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Robust Synchronization of Uncertain Linear Multi-Agent Systems Using LMI's

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Abstract

This thesis is about the problem of robust synchronization of uncertain multi-agent systems. The system of each agent has an uncertainty which we assume to be stable and bounded in H_∞-norm by a certain given tolerance. We found a dynamic protocol of which we think it robustly synchronizes the multi-agent systems, based on [4]. We investigated this problem, but did not manage to prove it when using LMI's as in [6]. Therefore we simplified the multi-agent system and had to assume that each agent has the same uncertainty. We derived a proof that a simplified protocol robustly synchronizes these multi-agent systems using LMI's. It is interesting for future research to study the assumptions we made and try to prove that the dynamic protocol does indeed work.

Contents

1 Introduction

In this master thesis we look at the problem of robust synchronization of linear, time-invariant, finite dimensional multi-agent systems. Multi-agent systems are, as the name already says, systems that describe multiple agents who can exchange information with each other. Each agent is represented by an input/output system. These multi-agent systems can appear in nature. Consider for example a large flock of birds. Each of these birds can be represented by the same system, because their flying techniques work the same. However, there are some differences. Every bird is in a different state and receives different information, because they can only exchange information with their neighbors.

Another example of a multi-agent system is a group of aircrafts required to fly in formation. This is an example of an engineering system. As with the birds, all the aircrafts can be represented by the same system, only differing in state and information exchange. We visualize the exchange of information by a graph, where each vertex represents an agent and the edges represent the interaction between two neighbors. In this thesis we only consider undirected graphs.

In the case of engineering multi-agent systems we can try to synchronize these systems, meaning that the difference between the states of two agents goes to zero when times goes to infinity for every combination of agents. Think of the aircrafts that are required to fly in formation. This can be done by finding a protocol of how the agents should interact with each other. This protocol can be seen as a network of feedback controllers which can be interconnected with the multi-agent system, such that the interconnected system is synchronized by the protocol.

The systems in a network of multiple agents are not exactly the same. There can be small perturbations in each system, which means that the systems are uncertain. Each agent has an uncertainty $\Delta_i \in \mathcal{RH}_\infty$ with $i = 1, \ldots, p$, meaning that Δ_i is stable and Δ_i also has H_∞-norm less than a certain given tolerance. Synchronizing these systems now comes down to finding a protocol that synchronizes the multi-agent network for all such perturbations. So, if we interconnect the protocol with the uncertain multi-agent system, we want the interconnected system to be synchronized. We call this the robust synchronization problem.

We try to solve the robust synchronization problem using linear matrix inequalities (LMI's) based on [6]. That is, we try to show that a certain protocol works for all perturbations Δ_i with $i = 2, \ldots, p$. It took too much time to finish the proof within the time available for this thesis, so we were not able to prove the robust synchronization problem for every multi-agent system. We had to make a few assumptions. Namely, we had to assume that all Δ_i's are the same. We also had to simplify the multi-agent system by setting the disturbance in the measurement output of each agent to zero and also setting the input for the system output to zero. We were not able to solve the problem without these assumptions within the time available for this thesis.

The outline of this thesis is as follows. We start with some preliminaries on graph theory, multi-agent systems and mathematical control theory in Section 2. Then we state the problem of robust synchronization of multi-agent systems in Section 3. At a certain point in this section we introduce the assumptions mentioned above. This is followed by a proof of the robust synchronization problem for the simplified multi-agent system. In Section 4 we will give some remarks on the assumptions we made and in Section 5 we give our conclusions.

2 Preliminaries

In this section we discuss some preliminary notation regarding graph theory, multi-agent systems and mathematical control theory.

2.1 Graph theory

How the agents in a network interconnect is described by an undirected unweighted graph. A graph can be represented as a pair $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. $\mathcal{V} = \{1, \ldots, p\}$ is a finite set of vertices and $\mathcal{E}$ is a finite set of edges that contains the pair (i, j) if there is an edge from i to j with $i, j \in \mathcal{V}$ and $i \neq j$. A graph is called *undirected* if whenever $(i, j) \in \mathcal{E}$, also $(j, i) \in \mathcal{E}$.

Let $i \in \mathcal{V}$ be a vertex. Then $\mathcal{N}_i$ is the *neighboring set* of i, given by

$$\mathcal{N}_i := \{j \in \mathcal{V} \mid (i, j) \in \mathcal{E}\}.$$

This set contains all elements $j \neq i \in \mathcal{V}$ that are connected to i. We can now define the *adjacency matrix* $\mathcal{A}$ for a given graph.

Definition 1 (Adjacency matrix)**.** *Given a graph $\mathcal{G}$, the adjacency matrix $\mathcal{A} = (a_{ij})$ of $\mathcal{G}$ is the $n \times n$ matrix defined by*

$$a_{ij} = \begin{cases} 0, & (i,j) \notin \mathcal{E} \\ 1, & (i,j) \in \mathcal{E} \end{cases}.$$

Since we only consider undirected graphs, the adjacency matrix is symmetric.

We also define the *Laplacian matrix* $\mathcal{L}$ of the graph.

Definition 2 (Laplacian matrix)**.** *Given a graph $\mathcal{G}$, the Laplacian matrix $\mathcal{L} = (l_{ij})$ of $\mathcal{G}$ is the $n \times n$ matrix defined by*

$$l_{ij} = \begin{cases} \sum_{i \neq j} a_{ij}, & i = j \\ -a_{ij}, & i \neq j \end{cases}.$$

The Laplacian matrix thus equals $\mathcal{D} - \mathcal{A}$, where $\mathcal{D}$ is the *degree matrix* defined as $\mathcal{D} = (d_{ij})$ with $d_{ii} = |\mathcal{N}_i|$ and $d_{ij} = 0$ for $i \neq j$.

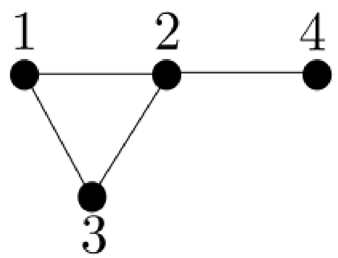

Figure 1: Undirected graph

Example 1. *Consider the undirected graph $\mathcal{G}$ as given in Figure 1. We see that $\mathcal{V} = \{1, 2, 3, 4\}$ and that $\mathcal{E} = \{(1, 2), (2, 1), (1, 3), (3, 1), (2, 3), (3, 2), (2, 4), (4, 2)\}$. Vertex 2 and 3 are adjacent to vertex 1, so the neighboring set of vertex 1 is $\mathcal{N}_1 = \{2, 3\}$. Vertex 4 is not adjacent to vertex 1 and is therefore not included in $\mathcal{N}_1$. By the same reasoning we find $\mathcal{N}_2 = \{1, 3, 4\}$, $\mathcal{N}_3 = \{1, 2\}$ and $\mathcal{N}_4 = \{2\}$. The degree d_1 of vertex 1 is now the* *cardinality of the set $\mathcal{N}_1$ which is 2. We find that that the other degrees are $d_2 = 3$, $d_3 = 2$ and $d_4 = 1$. The degree matrix is thus given by*

$$\mathcal{D} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The adjacency matrix is now given as follows:

$$\mathcal{A} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

The adjacency matrix is symmetric as expected. We can now compute the Laplacian matrix by subtracting $\mathcal{A}$ from $\mathcal{D}$. The Laplacian then becomes

$$\mathcal{L} = \mathcal{D} - \mathcal{A} = \begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 2 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}.$$

We immediately see that every row sum and every column sum of the Laplacian matrix is equal to zero. From this we can conclude that zero is always an eigenvalue of the Laplacian, because if we multiply $\mathcal{L}$ with the vector $v = (1, 1, \ldots, 1, 1)$, it becomes the zero vector.

Since the graphs are undirected, $\mathcal{L}$ is a positive semi-definite real symmetric matrix. The positive semi-definiteness follows from the fact that $\mathcal{L}$ can be written as $\mathcal{L} = \mathcal{M}\mathcal{M}^T$. In this case $\mathcal{M}$ is an oriented incidence matrix. The incidence matrix $\mathcal{M}$ is an $n \times m$ matrix with n the number of vertices and m the number of edges. It is defined as $\mathcal{M} = (m_{ij})$, where $m_{ij} = 1$ if vertex i is incident to edge j and the other vertex incident to j is still free, $m_{ij} = -1$ if vertex i is incident to edge j and the other vertex incident to j is already used and $m_{ij} = 0$ otherwise. We see that it indeed holds that $\mathcal{L} = \mathcal{M}\mathcal{M}^T$. It follows immediately that $\mathcal{L} \geq 0$, since $x^T L x = ||M^T x||^2 \geq 0$. This means that all eigenvalues of $\mathcal{L}$ are non-negative and real.

We call an undirected graph *connected* if there exists a finite path from i to j for every pair of vertices. If a graph is disconnected it consists of multiple components that are not connected to each other. Each of these components is connected.

Example 2. *Consider the undirected disconnected graph $\mathcal{G}$ as given in Figure 2. This graph consists of two connected components $\mathcal{G}_1 = (\mathcal{V}_1, \mathcal{E}_1)$ and $\mathcal{G}_2 = (\mathcal{V}_2, \mathcal{E}_2)$. $\mathcal{V}_1 = \{1, 2\}$, $\mathcal{V}_2 = \{3, 4\}$, $\mathcal{E}_1 = \{(1, 2), (2, 1)\}$ and $\mathcal{E}_2 = \{(3, 4), (4, 3)\}$. The Laplacian matrix of $\mathcal{G}$ is given by*

$$\mathcal{L} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}.$$

In this case the Laplacian is a block diagonal matrix where each block represents a component of the graph.

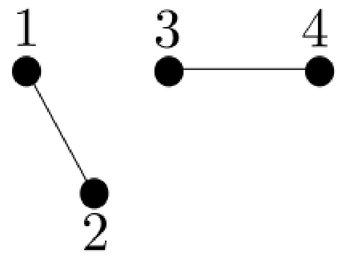

Figure 2: Disconnected graph

With disconnected graphs it is always the case that the Laplacian can be written as a block diagonal matrix. Each block represents a component of the graph. Sometimes the vertices need to be reordered to get the block diagonal form.

The following lemma leads to an important fact about the rank of the Laplacian.

Lemma 1. *Given a graph $\mathcal{G}$ and the eigenvalues $0 \leq \lambda_2 \leq \cdots \leq \lambda_p$ of the corresponding Laplacian, $\mathcal{G}$ is connected if and only if $\lambda_2 > 0$.*

Proof. ($\Rightarrow$) We assume that $\mathcal{G}$ is connected. Let $v = (v_1, \ldots, v_p)^T$ be an eigenvector of the Laplacian $\mathcal{L}$ corresponding to the eigenvalue 0. Then $\mathcal{L}v = 0$ and thus $v^T \mathcal{L} v = 0$. If we write this out we get that this equals $\sum_{(i,j)\in\mathcal{E}} (v_i - v_j)^2$. From this it follows that $v_i - v_j = 0$ for each pair $(i, j) \in \mathcal{E}$. Since $\mathcal{G}$ is connected, every pair of vertices is connected by a path. Thus it follows that $v_i - v_j = 0$ for each $i, j \in \mathcal{V}$. Thus, v must be a constant times the vector of all ones, which implies that the dimension of the eigenspace associated with the eigenvalue 0 is equal to 1. Therefore, $\lambda_2 > 0$.

($\Leftarrow$) Suppose that $\mathcal{G}$ is not connected, say it contains two connected components. We can write the Laplacian as a block diagonal matrix. This gives two eigenvectors which both correspond to the eigenvalue 0 and thus $\lambda_2 = 0$. This contradicts the fact that $\lambda_2 > 0$ and therefore $\mathcal{G}$ is connected. □

It follows from this lemma that, under the assumption that the graph is connected, the Laplacian $\mathcal{L}$ has rank $p - 1$, since $\mathcal{L}$ has exactly $p - 1$ independent eigenvectors.

2.2 Multi-agent systems

In this thesis we work with multi-agent systems. Consider a multi-agent system with p agents. Such a network can be modeled by an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. The vertex set consists of p nodes that represent the agents, i.e. $\mathcal{V} = \{1, \ldots, p\}$. The edge set represents the exchange of information between the agents. Let the dynamics of agent i be given by

$$\dot{x}_i = Ax_i + Bu_i \tag{1}$$

where $x_i \in \mathbb{R}^n$ is the state of agent i and $u_i \in \mathbb{R}^m$ is the control input of agent i with $i = 1, \ldots, p$. The information that is available to agent i is given by

$$\sum_{j\in\mathcal{N}_i} (x_j - x_i), \tag{2}$$

where $\mathcal{N}_i$ is the neighboring set of node i and $i, j = 1, \ldots, p$. To control the agent dynamics in (1) we write down the distributed control law as

$$u_i = F \sum_{j \in \mathcal{N}_i} (x_j - x_i), \tag{3}$$

with $i, j = 1, \ldots, p$. Interconnecting this control law with the dynamics of each agent gives

$$\dot{x}_i = Ax_i + BF \sum_{j \in \mathcal{N}_i} (x_j - x_i), \tag{4}$$

with $i, j = 1, \ldots, p$.

Example 3. *Consider the multi-agent system represented by the graph in Figure 1. We know that the vertex set is $\mathcal{V} = \{1, 2, 3, 4\}$, so the system consists of 4 agents. The graph is connected, so we can write down the information that is available for each agent.*

$$\begin{aligned}
\dot{x_1} &= Ax_1 + BF((x_2 - x_1) + (x_3 - x_1)) \\
\dot{x_2} &= Ax_2 + BF((x_1 - x_2) + (x_3 - x_2) + (x_4 - x_2)) \\
\dot{x_3} &= Ax_3 + BF((x_1 - x_3) + (x_2 - x_3)) \\
\dot{x_4} &= Ax_4 + BF(x_2 - x_4).
\end{aligned}$$

If we rewrite these equations, we see that this set of equations represents the network

$$\dot{x} = (I \otimes A - \mathcal{L} \otimes BF)x$$

where $x = (x_1, x_2, x_3, x_4)^T$, $\mathcal{L}$ is the Laplacian matrix as computed in Example 1 and $\otimes$ is the Kronecker product.

Since we will look at the synchronization of multi-agent systems, we will define what it means for a network to be synchronized.

Definition 3 (Synchronization)**.** *The network Σ is synchronized by the protocol (3) if for all $i, j = 1, \ldots, p$ we have $x_i(t) - x_j(t) \to 0$ as $t \to \infty$.*

In the example above we call F the *protocol matrix* that synchronizes the network Σ if for all $i, j = 1, \ldots, p$ we have $x_i(t) - x_j(t) \to 0$ as $t \to \infty$.

2.3 Mathematical control theory

In this section we will state some important definitions and lemmas concerning mathematical control theory. In this thesis we will work mostly with linear matrix inequalities (LMI's). They are closely related to the Schur complement of a matrix.

2.3.1 Schur complement

Definition 4 (Schur complement)**.** *Consider the block matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ where A and D are square matrices. If A^{-1} exists, then its Schur complement is defined as $D - CA^{-1}B$. If D^{-1} exists, then its Schur complement is defined as $A - BD^{-1}C$.*

The following lemma is called the Schur complement lemma.

Lemma 2. *Let A be a symmetric block matrix $\begin{pmatrix} A_1 & A_2 \\ A_2^T & A_3 \end{pmatrix}$ with $A_3 < 0$ (> 0). Then $A < 0$ (> 0) if and only if the Schur complement $A_1 - A_2 A_3^{-1} A_2^T < 0$ (> 0).*

In this thesis we also consider *quadratic matrix inequalities* of the form

$$A^TP + PA + PBR^{-1}B^TP + Q < 0$$

with A, B, $Q = Q^T > 0$ and $R = R^T > 0$ given matrices and $P = P^T$ the unknown symmetric solution of the inequality. This quadratic matrix inequality can be written as an LMI of the form

$$\begin{pmatrix} A^TP + PA + Q & PB \\ B^TP & -R \end{pmatrix} < 0.$$

2.3.2 Finsler's lemma

An important lemma we will use in this thesis is Finsler's lemma as stated below.

Lemma 3 (Finsler's Lemma). *Let $Q \in \mathbb{R}^{n\times n}$ symmetric and $M \in \mathbb{R}^{n\times m}$. Then $x^TQx < 0$ for all $x \in \mathbb{R}^n$ with $x \neq 0$ such that $x^TM = 0$ if and only if there exists some scalar $\sigma > 0$ such that $Q - \sigma MM^T < 0$.*

Finsler's lemma as stated above is not the complete lemma, there are more statements equivalent to it, but we only use this equivalence. A proof of Finsler's lemma can be found in [1].

2.3.3 Annihilator of a matrix

For a given matrix $M \in \mathbb{R}^{n\times m}$ with $rank(M) = m$ and $m < n$, the rows are linearly dependent. This implies that there exists a row vector $v \neq 0$ such that $vM = 0$. There even exists an $(n-m)\times n$ matrix with linearly independent rows, $M^\perp$, such that $M^\perp M = 0$. Below we will give the definition of the annihilator of a matrix.

Definition 5. *Let M be an $n \times m$ matrix of rank m with $m < n$. We denote by $M^\perp \in \mathbb{R}^{(n-m)\times n}$ any matrix of rank $n - m$ such that $M^\perp M = 0$. Any such $M^\perp$ is called an annihilator of M.*

Why does such an annihilator exist? Write out the singular value decomposition of M as

$$M = U \left(\begin{array}{c|c} \Sigma & 0 \\ \hline 0 & 0 \end{array}\right) V$$

where $U \in \mathbb{R}^{n\times n}$ and $V \in \mathbb{R}^{m\times m}$ are orthogonal matrices and Σ has the singular values $\sigma_1 \geq \cdots \geq \sigma_m > 0$ on the diagonal. Let T be any nonsingular $(n - m) \times (n - m)$ matrix and define

$$M^\perp := \left(\begin{array}{c|c} 0 & T \end{array}\right) U^T.$$

Then $M^\perp$ is an $(n - m) \times n$ matrix of rank $n - m$ and $M^\perp M = 0$.

2.3.4 Small Gain Theorem

An important theorem we use in this thesis is the Small Gain Theorem. Before stating this theorem we will give some definitions. Let Σ be a system given by

$$\Sigma : \begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du. \end{cases}$$

Here A is *Hurwitz*, i.e. $\sigma(A) \in \mathbb{C}^-$. Since we only consider linear systems, it holds that this is equivalent to Σ being *internally stable*. The *transfer matrix* $G(s)$ of this system Σ

is given by $G(s) = C(sI - A)^{-1}B + D$. $G(s)$ is proper and has all its poles in $\mathbb{C}^-$ since A is Hurwitz. A transfer matrix is called *proper* if for each element in the matrix the degree of the numerator is less than or equal to the degree of the denominator. $G(s)$ is well-defined for all $s = i\omega$ with $\omega \in \mathbb{R}$.

A feedback interconnection between two systems is *well posed* if at least one of the two systems has a strictly proper transfer matrix, i.e. for each element in the matrix the degree of the numerator is strictly less than the degree of the denominator.

Definition 6 (Operator norm)**.** *The operator norm of a matrix M is defined as*

$$||M|| = \ max\{||\ Mx\ ||\ \big| x \in \mathbb{R}^n, ||x|| = 1\}.$$

Note that this is not the standard Euclidean norm given by $||x|| = (x^T x)^{1/2}$. If we write out the definition and use that the largest singular value of M, σ_1, is equal to $\sqrt{\lambda_{\max}(M^T M)}$, we find that $||M|| = \sigma_1(M)$. For all ω we can now consider the complex matrix $G(i\omega)$ with operator norm $||G(i\omega)|| = \sigma_1(G(i\omega))$. We can now define the H_∞-norm.

Definition 7 (H_∞-norm)**.** *The H_∞-norm of the transfer matrix $G(s)$ is given by*

$$||G||_\infty := \ sup_{w\in\mathbb{R}}||G(i\omega)||.$$

We can now state the Small Gain Theorem:

Theorem 1 (Small Gain Theorem)**.** *Let $\gamma > 0$. Let Σ_1 and Σ_2 be two internally stable systems and let $G_1(s)$ and $G_2(s)$ be the transfer matrices of respectively Σ_1 and Σ_2. The feedback interconnection of Σ_1 and Σ_2 is well posed and internally stable for all Σ_2 with $G_2(s)$ satisfying $||G_2||_\infty \leq \gamma$ if and only if $||G_1||_\infty < \frac{1}{\gamma}$.*

2.3.5 Bounded real lemma

Another important lemma is the Bounded real lemma.

Lemma 4 (Bounded real lemma)**.** *Let $\gamma > 0$. Let Σ be a linear system and let $G(s) = C(sI - A)^{-1}B + D$ be its transfer matrix. Then $\sigma(A) \subset \mathbb{C}^-$ and $||G||_\infty < \gamma$ if and only if there exists an $Y > 0$ such that*

$$\begin{pmatrix} YA + A^T Y + C^T C & YB + C^T D \\ B^T Y + D^T C & -\gamma^2 I + DD^T \end{pmatrix} < 0. \tag{5}$$

3 Robust synchronization of multi-agent systems

We consider linear, time-invariant, finite dimensional multi-agent systems with p agents. We assume that there is an uncertainty $\Delta_i \in \mathcal{RH}_\infty$ in the dynamics of each agent i, with $i = 1, \dots, p$. $\Delta_i \in \mathcal{RH}_\infty$ means that Δ_i is stable. We can write these systems as closed loop feedback interconnections from d_i to z_i as follows:

$$\begin{cases} \dot{x}_i = Ax_i + Bu_i + Ed_i \\ y_i = Cx_i + Dd_i \\ d_i = \Delta_i z_i \\ z_i = Hx_i + Ju_i. \end{cases} \tag{6}$$

Here $i = 1, \dots, p$, $x_i \in \mathbb{R}^n$ is the state of agent i, $u_i \in \mathbb{R}^m$ is the input, $y_i \in \mathbb{R}^q$ is the measurement output of agent i, $z_i \in \mathbb{R}^k$ is the system output, d_i is the disturbance and $||\Delta_i||_\infty \leq \gamma$ where $\gamma > 0$ is a given uncertainty radius. Figure 3 gives a representation of the closed loop system.

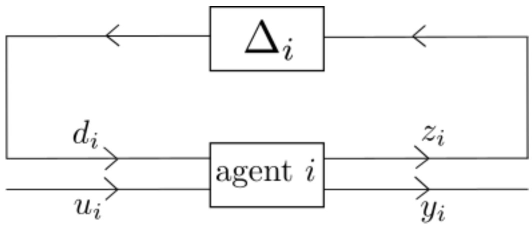

Figure 3: Closed loop system

The idea is to find a protocol that synchronizes these p systems for all Δ_i's. Based on [4] we will explain that dynamic protocols of the following form work.

$$\begin{cases} \dot{w}_i = Kw_i + L\sum_{j\in\mathcal{N}_i}(w_i - w_j) + M\sum_{j\in\mathcal{N}_i}(y_i - y_j) \\ u_i = Nw_i + R\sum_{j\in\mathcal{N}_i}(y_i - y_j) \end{cases} \tag{7}$$

We have that K, L, M, N and R are protocol matrices of appropriate sizes. The structure of these protocols follows from the fact that agent i receives information from the output of its neighbors. Thus $\sum_{j\in\mathcal{N}_i}(y_i - y_j)$ represents the sum of the relative outputs with respect to its neighbors. $\sum_{j\in\mathcal{N}_i}(w_i - w_j)$ is the sum of the relative estimated values of the states. Agent i receives information about the estimated value of the state of his neighbors. He also has information about the estimated value of his own state. The second equation in (7) is feeding back the estimate of the state back to agent i together with the sum of the relative outputs with respect to its neighbors.

The definition of synchronization as stated in Definition 3 slightly changes for the protocol (7).

Definition 8. *The network (6) is synchronized by the protocol (7) if for all $i, j = 1, \dots, p$ we have $x_i(t) - x_j(t) \to 0$ and $w_i(t) - w_j(t) \to 0$ as $t \to \infty$.*

In this definition of synchronization we also need the relative estimated values of the states to go to zero when time goes to infinity. This is the definition of synchronization that we will use from now on.

For each agent i, $i = 1, \dots, p$, we interconnect the protocol (7) with the system (6), so that we obtain the dynamics of the complete network. To do this we write (6) and (7) in vector form by defining $\mathbf{x} = (x_1, \ldots, x_p)^T$ and by defining $\mathbf{u}$, $\mathbf{y}$, $\mathbf{w}$, $\mathbf{z}$ and $\mathbf{d}$ in a similar way. Then we obtain

$$\begin{cases} \dot{\mathbf{x}} = (I \otimes A)\mathbf{x} + (I \otimes B)\mathbf{u} + (I \otimes E)\mathbf{d} \\ \mathbf{y} = (I \otimes C)\mathbf{x} + (I \otimes D)\mathbf{d} \\ \mathbf{z} = (I \otimes H)\mathbf{x} + (I \otimes J)\mathbf{u} \\ \mathbf{d} = \begin{pmatrix} \Delta_1 & & \\ & \ddots & \\ & & \Delta_p \end{pmatrix} \mathbf{z} \end{cases} \tag{8}$$

and

$$\begin{cases} \dot{\mathbf{w}} = (I \otimes K)\mathbf{w} + (\mathcal{L} \otimes L)\mathbf{w} + (\mathcal{L} \otimes M)\mathbf{y} \\ \mathbf{u} = (I \otimes N)\mathbf{w} + (\mathcal{L} \otimes R)\mathbf{y} \end{cases} \tag{9}$$

where $\mathcal{L}$ is the Laplacian matrix that corresponds to the undirected and connected network graph. The eigenvalues of $\mathcal{L}$ are $\lambda_1 = 0 < \lambda_2 < \cdots < \lambda_p$. Interconnecting (8) and (9) leads to the dynamics of the complete network

$$\begin{cases} \begin{pmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{w}} \end{pmatrix} = \begin{pmatrix} I \otimes A + \mathcal{L} \otimes BRC & I \otimes BN \\ \mathcal{L} \otimes MC & I \otimes K + \mathcal{L} \otimes L \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{w} \end{pmatrix} + \begin{pmatrix} I \otimes E + \mathcal{L} \otimes BRD \\ \mathcal{L} \otimes MD \end{pmatrix} \mathbf{d} \\ \mathbf{z} = \begin{pmatrix} I \otimes H + \mathcal{L} \otimes JRC & I \otimes JN \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{w} \end{pmatrix} + (\mathcal{L} \otimes JRD)\mathbf{d} \\ \mathbf{d} = \begin{pmatrix} \Delta_1 & & \\ & \ddots & \\ & & \Delta_p \end{pmatrix} \mathbf{z}. \end{cases} \tag{10}$$

Since, the network graph is undirected, the Laplacian is a real symmetric matrix. This means that there exists an orthogonal $p \times p$ matrix U that diagonalizes $\mathcal{L}$ as $U^T \mathcal{L} U = \Lambda = \mathrm{diag}(0, \lambda_2, \ldots, \lambda_p)$. We need this diagonalization, so that we can apply the state transformation

$$\begin{pmatrix} \tilde{\mathbf{x}} \\ \tilde{\mathbf{w}} \end{pmatrix} = \begin{pmatrix} U^T \otimes I & 0 \\ 0 & U^T \otimes I \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{w} \end{pmatrix} \tag{11}$$

together with the transformations $\tilde{\mathbf{d}} = (U^T \otimes I)\mathbf{d}$ and $\tilde{\mathbf{z}} = (U^T \otimes I)\mathbf{z}$. From this we obtain the transformed network dynamics

$$\begin{cases} \begin{pmatrix} \dot{\tilde{\mathbf{x}}} \\ \dot{\tilde{\mathbf{w}}} \end{pmatrix} = \begin{pmatrix} I \otimes A + \Lambda \otimes BRC & I \otimes BN \\ \Lambda \otimes MC & I \otimes K + \Lambda \otimes L \end{pmatrix} \begin{pmatrix} \tilde{\mathbf{x}} \\ \tilde{\mathbf{w}} \end{pmatrix} + \begin{pmatrix} I \otimes E + \Lambda \otimes BRD \\ \Lambda \otimes MD \end{pmatrix} \tilde{\mathbf{d}} \\ \tilde{\mathbf{z}} = \begin{pmatrix} I \otimes H + \Lambda \otimes JRC & I \otimes JN \end{pmatrix} \begin{pmatrix} \tilde{\mathbf{x}} \\ \tilde{\mathbf{w}} \end{pmatrix} + (\Lambda \otimes JRD)\tilde{\mathbf{d}} \\ \tilde{\mathbf{d}} = (U^T \otimes I) \begin{pmatrix} \Delta_1 & & \\ & \ddots & \\ & & \Delta_p \end{pmatrix} (U \otimes I)\tilde{\mathbf{z}}. \end{cases} \tag{12}$$

If we write out the last equation of (12), we get

$$\tilde{\mathbf{d}} = (U^T \otimes I)\begin{pmatrix} \Delta_1 & & \\ & \ddots & \\ & & \Delta_p \end{pmatrix}(U \otimes I)\tilde{\mathbf{z}} = \begin{pmatrix} \Delta_{11} & \cdots & \Delta_{1p} \\ \vdots & \ddots & \vdots \\ \Delta_{p1} & \cdots & \Delta_{pp} \end{pmatrix}\tilde{\mathbf{z}} \tag{13}$$

for certain Δ_{ij} where $i, j = 1, \ldots, p$. From this we conclude that the disturbance of agent i depends on $z_1, \ldots, z_n$. Thus, we cannot express $\tilde{d}_i$ in terms of $\tilde{z}_i$ only. We will see in the proof of Theorem 2 that this leads to a problem.

We can write out these dynamics from d to z for each agent $i = 1, \ldots, p$ as follows:

$$\begin{cases} \begin{pmatrix} \dot{\tilde{x}}_i \\ \dot{\tilde{w}}_i \end{pmatrix} = \begin{pmatrix} A + \lambda_i BRC & BN \\ \lambda_i MC & K + \lambda_i L \end{pmatrix}\begin{pmatrix} \tilde{x}_i \\ \tilde{w}_i \end{pmatrix} + \begin{pmatrix} E + \lambda_i BRD \\ \lambda_i MD \end{pmatrix}\tilde{d}_i \\ \tilde{z}_i = \begin{pmatrix} H + \lambda_i JRC & JN \end{pmatrix}\begin{pmatrix} \tilde{x}_i \\ \tilde{w}_i \end{pmatrix} + \lambda_i JRD\tilde{d}_i. \end{cases} \tag{14}$$

For notational ease we introduce the following extended matrices:

$A_e := \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$, $B_e := \begin{pmatrix} B & 0 \\ 0 & I \end{pmatrix}$, $C_e := \begin{pmatrix} C & 0 \\ 0 & I \end{pmatrix}$, $D_e := \begin{pmatrix} D \\ 0 \end{pmatrix}$, $E_e := \begin{pmatrix} E \\ 0 \end{pmatrix}$, $H_e := \begin{pmatrix} H & 0 \end{pmatrix}$, $J_e := \begin{pmatrix} J & 0 \end{pmatrix}$, $H_i := \begin{pmatrix} \lambda_i R & N \\ \lambda_i M & K + \lambda_i L \end{pmatrix}$, $x_e := \begin{pmatrix} \tilde{x}_i \\ \tilde{w}_i \end{pmatrix}$, $d := d_i$ and $z := z_i$.

Now we can write the extended system (14) as

$$\begin{cases} \dot{x}_e = (A_e + B_e H_i C_e)x_e + (E_e + B_e H_i D_e)d \\ z = (H_e + J_e H_i C_e)x_e + (J_e H_i D_e)d \end{cases} \tag{15}$$

and we can write it even more simple as

$$\begin{cases} \dot{x}_e = A_i x_e + B_i d \\ z = C_i x_e + D_i d \end{cases} \tag{16}$$

where $A_i := A_e + B_e H_i C_e$, $B_i := E_e + B_e H_i D_e$, $C_i := H_e + J_e H_i C_e$ and $D_i := J_e H_i D_e$.

This transformation was useful, since according to Lemma 3.2 in [4] we have that $x_i(t) - x_j(t) \to 0$ and $w_i(t) - w_j(t) \to 0$ for all i, j as $t \to \infty$ if and only if $\tilde{x}_i(t) \to 0$ and $\tilde{w}_i(t) \to 0$ for $i = 2, 3, \ldots, p$. This means that we only have to show stability of the transformed network instead of synchronization of the original network. This is also used in Theorem 2 below. First the definition of the robust synchronization problem is given.

Definition 9 (Robust synchronization)**.** *Let γ be a given uncertainty radius. The robust synchronization problem is to find a dynamic protocol of the form (7) such that for all i and for all $\Delta_i \in \mathcal{RH}_\infty$ with $||\Delta_i||_\infty \leq \gamma$, the network (10) is synchronized.*

As mentioned before we get a problem when proving Theorem 2 below. So, from now on we assume that all Δ's are the same, i.e. $\Delta_i = \Delta$ for all i. It is unfortunate that we have to make this assumption. In Section 4 it is explained why we make this assumption. Instead of (13) we now get the simpler equation

$$\tilde{\mathbf{d}} = (U^T \otimes I)(I \otimes \Delta)(U \otimes I)\tilde{\mathbf{z}} = (I \otimes \Delta)\tilde{\mathbf{z}}$$

which means that $\tilde{d}_i = \Delta\tilde{z}_i$. This assumption must be made to be able to prove the next theorem which is stated for a different network in [4].

Theorem 2. *Consider the network (10) and assume that the network graph is undirected and connected. Let $\gamma > 0$. The following statements are equivalent:*

1. *The dynamic protocol (7) synchronizes the network (10) for all $i = 1, \ldots, p$ and for all $\Delta_i = \Delta \in \mathcal{RH}_\infty$ with $||\Delta||_\infty \leq \gamma$*

2. *The perturbed linear system*

$$\Sigma : \begin{cases} \dot{x} = Ax + Bu + Ed \\ y = Cx + Dd \\ d = \Delta z \\ z = Hx + Ju. \end{cases} \tag{17}$$

   is internally stabilized for all $\Delta \in \mathcal{RH}_\infty$ with $||\Delta||_\infty \leq \gamma$ by all $p-1$ controllers

$$\Gamma_i : \begin{cases} \dot{w} = (K + \lambda_i L)w + \lambda_i My \\ u = Nw + \lambda_i Ry \end{cases} \tag{18}$$

   for $i = 2, 3, \ldots, p$.

A proof of this theorem can be found in [4] for a special case of the network that we use here.

Let $G_{\Sigma\times\Gamma_i}$ be the transfer matrix of $\Sigma\times\Gamma_i$. Then it follows from the small gain theorem that statement 2 in Theorem 2 is equivalent to $||G_{\Sigma\times\Gamma_i}||_\infty < \frac{1}{\gamma}$ and $\sigma(A_e + B_e H_i C_e) \subset \mathbb{C}^-$ for $i = 2, \ldots, p$. According to the bounded real lemma this is equivalent to saying that for given $\gamma > 0$ there exist $P_i > 0$ such that for $i = 2, \ldots, p$

$$\begin{pmatrix} P_i A_i + A_i^T P_i + C_i^T C_i & P_i B_i + C_i^T D_i \\ B_i^T P_i + D_i^T C_i & -\frac{1}{\gamma^2} I + D_i^T D_i \end{pmatrix} < 0. \tag{19}$$

We can now state the following conjecture using LMI's. This conjecture gives us suitable K, L, M, N, R and P_i for $i = 2, \ldots, p$ which assure that the corresponding protocol (7) indeed synchronizes the network (10) for all $\Delta_i = \Delta \in \mathcal{RH}_\infty$ with $||\Delta||_\infty \leq \gamma$.

Conjecture 1. *Let $\gamma > 0$ and let n_c be a positive integer. There exist $K \in \mathbb{R}^{n_c\times n_c}$, $L \in \mathbb{R}^{n_c\times n_c}$, $M \in \mathbb{R}^{n_c\times q}$, $N \in \mathbb{R}^{m\times n_c}$, $R \in \mathbb{R}^{m\times q}$ and $P_i > 0$ such that for $i = 2, \ldots, p$*

$$\begin{pmatrix} P_i A_i + A_i^T P_i + C_i^T C_i & P_i B_i + C_i^T D_i \\ B_i^T P_i + D_i^T C_i & -\frac{1}{\gamma^2} I + D_i^T D_i \end{pmatrix} < 0 \tag{20}$$

if and only if there exist matrices $X_1 > 0$ and $Y_1 > 0$ of size $n \times n$ such that the following three inequalities hold:

$$\begin{pmatrix} B \\ J \end{pmatrix}^\perp \begin{pmatrix} AX_1 + X_1 A^T + EE^T & X_1 H^T \\ HX_1 & -\frac{1}{\gamma^2} I \end{pmatrix} \begin{pmatrix} B \\ J \end{pmatrix}^{\perp T} < 0, \tag{21}$$

$$\begin{pmatrix} C^T \\ D^T \end{pmatrix}^\perp \begin{pmatrix} Y_1 A + A^T Y_1 + H^T H & Y_1 E \\ E^T Y_1 & -\frac{1}{\gamma^2} I \end{pmatrix} \begin{pmatrix} C^T \\ D^T \end{pmatrix}^{\perp T} < 0, \tag{22}$$

$$Y_1 - \frac{1}{\gamma^2} X_1^{-1} > 0. \tag{23}$$

Unfortunately we were not able to find these K, L, M, N, R and P_i within the time available for this thesis. In what follows we assume that $J = 0$ and $D = 0$. If J and D are arbitrary matrices, we are not able to show that Conjecture 1 holds. A more detailed analysis of the problem that arises with this conjecture will be given in Section 4.

A consequence of taking J and D equal to zero is that the annihilators in (21) and (22) simplify to

$$\begin{pmatrix} B \\ 0 \end{pmatrix}^{\perp} \text{ and } \begin{pmatrix} C^T \\ 0 \end{pmatrix}^{\perp}. \tag{24}$$

It also means that the network in (16) is changed. A_i remains the same (with a different H_i), but $B_i = E_e$, $C_i = H_e$ and $D_i = 0$. Since $D_i = 0$ it follows that the protocol matrix R has become unnecessary. The general form of the protocol becomes

$$\begin{cases} \dot{w}_i = Kw_i + L\sum_{j\in\mathcal{N}_i}(w_i - w_j) + M\sum_{j\in\mathcal{N}_i}(y_i - y_j) \\ u_i = Nw_i. \end{cases} \tag{25}$$

The theorem as we will prove it is stated below.

Theorem 3. *Let $\gamma > 0$ and let n_c be a positive integer. There exist $K \in \mathbb{R}^{n_c\times n_c}$, $L \in \mathbb{R}^{n_c\times n_c}$, $M \in \mathbb{R}^{n_c\times q}$, $N \in \mathbb{R}^{m\times n_c}$ and $P_i > 0 \in \mathbb{R}^{n_c\times n_c}$ such that for $i = 2,\dots,p$*

$$\begin{pmatrix} P_iA_i + A_i^TP_i + H_e^TH_e & P_iE_e \\ E_e^TP_i & -\dfrac{1}{\gamma^2}I \end{pmatrix} < 0 \tag{26}$$

if and only if there exist matrices $X_{11} > 0$ and $Y_{11} > 0$ of size $n\times n$ such that the following three inequalities hold:

$$\begin{pmatrix} B \\ 0 \end{pmatrix}^{\perp}\begin{pmatrix} AX_{11} + X_{11}A^T + EE^T & X_{11}H^T \\ HX_{11} & -\dfrac{1}{\gamma^2}I \end{pmatrix}\begin{pmatrix} B \\ 0 \end{pmatrix}^{\perp T} < 0, \tag{27}$$

$$\begin{pmatrix} C^T \\ 0 \end{pmatrix}^{\perp}\begin{pmatrix} Y_{11}A + A^TY_{11} + H^TH & Y_{11}E \\ E^TY_{11} & -\dfrac{1}{\gamma^2}I \end{pmatrix}\begin{pmatrix} C^T \\ 0 \end{pmatrix}^{\perp T} < 0, \tag{28}$$

$$Y_{11} - \frac{1}{\gamma^2}X_{11}^{-1} > 0. \tag{29}$$

Here, K, L, M, N and P_i can be obtained as follows.

- *Choose σ_1 such that*

$$\begin{pmatrix} AX_{11} + X_{11}A^T + EE^T - 2\sigma_1\lambda_2BB^T & X_{11}H^T \\ HX_{11} & -\dfrac{1}{\gamma^2}I \end{pmatrix} < 0. \tag{30}$$

- *Choose σ_2 such that*

$$\begin{pmatrix} Y_{11}A + A^TY_{11} + H^TH - 2\sigma_2\lambda_2C^TC & Y_1E \\ E^TY_1 & -\dfrac{1}{\gamma^2}I \end{pmatrix} < 0. \tag{31}$$

- *Define $F := -\sigma_1B^TX_{11}^{-1}$ and $G := -\sigma_2Y_{11}^{-1}C^T$.*

- *Define for* $i = 2, \ldots, p$

$$R_F^i := (A + \lambda_i BF)^T X_{11}^{-1} + X_{11}^{-1}(A + \lambda_i BF) + X_{11}^{-1} EE^T X_{11}^{-1} + \gamma^2 H^T H$$

and

$$R_G^i := (A + \lambda_i GC) Y_{11}^{-1} + Y_{11}^{-1}(A + \lambda_i GC)^T + Y_{11}^{-1} H^T H Y_{11}^{-1} + \gamma^2 EE^T.$$

- *Choose a real number* $k \in (0, 1)$ *such that*

$$Y_{11} R_G^p Y_{11} < \frac{1}{\gamma^2}(1 - k) R_F^p.$$

- *Define* $Z := Y_{11} - \frac{1}{\gamma^2} X_{11}^{-1}$.

- *Define*

$$\Omega_1 := \frac{k}{\gamma^2} Z^{-1}(A^T X_{11}^{-1} + X_{11}^{-1} A + X_{11}^{-1} EE^T X_{11}^{-1} + \gamma^2 H^T H)$$

and

$$\Omega_2 := -\frac{1}{\gamma^2} Z^{-1}((1 - k) F^T B^T X_{11}^{-1} - k X_{11}^{-1} BF).$$

- *Choose*

$$\begin{aligned} K &:= A + EE^T X_{11}^{-1} + \lambda_i Z^{-1} Y_{11} GC + \Omega_1, \\ L &:= BF + \Omega_2, \\ M &:= -Z^{-1} Y_{11} G, \\ N &:= \lambda_i F \end{aligned}$$

and

$$P_i = P := \begin{pmatrix} Y_{11} & -Z \\ -Z & Z \end{pmatrix}.$$

- *Define* $n_c := n$.

Proof. ($\Rightarrow$) Let $\gamma > 0$, $n_c > 0$ and assume there exist matrices K, L, M, N and $P_i = P > 0$ such that for $i = 2, \ldots, p$ (26) holds. Define $X := \frac{1}{\gamma^2} P^{-1} > 0$ and $Y := P > 0$. From this we can immediately conclude that for $i = 2, \ldots, p$

$$\begin{pmatrix} YA_i + A_i^T Y + H_e^T H_e & YE_e \\ E_e^T Y & -\frac{1}{\gamma^2} I \end{pmatrix} < 0. \tag{32}$$

After rewriting and by applying the Schur complement, it follows that $X := \frac{1}{\gamma^2} P^{-1}$ satisfies

$$\begin{pmatrix} A_i X + X A_i^T + E_e E_e^T & X H_e^T \\ H_e X & -\frac{1}{\gamma^2} I \end{pmatrix} < 0. \tag{33}$$

We want the terms in (32) and (33) that contain i to vanish. Therefore, we pre- and post multiply (32) with the annihilator $\begin{pmatrix} B_e \\ 0 \end{pmatrix}^\perp$ and (33) with the annihilator $\begin{pmatrix} C_e^T \\ 0 \end{pmatrix}^\perp$. This gives us the following inequalities:

$$\begin{pmatrix} B_e \\ 0 \end{pmatrix}^{\perp} \begin{pmatrix} YA_e + A_e^TY + H_e^TH_e & YE_e \\ E_e^TY & -\frac{1}{\gamma^2}I \end{pmatrix} \begin{pmatrix} B_e \\ 0 \end{pmatrix}^{\perp T} < 0. \tag{34}$$

$$\begin{pmatrix} C_e^T \\ 0 \end{pmatrix}^{\perp} \begin{pmatrix} A_eX + XA_e^T + E_eE_e^T & XH_e^T \\ H_eX & -\frac{1}{\gamma^2}I \end{pmatrix} \begin{pmatrix} C_e^T \\ 0 \end{pmatrix}^{\perp T} < 0. \tag{35}$$

Let us partition X and Y into $X = \begin{pmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{pmatrix}$ and $Y = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{12}^T & Y_{22} \end{pmatrix}$. After some computation we find out that there exist $X_{11} > 0$ and $Y_{11} > 0$ such that (27) and (28) hold.

To finish this part of the proof we must show that the inequality (29) holds. This follows from the fact that $XY = \frac{1}{\gamma^2}P^{-1}P = \frac{1}{\gamma^2}I$. If we write this out using the partitioned matrices X and Y we see that

$$X_{11}Y_{11} + X_{12}Y_{12}^T = \frac{1}{\gamma^2}I \tag{36}$$

and

$$X_{11}Y_{12} + X_{12}Y_{12} = 0. \tag{37}$$

Equation (36) is equivalent to $Y_{11} - \frac{1}{\gamma^2}X_{11}^{-1} = -X_{11}^{-1}X_{12}Y_{12}^T$. If we compute X_{11}^{-1} from equation (37) and substitute it into the right hand side of the equivalence of (36), we get that $Y_{11} - \frac{1}{\gamma^2}X_{11}^{-1} = Y_{12}Y_{22}^{-1}Y_{12}^T$. Since $Y > 0$, it holds that its Schur complement $Y_{11} - Y_{12}Y_{22}^{-1}Y_{12}^T = \frac{1}{\gamma^2}X_{11}^{-1} > 0$. Furthermore, it holds that $Y_{12}Y_{22}^{-1}Y_{12}^T \geq 0$. Thus, $Y_{11} - \frac{1}{\gamma^2}X_{11}^{-1} \geq 0$. We can slightly perturb X_{11} and Y_{11} such that (27) and (28) still hold, because (27) and (28) are strictly less than 0. We perturb X_{11} and Y_{11} such that (29) also holds. Thus the third inequality $Y_1 - \frac{1}{\gamma^2}X_1^{-1} > 0$ holds. This finishes this part of the proof.

($\Leftarrow$) We now assume that (27), (28) and (29) hold. By Finsler's lemma together with (27) we know that there exists a scalar $\sigma_1 > 0$ such that (30) holds. By taking the Schur complement of this we get

$$AX_{11} + X_{11}A^T + EE^T - 2\sigma_1\lambda_2BB^T + \gamma^2X_{11}H^THX_{11} < 0,$$

and by multiplying both sides with X_1^{-1} we get

$$X_{11}^{-1}A + A^TX_{11}^{-1} + X_{11}^{-1}EE^TX_{11}^{-1} - 2\sigma_1\lambda_2X_{11}^{-1}BB^TX_{11}^{-1} + \gamma^2H^TH < 0.$$

We can write this as

$$(A + \lambda_2BF)^TX_{11}^{-1} + X_{11}^{-1}(A + \lambda_2BF) + X_{11}^{-1}EE^TX_{11}^{-1} + \gamma^2H^TH < 0.$$

Note that this is R_F^2. We want that $R_F^i \leq R_F^2 < 0$ for all $i = 2, \ldots, p$. Remember that $0 < \lambda_2 < \cdots < \lambda_p$. Let $\lambda_i < \lambda_j$, then

$$\lambda_i(BF)^T = -\lambda_i\sigma_1X_{11}^{-1}BB^T < 0$$

and

$$\lambda_j (BF)^T = -\lambda_j \sigma_1 X_{11}^{-1} BB^T < 0.$$

This implies that $\lambda_j(BF)^T < \lambda_i(BF)^T$. Similarly, we find that $\lambda_j BF < \lambda_i BF$. Therefore, $R_F^i \leq R_F^2 < 0$ for all $i = 2, \ldots, p$.

We also know by Finsler's lemma together with (28) that there exists a scalar $\sigma_2 > 0$ such that (31) holds. If we again take the Schur complement and multiply both sides with Y_{11}^{-1}, we get

$$AY_{11}^{-1} + Y_{11}^{-1}A^T + Y_{11}^{-1}H^T HY_{11}^{-1} - 2\sigma_2\lambda_2 Y_{11}^{-1}C^T CY_{11}^{-1} + \gamma^2 EE^T < 0.$$

We can write this as

$$(A + \lambda_2 GC)Y_{11}^{-1} + Y_{11}^{-1}(A + \lambda_2 GC)^T + Y_{11}^{-1}H^T HY_{11}^{-1} + \gamma^2 EE^T < 0.$$

Note that this is R_G^2. By the same reasoning as with R_F^i, we have that $R_G^i \leq R_G^2 < 0$. It even holds that $R_F^p \leq R_F^{p-1} \leq \cdots \leq R_F^2$ and $R_G^p \leq R_G^{p-1} \leq \cdots \leq R_G^2$.

We choose $k \in (0, 1)$ such that $Y_{11}R_G^p Y_{11} < \frac{1}{\gamma^2}(1-k)R_F^p$. To show that such k always exists, note that since $Y_{11}R_G^p Y_{11} < 0$ there exists a $\delta > 0$ such that $Y_{11}R_G^p Y_{11} < -\delta I$. This means that all eigenvalues μ_j of $Y_{11}R_G^p Y_{11}$ are negative and $\mu_j < -\delta$ for all j. If we then choose $k \in (0, 1)$ close to 1 such that $\frac{1}{\gamma^2}(1-k)R_F^p > -\delta I$, all eigenvalues η_j of $\frac{1}{\gamma^2}(1-k)R_F^p$ are negative (since $\frac{1}{\gamma^2}(1-k)R_F^p < 0$) and $\eta_j > -\delta$ for all j. This means that $\mu_j < \delta < \eta_j$, i.e. all eigenvalues of $Y_{11}R_G^p Y_{11}$ are less than all eigenvalues of $\frac{1}{\gamma^2}(1-k)R_F^p$. We thus have a strict inequality $\mu_j < \eta_j$ and therefore $Y_{11}R_G^p Y_{11} < \frac{1}{\gamma^2}(1-k)R_F^p$ for some k. This k always exists.

From $Y_{11}R_G^p Y_{11} < \frac{1}{\gamma^2}(1-k)R_F^p$ it follows that $Y_{11}R_G^i Y_{11} < \frac{1}{\gamma^2}(1-k)R_F^i$ for all $i = 2, \ldots, p$, since $R_F^p \leq R_F^{p-1} \leq \cdots \leq R_F^2$ and $R_G^p \leq R_G^{p-1} \leq \cdots \leq R_G^2$.

We now have to show that (26) holds. The Schur complement of (26) is

$$P_i A_i + A_i^T P_i + H_e^T H_e + \gamma^2 P_i E_e E_e^T P_i < 0. \tag{38}$$

We can write P instead of P_i. This is what we will use in the rest of this proof. If we perform a similarity transformation on (38), we get that it is equivalent to

$$\tilde{P}\tilde{A}_i + \tilde{A}_i^T \tilde{P} + \tilde{H}_e^T \tilde{H}_e + \gamma^2 \tilde{P}\tilde{E}_e \tilde{E}_e^T \tilde{P} < 0 \tag{39}$$

where $\tilde{P} = S^T PS = \begin{pmatrix} \frac{1}{\gamma^2}X_{11}^{-1} & 0 \\ 0 & Z \end{pmatrix}$, $\tilde{A}_i = S^{-1}A_i S$, $\tilde{E}_e = S^{-1}E_e$ and $\tilde{H}_e = H_e S$ for $S = \begin{pmatrix} I & 0 \\ I & -I \end{pmatrix}$. If we can show that (39) holds, we have shown that (38) holds which finishes the proof. This follows from the similarity transformation.

We will compute $\tilde{A}_i$, $\tilde{E}_e$ and $\tilde{H}_e$:

$$\tilde{A}_i = S^{-1}A_i S = \begin{pmatrix} A + BN & -BN \\ A + BN - \lambda_i MC - K - \lambda_i L & -BN + K + \lambda_i L \end{pmatrix},$$

$\tilde{E}_e = S^{-1}E_e = \begin{pmatrix} E \\ E \end{pmatrix}$ and $\tilde{H}_e = H_e S = \begin{pmatrix} H & 0 \end{pmatrix}$. We can now compute the left hand side of (39). We start with

$$\begin{aligned}
\tilde{P}\tilde{A}_i &= \begin{pmatrix} \frac{1}{\gamma^2}X_{11}^{-1} & 0 \\ 0 & Z \end{pmatrix} \begin{pmatrix} A+BN & -BN \\ A+BN-\lambda_i MC - K - \lambda_i L & -BN+K+\lambda_i L \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{\gamma^2}X_{11}^{-1}(A+BN) & -\frac{1}{\gamma^2}X_{11}^{-1}BN \\ Z(A+BN-\lambda_i MC-K-\lambda_i L) & Z(-BN+K+\lambda_i L) \end{pmatrix}.
\end{aligned}$$

$\tilde{A}_i^T\tilde{P}$ is the transpose of $\tilde{P}\tilde{A}_i$, so that becomes

$$\tilde{A}_i^T\tilde{P} = \begin{pmatrix} \frac{1}{\gamma^2}(A+BN)^T X_{11}^{-1} & (A+BN-\lambda_i MC-K-\lambda_i L)^T Z \\ -\frac{1}{\gamma^2}N^T B^T X_{11}^{-1} & (-BN+K+\lambda_i L)^T Z \end{pmatrix}.$$

Then we compute $\tilde{H}_e^T\tilde{H}_e$:

$$\tilde{H}_e^T\tilde{H}_e = \begin{pmatrix} H^T \\ 0 \end{pmatrix}\begin{pmatrix} H & 0 \end{pmatrix} = \begin{pmatrix} H^T H & 0 \\ 0 & 0 \end{pmatrix}.$$

Finally, we compute $\gamma^2\tilde{P}\tilde{E}_e\tilde{E}_e^T\tilde{P}$:

$$\begin{aligned}
\gamma^2\tilde{P}\tilde{E}_e\tilde{E}_e^T\tilde{P} &= \gamma^2 \begin{pmatrix} \frac{1}{\gamma^2}X_{11}^{-1} & 0 \\ 0 & Z \end{pmatrix}\begin{pmatrix} E \\ E \end{pmatrix}\begin{pmatrix} E^T & E^T \end{pmatrix}\begin{pmatrix} \frac{1}{\gamma^2}X_{11}^{-1} & 0 \\ 0 & Z \end{pmatrix} \\
&= \gamma^2 \begin{pmatrix} \frac{1}{\gamma^2}X_{11}^{-1} & 0 \\ 0 & Z \end{pmatrix}\begin{pmatrix} EE^T & EE^T \\ EE^T & EE^T \end{pmatrix}\begin{pmatrix} \frac{1}{\gamma^2}X_{11}^{-1} & 0 \\ 0 & Z \end{pmatrix} \\
&= \begin{pmatrix} X_{11}^{-1} & 0 \\ 0 & \gamma^2 Z \end{pmatrix}\begin{pmatrix} \frac{1}{\gamma^2}EE^T X_{11}^{-1} & EE^T Z \\ \frac{1}{\gamma^2}EE^T X_{11}^{-1} & EE^T Z \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{\gamma^2}X_{11}^{-1}EE^T X_{11}^{-1} & X_{11}^{-1}EE^T Z \\ ZEE^T X_{11}^{-1} & \gamma^2 ZEE^T Z \end{pmatrix}.
\end{aligned}$$

We will now try to find expressions for each of the four blocks on the left hand side of (39). The (1,1)-block of (39) is

$$\frac{1}{\gamma^2}X_{11}^{-1}(A+BN) + \frac{1}{\gamma^2}(A+BN)^T X_{11}^{-1} + H^T H + \frac{1}{\gamma^2}X_{11}^{-1}EE^T X_{11}^{-1} \tag{40}$$

and can be expressed as $\frac{1}{\gamma^2}R_F^i$. The (1,2)-block of (39) is

$$\begin{aligned}
&-\frac{1}{\gamma^2}X_{11}^{-1}BN + (A+BN-\lambda_i MC-K-\lambda_i L)^T Z + X_{11}^{-1}EE^T Z \\
=\; &-\frac{\lambda_i}{\gamma^2}X_{11}^{-1}BF + (A+\lambda_i BF-\lambda_i MC-K-\lambda_i L)^T Z + X_{11}^{-1}EE^T Z. \qquad (41)
\end{aligned}$$

By substituting K, L and M into (41) we obtain that the (1,2)-block is equal to $-\frac{k}{\gamma^2}R_F^i$. Since the (2,1)-block is the transpose of the (1,2)-block, we know that the (2,1)-block of (39) also equals $-\frac{k}{\gamma^2}R_F^i$.

The (2,2)-block of (39) is

$$Z(-BN+K+\lambda_i L)+(-BN+K+\lambda_i L)^T Z+\gamma^2 ZEE^T Z. \tag{42}$$

If we again substitute K, L, M and N we obtain that (42) is equal to

$$Y_{11}R_G^i Y_{11}+\frac{2k}{\gamma^2}R_F^i-\frac{1}{\gamma^2}R_F^i.$$

The computation of the (2,2)-block can be found in Appendix A. The left hand side of (39) thus equals

$$\begin{pmatrix} \frac{1}{\gamma^2}R_F^i & -\frac{k}{\gamma^2}R_F^i \\ -\frac{k}{\gamma^2}R_F^i & Y_{11}R_G^i Y_{11}+\frac{2k}{\gamma^2}R_F^i-\frac{1}{\gamma^2}R_F^i \end{pmatrix} \tag{43}$$

for every $i=2,\ldots,p$. We can write this as

$$\begin{pmatrix} \frac{1}{\gamma^2}(1-k)R_F^i & 0 \\ 0 & Y_{11}R_G^i Y_{11}+\frac{1}{\gamma^2}(1-k)R_F^i \end{pmatrix}+\frac{k}{\gamma^2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\otimes R_F^i \tag{44}$$

for every $i=2,\ldots,p$. We must show that this is negative definite. Since we have chosen $k\in(0,1)$ such that $Y_{11}R_G^p Y_{11}<\frac{1}{\gamma^2}(1-k)R_F^p$, we conclude that the first matrix in (44) is negative definite for all $i=2,\ldots,p$ (according to Lemma 2). The matrix $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ is positive semi-definite and R_F^i is negative definite for all $i=2,\ldots,p$. This means that the second term in (44) is negative semi-definite for all $i=2,\ldots,p$. From this we conclude that (44) is negative definite for all $i=2,\ldots,p$.

We have shown that (44) is negative definite and therefore (39) holds. Since we performed a similarity transformation we also have shown that (38) holds and therefore we have proven that there exist K, L, M, N and P_i for $i=2,\ldots,p$ such that

$$\begin{pmatrix} P_iA_i+A_i^TP_i+H_e^TH_e & P_iE_e \\ E_e^TP_i & -\frac{1}{\gamma^2}I \end{pmatrix}<0.$$

□

With finishing this proof we have shown that the dynamic protocol (25) synchronizes the network (10), with $D=0$ and $J=0$, for all $i=1,\ldots,p$ and for all $\Delta_i=\Delta\in RH_\infty$ with $||\Delta||_\infty\leq\gamma$ **if and only if** there exist matrices $X_{11}>0$ and $Y_{11}>0$ of size $n\times n$ such that (27), (28) and (29) hold.

4 Remarks on assumptions

4.1 Remark on $\Delta_i = \Delta$ for all i

Consider the system given in (6). One of the assumptions we made was that all Δ's are equal, i.e. $\Delta_i = \Delta$ for all i. We made this assumption, because from (13) we concluded that the disturbance of agent i depends on $z_1, \ldots, z_n$. So, we were not able to express $\tilde{d}_i$ in terms of $\tilde{z}_i$ only and this lead to a problem in Theorem 2 if every Δ_i was different. We were not able to find a solution for this problem, because z_i depends on x_i and u_i, whereas in [4] z_i only depends on u_i. The x_i's are causing a problem when proving Theorem 2 with different Δ_i's.

The problem lies in the part when proving that $2 \Rightarrow 1$. Suppose we do not make the assumption that $\Delta_i = \Delta$ for all i. Then we have to show that the protocol (7) synchronizes the perturbed network (10) for all Δ_i with $||\Delta_i||_\infty \leq \gamma$ and $i = 1, \ldots, p$. Since we performed a state transformation in (11), it means that we must show that for $i = 2, \ldots, p$ we have $\tilde{x}_i(t) \to 0$ and $\tilde{w}_i(t) \to 0$ as $t \to \infty$. First define $\bar{\mathbf{x}} = \mathrm{col}(\tilde{x}_2, \ldots, \tilde{x}_p)$ and define $\bar{\mathbf{w}}$, $\bar{\mathbf{z}}$ and $\bar{\mathbf{d}}$ in a similar way. We can write out the network dynamics as follows:

$$
\begin{aligned}
\begin{pmatrix} \dot{\bar{\mathbf{x}}} \\ \dot{\bar{\mathbf{w}}} \end{pmatrix} &= \begin{pmatrix} I_{p-1} \otimes A + \Lambda_1 \otimes BRC & I_{p-1} \otimes BN \\ \Lambda_1 \otimes MC & I_{p-1} \otimes K + \Lambda_1 \otimes L \end{pmatrix} \begin{pmatrix} \bar{\mathbf{x}} \\ \bar{\mathbf{w}} \end{pmatrix} \\
&\quad + \begin{pmatrix} I_{p-1} \otimes E + \Lambda_1 \otimes BRD \\ \Lambda_1 \otimes MD \end{pmatrix} \bar{\mathbf{d}} \\
\bar{\mathbf{z}} &= \begin{pmatrix} I_{p-1} \otimes H + \Lambda_1 \otimes JRC & I_{p-1} \otimes JN \end{pmatrix} \begin{pmatrix} \bar{\mathbf{x}} \\ \bar{\mathbf{w}} \end{pmatrix} + (\Lambda_1 \otimes JRD)\bar{\mathbf{d}} \\
\bar{\mathbf{d}} &= \Delta' \bar{\mathbf{z}} + \begin{pmatrix} \Delta_{21} \\ \vdots \\ \Delta_{p1} \end{pmatrix} \tilde{z}_1.
\end{aligned}
$$

Here $\Lambda_1 := \mathrm{diag}(\lambda_2, \ldots, \lambda_p)$, I_{p-1} is the identity matrix of size $(p-1) \times (p-1)$ and

$$
\Delta' = \begin{pmatrix} \Delta_{11} & \cdots & \Delta_{1p} \\ \vdots & \ddots & \vdots \\ \Delta_{p1} & \cdots & \Delta_{pp} \end{pmatrix}.
$$

In this system the transfer matrix from $\bar{\mathbf{d}}$ to $\bar{\mathbf{z}}$ is equal to

$$
G_{\Sigma \times \Gamma} := \mathrm{blockdiag}(G_{\Sigma \times \Gamma_2}, \ldots, G_{\Sigma \times \Gamma_p}),
$$

with $||G_{\Sigma \times \Gamma_i}||_\infty < \dfrac{1}{\gamma}$ for all $i = 2, \ldots, p$. Thus, $||G_{\Sigma \times \Gamma}||_\infty < \dfrac{1}{\gamma}$.

The problem lies in the fact that $\bar{\mathbf{d}}$ must go to zero as time goes to infinity. We know from [4] that $||\Delta'||_\infty \leq \gamma$, so $(\Delta' \bar{\mathbf{z}}) \to 0$ as $t \to \infty$.

We also need that $\tilde{z}_1 = H\tilde{x}_1 + JN\tilde{w}_1$ goes to zero as time goes to infinity, but this leads to a problem. Look at the system for agent 1 (with $\lambda_1 = 0$):

$$
\begin{cases} \dot{\tilde{x}}_1 &= A\tilde{x}_1 + BN\tilde{w}_1 + E\tilde{d}_1 \\ \dot{\tilde{w}}_1 &= K\tilde{w}_1 \\ \tilde{z}_1 &= H\tilde{x}_1 + JN\tilde{w}_1 \\ \tilde{d}_1 &= \begin{pmatrix} \Delta_{11} & \ldots & \Delta_{1p} \end{pmatrix} \tilde{z}. \end{cases} \tag{45}
$$

If we assume that K is Hurwitz we can conclude that $\dot{\tilde{w}}_1 = K\tilde{w}_1$ is stable, which implies that $\tilde{w}_1 \to 0$, but we can't say anything about $\tilde{x}_1$. So, $\tilde{z}_1$ does not need to go to zero and therefore $\bar{\mathbf{d}}$ does not need to go to zero. Thus, we are not able to show that for $i = 2, \ldots, p$ we have $\tilde{x}_i(t) \to 0$ and $\tilde{w}_i(t) \to 0$ as $t \to \infty$. It can only be shown if we make the assumption that $\Delta_i = \Delta$ for all i.

4.2 Remark on $J = 0$ and $D = 0$

When we tried to prove Conjecture 1 we were not able to find suitable K, L, M, N, R and P. We believe it should be possible to prove it, but it takes too much time to finish the proof in this thesis. We were able to prove the 'only if' direction in a similar way as we did in Theorem 3. We only need different annihilators:

$$\begin{pmatrix} B_e \\ J_e \end{pmatrix}^{\perp} \text{ and } \begin{pmatrix} C_e^T \\ D_e^T \end{pmatrix}^{\perp}. \tag{46}$$

The problem lies in the 'if' direction. Let us look at inequality (21) and define $P := \frac{1}{\gamma^2} X_{11}^{-1}$. It follows that (21) is equivalent to

$$\begin{pmatrix} B \\ J \end{pmatrix}^{\perp} \begin{pmatrix} AP^{-1} + P^{-1}A^T + \gamma^2 EE^T & P^{-1}H^T \\ HP^{-1} & -I \end{pmatrix} \begin{pmatrix} B \\ J \end{pmatrix}^{\perp T} < 0.$$

Applying Finsler's lemma to this means that there exists a $\sigma > 0$ such that

$$\begin{pmatrix} AP^{-1} + P^{-1}A^T + \gamma^2 EE^T & P^{-1}H^T \\ HP^{-1} & -I \end{pmatrix} - \sigma \begin{pmatrix} B \\ J \end{pmatrix} \begin{pmatrix} B^T & J^T \end{pmatrix} < 0. \tag{47}$$

We know from [5] that there should exist an F such that $R_F^i < 0$ is of the following form:

$$R_F^i := (A + \lambda_i BF)^T P + P(A + \lambda_i BF) + \gamma^2 PEE^T P + (H + JF)^T (H + JF). \tag{48}$$

According to [3], this F should be of the following form:

$$F := -\sigma B^T P + \sigma J^T (I + \sigma JJ^T)^{-1}(-H + \sigma JB^T P). \tag{49}$$

Define $R := I + \sigma JJ^T$. A short computation tells us that $R^{-1} = I - \sigma JJ^T R^{-1}$ (or $R^{-1} = I - \sigma R^{-1} JJ^T$). The problem is now that if we substitute F into R_F^i, we can't conclude that R_F^i is less than zero.

We know, since (47) holds, that the Schur complement of the left hand side of (47) is also less than zero. If we pre- and post multiply this Schur complement with $P > 0$ and write it out, we get

$$\begin{aligned} &PA + A^T P + \gamma^2 PEE^T P - \sigma PBB^T P + H^T R^{-1} H \\ &\quad - \sigma PBJ^T R^{-1} H - \sigma H^T R^{-1} JB^T P + \sigma^2 PBJ^T R^{-1} JB^T P < 0 \end{aligned} \tag{50}$$

The idea is now that if we substitute F into R_F^i we get something similar that is also less than zero. The whole computation can be found in Appendix B. The result is

$$\begin{aligned} &A^T P + PA - 2\sigma PBB^T P + \gamma^2 PEE^T P + 2\sigma^2 PBJ^T R^{-1} JB^T P \\ &\quad - \sigma H^T R^{-1} JB^T P - \sigma PBJ^T R^{-1} H + H^T R^{-2} H \\ &\quad - \sigma H^T R^{-2} JB^T P - \sigma PBJ^T R^{-2} H + \sigma^2 PBJ^T R^{-2} JB^T P. \end{aligned} \tag{51}$$

If we compare (50) with (51), we see that we have too many terms in (51). We notice that the 2 in front of σPBB^TP does not have much influence, because the term is already negative definite. Furthermore, we know that $R^{-1} \geq R^{-2}$, since $I \geq R^{-1}$. The terms that remain problematic are

$$\sigma^2 PBJ^T R^{-1} JB^T P - \sigma H^T R^{-2} JB^T P - \sigma PBJ^T R^{-2} H + \sigma^2 PBJ^T R^{-2} JB^T P.$$

We have no knowledge about any of these terms, so we cannot conclude that (51) is less than zero. Unfortunately we did not have enough time to solve this problem properly, so we assumed that the matrices J and D are zero. If J is zero, the second term of F vanishes which caused the problem. D had to be zero because of the dual problem.

Although the literature shows that this F should work, we were not able to prove Conjecture 1 using this method with LMI's. We are not sure whether the conjecture holds if J and D are arbitrary matrices, but we believe it should be possible to prove it in a different way. The problem to check whether Conjecture 1 admits a proof is left for future research.

5 Conclusions

5.1 Summary

In this thesis we have looked at the robust synchronization problem for linear, time-invariant, finite dimensional multi-agent systems. We have considered agents with identical systems except for a perturbation Δ_i. Δ_i is stable and has H_∞-norm bounded by a certain given tolerance. The closed loop feedback system we have considered is given in (6). We have found a protocol (given in (7)) that synchronizes the systems in the multi-agent network (10). This protocol is based on a protocol in [4]. We needed an extra matrix R in the protocol, because otherwise we could not find the annihilators needed in the proof of Conjecture 1.

We have tried to solve the robust synchronization problem using LMI's as stated in Conjecture 1. So, we have tried to show that the protocol (7) works if and only if the three LMI's given in (21), (22) and (23) are satisfied. This appeared to be harder than we initially thought and because of that we were not able to prove the conjecture. We had to make a few assumptions to simplify the problem. The first thing we had to assume was that all Δ_i's are equal, that is $\Delta_i = \Delta$ for all i. This assumption had to be made, because with different Δ_i's we were not able to prove Theorem 2.

We also had to simplify the multi-agent system by setting $D = 0$ and $J = 0$. Without this assumption we were not able to prove Conjecture 1 within the time available for this thesis. We have struggled with the proof without these assumptions, but we still believe it should be possible to prove the conjecture (with or without taking the Δ_i's equal) for multi-agent systems.

5.2 Future research

For future research it would be interesting to look at the assumptions we made. The assumption that every agent has the same perturbation Δ can be studied. It is interesting to find out why this does not work for all multi-agent systems in this case. Maybe a different proof than the one in [4] can be found to show that it is not necessary to make this assumption.

It is also interesting for future research to try finding a proof of Conjecture 1. We think it should be possible to prove this theorem, but maybe it should be done in a different way. One could try to prove it using another method than the one we tried (based on [6]).

Appendices

A Computation of (2,2)-block of the left hand side of (39)

$$
\begin{aligned}
&Z(-BN+K+\lambda_i L)+(-BN+K+\lambda_i L)^T Z+\gamma^2 ZEE^TZ \\
&\quad = Z(-\lambda_i BF + A + EE^T X_{11}^{-1} + \lambda_i Z^{-1} Y_{11} GC + \Omega_1 + \lambda_i (BF+\Omega_2)) \\
&\qquad + (-\lambda_i BF + A + EE^T X_{11}^{-1} + \lambda_i Z^{-1} Y_{11} GC + \Omega_1 + \lambda_i (BF+\Omega_2))^T Z + \gamma^2 ZEE^TZ \\
&\quad = ZA + A^TZ + ZEE^TX_{11}^{-1} + X_{11}^{-1}EE^TZ + \gamma^2 ZEE^TZ + \lambda_i Y_{11}GC + \lambda_i C^TG^TY_{11} \\
&\qquad + Z(\Omega_1+\lambda_i\Omega_2) + (\Omega_1+\lambda_i\Omega_2)^TZ \\
&\quad = ZA + A^TZ + (Y_{11} - \frac{1}{\gamma^2}X_{11}^{-1})EE^TX_{11}^{-1} + X_{11}^{-1}EE^T(Y_{11} - \frac{1}{\gamma^2}X_{11}^{-1}) \\
&\qquad + \gamma^2(Y_{11} - \frac{1}{\gamma^2}X_{11}^{-1})EE^T(Y_{11} - \frac{1}{\gamma^2}X_{11}^{-1}) + \lambda_i Y_{11}GC + \lambda_i C^TG^TY_{11} \\
&\qquad + \frac{k}{\gamma^2}(A^TX_{11}^{-1} + X_{11}^{-1}A + X_{11}^{-1}EE^TX_{11}^{-1} + \gamma^2 H^TH) - \frac{\lambda_i}{\gamma^2}((1-k)F^TB^TX_{11}^{-1} - kX_{11}^{-1}BF) \\
&\qquad + \frac{k}{\gamma^2}(A^TX_{11}^{-1} + X_{11}^{-1}A + X_{11}^{-1}EE^TX_{11}^{-1} + \gamma^2 H^TH) - \frac{\lambda_i}{\gamma^2}((1-k)X_{11}^{-1}BF - kF^TB^TX_{11}^{-1}) \\
&\quad = ZA + A^TZ + Y_{11}EE^TX_{11}^{-1} - \frac{1}{\gamma^2}X_{11}^{-1}EE^TX_{11}^{-1} + X_{11}^{-1}EE^TY_{11} - \frac{1}{\gamma^2}X_{11}^{-1}EE^TX_{11}^{-1} \\
&\qquad + (\gamma^2 Y_{11}EE^T - X_{11}^{-1}EE^T)(Y_{11} - \frac{1}{\gamma^2}X_{11}^{-1}) + \lambda_i Y_{11}GC + \lambda_i C^TG^TY_{11} + \frac{2k}{\gamma^2}R_F^i \\
&\qquad - \frac{\lambda_i}{\gamma^2}F^TB^TX_{11}^{-1} - \frac{\lambda_i}{\gamma^2}X_{11}^{-1}BF \\
&\quad = ZA + A^TZ + \gamma^2 Y_{11}EE^TY_{11} - \frac{1}{\gamma^2}X_{11}^{-1}EE^TX_{11}^{-1} + \lambda_i Y_{11}GC + \lambda_i C^TG^TY_{11} \\
&\qquad + \frac{2k}{\gamma^2}R_F^i - \frac{\lambda_i}{\gamma^2}F^TB^TX_{11}^{-1} - \frac{\lambda_i}{\gamma^2}X_{11}^{-1}BF \\
&\quad = (Y_{11} - \frac{1}{\gamma^2}X_{11}^{-1})A + A^T(Y_{11} - \frac{1}{\gamma^2}X_{11}^{-1}) + \gamma^2 Y_{11}EE^TY_{11} - \frac{1}{\gamma^2}X_{11}^{-1}EE^TX_{11}^{-1} \\
&\qquad + \lambda_i Y_{11}GC + \lambda_i C^TG^TY_{11} + \frac{2k}{\gamma^2}R_F^i - \frac{\lambda_i}{\gamma^2}F^TB^TX_{11}^{-1} - \frac{\lambda_i}{\gamma^2}X_{11}^{-1}BF + H^TH - H^TH \\
&\quad = Y_{11}A + A^TY_{11} + \gamma^2 Y_{11}EE^TY_{11} + \lambda_i Y_{11}GC + \lambda_i C^TG^TY_{11} + H^TH + \frac{2k}{\gamma^2}R_F^i - \frac{1}{\gamma^2}R_F^i \\
&\quad = Y_{11}R_G^iY_{11} + \frac{2k}{\gamma^2}R_F^i - \frac{1}{\gamma^2}R_F^i.
\end{aligned}
$$

B Computation of filling in F in R_F^i

$$
\begin{aligned}
&(A - \sigma BB^TP + \sigma BJ^TR^{-1}(-H + \sigma JB^TP))^TP \\
&\quad + P(A - \sigma BB^TP + \sigma BJ^TR^{-1}(-H + \sigma JB^TP)) + \gamma^2 PEE^TP \\
&\quad + (H - \sigma JB^TP + \sigma JJ^TR^{-1}(-H + \sigma JB^TP))^T(H - \sigma JB^TP + \sigma JJ^TR^{-1}(-H + \sigma JB^TP)) \\
&\quad = (A - \sigma BB^TP - \sigma BJ^TR^{-1}H + \sigma^2 BJ^TR^{-1}JB^TP))^TP \\
&\quad + P(A - \sigma BB^TP - \sigma BJ^TR^{-1}H + \sigma^2 BJ^TR^{-1}JB^TP) + \gamma^2 PEE^TP \\
&\quad + (H - \sigma JB^TP - \sigma JJ^TR^{-1}H + \sigma^2 JJ^TR^{-1}JB^TP))^T \\
&\quad \times (H - \sigma JB^TP - \sigma JJ^TR^{-1}H + \sigma^2 JJ^TR^{-1}JB^TP)) \\
&\quad = A^TP + PA - 2\sigma PBB^TP + \gamma^2 PEE^TP + 2\sigma^2 PBJ^TR^{-1}JB^TP - \sigma H^TR^{-1}JB^TP \\
&\quad - \sigma PBJ^TR^{-1}H + (H - \sigma PBJ^T - \sigma H^TR^{-1}JJ^T + \sigma^2 PBJ^TR^{-1}JJ^T)) \\
&\quad \times (H - \sigma JB^TP - \sigma JJ^TR^{-1}H + \sigma^2 JJ^TR^{-1}JB^TP)).
\end{aligned}
$$

Use the fact that $R^{-1} = I - \sigma JJ^TR^{-1}$ and $R^{-1} = I - \sigma R^{-1}JJ^T$.

$$
\begin{aligned}
&= A^TP + PA - 2\sigma PBB^TP + \gamma^2 PEE^TP + 2\sigma^2 PBJ^TR^{-1}JB^TP - \sigma H^TR^{-1}JB^TP \\
&\quad - \sigma PBJ^TR^{-1}H + (H^T(I - \sigma R^{-1}JJ^T) - \sigma PBJ^T(I - \sigma R^{-1}JJ^T)) \\
&\quad \times ((I - \sigma JJ^TR^{-1})H - \sigma(I - \sigma JJ^TR^{-1})JB^TP) \\
&= A^TP + PA - 2\sigma PBB^TP + \gamma^2 PEE^TP + 2\sigma^2 PBJ^TR^{-1}JB^TP - \sigma H^TR^{-1}JB^TP \\
&\quad - \sigma PBJ^TR^{-1}H + (H^TR^{-1} - \sigma PBJ^TR^{-1})(R^{-1}H - \sigma R^{-1}JB^TP) \\
&= A^TP + PA - 2\sigma PBB^TP + \gamma^2 PEE^TP + 2\sigma^2 PBJ^TR^{-1}JB^TP \\
&\quad - \sigma H^TR^{-1}JB^TP - \sigma PBJ^TR^{-1}H + H^TR^{-2}H \\
&\quad - \sigma H^TR^{-2}JB^TP - \sigma PBJ^TR^{-2}H + \sigma^2 PBJ^TR^{-2}JB^TP.
\end{aligned}
$$

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