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Modeling turbulent flow

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Abstract

Fluid flow is mathematically modelled by the Navier-Stokes equations. In this thesis we will focus at turbulent flows. Therefore we will first look at the incompressible Navier-Stokes equations. To simulate turbulent flows with the incompressible Navier-Stokes, it is important that the kinetic energy of the flow is conserved. Therefore, we will derive the conservation of the kinetic energy. Then we will discretize the Navier-Stokes equations using the finite volume method and derive in this case the kinetic energy. At last we will derive a model called the QR model. Therefore we start with a large eddy simulation(LES), which means we will add a spatial filter to a solution of the incompressible Navier-Stokes equations. From this LES, we will derive an eddy-viscosity model called the QR model.

Contents

1 Introduction

The Navier-Stokes equations provide a model for turbulent flow. In this thesis we will follow the study of Henry Bandringa [1].

In section 2 we will look at the incompressible Navier-Stokes equations. Because the conservation of the kinetic energy is important, we will compute the evolution of the kinetic energy in time and see that the kinetic energy will decrease.

In section 3 we will discretize the incompressible Navier-Stokes equations and then compute the evolution of the kinetic energy in time of these discretized Navier-Stokes equation. We will see that the equation for this evolution of the kinetic energy the discrete version is of the equation we computed in the first section and that for the discrete version also the kinetic energy will decrease.

In section 4 we will look at large-eddy simulation (LES). For LES, we add a filter to the incompressible Navier-Stokes equations. This filter introduces a closure model τ. We will use scale-truncation to get a condition on this τ. Then we will introduce an eddy viscosity model for τ, and using the condition on τ, we will derive a model called the QR-model [3].

2 The incompressible Navier-Stokes equations

In this section we will see that the incompressible Navier-Stokes equations are

$$\begin{aligned}\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p - \nu \Delta u &= 0 \\ \nabla \cdot u &= 0\end{aligned}$$

From these equations, we will derive the evolution of the kinetic energy en see that the kinetic energy decreases.

2.1 The incompressible Navier-Stokes equations

In this subsection we will give the incompressible Navier-Stokes equations.

Therefore we introduce the following symbols: u is the velocity vector in $\mathbb{R}^3$, ρ the density, p the pressure and μ the dynamic viscosity. For the incompressible Navier-Stokes equations, we assume that the density, the temperature and viscosity are constant.

So the equation for conservation of mass

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0$$

becomes, using the constancy of the density,

$$\nabla \cdot u = 0 \tag{1}$$

The conservation of momentum is given by

$$\rho \frac{\partial u}{\partial t} + \rho \nabla \cdot (u \otimes u) + \nabla p - \mu \Delta u = 0$$

where $\otimes$ indicates the outer product, which is defined as $u \otimes v = uv^{\mathrm{T}}$ When we divide this equation by ρ and define $\nu = \frac{\mu}{\rho}$ and $\hat{p} = \frac{p}{\rho}$ and call this again p, we get

$$\frac{\partial u}{\partial t} + \nabla \cdot (u \otimes u) + \nabla p - \nu \Delta u = 0 \tag{2}$$

Now we rewrite $\nabla \cdot (u \otimes u)$:

$$\nabla \cdot (u \otimes u) = \sum_{ij} \frac{\partial}{\partial x_i}(u_i u_j) = \sum_{ij} \left(\frac{\partial u_i}{\partial x_i} u_j + u_i \frac{\partial u_j}{\partial x_i} \right)$$

using equation 1, $\sum_i \frac{\partial u_i}{\partial x_i} = \nabla \cdot u = 0$, so we get

$$\nabla \cdot (u \otimes u) = \sum_{ij} u_i \frac{\partial u_j}{\partial x_i} = (u \cdot \nabla) u \tag{3}$$

So we end up with the following incompressible Navier-Stokes equations:

$$\frac{\partial u}{\partial t} + (u \cdot \nabla) u + \nabla p - \nu \Delta u = 0 \tag{4}$$

$$\nabla \cdot u = 0 \tag{5}$$

2.2 The evolution of the kinetic energy

In this subsection we will compute the evolution of the kinetic energy

$$\frac{1}{2} \frac{d}{dt}(u, u) = -\nu \int_\Omega \nabla u : \nabla u \; \mathrm{d}\Omega + \text{boundary terms}$$

and see that if we neglect the boundary terms, the kinetic energy decreases.

We define the inner product $(\cdot, \cdot)$ as $(f, g) = \int_\Omega f \cdot g \; \mathrm{d}\Omega$. Then the time derivative kinetic energy is given by

$$\frac{1}{2} \frac{d}{dt} \int_\Omega u \cdot u \; \mathrm{d}\Omega = \frac{1}{2} \frac{d}{dt}(u, u) = \frac{1}{2} \left(\frac{\partial}{\partial t} u, u \right) + \frac{1}{2} \left(u, \frac{\partial}{\partial t} u \right)$$

Using $(f, g) = \int_\Omega f \cdot g \; \mathrm{d}\Omega = \int_\Omega g \cdot f \; \mathrm{d}\Omega = (g, f)$ we get

$$\frac{1}{2} \frac{d}{dt}(u, u) = \left(\frac{\partial u}{\partial t}, u \right)$$

Now we rewrite equation 4 as $\frac{\partial u}{\partial t} = -(u \cdot \nabla) u - \nabla p + \nu \Delta u$ and put it into the above equation. Then we get

$$\frac{1}{2} \frac{d}{dt}(u, u) = -((u \cdot \nabla) u, u) - (\nabla p, u) + \nu (\Delta u, u) \tag{6}$$

We will work this out with the following product rule

$$\nabla \cdot (f u) = \nabla f \cdot u + f (\nabla \cdot u) \tag{7}$$

When we replace f by $u \cdot u$ and use conservation of mass (equation 5), we get

$$\nabla \cdot ((u \cdot u) u) = \nabla (u \cdot u) \cdot u$$

Using $\nabla(u \cdot u) \cdot u = \sum_{ij} \frac{\partial}{\partial x_i}(u_j u_j) u_i = \sum_{ij} 2u_i \frac{\partial u_j}{\partial x_i} u_j = 2(u \cdot \nabla) u \cdot u$, we get

$$\frac{1}{2} \nabla \cdot ((u \cdot u) u) = (u \cdot \nabla) u \cdot u$$

Now we take the integral over Ω and get

$$\begin{aligned} \frac{1}{2} \int_\Omega \nabla \cdot ((u \cdot u) u) \; \mathrm{d}\Omega &= \int_\Omega (u \cdot \nabla) u \cdot u \; \mathrm{d}\Omega \\ &= ((u \cdot \nabla) u, u) \end{aligned}$$

Using Gauss's divergence theorem, we get

$$\int_\Omega \nabla \cdot ((u \cdot u) u) \; \mathrm{d}\Omega = \int_\Gamma (u \cdot u)(u \cdot n) \; \mathrm{d}\Gamma$$

where Γ is the boundary of Ω and n is the orthonormal vector at Γ. So when we combine the two above equations, we get

$$((u \cdot \nabla) u, u) = \frac{1}{2} \int_\Gamma (u \cdot u)(u \cdot n) \; \mathrm{d}\Gamma$$

So equation 6 becomes

$$\frac{1}{2} \frac{d}{dt}(u, u) = -\frac{1}{2} \int_\Gamma |u|^2 (u \cdot n) \; \mathrm{d}\Gamma - (\nabla p, u) + \nu(\Delta u, u) \tag{8}$$

where $|u|^2 = u \cdot u$.

Now we set in equation 7 $f = p$ and see that

$$(\nabla p, u) = \int_\Omega \nabla \cdot (pu) \; \mathrm{d}\Omega - (p, \nabla \cdot u) \tag{9}$$

When we now use conservation of mass (equation 5) and Gauss's divergence theorem, we get

$$(\nabla p, u) = \int_\Gamma (u \cdot n) p \; \mathrm{d}\Gamma$$

So equation 8 becomes

$$\frac{1}{2} \frac{d}{dt}(u, u) = -\int_\Gamma (u \cdot n)(\frac{1}{2}|u|^2 + p) \; \mathrm{d}\Gamma + \nu(\Delta u, u) \tag{10}$$

When we replace in equation 7 f by a tensor A, it becomes

$$\nabla \cdot (Au) = (\nabla \cdot A) \cdot u + A : \nabla u$$

Where : is the Frobenius inner product, which is defined by

$$A : B = \sum_{i,j} A_{ij} B_{ij} = tr(A^{\mathrm{T}} B) = tr(B^{\mathrm{T}} A)$$

When we fill in $A = \nabla u$, we get

$$\nabla \cdot ((\nabla u) u) = (\Delta u) \cdot u + \nabla u : \nabla u \tag{11}$$

Thus

$$\nu(\Delta u, u) = \nu \int_\Omega (\nabla \cdot ((\nabla u)u) - \nabla u : \nabla u) \; \mathrm{d}\Omega$$

When we now apply Gauss's divergence theorem, we get

$$\nu(\Delta u, u) = \nu \int_\Gamma ((\nabla u)u) \cdot n \; \mathrm{d}\Gamma - \nu \int_\Omega \nabla u : \nabla u \; \mathrm{d}\Omega$$

When we now use $((\nabla u)u) \cdot n = u \cdot \frac{\partial u}{\partial n}$, we end up with the following equation for the evolution of the kinetic energy in time

$$\frac{1}{2}\frac{d}{dt}(u, u) = -\int_\Gamma (u \cdot n)(\frac{1}{2}|u|^2 + p) \; \mathrm{d}\Gamma + \nu \int_\Gamma u \cdot \frac{\partial u}{\partial n} \; \mathrm{d}\Gamma - \nu \int_\Omega \nabla u : \nabla u \; \mathrm{d}\Omega \quad (12)$$

So if we use the no-slip boundary condition, the first two terms on the right hand side vanishes and we get

$$\frac{1}{2}\frac{d}{dt}(u, u) = -\nu \int_\Omega \nabla u : \nabla u \; \mathrm{d}\Omega$$

When we now use $\nu > 0$ and $\nabla u : \nabla u = \sum_{ij} \left(\frac{\partial u_i}{\partial x_j}\right)^2 \geq 0$, we see that $\frac{1}{2}\frac{d}{dt}(u, u) \leq 0$ and we get $\frac{1}{2}\frac{d}{dt}(u, u) = 0$ only if u is constant, in which case $u = 0$ due to the no-slip boundary condition. So if $u \neq 0$, the kinetic energy will decrease.

3 Discretization of the Navier-Stokes equation

In this section we will derive the folliwing discrete incompressible Navier-Stokes equations.

$$\frac{\partial u_i}{\partial t}\Omega_i + \sum_{j \in N_i} u_{ij} u_{ij} \cdot n_{ij} A_{ij} + \sum_{j \in N_i} p_{ij} n_{ij} A_{ij} - \sum_{j \in N_i} \nu \frac{u_j - u_i}{(x_j - x_i) \cdot n_{ij}} A_{ij} = 0$$

$$\sum_{j \in N_i} u_{ij} \cdot n_{ij} A_{ij} = 0$$

Furthermore we will show that for these discrete incompressible Navier-Stokes equations the kinetic energy decreases too.

3.1 Discretization of the Navier-Stokes equation

In this section, we will approximate the Navier-Stokes equations using the finite volume method. Furthermore we need a computational grid. We will use a body conforming unstructured grid. We will use the co-located cell-centered grid arrangement.

The conservation of mass in cell i is discretized by taking the integral of equation 5 over the control volume Ω_i:

$$\int_{\Omega_i} \nabla \cdot u \; \mathrm{d}\Omega_i = \int_{\Gamma_i} u \cdot n \; \mathrm{d}\Gamma_i = \sum_{j \in N_i} u_{ij} \cdot n_{ij} A_{ij} = 0 \quad (13)$$

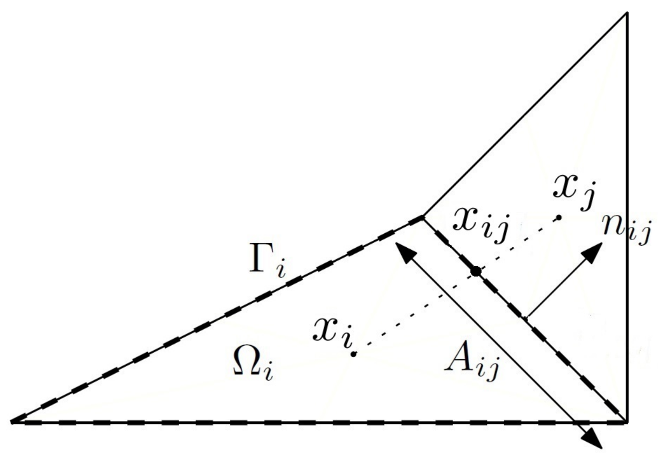

Figure 1: A part of an unstructured grid

Where N_i consist of indices j of cells adjacent to cell i, A_{ij} the surface between cells i and j and n_{ij} the normal vector to this surface A_{ij} in the direction of cell j as can be seen in figure 3.1. Furthermore Γ_i is the boundary of Ω_i and u_{ij} is the face velocity which is computed as

$$u_{ij} = \gamma_{ij} u_i + (1 - \gamma_{ij}) u_j$$

where u_i is the velocity vector in cell i and γ_{ij} is the geometric interpolation factor, defined as

$$\gamma_{ij} = \frac{(x_j - x_{ij}) \cdot n_{ij}}{(x_j - x_i) \cdot n_{ij}} \tag{14}$$

Integrating the momentum equation 2 over an arbitrary control volume Ω_i gives

$$\int_{\Omega_i} \frac{\partial u}{\partial t} \, \mathrm{d}\Omega_i + \int_{\Omega_i} \nabla \cdot (uu) \, \mathrm{d}\Omega_i + \int_{\Omega_i} \nabla p \, \mathrm{d}\Omega_i - \int_{\Omega_i} \nabla \cdot (\nu \nabla u) \, \mathrm{d}\Omega_i = 0 \tag{15}$$

Now we will discretize the four parts of this equation. When we discretize the first part, we get

$$\int_{\Omega_i} \frac{\partial u}{\partial t} \, \mathrm{d}\Omega_i \approx \frac{\partial u_i}{\partial t} \Omega_i \tag{16}$$

For the second part, we use first Gauss's divergence theorem and get

$$\int_{\Omega_i} \nabla \cdot (uu) \, \mathrm{d}\Omega_i = \int_{\Gamma_i} u(u \cdot n) \, \mathrm{d}\Gamma_i \approx \sum_{j \in N_i} u_{ij} u_{ij} \cdot n_{ij} A_{ij} \tag{17}$$

Before discretizing the third part, we again first apply Gauss's divergence theorem.

$$\int_{\Omega_i} \nabla p \, \mathrm{d}\Omega_i = \int_{\Gamma_i} pn \, \mathrm{d}\Gamma_i \approx \sum_{j \in N_i} p_{ij} n_{ij} A_{ij} \tag{18}$$

The last part is a little bit harder to discretize. When we start from $\int_{\Omega_i} \nabla \cdot (\nu \nabla u) \, \mathrm{d}\Omega_i$, we get, using Gauss's divergence theorem,

$$\int_{\Omega_i} \nabla \cdot (\nu \nabla u) \, \mathrm{d}\Omega_i = \int_{\Gamma_i} \nu \nabla u \cdot n \, \mathrm{d}\Gamma_i \approx \sum_{j \in N_i} \nu (\nabla u)_{ij} \cdot n_{ij} A_{ij} \tag{19}$$

When we now assume that the fluxes are normal to the faces, we get $(\nabla u)_{ij} \cdot n_{ij} = \left(\frac{\partial u}{\partial n}\right)_{ij} \approx \frac{u_j - u_i}{(x_j - x_i) \cdot n_{ij}}$, so equation 19 becomes

$$\int_{\Omega_i} \nabla \cdot (\nu \nabla u) \, \mathrm{d}\Omega_i \approx \sum_{j \in N_i} \nu (\nabla u)_{ij} \cdot n_{ij} A_{ij} \approx \sum_{i \in N_i} \nu \frac{u_j - u_i}{(x_j - x_i) \cdot n_{ij}} A_{ij} \tag{20}$$

Combining equations 15, 16, 17, 18 and 20, we get the following discrete form of the conservation of momentum equation

$$\frac{\partial u_i}{\partial t} \Omega_i + \sum_{j \in N_i} u_{ij} u_{ij} \cdot n_{ij} A_{ij} + \sum_{j \in N_i} p_{ij} n_{ij} A_{ij} - \sum_{j \in N_i} \nu \frac{u_j - u_i}{(x_j - x_i) \cdot n_{ij}} A_{ij} = 0 \tag{21}$$

3.2 The evolution of the kinetic energy

In this section we will compute evolution of the kinetic energy in time for the discretized incompressible Navier-Stokes equation and show that the kinetic energy decreases.

The evolution of the kinetic energy is given by

$$\sum_i u_i \cdot \frac{\partial u_i}{\partial t} \Omega_i$$

Using equation 21, this equation becomes

$$\begin{aligned} \sum_i u_i \cdot \frac{\partial u_i}{\partial t} \Omega_i = &- \sum_i u_i \cdot \sum_{j \in N_i} u_{ij} u_{ij} \cdot n_{ij} A_{ij} - \sum_i u_i \cdot \sum_{j \in N_i} p_{ij} n_{ij} A_{ij} \\ &+ \sum_i u_i \cdot \sum_{j \in N_i} \nu \frac{u_j - u_i}{(x_j - x_i) \cdot n_{ij}} A_{ij} \end{aligned} \tag{22}$$

We will now rewrite this equation using the following identity [1]

$$\phi_i \sum_{j \in N_i} \overline{\psi_{ij}} Q_{ij} + \psi_i \sum_{j \in N_i} \overline{\phi_{ij}} Q_{ij} = \sum_{j \in N_i} \widehat{\phi\psi}_{ij} Q_{ij} + \phi_i \psi_i \sum_{j \in N_i} Q_{ij} \tag{23}$$

Where ϕ and ψ are general variables, Q_{ij} is known on the cell face, $\overline{\phi_{ij}} = \gamma_{ij}\phi_i + (1-\gamma_{ij})\phi_j$, $\overline{\psi_{ij}} = \gamma_{ij}\psi_i + (1-\gamma_{ij})\psi_j$, with γ_{ij} the interpolation function defined in equation 14, and $\widehat{\phi\psi}_{ij} = \frac{1}{2}(\phi_i\psi_j + \psi_i\phi_j)$

Now we take in equation 23 $\phi = u$, $\psi = \phi$ and $Q_{ij} = u_{ij} \cdot n_{ij} A_{ij}$ and interpolate the face velocity as $u_{ij} = \frac{1}{2}(u_i + u_j)$. Then we get

$$\begin{aligned} &\sum_i u_i \cdot \sum_{j \in N_i} \phi_{ij} u_{ij} \cdot n_{ij} A_{ij} + \sum_i \phi_i \cdot \sum_{j \in N_i} \frac{1}{2}(u_i + u_j) u_{ij} \cdot n_{ij} A_{ij} \\ &= \sum_i \sum_{j \in N_i} \frac{1}{2}(u_i \cdot \phi_j + \phi_i \cdot u_j) u_{ij} \cdot n_{ij} A_{ij} + \sum_i u_i \phi_i \sum_{j \in N_i} u_{ij} \cdot n_{ij} A_{ij} \end{aligned} \tag{24}$$

Using equation 13, the last term vanishes. When we split the second term into two parts $\sum_i \frac{1}{2}\phi_i \cdot u_i \sum_{j \in N_i} u_{ij} \cdot n_{ij} A_{ij} + \sum_i \sum_{j \in N_i} \frac{1}{2}(\phi_i \cdot u_j) u_{ij} \cdot n_{ij} A_{ij}$ we see, using again equation 13, that the first part vanishes and the second part is also on the other hand of the equal sign in equation 24, so equation 24 becomes

$$\sum_i u_i \cdot \sum_{j \in N_i} \phi_{ij} u_{ij} \cdot n_{ij} A_{ij} = \sum_i \sum_{j \in N_i} \frac{1}{2}(u_i \cdot \phi_j) u_{ij} \cdot n_{ij} A_{ij} \tag{25}$$

When we now substitute $\phi = u$, $\psi = p$ and $Q_{ij} = n_{ij}A_{ij}$ into equation 23, we get

$$\sum_i u_i \cdot \sum_{j\in N_i} p_{ij}n_{ij}A_{ij} + \sum_i p_i \sum_{j\in N_i} u_{ij}\cdot n_{ij}A_{ij}$$
$$= \sum_i \sum_{j\in N_i} \widehat{up}_{ij}n_{ij}A_{ij} + \sum_i u_i p_i \cdot \sum_{j\in N_i} n_{ij}A_{ij}$$

Using equation 13, we see that the second term vanishes. Using geometric arguments, we see that $\sum_{j\in N_i} n_{ij}A_{ij} = 0$, so the last term vanishes also. Thus we end up with

$$\sum_i u_i \cdot \sum_{j\in N_i} p_{ij}n_{ij}A_{ij} = \sum_i \sum_{j\in N_i} \widehat{up}_{ij}n_{ij}A_{ij} \tag{26}$$

Putting equations 22, 25 (with $\phi = u$) and 26 together, we get the following equation for the kinetic energy

$$\sum_i u_i \cdot \frac{\partial u_i}{\partial t}\Omega_i = -\sum_i \sum_{j\in N_i} \frac{1}{2}(u_i\cdot u_j)u_{ij}\cdot n_{ij}A_{ij} - \sum_i\sum_{j\in N_i} \widehat{up}_{ij}n_{ij}A_{ij}$$
$$+\nu \sum_i u_i \cdot \sum_{j\in N_i} \frac{u_j - u_i}{(x_j - x_i)\cdot n_{ij}}A_{ij} \tag{27}$$

So this equation is the discrete version of equation 12. In this equation, we see that the first two terms on the right hand side vanishes at the interior. This is because $u_{ij} = u_{ji}$, $n_{ij} = -n_{ji}$, $A_{ij} = A_{ji}$ and $\widehat{up}_{ij} = \widehat{up}_{ji}$. The only relation here which is not trivial is $u_{ij} = u_{ji}$, so we will prove this relation. Therefore we will first show that $\gamma_{ij} = 1 - \gamma_{ji}$.

$$1-\gamma_{ji} = \frac{(x_i - x_j)\cdot n_{ji}}{(x_i - x_j)\cdot n_{ji}} - \frac{(x_i - x_{ji})\cdot n_{ji}}{(x_i - x_j)\cdot n_{ji}} = \frac{(x_j - x_{ij})\cdot n_{ij}}{(x_j - x_i)\cdot n_{ij}} = \gamma_{ij}$$

With this, we see that

$$u_{ij} = \gamma_{ij}u_i + (1-\gamma_{ij})u_j = (1-\gamma_{ji})u_i + \gamma_{ji}u_j = u_{ji}$$

So when we use the no-slip boundary condition, the first two terms of equation 27 vanishes and we end up with

$$\sum_i u_i \cdot \frac{\partial u_i}{\partial t}\Omega_i = \nu\sum_i u_i \cdot \sum_{j\in N_i} \frac{u_j - u_i}{(x_j - x_i)\cdot n_{ij}}A_{ij}$$

To see why this is non-positive, we use the fact that if $j \in N_i$, then $i \in N_j$, so for a term $\nu u_i \cdot \frac{u_j - u_i}{(x_j - x_i)\cdot n_{ij}}A_{ij}$ with $j \in N_i$, we get also a term $\nu u_j \cdot \frac{u_i - u_j}{(x_i - x_j)\cdot n_{ji}}A_{ji}$ with $i \in N_j$. So we get the sum over pairs

$$\sum_i u_i \cdot \frac{\partial u_i}{\partial t}\Omega_i = \nu\sum_i\sum_{j\in N_i} u_i \cdot \frac{u_j - u_i}{(x_j - x_i)\cdot n_{ij}}A_{ij}$$
$$= \frac{1}{2}\nu \sum_i\sum_{j\in N_i}\left(u_i \cdot \frac{u_j - u_i}{(x_j - x_i)\cdot n_{ij}}A_{ij} + u_j \cdot \frac{u_i - u_j}{(x_i - x_j)\cdot n_{ji}}A_{ji}\right)$$

When we now use $u_j \cdot \frac{u_i - u_j}{(x_i - x_j)\cdot n_{ji}} A_{ji} = -u_j \cdot \frac{u_j - u_i}{(x_j - x_i)\cdot n_{ij}} A_{ij}$ we get

$$\begin{aligned}
\sum_i u_i \cdot \frac{\partial u_i}{\partial t}\Omega_i &= \frac{1}{2}\nu \sum_i \sum_{j \in N_i} (u_i - u_j) \cdot \frac{u_j - u_i}{(x_j - x_i)\cdot n_{ij}} A_{ij} \\
&= \frac{1}{2}\underbrace{\nu}_{>0} \sum_i \sum_{j \in N_i} \underbrace{(u_i - u_j)\cdot(u_j - u_i)}_{\leq 0} \underbrace{\frac{1}{(x_j - x_i)\cdot n_{ij}}}_{>0} \underbrace{A_{ij}}_{>0} \\
&\leq 0
\end{aligned}$$

From this we see that the kinetic energy equals zero if $u_i = u_j$ for all pairs i and $j \in N_i$, which means that u is constant. Because we use the no-slip boundary condition, $u = 0$ in this case. So if $u \neq 0$, the kinetic energy will be negative. So the discrete and continuous form of the incompressible Navier-Stokes equations give the same conclusion.

4 QR model

In this section we will derive a model called the QR model. Therefore we will first give some notation and useful lemmas which we will use in the derivation of the OR model. Then we will start from large-eddy simulation (LES) and use scale-truncation to get the QR model

$$\tau - \frac{1}{3}\mathrm{tr}(\tau)I = -2C_\delta \frac{\max\{r(v), 0\}}{q(v)} S$$

4.1 notation and useful lemmas

In this subsection, we will give some notations and lemmas which we will use during this section.

4.1.1 notation

For clarity, we will first recall the definition of the Frobenius inner product

$$A : B = \sum_{i,j} A_{ij} B_{ij} = tr(A^{\mathrm{T}} B) = tr(B^{\mathrm{T}} A)$$

We will use the following notations

$$|A|^2 := A : A$$

$$|A|^3 := A^2 : A$$

$$||A||^2 := \int_{\Omega_\delta} A : A \,\mathrm{d}x = \int_{\Omega_\delta} |A|^2 \,\mathrm{d}x$$

$$||A||^3 := \int_{\Omega_\delta} A^2 : A \,\mathrm{d}x = \int_{\Omega_\delta} |A|^3 \,\mathrm{d}x$$

$$(\nabla v)_{ij} := \frac{\partial v_j}{\partial x_i}$$

$\overrightarrow{e_i}$ is the i^{th} standard basic vector in $\mathbb{R}^3$.

We define the Levi-Civita symbol ϵ_{ijk} as

$$\epsilon_{ijk} := \begin{cases} 1 & \text{if } (i,j,k) \text{ is } (1,2,3),\ (2,3,1) \text{ or } (3,1,2) \\ -1 & \text{if } (i,j,k) \text{ is } (1,3,2),\ (2,1,3) \text{ or } (3,2,1) \\ 0 & \text{if } i=j,\ i=k \text{ or } j=k \end{cases} \tag{28}$$

For the rest of this section, we will call S and T the symmetric and antisymmetric part of ∇v:

$$S := \frac{1}{2}\left(\nabla v + \nabla v^{\mathrm{T}}\right), \quad T := \frac{1}{2}\left(\nabla v - \nabla v^{\mathrm{T}}\right)$$

We define ω as the curl of v.

The QR model uses the invariants of S. We define $q(v)$ and $r(v)$ as minus the second and third invariant of S. Then $q(v)$ and $r(v)$ are given by

$$q(v) = \frac{1}{2}|S|^2, \qquad r(v) = -\frac{1}{3}|S|^3$$

We will show this in lemma 4.1.

In this section we will not write down the summation sign ($\sum$) for clarity reasons. So one should read for example $\sum_{ij} A_{ij}B_{ij}$ for $A_{ij}B_{ij}$. When we integrate over Ω_δ, we use periodic boundary conditions, so the boundary terms are equal to zero. So every time we use integration by parts, we will neglect the boundary terms.

4.1.2 useful lemmas

Lemma 4.1. *Let v be in $\mathbb{R}^3$ such that $\nabla \cdot v = 0$ and let S be the symmetric part of ∇v, then the second and third invariant of S, $q(v)$ and $r(v)$ are given by*

$$q(v) = \frac{1}{2}|S|^2, \qquad r(v) = -\frac{1}{3}|S|^3$$

Proof. Let $v \in \mathbb{R}^3$ be such that $\nabla \cdot v = 0$ and let S be the symmetric part of ∇v. Then the invariants of S are defined using the characteristic polynomial of S:

$$\lambda^3 + \mathrm{I}\lambda^2 + \mathrm{II}\lambda + \mathrm{III}$$

and are given by

$$\begin{aligned} \mathrm{I} &= \mathrm{tr}(S) \\ \mathrm{II} &= \frac{1}{2}\left(\mathrm{tr}(S)^2 - \mathrm{tr}(S^2)\right) \\ \mathrm{III} &= \det(S) = \frac{1}{6}\mathrm{tr}(S)^3 - \frac{1}{2}\mathrm{tr}(S^2)\mathrm{tr}(S) + \frac{1}{3}\mathrm{tr}(S^3) \end{aligned}$$

When we now use $\nabla \cdot v = 0$, these invariants become

$$\begin{aligned} \mathrm{I} &= \mathrm{tr}(S) = \nabla \cdot v = 0 \\ \mathrm{II} &= \frac{1}{2}\left(\mathrm{tr}(S)^2 - \mathrm{tr}(S^2)\right) = -\frac{1}{2}\mathrm{tr}(S^2) \\ \mathrm{III} &= \frac{1}{6}\mathrm{tr}(S)^3 - \frac{1}{2}\mathrm{tr}(S^2)\mathrm{tr}(S) + \frac{1}{3}\mathrm{tr}(S^3) = \frac{1}{3}\mathrm{tr}(S^3) \end{aligned}$$

When we now call $q(v)$ minus the second invariant and use the definition of the Frobenius inner product, we get

$$q(v) = \frac{1}{2}\mathrm{tr}(S^2) = \frac{1}{2}\mathrm{tr}(S^{\mathrm{T}}S) = \frac{1}{2}|S|^2$$

When we now call $r(v)$ minus the third invariant and use the definition of the Frobenius inner product, we get

$$r(v) = -\frac{1}{3}\mathrm{tr}(S^3) = -\frac{1}{3}\mathrm{tr}(S^{\mathrm{T}}S^2) = -\frac{1}{3}|S|^3$$

□

Lemma 4.2. *If A and B are respectively a symmetric and a skew-symmetric matrix, then $A : B = 0$*

Proof. Let A be a symmetric matrix and B an antisymmetric matrix. Then

$$\begin{aligned} A : B = A_{ij}B_{ij} = A_{ji}B_{ji} &= \left(A^T\right)_{ij}\left(B^T\right)_{ij} \\ &= A^T : B^T = A : (-B) = -A : B \end{aligned}$$

So $A : B = -A : B$, thus $A : B$ has to be zero □

Corollary 4.3. *If S is a symmetric matrix and A a matrix, then*

$$S : A = S : B$$

where B is the symmetric part of A

Lemma 4.4. *If v is a vector in $\mathbb{R}^3$ with $\nabla \cdot v = 0$, then $\Delta v = \nabla \cdot \nabla v = 2\nabla \cdot S$, where S is the symmetric part of ∇v*

Proof. Let $v = (v_1, v_2, v_3)$ and $\nabla \cdot v = \frac{\partial v_i}{\partial x_i} = 0$ then

$$\begin{aligned} 2\nabla \cdot S &= \nabla \cdot (\nabla v + \nabla v^{\mathrm{T}}) \\ &= \frac{\partial}{\partial x_i}\left(\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j}\right)\overrightarrow{e}_j \\ &= \frac{\partial^2 v_j}{\partial x_i^2}\overrightarrow{e}_j + \frac{\partial}{\partial x_j}\frac{\partial v_i}{\partial x_i}\overrightarrow{e}_j \end{aligned}$$

This last term is equal to zero because $\frac{\partial v_i}{\partial x_i} = 0$, so

$$2\nabla \cdot S = \frac{\partial^2 v_j}{\partial x_i^2}\overrightarrow{e_j} = \Delta v$$

□

Lemma 4.5. *If ω is the curl of a vector $v \in \mathbb{R}^3$, then we can write ω_i as*

$$\omega_i = \epsilon_{ijk}\frac{\partial v_k}{\partial x_j}$$

Proof. Let ω be the curl of v. Then, by definition,

$$\begin{aligned}
\omega &= \begin{pmatrix} \overrightarrow{e_1} & \overrightarrow{e_2} & \overrightarrow{e_3} \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ v_1 & v_2 & v_3 \end{pmatrix} \\
&= \left(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}\right)\overrightarrow{e_1} + \left(\frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}\right)\overrightarrow{e_2} + \left(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right)\overrightarrow{e_3} \\
&= \epsilon_{ijk}\frac{\partial v_k}{\partial x_j}\overrightarrow{e_i}
\end{aligned}$$

So $\omega_i = \epsilon_{ijk}\frac{\partial v_k}{\partial x_j}$ □

Lemma 4.6. *If v is a vector in $\mathbb{R}^3$, T the antisymmetric part of ∇v and ω the curl of v, then we can write T_{ij} as*

$$T_{ij} = \frac{1}{2}\epsilon_{ijk}\omega_k$$

Proof. Let $v \in \mathbb{R}^3$ and let T and ω be respectively the antisymmetric part and the curl of v. Then we get when we write T out

$$\begin{aligned}
T &= \frac{1}{2}\left(\nabla v - (\nabla v)^{\mathrm{T}}\right) \\
&= \frac{1}{2}\begin{pmatrix} 0 & \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} & \frac{\partial v_3}{\partial x_1} - \frac{\partial v_1}{\partial x_3} \\ \frac{\partial v_1}{\partial x_2} - \frac{\partial v_2}{\partial x_1} & 0 & \frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3} \\ \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1} & \frac{\partial v_2}{\partial x_3} - \frac{\partial v_3}{\partial x_2} & 0 \end{pmatrix}
\end{aligned}$$

When we now apply lemma 4.5, we get

$$T = \frac{1}{2}\begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & \epsilon_{123}\omega_3 & \epsilon_{132}\omega_2 \\ \epsilon_{213}\omega_3 & 0 & \epsilon_{231}\omega_1 \\ \epsilon_{132}\omega_2 & \epsilon_{321}\omega_1 & 0 \end{pmatrix}$$

So

$$T_{ij} = \frac{1}{2}\epsilon_{ijk}\omega_k$$

□

Lemma 4.7. *If a is a vector in $\mathbb{R}^3$ with $\nabla \cdot a = 0$, then*

$$\nabla \times (\nabla \times a) = -\Delta a$$

Proof. Let $a \in \mathbb{R}^3$ be such that $\nabla \cdot a = 0$, then we use 4.5 to see that the i^{th} element of a is given by

$$(\nabla \times (\nabla \times a))_i = \epsilon_{ijk}\frac{\partial}{\partial x_j}(\nabla \times a)_k = \epsilon_{ijk}\frac{\partial}{\partial x_j}\left(\epsilon_{klm}\frac{\partial a_m}{\partial x_l}\right)$$

Now we use $\epsilon_{klm} = \epsilon_{lmk}$ to get

$$(\nabla \times (\nabla \times a))_i = \epsilon_{ijk}\epsilon_{lmk}\frac{\partial}{\partial x_l}\frac{\partial a_m}{\partial x_j}$$

The product $\epsilon_{ijk}\epsilon_{lmk}$ is not zero if $i \neq j \neq k$ and $l \neq m \neq k$, in which cases $i = l$ and $j = m$ or $i = m$ and $j = l$. Thus

$$\begin{aligned}(\nabla \times (\nabla \times a))_i &= \epsilon_{ijk}\epsilon_{ijk}\frac{\partial}{\partial a_i}\frac{\partial v_j}{\partial x_j} + \epsilon_{ijk}\epsilon_{jik}\frac{\partial}{\partial x_j}\frac{\partial a_i}{\partial x_j} \\ &= \gamma_{ij}\frac{\partial}{\partial x_i}\frac{\partial a_j}{\partial x_j} - \gamma_{ij}\frac{\partial^2 a_i}{\partial x_j^2}\end{aligned}$$

where $\gamma_{ij} = \begin{cases} 0 & \text{if } i = j \\ 1 & \text{if } i \neq j \end{cases}$. When we now use $\nabla \cdot a = \frac{\partial a_j}{\partial x_j} = 0$, the first term becomes

$$\gamma_{ij}\frac{\partial}{\partial x_i}\frac{\partial a_j}{\partial x_j} = -\frac{\partial}{\partial x_i}\frac{\partial a_i}{\partial x_i} = -\frac{\partial^2 a_i}{\partial x_i^2}$$

so we end up with

$$(\nabla \times (\nabla \times a)) = \left(-\frac{\partial^2 a_i}{\partial x_i^2} - \gamma_{ij}\frac{\partial^2 a_i}{\partial x_j^2}\right)\overrightarrow{e_i} = -\frac{\partial^2 a_i}{\partial x_j^2}\overrightarrow{e_i} = -\Delta a$$

□

Lemma 4.8. *If a and b are vectors in $\mathbb{R}^3$, then, if we neglect boundary terms,*

$$\int (\nabla \times a) \cdot b \; dx = \int a \cdot (\nabla \times b) \; dx$$

Proof. Let $a, b \in \mathbb{R}^3$. Then

$$\int (\nabla \times a) \cdot b \, \mathrm{d}x = \int \epsilon_{ijk}\frac{\partial a_k}{\partial x_j} b_i \, \mathrm{d}x$$

Now we use integration by parts to get

$$\int (\nabla \times a) \cdot b \, \mathrm{d}x = -\int \epsilon_{ijk} a_k \frac{\partial b_i}{\partial x_j} \, \mathrm{d}x$$

where we neglect the boundary terms. Now we use $\epsilon_{ijk} = -\epsilon_{kji}$ to get

$$\int (\nabla \times a) \cdot b \, \mathrm{d}x = \int a_k \epsilon_{kji}\frac{\partial b_i}{\partial x_j} \, \mathrm{d}x = \int a_k (\nabla \times b)_k \, \mathrm{d}x = \int a \cdot (\nabla \times b) \, \mathrm{d}x$$

□

Corollary 4.9. *When we combine lemmas 4.7 and 4.8, we get*

$$\int (\nabla \times a) \cdot (\nabla \times b) \; dx = \int a \cdot (\nabla \times (\nabla \times b)) \; dx - \int a \cdot \Delta b \; dx$$

Lemma 4.10. *If a is a vector in $\mathbb{R}^3$ with $\nabla \cdot a = 0$, then*

$$\nabla \cdot (\Delta a) = 0$$

Proof. Let $a \in \mathbb{R}^3$ such that $\nabla \cdot a = \frac{\partial a_i}{\partial x_i} = 0$, then

$$\nabla \cdot (\Delta a) = \frac{\partial}{\partial x_i}\frac{\partial^2 a_i}{\partial x_j^2} = \frac{\partial^2}{\partial x_j^2}\frac{\partial a_i}{\partial x_i} = 0$$

□

Lemma 4.11. *If a is a vector in $\mathbb{R}^3$, S the symmetric part of ∇a and $\tilde{S}$ the symmetric part of $\nabla \Delta a$, then*

$$\tilde{S} = \Delta S$$

Proof. Let $a \in \mathbb{R}^3$, S be the symmetric part of ∇a and $\tilde{S}$ be the symmetric part of $\nabla \Delta a$. Then we get when we write $\tilde{S}_{ij} = \frac{1}{2}\left((\nabla \Delta a)_{ij} + ((\nabla \Delta a)^{\mathrm{T}})_{ij}\right)$ out

$$\tilde{S}_{ij} = \frac{1}{2}\left(\frac{\partial (\Delta a)_j}{\partial x_i} + \frac{\partial (\Delta a)_i}{\partial x_j}\right) = \frac{1}{2}\left(\frac{\partial}{\partial x_i}\frac{\partial^2 a_j}{\partial x_k^2} + \frac{\partial}{\partial x_j}\frac{\partial^2 a_i}{\partial x_k^2}\right)$$

When we now reorder terms we get

$$\tilde{S}_{ij} = \frac{\partial^2}{\partial x_k^2}\left(\frac{1}{2}\left(\frac{\partial a_j}{\partial x_i} + \frac{\partial a_j}{\partial x_i}\right)\right) = \frac{\partial^2}{\partial x_k^2} S_{ij} = (\Delta S)_{ij}$$

□

Lemma 4.12. *If v is a vector in $\mathbb{R}^3$ with $\nabla \cdot v = 0$ and we use the definition of S stated at the start of this subsection, then*

$$\int (v \cdot \nabla) v \cdot \Delta v \; dx = -\int \nabla v^T \nabla v : S \; dx$$

Proof. Let $v \in \mathbb{R}^3$ be such that $\nabla \cdot v = 0$ and let S be the symmetric part of ∇v.

Writing out $\int (v \cdot \nabla) v \cdot \Delta v \; \mathrm{d}x$ gives

$$\int (v \cdot \nabla) v \cdot \Delta v \; \mathrm{d}x = \int v_k \frac{\partial}{\partial x_k} v_j \frac{\partial^2}{\partial x_i^2} v_j \; \mathrm{d}x \tag{29}$$

Integration by parts gives

$$\begin{aligned} \int v_k \frac{\partial}{\partial x_k} v_j \frac{\partial^2}{\partial x_i^2} v_j \; \mathrm{d}x &= -\int \frac{\partial}{\partial x_i}\left(v_k \frac{\partial v_j}{\partial x_k}\right) \frac{\partial v_j}{\partial x_i} \; \mathrm{d}x \\ &= -\int \frac{\partial v_k}{\partial x_i} \frac{\partial v_j}{\partial x_k} \frac{\partial v_j}{\partial x_i} \; \mathrm{d}x - \int v_k \frac{\partial^2 v_j}{\partial x_i \partial x_k} \frac{\partial v_j}{\partial x_i} \; \mathrm{d}x \end{aligned} \tag{30}$$

We will look a little bit closer on this last term. For clarity, we call $a = \frac{\partial v_j}{\partial x_i}$, so we get

$$\int v_k \frac{\partial a}{\partial x_k} a \; \mathrm{d}x = \int \frac{\partial}{\partial x_k}(v_k a)\, a \; \mathrm{d}x - \int \frac{\partial v_k}{\partial x_k} a^2 \; \mathrm{d}x$$

Using $\nabla \cdot v = \frac{\partial v_k}{\partial x_k} = 0$ the last term vanishes, so

$$\int v_k \frac{\partial a}{\partial x_k} a \; \mathrm{d}x = \int \frac{\partial}{\partial x_k}(v_k a)\, a \; \mathrm{d}x$$

When we now use integration by parts we get

$$\int v_k \frac{\partial a}{\partial x_k} a \; \mathrm{d}x = -\int v_k a \frac{\partial a}{\partial x_k} \; \mathrm{d}x = -\int v_k \frac{\partial a}{\partial x_k} a \; \mathrm{d}x$$

So we conclude

$$\int v_k \frac{\partial a}{\partial x_k} a \; \mathrm{d}x = 0$$

Now, as we go back to equation 30, we get

$$\int v_k \frac{\partial}{\partial x_k} v_j \frac{\partial^2}{\partial x_i^2} v_j \, \mathrm{d}x = -\int \frac{\partial v_k}{\partial x_i} \frac{\partial v_j}{\partial x_k} \frac{\partial v_j}{\partial x_i} \, \mathrm{d}x \tag{31}$$

When we work this out, we get

$$\frac{\partial v_k}{\partial x_i} \frac{\partial v_j}{\partial x_k} \frac{\partial v_j}{\partial x_i} = (\nabla v)_{ik} (\nabla v)_{kj} (\nabla v)_{ij} = (\nabla v \nabla v)_{ij} (\nabla v)_{ij} = (\nabla v^2) : \nabla v$$

With the definition of the Frobenius inner product, we rewrite this as

$$\frac{\partial v_k}{\partial x_i} \frac{\partial v_j}{\partial x_k} \frac{\partial v_j}{\partial x_i} = (\nabla v^2) : \nabla v = \mathrm{tr}((\nabla v^2)^{\mathrm{T}} \nabla v) = \mathrm{tr}(\nabla v^{\mathrm{T}} (\nabla v^{\mathrm{T}} \nabla v)) = (\nabla v^{\mathrm{T}} \nabla v) : \nabla v$$

So equation 31 becomes, using corollary 4.3,

$$\int v_k \frac{\partial}{\partial x_k} v_j \frac{\partial^2}{\partial x_i^2} v_j \, \mathrm{d}x = \int (\nabla v^{\mathrm{T}} \nabla v) : \nabla v \, \mathrm{d}x = \int (\nabla v^{\mathrm{T}} \nabla v) : S \, \mathrm{d}x$$

Thus we get, using equation 31,

$$\int (v \cdot \nabla) v \cdot \Delta v \, \mathrm{d}x = -\int \left(\nabla v^{\mathrm{T}} \nabla v\right) : S \, \mathrm{d}x$$

□

Lemma 4.13. *If v is a vector in $\mathbb{R}^3$ with $\nabla \cdot v = 0$ and we use the definitions of ω and ϵ_{ijk} stated at the start of this subsection, then*

$$\nabla \times ((v \cdot \nabla) v) = -(\omega \cdot \nabla) v + (v \cdot \nabla) \omega$$

Proof. Let $v \in \mathbb{R}^3$ be such that $\nabla \cdot v = 0$ and let ω be the curl of v and ϵ_{ijk} be the Levi-Civita symbol.

Using lemma 4.5, we get

$$\begin{aligned} \nabla \times ((v \cdot \nabla) v) &= \nabla \times \left(v_i \frac{\partial v_j}{\partial x_i} \overrightarrow{e_j} \right) = \epsilon_{jkl} \frac{\partial}{\partial x_k} \left(v_i \frac{\partial v_l}{\partial x_i} \right) \overrightarrow{e_j} \\ &= \epsilon_{jkl} \frac{\partial v_i}{\partial x_k} \frac{\partial v_l}{\partial x_i} \overrightarrow{e_j} + \epsilon_{jkl} v_i \frac{\partial^2 v_l}{\partial x_i \partial x_k} \overrightarrow{e_j} \end{aligned} \tag{32}$$

We will work this two terms out. For $\epsilon_{jkl} \frac{\partial v_i}{\partial x_k} \frac{\partial v_l}{\partial x_i} \overrightarrow{e_j}$, the j^{th} component is given by $\epsilon_{jkl} \left(\frac{\partial v_i}{\partial x_k} \frac{\partial v_l}{\partial x_i} - \frac{\partial v_i}{\partial x_l} \frac{\partial v_k}{\partial x_i} \right)$, where k and l are chosen such that $k \neq l \neq j$. Now we write out the sum over i by filling in j, k and l for i and add $\frac{\partial v_j}{\partial x_k} \frac{\partial v_j}{\partial x_l} - \frac{\partial v_j}{\partial x_l} \frac{\partial v_j}{\partial x_k}$.

$$\begin{aligned} \epsilon_{jkl} \left(\frac{\partial v_i}{\partial x_k} \frac{\partial v_l}{\partial x_i} - \frac{\partial v_i}{\partial x_l} \frac{\partial v_k}{\partial x_i} \right) &= \epsilon_{jkl} \Bigg(\frac{\partial v_l}{\partial x_j} \frac{\partial v_j}{\partial x_k} - \frac{\partial v_k}{\partial x_j} \frac{\partial v_j}{\partial x_l} + \frac{\partial v_l}{\partial x_k} \frac{\partial v_k}{\partial x_k} - \frac{\partial v_k}{\partial x_k} \frac{\partial v_k}{\partial x_l} \\ &\quad + \frac{\partial v_l}{\partial x_l} \frac{\partial v_l}{\partial x_k} - \frac{\partial v_k}{\partial x_l} \frac{\partial v_l}{\partial x_l} + \frac{\partial v_j}{\partial x_k} \frac{\partial v_j}{\partial x_l} - \frac{\partial v_j}{\partial x_l} \frac{\partial v_j}{\partial x_k} \Bigg) \\ &= -\epsilon_{jkl} \Bigg(\left(\frac{\partial v_l}{\partial x_k} - \frac{\partial v_k}{\partial x_l} \right) \left(-\frac{\partial v_k}{\partial x_k} - \frac{\partial v_l}{\partial x_l} \right) \\ &\quad + \left(\frac{\partial v_j}{\partial x_l} - \frac{\partial v_l}{\partial x_j} \right) \frac{\partial v_j}{\partial x_k} + \left(\frac{\partial v_k}{\partial x_j} - \frac{\partial v_j}{\partial x_k} \right) \frac{\partial v_j}{\partial x_l} \Bigg) \end{aligned}$$

Now we use $\frac{\partial v_j}{\partial x_j} = -\frac{\partial v_k}{\partial x_k} - \frac{\partial v_l}{\partial x_l}$ to get

$$\epsilon_{jkl}\left(\frac{\partial v_i}{\partial x_k}\frac{\partial v_l}{\partial x_i} - \frac{\partial v_i}{\partial x_l}\frac{\partial v_k}{\partial x_i}\right)$$
$$= -\epsilon_{jkl}\left(\left(\frac{\partial v_l}{\partial x_k} - \frac{\partial v_k}{\partial x_l}\right)\frac{\partial v_j}{\partial x_j} + \left(\frac{\partial v_j}{\partial x_l} - \frac{\partial v_l}{\partial x_j}\right)\frac{\partial v_j}{\partial x_k} + \left(\frac{\partial v_k}{\partial x_j} - \frac{\partial v_j}{\partial x_k}\right)\frac{\partial v_j}{\partial x_l}\right)$$

When we now use lemma 4.5, we get

$$\epsilon_{jkl}\left(\frac{\partial v_i}{\partial x_k}\frac{\partial v_l}{\partial x_i} - \frac{\partial v_i}{\partial x_l}\frac{\partial v_k}{\partial x_i}\right) = -\left(\omega_j\frac{\partial v_j}{\partial x_j} + \omega_k\frac{\partial v_j}{\partial x_k} + \omega_l\frac{\partial v_j}{\partial x_l}\right) = -\left(\omega_i\frac{\partial}{\partial x_i}\right)v_j$$

Thus

$$\epsilon_{jkl}\frac{\partial v_i}{\partial x_k}\frac{\partial v_l}{\partial x_i}\overrightarrow{e_j} = -\left(\omega_i\frac{\partial}{\partial x_i}\right)v_j\overrightarrow{e_j} = -(\omega\cdot\nabla)v \tag{33}$$

For $\epsilon_{jkl}v_i\frac{\partial^2 v_l}{\partial x_i\partial x_k}\overrightarrow{e_j}$, we get, using again lemma 4.5,

$$\epsilon_{jkl}v_i\frac{\partial^2 v_l}{\partial x_i\partial x_k}\overrightarrow{e_j} = v_i\frac{\partial}{\partial x_i}\epsilon_{jkl}\frac{\partial v_l}{\partial x_k}\overrightarrow{e_j} = (v\cdot\nabla)\omega \tag{34}$$

Combining equations 32, 33 and 34 gives us

$$\nabla\times((v\cdot\nabla)v) = -(\omega\cdot\nabla)v + (v\cdot\nabla)\omega$$

□

Lemma 4.14. *If v is a vector in $\mathbb{R}^3$ with $\nabla\cdot v = 0$ and we use the definition of ω stated at the start of this subsection, then*

$$\int((v\cdot\nabla)\omega)\cdot\omega\ dx = 0$$

Proof. Let $v\in\mathbb{R}^3$ be such that $\nabla\cdot v = 0$ and let ω be the curl of v.

Writing out $\int((v\cdot\nabla)\omega)\cdot\omega$ dx gives

$$\int((v\cdot\nabla)\omega)\cdot\omega\ \mathrm{d}x = \int v_i\frac{\partial\omega_j}{\partial x_i}\omega_j\ \mathrm{d}x = \int\frac{\partial\omega_j}{\partial x_i}v_i\omega_j\ \mathrm{d}x$$

Now we use integration by parts

$$\int((v\cdot\nabla)\omega)\cdot\omega\ \mathrm{d}x = -\int\omega_j\frac{\partial v_i\omega_j}{\partial x_i}\ \mathrm{d}x$$
$$= -\int\omega_j\frac{\partial v_i}{\partial x_i}\omega_j\ \mathrm{d}x - \int\omega_j v_i\frac{\partial\omega_j}{\partial x_i}\ \mathrm{d}x$$

Now we use $\nabla\cdot v = \frac{\partial v_i}{\partial x_i} = 0$ and see that the first term is equal to zero. So we get

$$\int((v\cdot\nabla)\omega)\cdot\omega\ \mathrm{d}x = -\int v_i\frac{\partial\omega_j}{\partial x_i}\omega_j\ \mathrm{d}x$$
$$= -\int((v\cdot\nabla)\omega)\cdot\omega\ \mathrm{d}x$$

So we conclude

$$\int ((v\cdot\nabla)\omega)\cdot\omega \; \mathrm{d}x = 0$$

□

Lemma 4.15. *If v is a vector in $\mathbb{R}^3$ with $\nabla\cdot v = 0$ and we use the definitions of ω, ϵ_{ijk}, S and T stated at the start of this subsection, then*

$$((\omega\cdot\nabla)v)\cdot\omega = S_{ij}\omega_i\omega_j$$

Proof. Let $v\in\mathbb{R}^3$ be such that $\nabla\cdot v = 0$ and use the definitions of ω, ϵ_{ijk}, S and T stated at the start of this subsection. Then writing out $((\omega\cdot\nabla)v)\cdot\omega$ gives

$$((\omega\cdot\nabla)v)\cdot\omega = \left(\left(\omega_i\frac{\partial}{\partial x_i}\right)v_j\right)\omega_j = \omega_i\frac{\partial v_j}{\partial x_i}\omega_j$$

When we now use $\frac{\partial v_j}{\partial x_i} = (\nabla v)_{ij} = S_{ij} + T_{ij}$ we get

$$((\omega\cdot\nabla)v)\cdot\omega = S_{ij}\omega_i\omega_j + T_{ij}\omega_i\omega_j \tag{35}$$

Let $\omega_{123} = \omega_1\omega_2\omega_3$, then, using lemma 4.6 and $\sum_{ijk}\epsilon_{ijk} = 0$ we get

$$T_{ij}\omega_i\omega_j = \frac{1}{2}\epsilon_{ijk}\omega_k\omega_i\omega_j = \frac{1}{2}\epsilon_{ijk}\omega_{123} = 0$$

When we put this into equation 35, we get

$$((\omega\cdot\nabla)v)\cdot\omega = S_{ij}\omega_i\omega_j$$

□

Lemma 4.16. *If v is a vector in $\mathbb{R}^3$ with $\nabla\cdot v = 0$ and we use the definitions of ω, ϵ_{ijk}, S and T stated at the start of this subsection, then*

$$(\nabla v^T\nabla v) : S = |S|^3 - \frac{1}{4}S_{jk}\omega_j\omega_k$$

Proof. Let $v\in\mathbb{R}^3$ be such that $\nabla\cdot v = 0$ and use the definitions of ω, ϵ_{ijk}, S and T stated at the start of this subsection. We rewrite $(\nabla v^{\mathrm{T}}\nabla v) : S$ using $\nabla v = S + T$

$$\begin{aligned}(\nabla v^{\mathrm{T}}\nabla v) : S &= (S+T)^{\mathrm{T}}(S+T) : S\\ &= \left(S^{\mathrm{T}}S + S^{\mathrm{T}}T + T^{\mathrm{T}}S + T^{\mathrm{T}}T\right) : S\end{aligned}$$

Using the definition of the Frobenius inner product, we get

$$S^{\mathrm{T}}T : S + T^{\mathrm{T}}S : S = \mathrm{tr}(TS^{\mathrm{T}}S) + \mathrm{tr}(T^{\mathrm{T}}SS^{\mathrm{T}}) = \mathrm{tr}(TSS) - \mathrm{tr}(TSS) = 0$$

So

$$(\nabla v^{\mathrm{T}}\nabla v) : S = S^2 : S + T^{\mathrm{T}}T : S = |S|^3 + T_{ik}T_{ij}S_{kj} \tag{36}$$

We compute $S_{jk}T_{ik}T_{ij}$ using lemma 4.6:

$$S_{jk}T_{ik}T_{ij} = S_{jk}\left(\frac{1}{2}\epsilon_{ikm}\omega_m\right)\left(\frac{1}{2}\epsilon_{ijl}\omega_l\right) = S_{jk}\frac{1}{4}\epsilon_{ikm}\epsilon_{ijl}\omega_m\omega_l$$

For the product $\epsilon_{ikm}\epsilon_{ijl}$ we have two possibilities, namely

$$\epsilon_{ikm}\epsilon_{ijl} = \begin{cases} 1 & \text{if } k = j \text{ and } l = m \\ -1 & \text{if } l = k \text{ and } m = j \end{cases}$$

So we get

$$S_{jk}T_{ik}T_{ij} = \frac{1}{4}S_{jj}\omega_m\omega_m - \frac{1}{4}S_{jk}\omega_j\omega_k$$

Using $S_{jj} = \frac{\partial v_j}{\partial x_j} = 0$ this equation becomes

$$S_{jk}T_{ik}T_{ij} = -\frac{1}{4}S_{jk}\omega_j\omega_k$$

When we fill this into equation 36, we get

$$(\nabla v^{\mathrm{T}}\nabla v) : S = |S|^3 - \frac{1}{4}S_{jk}\omega_j\omega_k$$

□

4.2 LES

In this subsection, we introduce a methodology called large-eddy simulation (LES). For LES, we add a filter to the incompressible Navier-Stokes equations. In this filter, only the large eddies in a flow are resolved. So we get a filtered solution $\bar{u}$ which is much easier to compute by a computer.

We start again with the Navier-Stokes equations (equations 2 and 5)

$$\frac{\partial u}{\partial t} + \nabla \cdot (u \otimes u) + \nabla p - \nu\Delta u = 0$$

$$\nabla \cdot u = 0$$

Now we will use a filter, which we will denote by an overline. Because it will be a linear filter, we can apply it on each component:

$$\overline{\frac{\partial u}{\partial t}} + \overline{\nabla \cdot (u \otimes u)} + \overline{\nabla p} - \overline{\nu\Delta u} = \overline{0} = 0$$

When we now assume that $\overline{\nabla \cdot a} = \nabla \cdot \overline{a}$, we get

$$\frac{\partial \overline{u}}{\partial t} + \nabla \cdot \overline{u \otimes u} + \nabla\overline{p} - \nu\Delta\overline{u} = 0$$

When we rewrite this a little bit, we get the filtered incompressible Navier-Stokes equations:

$$\frac{\partial \bar{u}}{\partial t} + \nabla \cdot (\bar{u} \otimes \bar{u}) + \nabla\bar{p} - \nu\Delta\bar{u} = -\nabla \cdot (\overline{u \otimes u} - \bar{u} \otimes \bar{u})$$

$$\nabla \cdot \bar{u} = 0$$

with filtered velocity field $\bar{u}$. When we now call $\tau = \overline{u \otimes u} - \bar{u} \otimes \bar{u}$ and use the approximation $v \approx \bar{u}$ and $p \approx \bar{p}$, we get, using equation 3

$$\frac{\partial v}{\partial t} + (v \cdot \nabla)v - \nu \Delta v + \nabla p = -\nabla \cdot \tau \tag{37}$$

$$\nabla \cdot v = 0 \tag{38}$$

4.3 Finding the QR model

In this section we will find the QR model by splitting v into an average $\bar{v}$ and residue v' and find the following condition for which the residual field will be suppressed.

$$4 \int_{\Omega_\delta} r(v) \text{ d}x \leq \int_{\Omega_\delta} \tau : \Delta S \text{ d}x$$

We set $v = \bar{v} + v'$, where

$$\bar{v} = \frac{1}{|\Omega_\delta|} \int_{\Omega_\delta} v(x,t) \text{ d}x \tag{39}$$

is the average of v on a periodic box Ω_δ with diameter δ. The residual field in v is then v'. When we want to use this $\bar{v}$ in our model, the residual field should not become significant. Therefore we bound the kinetic energy from above

$$\frac{1}{2} \frac{d}{dt} ||v'||^2 \leq 0 \tag{40}$$

Note that when we satisfy this condition, and we add the initial condition $||v'||(t = 0) = 0$ and boundary condition $v' = 0$, we get $||v'|| = 0$ for all $t > 0$. In order to satisfy condition 41, we use the Poincaré inequality [3]

$$\frac{1}{2} ||v'||^2 \leq \frac{1}{2} C_\delta ||\nabla v||^2$$

where $C_\delta > 0$ is the Poincaré constant. So when we satisfy the condition

$$\frac{1}{2} \frac{d}{dt} ||\nabla v||^2 \leq 0 \tag{41}$$

we also satisfy our condition 40:

$$\frac{1}{2} \frac{d}{dt} ||v'||^2 \leq C_\delta \frac{1}{2} \frac{d}{dt} ||\nabla v||^2 \leq 0$$

With this Poincaré condition, we do not have to compute v' explicitly.

Now we will compute $\frac{1}{2} \frac{d}{dt} ||\nabla v||^2$ and see that it is given by

$$\frac{1}{2} \frac{d}{dt} ||\nabla v||^2 = -\frac{4}{3} ||S||^3 - 2\nu ||\nabla S||^2 - \int_{\Omega_\delta} \tau : \Delta S \text{ d}x$$

When we compute $\frac{1}{2} \frac{d}{dt} ||\nabla v||^2$, we get, using integration by parts,

$$\frac{1}{2} \frac{d}{dt} ||\nabla v||^2 = \int_{\Omega_\delta} \left(\nabla \frac{\partial v}{\partial t} \cdot \nabla v \right) \text{ d}x = \int_{\Omega_\delta} \left(-\frac{\partial v}{\partial t} \cdot \Delta v \right) \text{ d}x$$

Now we fill in equation 37 and get

$$\frac{1}{2}\frac{d}{dt}||\nabla v||^2 = \int_{\Omega_\delta} (v\cdot\nabla)v\cdot\Delta v \text{ d}x - \int_{\Omega_\delta} \nu\Delta v\cdot\Delta v \text{ d}x + \int_{\Omega_\delta} \nabla p\cdot\Delta v \text{ d}x + \int_{\Omega_\delta} (\nabla\cdot\tau)\cdot\Delta v \text{ d}x \tag{42}$$

We will first rewrite the first part ($\int_{\Omega_\delta}(v\cdot\nabla)v\cdot\Delta v \text{ d}x$) of equation 42 and see that it is equal to $-\frac{4}{3}||S||^3$.

We will start again with our filtered Navier-Stokes equations 37 and 38

$$\frac{\partial v}{\partial t} + (v\cdot\nabla)v - \nu\Delta v + \nabla p = -\nabla\cdot\tau \tag{43}$$

$$\nabla\cdot v = 0 \tag{44}$$

From this equations, we will compute $\frac{1}{2}\frac{d}{dt}||\nabla v||^2$ and $\frac{1}{2}\frac{d}{dt}||\omega||^2$ and see that these two quantities are the same and from this will follow that $\int_{\Omega_\delta}(v\cdot\nabla)v\cdot\Delta v \text{ d}x = -\frac{4}{3}||S||^3$

Computing $\frac{1}{2}\frac{d}{dt}||\nabla v||^2$ gives again equation 42:

$$\frac{1}{2}\frac{d}{dt}||\nabla v||^2 = \int_{\Omega_\delta} (v\cdot\nabla)v\cdot\Delta v dx + \int_{\Omega_\delta} \nabla p\cdot\Delta v \text{ d}x - \nu||\Delta v||^2 + \int_{\Omega_\delta} (\nabla\cdot\tau)\cdot\Delta v \text{ d}x$$

Using equation 53 $\left(\int_{\Omega_\delta}\nabla p\cdot\Delta v \text{ d}x = 0\right)$, this becomes

$$\frac{1}{2}\frac{d}{dt}||\nabla v||^2 = \int_{\Omega_\delta} (v\cdot\nabla)v\cdot\Delta v dx - \nu||\Delta v||^2 + \int_{\Omega_\delta} (\nabla\cdot\tau)\cdot\Delta v \text{ d}x \tag{45}$$

Now we take the curl of equation 43 and use $\nabla\times\nabla p = 0$ and get

$$\frac{\partial\omega}{\partial t} + \nabla\times((v\cdot\nabla)v) - \nu\nabla\times\Delta v = -\nabla\times(\nabla\cdot\tau)$$

Using lemma 4.13 this equation becomes

$$\frac{\partial\omega}{\partial t} - (\omega\cdot\nabla)v + (v\cdot\nabla)\omega - \nu\nabla\times\Delta v = -\nabla\times(\nabla\cdot\tau)$$

We will now compute $\frac{1}{2}\frac{d}{dt}||\omega||^2$ using the above equation

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}||\omega||^2 &= \int_{\Omega_\delta} \frac{\partial\omega}{\partial t}\cdot\omega \text{ d}x \\ &= \int_{\Omega_\delta} \big(-((v\cdot\nabla)\omega)\cdot\omega + ((\omega\cdot\nabla)v)\cdot\omega \\ &\quad + \nu(\nabla\times\Delta v)\cdot\omega - (\nabla\times(\nabla\cdot\tau))\cdot\omega\big) \text{ d}x\end{aligned}$$

Using lemma 4.14 this equation becomes

$$\frac{1}{2}\frac{d}{dt}||\omega||^2 = \int_{\Omega_\delta} \big(((\omega\cdot\nabla)v)\cdot\omega + \nu(\nabla\times\Delta v)\cdot\omega - (\nabla\times(\nabla\cdot\tau))\cdot\omega\big) \text{ d}x \tag{46}$$

We will now show that $-\int_{\Omega_\delta}(\nabla \times (\nabla \cdot \tau)) \cdot \omega \ \mathrm{d}x = \int_{\Omega_\delta}(\nabla \cdot \tau) \cdot \Delta v \ \mathrm{d}x$. Therefore we use corollary 4.9 and get

$$\begin{aligned} -\int_{\Omega_\delta}(\nabla \times (\nabla \cdot \tau)) \cdot \omega \ \mathrm{d}x &= -\int_{\Omega_\delta}(\nabla \times (\nabla \cdot \tau)) \cdot (\nabla \times v) \ \mathrm{d}x \\ &= \int_{\Omega_\delta}(\nabla \cdot \tau) \cdot (\Delta v) \ \mathrm{d}x \end{aligned}$$

Thus equation 46 becomes

$$\frac{1}{2}\frac{d}{dt}||\omega||^2 = \int_{\Omega_\delta}\big(((\omega \cdot \nabla)v) \cdot \omega + \nu(\nabla \times \Delta v) \cdot \omega + (\nabla \cdot \tau) \cdot \Delta v\big) \ \mathrm{d}x \tag{47}$$

Now we will show that $\int_{\Omega_\delta}(\nabla \times \Delta v) \cdot \omega \ \mathrm{d}x = -||\Delta v||^2$. Therefore we use corollary 4.9 to get

$$\begin{aligned} \int_{\Omega_\delta}(\nabla \times \Delta v) \cdot \omega \ \mathrm{d}x &= \int_{\Omega_\delta}(\nabla \times \Delta v) \cdot (\nabla \times v) \ \mathrm{d}x \\ &= -\int_{\Omega_\delta}(\Delta v) \cdot (\Delta v) \ \mathrm{d}x \\ &= -||\Delta v||^2 \end{aligned}$$

So equation 47 becomes

$$\frac{1}{2}\frac{d}{dt}||\omega||^2 = \int_{\Omega_\delta}((\omega \cdot \nabla)v) \cdot \omega \ \mathrm{d}x - \nu||\Delta v||^2 + \int_{\Omega_\delta}(\nabla \cdot \tau) \cdot \Delta v \ \mathrm{d}x \tag{48}$$

Now we will show that $||\nabla v||^2 = ||\omega||^2$ using again corollary 4.9

$$\begin{aligned} ||\omega||^2 &= \int_{\Omega_\delta}(\nabla \times v)(\nabla \times v) \ \mathrm{d}x \\ &= -\int_{\Omega_\delta} v \cdot (\Delta v) \ \mathrm{d}x \end{aligned}$$

Now we use integration by parts to get

$$\begin{aligned} ||\omega||^2 &= \int_{\Omega_\delta}\nabla v : \nabla v \ \mathrm{d}x \\ &= ||\nabla v||^2 \end{aligned}$$

So we can combine equations 45 and 48 to get

$$\int_{\Omega_\delta}(v \cdot \nabla)v \cdot \Delta v \ \mathrm{d}x = \int_{\Omega_\delta}(\omega \cdot \nabla)v \cdot \omega \ \mathrm{d}x \tag{49}$$

Using lemma 4.12, we get

$$-\int_{\Omega_\delta}(\nabla v^{\mathrm{T}}\nabla v) : S \ \mathrm{d}x = \int_{\Omega_\delta}(\omega \cdot \nabla)v \cdot \omega \ \mathrm{d}x$$

Using lemma 4.15 and lemma 4.16, we get

$$-||S||^3 + \frac{1}{4}\int_{\Omega_\delta} S_{jk}\omega_j\omega_k \ \mathrm{d}x = \int_{\Omega_\delta} S_{jk}\omega_j\omega_k \ \mathrm{d}x$$

thus

$$\int_{\Omega_\delta} S_{jk}\omega_k\omega_j \;\mathrm{d}x = -\frac{4}{3}||S||^3 \tag{50}$$

Now combining equation 49, lemma 4.15 and equation 50, we get

$$\int_{\Omega_\delta} (v\cdot\nabla)v\cdot\Delta v \;\mathrm{d}x = -\frac{4}{3}||S||^3 \tag{51}$$

which is what we wanted to show.

Now we rewrite the second part ($\int_{\Omega_\delta} -\nu\Delta v\cdot\Delta v \;\mathrm{d}x$) of equation 42 and we will see that it is equal to $-2\nu||\nabla S||^2$.

So, for $\int_{\Omega_\delta} -\nu\Delta v\cdot\Delta v \;\mathrm{d}x$ we get, when we use lemma 4.4,

$$\int_{\Omega_\delta} -\nu\Delta v\cdot\Delta v \;\mathrm{d}x = -2\nu\int_{\Omega_\delta} (\nabla\cdot S)\cdot\Delta v \;\mathrm{d}x$$

Integration by parts gives

$$\int_{\Omega_\delta} -\nu\Delta v\cdot\Delta v \;\mathrm{d}x = 2\nu\int_{\Omega_\delta} S:\nabla\Delta v \;\mathrm{d}x$$

Using corollary 4.3 gives

$$\int_{\Omega_\delta} -\nu\Delta v\cdot\Delta v \;\mathrm{d}x = 2\nu\int_{\Omega_\delta} S:\tilde{S} \;\mathrm{d}x$$

where $\tilde{S}$ is the symmetric part of $\nabla\Delta v$. Now we use lemma 4.11 and get

$$\int_{\Omega_\delta} -\nu\Delta v\cdot\Delta v \;\mathrm{d}x = 2\nu\int_{\Omega_\delta} S:\Delta S \;\mathrm{d}x$$

Using again integration by parts gives

$$\int_{\Omega_\delta} -\nu\Delta v\cdot\Delta v \;\mathrm{d}x = -2\nu||\nabla S||^2 \tag{52}$$

which is what we wanted to show.

Now we rewrite the third part ($\int_{\Omega_\delta} \nabla p\cdot\Delta v \;\mathrm{d}x$) of equation 42 and we will see that it is equal to zero.

Integration by parts gives

$$\int_{\Omega_\delta} \nabla p\cdot\Delta v \;\mathrm{d}x = -\int_{\Omega_\delta} p\,(\nabla\cdot(\Delta v)) \;\mathrm{d}x$$

When we now use lemma 4.10 we get

$$\int_{\Omega_\delta} \nabla p\cdot\Delta v \;\mathrm{d}x = 0 \tag{53}$$

At last we will rewrite the fourth part ($\int_{\Omega_\delta} (\nabla\cdot\tau)\cdot\Delta v \;\mathrm{d}x$) of equation 42 and we will see that it is equal to $\int_{\Omega_\delta} -\tau:\Delta S \;\mathrm{d}x$.

Integration by parts gives

$$\int_{\Omega_\delta} (\nabla\cdot\tau)\cdot\Delta v \;\mathrm{d}x = -\int_{\Omega_\delta} \tau:\nabla\Delta v \;\mathrm{d}x$$

Because τ is symmetric, we can use corollary 4.3 to get

$$\int_{\Omega_\delta} (\nabla \cdot \tau) \cdot \Delta v \; \mathrm{d}x = - \int_{\Omega_\delta} \tau : \tilde{S} \; \mathrm{d}x$$

where $\tilde{S}$ is the symmetric part of $\nabla \Delta v$. Using lemma 4.11 we get

$$\int_{\Omega_\delta} (\nabla \cdot \tau) \cdot \Delta v \; \mathrm{d}x = - \int_{\Omega_\delta} \tau : \Delta S \; \mathrm{d}x \tag{54}$$

which is what we wanted to show.

Now we have rewritten all the parts of equation 42 and we can put them together. So we combine equations 42, 51, 52, 53 and 54 and get

$$\frac{1}{2}\frac{d}{dt}||\nabla v||^2 = -\frac{4}{3}||S||^3 - 2\nu||\nabla S||^2 - \int_{\Omega_\delta} \tau : \Delta S \; \mathrm{d}x$$

With this, we see that in order to satisfy our condition 41, τ should satisfy

$$\int_{\Omega_\delta} \tau : \Delta S \; \mathrm{d}x \geq -\frac{4}{3}||S||^3 = 4\int_{\Omega_\delta} r(v) \; \mathrm{d}x$$

So the QR model get the condition

$$4\int_{\Omega_\delta} r(v) \; \mathrm{d}x \leq \int_{\Omega_\delta} \tau : \Delta S \; \mathrm{d}x \tag{55}$$

4.4 QR eddy viscosity

In this section, we will use the following eddy-viscosity model

$$\tau - \frac{1}{3}\mathrm{tr}(\tau)I = -2\nu_e S \tag{56}$$

and see what the QR model becomes with this.

When we put this model into the filtered Navier-Stokes equation 37, we get

$$\frac{\partial v}{\partial t} + (v \cdot \nabla)v - \nu\Delta v + \nabla p = \nabla \cdot (2\nu_e S) - \nabla \cdot (\frac{1}{3}\mathrm{tr}(\tau)I) \tag{57}$$

We will first say something about the term $\nabla \cdot (\frac{1}{3}\mathrm{tr}(\tau)I)$. We can rewrite this as

$$\nabla \cdot (\frac{1}{3}\mathrm{tr}(\tau)I) = \frac{1}{3}\nabla \cdot \begin{pmatrix} \mathrm{tr}(\tau) & 0 & 0 \\ 0 & \mathrm{tr}(\tau) & 0 \\ 0 & 0 & \mathrm{tr}(\tau) \end{pmatrix} = \frac{1}{3}\begin{pmatrix} \frac{\partial \mathrm{tr}(\tau)}{\partial x_1} \\ \frac{\partial \mathrm{tr}(\tau)}{\partial x_2} \\ \frac{\partial \mathrm{tr}(\tau)}{\partial x_3} \end{pmatrix} = \nabla \left(\frac{1}{3}\mathrm{tr}(\tau)\right)$$

So when we set $p' = p + \frac{1}{3}\mathrm{tr}(\tau)$ and call p' again p, equation 57 becomes

$$\frac{\partial v}{\partial t} + (v \cdot \nabla)v - \nu\Delta v + \nabla p = \nabla \cdot (2\nu_e S) \tag{58}$$

When we now use lemma 4.4, this equation becomes

$$\frac{\partial v}{\partial t} + (v \cdot \nabla)v - (\nu + \nu_e)\Delta v + \nabla p = 0$$

Because we only want to augment the viscosity, we assume that $\nu_e \geq 0$.

Comparing equation 58 and equation 37 with each other, we see that for τ we can fill in $-2\nu_e S$. When we now put this into condition 55, and use integration by parts, we get

$$4 \int_{\Omega_\delta} r(v) \, \mathrm{d}x \leq \int_{\Omega_\delta} \tau : \Delta S \, \mathrm{d}x = \int_{\Omega_\delta} -2\nu_e S : \Delta S \, \mathrm{d}x = 2\nu_e ||\nabla S||^2 \tag{59}$$

We will now use Poincaré inequality

$$\int_{\Omega_\delta} 2q(v) \, \mathrm{d}x = ||S||^2 \leq C_\delta ||\nabla S||^2$$

When we multiply both sides with $2\frac{\nu_e}{C_\delta}$, we get

$$4\frac{\nu_e}{C_\delta} \int_{\Omega_\delta} q(v) \, \mathrm{d}x \leq 2\nu_e ||\nabla S||^2$$

So in order to satisfy equation 59, we set

$$4 \int_{\Omega_\delta} r(v) \, \mathrm{d}x \leq 4\frac{\nu_e}{C_\delta} \int_{\Omega_\delta} q(v) \, \mathrm{d}x$$

Now we take the minimum viscosity ν_e that satisfies this condition

$$\nu_e = C_\delta \frac{\max\left\{\int_{\Omega_\delta} r(v) \, \mathrm{d}x, 0\right\}}{\int_{\Omega_\delta} q(v) \, \mathrm{d}x} = C_\delta \frac{\max\left\{\overline{r(v)}, 0\right\}}{\overline{q(v)}}$$

where $\overline{q(v)}$ and $\overline{r(v)}$ are the grid cell averages of q and r. When we use mid-point integration, we get

$$\nu_e = C_\delta \frac{\max\{r(v), 0\}}{q(v)}$$

When we put this into our eddy-viscosity model 56 we get the QR model

$$\tau - \frac{1}{3}\mathrm{tr}(\tau)I = -2C_\delta \frac{\max\{r(v), 0\}}{q(v)} S \tag{60}$$

In this model, we can see that $\tau = 0$ if $r(v) = 0$. It can be shown that for laminar flows, $r(v) = 0$ [1]. Furthermore, for two-dimensional flows, for example solid body rotation, at least one of the eigenvalues of S is zero, and thus $r(v) = 0$. So for laminar flows and two-dimensional flows, the QR model is not active.

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