha}}}{\kappa} - i\kappa(\psi\sigma^{\mu}\overline{\epsilon} - \epsilon\sigma^{\mu}\overline{\psi})\partial_{\mu}\overline{\psi}_{\dot{\alpha}}. \end{aligned}$$

These transformations satisfy the susy algebra, which for variations boils down to

$$[\delta^{Q}(\epsilon,\overline{\epsilon}), \delta^{Q}(\eta,\overline{\eta})] = -2i(\epsilon\sigma^{\mu}\overline{\eta} - \eta\sigma^{\mu}\overline{\epsilon})\partial_{\mu}.$$

We can see this by noting that $[\xi Q, \overline{\xi Q}] = 2\xi\sigma^{\mu}\overline{\xi}P_{\mu}$ and $[\xi Q, \xi Q] = 0$, for an anti-commuting parameter $\xi$ [16]. Then since $\delta(\epsilon,\overline{\epsilon})\psi = (\epsilon Q + \overline{\epsilon}\overline{Q})\psi$,

$$\begin{aligned} [\delta^{Q}(\epsilon,\overline{\epsilon}), \delta^{Q}(\eta,\overline{\eta})]\psi &= (\epsilon Q + \overline{\epsilon}\overline{Q})(\eta Q\psi + \overline{\eta}\overline{Q}\psi) - (\eta Q + \overline{\eta}\overline{Q})(\epsilon Q\psi + \overline{\epsilon}\overline{Q}\psi) \\ &= \epsilon Q\overline{\eta}\overline{Q}\psi + \overline{\epsilon}\overline{Q}\eta Q\psi - \eta Q\overline{\epsilon}\overline{Q}\psi - \overline{\eta}\overline{Q}\epsilon Q\psi \\ &= [\epsilon Q, \overline{\eta}\overline{Q}]\psi - [\eta Q, \overline{\epsilon}\overline{Q}]\psi \\ &= 2\epsilon\sigma^{\mu}\overline{\eta}P_{\mu}\psi - 2\eta\sigma^{\mu}\overline{\epsilon}P_{\mu}\psi = -2i(\epsilon\sigma^{\mu}\overline{\eta} - \eta\sigma^{\mu}\overline{\epsilon})\partial_{\mu}\psi. \end{aligned}$$

The explicit calculation for our particular variations is done in appendix 8.2.

Returning to the goal of formulating the VA-Lagrangian, it is useful to find differentials, invariant under the specified transformations. The differentials found from the coordinate transformations are not invariant themselves:

$$
\begin{aligned}
dx'^{\mu} &= dx^{\mu} + id\theta\sigma^{\mu}\overline{\epsilon} - i\epsilon\sigma^{\mu}d\overline{\theta} \\
d\theta' &= d\theta \\
d\overline{\theta}' &= d\overline{\theta},
\end{aligned}
$$

but they make it easy to see these are in fact invariant:

$$\omega^{\nu} = dx^{\nu} - id\theta\sigma^{\nu}\overline{\theta} + i\theta\sigma^{\nu}d\overline{\theta},$$

since

$$
\begin{aligned}
\omega^{\nu} &= dx'^{\nu} - id\theta'\sigma^{\nu}\overline{\theta}' + i\theta'\sigma^{\nu}d\overline{\theta}' \\
&= dx^{\nu} + id\theta\sigma^{\nu}\overline{\epsilon} - i\epsilon\sigma^{\nu}d\overline{\theta} - id\theta\sigma^{\nu}\overline{\theta} - id\theta\sigma^{\nu}\overline{\epsilon} + i\theta\sigma^{\nu}d\overline{\theta} + i\epsilon\sigma^{\nu}d\overline{\theta} \\
&= dx^{\nu} - id\theta\sigma^{\nu}\overline{\theta} + i\theta\sigma^{\nu}d\overline{\theta}.
\end{aligned}
$$

Now substitute in $\theta = \kappa\psi$ and $d\theta = \kappa\frac{\partial\psi}{\partial x^{\mu}}dx^{\mu}$, to get

$$
\begin{aligned}
\omega^{\nu} &= dx^{\nu} - i\kappa^2\partial_{\mu}\psi dx^{\mu}\sigma^{\nu}\overline{\psi} + i\kappa^2\psi\sigma^{\nu}\partial_{\mu}\overline{\psi}dx^{\mu} \\
&= dx^{\mu}[\delta_{\mu}^{\nu} + i\kappa^2(\psi\sigma^{\nu}\partial_{\mu}\overline{\psi} - \partial_{\mu}\psi\sigma^{\nu}\overline{\psi})] = dx^{\mu}W_{\mu}{}^{\nu}.
\end{aligned}
$$

The transformation of $W_{\mu}{}^{\nu}$ is given by:

$$\delta^{Q}(\epsilon,\overline{\epsilon})W_{\mu}{}^{\nu} = i\kappa(\epsilon\sigma^{\rho}\partial_{\mu}\overline{\psi} - \partial_{\mu}\psi\sigma^{\rho}\overline{\epsilon})W_{\rho}{}^{\nu} - i\kappa(\psi\sigma^{\rho}\overline{\epsilon} - \epsilon\sigma^{\rho}\overline{\psi})\partial_{\rho}W_{\mu}{}^{\nu},$$

as shown in appendix 8.3. With this transformation,

$$\mathcal{L} = -\frac{1}{2\kappa^2}\det W$$

gives rise to an invariant action. This can be shown as follows: since $W$ is an invertible matrix, $W^{-1} = (\det W)^{-1}\operatorname{adj} W$. Then Jacobi's formula $\frac{d}{dt}\det W = \operatorname{Tr}\big(\operatorname{adj}(W)\frac{dW}{dt}\big)$ tells us

$$\partial_{\mu}\det W = \det W\,\operatorname{Tr}\big(\partial_{\mu}WW^{-1}\big).$$

This implies

$$\delta^{Q}(\epsilon,\overline{\epsilon})\det W = \det W\,\operatorname{Tr}\big(\delta(\epsilon,\overline{\epsilon})WW^{-1}\big),$$

so we get

$$
\begin{aligned}
\delta^{Q}(\epsilon,\overline{\epsilon})\det W &= \det W\,\operatorname{Tr}\big(\delta^{Q}(\epsilon,\overline{\epsilon})WW^{-1}\big) \\
&= \det W\,\operatorname{Tr}\big(\big[i\kappa(\epsilon\sigma^{\rho}\partial_{\mu}\overline{\psi} - \partial_{\mu}\psi\sigma^{\rho}\overline{\epsilon})W_{\rho}{}^{\nu} - i\kappa(\psi\sigma^{\rho}\overline{\epsilon} - \epsilon\sigma^{\rho}\overline{\psi})\partial_{\rho}W_{\mu}{}^{\nu}\big]\,W^{-1}\big) \\
&= \det W\,\operatorname{Tr}\big(i\kappa(\epsilon\sigma^{\rho}\partial_{\mu}\overline{\psi} - \partial_{\mu}\psi\sigma^{\rho}\overline{\epsilon})\delta_{\rho}^{\nu}\big) - i\kappa(\psi\sigma^{\mu}\overline{\epsilon} - \epsilon\sigma^{\mu}\overline{\psi})\det W\,\operatorname{Tr}\big(\partial_{\mu}WW^{-1}\big) \\
&= -i\kappa\;\partial_{\mu}(\det W(\psi\sigma^{\mu}\overline{\epsilon} - \epsilon\sigma^{\mu}\overline{\psi})),
\end{aligned}
$$

making the transformation of the Lagrangian

$$\delta^Q(\epsilon,\overline{\epsilon})\mathcal{L} = \frac{i}{2\kappa}\partial_\mu(\det W(\psi\sigma^\mu\overline{\epsilon} - \epsilon\sigma^\mu\overline{\psi})),$$

which is a total derivative, so that the action will be invariant under the transformation.

## 5.2 The Lagrangian

To analyze the Lagrangian further, it is interesting to see the Lagrangian in the expanded form. Having defined the 'vierbein' $W_\mu{}^\nu = \delta_\mu^\nu + i\kappa^2(\psi\sigma^\nu\partial_\mu\overline{\psi} - \partial_\mu\psi\sigma^\nu\overline{\psi}) = \delta_\mu^\nu + \kappa^2 T_\mu{}^\nu$, the Lagrangian is given by

$$\begin{aligned}
\mathcal{L} = -\frac{1}{2\kappa^2}\det W = -\frac{1}{2\kappa^2}&\begin{vmatrix} 1+\kappa^2T_{11} & \kappa^2T_{12} & \kappa^2T_{13} & \kappa^2T_{14} \\ \kappa^2T_{21} & 1+\kappa^2T_{22} & \kappa^2T_{23} & \kappa^2T_{24} \\ \kappa^2T_{31} & \kappa^2T_{32} & 1+\kappa^2T_{33} & \kappa^2T_{34} \\ \kappa^2T_{41} & \kappa^2T_{42} & \kappa^2T_{43} & 1+\kappa^2T_{44} \end{vmatrix} \\
= -\frac{1}{2\kappa^2} - \frac{1}{2}\left(T_{11}+T_{22}+T_{33}+T_{44}\right) \\
- \frac{\kappa^2}{2}\left(T_{11}T_{22}+T_{11}T_{33}+T_{11}T_{44}+T_{22}T_{33}+T_{22}T_{44}+T_{33}T_{44}\right) \\
+ \frac{\kappa^2}{2}\left(T_{12}T_{21}+T_{13}T_{31}+T_{14}T_{41}+T_{23}T_{32}+T_{24}T_{42}+T_{34}T_{43}\right) \\
- \frac{\kappa^4}{2}\Sigma_p(-1)^pT_{11}T_{22}T_{33} - \frac{\kappa^4}{2}\Sigma_p(-1)^pT_{11}T_{22}T_{44} - \frac{\kappa^4}{2}\Sigma_p(-1)^pT_{11}T_{33}T_{44} \\
- \frac{\kappa^4}{2}\Sigma_p(-1)^pT_{22}T_{33}T_{44} - \frac{\kappa^6}{2}\Sigma_p(-1)^pT_{11}T_{22}T_{33}T_{44}
\end{aligned}$$

Where the sum over p is the sum over all permutations of the second index of the products of the tensors $T_{\mu\nu}$. The term proportional to $\frac{\kappa^2}{2}$ is simply half of $T_{\mu\mu}T_{\nu\nu}$, since it is counting only cases with $\mu < \nu$ and $T_{\mu\nu}$ and $T_{\nu\mu}$ commute. Similarly, the term proportional to $\frac{\kappa^2}{2}$ is half of $T_{\mu\nu}T_{\nu\mu}$. The terms proportional to $-\frac{\kappa^4}{2}$ are a fraction $\frac{1}{3!}$ of the sum $\Sigma_p(-1)^pT_{\mu\mu}T_{\nu\nu}T_{\rho\rho}$, since they contain all combinations of $\mu$, $\nu$, and $\rho$, but only one out of 3! orders for every combination. These orders of course do not matter since the $T_{\mu\nu}$ tensors commute. For the $-\frac{\kappa^6}{2}$ term, we get a factor $\frac{1}{4!}$ in front of $\Sigma_p(-1)^pT_{\mu\mu}T_{\nu\nu}T_{\rho\rho}T_{\sigma\sigma}$, since only one order of 4! possible arrangements is present. Putting this together we find the following:

$$\begin{aligned}
\mathcal{L} = -\frac{1}{2\kappa^2}\det W = -\frac{1}{2\kappa^2} - \frac{1}{2}T_{\mu\mu} - \frac{\kappa^2}{4}\left(T_{\mu\mu}T_{\nu\nu} - T_{\mu\nu}T_{\nu\mu}\right) \\
- \frac{\kappa^4}{12}\Sigma_p(-1)^pT_{\mu\mu}T_{\nu\nu}T_{\rho\rho} - \frac{\kappa^6}{48}\Sigma_p(-1)^pT_{\mu\mu}T_{\nu\nu}T_{\rho\rho}T_{\sigma\sigma}.
\end{aligned}$$

Here we can recognize a free theory in the first two terms, while the other terms make up the interaction part of the Lagrangian.

## 5.3 The equations of motion

To find the equations of motion, we use the usual variations with respect to $\psi$ and $\overline{\psi}$. In the latter case this looks like

$$\partial_\rho \left( \frac{\partial \mathcal{L}}{\partial(\partial_\rho \overline{\psi})} \right) - \frac{\partial \mathcal{L}}{\partial \overline{\psi}} = 0,$$

giving

$$\partial_\rho \left( -\frac{1}{2\kappa^2} \det W \operatorname{Tr} \left( \frac{\partial W}{\partial(\partial_\rho \overline{\psi})} W^{-1} \right) \right) + \frac{1}{2\kappa^2} \det W \operatorname{Tr} \left( \frac{\partial W}{\partial \overline{\psi}} W^{-1} \right) = 0,$$

or

$$\operatorname{Tr}\big(\partial_\rho W W^{-1}\big) \operatorname{Tr} \left( \frac{\partial W}{\partial(\partial_\rho \overline{\psi})} W^{-1} \right) + \partial_\rho \operatorname{Tr} \left( \frac{\partial W}{\partial(\partial_\rho \overline{\psi})} W^{-1} \right) - \operatorname{Tr} \left( \frac{\partial W}{\partial \overline{\psi}} W^{-1} \right) = 0.$$

In this, we must substitute the derivatives

$$\begin{aligned} &\frac{\partial W_\mu{}^\nu}{\partial(\partial_\rho \overline{\psi})} = i\kappa^2 \psi \sigma^\nu \delta^\rho_\mu, \quad \frac{\partial W_\mu{}^\nu}{\partial \overline{\psi}} = -i\kappa^2 \partial_\mu \psi \sigma^\nu, \\ &\partial_\rho W_\mu{}^\nu = i\kappa^2 \left( \partial_\rho \psi \sigma^\nu \partial_\mu \overline{\psi} - \partial_\mu \psi \sigma^\nu \partial_\rho \overline{\psi} + \psi \sigma^\nu \Box \overline{\psi} \eta_{\rho\mu} - \Box \psi \sigma^\nu \overline{\psi} \eta_{\rho\mu} \right), \end{aligned}$$

and the inverse $W^{-1}$ of the vierbein (derivation in appendix 8.4)

$$(W^{-1})_\nu{}^\tau = \delta^\tau_\nu - \kappa^2 T^\tau_\nu + \kappa^4 T^\chi_\nu T^\tau_\chi - \kappa^6 T^\lambda_\nu T^\chi_\lambda T^\tau_\chi + \kappa^8 T^\theta_\nu T^\lambda_\theta T^\chi_\lambda T^\tau_\chi.$$

Here again $T^\nu_\mu = i(\psi \sigma^\nu \partial_\mu \overline{\psi} - \partial_\mu \psi \sigma^\nu \overline{\psi})$. The equation of motion then becomes

$$\begin{aligned} &\operatorname{Tr}\bigg( i\kappa^2 (\partial_\rho \psi \sigma^\nu \partial_\mu \overline{\psi} - \partial_\mu \psi \sigma^\nu \partial_\rho \overline{\psi} + \psi \sigma^\nu \Box \overline{\psi} \eta_{\rho\mu} - \Box \psi \sigma^\nu \overline{\psi} \eta_{\rho\mu}) \\ &\cdot (\delta^\tau_\nu - \kappa^2 T^\tau_\nu + \kappa^4 T^\chi_\nu T^\tau_\chi - \kappa^6 T^\lambda_\nu T^\chi_\lambda T^\tau_\chi + \kappa^8 T^\theta_\nu T^\lambda_\theta T^\chi_\lambda T^\tau_\chi) \bigg) \\ &\cdot \operatorname{Tr} \left( \psi \sigma^\nu \delta^\rho_\mu (\delta^\tau_\nu - \kappa^2 T^\tau_\nu + \kappa^4 T^\chi_\nu T^\tau_\chi - \kappa^6 T^\lambda_\nu T^\chi_\lambda T^\tau_\chi + \kappa^8 T^\theta_\nu T^\lambda_\theta T^\chi_\lambda T^\tau_\chi) \right) \\ &+ \partial_\rho \operatorname{Tr} \left( \psi \sigma^\nu \delta^\rho_\mu (\delta^\tau_\nu - \kappa^2 T^\tau_\nu + \kappa^4 T^\chi_\nu T^\tau_\chi - \kappa^6 T^\lambda_\nu T^\chi_\lambda T^\tau_\chi + \kappa^8 T^\theta_\nu T^\lambda_\theta T^\chi_\lambda T^\tau_\chi) \right) \\ &+ \operatorname{Tr} \left( \partial_\mu \psi \sigma^\nu (\delta^\tau_\nu - \kappa^2 T^\tau_\nu + \kappa^4 T^\chi_\nu T^\tau_\chi - \kappa^6 T^\lambda_\nu T^\chi_\lambda T^\tau_\chi + \kappa^8 T^\theta_\nu T^\lambda_\theta T^\chi_\lambda T^\tau_\chi) \right) = 0. \end{aligned}$$

In the current expression, the highest order derivative is second order. This is interesting, because a free theory of fermions is only first order, as can be seen from the famous (four component spinor) Dirac equation:

$$(i\gamma^\mu \partial_\mu - m)\psi = 0.$$

This system has four degrees of freedom: one for each complex component (so one for every two real dimensions of phase space)[8]. Similarly, Weyl spinors, having two complex components, are described by the Weyl equations

$$i\sigma^\mu \partial_\mu \lambda = 0,$$

so that the system has just two degrees of freedom. These numbers of degrees of freedom are limited by the fact that the equation of motion is a first order equation, according to [8]. The question then naturally arises whether the second order equation of motion leads to more degrees of freedom. In the light of the next subsection, this question can have grave consequences for the theory.

## 5.4 Ostrogradsky's theorem

This highest order derivative in a theory is relevant, because for it to be remotely physical, the equation of motion is required to contain only derivatives of degree lower than or equal to two. That constraint is implied by Ostrogradsky's theorem [17]. Ostrogradsky derived in 1850 that including higher than first order derivatives in nondegenerate Lagrangians (so since $N^{th}$ order Lagrangian corresponds to an at most $2N^{th}$ order equation of motion, $2^{nd}$ order equation of motion) gives rise to Hamiltonians that are unbounded, from above and below. For odd ordered equations of motion, it was generalized by [18].

The theorem implies the possibility of the creation of enormous amounts of particles, for pairs of positive and negative energy particles (also referred to as 'ghosts',) can be created constantly. In fact, due to the basically infinite volume of phase space, there exists a very large entropy, or availability of states, in high energy (and correspondingly large negative energy) states. This means that not only there is a possibility of an ever increasing and large amount of particles; there is a certainty they will arise.

This is obviously problematic; we do not see such systems anywhere. A last objection to the disregarding of such theories might be time. After all, it could just be that the creation of particle pairs goes sufficiently slow for no one to notice. However, the same entropy making sure there will exist an infinite amount of particles also drives the 'decay rate' of the vacuum. Every moment in time there is an infinite amount of possibilities to decay, so decay will happen instantaneously. Inventing a cutoff for the phase space, so as to make it finite and thereby the decay time nonzero, would be a possible step in an effective theory, but not in a fundamental theory. Furthermore, ignoring high energy states might be a logical step in other theories, but the usual argument with which it is justified relies on the positiveness of the available energy states: When there are only positive energy states, exciting a high energy state is nearly impossible because the energy needs to come from de-exciting other modes.

It can be seen that there is no way of evading the disastrous consequences of the ghosts, other than violating an initial assumption [17]. This makes the theorem of Ostrogradsky about the most powerful restriction on fundamental Lagrangian field theories.

## 5.5 Ghosts in VA?

As mentioned, the equation of motion of VA theory is second order, meaning there is no direct reason to assume the existence of ghosts. However, the larger order with respect to the Dirac and Weyl equations does leave the possibility of a larger number of degrees of freedom, which, as we saw, might have far-reaching effects.

The answer to the question of whether VA theory houses ghosts, can be answered by using the concept of universality, which will be discussed at a bit more length in the next section. Universality means that two theories with the same (broken) symmetry, must be equivalent to each other. It was pointed out by the supervisor of this project (Diederik Roest), that because of universality, the answer could be found in an equivalent theory. The argument of Roest is as follows:

There exists a theory, having both linear and nonlinear susy. The linear susy means the degrees of freedom of the Goldstino and the bosons present in the theory are the same. Therefore, we can confidently use the theorem of Ostrogradsky on the bosonic field in attempting to determine whether the fermionic field has an Ostrogradskyan instability.

As mentioned, the theorem of Ostrogradsky holds for derivative orders larger than second order, meaning the second order derivatives, which the bosonic fields have, do not give rise to ghost-like degrees of freedom. Therefore, also the fermionic field will not propagate ghosts, since there would be no bosonic ghost to pair it with (this argument is also made in [19]). Universality of the broken susy theories than leads to the conclusion that VA too cannot house ghosts.

# 6 The Komargodski-Seiberg action

In their formalism presented in a paper in 2009 [20], Komargodski and Seiberg start out with the Ferrara-Zumino multiplet $\mathcal{J}_\mu$, which is a so called supercurrent multiplet. This multiplet is related to a chiral superfield $X$, in which, upon spontaneously breaking of the symmetry by constructing a general susy breaking Lagrangian, a Goldstino resides. Once the transformations of the component fields of $X$ are invoked, it appears that at low energies, in the realm of susy breaking, $X^2 = 0$ is satisfied. This leads to the conclusion that the constraint $X_{NL}^2 = 0$ does the trick in making the supersymmetry broken, or nonlinear. Throughout this section, [20] is used as reference.

## 6.1 The chiral superfield

The Ferrara-Zumino supermultiplet $\mathcal{J}_\mu$ contains the supersymmetry current $S_{\mu\alpha}$, the energy-momentum tensor $T_{\mu\nu}$ and the R-symmetry current $j_\mu$. The first two of these are conserved, meaning

$$\partial^\mu T_{\mu\nu} = \partial^\mu S_{\mu\alpha} = 0.$$

The superfield $\mathcal{J}_\mu$ also satisfies

$$\overline{D}^{\dot{\alpha}} \mathcal{J}_{\alpha\dot{\alpha}} = D_\alpha X,$$

where $\mathcal{J}_{\alpha\dot{\alpha}} = -2\sigma^{\mu}_{\alpha\dot{\alpha}}\mathcal{J}_{\mu}$. This can be solved to give the multiplet and, more importantly, the chiral superfield in terms of the currents

$$X = x(y) + \sqrt{2}\theta\psi(y) + \theta\theta f(y),$$
$$\psi_{\alpha} = \frac{\sqrt{2}}{3}\sigma^{\mu}_{\alpha\dot{\alpha}}\overline{S}^{\dot{\alpha}}_{\mu}, \quad f = \frac{2}{3}T + i\partial_{\mu}j^{\mu},$$

where as usual $y^{\mu} = x^{\mu} + i\theta\sigma^{\mu}\overline{\theta}$.

## 6.2 Breaking susy

Now we will analyze a free theory, breaking symmetry, to show the Goldstino is present in $X$. The Lagrangian of the theory is

$$\mathcal{L} = \overline{\phi}\phi|_{\theta^2\overline{\theta}^2} + c\phi|_{\theta^2} + \overline{c\phi}|_{\overline{\theta}^2},$$

with chiral superfield $\phi = z(y) + \sqrt{2}\theta\psi_z(y) + \theta\theta f_z(y)$. This includes the free fermion $(\psi_z(y))_{\alpha}$, which is the Goldstino $G_{\alpha}$. It can be shown that in this theory,

$$X = \frac{8}{3}c\phi,$$

where $c$ is the so called decay constant. So in case susy is broken, the chiral field $X$ related to the Ferrara-Zumino multiplet is proportional to the superfield $\phi$, containing the Goldstino. In fact, defining $X_{NL} = \frac{3}{8c}X$, the $\theta$ component is

$$X_{NL}|_{\theta^{\alpha}} = \sqrt{2}G_{\alpha}.$$

The $x$ component of $X_{NL}$ is clearly proportional to the free scalar $z(y)$. However, in general, the low-energy Goldstino does not arise alongside massless scalar. This means that the $x$ component cannot create a one particle state; $z(y)$ has to disappear somehow. The simplest bosonic state must instead be a two Goldstino state. This can be seen more directly from the susy transformations working on the component fields:

$$\begin{aligned}
[\epsilon Q, x_{NL}] &= \sqrt{2}\epsilon G, & [\overline{\epsilon}\overline{Q}, x_{NL}] &= 0,\\
[\epsilon Q, G_{\alpha}] &= \sqrt{2}\epsilon_{\alpha} f, & [\overline{\epsilon}\overline{Q}, G_{\alpha}] &= i\sqrt{2}\sigma^{\mu}_{\alpha\dot{\alpha}}\overline{\epsilon}^{\alpha}\partial_{\mu}x_{NL},\\
[\epsilon Q, f] &= 0, & [\overline{\epsilon}\overline{Q}, f] &= i\sqrt{2}\overline{\epsilon}_{\alpha}\sigma^{\mu\alpha\dot{\alpha}}\partial_{\mu}G_{\alpha}.
\end{aligned}$$

These equations give $x_{NL} = \frac{G^2}{2f}$ up to an added constant. So we can conclude

$$X_{NL} = \frac{G(y)^2}{2f(y)} + \sqrt{2}\theta G(y) + \theta\theta f(y),$$

which satisfies

$$
\begin{aligned}
X_{NL}^2 &= \frac{G^2}{2f}\left(\frac{G^2}{2f} + \sqrt{2}\theta G + \theta^2 f\right) + \sqrt{2}\theta G\left(\frac{G^2}{2f} + \sqrt{2}\theta G\right) + \theta^2 f\left(\frac{G^2}{2f}\right) \\
&= \frac{G^4}{4f^2} + \frac{G^2\theta G}{\sqrt{2}f} + \frac{G^2\theta^2 f}{2f} + \frac{\theta G^3}{\sqrt{2}f} + 2\theta G\theta G + \frac{\theta^2 f G^2}{2f} \\
&= \frac{G^2\theta^2 f}{2f} + \frac{\theta^2 G^2 f}{2f} + 2\theta G\theta G \\
&= \frac{G^2\theta^2 f}{f} + 2(-\frac{1}{2}\theta\theta GG) = 0.
\end{aligned}
$$

We have therefore derived that breaking susy is equivalent to constraining the chiral field to satisfy $X_{NL}^2 = 0$, since this guarantees the presence of a Goldstino only, that is, as long as no other symmetries are present.

## 6.3 The KS Action

We will now consider the most general susy Lagrangian using the constraint superfield $X_{NL}^2 = 0$ (and without derivative terms like $D_\alpha X_{NL}$):

$$
\mathcal{L} = \int d^2\theta d^2\overline{\theta}\ \overline{X}_{NL}X_{NL} + \int d^2\theta\ cX_{NL} + \int d^2\overline{\theta}\ c\overline{X}_{NL},
$$

where we can take $c$ to be real without loss of generality. This is almost exactly the same Lagrangian as the susy breaking free theory with field $\phi$, except that in this case, the massless scalar is substituted by a nonlinear term.

We want to substitute an explicit expression of $X_{NL}$ into this Lagrangian, so using a Taylor expansion we find

$$
\begin{aligned}
X_{NL}(y,\theta) = {}& \frac{G^2(x)}{2f(x)} + \sqrt{2}\theta G(x) + \theta^2 f(x) + \partial_\mu\left(\frac{G^2(x)}{2f(x)}\right)(i\theta\sigma^\mu\overline{\theta}) \\
&+ \partial_\mu(\sqrt{2}\theta G(x))(i\theta\sigma^\mu\overline{\theta}) + \frac{1}{2}\partial^2\left(\frac{G^2(x)}{2f(x)}\right)(i\theta\sigma\overline{\theta})^2 \\
={}& \frac{G^2}{2f} + \sqrt{2}\theta G + f\theta^2 + i\theta\sigma^\mu\overline{\theta}\partial_\mu\left(\frac{G^2}{2f}\right) - \frac{i}{\sqrt{2}}\theta^2\partial_\mu G\sigma^\mu\overline{\theta} - \frac{1}{4}\partial^2\left(\frac{G^2}{2f}\right)\theta^2\overline{\theta}^2
\end{aligned}
$$

and similarly

$$
\overline{X}_{NL}(\overline{y},\overline{\theta}) = \frac{\overline{G}^2}{2\overline{f}} + \sqrt{2}\overline{\theta G} + \overline{\theta}^2\overline{f} - i\theta\sigma^\mu\overline{\theta}\partial_\mu\left(\frac{\overline{G}^2}{2\overline{f}}\right) + \frac{i}{\sqrt{2}}\overline{\theta}^2\theta\sigma^\mu\partial_\mu\overline{G} - \frac{1}{4}\partial^2\left(\frac{\overline{G}^2}{2\overline{f}}\right)\theta^2\overline{\theta}^2.
$$

This gives for the Lagrangian

$$
\begin{aligned}
\mathcal{L} &= \overline{X}_{NL}X_{NL}|_{\theta^2\overline{\theta}^2} + cX_{NL}|_{\theta^2} + c\overline{X}_{NL}|_{\overline{\theta}^2} \\
&= -iG\sigma^\mu\partial_\mu\overline{G} + \overline{f}f + \frac{\overline{G}^2}{2\overline{f}}\partial^2\left(\frac{G^2}{2f}\right) + \frac{G^2}{2f}\partial^2\left(\frac{\overline{G}^2}{2\overline{f}}\right) + cf + c\overline{f}
\end{aligned}
$$

And we can get rid of the auxiliary fields $f$ and $\overline{f}$ by solving their equations of motion. These equations are

$$f + c - \frac{\overline{G}^2}{2\overline{f}^2}\partial^2\left(\frac{G^2}{2f}\right) = 0,$$
$$\overline{f} + c - \frac{G^2}{2f^2}\partial^2\left(\frac{\overline{G}^2}{2\overline{f}}\right) = 0,$$

and their solutions

$$f = -c\left(1 + \frac{\overline{G}^2}{4c^4}\partial^2 G^2 - \frac{3}{16f^8}G^2\overline{G}^2\partial^2 G^2\partial^2\overline{G}^2\right),$$
$$\overline{f} = -c\left(1 + \frac{G^2}{4c^4}\partial^2 \overline{G}^2 - \frac{3}{16f^8}G^2\overline{G}^2\partial^2 G^2\partial^2\overline{G}^2\right).$$

This gives the final Lagrangian as

$$\mathcal{L}_{KS} = -c^2 - iG\sigma^\mu\partial_\mu\overline{G} + \frac{\overline{G}^2}{4c^2}\partial^2 G^2 - \frac{1}{16f^6}G^2\overline{G}^2\partial^2 G^2\partial^2\overline{G}^2.$$

Again, we can see the first two terms as free theory, while the rest constitutes the interaction part. Furthermore it is worth noting that this Lagrangian is completely equivalent to the Volkov-Akulov Lagrangian.

## 6.4 Universality of the VA action

The fact that the VA and KS Lagrangians are equivalent to each other is not a unique or accidental occurrence. It is because all Lagrangians with broken supersymmetry at low enough energy scales only have one massless, spin one half particle: the Goldstino, and therefore they must all describe the same physics. This concept is known as universality. It is true for every set of theories with the same symmetries.

Universality has the consequence that the fields describing the different theories only look different mathematically. This means they can in principle all be related to each other by redefining the fields in a particular (nonlinear) way. For the Komargodski-Seiberg action this is done in [21]. In the notation used in this text, that means expressing the field $\psi$ used in the VA Lagrangian in terms of the field $G$, allowing the KS Lagrangian to be written down by substituting the expression into the VA one. This is however quite involving, while not being tremendously insightful, and is therefore not done here.

# 7 Conclusions

The aim of this project was to get acquainted with supersymmetry, and more specifically Volkov-Akulov theory. In the process, we tried to answer the question of the significance of VA theory in theoretical physics. This lead us to first consider symmetries and their importance in theory in general, before moving on to supersymmetry and the spontaneous breaking of symmetries. Analyzing VA, we looked both at the theory itself and at the Komargodski-Seiberg action, which is a theory with the same symmetry.

Since theories with the same broken symmetries can only describe the same physics, they are fundamentally the same theories. This is called universality. Universality implies that Volkov-Akulov theory is the same as every other theory one can find, with the same symmetry as VA. This means it is related to all those theories, such as Komargodski-Seiberg, by a (nonlinear) field redefinition.

VA theory is a nonlinear, global, supersymmetric theory and therefore describes a single fermionic field with no mass. Its equation of motion has a second order derivative as highest order, which is higher than the first order Dirac and Weyl equations. This does however not lead to problematic degrees of freedom, such as in Ostrogradsky's theorem, because of an argument relying on universality and the equality of the number of bosonic and fermionic degrees of freedom.

Although a lot of topics related to the significance of VA theory have been treated in this text, there are some that could be elaborated on. In our treatment of the Volkov-Akulov action for example, it would have been interesting to go into more depth on the topic of interaction terms and their physical significance. Also, the universality of the action is merely stated as a fact, along with a few arguments on why it should not be surprising, while a more thorough analysis of the concept and its causes would perhaps be enlightening.

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# 8 Appendix

## 8.1

The used Fierz identities are [5]

$$\begin{aligned}\theta\sigma^\mu\bar{\theta}\theta\sigma^\nu\bar{\theta} &= \frac{1}{2}\theta\theta\overline{\theta\theta}\eta^{\mu\nu},\\ \theta\psi\theta\chi &= -\frac{1}{2}\theta\theta\psi\chi,\\ \overline{\theta\psi\theta\chi} &= -\frac{1}{2}\overline{\theta\theta\psi\chi}.\end{aligned}$$

### 8.2

The transformations have the following commutation relation:

$$
\begin{aligned}
&[\delta^Q(\epsilon,\overline{\epsilon}),\delta^Q(\eta,\overline{\eta})]\psi^\alpha = \\
&\delta^Q(\epsilon,\overline{\epsilon})\left(\eta^\alpha - i(\psi\sigma^\mu\overline{\eta}-\eta\sigma^\mu\overline{\psi})\partial_\mu\psi^\alpha\right) - \delta^Q(\eta,\overline{\eta})\left(\epsilon^\alpha - i(\psi\sigma^\mu\overline{\epsilon}-\epsilon\sigma^\mu\overline{\psi})\partial_\mu\psi^\alpha\right) = \\
&-i\left[\left(\epsilon^\beta - i(\psi\sigma^\rho\overline{\epsilon}-\epsilon\sigma^\rho\overline{\psi})\partial_\rho\psi^\beta\right)\sigma^\mu_{\beta\dot{\beta}}\overline{\eta}^{\dot{\beta}} - \eta^\beta\sigma^\mu_{\beta\dot{\beta}}\left(\overline{\epsilon}^{\dot{\beta}} - i(\psi\sigma^\rho\overline{\epsilon}-\epsilon\sigma^\rho\overline{\psi})\partial_\rho\overline{\psi}^{\dot{\beta}}\right)\right]\partial_\mu\psi^\alpha \\
&-i(\psi\sigma^\mu\overline{\eta}-\eta\sigma^\mu\overline{\psi})\partial_\mu\left(\epsilon^\alpha - i(\psi\sigma^\rho\overline{\epsilon}-\epsilon\sigma^\rho\overline{\psi})\partial_\rho\psi^\alpha\right) \\
&+i\left[\left(\eta^\beta - i(\psi\sigma^\rho\overline{\eta}-\eta\sigma^\rho\overline{\psi})\partial_\rho\psi^\beta\right)\sigma^\mu_{\beta\dot{\beta}}\overline{\epsilon}^{\dot{\beta}} - \epsilon^\beta\sigma^\mu_{\beta\dot{\beta}}\left(\overline{\eta}^{\dot{\beta}} - i(\psi\sigma^\rho\overline{\eta}-\eta\sigma^\rho\overline{\psi})\partial_\rho\overline{\psi}^{\dot{\beta}}\right)\right]\partial_\mu\psi^\alpha \\
&+i(\psi\sigma^\mu\overline{\epsilon}-\epsilon\sigma^\mu\overline{\psi})\partial_\mu\left(\eta^\alpha - i(\psi\sigma^\rho\overline{\eta}-\eta\sigma^\rho\overline{\psi})\partial_\rho\psi^\alpha\right),
\end{aligned}
$$

which, grouping the right terms, gives

$$
\begin{aligned}
&[\delta^Q(\epsilon,\overline{\epsilon}),\delta^Q(\eta,\overline{\eta})]\psi^\alpha = \left[-i(\epsilon^\beta\sigma^\mu_{\beta\dot{\beta}}\overline{\eta}^{\dot{\beta}} - \eta^\beta\sigma^\mu_{\beta\dot{\beta}}\overline{\epsilon}^{\dot{\beta}}) + i(\eta^\beta\sigma^\mu_{\beta\dot{\beta}}\overline{\epsilon}^{\dot{\beta}} - \epsilon^\beta\sigma^\mu_{\beta\dot{\beta}}\overline{\eta}^{\dot{\beta}})\right]\partial_\mu\psi^\alpha \\
&+\left[-(\psi\sigma^\rho\overline{\epsilon}-\epsilon\sigma^\rho\overline{\psi})\partial_\rho\psi^\beta\sigma^\mu_{\beta\dot{\beta}}\overline{\eta}^{\dot{\beta}}\partial_\mu\psi^\alpha + (\psi\sigma^\mu\overline{\epsilon}-\epsilon\sigma^\mu\overline{\psi})\partial_\mu\psi^\beta\sigma^\rho_{\beta\dot{\beta}}\overline{\eta}^{\dot{\beta}}\partial_\rho\psi^\alpha\right] \\
&+\left[\eta^\beta\sigma^\mu_{\beta\dot{\beta}}(\psi\sigma^\rho\overline{\epsilon}-\epsilon\sigma^\rho\overline{\psi})\partial_\rho\overline{\psi}^{\dot{\beta}}\partial_\mu\psi^\alpha - (\psi\sigma^\mu\overline{\epsilon}-\epsilon\sigma^\mu\overline{\psi})\partial_\mu(\eta\sigma^\rho\overline{\psi})\partial_\rho\psi^\alpha\right] \\
&+\left[(\psi\sigma^\rho\overline{\eta}-\eta\sigma^\rho\overline{\psi})\partial_\rho\psi^\beta\sigma^\mu_{\beta\dot{\beta}}\overline{\epsilon}^{\dot{\beta}}\partial_\mu\psi^\alpha - (\psi\sigma^\mu\overline{\eta}-\eta\sigma^\mu\overline{\psi})\partial_\mu(\psi\sigma^\rho\overline{\epsilon})\partial_\rho\psi^\alpha\right] \\
&+\left[-\epsilon^\beta\sigma^\mu_{\beta\dot{\beta}}(\psi\sigma^\rho\overline{\eta}-\eta\sigma^\rho\overline{\psi})\partial_\rho\overline{\psi}^{\dot{\beta}}\partial_\mu\psi^\alpha + (\psi\sigma^\mu\overline{\eta}-\eta\sigma^\mu\overline{\psi})\partial_\mu(\epsilon\sigma^\rho\overline{\psi})\partial_\rho\psi^\alpha\right] \\
&+\left[-(\psi\sigma^\mu\overline{\eta})(\psi\sigma^\rho\overline{\epsilon})\partial_\mu\partial_\rho\psi^\alpha + (\psi\sigma^\mu\overline{\epsilon})(\psi\sigma^\rho\overline{\eta})\partial_\mu\partial_\rho\psi^\alpha\right] \\
&+\left[(\psi\sigma^\mu\overline{\eta})(\epsilon\sigma^\rho\overline{\psi})\partial_\mu\partial_\rho\psi^\alpha - (\epsilon\sigma^\mu\overline{\psi})(\psi\sigma^\rho\overline{\eta})\partial_\mu\partial_\rho\psi^\alpha\right] \\
&+\left[(\eta\sigma^\mu\overline{\psi})(\psi\sigma^\rho\overline{\epsilon})\partial_\mu\partial_\rho\psi^\alpha - (\psi\sigma^\mu\overline{\epsilon})(\eta\sigma^\rho\overline{\psi})\partial_\mu\partial_\rho\psi^\alpha\right] \\
&+\left[-(\eta\sigma^\mu\overline{\psi})(\epsilon\sigma^\rho\overline{\psi})\partial_\mu\partial_\rho\psi^\alpha + (\epsilon\sigma^\mu\overline{\psi})(\eta\sigma^\rho\overline{\psi})\partial_\mu\partial_\rho\psi^\alpha\right],
\end{aligned}
$$

where in all lines, except the first, the terms cancel each other to finally give:

$$[\delta^Q(\epsilon,\overline{\epsilon}),\delta^Q(\eta,\overline{\eta})]\psi^\alpha = -2i(\epsilon\sigma^\mu\overline{\eta}-\eta\sigma^\mu\overline{\epsilon})\partial_\mu\psi^\alpha.$$

## 8.3

Here we find the variation of the vierbein ${W_\mu}^\nu$:

$$\begin{aligned}
\delta^Q(\epsilon,\overline{\epsilon}){W_\mu}^\nu &= i\kappa^2\left(\frac{\epsilon^\alpha}{\kappa} - i\kappa(\psi\sigma^\rho\overline{\epsilon} - \epsilon\sigma^\rho\overline{\psi})\partial_\rho\psi^\alpha\right)\sigma^\nu_{\alpha\dot{\alpha}}\partial_\mu\overline{\psi}^{\dot{\alpha}} \\
&+ i\kappa^2\psi^\alpha\sigma^\nu_{\alpha\dot{\alpha}}\partial_\mu\left((\frac{\overline{\epsilon}^{\dot{\alpha}}}{\kappa} - i\kappa(\psi\sigma^\rho\overline{\epsilon} - \epsilon\sigma^\rho\overline{\psi})\partial_\rho\overline{\psi}^{\dot{\alpha}}\right) \\
&- i\kappa^2\partial_\mu\left(\frac{\epsilon^\alpha}{\kappa} - i\kappa(\psi\sigma^\rho\overline{\epsilon} - \epsilon\sigma^\rho\overline{\psi})\partial_\rho\psi^\alpha\right)\sigma^\nu_{\alpha\dot{\alpha}}\overline{\psi}^{\dot{\alpha}} \\
&- i\kappa^2\partial_\mu\psi^\alpha\sigma^\nu_{\alpha\dot{\alpha}}\left(\frac{\overline{\epsilon}^{\dot{\alpha}}}{\kappa} - i\kappa(\psi\sigma^\rho\overline{\epsilon} - \epsilon\sigma^\rho\overline{\psi})\partial_\rho\overline{\psi}^{\dot{\alpha}}\right) \\
&= i\kappa(\epsilon\sigma^\nu\partial_\mu\overline{\psi} - \partial_\mu\psi\sigma^\nu\overline{\epsilon}) \\
&+ \kappa^3(\psi\sigma^\rho\overline{\epsilon})\partial_\rho\psi\sigma^\nu\partial_\mu\overline{\psi} - \kappa^3(\epsilon\sigma^\rho\overline{\psi})\partial_\rho\psi\sigma^\nu\partial_\mu\overline{\psi} \\
&+ \kappa^3\psi^\alpha\sigma^\nu_{\alpha\dot{\alpha}}\partial_\mu\left[(\psi\sigma^\rho\overline{\epsilon})\partial_\rho\overline{\psi}^{\dot{\alpha}}\right] - \kappa^3\psi^\alpha\sigma^\nu_{\alpha\dot{\alpha}}\partial_\mu\left[(\epsilon\sigma^\rho\overline{\psi})\partial_\rho\overline{\psi}^{\dot{\alpha}}\right] \\
&- \kappa^3\partial_\mu\left[(\psi\sigma^\rho\overline{\epsilon})\partial_\rho\psi^\alpha\right]\sigma^\nu_{\alpha\dot{\alpha}}\overline{\psi}^{\dot{\alpha}} + \kappa^3\partial_\mu\left[(\epsilon\sigma^\rho\overline{\psi})\partial_\rho\psi^\alpha\right]\sigma^\nu_{\alpha\dot{\alpha}}\overline{\psi}^{\dot{\alpha}} \\
&- \kappa^3\partial_\mu\psi\sigma^\nu\partial_\rho\overline{\psi}(\psi\sigma^\rho\overline{\epsilon}) + \kappa^3\partial_\mu\psi\sigma^\nu\partial_\rho\overline{\psi}(\epsilon\sigma^\rho\overline{\psi}) \\
&= i\kappa(\epsilon\sigma^\nu\partial_\mu\overline{\psi} - \partial_\mu\psi\sigma^\nu\overline{\epsilon}) + \kappa^3(\psi\sigma^\rho\overline{\epsilon} - \epsilon\sigma^\rho\overline{\psi})\partial_\rho(\psi\sigma^\nu\partial_\mu\overline{\psi}) \\
&+ \kappa^3\psi\sigma^\nu\partial_\rho\overline{\psi}\partial_\mu(\psi\sigma^\rho\overline{\epsilon} - \epsilon\sigma^\rho\overline{\psi}) \\
&- \kappa^3(\psi\sigma^\rho\overline{\epsilon} - \epsilon\sigma^\rho\overline{\psi})\partial_\rho(\partial_\mu\psi\sigma^\nu\overline{\psi}) \\
&- \kappa^3\partial_\mu(\psi\sigma^\rho\overline{\epsilon} - \epsilon\sigma^\rho\overline{\psi})\partial_\rho\psi\sigma^\nu\overline{\psi} \\
&= i\kappa(\epsilon\sigma^\nu\partial_\mu\overline{\psi} - \partial_\mu\psi\sigma^\nu\overline{\epsilon}) + \kappa^3(\psi\sigma^\rho\overline{\epsilon} - \epsilon\sigma^\rho\overline{\psi})\partial_\rho(\psi\sigma^\nu\partial_\mu\overline{\psi} - \partial_\mu\psi\sigma^\nu\overline{\psi}) \\
&+ \kappa^3(\partial\psi\sigma^\rho\overline{\epsilon} - \epsilon\sigma^\rho\partial_\mu\overline{\psi})(\psi\sigma^\nu\partial_\rho\overline{\psi} - \partial_\rho\psi\sigma^\nu\overline{\psi}).
\end{aligned}$$

Using

$${W_\mu}^\nu = \delta^\nu_\mu - i\kappa^2\partial_\mu\psi\sigma^\nu\overline{\psi} + i\kappa^2\psi\sigma^\nu\partial_\mu\overline{\psi}$$

so that

$$\partial_\rho{W_\mu}^\nu = i\kappa^2\partial_\rho(\psi\sigma^\nu\partial_\mu\overline{\psi} - \partial_\mu\psi\sigma^\nu\overline{\psi}),$$

we find

$$\delta(\epsilon,\overline{\epsilon}){W_\mu}^\nu = i\kappa(\epsilon\sigma^\rho\partial_\mu\overline{\psi} - \partial_\mu\psi\sigma^\rho\overline{\epsilon}){W_\rho}^\nu - i\kappa(\psi\sigma^\rho\overline{\epsilon} - \epsilon\sigma^\rho\overline{\psi})\partial_\rho{W_\mu}^\nu.$$

## 8.4

We know that $W^{-1}$ must satisfy ${W_\mu}^\nu{(W^{-1})_\nu}^\tau = \delta^\tau_\mu$. Also, we have ${W_\mu}^\nu = \delta^\nu_\mu + \kappa^2 T^\nu_\mu$. Our first guess might be $W^{-1} = \delta^\tau_\nu$, so that

$${W_\mu}^\nu{(W^{-1})_\nu}^\rho = \delta^\tau_\mu + \kappa^2 T^\tau_\mu,$$

where we see the extra term $\kappa^2 T^\tau_\mu$. This needs to be compensated for, so we add an second term $-\kappa^2 T^\tau_\nu$ to $W^{-1}$, which itself creates an unneeded term in the product $WW^{-1}$, and so on. This process terminates when a term is found cancelling the previous one in a product with $\delta^\nu_\mu$, but not adding an unneeded term in the product with $\kappa^2 T^\nu_\mu$. This is the case when a product of four $T^\lambda_\chi$'s is added to $W^{-1}$, since this guarantees that in every term of the product with $T^\nu_\mu$, at least three spinors of the same kind are present, meaning it is zero.

We therefore have

$$(W^{-1})_\nu{}^\tau = \delta^\tau_\nu - \kappa^2 T^\tau_\nu + \kappa^4 T^\chi_\nu T^\tau_\chi - \kappa^6 T^\lambda_\nu T^\chi_\lambda T^\tau_\chi + \kappa^8 T^\theta_\nu T^\lambda_\theta T^\chi_\lambda T^\tau_\chi.$$

To check, let us write down the product $WW^{-1}$:

$$\begin{aligned} W_\mu{}^\nu (W^{-1})_\nu{}^\tau &= (\delta^\nu_\mu + \kappa^2 T^\nu_\mu)(\delta^\tau_\nu - \kappa^2 T^\tau_\nu + \kappa^4 T^\chi_\nu T^\tau_\chi - \kappa^6 T^\lambda_\nu T^\chi_\lambda T^\tau_\chi + \kappa^8 T^\theta_\nu T^\lambda_\theta T^\chi_\lambda T^\tau_\chi) \\ &= \delta^\tau_\mu + \kappa^2 T^\tau_\mu - \kappa^2 T^\tau_\mu - \kappa^4 T^\nu_\mu T^\tau_\nu + \kappa^4 T^\chi_\mu T^\tau_\chi + \kappa^6 T^\nu_\mu T^\chi_\nu T^\tau_\chi \\ &\quad - \kappa^6 T^\lambda_\mu T^\chi_\lambda T^\tau_\chi - \kappa^8 T^\nu_\mu T^\lambda_\nu T^\chi_\lambda T^\tau_\chi + \kappa^8 T^\theta_\nu T^\lambda_\theta T^\chi_\lambda T^\tau_\chi + \kappa^{10} T^\nu_\mu T^\theta_\nu T^\lambda_\theta T^\chi_\lambda T^\tau_\chi. \end{aligned}$$

In the last product, since $T^\nu_\mu = i(\psi\sigma^\nu\partial_\mu\overline{\psi} - \partial_\mu\psi\sigma^\nu\overline{\psi})$, necessarily either three or more $\psi$'s or $\overline{\psi}$'s are present in each resulting term. It must then be zero because for two component spinors. Using this and the obvious cancellations in the expression, we get the required

$$W_\mu{}^\nu (W^{-1})_\nu{}^\tau = \delta^\tau_\mu.$$

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