



university of
groningen

faculty of mathematics
and natural sciences

Poncelet Figures over Rational Numbers and over Real Quadratic Number Fields

Bachelor Project Mathematics

July 2017

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July 14, 2017

Abstract

Poncelet figures are polygons whose vertices lie on one conic section and its edges are tangent to a second one. In this thesis we construct examples of Poncelet figures with 7, 8, 9, 10 and 12 vertices over \mathbb{Q} and Poncelet figures with 11, 13, 14, 15, 16 and 18 vertices over real quadratic number fields.

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1 Introduction

Poncelet figures are polygons whose vertices lie on one conic section and its edges are tangent to another. Given a field $K \subset \mathbb{R}$, a Poncelet figure over K is a Poncelet figure for which both conic sections are given by an equation over K , and all vertices have coordinates in K . In his bachelor's thesis, J. Los [1] constructed Poncelet figures over \mathbb{Q} with 3, 4, 5, 6 and 7 vertices. Moreover, he shows that no Poncelet figures over \mathbb{Q} with $n = 11$ and $n \geq 13$ exist. In his conclusion he conjectures that the method he used to construct the Poncelet figure with 7 vertices will also construct figures with 8, 9, 10 and 12 vertices, which are all other possibilities over \mathbb{Q} . In this bachelor's thesis, we will check the validity of the conjecture of Los. We also consider the problem of constructing Poncelet figures over real quadratic number fields. The goal is to give an example of Poncelet figures with n vertices for $n \in \{7, \dots, 16, 18\}$, which are all the possible cases that occur over real quadratic number fields.

2 Preliminaries

An elliptic curve is a smooth projective curve, given by a third degree homogeneous polynomial and equipped with a distinguished point \mathcal{O} . On an elliptic curve, we can define a group law, making it into an abelian group, with \mathcal{O} as its unit element. Every elliptic curve has a so-called *torsion subgroup*. This is the group formed by all points of finite order on the elliptic curve. Given an elliptic curve E over \mathbb{Q} , its points with coordinates in \mathbb{Q} form a subgroup denoted $E(\mathbb{Q})$. A famous theorem of Mazur [2] (see [3] for an amazing historical overview) states that

$$E(\mathbb{Q})_{\text{tors}} \cong \begin{cases} \mathbb{Z}/n\mathbb{Z}, & n \in \{1, \dots, 10, 12\} \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2n\mathbb{Z}, & n \in \{1, 2, 3, 4\} \end{cases}$$

Any elliptic curve E where the point $P = (0, 0)$ has order larger than 3 can be put in the special form

$$E : y^2 + uxy - vy = x^3 - vx^2.$$

This has the property that the line $X = 0$ is tangent to E at P . Using this form, it is quite easy to find relations between u and v that have to hold to get a torsion point of the particular order you want. For example, if we want an elliptic curve with a torsion point of order 8, then we want to have

$4P = -4P$; Using the above form, we find that

$$\begin{aligned} 4P &= \left(\frac{-uv + v^2 + v}{u^2 - 2u + 1}, \frac{-u^2v^2 + 3uv^2 - v^3 - 2v^2}{u^3 - 3u^2 + 3u - 1} \right), \\ -4P &= \left(\frac{-uv + v^2 + v}{u^2 - 2u + 1}, \frac{u^2v - 2uv^2 - 2uv + v^3 + 2v^2 + v}{u^3 - 3u^2 + 3u - 1} \right) \end{aligned}$$

Of course, we don't want P to be a point of order 2 or 4, which means that we should have $P \neq -P$ and $2P \neq -2P$. Since $-P = (0, -v)$, we see that v is nonzero. Furthermore, we have $2P = (-v, uv - v)$ and $-2P = (-v, 0)$. This implies that u is not 1. So, if we want $4P = -4P$, then u and v should satisfy

$$-u^2v^2 + 3uv^2 - v^3 - 2v^2 = u^2v - 2uv^2 - 2uv + v^3 + 2v^2 + v$$

or, equivalently,

$$-u^2v + u^2 - 5uv - 2u + 2v^2 + 4v + 1 = 0$$

Using the programming language Magma, a parametrization in one variable of this curve is easy to find.

3 Poncelet Figures over Rational Numbers

We define the dual C^\vee of a conic section C to be the set of pairs (c, d) such that $y = cx + d$ is tangent to C . Now, we will consider the set $X = \{(P, \ell) \in C_1 \times C_2^\vee \mid P \in \ell\}$. If $C_1 \cap C_2^\vee$ consists (over some algebraic closure) of 4 distinct points and if a pair (P, ℓ) is given, then X can be viewed as an elliptic curve. For details on this we refer to [4] and [5]. The process of assigning to a pair (P, ℓ) the “other” intersection point P' of ℓ and C_1 , and the “other” line ℓ' containing P' and tangent to C_2 , we denote by $(P, \ell) \mapsto (P', \ell')$.

Now, we are going to do the opposite. Given an elliptic curve E with equation

$$E : y^2 + uxy + vy = x^3 + vx^2,$$

we want to construct C_1 and C_2 such that we can view E as such a set X , and moreover such that the process $(P, \ell) \mapsto (P', \ell')$ corresponds to the translation over $T = (0, 0)$ on the elliptic curve E . If we do this, then we can get a Poncelet figure by considering the case where T is a point of order n , and the pair (P, ℓ) we start with corresponds to a rational point P_0 on E . Because T was a point of order n , we will get back where we started after n steps. This means that we obtain a Poncelet figure with $\leq n$ vertices. We

will require that P_0 is a point of infinite order, because otherwise P_0 might depend on T and we will trace a Poncelet figure of smaller order multiple times.

To view E as a set of the form X , we consider new variables b and c given by

$$b = \frac{x+v}{y}, \quad c = \frac{x+v}{-y-ux-v}.$$

Rewriting gives

$$x = -v \frac{b+c+bc}{b+c+buc}, \quad y = \frac{v(u-1)c}{b+c+buc}.$$

Substituting this gives the curve

$$b^2c^2v + 2v(b^2c + bc^2) + bc(u^2 - u + 2v) + v(b^2 + c^2) + (u-1)(b+c) = 0.$$

Now, we set C_1 to be the parabola (b, b^2) , then we let C_2^\vee be parametrized by $(x(c), y(c))$ where

$$x(c) = -\frac{2vc^2 + (u^2 - u + 2v)c + u - 1}{vc^2 + 2vc + v}$$

and

$$y(c) = -\frac{vc^2 + (u-1)c}{vc^2 + 2vc + v}.$$

In other words, we have $b^2 = x(c)b + y(c)$.

We now find the general formula for C_2 by substituting $y = x(c)x + y(c)$ into the general formula $a_1x^2 + a_2xy + a_3y^2 + a_4x + a_5y + a_6$. Since this line should be a tangent line, we get that the discriminant is zero for all c . This gives us a nonlinear system of equations. Using the programming language Maple, we find that the general formula for C_2 is

$$(u^4 - 2u^3 + 4u^2v + u^2 - 12uv + 4v^2 + 8v)x^2 + 4v(u^2 - 2u + 1)xy + 2(u^2 - 2u + 1)ux + 4v(u - v - 1)y + u^2 - 2u + 1 = 0.$$

The process $(P, \ell) \mapsto (P', \ell')$ corresponds to $(b, c) \mapsto (b', c')$, described as follows. The new b' is, for given c , the “other” solution to the equation in b, c . So

$$b' = -b + x(c).$$

Next, substituting b' into the b, c -equation, c yields one solution and the other one is c' . Hence

$$c' = -c + x(b').$$

The first involution $(b, c) \mapsto (b', c)$ corresponds to the involution $P_0 \mapsto (-v, 0) - P_0$ on the elliptic curve. The second one $(b', c) \mapsto (b', c')$ corresponds to $P_0 \mapsto (-v, uv - v) - P_0$. Therefore the composition corresponds to translation over $(-v, uv - v) - (-v, 0) = [4](0, 0)$.

Hence if $(0, 0)$ is a torsion point of odd order N we obtain a Poncelet figure consisting of N vertices, and if the order is even we obtain a figure with less vertices.

Now that we have laid the ground work, all that is left is using the methods from the introduction to construct the u and v for the elliptic curve to have a point of order n and positive rank. Using the programming language Magma, we have created the following table of examples of the elliptic curves used to make the Poncelet figures in section 8. We remark that examples of pairs of conic sections defined over \mathbb{Q} with the property that a Poncelet figure with $N \in \{5, 8, 10, 12\}$ vertices based on these conic sections exist, were also constructed by Mirman [6, Section 3]. However, he did not insist that also the coordinates of all vertices in the figure are rational numbers. Moreover Los [1] constructed Poncelet figures over \mathbb{Q} with N vertices for each $N \in \{3, 4, 5, 6, 7\}$ and the same was done by Malyshev [7] for $N = 5$.

$\text{Ord}((0,0))$	u	v	Point of infinite order
7	-55	-48	$(30, 198)$
8	$-\frac{433}{50}$	$-\frac{483}{8}$	$(\frac{75}{8}, \frac{375}{4})$
9	13	84	$(6, 18)$
10	$-\frac{19}{11}$	$\frac{120}{121}$	$(\frac{12}{11}, \frac{252}{121})$
12	$\frac{463}{8}$	$-\frac{13195}{48}$	$(\frac{145}{48}, \frac{21025}{384})$

As discussed above, with $T = (0, 0)$ on the elliptic curve, the function b used here is invariant under the involution $P_0 \mapsto 2T - P_0$ and c is invariant under $P_0 \mapsto -2T - P_0$. As a consequence, the process $(P, \ell) \mapsto (P', \ell')$ in our case corresponds to translation over $4T$ on E .

4 Poncelet Figures over Real Quadratic Number Fields

We will now consider the possibilities over a real quadratic number field. These are extensions of degree 2 of the rational numbers, while still being a subfield of the real numbers. In other words, these are fields of the form $\mathbb{Q}(\sqrt{d}) = \{a + b\sqrt{d} \mid a, b \in \mathbb{Q}\}$, where d is a positive rational number that is

not a square. According to [8], an elliptic curve E over a quadratic number field K has a torsion subgroup isomorphic to one of the following groups

$$E(K)_{\text{tors}} \cong \begin{cases} \mathbb{Z}/n\mathbb{Z}, & n \in \{1, \dots, 16, 18\} \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2n\mathbb{Z}, & n \in \{1, \dots, 6\} \\ \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3n\mathbb{Z}, & n \in \{1, 2\} \\ \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}. \end{cases}$$

To obtain an example of such E/K having a point of order n , one starts with

$$y^2 + uxy + vy = x^3 + vx^2$$

as before. Then $(0, 0) \in E$ having order n yields a relation between u and v , and we want a solution $u, v \in K$ for this relation. For example, if u and v satisfy

$$v^2 + (u^3 - u^2 - 1)v - u^2 + u = 0,$$

then we can find constants r and s , given by

$$s = \frac{v + u^3 - u}{v + u^2 - u}, \quad r = \frac{-u^2 + u}{v}(1 - s) + s$$

and the elliptic curve given by

$$y^2 + (1 - s(r - 1))xy - (r - rs(r - 1))y = x^3 - (r - rs(r - 1))x^2$$

has $(0,0)$ as a point of order 13. The curve defined by this equation in u and v is called $X_1(13)$. In general, if we want an elliptic curve with torsion subgroup isomorphic to $\mathbb{Z}/n\mathbb{Z}$, then this curve is referred to as $X_1(n)$. A table of these $X_1(n)$ is given in [9] and [10]. In all the cases that we are interested in, namely $n \in \{11, 13, 14, 15, 16, 18\}$, these $X_1(n)$ are elliptic or hyperelliptic. This means that they can be written in the form of $v_1^2 = f(u_1)$, where f is a polynomial.

We want to find elliptic curves with a point of order n over some real quadratic number field. To find the number field, we substitute values for u_1 and define the corresponding elliptic curve over the field $\mathbb{Q}(\sqrt{f(u_1)})$. If this curve has positive rank, then we are done and we can apply the same process as in the previous section to find the corresponding Poncelet figures. The following table gives examples of the elliptic curves used to make the Poncelet figures in section 8. We have taken the example of the elliptic curve with a point of order 18 from [11] and transformed it so that $(0, 0)$ has order 18.

$\text{Ord}((0,0))$	u	v	Point of infinite order
11	$-\frac{14}{3} - \frac{1}{3}\sqrt{73}$	$-15 - 3\sqrt{73}$	$(16 + 2\sqrt{73}, \frac{28}{3} + \frac{8}{3}\sqrt{73})$
13	$\frac{151}{9} - \frac{10}{9}\sqrt{193}$	$\frac{1306}{3} - \frac{94}{3}\sqrt{193}$	$(125 - 9\sqrt{193}, -6974 + 502\sqrt{193})$
14	$\frac{2513}{728} + \frac{173}{728}\sqrt{105}$	$-\frac{555}{18928} - \frac{87}{18928}\sqrt{105}$	$(\frac{60}{4732} + \frac{3}{4732}\sqrt{105}, -\frac{135}{492128} - \frac{15}{492128}\sqrt{105})$
15	$\frac{993}{875} + \frac{2}{875}\sqrt{345}$	$\frac{11750}{109375} + \frac{146}{109375}\sqrt{345}$	$(\frac{650}{3125} - \frac{66}{3125}\sqrt{345}, -\frac{1299380}{2734375} + \frac{75428}{2734375}\sqrt{345})$
16	$121 + 39\sqrt{10}$	$-3510 - 1107\sqrt{10}$	$(-24 - 9\sqrt{10}, 9402 - 2970\sqrt{10})$
18	$\frac{10100}{13625} - \frac{21}{13625}\sqrt{26521}$	$-\frac{13179867}{37128125} - \frac{81003}{37128125}\sqrt{26521}$	$(\frac{188529}{1703125} + \frac{1161}{1703125}\sqrt{26521}, \frac{6481036062}{23205078125} + \frac{39796758}{23205078125}\sqrt{26521})$

5 General Method

As we saw above, the method described here runs into a problem when applied to finding a Poncelet figure of even order.

A solution to this is to choose more “natural” transformations. Before, we chose two new variables b and c and we got two involutions $X \rightarrow X$,

$$(P, \ell) \mapsto (P', \ell), \quad (P, \ell) \mapsto (P, \ell'),$$

where P' is the other point on ℓ and ℓ' is the other line that P is on. Now, we want to first choose these involutions and then find corresponding transformations. We choose the first involution to be

$$i_1 : P \mapsto -P$$

and we choose the second one to be

$$i_2 : P \mapsto (0, 0) - P$$

so that $i_2 \circ i_1$ maps P to $P + (0, 0)$. We now search for invariants of i_1 and i_2 in the sense that $i_k^* f := f \circ i_k = \text{id}$. The explicit formula for i_1 is

$$i_1(x, y) = (x, -y - ux - v).$$

We quickly see that x is invariant, because $i_1^*(x) = x$. The explicit formula for i_2 is

$$i_2(x, y) = \left(\frac{uvx + vy + v^2}{x^2}, \frac{uv^2x + v^2x^2 + v^2y + v^3}{x^3} \right).$$

Now, we can find that $\frac{y+v}{x}$ is invariant. To see this, we calculate

$$\begin{aligned} i_1^* \left(\frac{y+v}{x} \right) &= \frac{\frac{uv^2x+v^2x^2+v^2y+v^3}{x^3} + v}{\frac{uvx+vy+v^2}{x^2}} \\ &= \frac{uvx + vx^2 + vy + v^2 + x^3}{ux^2 + xy + vx} \\ &= \frac{uvx + vy + v^2 + y^2 + uxy + vy}{ux^2 + xy + vx} \\ &= \frac{(y+v)(y+ux+v)}{x(y+ux+v)} \\ &= \frac{y+v}{x}. \end{aligned}$$

Now, if we set

$$b := \frac{y+v}{x}, \quad c := x$$

and apply all the previous steps to get a Poncelet figure of order 8, we get Figure 6, which is exactly what we want. In this case, the dual conic C_2^\vee is parametrized as

$$(x(c), y(c)) = \left(\frac{v - uc}{c}, \frac{c^2 + vc + uv}{c} \right).$$

The conic C_2 is given by

$$u^2x^2 + y^2 + 2uxy + (-2uv - 4v)x - 2vy - v(4u - v) = 0.$$

6 Other Conic Sections

In the above discussion, we have used the parabola as one of the conic sections by default. Of course, we can use any other conic section. We will go over how to do this in this section.

We want to use the ellipse $\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1$ as one of our conics. This means that we want a rational parametrization for this conic. Such a parametrization is given by

$$\left(\frac{2b}{1+b^2}\alpha, \frac{1-b^2}{1+b^2}\beta \right),$$

where $\alpha, \beta \in \mathbb{Q}$. We now find functions $x(c)$ and $y(c)$ such that

$$\frac{1-b^2}{1+b^2}\beta = x(c)\frac{2b}{1+b^2}\alpha + y(c).$$

This condition can be rewritten to

$$b^2 = -\frac{2\alpha x(c)}{\beta + y(c)}b + \frac{\beta - y(c)}{\beta + y(c)}.$$

From the previous section, we find the equations

$$-\frac{2\alpha x(c)}{\beta + y(c)} = \frac{v - uc}{c}, \quad \frac{\beta - y(c)}{\beta + y(c)} = \frac{c^2 + vc + uv}{c}.$$

Solving these equations, we get

$$x(c) = \frac{\beta}{\alpha} \cdot \frac{uc - v}{c^2 + (v + 1)c + uv}, \quad y(c) = \beta \frac{-c^2 + (-v + 1)c - uv}{c^2 + (v + 1)c + uv}.$$

Using these functions, we find that the second conic is given by

$$\begin{aligned} C_2 : & -\beta u^2 x^2 + 2\alpha(uv + u + 2v)xy + \alpha^2(4uv - v^2 - 2v - 1)y^2 + \\ & 2\alpha\beta(uv - u + 2v)x + 2\alpha^2(4uv - v^2 + 1)y + \alpha^2\beta(4uv - v^2 + 2v - 1) = 0 \end{aligned}$$

We use these methods to construct examples of Poncelet figures of order 12 and order 14 defined over real quadratic fields, given in section 8.

We can do the same thing for the hyperbola $\frac{x^2}{\alpha^2} - \frac{y^2}{\beta^2} = 1$. A rational parametrization for this conic is

$$\left(\frac{1 + b^2}{2b}\alpha, \frac{1 - b^2}{2b}\beta \right),$$

where $\alpha, \beta \in \mathbb{Q}$. We now find functions $x(c)$ and $y(c)$ such that

$$\frac{1 - b^2}{2b}\beta = x(c)\frac{1 + b^2}{2b}\alpha + y(c).$$

This condition can be rewritten to

$$b^2 = -\frac{2y(c)}{\alpha x(c) + \beta}b + \frac{\beta - \alpha x(c)}{\beta + \alpha x(c)}.$$

From the previous section, we find the equations

$$-\frac{2y(c)}{\alpha x(c) + \beta} = \frac{v - uc}{c}, \quad \frac{\beta - \alpha x(c)}{\beta + \alpha x(c)} = \frac{c^2 + vc + uv}{c}.$$

Solving these equations, we get

$$x(c) = \frac{\beta}{\alpha} \cdot \frac{-a^2 + (-v + 1)a - uv}{a^2 + (v + 1)a + uv}, \quad y(c) = \beta \frac{ua - v}{a^2 + (v + 1)a + uv}.$$

Using these functions, we find that the second conic is given by

$$\begin{aligned} C_2 : & \beta^2(4uv - v^2 + 2v - 1)x^2 + 2\alpha\beta(4uv - v^2 + 1)xy + \alpha^2(4uv - v^2 - 2v - 1)y^2 + \\ & 2\alpha\beta^2(uv - u + 2v)x + 2\alpha^2\beta(uv + u + 2v)y - \alpha^2\beta^2u^2 = 0. \end{aligned}$$

We use these methods to construct an example of a Poncelet figure of order 16 defined over a real quadratic field, given in section 8.

7 Conclusion

We have seen how to construct Poncelet figures of a certain order using two methods and have actually constructed examples of these figures in all possible \mathbb{Q} -rational cases for the number of vertices, and also for all the possible new cases that arise when we allow the coordinates of the vertices to be defined in some real quadratic field.

The general method can likely be extended to number fields of higher degree. If one substitutes a certain value x_0 for x and find a root y_0 of the corresponding polynomial, then the elliptic curve will be defined over $\mathbb{Q}(y_0)$ and we can apply all the theory in the above sections. We invite the reader to research this.

8 Poncelet Figures

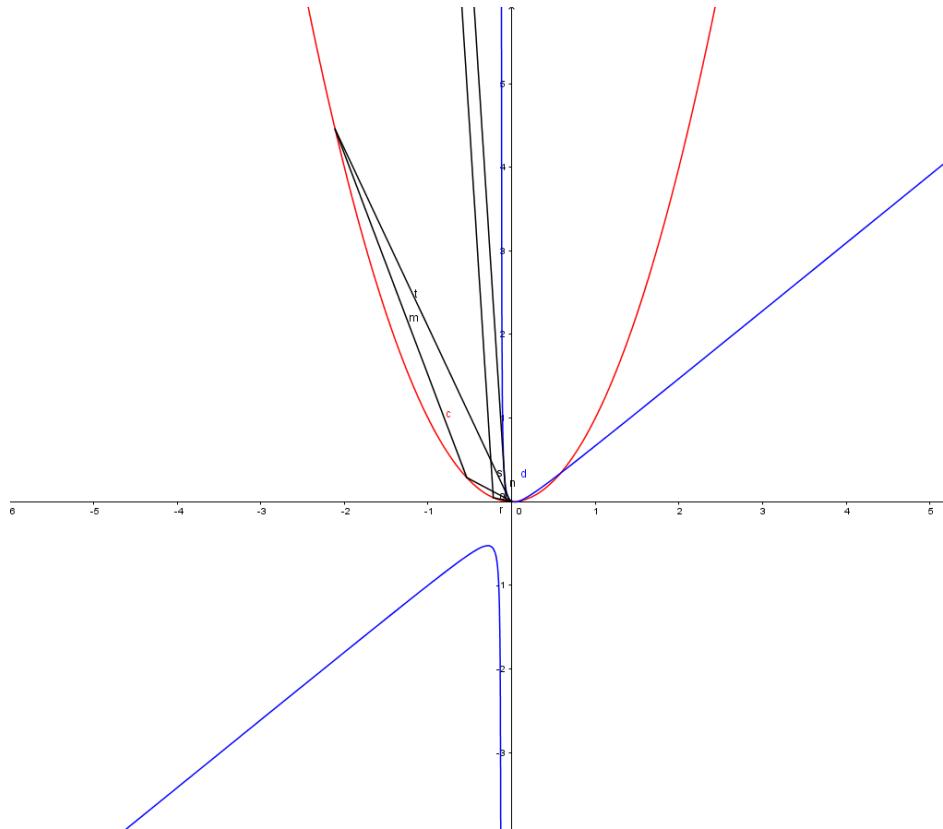


Figure 1: A Poncelet figure of order 7 over \mathbb{Q} and the conics $y = x^2$ and $4569152x^2 - 5619712xy - 344960x - 702464y + 3136 = 0$.

n	point	line
1	$(-\frac{19}{9}, \frac{361}{81})$	$y = -\frac{12943}{6084}x - \frac{209}{6084}$
2	$(-\frac{11}{676}, \frac{121}{456976})$	$y = -\frac{3151}{37856}x - \frac{165}{151424}$
3	$(-\frac{15}{224}, \frac{225}{50176})$	$y = -\frac{3473}{224}x - \frac{3705}{3584}$
4	$(-\frac{247}{16}, \frac{61009}{256})$	$y = -\frac{6263}{400}x - \frac{2717}{800}$
5	$(-\frac{11}{50}, \frac{121}{2500})$	$y = -\frac{3821}{18050}x + \frac{33}{18050}$
6	$(\frac{3}{361}, \frac{9}{130321})$	$y = -\frac{23102}{43681}x + \frac{195}{43681}$
7	$(-\frac{65}{121}, \frac{4225}{14641})$	$y = -\frac{2884}{1089}x - \frac{1235}{1089}$

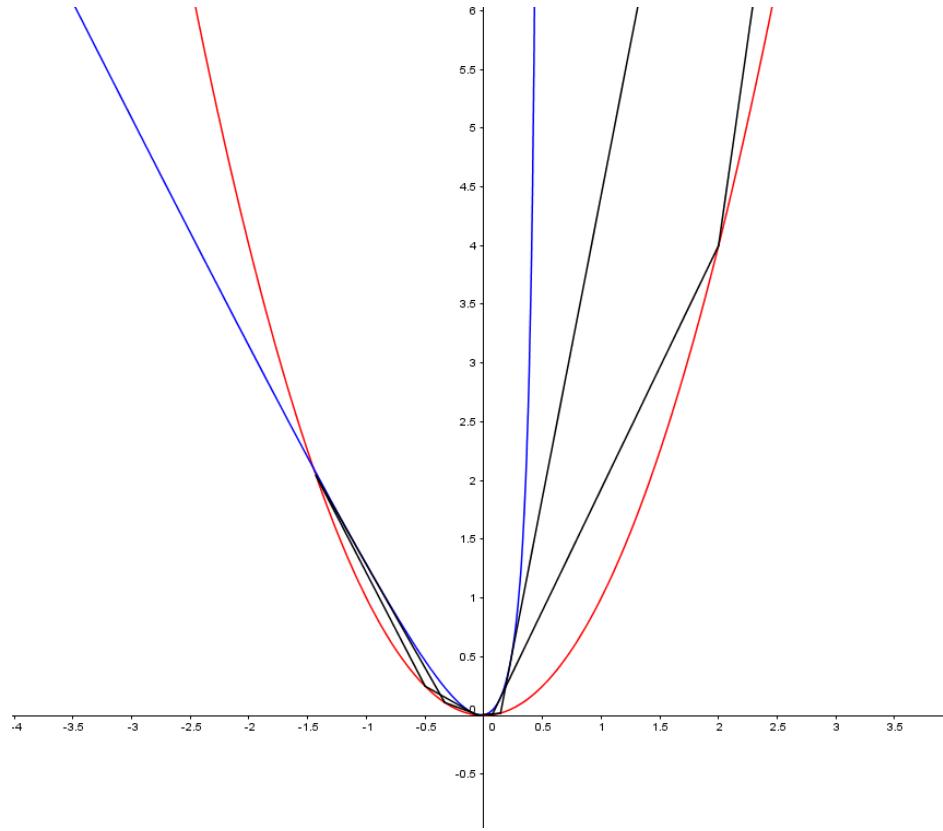


Figure 2: A picture of a Poncelet figure of order 9 over \mathbb{Q} and the conics $y = x^2$ and $96912x^2 + 48384xy + 3744x - 24192y + 144 = 0$.

number	point	line
1	$(5, 25)$	$y = \frac{36}{7}x - \frac{5}{7}$
2	$(\frac{1}{7}, \frac{1}{49})$	$y = \frac{18}{175}x + \frac{1}{175}$
3	$(-\frac{1}{25}, \frac{1}{625})$	$y = -\frac{28}{75}x - \frac{1}{75}$
4	$(-\frac{1}{3}, \frac{1}{9})$	$y = -\frac{37}{21}x - \frac{10}{21}$
5	$(-\frac{10}{7}, \frac{100}{49})$	$y = -\frac{27}{14}x - \frac{5}{7}$
6	$(-\frac{1}{2}, \frac{1}{4})$	$y = -\frac{9}{16}x - \frac{1}{32}$
7	$(-\frac{1}{16}, \frac{1}{256})$	$y = \frac{1}{112}x + \frac{1}{224}$
8	$(\frac{1}{14}, \frac{1}{196})$	$y = \frac{19}{14}x - \frac{1}{7}$
9	$(2, 4)$	$y = 7x + 10$

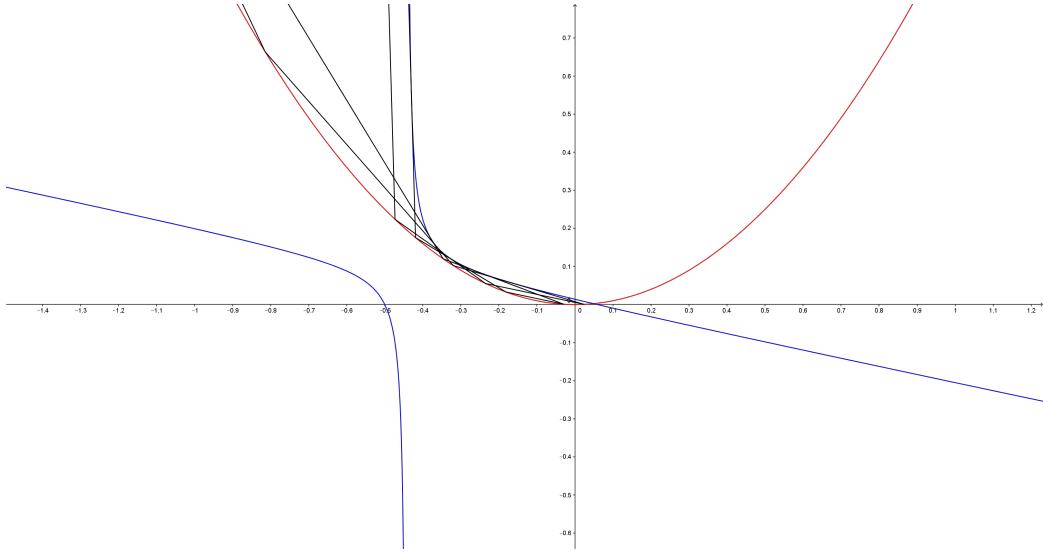


Figure 3: A Poncelet figure of order 11 over $\mathbb{Q}(\sqrt{73})$ and the conics $y = x^2$ and $(\frac{62086}{81} - \frac{-16142}{81}\sqrt{73})x^2 + (-\frac{17168}{3} - \frac{2128}{3}\sqrt{73})xy + (-\frac{15100}{27} - \frac{1676}{27})x + (-2896 - 27\sqrt{73})y + (\frac{362}{9} + \frac{34}{9}\sqrt{73}) = 0$.

number	point	line
1	$(-\frac{17}{36} + \frac{1}{36}\sqrt{73}, \frac{181}{648} - \frac{17}{648}\sqrt{73})$	$y = (-\frac{25}{24} + \frac{7}{72}\sqrt{73})x + (-\frac{59}{144} + \frac{7}{144}\sqrt{73})$
2	$(-\frac{41}{72} + \frac{5}{72}\sqrt{73}, \frac{1753}{2592} - \frac{205}{2592}\sqrt{73})$	$y = (-\frac{589}{576} + \frac{49}{576}\sqrt{73})x + (-\frac{259}{768} + \frac{31}{768}\sqrt{73})$
3	$(-\frac{29}{64} + \frac{1}{64}\sqrt{73}, \frac{457}{2048} - \frac{29}{2048}\sqrt{73})$	$y = (-\frac{175}{192} - \frac{5}{192}\sqrt{73})x + (-\frac{41}{256} - \frac{3}{256}\sqrt{73})$
4	$(-\frac{11}{24} - \frac{1}{24}\sqrt{73}, \frac{97}{288} + \frac{11}{288}\sqrt{73})$	$y = (-\frac{25}{24} - \frac{1}{8}\sqrt{73})x - (\frac{25}{48} - \frac{1}{16}\sqrt{73})$
5	$(-\frac{7}{12} - \frac{1}{12}\sqrt{73}, \frac{61}{72} + \frac{7}{72}\sqrt{73})$	$y = (\frac{-19}{48} - \frac{7}{48}\sqrt{73})x + (-\frac{13}{48} + \frac{-1}{48}\sqrt{73})$
6	$(\frac{3}{16} + \frac{-1}{16}\sqrt{73}, \frac{41}{128} + \frac{-3}{128}\sqrt{73})$	$y = (\frac{281}{48} + \frac{-35}{48}\sqrt{73})x + (\frac{-197}{48} + \frac{23}{48}\sqrt{73})$
7	$(\frac{17}{3} + \frac{-2}{3}\sqrt{73}, \frac{581}{9} + \frac{-68}{9}\sqrt{73})$	$y = (\frac{25}{3} + \frac{-1}{1}\sqrt{73})x + (\frac{-94}{3} + \frac{11}{3}\sqrt{73})$
8	$(\frac{8}{3} + \frac{-1}{3}\sqrt{73}, \frac{137}{9} + \frac{-16}{9}\sqrt{73})$	$y = (\frac{49}{18} + \frac{-7}{18}\sqrt{73})x + (\frac{-3}{2} + \frac{1}{6}\sqrt{73})$
9	$(\frac{1}{18} + \frac{-1}{18}\sqrt{73}, \frac{37}{162} + \frac{-1}{162}\sqrt{73})$	$y = (\frac{-305}{18} + \frac{-37}{18}\sqrt{73})x + (\frac{-43}{6} + \frac{-5}{6}\sqrt{73})$
10	$(\frac{-17}{1} + \frac{-2}{1}\sqrt{73}, \frac{581}{1} + \frac{68}{1}\sqrt{73})$	$y = (\frac{-935}{54} + \frac{-109}{54}\sqrt{73})x + (\frac{-145}{18} + \frac{-17}{18}\sqrt{73})$
11	$(\frac{-17}{54} + \frac{-1}{54}\sqrt{73}, \frac{181}{1458} + \frac{17}{1458}\sqrt{73})$	$y = (-\frac{85}{108} + \frac{1}{108}\sqrt{73})x - \frac{1}{9}$

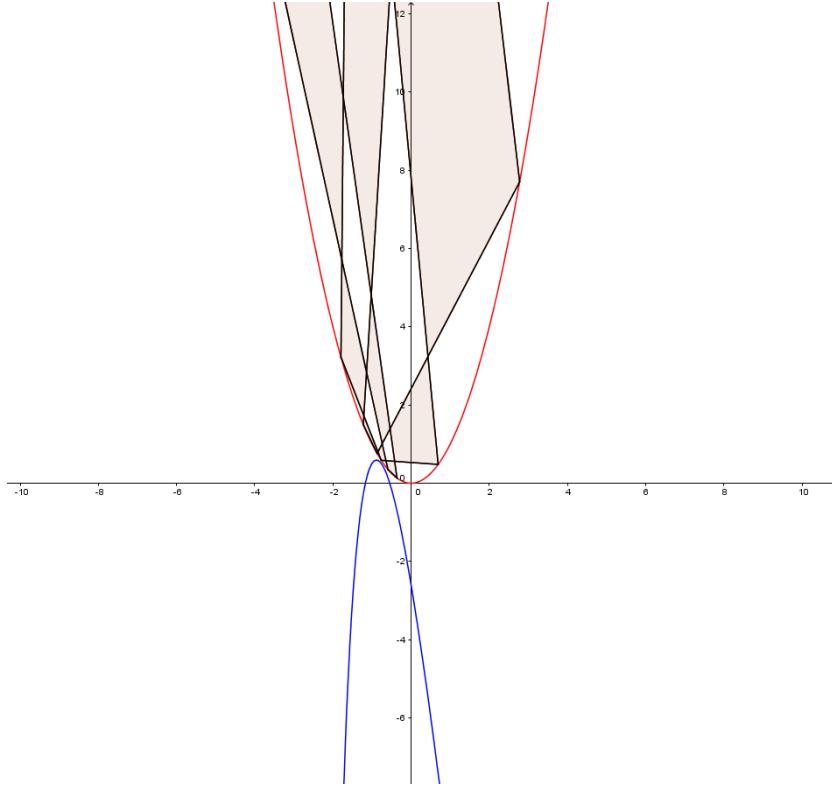


Figure 4: A Poncelet figure of order 13 over $\mathbb{Q}(\sqrt{193})$ and the conics $y = x^2$ and $(\frac{24037458536}{6561} - \frac{1730254040}{6561}\sqrt{193})x^2 + (\frac{412253056}{243} - \frac{29674624}{243}\sqrt{193})xy + (\frac{22880528}{729} - \frac{1646960}{729}\sqrt{193})x + (-\frac{39464320}{27} + \frac{2840704}{27}\sqrt{193})y + (\frac{39464}{81} - \frac{2840}{81}\sqrt{193}) = 0$.

n	point	line
1	$(\frac{11}{36} + \frac{1}{36}\sqrt{193}, \frac{157}{648}) + \frac{11}{648}\sqrt{193}$	$y = (-\frac{559}{1764} + \frac{31}{1764})x + (\frac{12}{49} + \frac{1}{49}\sqrt{193})$
2	$(-\frac{61}{98} - \frac{1}{98}\sqrt{193}, \frac{1957}{4802} + \frac{61}{4802}\sqrt{193})$	$y = (-\frac{2857}{1323} - \frac{38}{1323}\sqrt{193})x + (-\frac{146}{147} - \frac{4}{147}\sqrt{193})$
3	$(-\frac{83}{54} - \frac{1}{54}\sqrt{193}, \frac{3541}{1458} + \frac{83}{1458}\sqrt{193})$	$y = (\frac{2887}{54} + \frac{215}{54}\sqrt{193})x + (\frac{593}{6} + \frac{43}{6}\sqrt{193})$
4	$(55 + 4\sqrt{193}, 6113 + 440\sqrt{193})$	$y = (\frac{1907}{36} + \frac{139}{36}\sqrt{193})x + (\frac{875}{4} + \frac{63}{4}\sqrt{193})$
5	$(-\frac{73}{36} - \frac{5}{36}\sqrt{193}, \frac{5077}{648} + \frac{365}{648}\sqrt{193})$	$y = (-\frac{80701}{34596} - \frac{5525}{34596}\sqrt{193})x + (-\frac{4521}{3844} - \frac{325}{3844}\sqrt{193})$
6	$(-\frac{293}{961} - \frac{20}{961}\sqrt{193}, \frac{163049}{923521} + \frac{11720}{923521}\sqrt{193})$	$y = (-\frac{67096}{141267} - \frac{4862}{141267}\sqrt{193})x + (-\frac{5015}{47089} - \frac{362}{47089}\sqrt{193})$
7	$(-\frac{25}{147} - \frac{2}{147}\sqrt{193}, \frac{1397}{21609} + \frac{100}{21609}\sqrt{193})$	$y = (-\frac{2011}{588} - \frac{155}{588}\sqrt{193})x + (-\frac{237}{196} - \frac{17}{196}\sqrt{193})$
8	$(-\frac{13}{4} - \frac{1}{4}\sqrt{193}, \frac{181}{8} + \frac{13}{8}\sqrt{193})$	$y = (\frac{29}{4} + \frac{1}{4}\sqrt{193})x + (\frac{233}{4} + \frac{17}{4}\sqrt{193})$
9	$(\frac{21}{2} + \frac{1}{2}\sqrt{193}, \frac{317}{2} + \frac{21}{2}\sqrt{193})$	$y = (\frac{51}{4} + \frac{1}{4}\sqrt{193})x + (\frac{1}{2} + \frac{3}{2}\sqrt{193})$
10	$(\frac{9}{4} - \frac{1}{4}\sqrt{193}, \frac{137}{8} - \frac{9}{8}\sqrt{193})$	$y = (\frac{47}{24} - \frac{7}{24}\sqrt{193})x + (-\frac{65}{48} + \frac{1}{48}\sqrt{193})$
11	$(-\frac{7}{24} - \frac{1}{24}\sqrt{193}, \frac{121}{288} + \frac{7}{288}\sqrt{193})$	$y = (\frac{213}{192} + \frac{11}{192}\sqrt{193})x + (\frac{925}{768} + \frac{67}{768}\sqrt{193})$
12	$(\frac{269}{192} + \frac{19}{192}\sqrt{193}, \frac{71017}{18432} + \frac{5111}{18432}\sqrt{193})$	$y = (-\frac{817}{192} - \frac{59}{192}\sqrt{193})x + (\frac{4015}{256} + \frac{289}{256}\sqrt{193})$
13	$(-\frac{181}{32} - \frac{13}{32}\sqrt{193}, \frac{32689}{512} + \frac{2353}{512}\sqrt{193})$	$y = (-\frac{1541}{288} - \frac{109}{288}\sqrt{193})x + (\frac{125}{32} + \frac{9}{32}\sqrt{193})$

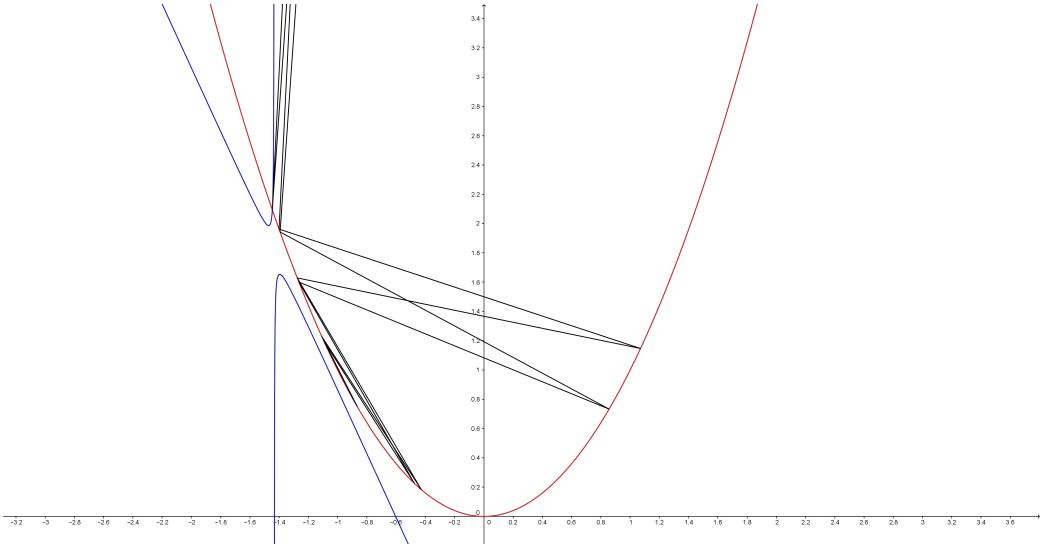


Figure 5: A Poncelet figure of order 15 over $\mathbb{Q}(\sqrt{345})$ and the conics $y = x^2$ and $(\frac{13523377736}{586181640625} + \frac{419344728}{586181640625}\sqrt{345})x^2 + (\frac{814386560}{83740234375} + \frac{31121536}{83740234375}\sqrt{345})xy + (\frac{31045104}{669921875} + \frac{998608}{669921875}\sqrt{345})x + (\frac{32390784}{2392578125} + \frac{1328000}{2392578125}\sqrt{345})y + (\frac{15304}{765625} + \frac{472}{765625}\sqrt{345}) = 0$.

number	point	line
1	$(\frac{-46575}{44521} + \frac{-850}{44521}\sqrt{345}, \frac{2418493125}{1982119441} + \frac{79177500}{1982119441}\sqrt{345})$	$y = (\frac{640874905}{69586323} + \frac{41189005}{69586323}\sqrt{345})x + (\frac{342217000}{23195441} + \frac{58104625}{69586323}\sqrt{345})$
2	$(\frac{16030}{1563} + \frac{955}{1563}\sqrt{345}, \frac{571609525}{2442969} + \frac{30617300}{2442969}\sqrt{345})$	$y = (\frac{376495909}{43590507} + \frac{27037449}{43590507}\sqrt{345})x + (\frac{212900620}{14530169} + \frac{12993215}{14530169}\sqrt{345})$
3	$(\frac{-451}{27889} + \frac{386}{27889}\sqrt{345}, \frac{89732101}{777796321} + \frac{22796532}{777796321}\sqrt{345})$	$y = (\frac{139046092}{23621983} + \frac{139046092}{23621983}\sqrt{345})x + (\frac{2031141}{3374569} + \frac{2281565}{3374569}\sqrt{345})$
4	$(\frac{6191}{847} + \frac{847}{847}\sqrt{345}, \frac{717409}{717409} + \frac{717409}{717409}\sqrt{345})$	$y = (\frac{4520816}{441287} + \frac{98256}{441287}\sqrt{345})x + (\frac{6702835}{441287} + \frac{161460}{441287}\sqrt{345})$
5	$(\frac{10705}{360} + \frac{-850}{360}\sqrt{345}, \frac{36309525}{13260609} + \frac{-181069}{13260609}\sqrt{345})$	$y = (\frac{43269005}{154764992} + \frac{-27773675}{154764992}\sqrt{345})x + (\frac{730570875}{154764992} + \frac{-29396125}{154764992}\sqrt{345})$
6	$(\frac{-5025}{42436} + \frac{2275}{42436}\sqrt{345}, \frac{13260609}{900407048} + \frac{13479375}{900407048}\sqrt{345})$	$y = (\frac{-406235575}{2091415824} + \frac{19988525}{232379536}\sqrt{345})x + (\frac{1082571875}{1394277216} + \frac{4182831648}{4182831648}\sqrt{345})$
7	$(\frac{-353875}{197136} + \frac{625}{2190}\sqrt{345}, \frac{68071765625}{19431301248} + \frac{-221717875}{2159033472}\sqrt{345})$	$y = (\frac{-197410545012}{759627302976} + \frac{3897075775}{84403033664}\sqrt{345})x + (\frac{-1091394912875}{675224269312} + \frac{331776240625}{6077018423808}\sqrt{345})$
8	$(\frac{-49560625}{61653056} + \frac{61653056}{61653056}\sqrt{345}, \frac{143163190625}{19004665765625} + \frac{180054965765625}{19004665765625}\sqrt{345})$	$y = (\frac{3341715}{1726285568} + \frac{147766568}{1726285568}\sqrt{345})x + (\frac{1017523368}{7891591168} + \frac{-769349375}{7891591168}\sqrt{345})$
9	$(\frac{147}{1792} + \frac{185}{1792}\sqrt{345}, \frac{6991625}{1605632} + \frac{-272875}{1605632}\sqrt{345})$	$y = (\frac{10925}{54208} + \frac{5193}{54208}\sqrt{345})x + (\frac{30761275}{277554496} + \frac{-310525}{277554496}\sqrt{345})$
10	$(\frac{-222175}{216832} + \frac{1613}{216832}\sqrt{345}, \frac{2512967046}{23508058112} + \frac{-358368275}{23508058112}\sqrt{345})$	$y = (\frac{-258221963}{112696474} + \frac{16225877}{564847360}\sqrt{345})x + (\frac{-486883698}{3615023104} + \frac{112764039}{3615023104}\sqrt{345})$
11	$(\frac{-42051}{33341} + \frac{3549}{33341}\sqrt{345}, \frac{485525837}{277553840} + \frac{-49238999}{277553840}\sqrt{345})$	$y = (\frac{-5412739}{2041296} + \frac{234461}{1620880}\sqrt{345})x + (\frac{-344129}{1315296} + \frac{16554}{1315296}\sqrt{345})$
12	$(\frac{1104}{1104} + \frac{1104}{1104}\sqrt{345}, \frac{699408}{699408} + \frac{699408}{699408}\sqrt{345})$	$y = (\frac{-3511445}{2041296} + \frac{20231}{2041296}\sqrt{345})x + (\frac{-1360864}{1360864} + \frac{4082592}{1360864}\sqrt{345})$
13	$(\frac{-2925}{3698} + \frac{20}{1849}\sqrt{345}, \frac{5957625}{13675204} + \frac{-46500}{3418801}\sqrt{345})$	$y = (\frac{-2473596275}{12960628366} + \frac{71343415}{6480314183}\sqrt{345})x + (\frac{-745578750}{925759169} + \frac{25855625}{1851518338}\sqrt{345})$
14	$(\frac{-4485500}{3504767} + \frac{375}{3504767}\sqrt{345}, \frac{203525025}{128330174289} + \frac{-4054152000}{128330174289}\sqrt{345})$	$y = (\frac{-1451746179125}{45169785727} + \frac{645203757275}{45169785727}\sqrt{345})x + (\frac{-451697857275}{3616876727} + \frac{3616876727}{3616876727}\sqrt{345})$
15	$(\frac{-78625}{902167} + \frac{56250}{902167}\sqrt{345}, \frac{1097783453125}{813905295889} + \frac{-8845312500}{813905295889}\sqrt{345})$	$y = (\frac{-45518891670}{40165377007} + \frac{1737464300}{40165377007}\sqrt{345})x + (\frac{12833353125}{40165377007} + \frac{2553012500}{40165377007}\sqrt{345})$

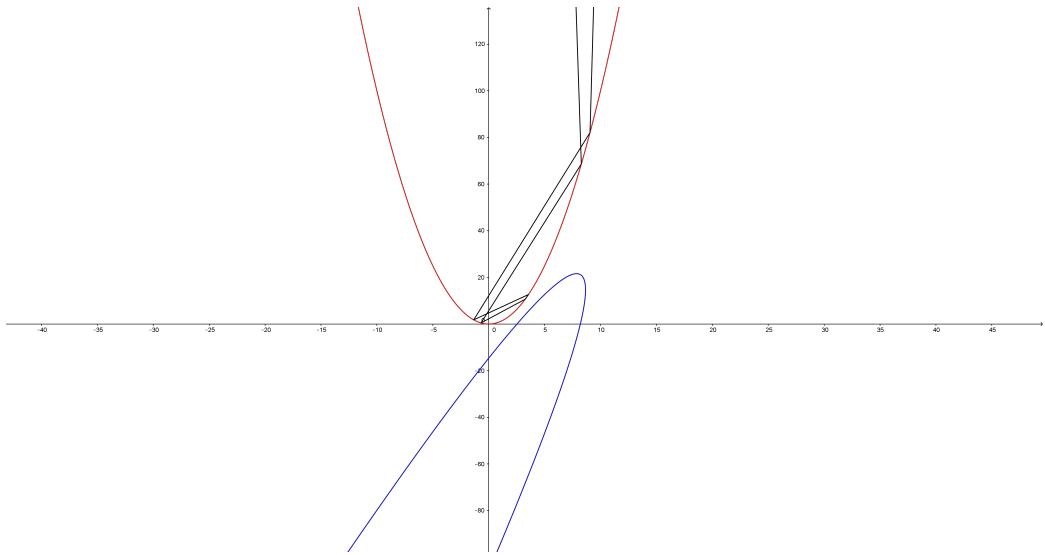


Figure 6: A Poncelet figure of order 8 over \mathbb{Q} and $y = x^2$ and $-11434224x^2 + 37746552xy - 101494849y + 532800x + 5442800y - 40000 = 0$.

number	point	line
1	$\left(\frac{89}{25}, \frac{7921}{625}\right)$	$y = \frac{111}{50}x + \frac{5963}{1250}$
2	$\left(\frac{-67}{50}, \frac{4489}{2500}\right)$	$y = \frac{3089}{400}x + \frac{1943}{160}$
3	$\left(\frac{145}{16}, \frac{21025}{256}\right)$	$y = \frac{8433}{50}x + \frac{-1851331}{1280}$
4	$\left(\frac{63839}{400}, \frac{4075417921}{160000}\right)$	$y = \frac{3463}{400}x + \frac{481792933}{20000}$
5	$\left(\frac{-7547}{50}, \frac{56957209}{2500}\right)$	$y = \frac{-135523}{950}x + \frac{5939489}{4750}$
6	$\left(\frac{787}{95}, \frac{619369}{9025}\right)$	$y = \frac{12917}{1700}x + \frac{17437559}{3068500}$
7	$\left(\frac{-22157}{32300}, \frac{490932649}{1043290000}\right)$	$y = \frac{2447}{950}x + \frac{24572113}{10982000}$
8	$\left(\frac{1109}{340}, \frac{1229881}{115600}\right)$	$y = \frac{11597}{1700}x + \frac{-98701}{8500}$

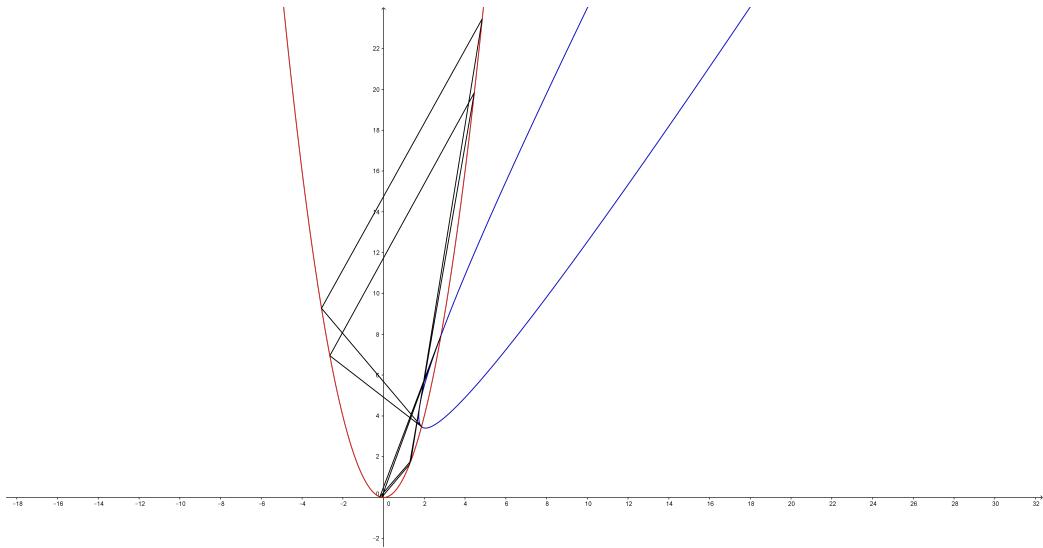


Figure 7: A Poncelet figure of order 10 over \mathbb{Q} and $y = x^2$ and $\frac{361}{121}x^2 - \frac{38}{11}xy - \frac{720}{1331}x + y^2 - \frac{240}{121}y + \frac{114720}{14641} = 0$.

number	point	line
1	$(\frac{31}{11}, \frac{961}{121})$	$y = \frac{29}{11}x + \frac{62}{121}$
2	$(\frac{-2}{11}, \frac{4}{121})$	$y = \frac{89}{77}x + \frac{206}{847}$
3	$(\frac{103}{77}, \frac{10609}{5929})$	$y = \frac{68}{11}x + \frac{-38419}{5929}$
4	$(\frac{373}{77}, \frac{139129}{5929})$	$y = \frac{277}{154}x + \frac{24991}{1694}$
5	$(\frac{-67}{22}, \frac{4489}{484})$	$y = \frac{-13}{11}x + \frac{2747}{484}$
6	$(\frac{41}{22}, \frac{1681}{484})$	$y = \frac{-17}{22}x + \frac{1189}{242}$
7	$(\frac{-29}{11}, \frac{841}{121})$	$y = \frac{20}{11}x + \frac{1421}{121}$
8	$(\frac{49}{11}, \frac{2401}{121})$	$y = \frac{63}{11}x + \frac{-686}{121}$
9	$(\frac{14}{11}, \frac{196}{121})$	$y = \frac{13}{11}x + \frac{14}{121}$
10	$(\frac{-1}{11}, \frac{1}{121})$	$y = \frac{30}{11}x + \frac{31}{121}$

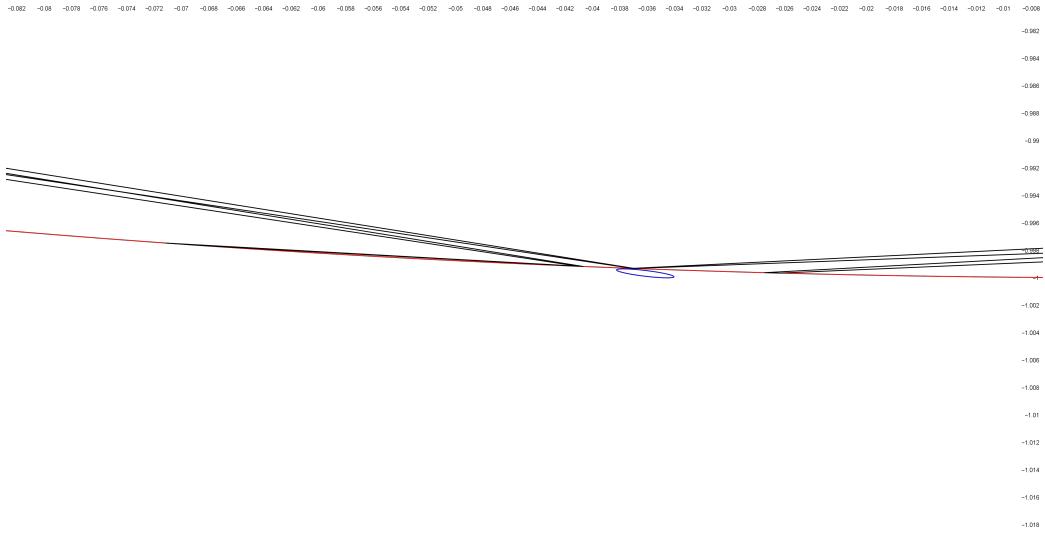


Figure 8: A Poncelet figure of order 12 over \mathbb{Q} , using the conics $x^2 + y^2 = 1$ and $-\frac{214369}{64}x^2 - \frac{6298181}{192}xy - \frac{6342629}{192}x - \frac{319466449}{2304}y^2 - \frac{320728561}{1152}y - \frac{321999889}{2304} = 0$.

number	point	line
1	$(\frac{-9328}{339953}, \frac{-339825}{339953})$	$y = \frac{-1191}{44300}x + \frac{-11079}{11075}$
2	$(\frac{-152}{5777}, \frac{-5775}{5777})$	$y = \frac{1393}{32780}x + \frac{-8183}{8195}$
3	$(\frac{20688}{186337}, \frac{-185185}{186337})$	$y = \frac{612}{16471}x + \frac{-279431}{280007}$
4	$(\frac{-15576}{421345}, \frac{-421057}{421345})$	$y = \frac{-5876}{36897}x + \frac{-12363}{12299}$
5	$(\frac{-912}{3313}, \frac{-3185}{3313})$	$y = \frac{-3600}{22337}x + \frac{-22465}{22337}$
6	$(\frac{-6288}{154513}, \frac{-154385}{154513})$	$y = \frac{-436}{7785}x + \frac{-44179}{44115}$
7	$(\frac{-8088}{113713}, \frac{-113425}{113713})$	$y = \frac{-181836}{2604875}x + \frac{-2611211}{2604875}$
8	$(\frac{-4086192}{59961817}, \frac{-59822425}{59961817})$	$y = \frac{-226633}{4145200}x + \frac{-259438}{259075}$
9	$(\frac{-11792}{287417}, \frac{-287175}{287417})$	$y = \frac{-114123}{753200}x + \frac{-47328}{47075}$
10	$(\frac{-518512}{2019137}, \frac{-1951425}{2019137})$	$y = \frac{-37508216}{250878775}x + \frac{-252097591}{250878775}$
11	$(\frac{-1182297888}{31868578513}, \frac{-31846639825}{31868578513})$	$y = \frac{18644256}{380832725}x + \frac{-379878869}{380832725}$
12	$(\frac{613728}{4561897}, \frac{-4520425}{4561897})$	$y = \frac{66904}{1243525}x + \frac{-1241221}{1243525}$

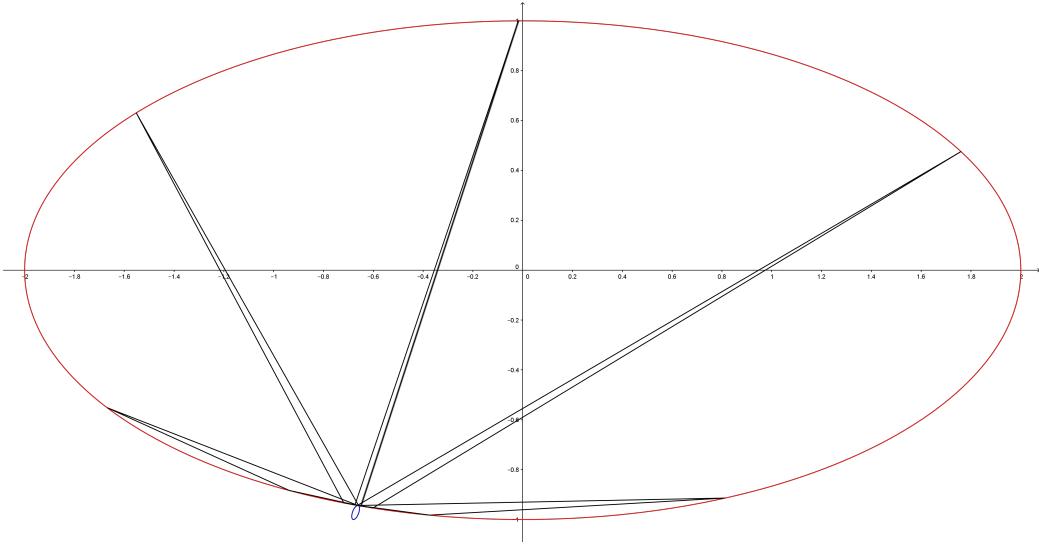


Figure 9: A Poncelet figure of order 14 over $\mathbb{Q}(\sqrt{105})$, using the conics $x^2 + \frac{y^2}{4} = 1$ and $-\frac{214369}{64}x^2 - \frac{6298181}{192}xy - \frac{6342629}{192}x - \frac{319466449}{2304}y^2 - \frac{320728561}{1152}y - \frac{321999889}{2304} = 0$.

number	point	line
1	($\frac{21922394}{291183181} + \frac{-15465450}{291183181}\sqrt{105}, \frac{-485751090}{291183181} + \frac{-2791880}{291183181}\sqrt{105}$)	$y = (\frac{-78214861}{86032185} + \frac{90429}{28677395}\sqrt{105})x + (\frac{-136117186}{86032185} + \frac{-1666496}{28677395}\sqrt{105})$
2	($\frac{-5545369841}{832317175} + \frac{271458675}{832317175}\sqrt{105}, \frac{416462333}{832317175} + \frac{4171274333}{832317175}\sqrt{105}$)	$y = (\frac{-2894081408}{35061355} + \frac{29961355}{35061355}\sqrt{105})x + (\frac{-295175258}{35061355} + \frac{28446196}{35061355}\sqrt{105})$
3	($\frac{-3683761011}{61254330586} + \frac{5618312175}{61254330586}\sqrt{105}, \frac{14569220455}{61254330586} + \frac{1420564635}{61254330586}\sqrt{105}$)	$y = (\frac{-1188648218}{123369642} + \frac{123369642}{123369642}\sqrt{105})x + (\frac{-12448738193}{35061355} + \frac{31179125367}{35061355}\sqrt{105})$
4	($\frac{-393791313769}{606206738194} + \frac{20653231725}{606206738194}\sqrt{105}, \frac{-30493836305}{303103303697} + \frac{-26710108005}{303103303697}\sqrt{105}$)	$y = (\frac{24563693}{310483955} + \frac{-73194983}{310483955}\sqrt{105})x + (\frac{-147916557}{186173410} + \frac{-146485493}{186173410}\sqrt{105})$
5	($\frac{1518942659}{6371571466} + \frac{6371571466}{6371571466}\sqrt{105}, \frac{14264055}{3185785733} + \frac{-4264055}{3185785733}\sqrt{105}$)	$y = (\frac{12920508}{317126} + \frac{86523}{1585630}\sqrt{105})x + (\frac{270702}{317126} + \frac{1585630}{317126}\sqrt{105})$
6	($\frac{-37299}{80194} + \frac{3565}{80194}\sqrt{105}, \frac{24335}{40097} + \frac{5451}{40097}\sqrt{105}$)	$y = (\frac{59127}{54269} + \frac{56859}{54269}\sqrt{105})x + (\frac{-205033}{54269} + \frac{31131}{54269}\sqrt{105})$
7	($\frac{-214185320}{32043841} + \frac{10094841}{32043841}\sqrt{105}, \frac{-20323112}{32043841} + \frac{229206}{32043841}\sqrt{105}$)	$y = (\frac{-599450520}{211660} + \frac{28205030}{211660}\sqrt{105})x + (\frac{-601125566}{211660} + \frac{5822320}{211660}\sqrt{105})$
8	($\frac{-2163597520}{3183686521} + \frac{99549768}{3183686521}\sqrt{105}, \frac{-2978111922}{3183686521} + \frac{-289291520}{3183686521}\sqrt{105}$)	$y = (\frac{-4319049}{328742} + \frac{541913}{986228}\sqrt{105})x + (\frac{-1917882}{164371} + \frac{341908}{164371}\sqrt{105})$
9	($\frac{-360094}{410629} + \frac{4050}{410629}\sqrt{105}, \frac{173400}{410629} + \frac{33645}{410629}\sqrt{105}$)	$y = (\frac{-4210955}{743158} + \frac{-3076527}{743158}\sqrt{105})x + (\frac{-1213099}{743159} + \frac{-939876}{743159}\sqrt{105})$
10	($\frac{2694658}{2715911986} + \frac{112611125}{2715911986}\sqrt{105}, \frac{1357955933}{2715911986} + \frac{137955933}{2715911986}\sqrt{105}$)	$y = (\frac{2032048}{87197185} + \frac{7540126}{87197185}\sqrt{105})x + (\frac{-12277161}{87197185} + \frac{-12277161}{87197185}\sqrt{105})$
11	($\frac{-245520241}{414688066} + \frac{16365525}{414688066}\sqrt{105}, \frac{-222831345}{20734033} + \frac{-18009845}{20734033}\sqrt{105}$)	$y = (\frac{87197185}{233900149} + \frac{87197185}{233900149}\sqrt{105})x + (\frac{-245655199}{87197185} + \frac{7308637}{87197185}\sqrt{105})$
12	($\frac{-32551421}{1006929056} + \frac{1066929056}{1006929056}\sqrt{105}, \frac{752340425}{548464078373} + \frac{-45464078373}{548464078373}\sqrt{105}$)	$y = (\frac{10918890}{5569210990} + \frac{3639630}{5569210990}\sqrt{105})x + (\frac{-10918890}{5569210990} + \frac{3639630}{5569210990}\sqrt{105})$
13	($\frac{47743875428081}{71763286849906} + \frac{234353311925}{71763286849906}\sqrt{105}, \frac{-3430103422054}{35881643424953} + \frac{-3261981946965}{35881643424953}\sqrt{105}$)	$y = (\frac{1152170778}{281805387} + \frac{3456512310}{281805387}\sqrt{105})x + (\frac{-165941683}{80628331} + \frac{1728256155}{80628331}\sqrt{105})$
14	($\frac{-66482171}{101075746} + \frac{-1737825}{101075746}\sqrt{105}, \frac{-7218295}{50537873} + \frac{160055}{50537873}\sqrt{105}$)	$y = (\frac{-20881803}{25316630} + \frac{-2448179}{25316630}\sqrt{105})x + (\frac{-27157014}{12658315} + \frac{-583752}{12658315}\sqrt{105})$

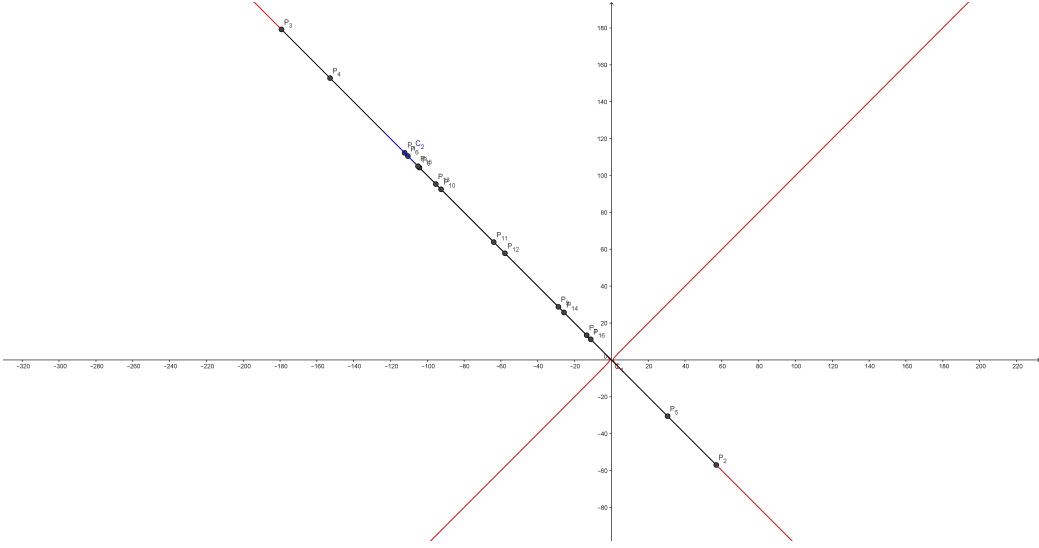


Figure 10: A Poncelet figure of order 16 over $\mathbb{Q}(\sqrt{10})$, using the conics $x^2 - y^2 = 1$ and $(-28007371 - 8856702\sqrt{10})x^2 + (-56000698 - 17708976\sqrt{10})xy + (-27993331 - 8852274\sqrt{10})y^2 + (-546180\sqrt{10} - 1727162)x + (-1726678 - 546024\sqrt{10})y - 29851 - 9438\sqrt{10} = 0$.

number	point	line
1	$(-\frac{313685}{5759} + \frac{-105336\sqrt{10}}{5759}, \frac{332652}{5759} + \frac{99330\sqrt{10}}{5759})$	$y = (\frac{125}{10271} + \frac{-13149\sqrt{10}}{41084}\sqrt{10})x + (\frac{-1177}{10271} + \frac{1545\sqrt{10}}{41084}\sqrt{10})$
2	$(\frac{95}{3} + \frac{8\sqrt{10}}{3}, \frac{-76}{3} + \frac{-10}{3}\sqrt{10})$	$y = (\frac{6688}{309751} + \frac{-100062\sqrt{10}}{309751}\sqrt{10})x + (\frac{-53852}{309751} + \frac{17616\sqrt{10}}{309751}\sqrt{10})$
3	$(\frac{-8154419}{92039} + \frac{-2637000\sqrt{10}}{92039}, \frac{8326500}{92039} + \frac{2582502\sqrt{10}}{92039})$	$y = (\frac{-213899}{3722805} + \frac{-221927\sqrt{10}}{744561}\sqrt{10})x + (\frac{-7979}{3722805} + \frac{3631}{744561}\sqrt{10})$
4	$(\frac{-5196136}{66991} + \frac{-1594305\sqrt{10}}{66991}, \frac{5049165}{66991} + \frac{1640712\sqrt{10}}{66991})$	$y = (\frac{-252775}{1409643} + \frac{-365786\sqrt{10}}{1409643}\sqrt{10})x + (\frac{-413155}{1409643} + \frac{136495}{1409643}\sqrt{10})$
5	$(\frac{3689}{234} + \frac{2185\sqrt{10}}{468}, \frac{-3565}{234} + \frac{-2261\sqrt{10}}{468})$	$y = (\frac{-150107}{204111} + \frac{-17068\sqrt{10}}{204111}\sqrt{10})x + (\frac{53662}{204111} + \frac{-16204}{204111}\sqrt{10})$
6	$(\frac{-71960}{1283} + \frac{-88389}{5132}\sqrt{10}, \frac{71736}{1283} + \frac{88665}{5132}\sqrt{10})$	$y = (\frac{-165509}{177333} + \frac{-3758\sqrt{10}}{177333}\sqrt{10})x + (\frac{-15029}{177333} + \frac{2407}{177333}\sqrt{10})$
7	$(\frac{-704}{129} + \frac{-325}{129}\sqrt{10}, \frac{715}{129} + \frac{320}{129}\sqrt{10})$	$y = (\frac{-17531}{16969} + \frac{879}{84845}\sqrt{10})x + (\frac{2809}{16969} + \frac{-5571}{84845}\sqrt{10})$
8	$(\frac{-68579}{1359} + \frac{-23120}{1359}\sqrt{10}, \frac{68680}{1359} + \frac{23086}{1359}\sqrt{10})$	$y = (\frac{-10582}{10947} + \frac{-116}{10947}\sqrt{10})x + (\frac{-502}{10947} + \frac{2}{267}\sqrt{10})$
9	$(\frac{-19871}{1277} + \frac{-26712}{6385}\sqrt{10}, \frac{19716}{1277} + \frac{26922}{6385}\sqrt{10})$	$y = (\frac{-191008}{227043} + \frac{-45635\sqrt{10}}{908172}\sqrt{10})x + (\frac{55880}{227043} + \frac{-77225}{908172}\sqrt{10})$
10	$(\frac{-8540665}{186667} + \frac{-13803768}{933335}\sqrt{10}, \frac{8618340}{186667} + \frac{13679358}{933335}\sqrt{10})$	$y = (\frac{-2057138}{7052643} + \frac{-1579906\sqrt{10}}{7052643}\sqrt{10})x + (\frac{-216729}{7052643} + \frac{655856}{7052643}\sqrt{10})$
11	$(\frac{-244069}{7449} + \frac{-73360}{7449}\sqrt{10}, \frac{233240}{7449} + \frac{76766}{7449}\sqrt{10})$	$y = (\frac{-6135}{63769} + \frac{-91141}{318845}\sqrt{10})x + (\frac{525}{63769} + \frac{-2491}{318845}\sqrt{10})$
12	$(\frac{-8996}{321} + \frac{-3025}{321}\sqrt{10}, \frac{9515}{321} + \frac{2860}{321}\sqrt{10})$	$y = (\frac{7579}{389397} + \frac{-125546}{389397}\sqrt{10})x + (\frac{-76241}{389397} + \frac{22399}{389397}\sqrt{10})$
13	$(\frac{-9295}{213} + \frac{-13937}{852}\sqrt{10}, \frac{11011}{213} + \frac{11765}{852}\sqrt{10})$	$y = (\frac{3443}{350041} + \frac{-111804}{350041}\sqrt{10})x + (\frac{-43318}{350041} + \frac{10968}{350041}\sqrt{10})$
14	$(\frac{-473}{42} + \frac{-55}{12}\sqrt{10}, \frac{605}{42} + \frac{43}{12}\sqrt{10})$	$y = (\frac{118855}{5916213} + \frac{-1908802}{5916213}\sqrt{10})x + (\frac{-926585}{5916213} + \frac{247769}{5916213}\sqrt{10})$
15	$(\frac{-11442236}{226641} + \frac{-3910115}{226641}\sqrt{10}, \frac{12350195}{226641} + \frac{3622652}{226641}\sqrt{10})$	$y = (\frac{278213}{17332785} + \frac{-5571419}{17332785}\sqrt{10})x + (\frac{-2657227}{17332785} + \frac{569191}{17332785}\sqrt{10})$
16	$(\frac{-130007}{35867} + \frac{-86040}{35867}\sqrt{10}, \frac{268980}{35867} + \frac{41586}{35867}\sqrt{10})$	$y = (\frac{14092}{717049} + \frac{-693892}{2151147}\sqrt{10})x + (\frac{-120008}{717049} + \frac{80414}{2151147}\sqrt{10})$

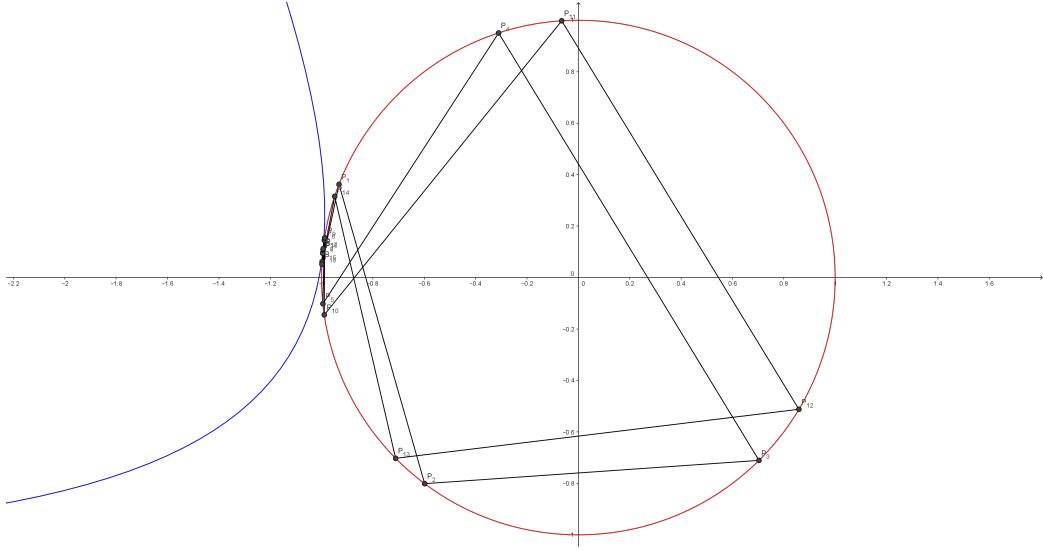


Figure 11: A Poncelet figure of order 18 over $\mathbb{Q}(\sqrt{26521})$, using the conic sections $x^2 + y^2 = 1$ and $(-\frac{113705761}{185640625} + \frac{424200}{185640625}\sqrt{26521})x^2 + (-\frac{144320156254}{505870703125} - \frac{7056750936}{505870703125}\sqrt{26521})xy + (\frac{-1644296406254}{505870703125} - \frac{393798436}{505870703125}\sqrt{26521})x + (-\frac{1706766368838553}{1378497666015625} - \frac{2020987228152}{1378497666015625}\sqrt{26521})y^2 + (\frac{143082928547894}{1378497666015625} - \frac{16071932493804}{1378497666015625}\sqrt{26521})y + (-\frac{3664141366676053}{1378497666015625} - \frac{14050945265652}{1378497666015625}\sqrt{26521}) = 0$.

number	point
1	$(\frac{-47105895121}{48763048217} + \frac{10164064}{48763048217}\sqrt{26521}, \frac{904847664}{48763048217} + \frac{52913596}{48763048217}\sqrt{26521})$
2	$(\frac{-1394781122436625}{1745240213769589} + \frac{2151304132500}{1745240213769589}\sqrt{26521}, \frac{-689161571847714}{1745240213769589} + \frac{-4353983906250}{1745240213769589}\sqrt{26521})$
3	$(\frac{159568342340000}{611570166287089} + \frac{1662256960875}{611570166287089}\sqrt{26521}, \frac{83389552978536}{611570166287089} + \frac{-3180777187504}{611570166287089}\sqrt{26521})$
4	$(\frac{-2038973631925}{3123695342794} + \frac{6564822975}{3123695342794}\sqrt{26521}, \frac{1640666154081}{3123695342794} + \frac{8158576875}{3123695342794}\sqrt{26521})$
5	$(\frac{-3000185301611577500}{3008059339897801709} + \frac{47740774340875}{3008059339897801709}\sqrt{26521}, \frac{-14093124770254834}{3008059339897801709} + \frac{-1015873431406250}{3008059339897801709}\sqrt{26521})$
6	$(\frac{-54996834815}{75395520187} + \frac{162580}{7913645741}\sqrt{26521}, \frac{919985534}{7913645741} + \frac{9719050}{55395520187}\sqrt{26521})$
7	$(\frac{-222429125700625}{384771540978841} + \frac{98584444000}{384771540978841}\sqrt{26521}, \frac{208439492411784}{384771540978841} + \frac{-1052014687500}{384771540978841}\sqrt{26521})$
8	$(\frac{-7734808050567325}{7752328099417834} + \frac{104190265575}{7752328099417834}\sqrt{26521}, \frac{411487453882791}{7752328099417834} + \frac{1958484268125}{7752328099417834}\sqrt{26521})$
9	$(\frac{-90673634826912205}{91221618084240394} + \frac{3343515943095}{91221618084240394}\sqrt{26521}, \frac{7436357739975831}{91221618084240394} + \frac{4076845287725}{91221618084240394}\sqrt{26521})$
10	$(\frac{-132057861052191625}{167578669314273809} + \frac{-20724373780000}{167578669314273809}\sqrt{26521}, \frac{55708685083741816}{167578669314273809} + \frac{-49127228437500}{167578669314273809}\sqrt{26521})$
11	$(\frac{-77414401848025}{156491316311581} + \frac{413151722100}{156491316311581}\sqrt{26521}, \frac{107881801620294}{156491316311581} + \frac{296471628750}{156491316311581}\sqrt{26521})$
12	$(\frac{-81819460135937500}{11605427443988615341} + \frac{66236790971572875}{11605427443988615341}\sqrt{26521}, \frac{-292344902061005966}{11605427443988615341} + \frac{-18537918996093750}{11605427443988615341}\sqrt{26521})$
13	$(\frac{-689473353579025}{805814338169074} + \frac{71289680275}{805814338169074}\sqrt{26521}, \frac{7436357739975831}{805814338169074} + \frac{-1799184268125}{805814338169074}\sqrt{26521})$
14	$(\frac{-324579151}{635848277} + \frac{-1711916}{635848277}\sqrt{26521}, \frac{417408914}{635848277} + \frac{-1331194}{635848277}\sqrt{26521})$
15	$(\frac{-2561001710278491025}{256388023516659746} + \frac{11840698735525}{256388023516659746}\sqrt{26521}, \frac{113360609431591379}{256388023516659746} + \frac{267500764724375}{256388023516659746}\sqrt{26521})$
16	$(\frac{-173025841571077691125}{173548163597360626306} + \frac{3171424494713625}{173548163597360626306}\sqrt{26521}, \frac{10192317507497826819}{173548163597360626306} + \frac{53838431914359375}{173548163597360626306}\sqrt{26521})$
17	$(\frac{-5089581699501775}{5263416927823594} + \frac{-865913528475}{5263416927823594}\sqrt{26521}, \frac{1189985907294531}{5263416927823594} + \frac{-3703520874375}{5263416927823594}\sqrt{26521})$
18	$(\frac{-112395265103305625}{1461069900995578666} + \frac{-2062023583492125}{1461069900995578666}\sqrt{26521}, \frac{653251944881874450}{1461069900995578666} + \frac{-3545739658359375}{1461069900995578666}\sqrt{26521})$

number	line
1	$y = \left(\frac{1226495093639}{25742078352} + \frac{-8082547901}{25742078352} \sqrt{26521} \right) x + \left(\frac{1234271007023}{25742078352} + \frac{-8035587125}{25742078352} \sqrt{26521} \right)$
2	$y = \left(\frac{278198125605125}{442492828554024} + \frac{-1519379687375}{442492828554024} \sqrt{26521} \right) x + \left(\frac{97272712149101}{442492828554024} + \frac{-2661122234375}{442492828554024} \sqrt{26521} \right)$
3	$y = \left(\frac{-2837492954225}{10282252882536} + \frac{-86010715925}{10282252882536} \sqrt{26521} \right) x + \left(\frac{439073823781}{541171204344} + \frac{-1227584375}{541171204344} \sqrt{26521} \right)$
4	$y = \left(\frac{139033724532475}{13898226578344} + \frac{174498936455}{13898226578344} \sqrt{26521} \right) x + \left(\frac{1798191880315903}{180676945518472} + \frac{-9457424003125}{180676945518472} \sqrt{26521} \right)$
5	$y = \left(\frac{75113422009955}{2241249893832} + \frac{174498936455}{2241249893832} \sqrt{26521} \right) x + \left(\frac{3933594799897}{117960520728} + \frac{-9057533125}{117960520728} \sqrt{26521} \right)$
6	$y = \left(\frac{4666132534595}{1003207842488} + \frac{2039941260575}{1003207842488} \sqrt{26521} \right) x + \left(\frac{4737178109387}{1003207842488} + \frac{21942101875}{1003207842488} \sqrt{26521} \right)$
7	$y = \left(\frac{415417130846675}{71606343844008} + \frac{2039941260575}{71606343844008} \sqrt{26521} \right) x + \left(\frac{417551993627867}{71606343844008} + \frac{2047837934375}{71606343844008} \sqrt{26521} \right)$
8	$y = \left(\frac{596429228526905}{11392166373912} + \frac{-3104049824795}{11392166373912} \sqrt{26521} \right) x + \left(\frac{596792405624213}{11392166373912} + \frac{-3102172664375}{11392166373912} \sqrt{26521} \right)$
9	$y = \left(\frac{57207125374680395}{534177894085128} + \frac{344923834700695}{534177894085128} \sqrt{26521} \right) x + \left(\frac{56571730227641747}{534177894085128} + \frac{340993759489375}{534177894085128} \sqrt{26521} \right)$
10	$y = \left(\frac{2496965362931275}{1993436657064648} + \frac{-203548544725}{1993436657064648} \sqrt{26521} \right) x + \left(\frac{2623704299582227}{1993436657064648} + \frac{-2916364478125}{1993436657064648} \sqrt{26521} \right)$
11	$y = \left(\frac{4864867672100425}{2937875736726576} + \frac{-59362068928075}{2937875736726576} \sqrt{26521} \right) x + \left(\frac{8588310540110409}{2937875736726576} + \frac{-36643635221875}{2937875736726576} \sqrt{26521} \right)$
12	$y = \left(\frac{27401398259175}{8974526839644} + \frac{-5027685215275}{8974526839644} \sqrt{26521} \right) x + \left(\frac{3519501194156503}{116668848915372} + \frac{-66159214121875}{116668848915372} \sqrt{26521} \right)$
13	$y = \left(\frac{-363470529805}{224321511366} + \frac{-3676366873}{224321511366} \sqrt{26521} \right) x + \left(\frac{-300786460241}{224321511366} + \frac{-3324878875}{224321511366} \sqrt{26521} \right)$
14	$y = \left(\frac{32938570179881}{10600935090318} + \frac{134177582921}{10600935090318} \sqrt{26521} \right) x + \left(\frac{33353869804307}{10600935090318} + \frac{134981060875}{10600935090318} \sqrt{26521} \right)$
15	$y = \left(\frac{1085193496386364175}{31860273707411076} + \frac{-4377187262450675}{31860273707411076} \sqrt{26521} \right) x + \left(\frac{31860273707411076}{31860273707411076} + \frac{-4373960496753125}{31860273707411076} \sqrt{26521} \right)$
16	$y = \left(\frac{1678662703685350}{230493659797393} + \frac{21933627631475}{230493659797393} \sqrt{26521} \right) x + \left(\frac{1685966046365107}{2074442938176537} + \frac{22235068850000}{2074442938176537} \sqrt{26521} \right)$
17	$y = \left(\frac{5015547448503400}{746106020118567} + \frac{25069583194100}{746106020118567} \sqrt{26521} \right) x + \left(\frac{10255928729984741}{1492212040237134} + \frac{49083515434375}{1492212040237134} \sqrt{26521} \right)$
18	$y = \left(\frac{175303261060489}{54191653439652} + \frac{474799197449}{54191653439652} \sqrt{26521} \right) x + \left(\frac{17677691889473}{54191653439652} + \frac{480928156625}{54191653439652} \sqrt{26521} \right)$

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Code

```

Q:=Rationals();
u:=char‘55; v:=char‘448;
E:=EllipticCurve([uchar‘ uchar‘ uchar‘ 0char‘ 0]);
P:=E! [0char‘ 0]; // This point will have order 7
P0:=E! [char‘4char‘ 32]; // This point will have infinite order

xc:=function(c)
  x:=(vchar‘u*c)/c;
  return x;
end function;
yc=function(c)
  y:=(c^2+v*c+u*v)/c;
  return y;
end function;
cxy:=function(xchar‘ y)
  c:=x;
  return c;
end function;
bxy:=function(xchar‘ y)
  b:=(y+v)/x;
  return b;
end function;
xbc:=function(bchar‘ c)
  x:=c;
  return x;
end function;
ybc:=function(bchar‘ c)
  y:=b*cchar‘v;
  return y;
end function;

Pts:=[P0+i*P : i in [0..(Order(P)char‘1)]]; 
K1:=[ [bxy(pt[1]char‘ pt[2]) char‘
        bxy(pt[1]char‘ pt[2]^2) : pt in Pts];
K2:=[ [xc(cxy(pt[1]char‘ pt[2])) char‘ yc(cxy(pt[1]char‘ pt[2]))] :
      pt in Pts];
Seq:=[ [ b[1]char‘ b[2]char‘ c[1]char‘ c[2] ] :
      b in K1char‘ c in K2 |
      b[2] eq c[1]*b[1]+c[2] ];

```

```

s0:=Seq[1];
("char' s0[1]char' "char' "char' s0[2]char' ")";
"y="char' s0[3]char' "*x+"char' s0[4];
s:=Seq[2];
while (s ne s0) do
  Exclude(~Seqchar' s);
  ("char' s[1]char' "char' "char' s[2]char' ")");
  T:=[r : r in Seq | (r[1] eq s[1]) and
    ([r[3]char' r[4]] ne [s[3]char' s[4]])];
  t:=T[1];
  S:=[r : r in Seq | (r[1] ne s[1]) and
    ([r[3]char' r[4]] eq [t[3]char' t[4]])];
  s:=S[1];
  "y="char' s[3]char' "*x+"char' s[4];
end while;

Pol[xchar' y]:=PolynomialRing(Qchar' 2);
K2 := u^2*x^2 + y^2 + 2*u*x*y char' ( 2*u*v + 4*v )*x char' 2*v*y char'-
v*(4*uchar' v); K2;

xc:=(vchar' u*c)/c;
yc:=(c^2+v*c+u*v)/c;
C2:=a1*x^2+a2*y^2+a3*x*y+a4*x+a5*y+a6;
pol:=collect(numer(discrim(subs(y=xc*x+ycchar' C2)char' x)))char' c);
solve({coeff(polchar' cchar' 4)char' {coeff(polchar' cchar' 3)char'
{coeff(polchar' cchar' 2)char' {coeff(polchar' cchar' 1)char'
{coeff(polchar' cchar' 0)char' {a1char' a2char' a3char' a4char' a5char' a6}});

```