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Chebyshev approximation

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Abstract

In this thesis, Chebyshev approximation is studied. This is a topic in approximation theory. We show that there always exists a best approximating polynomial $p(x)$ to the function $f(x)$ with respect to the uniform norm and that this polynomial is unique. We show that the best approximating polynomial is found if the set $f(x) - p(x)$ contains at least $n + 2$ alternating points. The best approximating polynomial can be found using four techniques: Chebyshev approximation, smallest norm, economization and an algorithm. For algebraic polynomials in the interval $[-1, 1]$, we assume that an orthogonal projection can be used too. We suppose that approximation of algebraic polynomials in $[-1, 1]$ with respect to the L^2-norm with inner product $\langle f, g\rangle = \int_{-1}^{1} \frac{f(x)g(x)}{\sqrt{1-x^2}} dx$ and approximation with respect to the uniform norm give the same best approximating polynomial.

Keywords: approximation theory, Chebyshev, L^2-norm, uniform norm, algebraic polynomial, error, economization, smallest norm, algorithm.

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Chapter 1

Introduction

This thesis is about *Chebyshev approximation.* Chebyshev approximation is a part of approximation theory, which is a field of mathematics about approximating functions with simpler functions. This is done because it can make calculations easier. Most of the time, the approximation is done using polynomials.

In this thesis we focus on algebraic polynomials, thus polynomials of the form $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$. We define $\mathcal{P}_n$ as the subspace of all algebraic polynomials of degree at most n in $C[a, b]$.

For over two centuries, approximation theory has been of huge interest to many mathematicians. One of them, and the first to approximate functions, was Pafnuty Lvovich Chebyshev. His contribution to approximation theory was so big, that this thesis only discusses his contributions.

Mathematicians before Chebyshev already did something with approximation theory, but far different than Chebyshev did. For example Archimedes, he approximated the circumference of a circle and therefore π, using polygons. Leonhard Euler approximated the proportion of longitudes and latitudes of maps to the real proportion of the earth and Pierre Simon Laplace approximated planets by ellipsoids [1].

Approximation theory is thus a subject with a long history, a huge importance in classical and contemporary research and a subject where many big names in mathematics have worked on. Therefore, it is a very interesting subject to study.

Chebyshev thus approximated functions and he did this in the uniform norm. We already know how approximation in the L^2-norm works: this is done using an orthogonal projection, as will be illustrated in chapter 3. This leads to the main question of this thesis:

How does approximation in the uniform norm work?

In order to answer this question, we need to answer four key questions in approximation theory:

1. Does there *exist* a best approximation in $\mathcal{P}_n$ for f?
2. If there exists a best approximation, is it *unique*?
3. What are the *characteristics* of a best approximation (i.e. how do you know a best approximation has been found)?
4. How do you *construct* the best approximation?

The goal of this thesis is thus to show how approximation in the uniform norm works. We therefore give answer to the four questions and actually solve approximation problems using different techniques. We also compare approximation in the uniform norm to the well-known approximation in the L^2-norm. This will give a complete overview of the subject.

Since this thesis is all about Chebyshev, in chapter 2 we will tell who Chebyshev was and why he was interested in approximation theory. In chapter 3 we show how approximation in the L^2-norm works, so that we can compare it to the uniform norm later. In chapter 4 we show how approximation in the uniform norm works. There, we will prove existence and uniqueness of the best approximating polynomial. In chapter 5 we will explain what Chebyshev polynomials are, since we need them to find the best approximating polynomial in chapter 6. In chapter 6 we show Chebyshev's solution to the approximation problem, compare this to the approximation in the L^2-norm, give some other techniques to solve the problem and show some utilities. At the end of this chapter, we show some examples using the different techniques. In chapter 7 we end this thesis with some conclusions about what we learnt in this thesis.

We end this introduction with a quote by Bertrand Russell, which shows the importance of this subject.

"*All exact science is dominated by the idea of approximation*"

Bertrand Russell

Chapter 2

Pafnuty Lvovich Chebyshev 1821-1894

Since this thesis is dedicated to Chebyshev approximation, we discuss in this chapter who Pafnuty Lvovich Chebyshev was and why he dealt with uniform approximation.

The information in this chapter is obtained from *The history of approximation theory* by K. G. Steffens [1].

2.1 Biography

Pafnuty Lvovich Chebyshev was born on May 4, 1821 in Okatovo, Russia. He could not walk that well, because he had a physical handicap. This handicap made him unable to do usual children things. Soon he found a passion: constructing mechanisms.

In 1837 he started studying mathematics at the Moscow University. One of his teachers was N. D. Brashman, who taught him practical mechanics. In 1841 Chebyshev won a silver medal for his 'calculation of the roots of equations'. At the end of this year he was called 'most outstanding candidate'. In 1846, he graduated. His master thesis was called 'An Attempt to an Elementary Analysis of Probabilistic Theory'. A year later, he defended his dissertation "About integration with the help of logarithms". With this dissertation he obtained the right to become a lecturer.

In 1849, he became his doctorate for his work 'theory of congruences'. A year later, he was chosen extraordinary professor at Saint Petersburg University. In 1860 he became here ordinary professor and 25 years later he became merited professor. In 1882 he stopped working at the University and started doing research.

He did not only teach at the Saint Petersburg University. From 1852 to 1858 he taught practical mechanics at the Alexander Lyceum in Pushkin, a suburb of Saint Petersburg.

Because of his scientific achievements, he was elected junior academician in 1856, and later an extraordinary (1856) and an ordinary (1858) member of the Imperial Academy of Sciences. In this year, he also became an honourable member of Moscow University.

Besides these, he was honoured many times more: in 1856 he became a member of the scientific committee of the ministry of national education, in 1859 he became ordinary membership of the ordnance department of the academy with the adoption of the headship of the "commission for mathematical questions according to ordnance and experiments related to the theory of shooting", in 1860 the Paris academy elected him corresponding member and full foreign member in 1874, and in 1893 he was elected honourable member of the Saint Petersburg Mathematical Society.

He died at the age of 73, on November 26, 1894 in Saint Petersburg.

Figure 2.1: Pafnuty Lvovich Chebyshev [Wikimedia Commons].

2.2 Chebyshev's interest in approximation theory

Chebyshev was since his childhood interested in mechanisms. The theory of mechanisms played in that time an important role, because of the industrialisation.

In 1852, he went to Belgium, France, England and Germany to talk with mathematicians about different subjects, but most important for him was to talk about mechanisms. He also collected a lot of empirical data about mechanisms, to verify his own theoretical results later.

According to Chebyshev, the foundations of approximation theory were established by the French mathematician Jean-Victor Poncelet. Poncelet approximated roots of the form $\sqrt{a^2+b^2}, \sqrt{a^2-b^2}$, and $\sqrt{a^2+b^2+c^2}$ uniformly by linear expressions (see [1] for Poncelet's Approximation Formulae).

Another important name in approximation theory was the Scottish mechanical engineer James Watt. His planar joint mechanisms were the most important mechanisms to transform linear motion into circular motion. The so called Watt's Curve is a tricircular plane algebraic curve of degree six. It is generated by two equal circles (radius b, centres a distance $2a$ apart). A line segment (length $2c$) attaches to a point on each of the circles, and the midpoint of the line segment traces out the Watt curve as the circles rotate (for more on Watt's Curve see [1]).

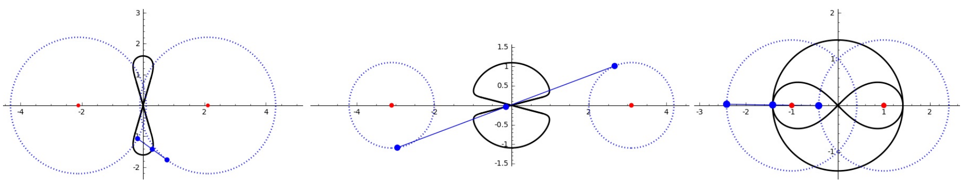

Figure 2.2: Watt's Curves for different values of a, b and c.

The Watt's Curve inspired Chebyshev to deal with the following: determine the parameters of the mechanism so that the maximal error of the approximation of the curve by the tangent on the whole interval is minimized.

In 1853, Chebyshev published his first solutions in his "Théorie des mécanismes, connus sous le nom de parallélogrammes". He tried to give mathematical foundations to the theory of mechanisms, because practical mechanics did not succeed in finding the mechanism with the smallest deviation from the ideal run. Other techniques did not work either. Poncelet's approach did work, but only for specific cases.

Chebyshev wanted to solve general problems. He formulated the problem as follows (translated word-by-word from French):

To determine the deviations which one has to add to get an approximated value for a function f*, given by its expansion in powers of* $x-a$*, if one wants to minimize the maximum of these errors between* $x=a-h$ *and* $x=a+h$*,* h *being an arbitrarily small quantity.*

The formulation of this problem is the start of approximation in the uniform norm.

Chapter 3

Approximation in the L^2-norm

The problem that Chebyshev wanted to solve is an approximation problem in the uniform norm. In this chapter we show how approximation in the L^2-norm works, so that we can compare it to approximation in the uniform norm later.

This chapter uses concepts of linear algebra, with which the reader should be familiar with. Some basic definitions that are needed can be found in appendix A.

The results in this chapter are derived from *Linear algebra with applications* by S. J. Leon [2], from *A choice of norm in discrete approximation* by T. Marošević [3] and from *Best approximation in the 2-norm* by E. Celledoni [4].

3.1 Best approximation in the L^2-norm

Let $f(x) \in C[a,b]$. We want to find the best approximating polynomial $p(x) \in \mathcal{P}_n$ of degree n to the function $f(x)$ with respect to the L^2-norm. We can restate this as follows

Problem 1. *Find the best approximating polynomial $p \in \mathcal{P}_n$ of degree n to the function $f(x) \in C[a,b]$ in the L^2-norm such that*

$$\|f - p\|_2 = \inf_{q \in \mathcal{P}_n} \|f - q\|_2.$$

The best approximating polynomial $p(x)$ always exists and is unique. We are not going to prove this, since this is out of the scope of this thesis. We will prove existence and uniqueness for approximation in the uniform norm in chapter 4.

To solve this problem, we want to minimize

$$E = \|f - p\|_2 = \left(\int_a^b |f(x) - p(x)|^2 dx \right)^{\frac{1}{2}},$$

since

$$\|f\|_2^2 = \int_a^b f(x)^2 dx.$$

Theorem 1. *The best approximating polynomial* $p(x) \in \mathcal{P}_n$ *is such that*

$$\|f - p\|_2 = \min_{q \in \mathcal{P}_n} \|f - q\|_2.$$

if and only if

$$\langle f - p, q \rangle = 0, \quad \textit{for all} \quad q \in \mathcal{P}_n,$$

where

$$\langle f, g \rangle = \int_a^b f(x)g(x)dx.$$

Thus, the integral is minimal if $p(x)$ is the orthogonal projection of the function $f(x)$ on the subspace $\mathcal{P}_n$. Suppose that $u_1, u_2, u_3, \dots, u_n$ form an orthogonal basis for P_n. Then

$$p(x) = \frac{\langle f, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1(x) + \frac{\langle f, u_2 \rangle}{\langle u_2, u_2 \rangle} u_2(x) + \cdots + \frac{\langle f, u_n \rangle}{\langle u_n, u_n \rangle} u_n(x)$$

Orthogonal polynomials can be obtained by applying the **Gram-Schmidt Process** to the basis for the inner product space V.

3.1.1 The Gram-Schmidt Process

Let $\{x_1, x_2, \dots, x_n\}$ be a basis for the inner product space V. Let

$$\begin{aligned} u_1 &= \left(\frac{1}{\|x_1\|} \right) x_1 \\ u_{k+1} &= \frac{1}{\|x_{k+1} - p_k\|} (x_{k+1} - p_k) \text{ for } k = 1, \dots, n-1. \\ p_k &= \langle x_{k+1}, u_1 \rangle u_1 + \langle x_{k+1}, u_2 \rangle u_2 + \dots \langle x_{k+1}, u_k \rangle u_k. \end{aligned}$$

Then p_k is the projection of x_{k+1} onto $\text{span}(u_1, u_2, \dots, u_n)$ and the set $\{u_1, u_2, \dots, u_n\}$ is an orthonormal basis for V.

3.1.2 Example

Find the best approximating quadratic polynomial to the function $f(x) = |x|$ on the interval $[-1, 1]$. Thus, we want to minimize

$$\|f-p\|_2 = \left(\int_a^b |f(x) - p(x)|^2 dx\right)^{\frac{1}{2}} = \||x|-p\|_2 = \left(\int_{-1}^1 ||x| - p(x)|^2 dx\right)^{\frac{1}{2}}.$$

This norm is minimal if if $p(x)$ is the orthogonal projection of the function $f(x)$ on the subspace of polynomials of degree at most 2.

We start with the basis $\{1, x, x^2\}$ for the inner product space V.

$$\begin{aligned}
u_1 &= \left(\tfrac{1}{\|x_1\|}\right) x_1 &&= \frac{1}{\sqrt{2}}\\
p_1 &= \langle x_2, u_1\rangle u_1 = \langle x, \tfrac{1}{\sqrt{2}}\rangle \tfrac{1}{\sqrt{2}} &&= 0\\
u_2 &= \tfrac{1}{\|x_2-p_1\|}(x_2 - p_1) = \tfrac{1}{\|x\|}x &&= \frac{\sqrt{3}}{\sqrt{2}}x\\
p_2 &= \langle x_3, u_1\rangle u_1 + \langle x_3, u_2\rangle u_2 = \langle x^2, \tfrac{1}{\sqrt{2}}\rangle \tfrac{1}{\sqrt{2}} + \langle x^2, \tfrac{\sqrt{3}}{\sqrt{2}}x\rangle \tfrac{\sqrt{3}}{\sqrt{2}}x &&= \frac{1}{3}\\
u_3 &= \tfrac{1}{\|x_3-p_2\|}(x_3 - p_2) = \tfrac{1}{\|x^2-\frac{1}{3}\|}(x^2 - \tfrac{1}{3}) &&= \frac{\sqrt{45}}{\sqrt{8}}(x^2 - \frac{1}{3}).
\end{aligned}$$

Then,

$$p(x) = \frac{\langle f, u_1\rangle}{\langle u_1, u_1\rangle}u_1(x) + \frac{\langle f, u_2\rangle}{\langle u_2, u_2\rangle}u_2(x) + \frac{\langle f, u_3\rangle}{\langle u_3, u_3\rangle}u_3(x).$$

So we have to calculate each inner product

$$\begin{aligned}
\langle u_1, u_1\rangle &= \textstyle\int_{-1}^1 \frac{1}{2}dx &&= 1\\
\langle f, u_1\rangle &= \tfrac{1}{\sqrt{2}}\textstyle\int_{-1}^1 |x|dx = \tfrac{2}{\sqrt{2}}\int_0^1 xdx &&= \frac{1}{\sqrt{2}}\\
\langle u_2, u_2\rangle &= \tfrac{3}{2}\textstyle\int_{-1}^1 x^2dx &&= 1\\
\langle f, u_2\rangle &= \tfrac{\sqrt{3}}{\sqrt{2}}\textstyle\int_{-1}^1 |x|xdx &&= 0\\
\langle u_3, u_3\rangle &= \textstyle\int_{-1}^1 (\frac{\sqrt{45}}{\sqrt{8}}(x^2 - \frac{1}{3}))^2 dx &&= 1\\
\langle f, u_3\rangle &= \tfrac{\sqrt{45}}{\sqrt{8}}\textstyle\int_{-1}^1 |x|(x^2 - \frac{1}{3})dx &&= \frac{\sqrt{5}}{4\sqrt{2}}.
\end{aligned}$$

Thus,

$$p(x) = \frac{1}{2} + \frac{15}{16}x^2 - \frac{5}{16} = \frac{15}{16}x^2 + \frac{3}{16},$$

with maximum error

$$E = \left(\int_{-1}^1 \left||x| - \frac{15}{16}x^2 - \frac{3}{16})\right|^2 dx\right)^{\frac{1}{2}} = \frac{1}{\sqrt{96}} \approx 0.1021.$$

3.1.3 Legendre polynomials

In fact, the polynomials that are orthogonal with respect to the inner product

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)dx$$

are called the **Legendre polynomials**, named after the French mathematician Adrien-Marie Legendre. The formula for finding the best approximating polynomial is then thus

$$p(x) = \frac{\langle f, P_0 \rangle}{\langle P_0, P_0 \rangle} P_0(x) + \frac{\langle f, P_1 \rangle}{\langle P_1, P_1 \rangle} P_1(x) + \cdots + \frac{\langle f, P_{n-1} \rangle}{\langle P_{n-1}, P_{n-1} \rangle} P_{n-1}(x).$$

The Legendre polynomials satisfy the recurrence relation

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_n - 1(x).$$

A list of the first six Legendre polynomials can be found in table B.1 in appendix B.

Chapter 4

Approximation in the uniform norm

Pafnuty Lvovich Chebyshev was thus the first who came up with the idea of approximating functions in the uniform norm. He asked himself at that time

Problem 2. *Is it possible to represent a continuous function $f(x)$ on the closed interval $[a, b]$ by a polynomial $p(x) = \sum_{k=0}^{n} a_k x^k$ of degree at most n, with $n \in \mathbb{Z}$, in such a way that the maximum error at any point $x \in [a, b]$ is controlled? I.e. is it possible to construct $p(x)$ so that the error $\max_{a \leq x \leq b} |f(x) - p(x)|$ is minimized?*

This thesis will give an answer to this question. In this chapter we show that the best approximating polynomial always exists and that it is unique.

The theorems, lemmas, corollaries, proofs and examples in this chapter are derived from *A Short Course on Approximation Theory* by N. L. Carothers [5], *Lectures on Multivariate Polynomial Interpolation* by S. De Marchi [6], *Oscillation theorem* by M. Embree [7] and *An introduction to numerical analysis* by E. Süli and D. F. Mayers [8].

4.1 Existence

In 1854, Chebyshev found a solution to the problem of best approximation. He observed the following

Lemma 1. *Let $f(x) \in C[a, b]$ and let $p(x)$ be a best approximation to $f(x)$ out of $\mathcal{P}_n$. Then there are at least two distinct points $x_1, x_2 \in [a, b]$ such that $f(x_1) - p(x_1) = -(f(x_2) - p(x_2)) = \|f(x) - p(x)\|_\infty$. That is, $f(x) - p(x)$ attains each of the values $\pm\|f - p\|_\infty$.*

Proof. This is a proof by contradiction. Write the error $E = \|f - p\|_\infty = \max_{a \le x \le b} |f - p|$. If the conclusion of the lemma is false, then we might suppose that $f(x_1) - p(x_1) = E$, for some x_1. But that

$$\epsilon = \min_{a \le x \le b} (f(x) - p(x)) > -E,$$

for all $x \in [a, b]$. Thus, $E + \epsilon \neq 0$, and so $q = p + \frac{E+\epsilon}{2} \in \mathcal{P}_n$, with $p \neq q$.

We now claim that $q(x)$ is a better approximation to $f(x)$ than $p(x)$. We show this using the inequality stated above.

$$\begin{aligned} E - \frac{E+\epsilon}{2} \ge f(x) - p(x) - \frac{E+\epsilon}{2} \ge \epsilon - \frac{E+\epsilon}{2} \\ -\frac{E-\epsilon}{2} \le f(x) - q(x) \le \frac{E-\epsilon}{2}, \end{aligned}$$

for all $x \in [a, b]$. That is,

$$\|f - q\|_\infty \le \frac{E-\epsilon}{2} < E = \|f - p\|_\infty$$

Hence, $q(x)$ is a better approximation to $f(x)$ than $p(x)$. This is a contradiction, since we have that $p(x)$ is a best approximation to $f(x)$. □

Corollary 1. *The best approximating constant to* $f(x) \in C[a, b]$ *is*

$$p_0 = \frac{1}{2} \left[\max_{a \le x \le b} f(x) + \min_{a \le x \le b} f(x) \right]$$

with error

$$E_0(f) = \frac{1}{2} \left[\max_{a \le x \le b} f(x) - \min_{a \le x \le b} f(x) \right].$$

Proof. This is again a proof by contradiction. Let x_1 and x_2 be such that $f(x_1) - p_0 = -(f(x_2) - p_0) = \|f - p_0\|_\infty$. Suppose d is any other constant. Then, $E = f - d$ cannot satisfy lemma 1. In fact,

$$\begin{aligned} E(x_1) &= f(x_1) - d \\ E(x_2) &= f(x_2) - d, \end{aligned}$$

showing that $E(x_1) + E(x_2) \neq 0$. This contradicts lemma 1. □

Next, we will generalize lemma 1 to show that a best linear approximation implies the existence of at least $n+2$ points, where n is the degree of the best approximating polynomial, at which $f - p$ alternates between $\pm\|f - p\|_\infty$.

We need some definitions to arrive at this generalization.
Definition 1. *Let* $f(x) \in C[a, b]$.

1. *$x \in [a, b]$ is called a* ***(+)point*** *for $f(x)$ if $f(x) = \|f\|_\infty$.*
2. *$x \in [a, b]$ is called a* ***(-)point*** *for $f(x)$ if $f(x) = -\|f\|_\infty$.*
3. *A set of distinct points $a \le x_0 < x_1 < \cdots < x_n \le b$ is called an* ***alternating set*** *for $f(x)$ if the x_i are alternately $(+)$points and $(-)$points; that is, if $|f(x_i)| = \|f\|_\infty$ and $f(x_i) = -f(x_{i-1})$ for all $i = 1, \ldots, n$.*

We use these notations to generalize lemma 1 and thus to characterize a best approximating polynomial.
Theorem 2. *Let $f(x) \in C[a, b]$, and suppose that $p(x)$ is a best approximation to $f(x)$ out of $\mathcal{P}_n$. Then, there is an alternating set for $f - p$ consisting of at least $n + 2$ points.*

Proof. We may suppose that $f(x) \notin \mathcal{P}_n$, since if $f(x) \in \mathcal{P}_n$, then $f(x) = p(x)$ and then there would be no alternating set. Hence, $E = \|f - p\|_\infty > 0$.

Consider the (uniformly) continuous function $\phi = f - p$ (continuous on a compact set is uniformly continuous). Our next step is to divide the interval $[a, b]$ into smaller intervals $a = t_0 < t_1 < \cdots < t_k = b$ so that $|\phi(x) - \phi(y)| < \frac{E}{2}$ whenever $x, y \in [t_i; t_{i+1}]$.

We want to do this, because if $[t_i; t_{i+1}]$ contains a $(+)$point for $\phi = f - p$, then ϕ is positive on the whole interval $[t_i; t_{i+1}]$

$$x, y \in [t_i; t_{i+1}] \quad \text{and} \quad \phi(x) = E \quad \Rightarrow \quad \phi(y) > \frac{E}{2}. \tag{4.1}$$

Similarly, if the interval $[t_i; t_{i+1}]$ contains a $(-)$point, then ϕ is negative on the whole interval $[t_i; t_i + 1]$. Hence, no interval can contain both $(+)$points and $(-)$points.

We call an interval with a $(+)$point a $(+)$interval, an interval with a $(-)$point a $(-)$interval. It is important to notice that no $(+)$interval can touch a $(-)$interval. Hence, the intervals are separated by a interval containing a zero for ϕ.

Our next step is to label the intervals

$I_1, I_2, \ldots, I_{k_1}$	$(+)$intervals
$I_{k_1+1}, I_{k_1+2}, \ldots, I_{k_2}$	$(-)$intervals
....	....
$I_{k_{m-1}+1}, I_{k_1+2}, \ldots, I_{k_m}$	$(-1)^{m-1}$intervals.

Let S denote the union of all signed intervals: $S = \bigcup_{j=1}^{k_m} I_j$. Let N denote the union of the remaining intervals. S and N are compact sets with $S \cup N = [a, b]$.

We now want to show that $m \geq n+2$. We do this, by letting $m < n+2$ and showing that this yields a contradiction.

The $(+)$intervals and $(-)$intervals are strictly separated, hence we can find points $z_1, \ldots, z_{m-1} \in N$ such that

$$\begin{array}{rcl} \max I_{k_1} < & z_1 & < \min I_{k_1+1} \\ \max I_{k_2} < & z_2 & < \min I_{k_2+1} \\ \cdots\cdots\cdots & & \cdots\cdots\cdots\cdots \\ \max I_{k_{m-1}} < & z_{m-1} & < \min I_{k_{m-1}+1}. \end{array}$$

We can now construct the polynomial which leads to a contradiction

$$q(x) = (z_1 - x)(z_2 - x)\ldots(z_{m-1} - x).$$

Since we assumed that $m < n+2$, $m-1 < n$ and hence $q(x) \in \mathcal{P}_n$.

The next step is to show that $p + \lambda q \in \mathcal{P}_n$ is a better approximation to $f(x)$ than $p(x)$.

Our first claim is that $q(x)$ and $f - p$ have the same sign. This is true, because $q(x)$ has no zeros on the $(\pm)$intervals, and thus is of constant sign. Thus, we have that $q > 0$ on $I_1, \ldots, I_{k_1}$, because $(z_j - x) > 0$ on these intervals. Consequently, $q < 0$ on $I_{k_1+1}, \ldots, I_{k_2}$, because $(z_1 - x) < 0$ on these intervals.

The next step is to find λ. Therefore, let $\epsilon = \max_{x \in N} |f(x) - p(x)|$. N is the union of all subintervals $[t_i; t_{i+1}]$, which are neither $(+)$intervals nor $(-)$intervals.

By definition, $\epsilon < E$. Choose $\lambda > 0$ in such a way, such that $\lambda \|q\| < \min\{E - \epsilon, \frac{E}{2}\}$.

Our next step is to show that $q(x)$ is a better approximation to $f(x)$ than $p(x)$. We show this for the two cases: $x \in N$ and $x \notin N$.

Let $x \in N$. Then,

$$\begin{aligned} |f(x) - (p(x) + \lambda q(x))| &\leq |f(x) - p(x)| + \lambda |q(x)| \\ &\leq \epsilon + \lambda \|q\|_\infty < E. \end{aligned}$$

Let $x \notin N$. Then x is in either a $(+)$interval or a $(-)$interval. From equation 4.1, we know that $|f - p > \frac{E}{2} > \lambda \|q\|_\infty$. Thus, $f - p$ and $\lambda q(x)$ have the same sign. Thus we have that

$$\begin{aligned} |f - (p + \lambda q)| &= |f - p| - \lambda |q| \\ &\leq E - \lambda \min_{x \in S} |q| < E, \end{aligned}$$

because $q(x)$ is non-zero on S.

So we arrived at a contradiction: we showed that $p + \lambda q$ is a better approximation to $f(x)$ than $p(x)$, but we have that $p(x)$ is the best approximation to $f(x)$. Therefore, our assumption $m < n+2$ is false, and hence $m \geq n+2$. $\square$

It is important to note that if $f - p$ alternates $n + 2$ times in sign, then $f - p$ must have at least $n + 1$ zeros. This means that $p(x)$ has at least $n + 1$ the same points as $f(x)$.

4.2 Uniqueness

In this section, we will show that the best approximating polynomial is unique.

Theorem 3. *Let $f(x) \in C[a, b]$. Then the polynomial of best approximation $p(x)$ to $f(x)$ out of $\mathcal{P}_n$ is unique.*

Proof. Suppose there are two best approximations $p(x)$ and $q(x)$ to $f(x)$ out of $\mathcal{P}_n$. We want to show that these $p(x), q(x) \in \mathcal{P}_n$ are the same.

If they are both best approximations, they satisfy $\|f-p\|_\infty = \|f-q\|_\infty = E$. The average $r(x) = \frac{p+q}{2}$ of $p(x)$ and $q(x)$ is then also a best approximation, because $f - r = f - \frac{p+q}{2} = \frac{f-p}{2} + \frac{f-q}{2}$. Thus, $\|f - r\|_\infty = E$.

By theorem 2, $f - r$ has an alternating set $x_0, x_1, \ldots, x_{n+1}$ containing of $n + 2$ points.

For each i,

$$(f - p)(x_i) + (f - q)(x_i) = \pm 2E \quad \text{(alternating)},$$

while

$$-E \leq (f - p)(x_i), \quad (f - q)(x_i) \leq E.$$

This means that

$$(f - p)(x_i) = (f - q)(x_i) = \pm E \quad \text{(alternating)},$$

for each i. Hence, $x_0, x_1, \ldots, x_{n+1}$ is an alternating set for both $f - p$ and $f - q$.

The polynomial $q-p = (f-p)-(f-q)$ has $n+2$ zeros. Because $q-p \in \mathcal{P}_n$, we must have $p(x) = q(x)$. This is what we wanted to show: if there are two best approximations, then they are the same and hence the approximating polynomial is unique. $\square$

We can finally combine our previous results in the following theorem.

Theorem 4. *Let $f(x) \in C[a,b]$, and let $p(x) \in \mathcal{P}_n$. If $f-p$ has an alternating set containing $n+2$ (or more) points, then $p(x)$ is the best approximation to $f(x)$ out of $\mathcal{P}_n$.*

Proof. This is a proof by contradiction. We want to show that if $q(x)$ is a better approximation to $f(x)$ than $p(x)$, then $q(x)$ must be equal to $p(x)$.

Therefore, let $x_0, \ldots, x_{n+1}$ be the alternating set for $f-p$. Assume $q(x) \in \mathcal{P}_n$ is a better approximation to $f(x)$ than $p(x)$. Thus, $\|f-q\|_\infty < \|f-p\|_\infty$.

Then we have

$$|f(x_i) - p(x_i)| = \|f-p\|_\infty > \|f-q\|_\infty \geq |f(x_i) - q(x_i)|,$$

for each $i = 0, \ldots, n+1$. Thus we have $|f(x_i) - p(x_i)| > |f(x_i) - q(x_i)|$.

This means that $f(x_i)-p(x_i)$ and $f(x_i)-p(x_i)-f(x_i)+q(x_i) = q(x_i)-p(x_i)$ must have the same sign ($|a| > |b|$, then a and $a-b$ have the same sign). Hence, $q-p = (f-p)-(f-q)$ alternates $n+2$ (or more) times in sign, because $f-p$ does too. This means that $q-p$ has at least $n+1$ zeros. Since $q-p \in \mathcal{P}_n$, we must have $q(x) = p(x)$. This contradicts the strict inequality, thus we conclude that $p(x)$ is the best approximation to $f(x)$ out of $\mathcal{P}_n$. □

Thus, from theorem 2 and theorem 4 we know that the polynomial $p(x)$ is the best approximation to $f(x)$ if and only if $f-p$ alternates in sign at least $n+2$ times, where n is the degree of the best approximating polynomial. Consequently, $f-p$ has at least $n+1$ zeros.

We can illustrate this theorem using an example.

Example 1. *Consider the function $f(x) = \sin(4x)$ in $[-\pi, \pi]$. Figure 4.1 shows this function together with the best approximating polynomial $p_0 = 0$.*

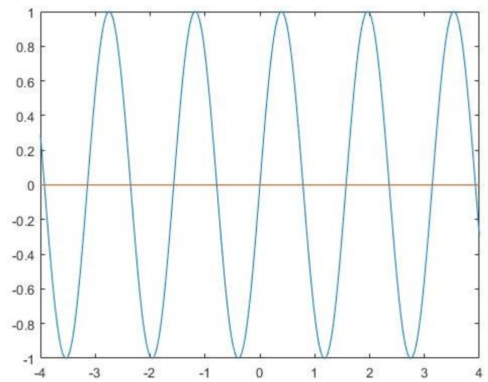

Figure 4.1: Illustration of the function $f(x) = \sin(4x)$ with best approximating polynomial $p_0 = 0$.

The error $E = f - p = \sin(4x)$ has 8 different alternating sets of 2 points. Using theorem 2 and theorem 4, we find that $p_0 = p_1 = p_2 = p_3 = p_4 = p_5 = p_6 = 0$ are best approximations.

This means that the best approximating polynomial of degree 0 is $p_0 = 0$. This is true since $f - p_0$ alternates 8 times in sign, much more than the required $n + 2 = 2$ times.

We can repeat this procedure: the best approximating polynomial in $\mathcal{P}_1$, is $p_1 = 0$, because then $f - p_1$ alternates again 8 times in sign, much more than the required $n + 2 = 3$ times.

The polynomial $p_7 = 0$ is not a best approximation, since $f - p_7$ only alternates 8 times in sign and it should alternate at least $n+2 = 9$ times in sign. So in $\mathcal{P}_7$ there exists a better approximating polynomial than $p_7 = 0$.

Example 2. *In this example we show that the function $p = x - \frac{1}{8}$ is the best linear approximation to the function $f(x) = x^2$ on $[0, 1]$ (techniques to find this polynomial will be discussed in chapter 6).*

The polynomial of best approximation has degree $n = 1$, so $f - p$ must alternate at least $2 + 1 = 3$ times in sign. Consequently, $f - p$ has at least $1 + 1 = 2$ zeros. We see this in figure 4.2. $f - p$ alternates in sign 3 times and has 2 zeros: $x = \frac{1}{2} \pm \frac{1}{4}\sqrt{2}$.

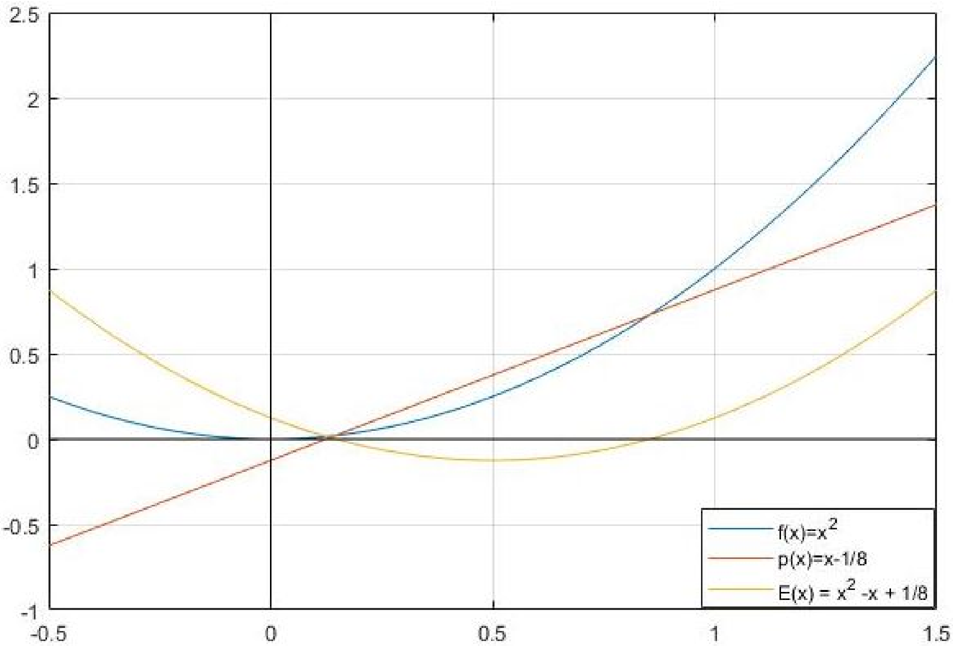

Figure 4.2: The polynomial $p(x) = x - \frac{1}{8}$ is the best approximation of degree 1 to $f(x) = x^2$, because $f - p$ changes sign 3 times.

We now know the characteristics of the best approximating polynomial. The next step is to find the maximum error between $f(x)$ and the best approximating function $p(x)$. De La Vallée Poussin proved the following theorem, which provides the lower bound for the error E.

Theorem 5 (De La Vallée Poussin)**.** *Let $f(x) \in C[a,b]$, and suppose that $q(x) \in \mathcal{P}_n$ is such that $f(x_i) - q(x_i)$ alternates in sign at $n+2$ points $a \leq x_0 < x_1 < \cdots < x_{n+1} \leq b$. Then*

$$E = \min_{p \in \mathcal{P}_n} \|f - p\|_\infty \geq \min_{i=0,\ldots,n+1} |f(x_i) - q(x_i)|.$$

Before proving this theorem, we show in figure 4.3 how this theorem works.

Suppose we want to approximate the function $f(x) = e^x$ with a quintic polynomial. In the figure, a quintic polynomial $r(x) \in \mathcal{P}_5$ is shown, that is chosen in such a way that $f - r$ changes sign 7 times. This is not the best approximating polynomial. The red curve shows the error for the best approximating polynomial $p(x)$, which also has 7 points for which the error changes in sign.

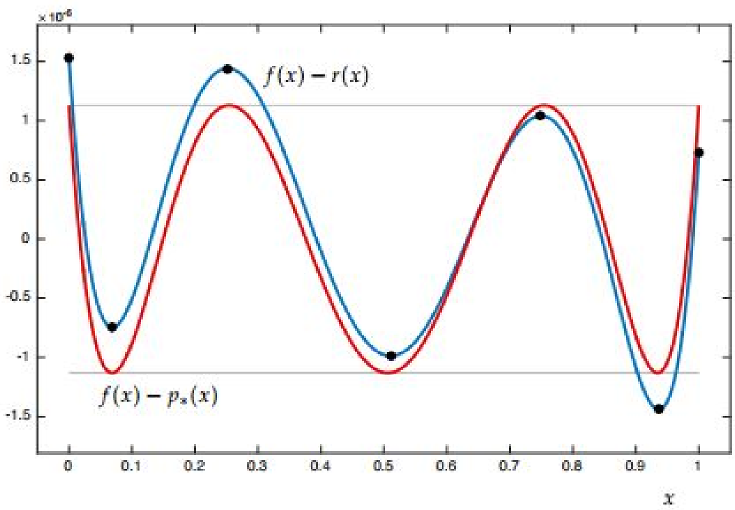

Figure 4.3: Illustration of de la Vallée Poussin's theorem for $f(x) = e^x$ and $n = 5$. Some polynomial $r(x) \in \mathcal{P}_5$ gives an error $f - r$ for which we can identify $n + 2 = 7$ points at which $f - r$ changes sign. The minimum value of $|f(x_i) - r(x_i)|$ gives a lower bound for the maximum error $\|f - p\|_\infty$ of the best approximating polynomial $p(x) \in \mathcal{P}_5$ [7].

The point of the theorem is the following:

Since the error $f(x) - r(x)$ changes sign $n+2$ times, the error $\pm\|f - p\|_\infty$ exceeds $|f(x_i) - r(x_i)|$ at one of the points x_i that give the changing sign.

So de la Vallée Poussin's theorem gives a nice mechanism for developing lower bounds on $\|f - p\|_\infty$.

We now prove theorem 5.

Proof. [De La Vallée Poussin.] This is a proof by contradiction. Assume that the inequality does not hold. Then, the best approximating polynomial $p(x)$ satisfies

$$\max_{0\leq i\leq n+1} |f(x_i) - p(x_i)| \leq E < \min_{0\leq i\leq n+1} |f(x_i) - q(x_i)|.$$

The middle part of the inequality is the maximum difference of $|f - p|$ over all $x \in [a, b]$, so it cannot be larger at $x_i \in [a, b]$. Thus,

$$|f(x_i) - p(x_i) < |f(x_i) - q(x_i)|, \quad \text{for all} \quad i = 0, \dots, n+1. \tag{4.2}$$

Now consider

$$p(x) - q(x) = (f(x) - q(x)) - (f(x) - p(x)),$$

which is a polynomial of degree n, since $p(x), q(x) \in \mathcal{P}_n$. Then from 4.2 we know that $f(x_i) - q(x_i)$ has always larger magnitude than $f(x_i) - p(x_i)$. Thus, the magnitude $|f(x_i) - p(x_i)|$ will never be large enough to overcome $|f(x_i) - q(x_i)|$. Hence,

$$sgn(p(x_i) - q(x_i)) = sgn(f(x_i)) - q(x_i)).$$

From the hypothesis we know that $f(x) - q(x)$ alternates in sign at least $n + 1$ times, thus the polynomial $p - q$ does too.

Changing sign $n+1$ times means $n+1$ roots. The only polynomial of degree n with $n+1$ roots is the zero polynomial. Thus, $(x)p = q(x)$. This contradicts the strict inequality. Hence, there must be at least one i for which

$$E_n(f) \geq |f(x_i) - q(x_i)|.$$

□

Chapter 5

Chebyshev polynomials

To show how Chebyshev was able to find the best approximating polynomial, we first need to know what the so called Chebyshev polynomials are.

The results in this chapter are derived from *Numerical Analysis* by R. L. Burden and J. Douglas Faires [9] and from *A short course on approximation theory* by N. L. Carothers [5].

Definition 2. *We denote the* ***Chebyshev polynomial*** *of degree n by $T_n(x)$ and it is defined as*

$$T_n(x) = \cos(n \arccos(x)), \quad \textit{for each} \quad n \geq 0.$$

This function looks trigonometric, and it is not clear from the definition that this defines a polynomial for each n. We will show that it indeed defines an algebraic polynomial.

$$\begin{aligned} &\text{For} \quad n = 0: \quad T_0(x) &&= \cos(0) = 1 \\ &\text{For} \quad n = 1: \quad T_1(x) &&= \cos(\arccos(x)) = x \end{aligned}$$

For $n \geq 1$, we use the substitution $\theta = \arccos(x)$ to change the equation to

$$T_n(\theta(x)) \equiv T_n(\theta) = \cos(n\theta), \quad \text{where} \quad \theta \in [0, \pi].$$

Then we can define a recurrence relation, using the fact that

$$T_{n+1}(\theta) = \cos((n+1)\theta) = \cos(\theta)\cos(n\theta) - \sin(\theta)\sin(n\theta)$$

and

$$T_{n-1}(\theta) = \cos((n-1)\theta) = \cos(\theta)\cos(n\theta) + \sin(\theta)\sin(n\theta).$$

If we add these equations, and use the variable $\theta = \arccos(x)$, we obtain

$$\begin{aligned} T_{n+1}(\theta) + T_{n-1}(\theta) &= 2\cos(n\theta)\cos(\theta) \\ T_{n+1}(\theta) &= 2\cos(n\theta)\cos(\theta) - T_{n-1}(\theta) \\ T_{n+1}(x) &= 2x\cos(n\arccos(x)) - T_{n-1}(x). \end{aligned}$$

That is,

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x). \quad (5.1)$$

Thus, the recurrence relation implies the following Chebyshev polynomials

$$\begin{array}{l} T_0(x) = 1 \\ T_1(x) = x \\ T_2(x) = 2xT_1(x) - T_0(x) = 2x^2 - 1 \\ T_3(x) = 2xT_2(x) - T_1(x) = 4x^3 - 3x \\ T_4(x) = 2xT_3(x) - T_2(x) = 8x^4 - 8x^2 + 1 \\ \dots \\ T_{n+1}(x) = 2xT_n(x) - T_{n-1(x)} \quad n \geq 1 \end{array}$$

Table 5.1: The Chebyshev polynomials (for a list of the first eleven Chebyshev polynomials see table C.1 in appendix C).

We see that if $n \geq 1$, $T_n(x)$ is a polynomial of degree n with leading coefficient 2^{n-1}.

In the next figure, the first five Chebyshev polynomials are shown.

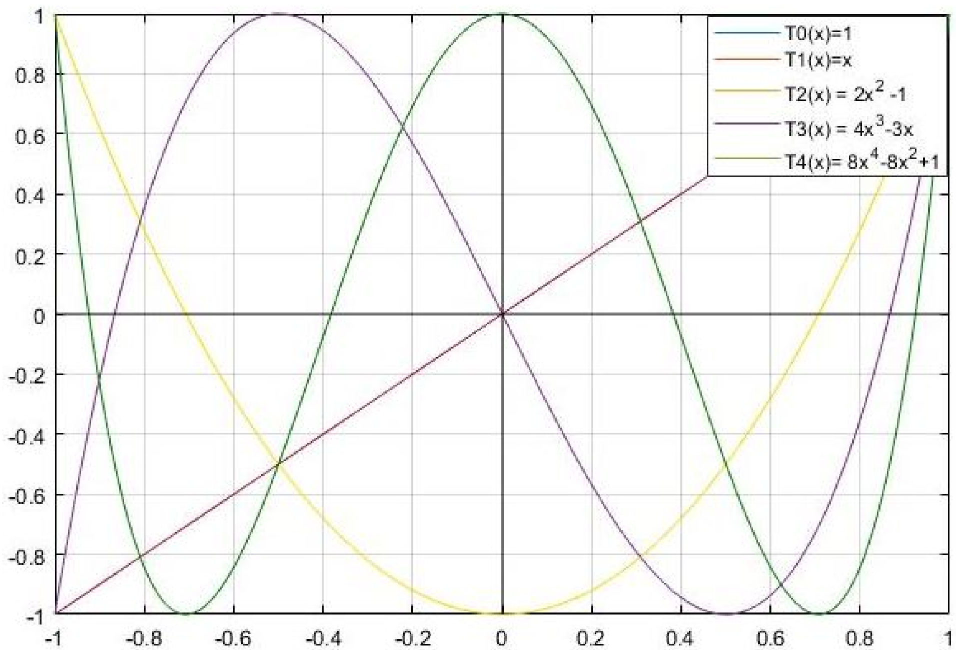

Figure 5.1: The first five Chebyshev polynomials.

5.1 Properties of the Chebyshev polynomials

The Chebyshev polynomials have a lot of interesting properties. A couple of them are listed below.

P 1: The Chebyshev polynomials are orthogonal on $(-1, 1)$ with respect to the weight function $w(x) = (1 - x^2)^{-\frac{1}{2}}$.

Proof. Consider

$$\int_{-1}^{1} \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}}\,dx = \int_{-1}^{1} \frac{\cos(n\arccos(x))\cos(m\arccos(x))}{\sqrt{1-x^2}}\,dx.$$

Using the substitution $\theta = \arccos(x)$, this gives

$$dx = -\sqrt{1-x^2}\,d\theta$$

and

$$\int_{-1}^{1} \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}}\,dx = -\int_{\pi}^{0} \cos(n\theta)\cos(m\theta)\,d\theta = \int_{0}^{\pi} \cos(n\theta)\cos(m\theta)\,d\theta.$$

Suppose $n \neq m$. Since

$$\cos(n\theta)\cos(m\theta) = \frac{1}{2}[\cos(n+m)\theta + \cos(n-m)\theta],$$

we have

$$\begin{aligned}\int_{-1}^{1} \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}}\,dx &= \frac{1}{2}\int_{0}^{\pi} \cos((n+m)\theta)\,d\theta + \frac{1}{2}\int_{0}^{\pi} \cos((n-m)\theta)\,d\theta \\ &= \left[\frac{1}{2(n+m)}\sin((n+m)\theta) + \frac{1}{2(n-m)}\sin((n-m)\theta)\right]_0^{\pi} = 0.\end{aligned}$$

Suppose $n = m$. Then

$$\begin{aligned}\int_{-1}^{1} \frac{[T_n(x)]^2}{\sqrt{1-x^2}}\,dx &= \int_{0}^{\pi} \cos^2(n\theta)\,d\theta \\ &= \left[\frac{2n\theta + \sin(2n\theta)}{4n}\right]_0^{\pi} = \begin{cases}\pi & \text{if} \quad n = 0 \\ \frac{\pi}{2} & \text{if} \quad n > 0.\end{cases}\end{aligned}$$

So we have

$$\int_{-1}^{1} \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}}\,dx = \begin{cases}0 & n \neq m \\ \frac{\pi}{2} & n = m \neq 0 \\ \pi & n = m = 0.\end{cases}$$

Hence we conclude that the Chebyshev polynomials are orthogonal with respect to the weight function $w = (1 - x^2)^{-\frac{1}{2}}$. □

P 2: The Chebyshev polynomial $T_n(x)$ of degree $n \geq 1$ has n simple zeros in $[-1, 1]$ at

$$\bar{x}_k = \cos\left(\frac{2k-1}{2n}\pi\right), \quad \text{for each} \quad k = 1, 2, \ldots, n.$$

Proof. Let

$$\bar{x}_k = \cos\left(\frac{2k-1}{2n}\pi\right).$$

Then

$$\begin{aligned} T_n(\bar{x}_k) &= \cos(n \arccos(\bar{x}_k)) \\ &= \cos\left(n \arccos\left(\cos\left(\frac{2k-1}{2n}\pi\right)\right)\right) \\ &= \cos\left(\frac{2k-1}{2}\pi\right) \\ &= 0. \end{aligned}$$

The $\bar{x}_k$ are distinct and $T_n(x)$ is a polynomial of degree n, so all the zeros must have this form. □

P 3: $T_n(x)$ assumes its absolute extrema at

$$\bar{x}'_k = \cos\left(\frac{k\pi}{n}\right), \quad \text{with} \quad T_n(\bar{x}'_k) = (-1)^k, \quad \text{for each} \quad k = 0, 1, \ldots, n.$$

Proof. Let

$$\bar{x}'_k = \cos\left(\frac{k\pi}{n}\right).$$

We have

$$\begin{aligned} T'_n(x) &= \frac{d}{dx}[\cos(n \arccos(x))] \\ &= \frac{n \sin(n \arccos(x))}{\sqrt{1-x^2}} \end{aligned}$$

and when $k = 1, 2, \ldots, n-1$ we have

$$\begin{aligned} T'_n(\bar{x}'_k) &= \frac{n \sin\left(n \arccos\left(\cos\left(\frac{k\pi}{n}\right)\right)\right)}{\sqrt{1 - \left[\cos\left(\frac{k\pi}{n}\right)\right]^2}} \\ &= \frac{n \sin(k\pi)}{\sin\left(\frac{k\pi}{n}\right)} \\ &= 0. \end{aligned}$$

Since $T_n(x)$ is of degree n, its derivative is of degree $n-1$. All zeros occur at these $n-1$ distinct points.

The other possibilities for extrema of $T_n(x)$ occur at the endpoints of the interval $[-1,1]$, so at $\bar{x}'_0 = 1$ and at $\bar{x}'_n = -1$.

For any $k = 0, 1, \dots, n$ we have

$$\begin{aligned} T_n(\bar{x}'_k) &= \cos\left(n \arccos\left(\frac{k\pi}{n}\right)\right) \\ &= \cos(k\pi) \\ &= (-1)^k. \end{aligned}$$

So we have a maximum at even values of k and a minimum at odd values of k. □

P 4: The monic Chebyshev polynomials (a polynomial of the form $x^n + c_{n-1}x^{n-1} + \cdots + c_2x^2 + c_1x + c_0$) $\widetilde{T}_n(x)$ are defined as

$$\widetilde{T}_0(x) = 1 \quad \text{and} \quad \widetilde{T}_n(x) = \frac{1}{2^{n-1}} T_n(x), \quad \text{for each} \quad n \geq 1.$$

The recurrence relation of the Chebyshev polynomials implies

$$\begin{aligned} \widetilde{T}_2(x) &= \quad x\widetilde{T}_1(x) - \tfrac{1}{2}\widetilde{T}_0(x) && \text{and} \\ \widetilde{T}_{n+1}(x) &= \quad x\widetilde{T}_n(x) - \tfrac{1}{4}\widetilde{T}_{n-1}(x) && \text{for each} \quad n \geq 2. \end{aligned}$$

Proof. We derive the monic Chebyshev polynomials by dividing the Chebyshev polynomials $T_n(x)$ by the leading coefficient 2^{n-1}. □

The first five monic Chebyshev polynomials are shown in figure 5.2.

P 5: The zeros of $\widetilde{T}_n(x)$ occur also at

$$\bar{x}_k = \cos\left(\frac{2k-1}{2n}\pi\right), \quad \text{for each} \quad k = 1, 2, \dots, n.$$

and the extrema of $\widetilde{T}_n(x)$ occur at

$$\bar{x}'_k = \cos\left(\frac{k\pi}{n}\right), \quad \text{with} \quad \widetilde{T}_n(\bar{x}'_k) = \frac{(-1)^k}{2^{n-1}}, \quad \text{for each} \quad n = 0, 1, 2, \dots, n.$$

Proof. This follows from the fact that $\widetilde{T}_n(x)$ is just a multiple of $T_n(x)$. □

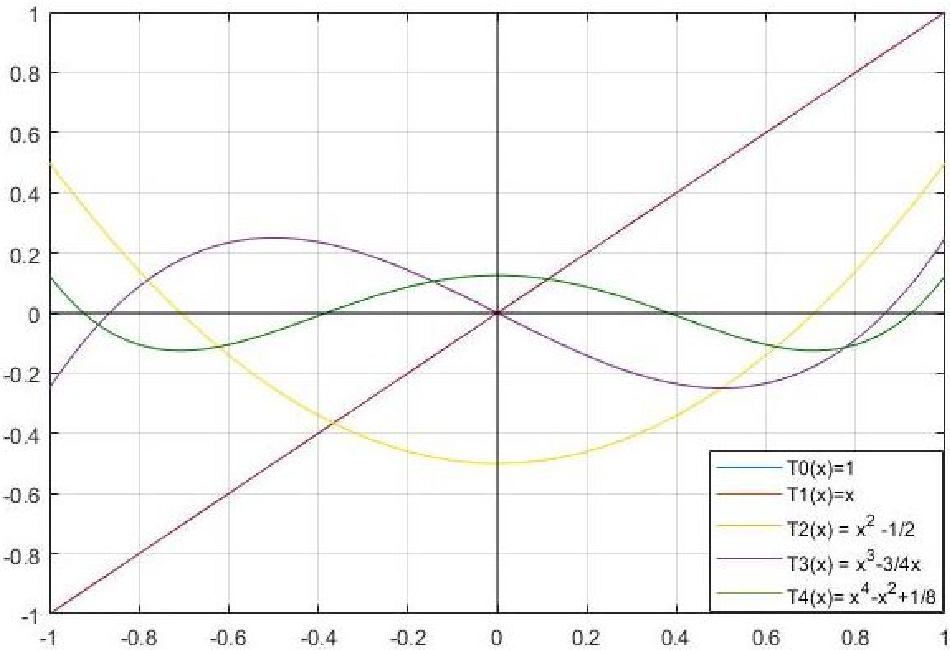

Figure 5.2: The first five monic Chebyshev polynomials.

P 6: Let $\widetilde{\prod}_n$ denote **the set of all monic poynomials of degree n**.

The polynomials of the form $\widetilde{T}_n(x)$, when $n \geq 1$, have the property that

$$\frac{1}{2^{n-1}} = \max_{x\in[-1,1]} |\widetilde{T}_n(x)| \leq \max_{x\in[-1,1]} |P_n(x)|, \quad \text{for all} \quad P_n(x) \in \widetilde{\prod}_n.$$

The equality only occurs if $P_n \equiv \widetilde{T}_n$.

Proof. This is a proof by contradiction. Therefore, suppose that $P_n(x) \in \widetilde{\prod}_n$ and that

$$\max_{x\in[-1,1]} |P_n(x)| \leq \frac{1}{2^{n-1}} = \max_{x\in[-1,1]} |\widetilde{T}_n(x)|.$$

We want to show that this does not hold. Let $Q = \widetilde{T}_n - P_n$. Since $\widetilde{T}_n$ and P_n are both monic polynomials of degree n, Q is a polynomial of degree at most $n-1$.

At the $n+1$ extreme points $\bar{x}'_k$ of $\widetilde{T}_n$, we have

$$Q(\bar{x}'_k) = \widetilde{T}_n(\bar{x}'_k) - P_n(\bar{x}'_k) = \frac{(-1)^k}{2^{n-1}} - P_n(\bar{x}'_k).$$

From our assumption we have

$$|P_n(\bar{x}'_k)| \leq \frac{1}{2^{n-1}} \quad \text{for each} \quad k = 0, 1, \ldots, n,$$

so we have

$$\begin{cases} Q(\bar{x}'_k) \leq 0 & \text{when } k \text{ is odd} \\ Q(\bar{x}'_k) \geq 0 & \text{when } k \text{ is even.} \end{cases}$$

Since Q is continuous, we can apply the Intermediate Value Theorem. This theorem implies that for each $j = 0, 1, \ldots, n-1$ the polynomial $Q(x)$ has at least one zero between $\bar{x}'_j$ and $\bar{x}'_{j+1}$. Thus, Q has at least n zeros in the interval $[-1, 1]$. But we have that the degree of $Q(x)$ is less than n, so we must have $Q \equiv 0$. This implies that $P_n \equiv \widetilde{T}_n$, which is a contradiction. □

P 7: $T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$ for $n \geq 2$.

Proof. This is the same as equation 5.1, but then with $n+1$ replaced by n. □

P 8: $T_m(x)T_n(x) = \frac{1}{2}\left[T_{m+n} + T_{m-n}\right]$ for $m > n$.

Proof. Using the trig identity $\cos(a)\cos(b) = \frac{1}{2}[\cos(a+b) + \cos(a-b)]$, we get

$$\begin{aligned} T_m(x) \cdot T_n(x) &= \cos(m \arccos(x)) \cos(n \arccos(x)) \\ &= \frac{1}{2}[\cos((m+n)\arccos(x)) + \cos((m-n)\arccos(x))] \\ &= \frac{1}{2}[T_{m+n}(x) + T_{m-n}(x)]. \end{aligned}$$

□

P 9: $T_m(T_n(x)) = T_{mn}(x)$.

Proof.

$$\begin{aligned} T_m(T_n(x)) &= \cos(m \arccos(\cos(n \arccos(x)))) \\ &= \cos(mn \arccos(x)) \\ &= T_{mn}(x). \end{aligned}$$

□

P 10: $T_n(x) = \frac{1}{2}\left[(x + \sqrt{x^2-1})^n + (x - \sqrt{x^2-1})^n\right]$.

Proof. Combining the binomial expansions of the right-hand side, makes the odd powers of $\sqrt{x^2-1}$ cancel. Thus, the right-hand side is a polynomial as well.

Let $x = \cos(\theta)$. Using the trig identity $\cos(x)^2 + \sin(x)^2 = 1$ we find

$$\begin{aligned} T_n(x) = T_n(\cos(\theta)) = \cos(n\theta) &= \frac{1}{2}(e^{in\theta} + e^{-in\theta}) \\ &= \frac{1}{2}\left[(\cos(\theta) + i\sin(\theta))^n + (\cos(\theta) - i\sin(\theta))^n\right] \\ &= \frac{1}{2}\left[(x + i\sqrt{1-x^2})^n + (x - i\sqrt{1-x^2})^n\right] \\ &= \frac{1}{2}\left[(x + \sqrt{x^2-1})^n + (x - \sqrt{x^2-1})^n\right], \end{aligned}$$

which is the desired result. These polynomials are thus equal for $|x| \leq 1$. □

P 11: For real x with $|x| > 1$, we get

$$\frac{1}{2}\left[(x + \sqrt{x^2-1})^n + (x - \sqrt{x^2-1})^n\right] = \cosh(n\cosh^{-1}(x)).$$

Thus,

$$T_n(\cosh(x)) = \cosh(nx) \quad \text{for all real} \quad x.$$

Proof. This follows from property 10. □

P 12: $T_n(x) \leq (|x| + \sqrt{x^2-1})^n$ for $|x| > 1$.

Proof. This follows from property 10. □

P 13: For n odd,

$$2^n x^n = \sum_{k=0}^{[n/2]} \binom{n}{k} 2T_{n-2k}(x).$$

For n even, $2T_0$ is replaced by T_0.

Proof. For $|x| \leq 1$, let $x = \cos(\theta)$. Using the binomial expansion we get

$$
\begin{aligned}
2^n x^n &= 2^n \cos^n(\theta) \\
&= (e^{i\theta} + e^{-i\theta})^n \\
&= e^{in\theta} + \binom{n}{1} e^{i(n-2)\theta} + \binom{n}{2} e^{i(n-4)\theta} + \cdots + \\
&+ \binom{n}{n-2} e^{-i(n-4)\theta} + \binom{n}{n-1} e^{-i(n-2)\theta} + e^{-in\theta} \\
&= 2\cos(n\theta) + \binom{n}{1} 2\cos((n-2)\theta) + \binom{n}{2} 2\cos((n-4)\theta) + \dots \\
&= 2T_n(x) + \binom{n}{1} 2T_{n-2}(x) + \binom{n}{2} 2T_{n-4}(x) + \dots \\
&= \sum_{k=0}^{[n/2]} \binom{n}{k} 2T_{n-2k}(x).
\end{aligned}
$$

If n is even, the last term in this last sum is $\binom{n}{n/2} T_0$ (because then the central term in the binomial expansion is not doubled). □

P 14: T_n and T_{n-1} have no common zeros.

Proof. Assume they do have a common zero. Then $T_n(x_0) = 0 = T_{n-1}(x_0)$. But then using property 7, we find that $T_{n-2}(x_0)$ must be zero too. If we repeat this, we find $T_k(x_0) = 0$ for every $k < n$, including $k = 0$. This is not possible, since $T_0(x) = 1$ has no zeros. Therefore, we conclude that T_n and T_{n-1} have no common zeros. □

P 15: $|T_n'(x)| \leq n^2$ for $|x| \leq 1$ and $|T_n'(\pm 1)| = n^2$.

Proof.

$$
\frac{d}{dx} T_n(x) = \frac{\frac{d}{d\theta} T_n(\cos(\theta))}{\frac{d}{d\theta} \cos(\theta)} = \frac{n \sin(n\theta)}{\sin(\theta)}.
$$

For $|x| \leq 1$, $|T_n'(x)| \leq n^2$, because $|\sin(n\theta)| \leq n|\sin(\theta)|$.
For $x = \pm 1$, $|T_n'(\pm)| = n^2$, because we can interpret the derivative as a limit: let $\theta \to 0$ and $\theta \to \pi$. Using the L'Hôpital rule we find $|T_n'(\pm 1)| = n^2$. □

Chapter 6

How to find the best approximating polynomial in the uniform norm

In this chapter we first show how Chebyshev was able to solve an approximation problem in the uniform norm. We will then compare this to approximation in the L^2-norm. After this, we will give some other techniques and utilities to find the best approximating function. We close this chapter with some examples.

6.1 Chebyshev's solution

In this section we show step by step an approximation problem that Chebyshev was able to solve. This section is derived from *A short course on approximation theory* by N. L. Carothers [5] and from *Best Approximation: Minimax Theory* by S. Ghorai [10].

The problem that Chebyshev wanted to solve is the following
Problem 3. *Find the best approximating polynomial $p_{n-1} \in \mathcal{P}_{n-1}$ of degree at most $n-1$ of $f(x) = x^n$ on the interval $[-1, 1]$.*

This means we want to minimize the error between $f(x) = x^n$ and $p_{n-1}(x)$, thus minimize $\max_{x \in [-1,1]} |x^n - p_{n-1}|$. Hence, we can restate the problem in the following way: *Find the monic polynomial of degree n of smallest norm in $C[-1, 1]$.*

We show Chebyshev's solution in steps.

Step 1: Simplify the notation. Let $E(x) = x^n - p$ and let $M = \|E\|_\infty$. We know that $E(x)$ has an alternating set: $-1 \leq x_0 < x_1 < \cdots < x_n \leq 1$ containing $(n-1)+2 = n+1$ points and $E(x)$ has at least $(n-1)+1 = n$ zeros. So $|E(x_i)| = M$ and $E(x_{i+1}) = -E(x_i)$ for all i.

Step 2: $E(x_i)$ is a relative extreme value for $E(x)$, so at any x_i in $(-1,1)$ we have $E'(x_i) = 0$. $E'(x_i)$ is a polynomial of degree $n-1$, so it has at most $n-1$ zeros. Thus,

$$\begin{aligned} x_i \in (-1,1) \quad \text{and} \quad E'(x_i) &= 0 \qquad \text{for} \quad i = 1,\ldots,n-1, \\ x_0 = -1, \quad E'(x_0) &\neq 0, \qquad x_n = 1, \quad E'(x_n) \neq 0. \end{aligned}$$

Step 3: Consider the polynomial $M^2 - E^2 \in \mathcal{P}_{2n}$. $M^2 - (E(x_i))^2 = 0$ for $i = 0, 1, \ldots, n$, and $M^2 - E^2 \geq 0$ on $[-1,1]$. Thus, $x_1, \ldots, x_{n-1}$ must be double roots of $M^2 - E^2$. So we already have $2(n-1)+2 = 2n$ roots. This means that $x_1, \ldots, x_{n-1}$ are double roots and that x_0 and x_n are simple roots. These are all the roots of $M^2 - E^2$.

Step 4: The next step is to consider $(E'(x))^2 \in \mathcal{P}_{2(n-1)}$. We already know from the previous steps that $x_1, \ldots x_{n-1}$ are double roots of $(E'(x))^2$. Hence $(1-x^2)(E'(x))^2$ has double roots at $x_1, \ldots, x_{n-1}$ and simple roots at x_0 and x_n. These are all the roots, since $(1-x^2)(E'(x))^2$ is in $\mathcal{P}_{2n}$.

Step 5: In the previous steps we found that $M^2 - E^2$ and $(1-x^2)(E'(x))^2$ are polynomials of the same degree and with the same roots. This means that these polynomials are the same, up to a constant multiple. We can calculate this constant. Since $E(x)$ is a monic polynomial, it has leading coefficient equal to 1. The derivative $E'(x)$ has thus leading coefficient equal to n. Thus,

$$\begin{aligned} M^2 - (E(x))^2 &= \frac{(1-x^2)(E'(x))^2}{n^2} \\ \frac{E'(x)}{\sqrt{M^2 - (E(x))^2}} &= \frac{n}{\sqrt{1-x^2}}. \end{aligned}$$

E' is positive on some interval, so we can assume that it is positive on $[-1, x_1]$ and therefore we do not need the $\pm$-sign. If we integrate our result, we get

$$\begin{aligned} \arccos\left(\frac{E(x)}{M}\right) &= n\arccos(x) + C \\ E(x) &= M\cos(n\arccos(x) + C). \end{aligned}$$

Since $E'(-1) \geq 0$, we have that $E(-1) = -M$. So if we substitute this we get

$$\begin{aligned}E(-1) = M\cos(n\arccos(-1)+C) &= -M\\ \cos(n\pi + C) &= -1\\ n\pi + C &= \pi + k2\pi\\ C &= m\pi \quad \text{with } n+m \text{ odd}\\ E(x) &= \pm M\cos(n\arccos(x)).\end{aligned}$$

From the previous chapter we know that $\cos(n\arccos(x))$ is the n-th Chebyshev polynomial. Thus, it has degree n and leading coefficient 2^{n-1}. Hence, the solution to problem 1 is

$$E(x) = 2^{-n+1}T_n(x).$$

We know that $|T_n(x)| \leq 1$ for $|x| < 1$, so the minimal norm is $M = 2^{-n+1}$.

Using theorem 4 and the characteristics of the Chebyshev polynomials, we can give a fancy solution.

Theorem 6. *For any $n \geq 1$, the formula $p(x) = x^n - 2^{-n+1}T_n(x)$ defines a polynomial $p \in \mathcal{P}_{n-1}$ satisfying*

$$2^{-n+1} = \max_{|x|\leq 1}|x^n - p(x)| < \max_{|x|\leq 1}|x^n - q(x)|$$

for any $q \in \mathcal{P}_{n-1}$.

Proof. We know that $2^{-n+1}T_n(x)$ has leading coefficient 1, so $p \in \mathcal{P}_{n-1}$.

Let $x_k = \cos\left((n-k)\frac{\pi}{n}\right)$ for $k = 0, 1, \ldots, n$. Then, $-1 = x_0 < x_1 < \cdots < x_n = 1$ and

$$\begin{aligned}T_n(x_k) &= T_n(\cos((n-k)\frac{\pi}{n}))\\ &= \cos((n-k)\pi)\\ &= (-1)^{n-k}.\end{aligned}$$

We have that $|T_n(x)| = |T_n(\cos(\theta))| = |\cos(n\theta)| \leq 1$ for $-1 \leq x \leq 1$.

This means that we have found an alternating set for $T_n(x)$ containing $n+1$ points.

So we have that $x^n - p(x) = 2^{-n+1}T_n(x)$ satisfies $|x^n - p(x)| \leq 2^{-n+1}$, and for each $k = 0, 1, \ldots, n$ has $x_k^n - p(x_k) = 2^{-n+1}T_n(x_k) = (-1)^{n-k}2^{-n+1}$.

Using theorem 4, we find that $p(x)$ must be the best approximating polynomial to x^n out of $\mathcal{P}_{n-1}$. □

Corollary 2. *The monic polynomial of degree exactly n having smallest norm in $C[a,b]$ is*

$$\frac{(b-a)^n}{2^n 2^{n-1}} \cdot T_n\left(\frac{2x-b-a}{b-a}\right).$$

Proof. Let $p(x)$ be the monic polynomial of degree n. Make the transformation $2x = (a+b)+(b-a)t$. Then we have $\tilde{p}(t) = p(\frac{a+b}{2} + \frac{b-a}{2}t)$. This is a polynomial of degree n with leading coefficient $\frac{(b-a)^n}{2^n}$, and

$$\max_{a\le x\le b} |p(x)| = \max_{-1\le t\le 1} |\tilde{p}(t)|.$$

We can write

$$\tilde{p}(t) = \frac{(b-a)^n}{2^n}\hat{p}(t),$$

where $\hat{p}(t)$ is a monic polynomial of degree n in $[-1,1]$ and

$$\max_{a\le x\le b} |p(x)| = \frac{(b-a)^n}{2^n} \max_{-1\le t\le 1} |\hat{p}(t)|.$$

Hence, for minimum norm, we must have $\hat{p}(t) = 2^{1-n}T_n(t)$.
Combining the results and substituting back $t = \frac{2x-a-b}{a-b}$ we get

$$\begin{aligned}\tilde{p} &= \frac{(b-a)^n}{2^n}2^{1-n}T_n(t)\\ &= \frac{(b-a)^n}{2^n 2^{n-1}} \cdot T_n\left(\frac{2x-a-b}{a-b}\right).\end{aligned}$$

□

We can generalize theorem 6 and corollary 2 to find the best approximating polynomial for functions $f(x)$ of degree n with leading term a_0x^n, by a polynomial $p(x)$ of degree at most $n-1$ in $[-1,1]$.

Corollary 3 (Generalizations)**.** *Given a function $f(x)$ of degree n with leading term a_0x^n,*

1. *the best approximation to $f(x)$ by a polynomial $p(x)$ of degree at most $n-1$ in $[-1,1]$ is such that*

$$p(x) = f(x) - a_0 2^{-n+1}T_n(x),$$

   *with maximum error $a_0 2^{-n+1}$. To refer to this technique, we will call this technique **Chebyshev approximation**.*

2. *the polynomial of degree exactly n having smallest norm in $C[a,b]$ is*

$$a_0 \cdot \frac{(b-a)^n}{2^n 2^{n-1}} \cdot T_n\left(\frac{2x-b-a}{b-a}\right).$$

   *To refer to this technique, we will call this technique **smallest norm**.*

So we now found two ways to calculate the best approximating polynomial $p(x) \in \mathcal{P}_n$ for functions with leading term $a_0 x^n$. Either it is found using Chebyshev approximation by the formula $p(x) = f(x) - a_0 \cdot 2^{-n+1} T_n(x)$ or by finding the polynomial of degree n having smallest norm in $C[a, b]$ by the formula $a_0 \cdot \frac{(b-a)^n}{2^n 2^{n-1}} \cdot T_n \left(\frac{2x-b-a}{b-a} \right)$.

In section 6.3 we will encounter two other techniques to find the best approximating polynomial.

6.2 Comparison to approximation in the L^2-norm

In this section we compare approximation in the L^2-norm to approximation in the uniform norm.

6.2.1 Differences between approximation in the L^2-norm and in the uniform norm

This subsection shows the *differences* between approximation in the L^2-norm and in the uniform norm.

This subsection is based on *Introduction to numerical analysis 1* by D. Levy [11] and on *A choice of norm in discrete approximation* by T. Marošević [3].

Approximation in the uniform norm and in the L^2-norm mean two different things:

- *Uniform norm*: minimize the largest absolute deviation, thus minimize $\|f(x) - p(x)\|_\infty = \max_{a \leq x \leq b} |f(x) - p(x)|$.
- *L^2-norm*: minimize the integral of the square of absolute deviations, thus minimize $\|f(x) - p(x)\|_2 = \sqrt{\int_a^b |f(x) - p(x)|^2 dx}$.

The question is now, how important is the choice of norm for the solution of the approximation problem?

The value of the norm of a function can vary substantially based on the function as well as the choice of norm.

If $\|f\|_\infty < \infty$, then

$$\|f\|_2 = \sqrt{\int_a^b |f|^2 dx} \leq (b - a)\|f\|_\infty.$$

However, it is also possible to construct a function with a small $\|f\|_2$ and a large $\|f\|_\infty$. Thus, the choice of norm has a significant impact on the solution of the approximation problem.

6.2.2 Similarity between approximation in the L^2-norm and in the uniform norm

This subsection shows a possible similarity between approximation in the L^2-norm and in the uniform norm.

In chapter 5 we saw that the Chebyshev polynomials are orthogonal on $(-1, 1)$ with respect to the weight function $w(x) = \frac{1}{\sqrt{1-x^2}}$.

On this interval, we assume that the best approximating polynomial in the L^2-norm with inner product $\langle f, g\rangle = \int_{-1}^{1} \frac{f(x)g(x)}{\sqrt{1-x^2}}$ and the uniform norm are the same.

It is important to note that we assume that this holds only for algebraic polynomials. We know that this does not hold for other types of functions, because in general

$$\int_{-1}^{1} \frac{f(x)g(x)}{\sqrt{1-x^2}} dx \neq \max_{x\in[-1,1]} |f(x) - g(x)|.$$

We illustrate that we get the same best approximating polynomial in the two norms for the function $f(x) = x^n$ on $[-1, 1]$.

We start with the basis $\{1, x, x^2, \ldots, x^n\}$ for the inner product space V and apply the Gram-Schmidt Process.

$$\begin{aligned}
u_1 &= \left(\tfrac{1}{\|x_1\|}\right) x_1 &&= \frac{1}{\sqrt{\pi}} \\
p_1 &= \langle x_2, u_1\rangle u_1 = \langle x, \tfrac{1}{\sqrt{\pi}}\rangle \tfrac{1}{\sqrt{\pi}} &&= 0 \\
u_2 &= \tfrac{1}{\|x_2 - p_1\|}(x_2 - p_1) = \tfrac{1}{\|x\|} x &&= \frac{\sqrt{2}}{\sqrt{\pi}} x \\
p_2 &= \langle x_3, u_1\rangle u_1 + \langle x_3, u_2\rangle u_2 = \langle x^2, \tfrac{1}{\sqrt{\pi}}\rangle \tfrac{1}{\sqrt{\pi}} + \langle x^2, \tfrac{\sqrt{2}}{\sqrt{\pi}} x\rangle \tfrac{\sqrt{2}}{\sqrt{\pi}} x &&= \frac{1}{2} \\
u_3 &= \tfrac{1}{\|x_3 - p_2\|}(x_3 - p_2) = \tfrac{1}{\|x^2 - \frac{1}{2}\|}(x^2 - \tfrac{1}{2}) &&= \frac{2\sqrt{2}}{\sqrt{\pi}}(x^2 - \frac{1}{2}).
\end{aligned}$$

Calculating the other terms gives the corresponding orthonormal system

$$\{\frac{1}{\sqrt{\pi}}, \frac{\sqrt{2}}{\sqrt{\pi}} T_1, \frac{\sqrt{2}}{\sqrt{\pi}} T_2, \ldots, \frac{\sqrt{2}}{\sqrt{\pi}} T_n\}.$$

Then,

$$p(x) = \frac{\langle f, u_1\rangle}{\langle u_1, u_1\rangle} u_1(x) + \frac{\langle f, u_2\rangle}{\langle u_2, u_2\rangle} u_2(x) + \cdots + \frac{\langle f, u_n\rangle}{\langle u_n, u_n\rangle} u_n(x).$$

Since the inner products $\langle u_i, u_i \rangle = 1$ for $i = 1, \ldots, n$, this reduces to

$$p(x) = \langle f, \frac{1}{\sqrt{\pi}} \rangle \frac{1}{\sqrt{\pi}} + \langle f, \frac{\sqrt{2}}{\sqrt{\pi}} T_1 \rangle \frac{\sqrt{2}}{\sqrt{\pi}} T_1 + \cdots + \langle f, \frac{\sqrt{2}}{\sqrt{\pi}} T_n \rangle \frac{\sqrt{2}}{\sqrt{\pi}} T_n.$$

So we have to calculate each inner product:

$$\begin{aligned}
\langle f, u_1 \rangle = &\quad \tfrac{1}{\sqrt{\pi}} \textstyle\int_{-1}^{1} \frac{x^n}{\sqrt{(1-x^2)}} dx = dx &&= \frac{((-1)^n + 1)\Gamma(\frac{n+1}{2})}{n\Gamma(\frac{n}{2})} \\
\langle f, u_2 \rangle = &\quad \tfrac{\sqrt{2}}{\sqrt{\pi}} \textstyle\int_{-1}^{1} \frac{x^{n+1}}{\sqrt{(1-x^2)}} dx &&= -\frac{((-1)^n - 1)n\Gamma(\frac{n}{2})}{2\sqrt{2}\Gamma(\frac{n+3}{2})} \\
\langle f, u_3 \rangle = &\quad \tfrac{\sqrt{2}}{\sqrt{\pi}} \textstyle\int_{-1}^{1} \frac{x^n(2x^2-1)}{\sqrt{(1-x^2)}} dx &&= \frac{((-1)^n + 1)n\Gamma(\frac{n+1}{2})}{2\sqrt{2}\Gamma(\frac{n}{2} + 2)} \\
\langle f, u_4 \rangle = &\quad \tfrac{\sqrt{2}}{\sqrt{\pi}} \textstyle\int_{-1}^{1} \frac{x^n(4x^3-3x)}{\sqrt{(1-x^2)}} dx &&= -\frac{((-1)^n - 1)(n-1)\Gamma(\frac{n}{2} + 1)}{2\sqrt{2}\Gamma(\frac{n+5}{2})} \\
\langle f, u_5 \rangle = &\quad \tfrac{\sqrt{2}}{\sqrt{\pi}} \textstyle\int_{-1}^{1} \frac{x^n(8x^4-8x^2+1)}{\sqrt{(1-x^2)}} dx &&= \frac{((-1)^n + 1)(n-2)n\Gamma(\frac{n+1}{2})}{4\sqrt{2}\Gamma(\frac{n}{2} + 3)} \\
\langle f, u_6 \rangle = &\quad \tfrac{\sqrt{2}}{\sqrt{\pi}} \textstyle\int_{-1}^{1} \frac{x^n(16x^5-20x^3+5x)}{\sqrt{(1-x^2)}} dx &&= -\frac{((-1)^n - 1)(n-3)(n-1)\Gamma(\frac{n}{2} + 1)}{4\sqrt{2}\Gamma(\frac{n+7}{2})} \\
\cdots\cdots\cdots &\quad \cdots\cdots\cdots\cdots\cdots\cdots &&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

Thus,

$$\begin{aligned}
p(x) = &\frac{((-1)^n + 1)\Gamma(\frac{n+1}{2})}{\sqrt{\pi}n\Gamma(\frac{n}{2})} - \frac{((-1)^n - 1)n\Gamma(\frac{n}{2})}{2\sqrt{\pi}\Gamma(\frac{n+3}{2})} x + \frac{((-1)^n + 1)n\Gamma(\frac{n+1}{2})}{2\sqrt{\pi}\Gamma(\frac{n}{2} + 2)} (2x^2 - 1) - \\
&- \frac{((-1)^n - 1)(n-1)\Gamma(\frac{n}{2} + 1)}{2\sqrt{\pi}\Gamma(\frac{n+5}{2})} (4x^3 - 3x) + \\
&+ \frac{((-1)^n + 1)(n-2)n\Gamma(\frac{n+1}{2})}{4\sqrt{\pi}\Gamma(\frac{n}{2} + 3)} (8x^4 - 8x^2 + 1) - \\
&- \frac{((-1)^n - 1)(n-3)(n-1)\Gamma(\frac{n}{2} + 1)}{4\sqrt{\pi}\Gamma(\frac{n+7}{2})} (16x^5 - 20x^3 + 5x) + \ldots
\end{aligned}$$

The best approximating polynomials for the first six powers of x are shown in table 6.1.

These are exactly the same best approximating polynomials as we would find using Chebyshev approximation. So if we want to find the best approximating polynomial for an algebraic polynomial in $[-1, 1]$, we assume that we can just apply the Gram-Schmidt Process with inner product $\langle f, g \rangle = \int_{-1}^{1} \frac{f(x)g(x)}{\sqrt{1-x^2}}$. If we want to approximate a function in a different interval, it is recommended to use Chebyshev approximation, since this gives usually a better approximation than approximation in the L^2-norm.

$n =$	$f(x) =$	Basis	$p(x) =$
1	x	$\{1\}$	0
2	x^2	$\{1, x\}$	$\frac{1}{2}$
3	x^3	$\{1, x, x^2\}$	$\frac{3}{4}x$
4	x^4	$\{1, x, x^2, x^3\}$	$x^2 - \frac{1}{4}$
5	x^5	$\{1, x, x^2, x^3, x^4\}$	$\frac{5}{4}x^3 - \frac{5}{16}x$
6	x^6	$\{1, x, x^2, x^3, x^4, x^5\}$	$\frac{3}{2}x^4 - \frac{9}{16}x^2 + \frac{1}{32}$

Table 6.1: The best approximating polynomial for the first six powers of x using the approximation in the L^2-norm.

In appendix D we show using an example that we find the same best approximating polynomial in the L^2-norm with inner product $\langle f, g \rangle = \int_{-1}^{1} \frac{f(x)g(x)}{\sqrt{1-x^2}}$ and in the uniform norm for another algebraic polynomial.

In this appendix we also show that this does not hold for functions other than algebraic polynomials.

6.3 Other techniques and utilities

In this section we will list two other techniques to find the best approximating polynomial and show why these techniques are useful, we show how we can find the solution for a different domain and we give the generating functions for $T_n(x)$ and x^n.

The six subsections follow from the book *Chebyshev Polynomials in Numerical Analysis* by L. Fox and I. B. Parker [12].

6.3.1 Economization

In table 5.1, we listed the first four Chebyshev polynomials. We can reverse the procedure, and express the powers of x in terms of the Chebyshev polynomials $T_n(x)$.

$$
\begin{aligned}
1 &= T_0 \\
x &= T_1 \\
x^2 &= \tfrac{1}{2}(T_0 + T_2) \\
x^3 &= \tfrac{1}{4}(3T_1 + T_3) \\
x^4 &= \tfrac{1}{8}(3T_0 + 4T_2 + T_4)
\end{aligned}
$$

Table 6.2: The powers of x in terms of the Chebyshev polynomials (for the first eleven powers of x see table E.1 in appendix E).

Writing the powers of x in terms of Chebyshev polynomials has some advantages for numerical computation. The power and the error of the approximation are immediately visible. Moreover, the Chebyshev form has much smaller coefficients and therefore a greater possibility for significant truncation with little error.

Example 3. *In this example we want to approximate the function* $f(x) = 1 - x + x^2 - x^3 + x^4$ *in the interval* $[-1, 1]$ *by a cubic polynomial* $p(x) \in \mathcal{P}_3$*. This can be done by writing each power of* x *in terms of Chebyshev polynomials. So we get*

$$\begin{aligned} 1 - x + x^2 - x^3 + x^4 &= T_0 - T_1 + \frac{1}{2}(T_0 + T_2) - \frac{1}{4}(3T_1 + T_3) + \frac{1}{8}(3T_0 + 4T_2 + T_4) \\ &= \frac{15}{8}T_0 - \frac{7}{4}T_1 + T_2 - \frac{1}{4}T_3 + \frac{1}{8}T_4. \end{aligned}$$

We want a cubic approximation, so we just drop the T_4 *term. So the best approximating function is* $p = \frac{7}{8} - x + 2x^2 - x^3$*. This gives an error at most* $\frac{1}{8}$*. If we did not use this technique, but just used the cubic approximation* $q(x) = 1 - x + x^2 - x^3$*, we would get a maximum error equal to* 1 *(see figure 6.1). This technique of writing the powers of* x *in terms of the Chebyshev polynomials is called* ***economization****.*

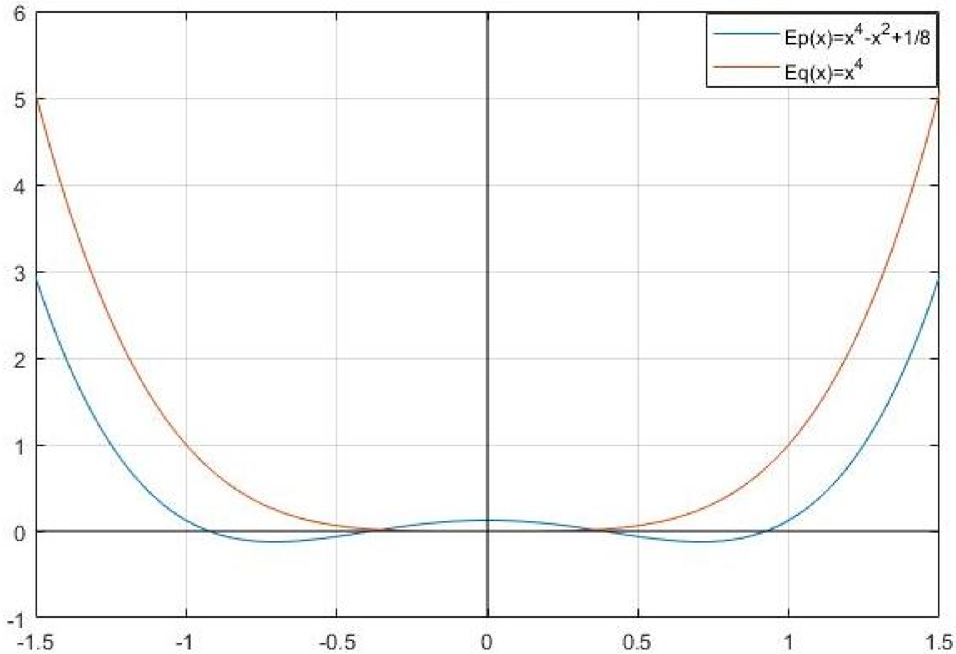

Figure 6.1: Illustration of the errors of the best approximating polynomial $p(x) = \frac{7}{8} - x + 2x^2 - x^3$ and of $q(x) = 1 - x + x^2 - x^3$.

How does economization work? The function $f(x)$ is a power series or can be expressed as a power series, so $f(x) \approx P_n(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots +$

$a_1x + a_0$. We want to approximate this by a lower degree polynomial, P_{n-1}. So the n-th degree polynomial Q_n with coefficient a_n in front of x^n will be removed. Then

$$\begin{aligned}\max|f(x) - P_{n-1}(x)| &= \max|f(x) - (P_n(x) - Q_n(x))| \\ &\le \max|f(x) - P_n(x)| + \max|Q_n(x)|,\end{aligned}$$

where the first term on the right-hand side is zero if $f(x)$ has the form of a power series. The loss of accuracy $Q_n(x) = \frac{a_n}{2^{n-1}}$.

6.3.2 Transformation of the domain

In the previous example we used the interval $[-1, 1]$. In some problems, we have a different domain. The good thing is that we can transform any finite domain $a \le y \le b$ to the basic domain $-1 \le x \le 1$ with the change of variable

$$y = \frac{1}{2}(b-a)x + \frac{1}{2}(b+a).$$

For the special domain $0 \le y \le 1$, we can write

$$y = \frac{1}{2}(x+1), \quad x = 2y - 1.$$

For this domain, we denote the Chebyshev polynomial by $T_n^\star(x) = T_n(2x - 1)$, $0 \le x \le 1$. Then we have the following Chebyshev polynomials and powers of x:

$T_0^\star(x) = 1$	$1 = T_0^\star$
$T_1^\star(x) = 2x - 1$	$x = \frac{1}{2}(T_0^\star + T_1^\star)$
$T_2^\star(x) = 8x^2 - 8x + 1$	$x^2 = \frac{1}{8}(3T_0^\star + 4T_1^\star + T_2^\star)$
$T_3^\star(x) = 32x^3 - 48x^2 + 18x - 1$	$x^3 = \frac{1}{32}(10T_0^\star + 15T_1^\star + 6T_2^\star + T_3^\star)$

Table 6.3: The Chebyshev polynomials and powers of x for the domain $[0, 1]$.

These formulas can be found by replacing x by $2x - 1$ or by using the recurrence relation $T_{n+1}^\star(x) = 2(2x-1)T_n^\star(x) - T_{n-1}^\star(x)$. In a smaller domain, the error is smaller than in a larger domain.

Example 4. *Consider the function $f(x) = x^3 + x^2$. First, we want to approximate this function with a quadratic polynomial in the interval $[-1, 1]$. We do this using economization.*

$$\begin{aligned}x^3 + x^2 &= \frac{1}{4}(3T_1 + T_3) + \frac{1}{2}(T_0 + T_2) \\ &= \frac{1}{2}T_0 + \frac{3}{4}T_1 + \frac{1}{2}T_2 + \frac{1}{4}T_3.\end{aligned}$$

We drop the T_3 term. We get a maximum error of $\frac{1}{4}$. The best approximating polynomial is $p(x) = x^2 + \frac{3}{4}x$.

Next, we are going to approximate the function $f(x)$ with a quadratic polynomial, but now in the interval $[0, 1]$. As said before, we expect a smaller error.

$$\begin{aligned} x^3 + x^2 &= \frac{1}{32}(10T_0^\star + 15T_1^\star + 6T_2^\star + T_3^\star) + \frac{1}{8}(3T_0^\star + 4T_1^\star + T_2^\star) \\ &= \frac{11}{16}T_0^\star + \frac{31}{31}T_1^\star + \frac{5}{16}T_2^\star + \frac{1}{32}T_3^\star \end{aligned}$$

Again, we drop the $T_3^\star$ term. We get a maximum error of $\frac{1}{32}$. The best approximating polynomial is $p(x) = \frac{1}{32} - \frac{9}{16}x + \frac{5}{2}x^2$.

Finally, we are going to approximate the function $f(x)$ with a quadratic polynomial in the interval $[-3, 3]$. We expect the largest error.

First, we need a change of variable

$$\begin{aligned} y &= \frac{1}{2}(b-a)x + \frac{1}{2}(b+a) \\ &= \frac{1}{2}(3+3)x + \frac{1}{2}(3-3) \\ &= 3x, \quad x = \frac{1}{3}y. \end{aligned}$$

The first four Chebyshev polynomials and powers of x become

$$\begin{array}{ll} T_0^* = 1 & 1 = T_0^* \\ T_1^* = \frac{1}{3}x & x = 3T_1^* \\ T_2^* = \frac{2}{9}x^2 - 1 & x^2 = \frac{9}{2}T_2^* + \frac{9}{2}T_0^* \\ T_3^* = \frac{4}{27}x^3 - x & x^3 = \frac{27}{4}T_3^* + \frac{81}{4}T_1^* \end{array}$$

Thus,

$$x^3 + x^2 = \frac{27}{4}T_3^* + \frac{81}{4}T_1^* + \frac{9}{2}T_2^* + \frac{9}{2}T_0^*$$

We drop the T_3^* term. This gives a maximum error of $\frac{27}{4}$. The best approximating polynomial is $p(x) = x^2 + \frac{27}{4}x$.

This example showed that the largest domain has the largest error.

6.3.3 Generating functions

In this section, we want to find the general expressions of table 5.1, table 6.2 and table 6.3. Therefore, we establish a generating function for the series $\sum_{n=0}^{\infty} a^n T_n(x)$. In terms of $x = \cos(\theta)$, this is the real part of $\sum_{n=0}^{\infty} a^n e^{in\theta}$.

We can write

$$\sum_{n=0}^{\infty} a^n e^{in\theta} = \sum_{n=0}^{\infty}(ae^{i\theta})^n = (1-ae^{i\theta})^{-1} = \{1-a(\cos(\theta)+i\sin(\theta)\}^{-1}$$

$$= \{1-ax-ia(1-x^2)^{\frac{1}{2}}\}^{-1} = \frac{1-ax+ia(1-x^2)^{\frac{1}{2}}}{(1-ax)^2+a^2(1-x^2)}.$$

We can now find the sum of the series $\sum_{n=0}^{\infty} a^n T_n(x)$, by taking the real part of the previous result.

$$\sum_{n=0}^{\infty} a^n T_n(x) = 1 + aT_1(x) + a^2 T_2(x) + \cdots = \frac{1-ax}{1-2ax+a^2}$$

$$= \left(1 - \frac{1}{2}a \cdot 2x\right)\{1 + a(2x-a) + a^2(2x-a)^2 + \ldots\}.$$

Evaluating the coefficient of a^n on the right-hand side gives for $n \geq 1$ the generating function

$$T_n(x) = \frac{1}{2}\Bigg[(2x)^n - \left\{2\binom{n-1}{1} - \binom{n-2}{1}\right\}(2x)^{n-2} + \\ + \left\{2\binom{n-2}{2} - \binom{n-3}{2}\right\}(2x)^{n-4} - \ldots\Bigg]. \qquad (6.1)$$

To find the generating function for x^n, we use the identity

$$x^n = \left(\frac{e^{i\theta + e^{-i\theta}}}{2}\right)^n.$$

This gives

$$x^n = \frac{1}{2^{n-1}}\left\{T_n(x) + \binom{n}{1}T_{n-2}(x) + \binom{n}{2}T_{n-4}(x) + \ldots\right\}. \qquad (6.2)$$

For even n, we take half the coefficient of $T_0(x)$.

For the domain $0 \leq x \leq 1$, we can replace x by $2x-1$ in 6.1 and 6.2. However, it is simpler to use the functional equation

$$T_m(T_n(x)) = T_n(T_m(x)) = \cos(nm\theta) = T_{nm}(x).$$

For $m = 2$, we find

$$T_n(T_2(x)) = T_n(2x^2-1) = T_2(T_n(x)) = 2T_n^2(x) - 1 = T_{2n}(x).$$

If we replace x^2 by x, we find

$$T_n(2x-1) = T_n^\star(x) = 2T_n^\star(x^{\frac{1}{2}}) - 1 = T_{2n}(x^{\frac{1}{2}}).$$

Using this result and 6.1 and 6.2 we find the generating functions

$$T_n^\star(x) = \frac{1}{2}\left[2^{2n}x^n - \left\{2\binom{2n-1}{1} - \binom{2n-2}{1}\right\}2^{2n-2}x^{n-1} + \dots\right] \quad (6.3)$$

$$x^n = \frac{1}{2^{2n-1}}\left\{T_n^\star(x) + \binom{2n}{1}T_{n-1}^\star(x) + \dots\right\} \quad (6.4)$$

Again, for even n we take half the coefficient of $T_0^\star(x)$.

6.3.4 Small coefficients and little error

Polynomials with a small norm on $[-1, 1]$ can have very large coefficients, which can be inconvenient. The good thing about Chebyshev polynomials, is that the coefficients are much smaller, and therefore have a greater possibility for significant truncation with little error. This is shown in the following example.

Example 5. *Consider the polynomial*

$$\begin{aligned}(1-x^2)^{10} =& 1 - 10x^2 + 45x^4 - 120x^6 + 210x^8 - 252x^{10} + \\ &+ 210x^{12} - 120x^{14} + 45x^{16} - 10x^{18} + x^{20}.\end{aligned}$$

This polynomial has large coefficients, but not in Chebyshev form

$$\begin{aligned}(1-x^2)^{10} =& \frac{1}{524288}\{92378T_0 - 167960T_2 + 125970T_4 - 77520T_6 + \\ &+ 38760T_8 - 15504T_{10} + 4845T_{12} - \\ &- 1140T_{14} + 190T_{16} - 20T_{18} + T_{20}\}.\end{aligned}$$

The largest coefficient is now ≈ 0.32, *much smaller than the largest coefficient before,* 252. *In addition, we can even drop the last three terms. This produces a maximum error of only* 0.0004.

6.3.5 Taylor expansion

The technique of economization can also be used to economize the terms of a Taylor expansion.

Example 6. *Consider the Taylor polynomial* $e^x = \sum_{k=0}^{n} \frac{x^k}{k!} + \frac{x^{n+1}}{(n+1)!}e^\xi$, *where* $-1 \le x$, $|\xi| \le 1$. *For* $n = 6$, *this series has a maximum error of* $\frac{e}{7!} = 0.0005$. *Economizing the first six terms gives*

$$\begin{aligned}\sum_{k=0}^{6}\frac{x^k}{k!} =& \frac{1}{720}\left(x^6 + 6x^5 + 30x^4 + 120x^3 + 360x^2 + 720x + 720\right) \\ =& 1.26606T_0 + 1.13021T_1 + 0.27148T_2 + \\ &+ 0.04427T_3 + 0.00547T_4 + 0.00052T_5 + 0.00004T_6.\end{aligned}$$

Dropping the last two terms gives an additional error of only 0.00056*. Adding this to the existing error gives an total error of only* 0.001*. This error is far smaller than the truncated Taylor expansion of degree* 4*, which is* $\frac{e}{5!} \approx 0.023$.

6.3.6 Algorithm to find the alternating set, the polynomial and the error

From theorem 2, we know that $E = f - p$ has an alternating set of maximum values $\pm M$ of at least $n + 2$ points in $[-1, 1]$.

There exists an algorithm to find these points, the best approximating polynomial p and the maximum error. It is an iterative process, consisting of the following steps:

Step 1: Start with an arbitrary selection $x_0, x_1, \ldots, x_{n+1}$ of $n + 2$ points in $[-1, 1]$. We then find the polynomial $p^{(1)}(x)$ for which $E_n^{(1)}(x)$ has equal and successive opposite values $\pm M_1$ at these points.

Step 2: We then find the position x_{n+2} of the maximum value

$$|E_n^{(1)}(x)| = |f(x) - p^{(1)}(x)|,$$

which is either at a terminal point or at a zero of $\frac{dE_n^{(1)}(x)}{dx}$.

Step 3: If this maximum exceeds M_1, we replace one of the $x_0, \ldots, x_{n+1}$ by x_{n+2} in such a way that $E_n^{(1)}(x)$ has successively opposite signs at the new set of $n + 2$ points.

Step 4: With this new set we repeat steps 1, 2 and 3. The process converges uniformly to the required solution.

We illustrate this algorithm with an example.
Example 7. *Consider the function* $f(x) = e^x$ *in* $[-1, 1]$*. We want to approximate this function with a linear polynomial, i.e. a function of the form* $y = a + bx$*. The error function* $E_1(x) = e^x - (a + bx)$ *must have an alternating set of at least* $2 + 1 = 3$ *points in* $[-1, 1]$*.*

Step 1: We choose the arbitrary points $x_0 = \frac{1}{2}, x_1 = 0, x_2 = -\frac{1}{2}$*. This yields a system of three equations with three unknowns, thus it is solvable (In fact, for every degree it is solvable. There are always n arbitrary points, which yields a system of n equations, with n unknowns.).*

$$e^{\frac{1}{2}} - (a + \frac{1}{2}b) = M, \quad 1 - a = -M, \quad e^{-\frac{1}{2}} - (a - \frac{1}{2}b) = M.$$

Solving this system gives $a = 1.0638$*,* $b = 1.0422$ *and* $M = 0.0638$*.*

Step 2: *Finding the maximum value gives*

$$\frac{dE_1^{(1)}(x)}{dx} = 0 \quad \Rightarrow \frac{d}{dx}\left[e^x - 1.0638 - 1.0422x\right] = e^x - 1.0422 = 0$$
$$\Rightarrow x = 0.04 \quad \Rightarrow E_1^{(1)}(0.04) = -0.065$$

This is not the maximum. The absolute maximum is at the end point $x = 1$*, with value* 0.61*. This exceeds the original* M*, so we go to the next step.*

Step 3: *Our new set of points is now* $x_0 = 1$*,* $x_1 = 0$ *and* $x_2 = -\frac{1}{2}$*. The new system of equations gives*

$$a = 1.1552, \quad b = 1.4079, \quad M = 0.1552.$$

Step 4: *The maximum of the new error* $E_n^{(2)}$ *is at the beginning point* $x = -1$ *with value* 0.620*. Our new points are* $1, 0, -1$*. This has the results* $a = 1.2715$*,* $b = 1.1752$ *and* $M = 0.2715$*. The maximum error is at* $x = 0.1614$ *with value* -0.2860*. The new points are* $1, 0.1614, -1$*. Continuing these steps gives the final result*

$$p(x) = 1.2643 + 1.1752x, \quad M = 0.2788.$$

6.4 Overview of the techniques to find the best approximating polynomial and error

In the previous sections, we found four ways to find the best approximating polynomial. The following table gives an overview of techniques to find the best approximating polynomial and the associated error.

	Technique	Maximum error
1.	Chebyshev approximation $p(x) = f(x) - a_0 2^{-n+1} T_n(x)$	$a_0 2^{-n+1}$
2.	Smallest norm $E = a_0 \cdot \frac{(b-a)^n}{2^n 2^{n-1}} \cdot T_n\left(\frac{2x-b-a}{b-a}\right) \Rightarrow p(x) = f(x) - E(x)$	Either at a terminal point or at a zero of $\frac{dE(x)}{dx}$.
3.	Economization	The coefficient before the term that you drop.
4.	Algorithm	The value of M at the last step.

Table 6.4: Overview of the techniques to find the best approximating polynomial and the associated maximum error.

6.5 Examples

In this section we show how the techniques called Chebyshev approximation and smallest norm work and how the maximum error can be calculated. After this, we show that the four techniques give the same best approximating polynomial and the same maximum error.

6.5.1 Chebyshev approximation

In this section, we show how the technique which we called Chebyshev approximation works.

Example 8. *Consider the function $f(x) = 2x^4 + 3x^3 + 4x^2 + 5x + 6$. We want to approximate this function with a polynomial of degree at most 3 in the interval $[-1, 1]$.*

We need the formula $p(x) = f(x) - a_0 2^{-n+1} T_n(x)$. Since $n = 4$ and $a_0 = 2$, we get

$$\begin{aligned} p(x) &= 2x^4 + 3x^3 + 4x^2 + 5x + 6 - 2 \cdot 2^{-4+1} T_4(x) \\ &= 2x^4 + 3x^3 + 4x^2 + 5x + 6 - \frac{1}{4}\left(8x^4 - 8x^2 + 1\right) \\ &= 3x^3 + 6x^2 + 5x + \frac{23}{4}. \end{aligned}$$

Thus, the best approximating polynomial is $p(x) = 3x^3 + 6x^2 + 5x + \frac{23}{4}$*, with maximum error* $a_0 2^{-n+1} = 2 \cdot 2^{-4+1} = \frac{1}{4}$.

6.5.2 Smallest norm

In this section, we show how the technique which we called smallest norm works.

Example 9. *Consider the function* $f(x) = 4x^4 + 3x^3 + 2x^2 + 1$*. We want to approximate this function with a polynomial of degree at most* 3 *in the interval* $[-2, 2]$.

We need the formula $E = a_0 \cdot \frac{(b-a)^n}{2^n 2^{n-1}} \cdot T_n \left(\frac{2x-b-a}{b-a}\right)$*. Since* $n = 4, a_0 = 4, a = -2, b = 2$*, we get*

$$\begin{aligned} E &= 4 \cdot \frac{(2+2)^4}{2^4 2^3} \cdot T_4 \left(\frac{2x-2+2}{2+2}\right) \\ &= 8T_4(\frac{1}{2}x) \\ &= 8\left(8\left(\frac{1}{2}x\right)^4 - 8\left(\frac{1}{2}x\right)^2 + 1\right) \\ &= 4x^4 - 16x^2 + 8. \end{aligned}$$

Since $p(x) = f(x) - E(x)$*, we get*

$$\begin{aligned} p(x) &= 4x^4 + 3x^3 + 2x^2 + 1 - (4x^4 - 16x^2 + 8) \\ &= 3x^3 + 18x^2 - 7. \end{aligned}$$

So we get $p(x) = 3x^3 + 18x^2 - 7$*. The maximum error is at an end point or at a zero of* $\frac{dE(x)}{dx}$*. This gives a maximum error of* 8.

6.5.3 Five techniques, same result

In this section we show that we find the same best approximating polynomial for an algebraic polynomial using the five techniques: Chebyshev approximation, smallest norm, economization, algorithm and using the L^2-norm approximation with inner product $\langle f, g \rangle = \int_{-1}^{1} \frac{f(x)g(x)}{\sqrt{1-x^2}}$. We do this for the function $f(x) = x^3 + x^2$. We want to find the best approximating polynomial of degree at most 2 in $[-1, 1]$.

Technique 1: We use $p(x) = f(x) - a_0 2^{-n+1} T_n(x)$. Then

$$\begin{aligned} p(x) &= x^3 + x^2 - 2^{-3+1} T_3(x) \\ &= x^3 + x^2 - \frac{1}{4}(4x^3 - 3x) \\ &= x^2 + \frac{3}{4}x. \end{aligned}$$

Thus, $p(x) = x^2 + \frac{3}{4}x$ with maximum error $E = a_0 2^{-n+1} = \frac{1}{4}$.

Technique 2: We use $E = a_0 \cdot \frac{(b-a)^n}{2^n 2^{n-1}} \cdot T_n\left(\frac{2x-b-a}{b-a}\right)$. Then

$$\begin{aligned} E &= \frac{2^3}{2^3 2^2} \cdot T_3\left(\frac{2x}{2}\right) \\ &= \frac{1}{4} T_3(x) \\ &= \frac{1}{4}\left(4x^3 - 3x\right) \\ &= x^3 - \frac{3}{4}x. \end{aligned}$$

Since $E = f - p$, $p = f - E$, thus $p(x) = x^3 + x^2 - (x^3 - \frac{3}{4}x) = x^2 + \frac{3}{4}x$. The maximum error is at an end point or at a zero of $\frac{dE(x)}{dx}$. This gives a maximum error of $\frac{1}{4}$.

Technique 3: Finding the best approximating polynomial for this function using economization is already done in the first part of example 4. There, we found that $p(x) = x^2 + \frac{3}{4}x$ with maximum error $E = \frac{1}{4}$.

Technique 4: We use the algorithm. We want to approximate the function $f(x) = x^3 + x^2$ with a quadratic polynomial, so a polynomial of the form $y = ax^2 + bx + c$. The error function $E(x) = x^3 - ax^2 - bx - c$ must have an alternating set of at least $2 + 2 = 4$ points in $[-1, 1]$.

Step 1: We choose the arbitrary points $x_0 = -1, x_1 = -\frac{1}{2}, x_2 = 0, x_3 = \frac{1}{2}$. This gives the system of equations

$$\begin{aligned} -a + b - c &= M \\ -\frac{1}{8} + \frac{1}{4} - \frac{1}{4}a + \frac{1}{2}b - c &= -M \\ -c &= M \\ \frac{1}{8} + \frac{1}{4} - \frac{1}{4}a - \frac{1}{2}b - c &= -M \end{aligned}$$

Solving this system gives $a = \frac{1}{4}, b = \frac{1}{4}, c = \frac{3}{32} M = -\frac{3}{32}$.

Step 2: The maximum value is at the end point $x = 1$ with value $\frac{45}{32}$. This exceeds the original M, so we go to the next step.

Step 3: Our new set of points are $x_0 = -1, x_1 = -\frac{1}{2}, x_2 = \frac{1}{2}, x_3 = 1$. The new system of equations gives $a = 1, b = \frac{3}{4}, c = 0, M = \frac{1}{4}$.

Step 4: The maximum of the new error is at the points $\pm\frac{1}{2}$ and at the endpoints ± 1, all with value $\frac{1}{4}$. These points do not exceed the original M, so we have found te best approximating polynomial.

Thus, the best approximating polynomial is $p(x) = x^2 + \frac{3}{4}x$ with maximum error $E = \frac{1}{4}$.

Technique 5: We need the following equation

$$p(x) = \langle f, \frac{1}{\sqrt{\pi}} \rangle \frac{1}{\sqrt{\pi}} + \langle f, \frac{\sqrt{2}}{\sqrt{\pi}} T_1 \rangle \frac{\sqrt{2}}{\sqrt{\pi}} T_1 + \langle f, \frac{\sqrt{2}}{\sqrt{\pi}} T_2 \rangle \frac{\sqrt{2}}{\sqrt{\pi}} T_2 + \langle f, \frac{\sqrt{2}}{\sqrt{\pi}} T_3 \rangle \frac{\sqrt{2}}{\sqrt{\pi}} T_3.$$

So we have to calculate each inner product:

$$\begin{aligned}
\langle f, u_1 \rangle &= \tfrac{1}{\sqrt{\pi}} \int_{-1}^{1} \tfrac{x^3+x^2}{\sqrt{(1-x^2)}} dx &&= \frac{\sqrt{\pi}}{2} \\
\langle f, u_2 \rangle &= \tfrac{\sqrt{2}}{\sqrt{\pi}} \int_{-1}^{1} \tfrac{(x^3+x^2)x}{\sqrt{(1-x^2)}} dx &&= \frac{3\sqrt{\frac{\pi}{2}}}{4} \\
\langle f, u_3 \rangle &= \tfrac{\sqrt{2}}{\sqrt{\pi}} \int_{-1}^{1} \tfrac{(x^3+x^2)(2x^2-1)}{\sqrt{(1-x^2)}} dx &&= \frac{\sqrt{\frac{\pi}{2}}}{2}.
\end{aligned}$$

Thus,

$$p(x) = \frac{1}{2} + \frac{3}{4}x + x^2 - \frac{1}{2} = x^2 + \frac{3}{4}x.$$

The maximum error is at

$$\sqrt{\int_{-1}^{1} \left| x^3 - \frac{3}{4}x \right|^2 dx} = \frac{1}{4}.$$

We see that we get the same best approximating polynomial and the same maximum error using the five techniques.

Chapter 7

Conclusion

In this thesis, we studied Chebyshev approximation. We found that Chebyshev was the first to approximate functions in the uniform norm. The problem he wanted to solve was to represent a continuous function $f(x)$ on the closed interval $[a, b]$ by an algebraic polynomial of degree at most n, in such a way that the maximum error $\max_{a \leq x \leq b} |f(x) - p(x)|$ is minimized.

We first looked at approximation in the L^2-norm, to compare it to the uniform norm later. In this norm, the solution to the approximation problem is just an orthogonal projection, which can be found by applying the Gram-Schmidt Process. This is just calculus, so it is an easy procedure.

We then looked at the actual problem: approximating functions in the uniform norm. We succeeded in answering the four questions asked in the introduction: there always *exists* a best approximating polynomial and this polynomial is *unique*. We found that a *necessary and sufficient condition* for the best approximating polynomial is the following:

The polynomial $p(x)$ is the best approximation to $f(x)$ if and only if $f(x) - p(x)$ alternates in sign at least $n+2$ times, where n is the degree of the best approximating polynomial. Then, $f - p$ has at least $n+1$ zeros.

We found that we can *construct* the best approximating polynomial using four techniques:

1. Chebyshev approximation: $p(x) = f(x) - a_0 2^{-n+1} T_n(x)$. The maximum error is $E = a_0 2^{-n+1}$.
2. Smallest norm: $p(x) = f(x) - E(x)$, where $E = a_0 \cdot \frac{(b-a)^n}{2^n 2^{n-1}} \cdot T_n \left(\frac{2x-b-a}{b-a} \right)$. The maximum error is either at a terminal point or at a zero of $\frac{dE(x)}{dx}$.
3. Economization. The maximum error is the coefficient before the term that you drop.

4. Algorithm. The maximum error the value of M at the last step.

Here, $T_n(x)$ is defined as $T_n(x) = \cos(n \arccos(x))$. These are the so called Chebyshev polynomials, defined by the recurrence relation $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$.

For intervals different then $[-1, 1]$, a change of variable is needed for techniques $1, 3$ and 4. This is done using the following formula: $y = \frac{1}{2}(b-a)x + \frac{1}{2}(b+a)$.

The difference between approximation in the L^2-norm and the uniform norm is that the uniform norm gives in general a smaller error. However, it is more difficult to find the solution. So if the difference between the error of both norms is negligible, then it is recommended to use the orthogonal projection.

We assume that for algebraic polynomials in the interval $[-1, 1]$, the best approximating polynomial in the two norms is the same, if the inner product $\langle f, g \rangle = \int_{-1}^{1} \frac{f(x)g(x)}{\sqrt{1-x^2}} dx$ is used. This is because the Chebyshev polynomials are orthogonal with respect to the weight function $w(x) = \frac{1}{\sqrt{1-x^2}}$. This does for sure not hold for functions other than algebraic polynomials, because in general $\int_{-1}^{1} \frac{f(x)g(x)}{\sqrt{1-x^2}} dx \neq \max_{x \in [-1,1]} |f(x) - g(x)|$. It is interesting to investigate this further.

This thesis gave a good overview of what Chebyshev approximation is. However, approximation theory is much broader than Chebyshev approximation. For example Weierstrass theorem, Bernstein polynomials and spline approximation are important in approximation theory. Therefore, we refer the interested reader to [5], [13], [14] and [15] to learn more about this interesting field of mathematics.

Appendices

Appendix A

Definitions from linear algebra

The information in this appendix is cribbed from *Linear algebra with applications* by S. J. Leon [2] and from *Best approximation in the 2-norm* by E. Celledoni [4].

Definition 3 (Vector space)**.** *A real* ***vector space*** V *is a set with a zero element and three operations.*

Addition $x, y \in V$ *then* $x + y \in V$.

Inverse *For all* $x \in V$ *there exists* $-x \in V$ *such that* $x + (-x) = 0$.

Scalar multiplication $\lambda \in \mathbb{R}$, $x \in V$ *then* $\lambda x \in V$.

Furthermore, the following axioms are satisfied.

A 1: $x + y = y + x$

A 2: $(x + y) + z = x + (y + z)$

A 3: $0 + x = x + 0 = x$

A 4: $0x = 0$

A 5: $1x = 1$

A 6: $(\lambda\mu)x = \lambda(\mu x)$

A 7: $\lambda(x + y) = \lambda x + \lambda y$

A 8: $(\lambda + \mu)x = \lambda x + \mu x$

Definition 4 (Inner product)**.** *Let* V *be a vector space over* $\mathbb{R}$. *A function* $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ *is called an* ***inner product*** *on* V *if*

1. $\langle \lambda f + \mu g, h \rangle = \lambda \langle f, h \rangle + \mu \langle g, h \rangle$ *for all* $f, g, h \in V$ *and* $\lambda, \mu \in \mathbb{R}$.

2. $\langle f, g\rangle = \langle g, f\rangle$ *for all* $f, g \in V$.

3. $\langle f, f\rangle \geq 0$ *with equality if and only if* $f = 0$.

A vector space with an inner product is called an ***inner product space****.*
Example 10. *An inner product on* $C[a, b]$ *is defined by*

$$\langle f, g\rangle = \int_a^b f(x)g(x)dx.$$

It is easily checked that the three conditions hold. If the ***weight function*** $w(x)$ *is a positive continuous function on* $[a, b]$*, then*

$$\langle f, g\rangle = \int_a^b f(x)g(x)w(x)dx$$

also defines an inner product on $C[a, b]$*. Thus we can define many different inner products on* $C[a, b]$*.*
Definition 5 (Projection)**.** *If* u *and* v *are vectors in an inner product space* V *and* $v \neq 0$*, then the* ***scalar projection*** *of* u *onto* v *is given by*

$$\alpha = \frac{\langle u, v\rangle}{\|v\|}$$

and the ***vector projection*** *of* u *and* v *is given by*

$$p = \alpha\left(\frac{1}{\|v\|}v\right) = \frac{\langle u, v\rangle}{\langle v, v\rangle}v.$$

Definition 6 (Orthogonality)**.** *If in an inner product space* V *we have that* $\langle f, g\rangle = 0$*, then we say that* f *is* ***orthogonal*** *to* g*.*
Definition 7 (Norm)**.** *A vector space* V *is called a* ***normed linear space*** *if, to each vector* $\boldsymbol{v} \in V$*, there is associated a real number* $\|\boldsymbol{v}\|$*, called the* ***norm*** *of* $\boldsymbol{v}$*, satisfying*

1. $\|\boldsymbol{v}\| \geq 0$ *with equality if and only if* $\boldsymbol{v} = \boldsymbol{0}$*.*

2. $\|\alpha\boldsymbol{v}\| = |\alpha|\|\boldsymbol{v}\|$ *for any scalar* α*.*

3. $\|\boldsymbol{v} + \boldsymbol{w}\| \leq \|\boldsymbol{v}\| + \|\boldsymbol{w}\|$ *for all* $\boldsymbol{v},\boldsymbol{w} \in V$ *(the triangle inequality).*

Any norm on V induces a *metric* or distance function by setting:

$$d = d(v, w) = \|v - w\| \tag{A.1}$$

In this thesis, we only deal with the L^2-norm and the uniform norm.

1. L^2-norm: $\|(x_k)_{k=1}^n\|_2 = (\sum_{k=1}^n |x_k|^2)^{\frac{1}{2}}$
2. Uniform norm: $\|(x_i)_{i=1}^n\|_\infty = \max_{1\leq i\leq n} |x_i|$

Definition 8 (Induced norm)**.** *In an inner product space we can define the norm*

$$\|f\|_2 := \langle\sqrt{\langle f, f\rangle}\rangle.$$

This is called norm induced by the inner product.

With the definitions of this section, we can now prove the following theorem

Theorem 7. *An inner product space V with the induced norm is a normed space.*

Example 11. *In example 10 we saw that $C[a, b]$ is an inner product space with inner product $\langle f, g\rangle = \int_a^b f(x)g(x)w(x)dx$. The induced norm is*

$$\|f\|_2 = \left(\int_a^b w(x)|f(x)|^2 dx\right)^{\frac{1}{2}}.$$

*This norm is called the **L^2-norm** on $C[a, b]$.*

Definition 9 (Orthonormality)**.** *An **orthonormal** set of vectors is an orthogonal set of unit vectors.*

Appendix B

Legendre polynomials

$$
\begin{aligned}
P_0(x) &= 1\\
P_1(x) &= x\\
P_2(x) &= \tfrac{1}{2}(3x^2 - 1)\\
P_3(x) &= \tfrac{1}{2}(5x^3 - 3x)\\
P_4(x) &= \tfrac{1}{8}(35x^4 - 30x^2 + 3)\\
P_5(x) &= \tfrac{1}{8}(63x^5 - 70x^3 + 15x)\\
P_6(x) &= \tfrac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5)
\end{aligned}
$$

Table B.1: The first six Legendre polynomials.

Appendix C

Chebyshev polynomials

$$
\begin{aligned}
&T_0(x) = 1\\
&T_1(x) = x\\
&T_2(x) = 2x^2 - 1\\
&T_3(x) = 4x^3 - 3x\\
&T_4(x) = 8x^4 - 8x^2 + 1\\
&T_5(x) = 16x^5 - 20x^3 + 5x\\
&T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1\\
&T_7(x) = 64x^7 - 112x^5 + 56x^3 - 7x\\
&T_8(x) = 128x^8 - 256x^6 + 160x^4 - 32x^2 + 1\\
&T_9(x) = 256x^9 - 576x^7 + 432x^5 - 120x^3 + 9x\\
&T_{10}(x) = 512x^{10} - 1280x^8 + 1120x^6 - 400x^4 + 50x^2 - 1\\
&T_{11}(x) = 1024x^{11} - 2816x^9 + 2816x^7 - 1232x^5 + 220x^3 - 11x
\end{aligned}
$$

Table C.1: The first eleven Chebyshev polynomials.

Appendix D

Approximation in L^2-norm and in uniform norm

In the first section of this appendix we show that we get the same best approximating polynomial using the approximation in the L^2-norm with inner product $\langle f, g\rangle = \int_{-1}^{1} \frac{f(x)g(x)}{\sqrt{1-x^2}}$ and in the uniform norm for an algebraic polynomial different than x^n.

In the second section we show that we do not get the same best approximating polynomial in the L^2-norm with inner product $\langle f, g\rangle = \int_{-1}^{1} \frac{f(x)g(x)}{\sqrt{1-x^2}}$ and in the uniform norm for other functions than algebraic polynomials.

D.1 Same result for algebraic polynomials

Consider the function $f(x) = 2x^4 + 3x^3 + 4x^2 + 5x + 6$. We want to find the best approximating polynomial p of degree less than 4 to this function $f(x)$ in the interval $[-1, 1]$. We do this using the L^2-norm approximation.

We need the following equation

$$p(x) = \langle f, \frac{1}{\sqrt{\pi}}\rangle \frac{1}{\sqrt{\pi}} + \langle f, \frac{\sqrt{2}}{\sqrt{\pi}}T_1\rangle \frac{\sqrt{2}}{\sqrt{\pi}}T_1 + \langle f, \frac{\sqrt{2}}{\sqrt{\pi}}T_2\rangle \frac{\sqrt{2}}{\sqrt{\pi}}T_2 + \\ + \langle f, \frac{\sqrt{2}}{\sqrt{\pi}}T_3\rangle \frac{\sqrt{2}}{\sqrt{\pi}}T_3 + \langle f, \frac{\sqrt{2}}{\sqrt{\pi}}T_4\rangle \frac{\sqrt{2}}{\sqrt{\pi}}T_4.$$

So we have to calculate each inner product

$$\begin{aligned}
\langle f, u_1\rangle &= \tfrac{1}{\sqrt{\pi}}\int_{-1}^{1}\tfrac{2x^4+3x^3+4x^2+5x+6}{\sqrt{(1-x^2)}}dx &&= \frac{35\sqrt{\pi}}{4}\\
\langle f, u_2\rangle &= \tfrac{\sqrt{2}}{\sqrt{\pi}}\int_{-1}^{1}\tfrac{(2x^4+3x^3+4x^2+5x+6)x}{\sqrt{(1-x^2)}}dx &&= \frac{29\sqrt{\frac{\pi}{2}}}{4}\\
\langle f, u_3\rangle &= \tfrac{\sqrt{2}}{\sqrt{\pi}}\int_{-1}^{1}\tfrac{(2x^4+3x^3+4x^2+5x+6)(2x^2-1)}{\sqrt{(1-x^2)}}dx &&= 3\sqrt{\frac{\pi}{2}}\\
\langle f, u_4\rangle &= \tfrac{\sqrt{2}}{\sqrt{\pi}}\int_{-1}^{1}\tfrac{(2x^4+3x^3+4x^2+5x+6)(4x^3-3x)}{\sqrt{(1-x^2)}}dx = dx &&= \frac{3\sqrt{\frac{\pi}{2}}}{4}
\end{aligned}$$

Thus,

$$\begin{aligned}
p(x) =& \frac{35}{4} + \frac{29}{4}x + 6x^2 - 3 + 3x^3 - \frac{9}{4}x\\
=& 3x^3 + 6x^2 + 5x + \frac{23}{4},
\end{aligned}$$

with maximum error

$$E = \sqrt{\int_{-1}^{1}\left|2x^4 - 2x^2 + \frac{1}{4}\right|^2 dx} = \frac{1}{4}.$$

This is the same result as we found using Chebyshev approximation in example 8. Thus, we assume that approximation in the L^2-norm and in the uniform norm are the same in the interval $[-1, 1]$ for algebraic polynomials.

D.2 Different result for other functions

Consider the function $f(x) = e^x$. We want to find the best approximating polynomial of degree 1 to this function $f(x)$ in the interval $[-1, 1]$. We do this using L^2-norm approximation.

We need the following equation

$$p(x) = \langle f, \frac{1}{\sqrt{\pi}}\rangle\frac{1}{\sqrt{\pi}} + \langle f, \frac{\sqrt{2}}{\sqrt{\pi}}T_1\rangle\frac{\sqrt{2}}{\sqrt{\pi}}T_1.$$

So we have to calculate each inner product:

$$\begin{aligned}
\langle f, u_1\rangle &= \tfrac{1}{\sqrt{\pi}}\int_{-1}^{1}\tfrac{e^x}{\sqrt{(1-x^2)}}dx &&= \sqrt{\pi}I_0(1)\\
\langle f, u_2\rangle &= \tfrac{\sqrt{2}}{\sqrt{\pi}}\int_{-1}^{1}\tfrac{(e^x)x}{\sqrt{(1-x^2)}}dx &&= \sqrt{2\pi}I_1(1),
\end{aligned}$$

where $I_n(z)$ is the modified Bessel function of the first kind.

Thus,

$$\begin{aligned} p(x) &= 2I_1(1)x + I_0(1) \\ &\approx 1.1303x + 1.2661, \end{aligned}$$

with maximum error

$$E = \sqrt{\int_{-1}^{1} |e^x - 2I_1(1)x - I_0(1)|^2 dx} \approx 0.2639.$$

In example 7 we approximated the function $f(x) = e^x$ with the use of the algorithm and found that $p(x) = 1.1752x + 1.2643$, with maximum error of $E_2(1) = e - 1.1752 - 1.2643 = 0.2788$. This is slightly different than the result that we found using the L^2-norm approximation.

Thus, we showed using a counterexample that we do not find the same best approximating polynomial using the L^2-norm approximation and the uniform approximation for functions different than algebraic polynomials.

Appendix E

Powers of x as a function of $T_n(x)$

$$
\begin{aligned}
1 &= T_0 \\
x &= T_1 \\
x^2 &= \tfrac{1}{2}(T_0 + T_2) \\
x^3 &= \tfrac{1}{4}(3T_1 + T_3) \\
x^4 &= \tfrac{1}{8}(3T_0 + 4T_2 + T_4) \\
x^5 &= \tfrac{1}{16}(10T_1 + 5T_3 + T_5) \\
x^6 &= \tfrac{1}{32}(10T_0 + 15T_2 + 6T_4 + T_6) \\
x^7 &= \tfrac{1}{64}(35T_1 + 21T_3 + 7T_5 + T_7) \\
x^8 &= \tfrac{1}{128}(35T_0 + 56T_2 + 28T_4 + 8T_6 + T_8) \\
x^9 &= \tfrac{1}{256}(126T_1 + 84T_3 + 36T_5 + 9T_7 + T_9) \\
x^{10} &= \tfrac{1}{512}(126T_0 + 210T_2 + 120T_4 + 45T_6 + 10T_8 + T_{10}) \\
x^{11} &= \tfrac{1}{1024}(462T_1 + 330T_3 + 165T_5 + 55T_7 + 11T_9 + T_{11})
\end{aligned}
$$

Table E.1: The first eleven powers of x in terms of the Chebyshev polynomials.

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