UNIVERSITY OF GRONINGEN

MASTER THESIS

Radiative Symmetry Breaking in minimal Conformally Invariant extensions of the Standard Model

Author:
Susan van der Woude
(S2368641)

Supervisor:
Prof. Dr. D. Boer

2nd examiner:
Dr. K. Papadodimas

Date:
August 1, 2017

A thesis submitted in partial fulfilment of the requirements for the degree of Master of Science

in the

Theoretical Particle Physics Group
Van Swinderen Institute for Particle Physics and Gravity

university of groningen

faculty of science and engineering

Abstract

The Standard Model is expected to be only an effective theory because, for one, the Standard Model does not describe gravity. Additionally the Standard Model can not describe Dark Matter and faces the Hierarchy Problem. Beyond the Standard Model (BSM) theories are devised to solve these and other problems of the Standard Model. However, these BSM theories face some strict limitations from experiments because up to now not a single particle physics experiment has measured any significant deviations from the Standard Model.

In this thesis a class of BSM theories called the **minimal conformally symmetric SM extensions** are discussed. These theories are promising BSM candidates because the conformal symmetry results in a theory which does not suffer from the Hierarchy Problem. The minimality ensures that the theory will closely resemble the Standard Model and thus not have a too big effect on experimental data.

Because conformally symmetric theories do not exhibit classical spontaneous symmetry breaking another mechanism for spontaneous symmetry breaking is necessary. Therefore a large part of this work is spent on explaining **Radiative Symmetry Breaking**, which describes spontaneous symmetry breaking due to quantum corrections. The concept of Radiative Symmetry Breaking was first introduced by S. Coleman and E. Weinberg. Some minimal conformal extensions of the Standard Model are discussed, one of which is the extension of the Standard Model with one scalar. The phenomenology but also the possible problems of these extensions are discussed to determine which extensions are good candidates to be a BSM theory. It will be argued that already a two scalar extension is able to provide a theory which is reliable up to the Planck Scale.

The idea of **Asymptotic Safety** is briefly discussed to introduce the reader to another possible way in which the Standard Model can be extended.

Contents

Chapter 1

Introduction

The Standard Model (SM) is a beautiful theory capable of explaining almost all of particle physics. The SM was finally completed in 2012 with the detection of its last missing piece, the Higgs boson. The SM describes three of the four fundamental interactions, while the fourth interaction is described by the theory of General Relativity (GR). Together the SM and GR are able to describe physical phenomena remarkably well. Precision measurements of for example the electron magnetic moment but also the gravitational waves experiment LIGO have not been able to measure any significant deviations from either SM or GR predictions. Additionally, particle colliders like the LHC have not (yet) measured anything that is not compatible with the SM framework.

Despite the extremely good agreement between theory and experiment there are still some things which can not be understood within the current framework of theoretical physics. For example, the nature of Dark Matter and Dark Energy but also the Hierarchy Problem and the Stability Problem can not be explained by either the SM or GR. Apart from these problems the SM has one fundamental shortcoming - it does not include a quantized theory of gravity. General Relativity is only a classical theory, to describe physics on the smallest length scales - where gravity becomes a non-negligible interaction - a quantum theory of gravity is needed. Unfortunately gravity can not be put in the same framework as the SM without encountering serious renormalization problems.

The problems outlined above are strong indications that the SM is only an effective field theory - at some scale the SM will break down. An extended SM is therefore necessary to gain a more complete picture of the fundamental physics which defines the world we live in.

To develop possible SM extension one needs to look at theories solving one or more of the mentioned SM problems. In this thesis the focus will be on the possible solutions of the Hierarchy problem. The solutions discussed might also be able to explain Dark Matter and even solve the Stability Problem. What we will however not attempt in this thesis is to provide a possible quantum theory of gravity. Therefore the goal of this work is to discuss viable theories which are able to describe particle physics up to the Planck scale. The extensions discussed will still be effective theories, since a quantum theory of gravity is expected to play a role at energies bigger than the Planck scale.

An interesting and elegant class of solutions to the Hierarchy Problem are the so-called conformally invariant extensions of the SM. In these extensions the mass term of the scalar potential is set to zero, thereby resulting in a theory which only has dimensionless fundamental parameters. Apart from being conformally invariant the SM extensions discussed in this thesis will be minimal, in the sense that only *hidden* scalars are added to the particle spectrum. The term hidden scalar means that the added scalars only couple to the Higgs boson and not to any of the other SM particles. Arguments will be provided to convince the reader that these kind of extensions are especially interesting to look at.

This thesis is mostly a review on the work done on minimal conformally invariant extensions of the SM. This review will hopefully provide a clear and comprehensive basis from which new research can be conducted. The goal is to determine which extensions might be viable Beyond the SM theories and determine in which direction subsequent research on this topic can be done.

The outline of the thesis is as follows:

- Chapter 2 will start with a short summary of the SM. Because the Hierarchy problem originates in the electroweak sector the chapter will extensively discuss the electroweak interactions and the Higgs mechanism.
- Chapter 3 discusses the problems of the SM, mentioned in this introduction, in more detail. Special attention will be paid to the Hierarchy Problem. Also, a short summary of some experiments will be given to determine where there is still room to improve the SM. This will allow us to establish which properties are interesting to impart on a SM extension. We will argue that combining minimal models with conformal symmetry gives rise to viable SM extensions.
- Chapter 4 introduces the concept of Radiative Symmetry Breaking, this mechanism is necessary to preserve Electroweak Symmetry Breaking in theories with conformal symmetry. In these theories the *classical* Higgs Mechanism does not function, therefore a *quantum* version of Spontaneous Symmetry Breaking is introduced - called the Coleman-Weinberg mechanism. Since the classical Higgs mechanism has not been verified by experiments this option is still a possible cause of Electroweak Symmetry Breaking. The Coleman-Weinberg mechanism will be explained through some simple examples before deriving the general formula of radiative symmetry breaking.
- Chapter 5 focusses on implementing radiative symmetry breaking into the SM and its minimal extensions. Since these extension will contain multiple scalars an extension of the Coleman-Weinberg mechanism developed by E. Gildener and S. Weinberg will be introduced. Using a simple theory, supplemented by clear graphs the Gildener-Weinberg method will be explained in an intuitive and non-abstract way. In the end some experimental signatures of the discussed extensions will be determined. From this we will be able to see how the minimal conformally invariant extensions differ from the SM.
- In chapter 6 an alternative way to extend the SM - called Asymptotic Safety - will be briefly mentioned.
- Chapter 7 will give a short summary of the things discussed in this thesis.
- Chapter 8 will end with an outlook on possible further research.

Conventions

The conventions used throughout this thesis are as follows:

- Natural units are used, e.g. $c = \hbar = 1$.
- The metric used is the mostly-minus four dimensional Minkowski metric: $\eta_{\mu\nu} = \text{diag}(+, -, -, -)$.
- Greek Indices run over spacetime dimensions $(0, 1, \cdots, d-1)$
- Roman indices run over space dimensions $(1, 2, \cdots, d-1)$.

The figures in this thesis are all made using MATHEMATICA, unless specified otherwise.

Chapter 2

The Higgs Mechanism and the Standard Model

2.1 The Standard Model

The Standard Model (SM) is described by the gauge symmetry group $SU(3)_C \times SU(2)_L \times U(1)_Y$. The $SU(3)_C$ gauge symmetry group describes the strong interaction between the gluons and quarks whereas the electroweak interactions are described by the $SU(2)_L \times U(1)_Y$ gauge group. The theory of Electroweak interactions is called the Glashow-Salam-Weinberg (GSW) model (see for example [1]). The GSW model contains the Higgs mechanism which requires the existence of a scalar; the Higgs boson.

Apart from the Higgs boson the elementary particles of the SM include the leptons and quarks. The detection of the Higgs Boson in 2012 at the LHC in CERN means that all elementary particles of the Standard Model are now measured.

The elementary particles of the SM are summarized in table 2.1. Only the first generation is shown, the other two are exact copies with respect to the symmetries and quantum numbers. Q is the electromagnetic charge, Y is the hypercharge related to the $U(1)_Y$ gauge symmetry and T_3 is the third component of the weak isospin, related to the $SU(2)$ symmetry. The quantum numbers are related through: $Y = 2(Q - T_3)$.

The SM is left-right asymmetric in the sense that only the left handed fermions interact via the weak interaction. This asymmetry is also visible in table 2.1. From this table we see that the left handed fermions combine into doublets with respect to the $SU(2)_L \otimes U(1)_Y$ symmetry whereas the right handed fermions are singlets with respect to this symmetry. The fermion is decomposed into left- and right-handed parts through: $\Psi = \Psi_L + \Psi_R$. This results in $\overline{\Psi}\Psi = \overline{\Psi}_L\Psi_R + \text{h.c.}$.

Φ is the complex Higgs doublet with components:

$$\Phi = \begin{pmatrix} \Phi^+ \\ \Phi^0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_0 + i\phi_3 \end{pmatrix} \tag{2.1}$$

Before introducing the concept of spontaneous symmetry breaking the necessity of the Higgs mechanism will be explained.

		Q	Y	T_3
$Q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}$	u_L	$+\frac{2}{3}$	$+\frac{1}{3}$	$+\frac{1}{2}$
	d_L	$-\frac{1}{3}$	$+\frac{1}{3}$	$-\frac{1}{2}$
u_R		$+\frac{2}{3}$	$+\frac{4}{3}$	0
d_R		$-\frac{1}{3}$	$-\frac{2}{3}$	0
$E_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$	ν_L	0	-1	$+\frac{1}{2}$
	e_L	-1	-1	$-\frac{1}{2}$
e_R		-1	-2	0
$\Phi = \begin{pmatrix} \Phi^+ \\ \Phi_0 \end{pmatrix}$	Φ^+	$+1$	$+1$	$+\frac{1}{2}$
	Φ_0	0	$+1$	$-\frac{1}{2}$

TABLE 2.1: The electroweak quantum numbers of the Standard Model elementary particles.

2.2 Why is the Higgs boson necessary?

An essential part of the GSW model is the Higgs mechanism, which described spontaneous symmetry breaking of the $SU(2)_L \times U(1)_Y$ group into the $U(1)_Q$ group. Without the Higgs mechanism the theory of electroweak interactions suffers one big problem: all gauge bosons and fermions are necessarily massless, which is in clear contradiction with experiments.

The naive way of giving particles a mass would be to add explicit mass terms to the Lagrangian, however for fermions and gauge bosons this is impossible without breaking the $SU(2)_L \otimes U(1)_Y$ gauge symmetry explicitly.

In the case of gauge bosons a mass term cannot be added explicitly due to gauge invariance. The simplest theory for which this can be shown is a $U(1)$ gauge theory, for which the gauge field transforms as:

$$A_\mu \to A_\mu - \frac{1}{e}\partial_\mu \theta \tag{2.2}$$

Adding a mass term of the form $m^2 A_\mu A^\mu$ to the Lagrangian will clearly break the $U(1)$ gauge symmetry. The same is true for general ($SU(N)$) gauge theories.

For fermions we run into another problem when constructing an invariant mass term. The root of the problem lies in the asymmetry of the Standard Model. The left handed fermions transform as doublets under $SU(2)_L$ whereas the right handed fermions transform as singlets under $SU(2)_L$. Mass terms for fermions are of the form $m^2\overline{\psi}\psi$ which can be decomposed as $m^2\overline{\psi}_L\psi_R$+h.c.. Due to the different $SU(2)_L$ transformation properties of the left- and right-handed spinor this term cannot be invariant under $SU(2)_L$ transformations and is therefore not a suitable mass term for the fermion.

The examples above show that explicit mass terms in the SM Lagrangian are not allowed for fermions and gauge bosons. To achieve mass generation the Higgs Mechanism is needed. The Higgs mechanism describes spontaneous symmetry breaking (SSB) of local/gauge symmetries through which particles can obtain a mass, even though the Lagrangian does not allow for any explicit mass terms for these particles.

In the next sections SSB of a global symmetry will be explained first, before moving on to the Higgs mechanism in which local/gauge symmetries are spontaneously broken. A sketch of the mechanism and how it gives mass to the gauge bosons as well as the fermions in the SM will be given.

The theory of SSB and its implementation into the SM presented in this chapter can be found in most standard textbooks on quantum field theory, eg. [1] and [2].

2.3 Spontaneous Symmetry Breaking

Spontaneous symmetry breaking is the phenomenon that even though the Lagrangian is invariant under some symmetry the ground state is not. A well known example is that of magnetism, the Hamiltonian does not have a preferred direction, but the ground state is a system in which all spins point in the same direction. When the system selects a particular ground state the rotational invariance is broken.

Brout, Englert and Higgs applied the phenomenon of SSB to the SM in order to explain how fermions and gauge bosons acquire mass. This model of electroweak symmetry breaking (EWSB) requires the existence of a neutral scalar particle, the Higgs boson, which was finally found in 2012 by the Large Hadron Collider.

First a simple theory will be discussed to give an example of SSB, however this example only shows how SSB can be achieved; it does not result in massive gauge bosons. For gauge bosons to acquire mass a somewhat more complicated theory is needed, which will be discussed in the next section.

The phenomenon of SSB can be discussed using the following (global) $U(1)$ invariant Lagrangian:

$$\mathcal{L} = \partial_\nu \phi^* \partial^\nu \phi - V(\phi\phi^*) \tag{2.3a}$$

$$V(\phi\phi^*) = \mu^2 \phi\phi^* + \frac{1}{6}\lambda(\phi\phi^*)^2 \tag{2.3b}$$

This theory is equivalent to a real scalar field theory with two real scalars, where $\phi = (\phi_1 + i\phi_2)/\sqrt{2}$.

The scalar field theory exhibits qualitatively different behaviour depending on the sign of μ^2.

For positive μ^2 this Lagrangian describes a massive scalar field theory with quartic interactions, the minimum of the potential is at $\phi\phi^* = 0$.

For negative μ^2, on the other hand, the theory behaves completely different. The negative mass results in a potential which has the shape of a "Mexican hat" (see fig. 2.1). The true minimum (vacuum expectation value or VEV) of the theory is not at $\phi\phi^* = 0$, but is described by a continuous family of vacua: $\phi\phi^* = \frac{-3\mu^2}{\lambda} \equiv \frac{v^2}{2}$.

To accurately describe the (low energy) physics of this Lagrangian, the Lagrangian needs to be expanded around a ground state. Due to the global $U(1)$ symmetry a ground state can be selected freely as long as it satisfies $\phi\phi^* = \frac{v^2}{2}$.

Eq. 2.3 can now be expanded around the ground state in the following way:

$$\phi(x) = \frac{v + \eta(x)}{\sqrt{2}} e^{i\sigma(x)} \tag{2.4}$$

The ground state is defined by $\eta(x) = \sigma(x) = 0$. η describes oscillations in the radial direction, σ describes fluctuations in the polar direction. Plugging this into the Lagrangian (eq. 2.3) results in:

$$\mathcal{L} = \frac{1}{2}\partial_\nu\eta\partial^\nu\eta + \frac{1}{2}\partial_\nu(v\sigma)\partial^\nu(v\sigma) + \mu^2\eta^2 + \text{interactions} + \text{constants} \tag{2.5}$$

The interactions include tadpoles as well as quartic and cubic interactions. For clarity they are not written down explicitly since our interest at the moment is in the generation of mass by SSB.

After SSB the particle spectrum contains one massless scalar, $v\sigma$ - the Goldstone boson of broken $U(1)$ symmetry - and one massive scalar η with $m^2 = -2\mu^2$. Looking back at the figure of the Mexican hat potential we can understand why one scalar is massless while the other is massive. The angular direction is flat and therefore the scalar $v\sigma$ is massless. A scalar fluctuation in the radial direction feels a quadratic potential thus η is massive.

SSB of a global symmetry thus results in a particle spectrum with one massive scalar and a number of massless scalars. In a relativistic theory the number of massless scalars can be determined using Goldstone's theorem:

"For every generator of a broken continuous symmetry a massless scalar particle is present in the particle spectrum."

This means that for a field theory with symmetry group G (M generators) undergoing SSB into a vacuum state which is invariant under the group H (N generators) a total of $M - N$ Goldstone bosons is present, one for each broken generator.

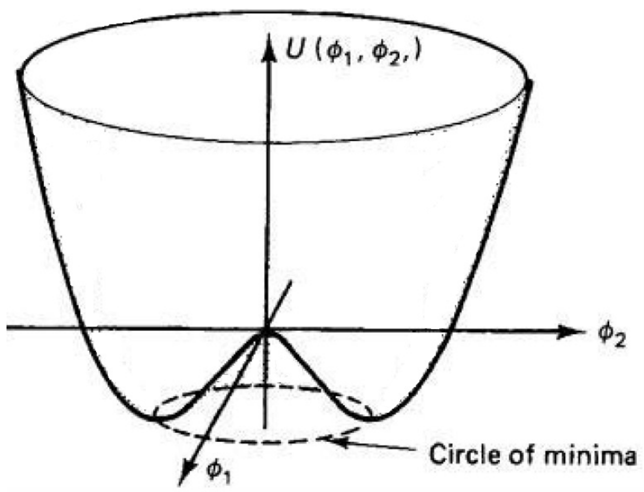

FIGURE 2.1: The potential as function of ϕ_1 and ϕ_2 for negative μ^2 [3]

2.4 Abelian Higgs

The simplest theory in which SSB occurs and a gauge boson does acquire mass is the Abelian $U(1)$ gauge theory. We will start with this theory to explain the idea and then extend the model to incorporate it into the SM.

The Lagrangian of eq. 2.3 has a global $U(1)$ symmetry but is not invariant under local $U(1)$ transformations. The difference between local(gauge) and global symmetries is that global symmetries act on ϕ independent of the spacetime coordinate of ϕ whereas for a local symmetry the change of ϕ under the transformation is dependent on the spacetime coordinates. Mathematically these statements are described by the following equations:

$$\phi \to U\phi = e^{i\alpha}\phi \qquad \textbf{global}$$
$$\phi \to U(x)\phi = e^{i\alpha(x)}\phi \qquad \textbf{local}$$

Due to the x dependence of the local transformation the kinetic term in the Lagrangian is not invariant under local $U(1)$ transformations. By postulating a new field A_μ and coupling this to the field ϕ the Lagrangian can be made invariant under local $U(1)$ transformations:

$$\mathcal{L} = D_\nu\phi D^\nu\phi^* - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \mu^2\phi\phi^* - \frac{1}{6}\lambda(\phi\phi^*)^2 \tag{2.6a}$$

with

$$D_\nu\phi = \partial_\nu\phi + ieA_\nu\phi \tag{2.6b}$$

$F_{\mu\nu}$ is the kinetic term of the field A_ν and is invariant under local $U(1)$ transformations.

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2.6c}$$

The field ϕ transforms as:

$$\phi(x) \to e^{i\alpha(x)}\phi(x) \tag{2.6d}$$

To make the Lagrangian invariant under local $U(1)$ transformations the following transformation is imposed on $D_\mu\phi$:

$$D_\mu\phi \to e^{i\alpha(x)}D_\mu\phi \tag{2.6e}$$

Combining these transformations it can be seen that A_μ has to transforms as:

$$A_\mu(x) \to A_\mu(x) - \frac{1}{e}\partial_\mu\alpha(x) \tag{2.6f}$$

The equations above, for a $U(1)$ gauge theory, can be extended to general $SU(N)$ gauge invariant theories.

From this example it can be concluded that imposing a local/gauge symmetry requires the existence of gauge bosons. Imposing local $SU(N)$ invariance results in N^2-1 gauge fields, one for every generator. These gauge fields are however still massless, to give them mass SSB is required. From the simple example fo a global symmetry we have seen that this requires μ^2 to be negative. The scalar potential has not changed, so for negative μ^2 the potential will have a VEV at $\phi\phi^* \neq 0$.

To investigate the consequences of a non zero VEV the Lagrangian (2.6) is expanded around the ground state (eq. 2.4) :

$$\begin{aligned}\mathcal{L} &= \partial_\nu\phi\partial^\nu\phi^* - \mu^2\phi\phi^* - \frac{1}{6}\lambda(\phi\phi^*)^2 + e^2 A_\nu A^\nu \phi\phi^* + ieA^\nu(\phi\partial_\nu\phi^* - \phi^*\partial_\nu\phi) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ &= \frac{1}{2}\partial_\nu\eta\partial^\nu\eta + \mu^2\eta^2 + \frac{1}{2}\partial_\nu(v\sigma)\partial^\nu(v\sigma) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}v^2e^2A_\nu A^\nu + ev^2A^\nu\partial_\nu\sigma \\ &\quad + \text{interactions} + \text{constants} \end{aligned} \tag{2.7}$$

The interaction term includes self-interactions as well as interactions between the fields A_μ, σ and η.

The Lagrangian contains a term $\propto A^\nu\partial_\nu\sigma$, since it is only second order in the fields it is not a normal interaction term but more similar to a mass term. This term mixes the two fields A_μ and σ. By selecting the unitary gauge we can get rid of this mixing term and find the physical degrees of freedom. In the unitary gauge $A_\mu = A'_\mu - \frac{1}{e}\partial_\mu\sigma$. The Lagrangian in the unitary gauge is:

$$\mathcal{L} = \frac{1}{2}\partial_\nu\eta\partial^\nu\eta + \mu^2\eta^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}v^2e^2A_\nu A^\nu + \text{interactions} + \text{constants} \tag{2.8}$$

with the prime of the gauge field omitted. Not only does the mixing term disappear, all terms in the Lagrangian - interactions as well as the kinetic term - containing the σ field disappear. The σ field is thus not a physical field but an artefact of gauge invariance (unphysical degree of freedom).

Summarizing; after SSB of the $U(1)$ gauge theory and selecting the unitary gauge we are left with a theory describing a massive scalar with mass $m^2 = -2\mu^2$ and a massive gauge boson with mass $M^2 = v^2e^2$. The degrees of freedom are conserved; before symmetry breaking the theory has $2+2$ degrees of freedom, two for the complex scalar and two for the massless boson. After symmetry breaking the theory contains $1+3$ degrees of freedom, one for the massive real scalar and three for the massive gauge boson, respectively.

There is a crucial difference between breaking a local and a global symmetry. When a local symmetry is broken, instead of obtaining a massless Goldstone boson, the gauge boson becomes massive. A common way to describe this process is that the Goldstone boson has been "eaten" by the gauge boson.

In general; for a field theory which is locally symmetric under transformations of the group G (M generators), undergoing SSB into a vacuum state, which is locally invariant under the group H (N generators), there is a total of $M-N$ massive gauge bosons, one for each broken generator. There are also N massless gauge bosons, one for each unbroken generator.

2.5 SM Higgs

To see how the theory of spontaneous symmetry breaking can be applied to the SM we need to take another look at the electroweak part of the SM and its symmetry group.

Before symmetry breaking the gauge group is the $SU(2)_L \otimes U(1)_Y$ group, which describes four gauge bosons; A^1_μ, A^2_μ, A^3_μ and B_μ.

After symmetry breaking the particle spectrum contains three massive gauge bosons; W^+, W^-, Z and one massless gauge boson; A_μ. The existence of three massive gauge bosons implies that three generators are broken. It might be tempting to propose that the $SU(2)_L \otimes U(1)_Y$ symmetry is broken into the $U(1)_Y$ symmetry, this would be in accordance with Goldstone's theorem. However, we know experimentally that this cannot be the case because the neutrinos do not couple to the photon whereas the electrons do couple to the photon (for $U(1)_Y$ the electron and neutrino are the same).

The way in which symmetry breaking does occur in the SM is that the $SU(2)_L \otimes U(1)_Y$ symmetry is broken into a $U(1)_Q$ symmetry. The unbroken generator corresponds to the massless gauge boson - the photon. The three massive gauge bosons are the consequence of the three broken generators. It turns out that Z and A_μ are composed out of linear combinations of the A^3_μ and B_μ gauge boson and $W^\pm$ are composed out of linear combinations of A^1_μ and A^2_μ. The linear combinations are chosen such that the neutrino only feels the weak interaction and the electron couples to the electromagnetic gauge boson with coupling strength $-e$ (charge of electron).

To break the $SU(2)_L \times U(1)_Y$ symmetry we take a scalar complex doublet (2.1). A complex doublet is taken because it transforms according to the $SU(2)_L \times U(1)_Y$ symmetry group, this is necessary because the scalar field needs to couple to the gauge bosons. The scalar potential is given by:

$$V(\Phi) = \mu^2 \Phi^\dagger \Phi + \lambda(\Phi^\dagger \Phi)^2, \tag{2.9}$$

which has a non-zero VEV (v) for negative μ^2:

$$\phi_0 = h + v, \text{ with } v^2 = \frac{-\mu^2}{\lambda}. \tag{2.10}$$

h is the physical Higgs boson.

The coupling of the gauge bosons with the Higgs doublet, through the gauge invariant derivative, will give the gauge bosons its mass. This happens in a way similar to the example shown in the previous section. The three degrees of freedom of the complex doublet which are not equated to the physical Higgs boson h are the would-be Goldstone bosons. As shown in previous examples these degrees of freedom are "eaten" by the gauge bosons.

The specific form of the SM Lagrangian before and after symmetry breaking can be found elsewhere (e.g. [2]). The resulting mass spectrum of the SM gauge bosons and the Higgs boson is summarized in table 2.2, with g the $SU(2)_L$ coupling constant and g' the $U(1)_Y$ coupling constant.

Particle	**Mass**
Higgs	$m_h^2 = -2\mu^2 4$
$W^\pm$	$m_W^2 = \frac{1}{4}g^2 v^2$
Z	$m_Z^2 = \frac{1}{4}(g^2 + g'^2)v^2$
photon	$m_A^2 = 0$

TABLE 2.2: The mass of the gauge bosons and Higgs boson

Fermion mass

The preceding has shown how a gauge boson mass can be generated through the phenomenon of spontaneous symmetry breaking. However, we have not yet shown how fermion mass is generated. As stated in the introduction an explicit mass term for fermions is not invariant since a mass term couples a doublet to a singlet. Another doublet is thus needed to couple to the fermion doublet in order to get an invariant term. Taking another look at table 2.1 we see that the scalar Higgs doublet introduced to give mass to gauge bosons has the right properties. Fermions can thus be coupled to the scalar doublet via a term:

$$\mathcal{L} \supset -y\bar{\Psi}_L \Phi \Psi_R + h.c., \tag{2.11}$$

which is $SU(2)_L \otimes U(1)_Y$ invariant and therefore an allowed term in the SM Lagrangian. This kind of interaction is called a Yukawa interaction. From table 2.1 it can be read of which elementary particles should be assigned to the doublet Ψ_L and which to the singlet Ψ_R.

As an example we take a look at the quarks, taking:

$$\Psi_L = \begin{pmatrix} \psi_u \\ \psi_d \end{pmatrix}_L \tag{2.12}$$

and

$$\Psi_R = (\psi_d)_R. \tag{2.13}$$

The Yukawa interaction terms 2.11 can now be expanded around the vacuum in unitary gauge:

$$\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v+h \end{pmatrix}. \tag{2.14}$$

A mass term for fermions is formed:

$$\mathcal{L} \supset -M_f \bar{\psi}_d \psi_d \quad \text{with } M_f = y\frac{v}{\sqrt{2}} \tag{2.15}$$

The above Yukawa interaction (2.11) only gives mass to the lower component of a $SU(2)_L$ doublet, because only the lower component of the Higgs doublet acquires a VEV. Thus this term only gives mass to the charged leptons ($e,\ \mu,\ \tau$) and the "down"-type quarks ($d,\ s,\ b$). The "up"-type quarks ($u,\ c,\ t$) remain massless, which is in contradiction with experimental results. Therefore another term has to be added to the Lagrangian to give mass to the "up"-type quarks. This Yukawa term turns out to be:

$$y\bar{\Psi}_L \Phi^c \Psi_R + \text{h.c.}. \tag{2.16}$$

Φ^c is the charge conjugate of the Higgs doublet:

$$\Phi^c = \frac{1}{\sqrt{2}} \begin{pmatrix} v+h \\ 0 \end{pmatrix} \tag{2.17}$$

Expanding around the VEV will give a similar mass term for the "up"-type quarks (eq. 2.15). In general this term will have a different Yukawa coupling.

Interactions

The expanded Lagrangian not only contains the mass terms derived above but also interaction terms.

- *Fermions*
  To determine which interactions originate from the Yukawa terms (eq. 2.11 and eq. 2.16) we look at the example of the lepton doublet:

$$\Psi = \begin{pmatrix} \psi_\nu \\ \psi_e \end{pmatrix}$$

  After symmetry breaking this results in an interaction with the physical Higgs boson:

$$\mathcal{L} \supset \frac{1}{\sqrt{2}} y h \bar{\psi}_{eL} \psi_{eR} + \text{h.c.} = \frac{1}{\sqrt{2}} y h \bar{\psi}_e \psi_e, \tag{2.18}$$

  This term describes an interaction between two electrons and the Higgs boson with strength $\lambda_f = \frac{1}{\sqrt{2}} y$ as depicted in the Feynman diagram of figure 2.2.

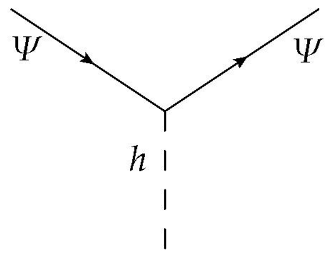

FIGURE 2.2: Feynman diagram of a Yukawa interaction.

- *Gauge bosons*
  The gauge invariant Lagrangian also results in interaction terms between the gauge boson and the Higgs boson. Since the terms in the Lagrangian are quartic in the fields there will be cubic as well quartic interactions between the Higgs boson and the gauge boson. Looking back at the scalar QED Lagrangian (eq. 2.6a) and expanding it around the VEV the interaction terms can be determined in the unitary gauge, resulting in the following interaction terms:

$$\mathcal{L}_{int} \supset \frac{1}{2} e^2 h^2 A_\nu A^\nu + e^2 v h A_\nu A^\nu \tag{2.19}$$

  Thus the theory describes a cubic interaction with strength $\sim e^2 v$ and a quartic interaction with strength $\sim e^2$. In the SM the interaction terms are similar but with slightly different forms due to the mixing of the gauge bosons. The Feynman diagrams of the interactions are shown in figure 2.3.

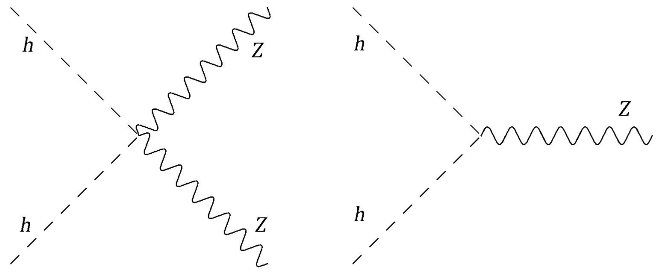

FIGURE 2.3: Feynman diagrams of Higgs interaction with gauge boson(s).

- *Higgs self-interaction*
  Looking at the scalar potential of eq. 2.9 we see that the Higgs boson also interacts with itself when the potential is expanded around the VEV. The self-interactions are cubic as well as quartic. Figure 2.4 shows the Feynman diagrams of these self-interactions.

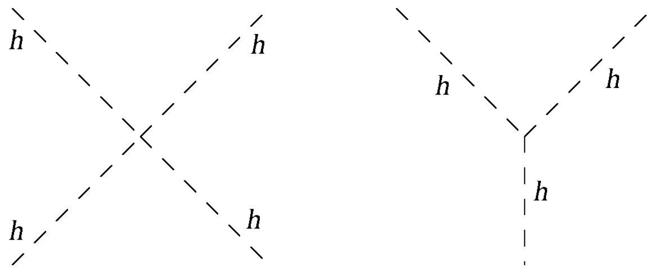

FIGURE 2.4: Feynman diagrams of Higgs self-interaction.

To summarize this section we end with a table with the interactions of the Higgs boson with the SM particles and itself.

Interaction	**Strength**
$W^{\pm}W^{\pm}hh$	$\frac{1}{8}g^2$
$W^{\pm}W^{\pm}h$	$\frac{1}{4}g^2v$
$ZZhh$	$\frac{1}{8}(g^2+g'^2)$
ZZh	$\frac{1}{4}(g^2+g'^2)v$
ffh	$\frac{1}{\sqrt{2}}y$
hhh	λv
$hhhh$	$\frac{1}{4}\lambda$

TABLE 2.3: Table of the different interactions and their strengths. f is a fermion field.

An important point to notice is that the coupling strength between a particle and the Higgs boson is proportional to the mass (fermions) or mass squared (gauge bosons) of that particle. Thus, the heavier a particle the stronger its interaction with the Higgs boson. Consequently, the interaction strength between the Higgs boson and a particle can be predicted precisely in the SM framework once the mass of the particle is known.

The relation between mass and coupling strength is a general consequence of spontaneous symmetry breaking, because SSB results in interaction terms and mass terms with the same origin. By measuring both the coupling strength and the mass independently the SM can be tested.

2.6 Confirming Spontaneous Symmetry Breaking

The relation between mass and coupling strength can be examined through the measurement of the cross section, which can be predicted precisely in the SM framework once the mass is known. The graph of figure 2.5 shows the measured as well as the predicted cross section for some decay processes. This plot does not show any significant deviations from the SM values, thereby confirming that the SM is a remarkable good model of particle physics.

Additionally, these measurements indicate that spontaneous symmetry breaking of the electroweak sector is a good description of nature, any other theory of mass generation would in general not predict the same relation between coupling strength and mass.

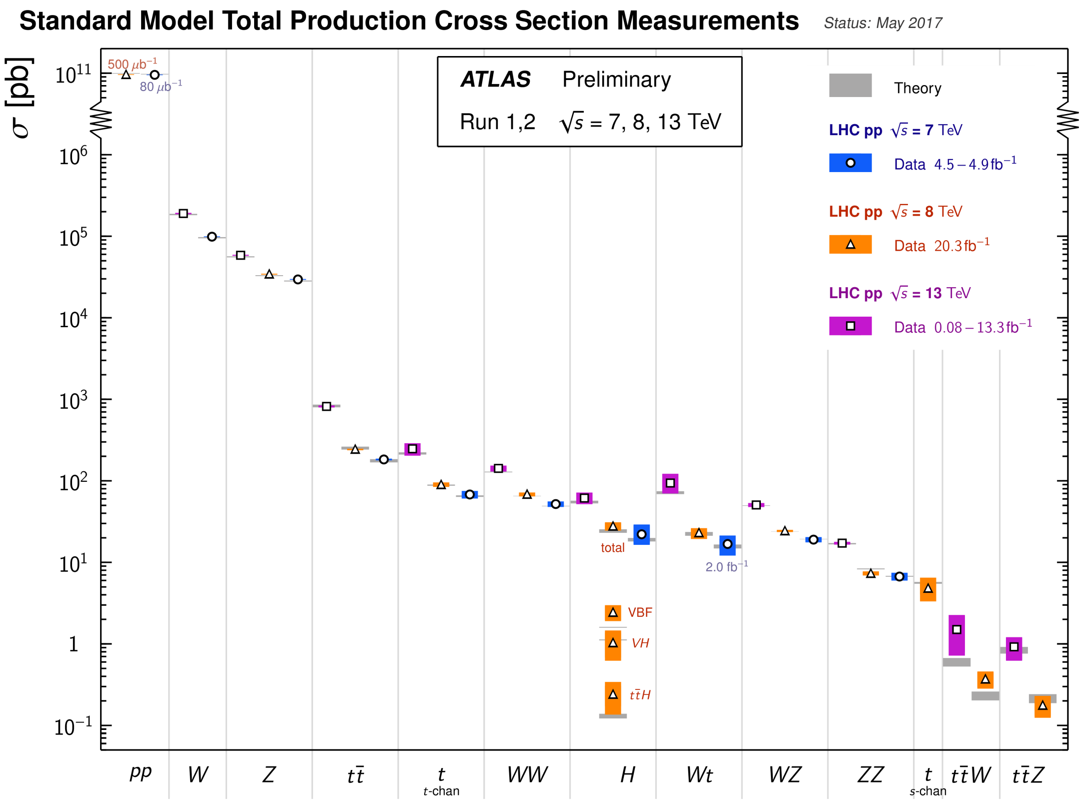

FIGURE 2.5: Summary plot from [4] of the ATLAS experiment at the LHC, comparing measured and predicted cross sections.

Chapter 3

Beyond the Standard Model

The SM, as introduced in the previous chapter, is able to explain almost all particle physics seen in experiments. The observation of the Higgs boson in 2012 was a big success for the SM.

Despite the success of the Standard Model there are some things which cannot be explained within the SM framework. To understand how we should work towards an improved SM first some of the problems of the SM and possible solutions will be discussed. In the second part of this chapter it will be argued why especially a minimal conformally symmetric SM extension is interesting to look at when developing a BSM theory.

3.1 Shortcomings of the SM

In the following the main problems of the SM will be discussed. We will start with a short summary of these problems. The last two problems of the upcoming list; the Stability Problem and the Hierarchy problem are both related to the electroweak interactions. Because the focus of this thesis is on conformally symmetric BSM theories, which have an adapted electroweak sector, these two problems will be discussed in more detail.

The main SM problems and their possible solutions can be summarized as follows:

- **Dark matter & Dark Energy**
  From cosmological measurements of for example rotation curves and the expansion of the universe it seems that baryonic matter - the SM particles - only account for approximately 5% of the energy density of the universe. The other 95% is composed out of so called Dark Matter and Dark Energy and it is unclear what either of them exactly is.
  Possible solution: Theories which attempt to explain Dark Matter in general introduce additional particles. Dark energy can be explained by introducing a cosmological constant, however the origin of such a constant is still unknown.

- **Baryon Asymmetry of the Universe**
  The universe consists mainly of normal matter. This means that at the Big Bang slightly more matter than anti-matter was produced. The CP-violation necessary to produce this Baryon Asymmetry of the Universe can not be explained within the SM.
  Possible solution: SM extensions can provide additional CP violation in order to explain the observed matter/anti-matter asymmetry.

- **Gravity**
  Only three of the four fundamental forces are described by the SM, gravity is left out. General Relativity (GR) does of course a great job in describing how gravity works, however, GR is only a classical theory. At the Planck scale gravitational forces will be of the same strength as SM forces. Thus, to describe physics beyond the Planck scale one needs a quantum theory of gravity. Therefore it is usually thought that the SM can only be an effective field theory which will break down at the latest at Planck scale.
  Possible solution: At the moment string theory is one of the only theories which is able to provide a quantum theory of gravity.

- **Randomness**
  The SM contains about 20 free parameters which have to be determined by experiments. The value of these parameters seems to be random [1]. Also the fact that the SM contains three generations is unexplained. Even though the above does not pose any fundamental problems for the SM the general feeling is that the true theory of physics should have less free parameters. This situation is in some sense similar to the state of affairs before the discovery of quarks. Back then particle colliders created a lot of seemingly useless particles. Eventually this "particle zoo" was reduced to only six fundamental quarks by the introduction of the $SU(3)_C$ symmetry.
  Possible solution: Physicists hope to reduce the number of Standard Model parameters in a similar way, realized in the Grand Unified Theories (GUTs) [5]. GUTs are theories with a large symmetry group which at some scale breaks down into the SM symmetry group. Due to the unification of the SM gauge groups into a bigger group the SM parameters will have specific relations between each other, thereby reducing the number of free parameters.

- **The Stability Problem**
  The running self-coupling λ turns negative at large energy scales. Consequently the SM vacuum appears to be metastable. See section 3.1.1.

- **The Hierarchy Problem**
  Compared to the Planck scale the mass of the Higgs is extremely small. This large hierarchy requires fine-tuning up to a high degree, making the SM unnatural in a sense. See section 3.1.2.

3.1.1 Stability Problem

To determine how coupling constants behave at different energies the β function has to be determined. An overview of the theory of β functions and how they can be calculated is given in Appendix A. This Appendix also gives the one-loop SM β functions.

Using these β functions the running of the SM couplings can be calculated. This is shown in figure 3.1. From the graph it is clear that the scalar coupling λ turns negative at some energy.

[1] Although random, the specific values of some of these parameters are necessary for life to have developed. This fascinating observation is for example related to the anthropic principle and the many-world hypothesis but will not be treated here.

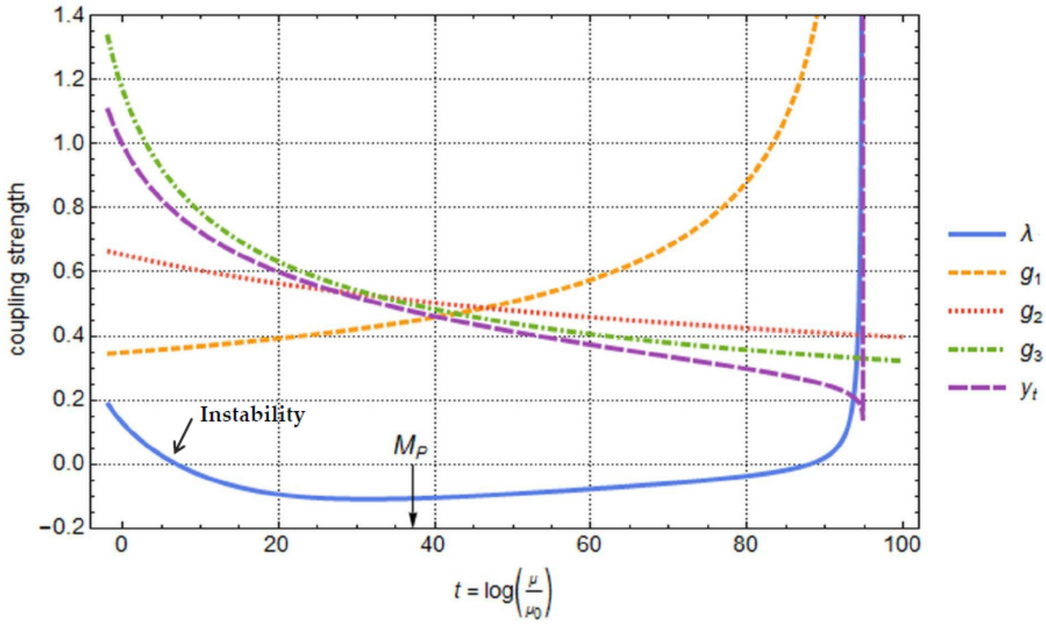

FIGURE 3.1: The running of the SM couplings. β functions and initial conditions can be found in Appendix A. λ is the scalar coupling. $g_1,\ g_2,\ g_3$ are the gauge couplings of $U(1)_Y,\ SU(2)_L,\ SU(3)_C$, respectively and y_t is the Yukawa coupling of the top quark. $\mu_0 = m_t = 173$GeV. M_P is the Planck scale.

The negative λ indicates the existence of another vacuum. When this second vacuum is deeper than the vacuum our universe resides in now (at 246 GeV) our universe could tunnel into this new vacuum state. At this point two different situations can be distinguished, metastability and instability [6]. When the tunnelling time is longer than the lifetime of the universe the SM vacuum is called metastable, otherwise the term instability is used. Figure 3.2 shows the stability plot of the SM, from it we see that within current experimental and theoretical errors our universe is exactly in between stability and instability.

The plot of figure 3.2 also shows that the scalar beta-function is extremely sensitive to the specific values of the Higgs mass and top quark mass. More sensitive measurements of both of these - but especially the top mass - might push the SM vacuum into either the unstable or the stable region. For the moment experimental data seems to favour a metastable vacuum with a small chance of the vacuum being actually stable. Whereas an instability would pose a serious treat to the SM theory a metastable vacuum is less of a problem. However, it does pose a problem in the sense that it is unclear why our universe resides in the electroweak vacuum at 246 GeV [7]. A stable electroweak vacuum is therefore a preferable situation.

Possible solution: Extending the Standard Model will in general have some effect on the running of the scalar coupling. Choosing the appropriate extension might lead to a stable vacuum. For example in [8] it is shown that an extra scalar is already able to stabilize the electroweak vacuum.

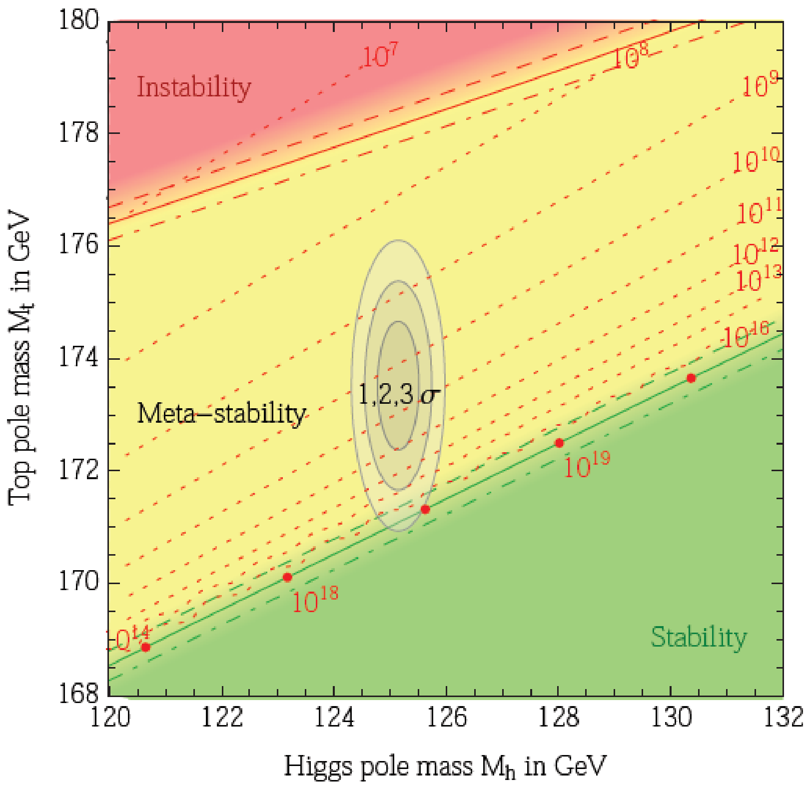

FIGURE 3.2: Stability plot of the Standard Model. The shaded area indicates the experimental value including the 1, 2 and 3σ uncertainties. Plot from [9]

3.1.2 Hierarchy Problem

This section is partly based on lectures given by F. Brümmer in the 2016 DESY Summerstudent Programme [10]. A review of the topic is also given in for example [11].

Even though the introduction of a scalar particle is an elegant way to solve the problem of mass creation in the SM there is one huge problem connected to a scalar as an elementary particle; the mass of a fundamental scalar is not protected from large quantum corrections by any symmetry [12] [13]. This "protection" has to be understood in terms of the concept of *naturalness* first proposed by 't Hooft ([13]):

"A quantity is technically natural if a symmetry is enhanced by setting the quantity to zero"

Using the concept of naturalness a mass which is small relative to some UV scale can be explained if a symmetry is enhanced when the mass is set to zero.

The relation between naturalness and mass protection can be understood by looking at the quantum corrections to the mass. To calculate quantum corrections it is easiest to work with a UV cut-off. As explained in the beginning of this chapter the SM can not be the complete theory and therefore has to be viewed as an effective field theory. At the latest around the Planck scale the SM will break down and needs to be replaced by a new theory. The energy scale at which the SM breaks down defines a cut-off and is expected to be relatively large. Assuming some UV cut-off Λ the quantum correction to the bare mass is in general of the form:

$$m^2 = m_0^2 - \delta m^2 \text{ with } \delta m^2 = \Lambda^2 + m_0^2 \log\left(\frac{m_0}{\Lambda}\right). \tag{3.1}$$

m is the physical mass and m_0 is the bare mass.

These quantum corrections are for example the result of a one-loop interaction shown in the Feynman diagram of figure 3.3. This diagram show the interaction of the Higgs boson with a new heavy fermion mass state χ. The mass of χ is of the order of the UV cut-off Λ.

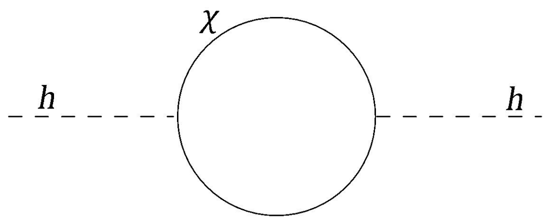

FIGURE 3.3: BSM fermion interacting with Higgs.

For a large cut-off the quantum corrections will be large as well, due to its quadratic dependence on the cut-off scale. This means that for a small physical mass m the bare mass m_0 needs to cancel almost exactly with the quantum corrections. This fine-tuning is in a sense unnatural and something which should be avoided in theoretical models. However when a symmetry is present this fine-tuning problem can be avoided in the following way:

Because quantum corrections in general respect the symmetries of the classical theory [2]we know that if for $m^2 \to 0$ a symmetry is restored at the same time also $\delta m^2 \to 0$. Thus, $\delta m^2 \sim m^2$ for small m^2 when the theory has a protecting symmetry. Looking back at eq. 3.1 it can be concluded that the Λ^2 term is not present in the radiative corrections when the theory has a protecting symmetry. Thus:

$$m^2 = m_0^2 - m_0^2 \log\left(\frac{m_0}{\Lambda}\right) \tag{3.2}$$

when a protecting symmetry is present. Since now the quantum corrections are only logarithmic instead of quadratic in the cut-off scale Λ this equation does not exhibit extreme fine-tuning. Thus symmetries protect masses from large quadratic quantum corrections resulting in *technically natural* small parameters.

Using the concept of naturalness the smallness of fermion masses as well as gauge boson masses can be explained. As explained in the previous chapter a fermion mass term couples the left-handed spinor to the right-handed spinor, thus the presence of a fermion mass breaks the chiral symmetry. If the fermion mass is set to zero the chiral symmetry is restored. Therefore, the chiral symmetry protects the fermion mass from large quantum corrections. This however does not explain *why* there is such a huge difference in fermion masses it only means no fine-tuning is necessary and therefore does not pose a fundamental problem. In a similar way the gauge boson mass is protected by the gauge symmetry - when the energy is increased gauge symmetries will be restored and gauge bosons will becomes massless.

[2]An exception of this statement is the anomalous symmetry breaking.

The Higgs boson

Whereas the fermions and the gauge bosons both have symmetries protecting their masses from large quantum corrections scalars do not possess such a symmetry. This would still not be a problem when the SM would have been the complete theory, valid up to arbitrary scale, since the Higgs mass is the only scale in the SM. However, due to the problems listed in the first section of this chapter, it is assumed that the SM is only an effective field theory which will break down at scale Λ.

New states with mass M from SM extensions or some new theory valid in the UV will contribute to the Higgs mass according to eq. 3.1. However experiments have determined the Higgs mass to be about 125 GeV. To obtain such a small physical Higgs mass in the presence of new heavy states fine-tuning is needed. This is the core of the Hierarchy problem.

Possible solutions: Introducing a protecting symmetry for scalars would result in a technically natural small Higgs mass [13]. A proposed symmetry is supersymmetry (SUSY). SUSY predicts that SM fermions have a supersymmetric bosonic partner and that SM bosons have a supersymmetric fermionic partner. Due to supersymmetry the quantum corrections of the superpartners cancel against the quantum corrections of the SM particles. However experiments haven't seen any supersymmetric particles yet, pushing the SUSY breaking scale well above the weak scale. This results in SUSY particles which are a lot heavier than there SM partners, therefore the bosonic and fermionic terms do not cancel exactly any more. This in turn decreases the ability of SUSY to solve the Hierarchy problem [5].

Another proposed solution to the Hierarchy Problem employs the observation that the SM Lagrangian is conformally symmetric apart from the Higgs mass term. By imposing classical conformal symmetry of the SM Lagrangian and thereby setting the Higgs mass term to zero it is possible to avoid the Hierarchy problem [14] [3].

A last note on Standard Model shortcomings

Despite these seemingly huge problems there are no significant *experimental* indications that anything is wrong with the Standard Model. Particle physics experiments like the LHC have not been able to measure any new particles or significant deviations from the SM predictions. An example of this was already shown in figure 2.5 which summarized the measurements of cross sections and did not show any significant differences between SM predictions and the experimental values.

Thus even though the SM has severe problems, at the same time it works very well. Lacking experimental evidence to give direction to possible BSM models the development of new theories is mostly guided by the solutions to the SM problems.

[3] In the next chapter we will see how this happens exactly, for now this statement has to be taken at face value.

3.2 Minimal Conformal Extension

Knowing the problems of the Standard Model and keeping in mind that the lack of experimental evidence for new physics puts severe restrictions on BSM theories it is now possible to explain why an extension which is minimal as well as conformally symmetric is such a promising option.

First of all the minimality, minimal extensions aim to add as little extra degrees of freedom to the SM as possible. Additionally these extensions minimize the interactions of the new particle with the SM particles. The idea of minimal extensions is that they only slightly change the Standard Model phenomenology, thereby leaving the low energy predictions of the SM largely unchanged. In the light of the almost perfect agreement between SM predictions and experimental results it has become increasingly interesting to look at minimal SM extensions.

Secondly conformal symmetry; why is conformal symmetry a nice feature to have in a BSM theory? To see why this is the case we have to look back at the problems of the SM. On the one hand we have the problems of Dark Energy and Quantum Gravity which can probably only be solved by considering a unification of the SM with gravity. On the other hand, the Stability Problem, the Hierarchy Problem and the unexplained nature of Dark Matter might be solved within a theory of particle physics, without introducing gravity. Additionally the "randomness" problem also has solutions within particle physics. Since most of the problems have the potential to be solved within a theory of particle physics it is interesting to look at (effective) theories which solve the particle physics problems. Of the particle physics the Hierarchy Problem is probably the most severe. As already mentioned conformal symmetry is a possible solution of the Hierarchy Problem. Even more interesting, looking at it naively we can expect that a conformal model with additional scalars also solves the Stability Problem as well as provide a Dark Matter candidate.

Apart from the appeal of conformally symmetric models there is one minor problem connected to them. At first sight setting the mass term of the scalar potential to zero will destroy spontaneous symmetry breaking and thus the generation of mass will not happen. This poses a big problem because mass generation but also the mass-coupling relation is hard to explain without Electroweak Symmetry Breaking (EWSB). However the cause of EWSB does not have to be a negative mass term in the classical potential. Another mechanism for EWSB is Radiative Symmetry Breaking, which is a mechanism in which spontaneous symmetry breaking is induced through quantum corrections.

Thus minimal conformally symmetric models are an interesting group of BSM theories because they can solve several of the SM problems at once without introducing too large an effect on the SM predictions for low energy phenomena. In order to preserve EWSB, a necessary constituent of these models is Radiative Symmetry Breaking, which will be the topic of the next chapter. Conformally symmetric models will be the topic of chapter 5.

Chapter 4

Radiative Symmetry Breaking

4.1 Electroweak Symmetry Breaking

As discussed in the previous chapters the SM uniquely predicts the coupling strength of a particle to the Higgs boson once the mass of that particle is known. The experimental confirmation of this relation (figure 2.5) is an indication that EWSB is the true cause of mass generation in our world.

However the exact nature of electroweak symmetry breaking is not known. Spontaneous symmetry breaking can be of either a **classical** or of a **quantum** nature. The SM features **classical** SSB in which a negative mass term in the scalar potential results in SSB. This is however an *ad hoc* assumption, there is no theoretical reason why μ^2 should be negative in the first place.

Classical spontaneous symmetry breaking has not been confirmed by experiments like the LHC, because it is at the moment not yet possible to measure the Higgs self coupling λ with significant precision. Without this measurement the shape of the scalar potential can not be verified experimentally. A proposed experiment which will be able to measure λ accurately is the INTERNATIONAL LINEAR COLLIDER [15]. This experiment is designed to collide electrons and positrons at high energies. Since electrons and positrons are elementary particles - contrary to the LHC which collides protons - the background in these experiments is expected to be much smaller. This in turn will increase the precision of experiments, making it possible to determine for example λ with significant precision.

In case of conformally symmetric models classical spontaneous symmetry breaking does not apply, therefore the **quantum** version of SSB is needed. This was first introduced by S. Coleman and E. Weinberg [16] and is therefore often called the Coleman-Weinberg (CW) mechanism. They showed that, in the absence of a mass term in the scalar potential, quantum corrections can still result in a potential with a symmetry breaking minimum, resulting in SSB.

A general way to determine if quantum corrections result in SSB is by calculating the effective potential V. The effective potential is defined as the first term of the effective action Γ expanded around the external momenta [16]:

$$\Gamma = \int d^4x \left[-V(\phi_c) + \frac{1}{2}(\partial_\mu \phi_c)^2 Z(\phi_c) + \cdots \right], \tag{4.1}$$

with ϕ_c the classical field and $Z(\phi_c)$ the field renormalization term (see Appendix B). The effective action can also be expanded as a Taylor series:

$$\Gamma = \sum_n \frac{1}{n!} \int d^4x_1 \cdots d^4x_n \Gamma^{(n)}(x_1 \cdots x_n)\phi_c(x_1) \cdots \phi_c(x_n) \tag{4.2}$$

with $\Gamma^{(n)}$ the sum of all 1PI [1] Feynman diagrams with n external momenta. Comparing these two expansions it is apparent that the effective potential includes all 1PI graphs with external momenta set to zero.

The calculation of the effective potential is in general done via a loop expansion, taking into account only the diagrams up to first order in loops. Because the external momenta are zero we have $\partial_\mu \phi_c = 0$, therefore only the non-derivative terms of the Lagrangian contribute to the effective potential.

The contribution of a diagram with E external lines to the effective potential (V) is given by:

$$V_E = i \cdot \text{"diagram"} \cdot \frac{1}{E!}\phi_c^E,$$

From this effective potential the true minimum of a theory can be determined.

A more complete discussion on the effective potential and its connection to the Green's functions can be found in for example [17].

In the coming sections some simple theories will be investigated to show that radiative corrections can indeed result in SSB. It will also been shown that conformal models do not suffer from the Hierarchy Problem. To apply the Coleman-Weinberg method to general theories the effective potential of a general theory with any number of fermions, scalars and gauge bosons will be given.

4.2 ϕ^4 theory

The simplest theory for which the CW mechanism can be explained is that of massless ϕ^4 theory. Here we follow the calculations done in [16] and [18].

We start with the renormalized Lagrangian for a massless ϕ^4 theory:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{\lambda}{4!}\phi^4 + \frac{1}{2}A(\partial_\mu\phi)^2 - \frac{1}{2}B\phi^2 - \frac{C}{4!}\phi^4 \tag{4.3}$$

The constants A, B and C have to be determined by choosing appropriate renormalization conditions (see Appendix B for details). In this case the following conditions are selected:

$$\frac{d^2V}{d\phi_c^2} = 0 \text{ at } \phi_c = 0 \tag{4.4a}$$

$$\frac{d^4V}{d\phi_c^4} = \lambda \text{ at } \phi_c = M \tag{4.4b}$$

$$A = 0 \text{ at } \phi_c = M \tag{4.4c}$$

[1] 1PI stands for one-particle-irreducible, these are the connected Feynman diagrams which can not be split into two separate diagrams by cutting only one line.

When determining the effective potential in a loop expansion these renormalization conditions should hold for every order.

At tree-level (zero loop) the diagrams contributing to the effective potential are shown in the top row of fig. 4.1. These contribute the following to the effective potential:

$$V^{(0)} = i\left(-i\lambda - iC^{(0)}\right)\frac{1}{4!}\phi_c^4 + i\left(-iB^{(0)}\right)\frac{1}{2}\phi_c^2 \tag{4.5}$$

Fixing $B^{(0)}$ and $C^{(0)}$ using the renormalization conditions (Eq. 4.4) we obtain:

$$B^{(0)} = C^{(0)} = 0, \tag{4.6}$$

thus at tree-level the renormalization terms do not contribute. This is as expected since renormalization terms are only needed when considering quantum corrections. The tree level effective potential is therefore just the classical potential: $V^{(0)} = \frac{\lambda}{4!}\phi_c^4$.

At one-loop level, aside from the renormalization terms $B^{(1)}$ and $C^{(1)}$, an infinite number of polygons contribute to the effective potential. These diagrams are depicted in the bottom row of figure 4.1. The one-loop effective potential is given by:

$$V^{(1)} = i\left(-iC^{(1)}\right)\frac{1}{4!}\phi_c^4 + i\left(-iB^{(1)}\right)\frac{1}{2}\phi_c^2 + \text{polygons}. \tag{4.7}$$

The contribution of a polygon with n vertices, $2n$ external lines and n internal lines is given by:

$$i\frac{1}{(2n)!}\phi_c^{2n}\cdot\frac{1}{n!}\left(\frac{-i\lambda}{4!}\right)^n\cdot(4\cdot 3)^n\cdot 2^{(n-1)}\cdot n!\cdot\frac{(2n)!}{2^n}\cdot\frac{1}{n}\cdot\int\frac{d^4k}{(2\pi)^4}\frac{i^n}{k^{2n}} \tag{4.8}$$

The first term comes from the definition of V. The second is for the n vertices. The third term is for the different ways to combine two external lines with a vertex. The fourth term is for attaching the vertices to each other. The fifth term is for permutations of the n vertices. The 6th term is for the different ways to permutate the external lines ($2n$ external lines but switching two external lines connected to the

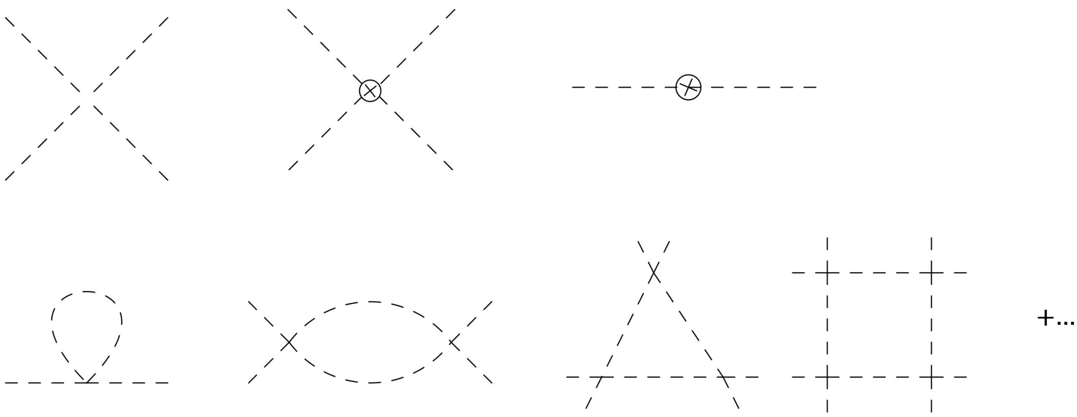

FIGURE 4.1: The first row shows the tree level diagram and the renormalization terms, the second row the 1-loop diagrams contributing to the effective potential.

same vertex is indistinguishable ($1/2^n$)). The 7th term is because rotating the pentagon gives an equivalent diagram ($1/n$). The last term is the integral over internal momenta, since all external lines have zero momentum the internal lines all have the same momentum (if defined in either clockwise or anti-clock wise way), there is thus one independent internal momentum.

The effective potential up to and including one-loop diagrams is thus:

$$V \approx V^{(0)} + V^{(1)} = +\frac{\lambda}{4!}\phi_c^4 + \frac{1}{4!}C^{(1)}\phi_c^4 + \frac{1}{2}B^{(1)}\phi_c^2 + \sum_{n=1}^{\infty} i \int \frac{d^4k}{(2\pi)^4}\frac{1}{2n}\left(\frac{\lambda/2}{k^2}\right)^n \phi_c^{2n}$$

The polygon term can be simplified using the geometric series (Appendix C), after this the integral can be computed using a UV cut-off:

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{i}{2}\int\frac{d^4k}{(2\pi)^4}\frac{1}{n}\left(\frac{1/2\cdot\lambda\phi_c^2}{k^2}\right)^n &= -\frac{i}{2}\int\frac{d^4k}{(2\pi)^4}\log\left(1-\frac{\lambda\phi_c^2}{2k^2}\right)\\ &= \frac{1}{2^6\pi^2}\left[\lambda\phi_c^2\Lambda^2 + \frac{\lambda^2\phi_c^4}{4}\left(\log\left(\frac{\lambda\phi_c^2}{2\Lambda^2}\right)-\frac{1}{2}\right)\right] \qquad (4.9)\end{aligned}$$

The integral is calculated in Appendix C.

The effective potential up to one-loop can now be written as:

$$V = +\frac{\lambda}{4!}\phi_c^4 + \frac{1}{4!}C^{(1)}\phi_c^4 + \frac{1}{2}B^{(1)}\phi_c^2 + \frac{1}{2^6\pi^2}\left[\lambda\phi_c^2\Lambda^2 + \frac{\lambda^2\phi_c^4}{4}\left(\log\left(\frac{\lambda\phi_c^2}{2\Lambda^2}\right)-\frac{1}{2}\right)\right]$$

$B^{(1)}$ and $C^{(1)}$ can again be determined by the renormalization conditions in eq. 4.4. Applying these conditions we obtain:

$$\begin{aligned}B^{(1)} &= -\frac{\lambda\Lambda^2}{32\pi^2}\\ C^{(1)} &= -\frac{3\lambda^2}{32\pi^2}\left(\log\left(\frac{\lambda M^2}{2\Lambda^2}\right)+\frac{11}{3}\right)\end{aligned}$$

Combining results in the one-loop renormalized effective potential we obtain:

$$V = +\frac{\lambda}{4!}\phi_c^4 + \frac{\lambda^2\phi_c^4}{2^8\pi^2}\left[\log\left(\frac{\phi_c^2}{M^2}\right)-\left(\frac{25}{6}\right)\right] + \mathcal{O}(\lambda^3) \qquad (4.10)$$

The classical potential and the effective potential of eq. 4.10 are shown in figure 4.2. For comparison the (Abelian) Higgs potential has also been added to this graph.

From these potentials we see that the classical potential $V = \frac{\lambda}{4!}\phi_c^4$ has its ground state at $\phi_c = 0$, whereas the quantum corrections result in a slightly perturbed potential with a minimum ($\langle\phi\rangle$) at:

$$\frac{dV}{d\phi_c} = 0 \rightarrow \lambda\log\left(\frac{\langle\phi\rangle^2}{M^2}\right) = -\frac{32}{3}\pi^2 + \frac{22\lambda}{6} + \mathcal{O}(\lambda^2) \rightarrow \langle\phi\rangle^2 = M^2 e^{-\frac{32\pi^2}{3}\frac{1}{\lambda}+\frac{22}{6}+\mathcal{O}(\lambda)} \qquad (4.11)$$

Including the effect of quantum corrections can thus result in the appearance of a VEV and as a consequence of this VEV there will be SSB.

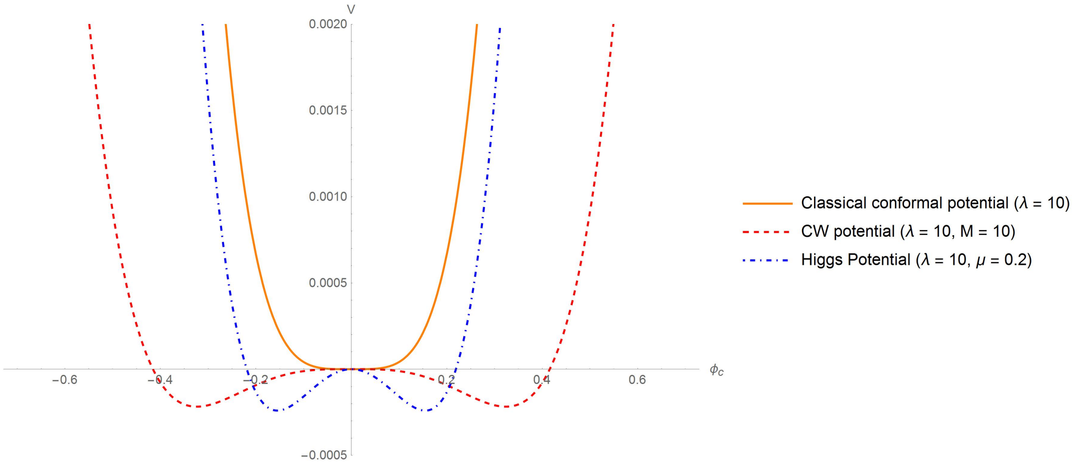

FIGURE 4.2: The effective potential with and without quantum corrections. For comparison also a "Higgs"-like potential is given with potential: $V = -\frac{\mu^2}{2}\phi_c^2 + \frac{\lambda}{4!}\phi_c^4$.

Another observation which can be taken from this plot is that the Coleman-Weinberg potential is not equivalent to the Higgs potential. However, its most important feature, the existence of a symmetry breaking ground state is mimicked.

After SSB the phenomenology of the Lagrangian changes completely, whereas the theory contains a massless scalar before symmetry breaking this scalar will get a mass due to quantum corrections. The mass can be determined in two equivalent ways. One way is to expand the potential around the VEV and find the quadratic term since a mass term in the potential is always of the form $\sim \frac{1}{2}m_S^2\phi^2$ therefore the mass of the scalar can be extracted. The second way to determine the scalar mass is by using its definition:

$$m_S^2 = \left.\frac{\partial^2 V}{\partial \phi^2}\right|_{\phi=\langle\phi\rangle} \tag{4.12}$$

Applying this equation to the effective potential of eq. 4.10 the scalar mass can be determined:

$$m_S^2 = \frac{\lambda^2\langle\phi\rangle^2}{2^5\pi^2} \tag{4.13}$$

Perturbative Validity

This simple theory is the first example where quantum corrections result in a symmetry breaking vacuum. However this result, obtained by a loop expansion, is not perturbatively valid.

For the theory to be perturbatively valid the one-loop contribution must dominate over the higher loop corrections. The nth order loop corrections result in terms of the form $\lambda\left[\lambda\log\left(\frac{\phi_c^2}{M^2}\right)\right]^n$ [16]. Thus λ as well as the λ times the logarithm should be small for perturbation theory to be valid. Or equivalently $\lambda \ll 1$ and $\log \sim \mathcal{O}(1)$.

Looking back at eq. 4.11 we see that in order to have a minimum the first derivative should be zero. However to achieve this a term of order λ has to cancel with a large constant and a term of order $\lambda \log\left(\frac{\phi_c^2}{M^2}\right)$. However, this is clearly not possible when both λ and $\lambda \log\left(\frac{\phi_c^2}{M^2}\right)$ are small.

Thus, the minimum found for the ϕ^4 theory may be just an "artefact of perturbation theory". It can be shown, by exploiting renormalization theory, that the minimum is indeed a maximum [16]. Thus ϕ^4 theory does not exhibit spontaneous symmetry breaking through quantum corrections.

4.3 Scalar QED

Another simple QFT for which the effective potential can be calculated is scalar QED, again the discussion of [16] is followed. We will show that, contrary to the ϕ^4 theory, this is a theory for which the minimum at $\phi \neq 0$ is perturbatively reliable. The renormalized Lagrangian of massless scalar QED, a $U(1)$ gauge theory, is given by:

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\left(\partial_\mu\phi_1 - eA_\mu\phi_2\right)^2 + \frac{1}{2}\left(\partial_\mu\phi_2 + eA_\mu\phi_1\right)^2 - \frac{\lambda}{4!}\left(\phi_1^2+\phi_2^2\right)^2 \\ &+ \text{renormalization terms}\end{aligned} \tag{4.14}$$

This Lagrangian has a $U(1)$ global symmetry (more clear in $\phi,\ \phi^*$ basis). The effective potential should also respect this symmetry, which means it has to depend on $\phi\phi^* \propto \phi_{1c}^2 + \phi_{2c}^2 \equiv \phi_c^2$. Thus to calculate the effective potential it is sufficient to calculate all diagrams with only ϕ_1 external lines and then replace ϕ_{1c} by ϕ_c.

The computation of the effective potential can be simplified further if we work in Landau gauge. Diagrams with a photon and scalar as internal lines and one scalar as external line do not contribute in this gauge. This can be understood by looking at the photon propagator in Landau gauge: $D_{\mu\nu} = -i\frac{g_{\mu\nu} - k_\mu k_\nu/k^2}{k^2+i\epsilon}$. Since the external momenta are set to zero when calculating the effective potential the vertex will give another factor of k_μ for the $\partial_\mu\phi$ term. However since these momenta have to be the same a contraction leads to the diagram being zero. Thus diagrams with mixed internal lines do not contribute. Terms which do contribute in the Landau gauge are the following.

At tree level only the quartic vertex contributes to the effective potential: $V^{(0)} = \frac{\lambda}{4!}\phi_c^4$. The renormalization terms do not contribute, as explained previously.

At one-loop order the following diagrams contribute to the effective potential:

(a) ϕ_1 external ϕ_1 internal

(b) ϕ_1 external ϕ_2 internal

(c) ϕ_1 external A_μ internal

(d) renormalization terms: $-\frac{C}{4!}(\phi_1^2+\phi_2^2)^2 - \frac{B}{2}(\phi_1^2+\phi_2^2)$

The set of diagrams (a) are exactly the same as those in fig. 4.1. Thus the contribution to the effective potential is given by:

$$V_{\mathrm{a}} = +\frac{1}{2^6\pi^2}\left[\lambda\phi_{1c}^2\Lambda^2 + \frac{\lambda^2\phi_{1c}^4}{4}\left(\log\left(\frac{\lambda\phi_{1c}^2}{2\Lambda^2}\right) - \frac{1}{2}\right)\right] \tag{4.15}$$

The set of diagrams (b) are similar to those in fig. 4.1. However counting equivalent diagrams is a bit different and the vertex term is now given by $-2i\lambda/4!$. The polygon contribution for a polygon with n vertices, $2n$ external ϕ_1 lines and n internal ϕ_2 lines is given by:

$$i\frac{1}{(2n)!}\phi_{1c}^{2n}\cdot\frac{1}{n!}\left(\frac{-2i\lambda}{4!}\right)^n\cdot(2\cdot 1)^n\cdot 2^{(n-1)}\cdot n!\cdot\frac{(2n)!}{2^n}\cdot\frac{1}{n}\cdot\int\frac{d^4k}{(2\pi)^4}\frac{i^n}{k^{2n}} \tag{4.16}$$

in which the blue terms are different in comparison to the term for the (a) diagrams. The second term is slightly different due to the extra factor of two in the interaction term. The third term is different due to the combinatorics of attaching external lines to vertices (since the vertices contain only two ϕ_1's per vertex instead of four). Working this out it can be determined that the (a) and (b) diagrams are equivalent with as only change $\lambda \to \lambda/3$.The remainder of the computation is exactly equivalent, resulting in:

$$V_{\mathrm{b}} = +\frac{1}{2^6\pi^2}\left[\frac{\lambda}{3}\phi_{1c}^2\Lambda^2+\frac{(\lambda/3)^2\phi_{1c}^4}{4}\left(\log\left(\frac{\lambda\phi_{1c}^2}{6\Lambda^2}\right)-\frac{1}{2}\right)\right] \tag{4.17}$$

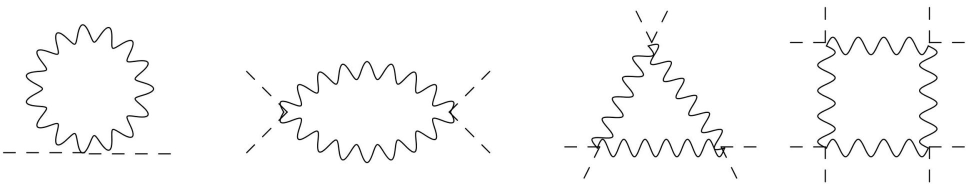

FIGURE 4.3: The 1-loop diagrams with a photon running around the loop contributing to the effective potential.

The set of diagrams (c) requires some more attention. The polygon diagrams with photons as internal lines are shown in figure 4.3. Again an equation can be found for a graph with $2n$ external scalars, n vertices and n internal photons. The vertex term can be read off from the Lagrangian and is given by $\frac{i}{2}e^2g^{\mu\nu}$.

$$V_{\mathrm{c}} = \frac{i}{(2n)!}\phi_{1c}^{2n}\cdot\frac{1}{n!}\left(\frac{i}{2}e^2g^{\mu\nu}\right)^n\cdot 2^n\cdot 2^{n-1}\cdot n!\cdot\frac{1}{n}\frac{(2n)!}{2^n}\cdot\int\frac{d^4k}{(2\pi)^4}\left(\frac{-i(g_{\mu\nu}-k_\mu k_\nu/k^2)}{k^2}\right)^n \tag{4.18}$$

All terms have the same meaning as previously. The integral is slightly changed since the propagator of a photon is different from that of a scalar. The somewhat sloppy notation in indices has to be explained a bit more; every vertex is contracted with one index of each of the two propagators connected to it, thereby resulting in a polygon graph. Because of this specific way of contracting (in a loop) the contraction can be written as a trace over the Lorentz indices:

$$\begin{aligned}(g^{\mu\nu})^n(g_{\mu\nu}-k_\mu k_\nu/k^2)^n &\equiv \mathrm{Tr}(g\Pi\cdots g\Pi)\\ &= g^{\mu\nu}(g_{\nu\alpha}-k_\nu k_\alpha/k^2)\cdot g^{\alpha\beta}(g_{\beta\gamma}-k_\beta k_\gamma/k^2)\\ &\quad\cdots{}_\eta^\gamma\cdot g^{\eta\zeta}(g_{\zeta\mu}-k_\zeta k_\mu/k^2)\\ &= 3\end{aligned} \tag{4.19}$$

Π is defined as: $\Pi_{\mu\nu} = g_{\mu\nu} - k_\mu k_\nu/k^2$. The factor of three can be understood as the number of independent polarization directions of the (massive) photon.

Calculating V_c by combining eq. 4.18 and eq. 4.19, summing the geometric series and performing the integral we obtain:

$$\begin{aligned} V_c &= \sum_{n=1}^{\infty} 3i \int \frac{d^4k}{(2\pi)^4} \frac{1}{2n} \left(\frac{e^2 \phi_{1c}^2}{k^2} \right)^n \\ &= \frac{3}{2^6\pi^2} \left[2e^2\phi_{1c}^2\Lambda^2 + e^4\phi_{1c}^4 \left(\log\left(\frac{e^2\phi_{1c}^2}{\Lambda^2} \right) - \frac{1}{2} \right) \right] \end{aligned} \tag{4.20}$$

The integral is exactly the same as in the scalar case, the calculation is done in Appendix C, eq. C.1.

Combining all terms plus the renormalization terms - keeping in mind to replace ϕ_{1c} by ϕ_c - results in the effective potential up to and including one-loop order:

$$\begin{aligned} V = \frac{\lambda}{4!}\phi_c^4 &+ \frac{1}{2^6\pi^2} \left[\lambda\phi_c^2\Lambda^2 + \frac{\lambda^2\phi_c^4}{4} \left(\log\left(\frac{\lambda\phi_c^2}{2\Lambda^2} \right) - \frac{1}{2} \right) \right] \\ &+ \frac{1}{2^6\pi^2} \left[\frac{\lambda}{3}\phi_c^2\Lambda^2 + \frac{(\lambda/3)^2\phi_c^4}{4} \left(\log\left(\frac{\lambda\phi_c^2}{6\Lambda^2} \right) - \frac{1}{2} \right) \right] \\ &+ \frac{3}{2^6\pi^2} \left[2e^2\phi_c^2\Lambda^2 + e^4\phi_c^4 \left(\log\left(\frac{e^2\phi_c^2}{\Lambda^2} \right) - \frac{1}{2} \right) \right] \\ &- \frac{C}{4!}\phi_c^4 - \frac{B}{2}\phi_c^2 \end{aligned} \tag{4.21}$$

Applying the same renormalization conditions as for the scalar case (eq. 4.4) results in:

$$\begin{aligned} B &= \frac{1}{2^5\pi^2} \left[\lambda\Lambda^2 + \frac{\lambda}{3}\Lambda^2 + 6e^2\Lambda^2 \right] \\ C &= + \frac{1}{2^6\pi^2} \frac{\lambda^2}{4} \left[24 \log\left(\frac{\lambda M^2}{2\Lambda^2} \right) + 88 \right] \\ &\quad + \frac{1}{2^6\pi^2} \frac{\lambda^2}{4\cdot 9} \left[24 \log\left(\frac{\lambda M^2}{6\Lambda^2} \right) + 88 \right] \\ &\quad + \frac{3}{2^6\pi^2} e^4 \left[24 \log\left(\frac{e^2 M^2}{\Lambda^2} \right) + 88 \right] \end{aligned}$$

The renormalized 1-loop effective potential for massless QED is thus:

$$V = \frac{\lambda}{4!}\phi_c^4 + \left(\frac{5}{1152\pi^2}\lambda^2 + \frac{3}{64\pi^2}e^4 \right) \phi_c^4 \left[\log\left(\frac{\phi_c^2}{M^2} \right) - \frac{25}{6} \right] + \mathcal{O}(e^8) + \mathcal{O}(\lambda^4) \tag{4.22}$$

In order to not run into the same perturbative validity problems as we did for the ϕ^4 theory, the e^4 term needs to dominate the λ^2 term. This is achieved by assuming $\lambda \propto e^4 \ll 1$, λ^2 can therefore be neglected.

The effective potential can be simplified further by setting M equal to the minimum of the effective potential; $\langle\phi\rangle$. Since M is an arbitrary scale this is allowed. Because the first derivative of the potential has to be zero at the minimum $\langle\phi\rangle$, this choice of M gives a relation between λ and e: $\lambda = \frac{33e^4}{8\pi^2}$.

Thus, the simplified effective potential is given by:

$$V = \frac{3}{64\pi^2} e^4 \phi_c^4 \left[\log \left(\frac{\phi_c^2}{\langle \phi \rangle^2} \right) - \frac{1}{2} \right] + \mathcal{O}(e^8). \tag{4.23}$$

This potential has a minimum away from the origin, resulting in SSB.

Due to SSB the scalar boson as well as the gauge boson will acquire a mass. The scalar mass is again given by eq. 4.12, resulting in:

$$m_S^2 = \frac{3e^4 \langle \phi \rangle^2}{8\pi^2} \tag{4.24}$$

The mass of the gauge boson can be determined by expanding the Lagrangian around the minimum: $m_V^2 = e^2 \langle \phi \rangle^2$.

Perturbative Validity
To check if this minimum is perturbatively it is necessary to return to the effective potential of eq. 4.22, keeping M arbitrary but neglecting λ^2. The minimum of the effective potential is given by:

$$\langle \phi \rangle^2 = M^2 e^{-\frac{16\pi^2}{18} \frac{\lambda}{e^4} + \frac{22}{6} + \mathcal{O}(\lambda) + \mathcal{O}(e^8/\lambda)} \tag{4.25}$$

From this equation it is clear that for small λ and e^4 the 1-loop correction dominates over the higher loop corrections. Thus, the perturbatively found minimum can be trusted and is a true minimum of the theory.

Compared to the ϕ^4 theory, the main difference is the presence of a second term ($\sim e^4$). Instead of balancing terms of λ with $\lambda^2 \log \left(\frac{\phi_c^2}{M^2} \right)$ a term of λ is balanced with a term $e^4 \log \left(\frac{\phi_c^2}{M^2} \right)$. Since e and λ are independent they can both be small without the need for a large logarithm, thus avoiding problems with the perturbative validity.

Dimensional Transmutation
A curious thing has happened, by setting the renormalization scale equal to the minimum, two dimensionless parameters (e and λ) are replaced by one dimensionful and one dimensionless parameter (e and $\langle \phi \rangle$). This phenomenon is called "dimensional transmutation" [16] and has the effect that all dimensionless quantities only depend on e.

Looking back at our first example of ϕ^4 we can also set $M = \langle \phi \rangle$, in this case dimensional transmutation would result in: $\lambda = \frac{64\pi^2}{22}$, $\lambda = 0$ or $\langle \phi \rangle = 0$. The second solution is unphysical whereas the third solution would not result in a VEV. Only the first solution is able to accommodate a VEV, however λ in this case is clearly not small and thus the perturbative calculation is not valid. This is another indication that the VEV found for the scalar theory can not be trusted.

The previous two examples of the Coleman-Weinberg mechanism show that already for simple theories radiative corrections can result in SSB. However one has to take care of the perturbative validity. In the presence of multiple couplings and when setting the scalar coupling equal to the fourth power of the gauge coupling ($\lambda \sim e^4$) perturbative validity is in general satisfied. Since we would like to apply the theory of radiative symmetry breaking to more complicated theories it is useful to calculate the effective potential for a general field theory. This is topic of the next section.

4.4 General case

For a general gauge invariant [2] theory with scalars, gauge bosons and fermions with the classical potential given by:

$$V = V_0 + \frac{1}{2}\sum_{ab}(\mu_\phi^2)_{ab}A_{\mu a}A_b^\mu + \sum_{ab}\overline{\Psi}_a(m_\phi)_{ab}\Psi_b, \tag{4.26}$$

the effective potential (only scalars as external lines) up to and including one loop order is given by [16]:

$$\begin{aligned} V(\phi_c) = &+ V_0(\phi_c) \\ &+ \frac{1}{64\pi^2}\mathrm{Tr}\left[W_\phi^2(\log W_\phi - \tfrac{1}{2})\right] \\ &+ \frac{3}{64\pi^2}\mathrm{Tr}\left[\mu_\phi^4(\log \mu_\phi^2 - \tfrac{1}{2})\right] \\ &- \frac{1}{64\pi^2}\mathrm{Tr}\left[(m_\phi m_\phi^\dagger)^2(\log m_\phi m_\phi^\dagger - \tfrac{1}{2})\right] \\ &+ \text{coupling renormalization} \end{aligned} \tag{4.27}$$

- This calculation is done using an UV cut-off, using for example the $\overline{\text{MS}}$ scheme will result in a slightly different effective potential [19]. The choice of renormalization scheme does not affect the existence of a VEV.
- The quadratic terms are cancelled completely by the mass renormalization term (using the same renormalization condition as previously) and have thus been omitted for clarity. The $1/\Lambda^2$ term in the logarithm has also been omitted.
- V_0 is the tree level term and describes the scalar interaction between one or multiple scalars.
- W_ϕ, μ_ϕ^2 and m_ϕ^2 are the mass matrices of the scalars, gauge bosons and fermions respectively in the presence of a VEV ϕ_c. They are quadratic in ϕ_c and equal to the true mass matrices for a VEV $\phi_c = \langle\phi\rangle$:

  $W_\phi = W\phi_c^2/\langle\phi\rangle^2$,

  $\mu_\phi^2 = \mu^2\phi_c^2/\langle\phi\rangle^2$,

  $m_\phi m_\phi^\dagger = m^2\phi_c^2/\langle\phi\rangle^2$,

  with W, μ^2 and m^2 the true quadratic mass matrices of the scalars, vectors and fermions, respectively.
- W_ϕ is defined as: $(W_\phi)_{ab} = \frac{\partial^2 V_0}{\partial\phi_a\partial\phi_b}|_{\phi=\phi_c}$, it gives the vertex term for scalars. The trace is over the scalar mass states.
- μ_ϕ^2 is defined as: $(\mu_\phi^2)_{ab} = g_a g_b(T_a\vec{\phi}, T_b\vec{\phi})$, it gives the vertex term for the interaction between two scalars and two gauge bosons. T are the different generators determined by the specific gauge group under which the gauge bosons

[2] Total Lagrangian is gauge invariant.

transform. For example in the case of a complex scalar doublet the generators are given by the Pauli matrices. The trace is over the gauge boson mass states.

- m_ϕ is defined by: $(m_\phi)_{ab} = A_{ab} + iB_{ab}\gamma_5$ it gives the vertex term of the Yukawa interaction - between one scalar and two fermions. A and B are both hermitian matrices. In case of the SM the imaginary part of m_ϕ is always zero. The trace is over the fermion mass eigenstates as well as over the internal indices. The trace over the spinor indices will in general give a factor of 4.
- The renormalization term is given by $C\phi_c^4$. Renormalization conditions are the same as used previously.

The scalar and gauge boson contribution to the effective potentials are a generalization of the terms already derived in order to determine the effective potential for scalar QED and ϕ^4 theory. We will therefore not go into detail on te derivation of these terms. Since we have not seen how fermions affect the effective potential the derivation of this term will be done in detail below. After that some consequences of this effective potential will be discussed.

Fermion contribution

To derive the 1-loop effective potential for fermions the 1-loop diagrams with fermions running in the loop and scalars as external lines need to be calculated. These diagrams are shown in Fig. 4.4

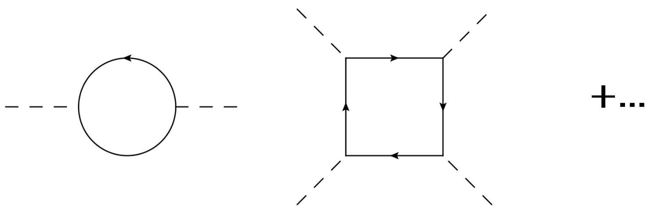

FIGURE 4.4: The 1-loop diagrams with a fermion running around the loop contributing to the effective potential.

The contribution to the effective potential for a diagram with n external scalars, n internal fermions and n vertices is given by:

$$i\frac{1}{n!}\phi_c^n \cdot \frac{1}{n!}\left(\frac{-i(m_\phi)_{ab}}{\phi_c}\right)^n \cdot 1 \cdot 1 \cdot n! \cdot n! \cdot \frac{1}{n} \cdot \int \frac{d^4k}{(2\pi)^4}\left(\frac{i\not{k}}{k^2}\right)^n \cdot (-1) \tag{4.28}$$

All terms have the same origin as previously. The extra factor of -1 is due to the fact that we are dealing with fermions. In the following the subscript ϕ will be left out to avoid cluttering of indices. Again we have to be careful with the indices in the term $(m_{ab}\not{k})^n = (m_{ab}k_\mu\gamma^\mu)^n$ the contraction of the spinor indices is done as follows:

$$\begin{aligned}(mk_\mu\gamma^\mu)^n &= m_{ij}k_\mu(\gamma^\mu)^{jk}\cdots{}_k^l\, m_{lm}k_\nu(\gamma^\nu)^{mi} \\ &= \mathrm{Tr}(m\gamma^\mu\cdots m\gamma^\nu)k_\mu\cdots k_\nu\end{aligned}$$

with the trace over spinor indices. Some relevant identities related to the Gamma matrices are listen in Appendix C. The Roman indices $i\cdots m$ are the spinor indices and μ, ν the Lorentz indices. The fermion contribution to the effective potential (V_f) can now be written as:

$$
\begin{aligned}
V_f &= -i \cdot \sum_{n=1}^{\infty} \frac{1}{n} \cdot \int \frac{d^4k}{(2\pi)^4} \left(\frac{1}{k^2}\right)^n \mathrm{Tr}(m\gamma^{\mu_1} \cdots m\gamma^{\mu_n}) k_{\mu_1} \cdots k_{\mu_n} \\
&= \sum_{n=1}^{\infty} \frac{-i}{2n} \cdot \int \frac{d^4k}{(2\pi)^4} \left(\frac{1}{k^4}\right)^n \mathrm{Tr}(m\gamma^{\mu_1} m\gamma^{\nu_1} \cdots m\gamma^{\mu_n} m\gamma^{\nu_n}) k_{\mu_1} k_{\nu_1} \cdots k_{\mu_n} k_{\nu_n} \\
&= \frac{i}{2} \mathrm{Tr} \int \frac{d^4k}{(2\pi)^4} \log\left(1 - \frac{1}{k^4} m\gamma^{\mu} m\gamma^{\nu} k_{\mu} k_{\nu}\right)
\end{aligned}
$$

To go from the first to the second line we have used the property that the trace over an uneven number of gamma matrices is zero, thus only the even terms in the sum contribute to the effective potential. The trace is over spinor indices as well as over the internal indices and the mass matrix indices. $m_{ab}\gamma^{\mu} m_{ab}\gamma^{\nu}$ can be simplified as $mm^{\dagger}\gamma^{\mu}\gamma^{\nu}$ (by the anti-commutation relations of gamma matrices and the definition of m).

$$
V_f = \frac{i}{2} \mathrm{Tr} \int \frac{d^4k}{(2\pi)^4} \log\left(1 - \frac{1}{k^4} mm^{\dagger}\gamma^{\mu}\gamma^{\nu} k_{\mu} k_{\nu}\right)
$$

To simplify matters further we observe that: $\gamma^{\mu}\gamma^{\nu}k_{\nu}k_{\mu} = \frac{1}{2}\left(\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu}\right)k_{\nu}k_{\mu} = \frac{1}{2}\eta^{\mu\nu}k_{\nu}k_{\mu}\mathrm{I} = k^2\mathrm{I}$. The integral can now also be computed easily resulting in the following effective potential:

$$
\begin{aligned}
V_f &= \frac{i}{2} \mathrm{Tr} \int \frac{d^4k}{(2\pi)^4} \log\left(1 - \frac{1}{k^2} mm^{\dagger}\right) \\
&= -\frac{1}{64\pi^2} \mathrm{Tr}\left[\left(mm^{\dagger}\right)^2 \left(\log\left(\frac{mm^{\dagger}}{\Lambda^2}\right) - \tfrac{1}{2}\right)\right]
\end{aligned}
$$

with the trace over spinor, internal and mass matrix indices. Reinstating the subscript ϕ we see that this is the same result as stated in the beginning of this section.

Consequences

From Eq. 4.27 we can conclude that whereas bosons and scalars can result in a symmetry breaking minimum the fermion contribution makes the minimum shallower or even results in the complete disappearance of the minimum, turning it into a maximum.

After renormalization (eq. 4.4) the effective potential of eq. 4.27 can be written as:

$$
\begin{aligned}
V =& V_0 + \frac{1}{64\pi^2}\left[\mathrm{Tr}\ W_{\phi}^2 + 3\mathrm{Tr}\ \mu_{\phi}^4 - 4N_c \mathrm{Tr}\ (m_{\phi}m_{\phi}^{\dagger})^2\right]\left[\log\left(\frac{\phi_c^2}{M^2}\right) - \frac{25}{6}\right] \\
=& V_0 + \frac{1}{64\pi^2\langle\phi\rangle^4}\phi_c^4\left[\mathrm{Tr}\ W^2 + 3\mathrm{Tr}\ \mu^4 - 4N_c \mathrm{Tr}\ (mm^{\dagger})^2\right]\left[\log\left(\frac{\phi_c^2}{M^2}\right) - \frac{25}{6}\right]
\end{aligned}
\tag{4.29}
$$

The trace is only over the mass matrix. N_c is the number of colours and the factor of 4 is due to the trace over the spinor indices.

From this equation we can conclude that the effective potential always has the form: $V = V_0 + A\phi_c^4 + B\phi_c^4 \log\left(\frac{\phi_c^2}{M^2}\right)$, from this general form the scalar mass, induced by quantum corrections can be determined using equation 4.12. The scalar mass is found to be:

$$m_S^2 = 8B\langle\phi\rangle^2 \tag{4.30}$$

If this formula gives a negative value for m_S^2 it can be concluded that the extremal value was in fact not a minimum. The true ground state in this case will be $\phi_c = 0$ resulting in a scalar with zero mass; the potential will not exhibit SSB. The arbitrariness of the renormalization scale M can be exploited to set a specific value for the VEV - in the case of the SM we need to set the VEV to 246 GeV.

4.5 Avoiding the Hierarchy Problem

It has now been stated several times that the Hierarchy Problem does not occur in conformally symmetric models. Now that we have treated the subject of radiative symmetry breaking it is possible to show why this is the case [14].

For this we look back at the theory of scalar QED. The β function of the gauge coupling e is in general given by:

$$\frac{de}{dt} \sim \frac{e^3}{\pi^2},$$

with $t = \log M$. It is now possible to integrate this differential equation from $M = \langle\phi\rangle$ to $M = \Lambda$, with Λ the UV cut-off of the effective potential. This results in:

$$-\frac{1}{2}\left(\frac{1}{e^2(\Lambda)} - \frac{1}{e^2(\langle\phi\rangle)}\right) = \frac{1}{\pi^2}\log\left(\frac{\Lambda}{\langle\phi\rangle}\right)$$

Because $e(\Lambda)$ is an increasing function for increasing Λ the term with $e(\Lambda)$ can be neglected compared to the term with $e(\langle\phi\rangle)$. The above equation can now be solved for $\langle\phi\rangle$:

$$\langle\phi\rangle = M \exp\left(-\frac{\pi^2}{2e^2(\langle\phi\rangle)}\right)$$

Small e has been assumed since the theory needs to be perturbatively valid at the VEV. From the equation it is apparent that for small e the VEV is perturbatively suppressed with respect to the cut-off scale M. This shows that a small VEV is completely natural. Since all masses are proportional to the VEV the naturalness of a small VEV also implies the naturalness of small scalar masses.

In this chapter we have seen that a theory which is conformally invariant can exhibit SSB, which is crucial for EWSB in the SM. Additionally a conformally symmetric theory has more predictive power than its massive counterpart since the scalar mass is no longer a free parameter but is determined by the parameters of the theory. It has also been shown that conformal symmetric models indeed do not suffer from the Hierarchy Problem.

In the following chapter the Coleman-Weinberg mechanism will be applied to some conformally symmetric extensions of the SM.

Chapter 5

Minimal Conformal Extensions of the Standard Model

In this chapter we will look at conformally invariant extensions of the SM. The extensions proposed will be minimal to keep the changes with respect to the Standard Model as small as possible. To make sure the theory closely resembles the SM theory we assume that the Higgs doublet still couples to the other SM particles in the same way. Furthermore, EWSB is imposed, although in these conformal models SSB will be caused by quantum corrections instead of a negative mass term. To keep all masses and coupling constants the same the VEV of the Higgs doublet is set to $\langle\phi_0\rangle = v = 246$ GeV. Another constraint on the extensions is that the predicted scalar Higgs mass has to agree with the experimental value of 125 GeV. These conditions are necessary in order for the new model to agree as much as possible with experiments.

As a first extension the Coleman-Weinberg Mechanism will be applied to the SM, μ^2 will be set to zero and the scalar mass according to this model will be predicted. It will be shown that the predictions of this model are at odds with experimental results. After this we will take a look at the generalized version of the Coleman-Weinberg Mechanism developed by E. Gildener and S. Weinberg [20] in 1976. Some SM extensions which have been proposed in order to predict realistic values for the Higgs mass will also be discussed [21] [19].

5.1 Conformal Standard Model

The one-loop effective potential of eq. 4.27 can be applied to the SM, which will give us a prediction of the Higgs mass. The only scalar in the Standard Model is the Higgs bosons, the vector bosons are the $W^{\pm}$, Z and the photon, γ. Additionally we have the quarks and leptons contributing to the fermion effective potential. Starting from this SM particle content the scalar, gauge and fermion contribution to the effective potential can be determined.

Scalars
The scalar interaction is: $V_0 = \lambda(\Phi\Phi^\dagger)^2$, with Φ a complex doublet with components:

$$\Phi = \frac{1}{\sqrt{2}}\begin{pmatrix}\phi_1 + i\phi_2 \\ \phi_0 + i\phi_3\end{pmatrix},$$

with ϕ_0, ϕ_1, ϕ_2 and ϕ_3 real scalar fields. ϕ_0 is the field which will obtain the non-zero VEV $\langle\phi_0\rangle = v$. Using this, V_0 can be written as: $V_0 = \frac{1}{4}\lambda\left(\phi_0^2 + \phi_1^2 + \phi_2^2 + \phi_3^2\right)^2$. The tree level term contributing to the effective potential is therefore $V_0(\phi_c) = \frac{1}{4}\lambda\phi_c^4$ with $\phi_c^2 = \sum_{i=0}^{3}\phi_{ic}^2$. W_ϕ can now be determined using its definition, this will give a 4×4 mass matrix. Taking the square and then taking the trace results in Tr $W_\phi^2 = 12\lambda^2\phi_c^4$.

Gauge bosons
Since we already know the masses of the vector bosons, μ^2 is known, thus μ_ϕ^2 can be simply found to be:

$$\mu_\phi^2 = \begin{pmatrix} M_W^2\frac{\phi_c^2}{\langle\phi_0\rangle^2} & 0 & 0 & 0 \\ 0 & M_W^2\frac{\phi_c^2}{\langle\phi_0\rangle^2} & 0 & 0 \\ 0 & 0 & M_Z^2\frac{\phi_c^2}{\langle\phi_0\rangle^2} & 0 \\ 0 & 0 & 0 & M_\gamma^2\frac{\phi_c^2}{\langle\phi_0\rangle^2}\end{pmatrix} = \begin{pmatrix}\frac{1}{4}g^2\phi_c^2 & 0 & 0 & 0 \\ 0 & \frac{1}{4}g^2\phi_c^2 & 0 & 0 \\ 0 & 0 & \frac{1}{4}(g^2+g'^2)\phi_c^2 & 0 \\ 0 & 0 & 0 & 0\end{pmatrix} \tag{5.1}$$

Fermions
Of all the fermions the top quark has the highest mass, its mass is already a factor of 40 higher than the bottom quark. The fermion mass and/or Yukawa coupling enters the effective potential as y_t^4, therefore the contribution of the top quark will by far overshadow the contributions of the other quarks. Thus we will only take the top quark into account and ignore all other fermions.

The mass of the top quark is given by $m_t = y_t\frac{1}{\sqrt{2}}\langle\phi_0\rangle$. Thus $m_\phi m_\phi^\dagger = y_t^2\frac{1}{2}\phi_c^2$. The $SU(3)_c$ symmetry of the Standard Model gives $N_c = 3$ for the three colours.

Combining all these terms into Eq. 4.29 we obtain the effective potential of the SM:

$$V = \frac{1}{4}\lambda\phi_c^4 + \frac{1}{64\pi^2}\phi_c^4\left[12\lambda^2 + \frac{3}{16}(3g^4 + 2g^2g'^2 + g'^4) - 3y_t^4\right]\left[\log\left(\frac{\phi_c^2}{M^2}\right) - \frac{25}{6}\right] \tag{5.2}$$

From this effective potential B can be determined plugging in the SM coupling constants (see Appendix A) and using eq. 4.30 the Higgs mass can be predicted:

$$m_h^2 = m_S^2 = \frac{1}{8\pi^2}\langle\phi\rangle^2\left[12\lambda^2 + \frac{3}{16}(3g^4 + 2g^2g'^2 + g'^4) - 3y_t^4\right] \approx -2\cdot10^3\ \text{GeV}^2 \tag{5.3}$$

The negative sign of the scalar mass squared shows that for this potential there is no non-zero VEV. The extremal point at $\langle\phi_0\rangle$ is in fact a maximum and not a minimum. Thus, the CW mechanism can not be employed to give the "conformal SM" a VEV.

The above calculation of the Higgs mass shows that extra scalars or gauge bosons need to be added in order to make radiative symmetry breaking a realistic option for conformal extensions of the SM. The SM quantum corrections are not able to induce SSB, mainly because the top quark is too heavy.

Since the addition of gauge bosons also implies a new fundamental interaction - which is an interesting but rather radical change of the SM - we will in this work focus on extending the SM with extra scalars. Adding extra scalars has as an additional advantage that the change to the SM is minor, also scalars are already present in a lot of BSM models which try to explain phenomena like Dark Matter. Additional scalars are thus a simple and elegant way to extend the SM.

In order to describe spontaneous symmetry breaking in the presence of multiple scalars a more general version of the Coleman-Weinberg mechanism needs to be used. This generalization, developed by E. Gildener and S. Weinberg [20] will be the topic of the next section.

5.2 Gildener- Weinberg generalization

One of the difficulties related to the addition of extra scalars can be explained by looking at a simple two scalar potential:

$$V = aS^4 + bR^4 + cS^2R^2. \tag{5.4}$$

Now assume that the scalar S couples to some gauge bosons and thus obtains a VEV due to quantum corrections. For a negative coupling constant c this will also result in a VEV for the scalar R, because the potential will contain a negative mass term for the scalar R. Expanding around these VEVS results in cross terms which mixes the two scalars. Therefore a basis transformation needs to be applied to find the mass eigenstates (physical scalars). For more general theories this will lead to difficult and messy calculations making it difficult to determine if radiative corrections lead to spontaneous symmetry breaking.

Another problem related to Radiative Symmetry Breaking is the perturbative validity. We have seen that for a theory with one scalar (ϕ^4 theory) the symmetry breaking minimum found is perturbatively not valid. The example of scalar QED does produce a minimum which is perturbatively valid. However in this case a necessary assumption is that the scalar coupling is of the order of e^4, with e the gauge coupling.

The Coleman-Weinberg mechanism can not be applied in the case of $\lambda \sim e^2$. This can be shown by looking at the most general classical scalar potential describing at least two scalars:

$$V_0(\Phi) \sim f_{ijkl}\Phi_i\Phi_j\Phi_k\Phi_l \tag{5.5}$$

with f_{ijkl} the scalar coupling; also called λ; $f_{ijkl} \sim e^2 \ll 1$

Loop corrections will be of order $e^4\Phi^4\log(\Phi/M)$. However, as already explained in the case of the ϕ^4 theory this will lead to a symmetry breaking minimum which is perturbatively not valid, because a term of order $\sim e^2$ has to be balanced with a term of order $\sim e^4\log(\Phi/M)$. As we will see the Gildener-Weinberg method provided a way to determine if SSB occurs for $\lambda \sim e^2$.

The Gildener-Weinberg method [20] can be used to determine if spontaneous symmetry breaking occurs in the presence of multiple scalars. Additionally this method is already perturbatively valid for $\lambda \sim e^2$. The method exploits the freedom in choosing the renormalization conditions.

The first step of the Gildener-Weinberg method is concerned with the classical potential. Classically $\Phi = 0$ will in general be the only minimum of the scalar potential of eq. 5.5. However, it is possible to select the scalar couplings in such a way that the classical potential has a flat direction in field space. This is possible because the theory has one free parameter, the renormalization scale M, which can be used to impose one condition on the coupling constants f_{ijkl}.

The flat direction is determined by imposing the following condition on the effective potential:

$$\min_{N_iN_i=1}(f_{ijkl}N_iN_jN_kN_l) = 0 \text{ at } M = M_W \tag{5.6}$$

where $N_iN_i = 1$ [1] is the unit sphere, with N_i the normalized fields and $\Phi_i = N_i\phi$. Defining the point of this minimum as n_i we can conclude that if $V(n) = 0$ then also $V(n\phi) = 0$. Thus the classical effective potential has a 'ray' of minima in the direction n_i at the scale M_W. Additionally eq. 5.6 ensures that the potential along this ray of minima is zero.

In general the number of possible flat directions depends on the number of scalar fields. After choosing a flat direction it is possible to do a change of basis of the scalar particles. This is useful because as explained in some cases multiple scalars can obtain a VEV, resulting in mixing terms between the scalars. The change of basis can be done in such a way that one of the scalar corresponds to the flat direction, whereas the other perpendicular directions correspond to massive scalars.

In the third and last step of the Gildener-Weinberg method the quantum corrections are turned on. Due to these quantum corrections the flat direction will obtain a non-zero VEV, whereas the other scalars will not be affected significantly by the quantum corrections. This is because the other scalars already have a classical mass, the quantum corrections only contribute slightly to this mass. The above is exactly the reason why changing to the new basis is so useful, since only the flat direction will obtain a non-zero VEV there will be no mixing between the scalars, thus making the calculations less complicated.

To see how the flat direction gets a VEV quantum corrections (δV) need to be added to the classical effective potential. Due to these quantum corrections the flat direction will curve, resulting in a non-zero VEV ($\langle\phi\rangle$). $\langle\phi\rangle$ can be determined by solving [2]:

$$\left.\frac{\partial \delta V(n\phi)}{\partial \phi}\right|_{\langle\phi\rangle} = 0. \tag{5.7}$$

Due to this non-zero VEV the classically massless scalar along the flat direction will become massive. The scalar corresponding to the flat direction is therefore the pseudo-Goldstone boson (PGB) of broken conformal invariance. Similar to the Coleman-Weinberg mechanism the mass of the scalar along the classically flat direction is given by equation 4.30. In this case the minimum $\langle\phi\rangle$ is the minimum along the classically flat direction.

[1] It is not necessary to normalize the fields such that $N_iN_i = 1$. As long as we find a minimum on some surface - not necessarily a sphere - there will be a ray of minima.

[2] Again this condition only ensures an extreme point not a minimum. However as before when the scalar mass turns out to be positive the extreme point $\langle\phi\rangle$ is a true minimum. Negative scalar masses are unphysical.

Perturbative validity
The advantage of first defining a flat direction is that along this flat direction the effective potential is zero. Thus along the flat direction the only terms in the effective potential are the quantum corrections. Therefore there is no need to balance classical terms with quantum corrections. Because of this the condition $\lambda \sim e^4$ which had to be imposed in the case of scalar QED can be loosened a bit to $\lambda \sim e^2$. This can be seen by looking back at the effective potential of scalar QED 4.22, without the classical potential this has the form:

$$V = \delta V \sim (\lambda^2 + e^4)\phi_c^4 \log\left(\frac{\phi_c}{M}\right) - (\lambda^2 + e^4)\phi_c^4.$$

Again, for perturbation theory to be valid we need the coupling constants to be small and the logarithms to be of $\mathcal{O}(1)$. Calculating the minimum using eq. 5.7 we see that for $\lambda \sim e^2$ a term of order e^4 has to be balanced against a term of order $e^4 \log(\phi_c/M)$. This does not pose any perturbative problems since e can be small without the need for a large logarithm. Therefore the Gildener-Weinberg method is already valid for scalar couplings which are of the order of the gauge couplings squared: $\lambda \sim e^2$.

To make the above less abstract an example of the Gildener Weinberg mechanism will be discussed in the next section. This example will also be the first Conformally Symmetric SM extension which does exhibit SSB.

5.3 One extra scalar

The simplest way to extend a conformally symmetric SM is the addition of one extra scalar. This is exactly the model which was looked into by R. Foot, A. Kobakhidze and R. Volkas [21]. Here we will briefly explain their ideas and results. In the end of this section the implications and phenomenology of this extension will be discussed.

Since we are now working with multiple scalars a small comment on the notation is in order. In the following the symbol for the SM Higgs boson h will be changed to $\phi_0' = \phi_0 - \langle\phi_0\rangle$ whereas h will be reserved for the scalar identified with the flat direction.

In their 2007 paper Foot et al. [21] start out with a classical scalar potential given by:

$$V_0(\Phi, S) = \frac{\lambda_1}{2}(\Phi^\dagger\Phi)^2 + \frac{\lambda_2}{8}S^4 + \frac{\lambda_3}{2}(\Phi^\dagger\Phi)S^2 \tag{5.8}$$

with Φ the complex SM doublet, coupled to the SM particles in the usual way. S is the new scalar and is assumed to be a real singlet field. The S scalar only couples to the SM doublet. Since there are multiple scalars involved the Gildener-Weinberg mechanism needs to be employed, which means a flat direction exists.

In order to find the flat direction we first parametrize the fields. From the Higgs mechanism we already know that only one of the four real scalar fields composing the complex doublet describes a physical particle. The other three are eaten by the gauge bosons, which thereby obtain a non-zero mass. Taking this into account the fields Φ and S can be parametrized as:

$$\Phi = \frac{r}{\sqrt{2}} \begin{pmatrix} 0 \\ \cos\omega \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \phi_0 \end{pmatrix}$$

and

$$S = r \sin\omega$$

ω is the polar angle, with $\tan\omega = \frac{S}{\phi_0}$ and r is the radius: $r^2 = S^2 + \phi_0^2$.
Using this parametrization we obtain:

$$V_0(r,\omega) = r^4 \left(\frac{\lambda_1}{8} \cos\omega^4 + \frac{\lambda_2}{8} \sin\omega^4 + \frac{\lambda_3}{4} \cos\omega^2 \sin\omega^2 \right) \tag{5.10}$$

Clearly, if we minimize the potential with respect to ω, this minimum will be a ray of minima - a flat direction - since the existence of the minimum is independent of the value of r. The independence of the minimum on r indicates that the flat direction is in the radial direction.

Following the GW approach we also impose that the potential is zero at this minimum. The minimum $\langle\omega\rangle$, for some r_0 is thus found by solving the following system of equations:

$$V_0(\omega, r_0) = 0 \text{ at } \omega = \langle\omega\rangle \tag{5.11a}$$

$$\frac{\partial}{\partial\omega} V_0(\omega, r_0) = 0 \text{ at } \omega = \langle\omega\rangle \tag{5.11b}$$

$$\frac{\partial^2}{\partial\omega^2} V_0(\omega, r_0) > 0 \text{ at } \omega = \langle\omega\rangle \tag{5.11c}$$

The second and third equation will allow us to find the minimum, the first is to set the potential to zero at the minimum. Solutions which results in $r = 0$ (unphysical) or $\cos\omega = 0$ (no VEV of Higgs doublet) are discarded since we are looking for solutions which allow for a VEV of Φ and as a result cause SSB of the electroweak sector.

The solutions of this system of equations consists of two minima [21], [22]:
with $\epsilon = \sqrt{\frac{\lambda_1(M_W)}{\lambda_2(M_W)}}$.

Min. 1	$\langle\sin\omega_1\rangle = 0$	for	$\lambda_1(M_W) = 0$
		and	$\lambda_3(M_W) > 0$
Min. 2	$\langle\tan^2\omega_2\rangle = \epsilon$	for	$\lambda_3(M_W) + \sqrt{\lambda_1(M_W)\lambda_2(M_W)} = 0$
		and	$\lambda_3(M_W) < 0$

For both minima the SM doublet obtains a non-zero VEV due to its coupling with the SM particles. For *minimum 1* λ_3 is positive, therefore a non-zero VEV of Φ will not result in a negative mass term in the classical potential of the scalar S. Thus, no spontaneous symmetry breaking will be induced for the scalar S. For *minimum 2* things are different. The negative λ_3 together with the non-zero VEV of Φ induces a negative mass term in the scalar potential of the scalar S. Therefore S also obtains a VEV.

The shape of the potential

The flat direction described by the first minimum is simply in the direction $\langle\omega_1\rangle = 0$ rad, independent of the exact values of the coupling constants. To plot the potential described by this minimum the values of the coupling constants are selected as: $\lambda_1 = 0$, $\lambda_2 = 0.02$ and $\lambda_3 = 0.02$.

The flat direction of the second minimum depends on the value of the coupling constants. Selecting for example $\lambda_1 = 0.01$ and $\lambda_2 = 0.04$ results in: $\epsilon = 0.5$ and $\lambda_3 = -0.02$. In this case the flat direction is specified by $\langle\omega_2\rangle = n\pi \pm 0.615$ rad, with n integer. Thus in this case the flat direction consists out of two rays of minima.

For the chosen values both minima are shown in fig. 5.1. The resulting potentials are depicted in fig. 5.2. The flat directions are clearly recognizable and are in the radial direction, as expected. For "minimum 1" it is clear that the flat direction is in the ϕ_0 direction, thus ϕ_0 will be classically massless and obtain its mass due to quantum corrections. The scalar S will have a non-zero classical mass. In case of "minimum 2" the flat direction is along a linear combination of S and ϕ_0. To describe the physical scalars one has to look at the mass eigenstates. One of the eigenstates will be along the flat direction and is classically massless, again it will obtain its mass via quantum corrections. The other (perpendicular) mass eigenstate will have a non-zero classical mass. A more mathematical treatment of the above is given in the next section.

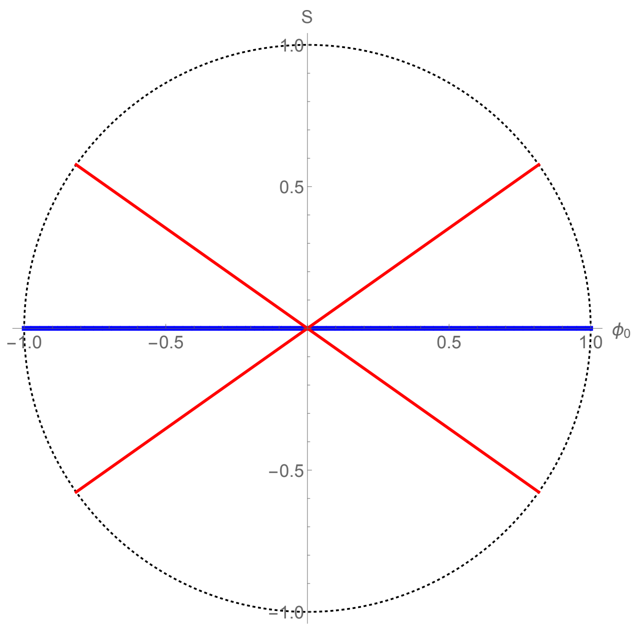

FIGURE 5.1: The two minima, plotted w.r.t. ϕ_0 and S. $\langle\omega_1\rangle$ is shown in dashed-blue and $\langle\omega_2\rangle$ in solid-red. The dotted circle is the unit circle with $r = 1$

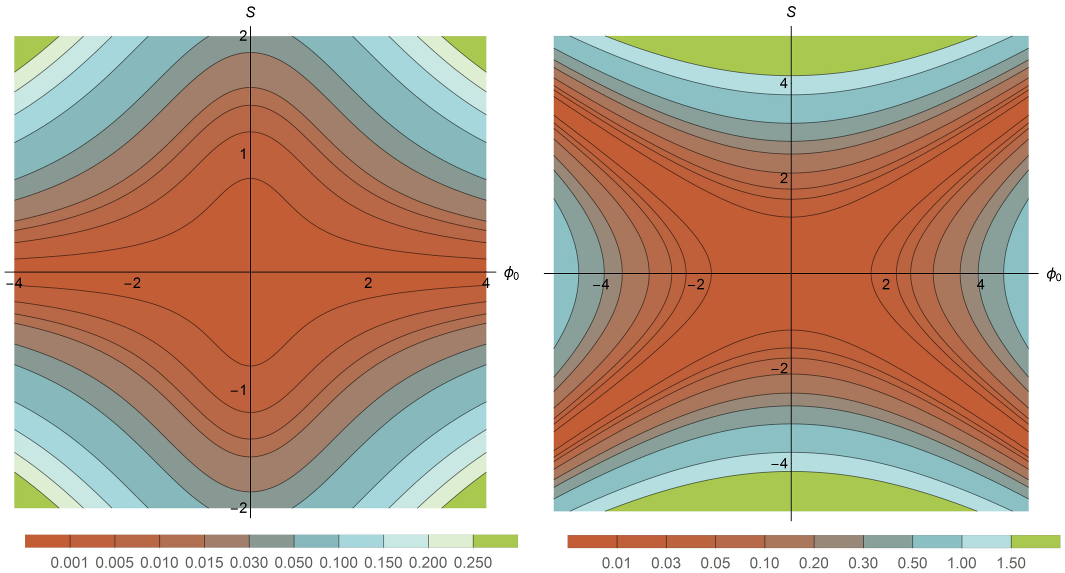

FIGURE 5.2: Contour plot of the potential $V(\phi_0, S)$ for each minimum. On the left with the minimum given by $\langle\omega_1\rangle$ and on the right with the minimum given by $\langle\omega_2\rangle$.

Mass spectrum

For both these minima the mass spectrum can be determined. The mass spectrum contains the PGB (h), which is a classically massless scalar and one massive scalar (H). For the time being it is assumed that the PGB - the flat direction - corresponds to the physical Higgs boson. Since the PGB does not have a classical mass it does not contribute to the effective potential. The massive scalar H contributes to the effective potential with $W = m_H^2$.

When quantum corrections are taken into account h will acquire a mass which can be determined using eq. 4.30. In this formula $\langle\phi\rangle$ is the VEV along the flat direction - the VEV of h. H will not acquire a non-zero VEV. Furthermore we will set $\langle\phi_0\rangle = v = 246$ GeV to match the model with the SM.

Minimum 1
The classically flat direction is in the ϕ_0 direction and thus the Higgs field is simply given by $h = \phi_0$. The massive scalar H corresponds to the field S, its mass can be determined by expanding the Lagrangian around the VEV of ϕ_0 resulting in:

$$m_H^2 = m_S^2 = \frac{1}{2}\lambda_3\langle\phi_0\rangle^2 = \frac{1}{2}\lambda_3 v^2.$$

From eq. 4.30 the mass of h can be determined:

$$m_h^2 = 8B\langle\phi_0\rangle^2 = \frac{1}{8\pi^2\langle\phi_0\rangle^2}\left[m_H^4 + 6M_W^4 + 3M_Z^4 - 12m_t^4\right]. \tag{5.12}$$

For this model to predict the correct Higgs mass of $m_h = 125$ GeV, we need a heavy scalar mass of $m_H \sim 540$ GeV, which requires $\lambda_3 = 9.6$.

Minimum 2

When quantum corrections are turned on both S and ϕ_0 will develop a non-zero VEV: $S = S' + \langle S\rangle$ and $\phi_0 = \phi_0' + \langle\phi_0\rangle$. Setting $\langle\phi_0\rangle = v$ to match this theory with the SM results in $\langle S\rangle = v\langle\tan\omega\rangle$. The flat direction is some linear combination of S' and ϕ_0'. By looking back at fig. 5.1 it is clear that the flat direction is given by:

$$h = \langle\cos\omega_2\rangle\phi_0' + \langle\sin\omega_2\rangle S' \tag{5.13a}$$

The massive scalar is now given by the direction perpendicular to this:

$$H = -\langle\sin\omega_2\rangle\phi_0' + \langle\cos\omega_2\rangle S' \tag{5.13b}$$

The classical masses of h and H can be determined by expanding the Lagrangian (eq. 5.8) around the VEVS of S and ϕ_0. The quadratic terms gives the mass matrix in the (ϕ_0', S') basis:

$$M_{ij}^2 = \begin{pmatrix} \frac{3}{2}\lambda_1\langle\phi_0\rangle^2 + \frac{1}{2}\lambda_3\langle S\rangle^2 & \lambda_3\langle\phi_0\rangle\langle S\rangle \\ \lambda_3\langle\phi_0\rangle\langle S\rangle & \frac{3}{2}\lambda_2\langle S\rangle^2 + \frac{1}{2}\lambda_3\langle\phi_0\rangle^2 \end{pmatrix} = v^2\begin{pmatrix} \lambda_1 & -\lambda_1^{3/4}\lambda_2^{1/4} \\ -\lambda_1^{3/4}\lambda_2^{1/4} & \sqrt{\lambda_1\lambda_2} \end{pmatrix}.$$

The second equality is obtained by setting the VEVs to the values stated earlier and applying the condition $\lambda_3(M_W) + \sqrt{\lambda_1(M_W)\lambda_2(M_W)} = 0$ [3]. Diagonalizing this mass matrix gives a matrix in the (h, H) basis:

$$D_{ij}^2 = \begin{pmatrix} 0 & 0 \\ 0 & (\lambda_1 + \sqrt{\lambda_1\lambda_2})v^2 \end{pmatrix}.$$

The classical masses of h and H are now simply given by the diagonal values of this matrix:

$$m_h^2 = 0 \text{ and } m_H^2 = \left(\lambda_1 + \sqrt{\lambda_1\lambda_2}\right)v^2 \tag{5.14}$$

Taking quantum corrections into account the scalar mass of h is determined to be4.30:

$$m_h^2 = 8B\langle h\rangle^2 = \frac{1}{8\pi^2\langle h\rangle^2}\left[m_H^4 + 6M_W^4 + 3M_Z^4 - 12m_t^4\right] \tag{5.15}$$

with

$$\langle h\rangle^2 = \langle\phi_0\rangle^2 + \langle S\rangle^2 = \frac{v^2}{\cos^2\omega_2} = (1+\epsilon)v^2 = \left(1 + \sqrt{\frac{\lambda_1}{\lambda_2}}\right)v^2. \tag{5.16}$$

Combining the previous two equations we obtain:

$$m_h^2 = \frac{1}{8\pi^2 v^2}\frac{1}{1+\sqrt{\lambda_1/\lambda_2}}\left[m_H^4 + 6M_W^4 + 3M_Z^4 - 12m_t^4\right] \tag{5.17}$$

The above equation can be solved such that this model predicts the correct Higgs mass of $m_h = 125$ GeV. However, since the equation contains two unknown parameters; λ_1 and λ_2, there will be a solution for every value of λ_1. This is shown in the left graph of fig. 5.3. It shows the solution of eq. 5.17 in the parameter space $(\lambda_1,\ \lambda_2)$. Using these values of the coupling constant it is possible to calculate the mass of the

[3]The scale M_W is implicit in the following equations

heavy Higgs (eq. 5.14) for each set of coupling constants (λ_1, λ_2). This is shown in the right graph of figure 5.3 [4].

Clearly small λ_1 and small λ_2 are mutually exclusive. Choosing for example $\lambda_1 = 3.5$ it is necessary to have $\lambda_2 = 2.98$. These values result in a mass of the heavy scalar of approximately 640 GeV.

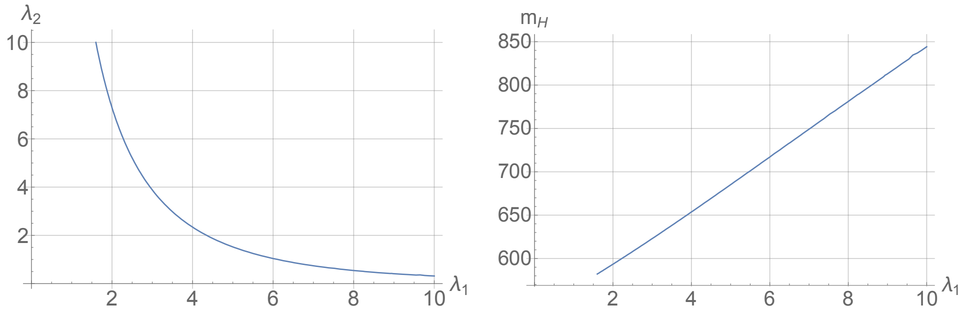

FIGURE 5.3: The coupling constants λ_1, λ_2 which are consistent with $m_h = 125$ GeV (left). The corresponding mass of the heavy scalar (m_H) is shown on the right.

PGB is not the physical Higgs

Another options which has not been discussed yet is that the physical Higgs boson is not identified with the PGB[19]. To match with SM the physical Higgs boson should predominantly be the SM Higgs ϕ_0' [5]. Therefore this is not an option for minimum 1 since here the non-PGB would be S which does not contain any ϕ_0'.

For minimum 2 it is possible to identify the non-PGB as the physical Higgs because both physical scalars are a superposition of S' and ϕ_0'. When the physical Higgs is identified with the non-PGB scalar H the mass of the PGB can be calculated using eq. 5.17. However, the Higgs mass is not large enough to cancel the top quark contribution to the effective potential. The calculated PGB is negative, which is unphysical. Thus in the case of one additional scalar the PGB always needs to be associated with the physical Higgs boson.

Implications & Experimental signatures

We have seen that the one-scalar extension contains two scalar degrees of freedom. One of the scalars (h) corresponds to the flat direction and is identified as the PGB of broken conformal invariance. This PGB h gains a mass due to quantum corrections. The other scalar (H) corresponds to the direction perpendicular to the flat direction, it has a classical mass. Additionally to match this extension with the SM the scalar h needs to be identified with the physical Higgs boson.

[4]The approximate linearity of this graph seems to be a result of the small range which is taken. Doing the same calculations for a larger range in coupling constants does indeed result in deviations from this linear behaviour.

[5]Measurements indicate that the mixing angle of the SM Higgs ϕ_0' with another scalar S it bounded by $\frac{S'}{\phi_0'} \tan\omega < 0.65$ [21].

The model by Foot et al. allows for two solutions, which both give a non-zero VEV to Φ. The mass of the physical Higgs boson h is known, therefore it is possible to determine the mass of the scalar H. We have seen that both of these models require big coupling constants to predict the correct Higgs mass. On first sight this might indicate that the perturbative expansion is not valid, however since the true expansion parameter seems to be $\frac{1}{4\pi}\lambda$, with λ the scalar coupling. Therefore these large couplings are just on the boundary of perturbative validity.

The one-scalar extension has certain experimental signatures, the most obvious being the prediction of a heavy scalar. The presence of another scalar in the particle spectrum results in some experimental signatures. One of these is the mixing of the scalars in the case of "minimum 2". The mixing of scalars means that the complex scalar doublet is not identified with the physical Higgs, instead the physical Higgs is a linear superposition of the SM Higgs boson ϕ_0' and the new scalar S. This has an effect on the coupling of SM particles with the physical Higgs boson [21].

To see which effect we take a look at the Yukawa coupling of the SM Higgs with for example the electron:

$$\mathcal{L} \sim y\phi_0'\overline{\psi_e}\psi_e. \tag{5.18}$$

When scalars mix this term can be written out in terms of the new physical fields (eq. 5.13) resulting in:

$$\mathcal{L} \sim y\langle\cos\omega\rangle h\overline{\psi}_e\psi_e - y\langle\sin\omega\rangle H\overline{\psi}_e\psi_e. \tag{5.19}$$

It can be read off from the Lagrangian that the physical Higgs h interacts with the electrons with a strength $\sim y\langle\cos\omega\rangle$. In comparison, the SM Higgs ϕ_0' interacts with the electrons with a strength $\sim y$. Measuring the interacting strengths can give bounds on the mixing angle ω. In the future it might be possible to measure the mixing angle precise enough to either prove or disprove mixing of the scalar doublet with another scalar.

Another experimental signature of the one-scalar extension is related to the interaction term between the S and ϕ_0 scalar. Due to these interaction terms it is in principle possible for the heavy scalar to decay into two light scalars. To see under which conditions this is possible we have to look separately at both possible minima.

In case of "minimum 1" the potential of eq. 5.8 can be expanded around the ground state - assuming only ϕ_0 gets a non-zero VEV. The following interaction terms are present:

$$V_{int} = \frac{\lambda_3}{4}\phi_0'\phi_0'SS + \frac{\lambda_3}{2}\langle\phi_0\rangle\phi_0'SS. \tag{5.20}$$

These interaction terms show that it is theoretically possible for a ϕ_0' scalar to decay into two S scalars. However we already know that this decay channel is excluded when considering energy conservation, since the S scalar is heavier than the physical Higgs boson.

Things get more complicated when both ϕ_0 and S obtain a VEV. Expanding the potential around the ground state and doing a change of basis to the (h, H) basis results in for example an interaction term like $\sim hhH$. Thus in this case there is the theoretical possibility for the heavy scalar H to decay into two (lighter) Higgs bosons. This is possible in practice as long as the mass of H is at least 250 GeV. Looking back at the possible scalar masses in figure 5.3 it can be concluded that this condition is satisfied for a large range of coupling constants. Thus in case of "minimum 2" the decay of the heavy scalar into two Higgs bosons is a clear signature of the model.

Apart from the decay signatures one can expect that these interaction terms also have some effect on cross sections. These calculations are not part of this thesis but will be interesting for testing this one-scalar extension.

The signatures discussed here show that whereas "minimum 2" has some clear experimental signatures - the decay of the heavy scalar and altered coupling constants - minimum 1 is almost completely hidden. This is related to the fact that the scalar S only couples to the SM Higgs doublet and not to any of the other SM particles.

5.4 UV behaviour

Minimal conformally symmetric models can only be expected to be valid up to the Planck since gravity is not included in these models. Therefore an interesting additional condition to put on these models it that they have to be stable up to the Planck scale. Which means that the theory should not have any Landau poles before the Planck scale.

To see how the model of Foot et al. behaves as a function of energy the β functions of the coupling constants are needed. The β functions of the scalar couplings, the gauge couplings and the Yukawa coupling can be found in for example [23] and [19].

The running of the one scalar extension has been calculated to one loop order, the results are shown in figure 5.4. In this graph it is clear to see that the scalar couplings turn large already at a small energy scale. This shows that one scalar extensions contain a Landau pole before the Planck scale, making these models unreliable all the way up to the Planck scale. Calculations done in for example [19] also conclude that one scalar extensions suffer from a small Landau pole. Thus, to develop a conformally symmetric theory which does not have a Landau pole before the Planck scale further extensions are needed. Some of these possible extensions will be discussed in the next section.

Comparing the graph in figure 5.4 to the running of coupling constants of the SM (figure 3.1) it can be concluded that the addition of a single scalar has a huge effect on the position of the Landau pole. Whereas the Landau pole of the SM lies far beyond the Planck scale the extra scalar pushes the Landau pole to small energy scales. The reason for this lies in the assumption of conformal symmetry, because of the necessarily large scalar couplings the β functions get a large positive contribution. This in turn results in accelerated growth of the scalar coupling constants, pushing the Landau pole to small energy scales.

Another interesting thing which can be observed in this plot is that the scalar coupling constants do not become negative at large scale, thus these models do not seem to suffer from the Instability Problem.

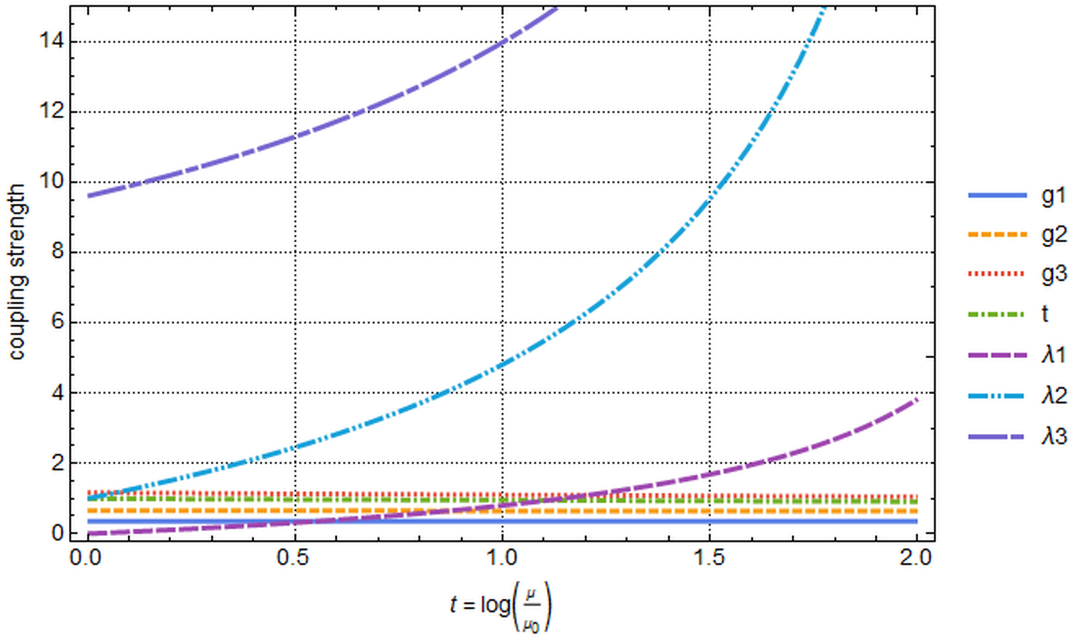

FIGURE 5.4: The running of the coupling constants in the one scalar model, for minimum 1. With $\lambda_1(0) = 0$, $\lambda_2(0) = 0.1$ and $\lambda_3(0) = 9.6$. The initial values of the gauge and Yukawa coupling constants are the same as before (Appendix A). $\mu_0 = 173$ GeV.

5.5 Additional scalars

Foot et al. [21] also consider adding an $SO(N)$ scalar multiplet instead of one scalar. The potential of such a model is given by:

$$V_0(\Phi, S) = \frac{\lambda_1}{2}(\Phi^\dagger\Phi)^2 + \frac{\lambda_2}{8}(S^T S)^2 + \frac{\lambda_3}{2}(\Phi^\dagger\Phi)(S^T S) \tag{5.21}$$

with S the $SO(N)$ multiplet given by $S = (S_1, \cdots S_N)^T$. The potential is very similar to the one-scalar extension. Following the Gildener-Weinberg method several minima can be found, again it is assumed that Φ should have a non-zero VEV. One of the minima found is the case where only Φ obtains a non-zero VEV. This situation is exactly the same as "minimum 1" in the one-scalar extension therefore the same conditions apply to the coupling constants. The other minima describe situations in which both Φ and one (or more) of the scalars in the multiplet obtains a non-zero VEV.

To be able to match the model with the SM only "minimum 1" is a possibility since the other minima require the multiplet to obtain a VEV. A VEV of the multiplet would break the rotational symmetry, resulting in massless Goldstone bosons. Since we do not observe these particles this option is excluded [21]. Investigating "minimum 1" in the case of a scalar multiplet it is clear that the mass of each scalar in the $SO(N)$ multiple will be:

$$m_H^2 = \frac{1}{2}\lambda_3 v^2. \tag{5.22}$$

Thus the contribution of the heavy scalars to the PGB mass 4.30 is now Nm_H^4 instead of m_H^4. In order for the model to generate a physical Higgs mass of 125 GeV we need to have:

$$Nm_H^4 = 540^4\text{GeV}^4.$$

Larger N therefore requires a smaller coupling constant λ_3. Requiring smaller scalar coupling constants is a nice feature regarding the perturbative validity, however these models still have a Landau pole at small scales [19]. Even more elaborate extensions are needed to obtain a Landau pole beyond the Planck scale.

Another possible extension is an extension with two separate scalars, with potential [19]:

$$V = \lambda_1(\Phi^\dagger\Phi)^2 + \lambda_2 S^4 + \lambda_3 R^4 + \kappa_1(\Phi^\dagger\Phi)S^2 + \kappa_2(\Phi^\dagger\Phi)R^2 + \kappa_3 S^2 R^2. \tag{5.23}$$

An additional global $\mathbb{Z}_2$ symmetry has been imposed to remove all terms which have an odd number S or R fields.

This potential has a number of different flat directions. Only the flat direction will acquire a non-zero VEV. When turning on quantum corrections we still require Φ to have a non-zero VEV, therefore only solutions with a flat direction (partly) along Φ are physically relevant. The following flat directions remain:

(a) Flat direction along Φ

(b) Flat direction in the $(\Phi,\ S)$ plane

(c) Flat direction in the $(\Phi,\ R)$ plane (equivalent to case (b))

(d) Flat direction in the $(\Phi,\ S,\ R)$ space

Each flat direction will impose specific conditions on the coupling constants, similar to the conditions determined for the flat directions of the one-scalar extension.

When turning on quantum corrections the PGB (or flat direction) will obtain a minimum away from the origin. By expanding the potential around this VEV the following mass spectrum will appear for each choice of flat direction:

(a) Φ will be the PGB which becomes massive due to quantum corrections. The scalars S and R will have a classical mass.

(b) Φ and S will both have a non-zero VEV, therefore the mass eigenstates have to be determined by doing a change of basis. This will result in one PGB h and one heavy scalar H. Additionally the scalar R will have a non-zero classical mass.

(c) Equivalent to case (b) with $S \leftrightarrow R$.

(d) All scalars will have a non-zero VEV. Again the physical scalars can be determined by doing a change of basis resulting in one PGB h and two heavy scalars H_1 and H_2.

Case (a) is similar to "minimum 1" of the one-scalar extension whereas case (b) and (c) are similar to "minimum 2" of the one-scalar extension. Case (d) will not be discussed any further in this thesis, because as we will see cases (b) and (c) already allow for interesting physics.

Taking a look at case (a) and expanding around the VEV of Φ it can be determined that the classical mass of the scalars S and R is given by:

$$m_S^2 = \kappa_1\langle\phi_0\rangle^2 \text{ and } m_R^2 = \kappa_2\langle\phi_0\rangle^2.$$

These classical masses will contribute to the mass of the PGB. Setting the mass of the PGB to the physical Higgs mass of 125 GeV it can be determined from eq. 4.30 that $m_S^4 + m_R^4 = 540^4$ GeV^4. Even though this condition will in general require smaller scalar couplings it is shown in [19] that the Landau pole of this model is still below the Planck scale.

In the same manner as for "minimum 2" of the one-scalar extension the masses of the scalars in case (b) (and equivalently, case (c)) can be determined. The calculations will not be done here but can be found in for example [19]. In this paper it is also shown that the two-scalar extension with a non-zero VEV of S allows for a Landau pole beyond the Planck scale. The paper assumes that the physical Higgs boson is identified with the classically massive (non-PGB) scalar H.

Thus a two scalar extension where both Φ and one of the new scalars obtain a VEV appears to be a viable **Minimal Conformal Model** [19] in the sense that it can preserve EWSB, predict a Higgs mass of 125 GeV and be stable up to the Planck scale.

An additional benefit of this two-scalar extension where both Φ and S obtain a non-zero VEV but R does not is that the scalar R is completely stable and therefore an interesting dark matter candidate. The stability of R can be inferred from the potential of eq. 5.23. The scalar R can only decay when a term in the potential is present which contains a single R field. However, when expanding the potential around the non-zero VEVs of S and Φ all terms containing R will still be either quartic or quadratic in R. Thus the theory does not allow for any decay of R into other scalars, thereby making it a stable scalar particle. The scalar R can only disappear through interactions or by annihilating with another R scalar.

The previous chapters have introduced conformally symmetric SM extensions as a possible class of BSM theories. We have shown that conformal models are still a theoretical option because the scalar potential has not been measured precise enough. The lack of knowledge about the scalar potential makes theories with an adapted scalar potential interesting to look at. However, conformal symmetry which sets $\mu^2 = 0$ is not the only possible modification. Instead of taking the limit $\mu^2 \to 0$ one can also take the limit $\mu^2 \to \infty$. In this limit the Higgs potential - which is a linear sigma model (LSM) - changes into a non-linear sigma model (NLSM).

The next chapter will introduce the NLSM and show how it is related to the Higgs potential. A fascinating property of the NLSM is that the model exhibits Asymptotic Safety, what this is and why it might be an interesting property for a BSM theory will also be discussed in the next chapter.

Chapter 6

The Non-Linear Sigma Model & Asymptotic Safety

6.1 NLSM vs LSM

To see how the scalar part of the SM is related to the NLSM one needs to look at the Lagrangian of both theories. The scalar part of the SM Lagrangian is given by:

$$\mathcal{L} = (\partial_\nu \Phi)^\dagger (\partial^\nu \Phi) - \mu^2 (\Phi^\dagger \Phi)^2 - \lambda (\Phi^\dagger \Phi)^2, \tag{6.1}$$

with $\mu^2 < 0$. Using the definition of the complex doublet this Lagrangian can also be written as:

$$\mathcal{L} = \frac{1}{2}(\partial_\nu \phi_i)^2 - \frac{\mu^2}{2}\phi_i^2 - \frac{\lambda}{4}(\phi_i^2)^2, \tag{6.2}$$

with $i = 0, 1, 2, 3$. This is equivalent to an $O(4)$ LSM. The Lagrangian of a simple NLSM can be written as:

$$\mathcal{L} = \frac{1}{2g^2}(\partial_\mu \varphi_i)^2 \qquad \text{with } \varphi_i^2 = 1, \tag{6.3}$$

Again with $i = 0, 1, 2, 3$. The addition of a constant to the Lagrangian does not change the physics, therefore the Lagrangian of the LSM can also be written as:

$$\mathcal{L} = \frac{1}{2}(\partial_\nu \phi_i)^2 - \frac{\lambda}{4}\left(\phi_i^2 + \frac{\mu^2}{\lambda}\right)^2 \tag{6.4}$$

Taking the limits $\lambda \to \infty$ and $\mu^2 \to -\infty$ it is clear that in order for this Lagrangian to be finite we must have $\phi_i^2 = -\frac{\mu^2}{\lambda}$. Comparing this limit of the LSM to the NLSM it can be seen that these models are equivalent for $\phi_i = \frac{1}{g}\varphi_i$ and $g^2 = -\frac{\lambda}{\mu^2}$. g^2 has the correct sign for negative μ^2.

The NLSM can thus be seen as the strong coupling (and infinite mass) limit of the LSM. To see why a NLSM is interesting we have to take a look at its UV behaviour.

6.2 UV behaviour

The behaviour of a theory in the UV can be determined from the β functions. The theory of β functions can be found in most textbooks (e.g. [1]) and is also summarized in Appendix A. For clarity the definition of the β function will be stated here:

$$\beta = M\frac{\partial g}{\partial M}, \tag{6.5}$$

with M the energy scale and g a coupling constant. The running of the coupling constants in the SM (fig. 3.1) has already shown two ways for the coupling constant to behave in the UV. The third UV behaviour is called asymptotic safety. We start with a short summary of all three main UV behaviours:

(a) Asymptotic freedom
This occurs when the β function is negative for all values of the coupling constant g. Increasing the energy scale M thus requires g to decrease. Running the coupling all the way up to the UV it is clear that the coupling constant goes to zero. Thus a theory with a negative β function is free or non-interacting in the UV. An example of this was already encountered in the SM in for example the theory of the strong interactions; QCD.

(b) Landau pole
When the β function is positive for all g a Landau pole will occur. Thus, when M increases also g should increase. When the coupling constant increases fast enough the coupling becomes infinite already at a finite scale - this is a Landau pole. The theory can not be trusted at and beyond a Landau pole because the theory is no longer perturbative for large coupling constants. Also this behaviour was already encountered in the SM, for example in the theory of QED.

(c) Asymptotic safety
In some theories the β function can be positive for small values of g whereas it is negative for large values of g. For small g increasing M means that also g should increase, but for large g increasing M means decreasing g. Thus in the UV the coupling constant flows to a non-zero fixed point. This fixed point can be at large coupling constants (non-perturbative) as well as at small coupling constants (perturbative). Asymptotic safety is in some sense a generalized version of asymptotic freedom.

The β function as a function of the coupling constant g is shown in figure 6.1 for all three UV behaviours.

Asymptotic Safety

Of these three UV behaviours asymptotic safety is probably the most interesting one. Asymptotic safety was first proposed in the search of a renormalizable theory of gravity [23]. It is well know that quantum theories of gravity have the problem of being non-renormalizable due to the negative mass dimension of the gravitational

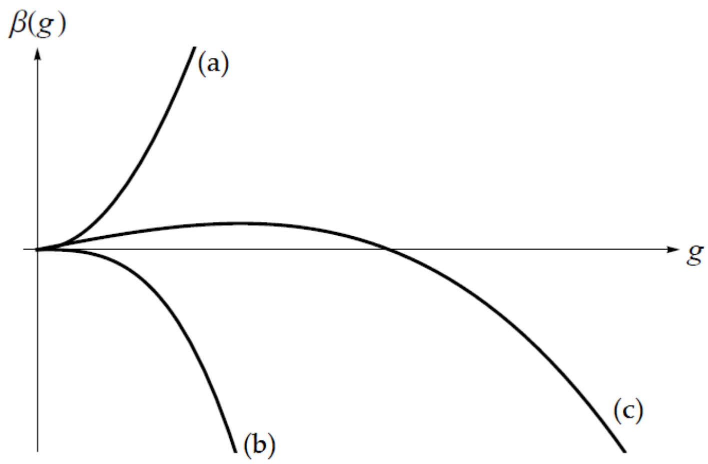

FIGURE 6.1: The three different UV behaviours.

coupling constant. However this non-renormalizability refers to *perturbative* renormalizability, in the sense that every order in perturbation theory can be renormalized. Since most techniques physicists have developed to calculate physical processes like cross-sections rely on perturbation theory perturbative renormalizability is usually seen as a necessary aspect of any QFT.

However, a theory can be renormalizable in a non-perturbative sense [24]. This happens when the theory contains a non-trivial UV fixed point - all coupling constants flow towards a constant (large) non-zero value in the UV [1]. This phenomenon is called asymptotic safety. In the presence of a non-trivial UV fixed point a theory is well defined up to arbitrary energies, resulting in a renormalizable theory. Due to the non-perturbative nature of these theories calculating for example the fixed points is in general not an easy task.

A simple theory for which asymptotic safety can be shown is the NLSM. In the next section the β function of the NLSM will be calculated, showing that the theory is indeed asymptotic safe.

6.3 Non-Linear Sigma Model

The NLSM is a theory closely related to the LSM but, as we will see shortly, with completely different UV behaviour.

We will start with the UV behaviour of the LSM. The scalar part of the SM is equivalent to the $O(4)$ LSM. Therefore one can determine the β function of this LSM from the scalar β function of the SM (Appendix A). Ignoring all contributions from the other SM particles the β function is given by:

$$\beta_\lambda = \frac{1}{16\pi^2} 24\lambda^2. \tag{6.6}$$

[1] A UV fixed point is called trivial or Gaussian when all coupling constants flow towards zero in the UV, resulting in a non-interacting theory. This is called asymptotic freedom. In the presence of a Landau pole the coupling constants also flows to zero in the UV beyond the Landau pole.

This β function is positive and therefore the $O(4)$ LSM has a Landau pole in the UV. Generalizing the β function for a general $(O(N))$ LSM [1], shows that the β function is positive for general N, thus $O(N)$ LSM suffer from a Landau pole.

To determine the UV behaviour of the NLSM the β function needs to be calculated. We will calculate the β function in the manner outlined in Appendix A using the Callan-Symanzik (CZ) equations and calculating Green's functions.

The NLSM is perturbatively renormalizable in $d = 2$ but not in $d = 4$. The Callan-Symanzik equations are only defined for renormalized theories, thus to determine the β function in $d = 4$ the β function in $d = 2$ needs to be determined first. This can be done using the CZ equations of Appendix A. Here only an outline of the calculation will be given, details can be found in for example [1]. The general Lagrangian of a NLSM with N scalar fields is given by:

$$\mathcal{L} = \frac{1}{2g^2}(\partial_\mu \phi_i)^2 \text{ with } \phi_i^2 = 1, \tag{6.7}$$

with $i = 1, \dots N$. This can be rewritten by defining $\phi_i = (\pi_1, \dots, \pi_{(N-1)}, \sigma)$, resulting in:

$$\phi_i^2 = \pi_i^2 + \sigma^2 = 1 \to \sigma^2 = 1 - \pi_i^2 \tag{6.8}$$

By expanding in powers of π_i the Lagrangian can now be written as:

$$\mathcal{L} = \frac{1}{2g^2}(\partial_\mu \pi_i)^2 + \frac{1}{2g^2}\frac{(\pi_i \partial_\mu \pi_i)^2}{1 - \pi_i^2} \approx \frac{1}{2g^2}(\partial_\mu \pi_i)^2 + \frac{1}{2g^2}(\pi_i \partial_\mu \pi_i)^2 + \cdots \tag{6.9}$$

To determine the β function using the CZ equations two Green's functions need to be calculated, in this case $G^{(1)}(\sigma)$ and $G^{(2)}(\pi_1, \pi_1)$ can be used. Figure 6.2 shows the Feynman diagrams for both Green's functions up to 1-loop order.

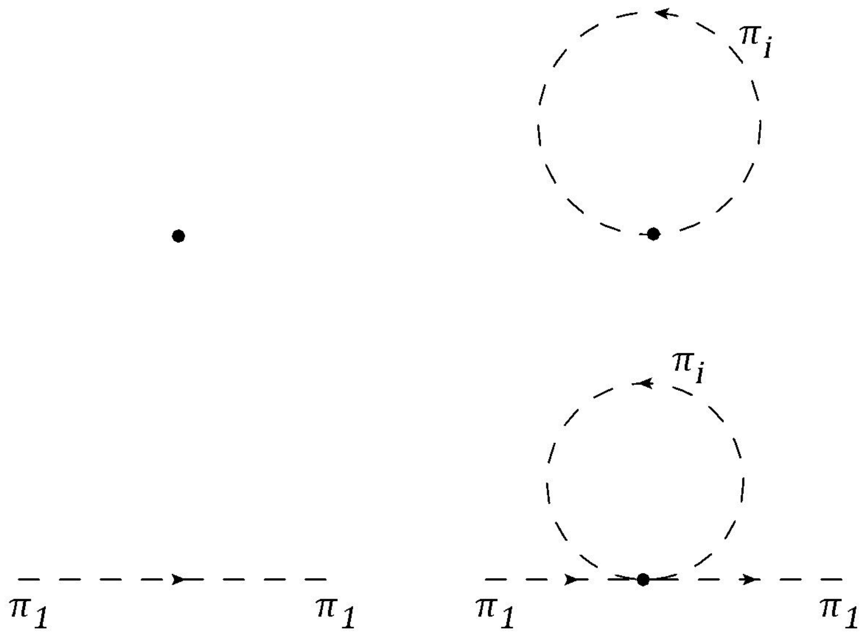

FIGURE 6.2: Diagrams contributing to the Green's functions $G^{(1)}$ (above) and $G^{(2)}$ (below). $i = 1, \cdots, N - 1$.

Using the following renormalization conditions:

$$
\begin{aligned}
G^{(1)} &= \langle \sigma(x) \rangle = 1 && \text{at } \mu^2 = M^2 \\
G^{(2)} &= \frac{ig^2}{k^2} && \text{at } \mu^2 = M^2
\end{aligned}
\tag{6.10}
$$

both Green's functions can be determined. Resulting in:

$$
G^{(1)}(\sigma) = \langle \sigma(0) \rangle = 1 - \frac{g^2(N-1)}{8\pi} \lim_{\mu^2 \to 0} \log(M^2/\mu^2) \tag{6.11}
$$

and

$$
G^{(2)}(\pi_1, \pi_1) = \frac{ig^2}{k^2} - \frac{ig^4}{4\pi k^2} \lim_{\mu^2 \to 0} \log(M^2/\mu^2). \tag{6.12}
$$

For both of these Green's functions the first term is the tree level term whereas the second term is the one-loop correction: $G = G_{tree} + G_{loop}$.

The CZ equation for $G^{(1)}$ is given by:

$$
\left[M \frac{\partial}{\partial M} + \beta \frac{\partial}{\partial g} + \gamma \right] G^{(1)} = 0. \tag{6.13}
$$

This equation can be split up in a tree level and a one-loop level equation. Since both β and γ are zero at tree level only the first order equation is relevant:

$$
M \frac{\partial}{\partial M} G^{(1)}_{loop} = - \left[\beta \frac{\partial}{\partial g} + \gamma \right] G^{(1)}_{tree}. \tag{6.14}
$$

Plugging in $G^{(1)}_{tree}$ and $G^{(1)}_{loop}$ gives:

$$
\gamma = \frac{g^2(N-1)}{4\pi} \tag{6.15}
$$

Doing the same for $G^{(2)}$ and using the already determined value of γ, the β function of the NLSM in $d = 2$ can be determined:

$$
\beta = -\frac{g^3}{4\pi}(N-2) \tag{6.16}
$$

From this β function it can be concluded that the NLSM is asymptotically safe in $d = 2$ for $N > 2$.

To determine the β function in $d > 2$ the general rule derived in Appendix A.2, eq. A.16) can be used. To apply this a rule a dimensionless coupling constant is needed. From the Lagrangian it can be determined that g^2 has dimension $(2-d)$. A dimensionless coupling constant λ can thus be defined as: $g^2 = \lambda M^{2-d}$. Applying the general equation we obtain:

$$
\beta^{(d)}(\lambda) = -(2-d)\lambda + \beta^{(2)}(\lambda), \tag{6.17}
$$

with:

$$
\beta^{(2)}(\lambda) = \beta^{(2)}(g^2) = M \frac{\partial g^2}{\partial M} = 2g\beta^{(2)}(g) = -\frac{g^4}{2\pi}(N-2) = -\frac{\lambda^2}{2\pi}(N-2). \tag{6.18}
$$

Combining the previous two equations gives:

$$\beta^{(d)}(\lambda) = -(2-d)\lambda - \frac{\lambda^2(N-2)}{2\pi} \tag{6.19}$$

The β function of the NLSM in $d = 4$ is thus:

$$\beta_\lambda^{(4)} = 2\lambda - \frac{\lambda^2(N-2)}{2\pi} \tag{6.20}$$

For $d = 4$ this theory is positive for small λ and negative for large λ. This means that the theory has an UV fixed point at: $\lambda^* = \frac{2\pi(d-2)}{N-2}$.

Concluding, the NLSM in $d = 4$ is an asymptotic safe theory and is therefore expected to be renormalizable in a non-perturbative sense.

6.4 Possible application to a SM extension

A NLSM has the advantage of being asymptotic safe and therefore renormalizable. Asymptotic Safe models do not have any Landau poles, which makes Asymptotic Safety an interesting property to look at when developing BSM theories.

Unfortunately, implementing the NLSM into the scalar part of the SM poses severe problems regarding the phenomenology of the model. For one, taking the limits $\lambda \to \infty$ and $\mu^2 \to \infty$ of the LSM would naively result in an infinitely heavy scalar mass, since the scalar mass is proportional to μ. Calculations do seem to indicate that the scalar mass in a NLSM is bounded from above [25]. Also, it is not entirely clear how a NLSM would produce a VEV in order for the other SM particles to obtain a mass via spontaneous symmetry breaking. Additionally it is not certain if such a theory would contain a massive Higgs boson. In for example [26] the following statement is made:

"Replacing the complex Higgs doublet by a S^3 NLSM results in a "Higgsless" theory."

An S^3 NLSM is described by the Lagrangian of eq. 6.7 with $i = 1, \ldots, 4$. This theory would be the natural replacement of the LSM in the SM since it also contains four scalar fields. However, in the NLSM due to the condition ϕ_i^2 = constant only three fields are independent.

Another problem related to implementing a NLSM into a SM extension is the fact that the scalar β function will get contributions from the SM fermions and gauge bosons. It is shown in for example [27] that this can result in the disappearance of the non-trivial fixed point.

In the literature not much can be found regarding the implementation of the NLSM into the scalar part of a SM extension. The problems outlined above seem to make it difficult to find such an implementation.

Still, because of its renormalizability and relation to theories of quantum gravity, Asymptotic Safety is an intriguing property for a BSM theory. Whereas the NLSM - one of the simplest Asymptotic Safe theories - faces some problems when trying to implement it in the SM there is a lot of research focussed on SM extensions which exhibit Asymptotic Safety (see for example [27] and [28]). One fascinating result regarding Asymptotic Safety and gravity can be found in the paper by M. Shaposhnikov and C. Wetterich [23]. In this article from 2009 - before the discovery of the Higss boson - the Higgs mass is predicted to be about 126 GeV. Surprisingly, this value is consistent with the experimentally determined Higgs mass. The key assumption made in this calculation is that the SM plus gravity is asymptotically safe.

Asymptotic Safe extensions are by themselves a complete topic for a Master Thesis and can therefore not be discussed any further in this work.

Chapter 7

Conclusions

In this thesis previous work done on minimal conformally invariant extensions of the Standard Model has been explained. To get an in-depth understanding of what is physically happening part of the results obtained in articles related to this topic have been recalculated. Figures have been added to clarify the sometimes abstract calculations.

We have argued that minimal extensions are becoming increasingly interesting due to the lack of evidence of new physics in experiments. Conformally symmetric models are fascinating because these models do not suffer from the Hierarchy Problem, thereby solving one of the biggest SM problems. Thus, combining the idea of a minimal extension with conformal symmetry allows for a theory which does not suffer from the Hierarchy Problem while most of the Standard Model predictions remain largely unchanged.

The problem of conformally symmetric theories is that they do not exhibit classical spontaneous symmetry breaking, of which the Higgs mechanism is an example. Since Electroweak Symmetry Breaking is a crucial part of the Standard Model this seems to make conformal symmetry unsuited as a BSM property. However, it is shown that apart from classical spontaneous symmetry breaking there exists a second way to induce Electroweak Symmetry Breaking. This phenomenon is called Radiative Symmetry Breaking or the Coleman-Weinberg mechanism. The theory of Radiative Symmetry Breaking shows that a VEV can be created through quantum corrections. The theory of Radiative Symmetry Breaking has been explained using some simple examples like the ϕ^4 theory. These examples also show that conformally symmetric models indeed do not face the Hierarchy Problem.

Conformal Symmetry has been applied to the SM, unfortunately we have seen that the SM by itself can not be conformally symmetric. However, it is shown that simple scalar extensions can already result in viable BSM theories, which do exhibit Radiative Symmetry Breaking. Whereas the one-scalar extension suffers from a small Landau pole, the two-scalar extension can have a Landau pole beyond the Planck scale. Large Landau poles are preferential because they reduce the need for any new physics up to the Planck scale, at which the theory needs to be replaced by a theory containing gravity in any case. Additionally it is shown that the two-scalar extension allows for a Dark Matter candidate.

The end of the thesis has briefly touched upon the topic of Asymptotic Safety. This property might be a solution to the problem of the perturbative non-renormalizability of quantum theories of gravity. The simplest Asymptotic Safe theory, the non-linear sigma model, has been introduced and the property of Asymptotic Safety is proven.

Despite the similarities between the scalar Higgs potential and the non-linear sigma model it seems unlikely that the SM scalar potential can be exchanged by a non-linear sigma model without changing the SM phenomenology to much. Asymptotic Safety itself remains interesting in the development of SM extension.

Chapter 8

Outlook

Future theoretical work consists for example in determining precisely the parameters which can be used to differentiate the Standard Model from the proposed scalar extensions. These parameters are for example the interaction strengths and cross sections. We have already encountered a prediction made by some scalar extensions; the non-zero mixing angle between scalars. By measuring this mixing angle through the coupling between the Higgs boson and the SM particles it is possible to determine the validity of these scalar extensions. Another experimental challenge is the measurement of the Higgs self-coupling. This measurement is crucial because it will give the clearest indication to decide if Spontaneous Symmetry Breaking is a classical or quantum phenomenon.

While anticipating these experimental results more theoretical research on the concept of conformal extensions should be done, specifically research on the addition of extra gauge sectors. This has an interesting application in the context of Grand Unified Theories (GUTs) [5].

Symmetry breaking in these GUTs is usually induced by a scalar potential, similar to the Higgs mechanism of Electroweak symmetry breaking. Instead of this classical symmetry breaking it is interesting to determine if it is possible to break the GUT symmetry through radiative symmetry breaking. A BSM theory which combines the predictive power of a GUT with the elegance of conformal symmetry will result in rich phenomenology while hopefully having less problems than the current Standard Model.

An interesting research question for future work is thus:
"Can a GUT be broken into the SM symmetry group through radiative symmetry breaking?"

To give a complete answer to this research question several things have to be considered. An essential first step is to look at radiative symmetry breaking in simple SM extension with an extra gauge sector and determine the effect of extra gauge sectors on the phenomenology of the theory. After researching some simple gauge sector extensions more elaborate extensions can be researched.

To determine which gauge sector extensions are interesting for the specified research question it is necessary to take a look at viable GUTs. In general these GUTs predict extra scalars as well as additional gauge sectors. These GUTs reduce to the SM via specific symmetry breaking patterns.

These two research lines can subsequently be combined to determine if it possible to break a GUT symmetry through radiative symmetry breaking instead of classical spontaneous symmetry breaking.

A specific GUT to which this line of research can be applied is the $SO(10)$ gauge theory. It is already known that, by breaking the $SO(10)$ gauge group, the Standard Model symmetry group can in principle be recovered. Among the different intermediate steps is the possibility for $SO(10)$ to break down into a so-called left-right symmetric model. Left-right symmetric models contain an extra $SU(2)_R$ gauge group and are interesting because they avoid the unexplained handedness of the Standard Model.

Appendix A

Running coupling constants

In this Appendix the theory of β functions will be discussed for massless as well as massive theories. β functions can be calculated using the Callan-Symanzik equation, however in their simplest form they are only valid for dimensionless coupling constants. The Callan-Symanzik equation for dimensionless coupling constants is the topic of the first section. In the second section this theory will be generalized for dimensionful coupling constants, which will lead to a β function as function of the space-time dimension. In the third and last section the running of the SM will be discussed.

A.1 Callan-Symanzik equation

The Callan-Symanzik equation describes the idea that the renormalization scale (M) is an arbitrary scale, measurable quantities like the renormalized Green's functions ($G^{(n)}(x_1, \cdots, x_n)$) should not depend on it. This notion can be made precise by looking at the effect of an infinitesimal shift in M on $G^{(n)}$. For an interacting massless scalar field theory with a dimensionless coupling constant λ. The connected Green's function is defined as:

$$G^{(n)}(x_1, \cdots, x_n) = \langle\Omega|T\phi(x_1)\cdots\phi(x_n)|\Omega\rangle_c \tag{A.1}$$

This is the n-point correlation function. It describes the vacuum to vacuum amplitude for n fields. An infinitesimal change in M has to be accompanied by infinitesimal changes in ϕ and λ in order to make $G^{(n)}$ independent of the renormalization scale. We define the following infinitesimal shifts:

$$M \to M + \delta M \tag{A.2a}$$

$$\lambda \to \lambda + \delta\lambda \tag{A.2b}$$

$$\phi \to \phi(1 + \delta\eta) \tag{A.2c}$$

Since $G^{(n)}$ depends on λ as well as on M, an infinitesimal change in $G^{(n)}$ is defined by:

$$dG^{(n)} = \frac{\partial G^{(n)}}{\partial M}\delta M + \frac{\partial G^{(n)}}{\partial \lambda}\delta\lambda \tag{A.3}$$

Combining Eq. A.1 and Eq. A.2 the infinitesimal change in $G^{(n)}$ can also be written as:

$$dG^{(n)} = n\delta\eta G^{(n)} \tag{A.4}$$

Now combining the previous two equations with the notion that the Green's function should be independent of the renormalization scale we find the Callan-Symanzik equation:

$$\left[M\frac{\partial}{\partial M}+\beta\frac{\partial}{\partial\lambda}+n\gamma\right]G^{(n)}=0 \tag{A.5}$$

with

$$\beta\equiv M\frac{\delta\lambda}{\delta M} \tag{A.6a}$$

and

$$\gamma\equiv -M\frac{\delta\eta}{\delta M} \tag{A.6b}$$

The infinitesimal β function can be generalized for finite changes:

$$\beta = M\frac{\partial\lambda}{\partial M} \tag{A.7}$$

Equation A.5 shows that the β function can be determined by calculating the renormalized Greens functions.

A.2 Massive operators and different dimensions

The Callan-Symanzik equation derived above is valid for massless field theories. When the Lagrangian contains massive operators the CZ equation needs to be modified. The general idea remains the same:

"A change in renormalization scale needs to be accompanied by a change in coupling constants and field strengths such that physical quantities like the Green's function are invariant under changes in the renormalization scale."

In the following this principle will be applied to a massive renormalized field theory in d dimensions of the form:

$$\mathcal{L}=\mathcal{L}_M+\rho_i M^{d-di}\mathcal{O}^i_M, \tag{A.8}$$

with $\mathcal{L}_M$ the massless renormalized field theory, renormalized at a scale M. The dimension of the operator $\mathcal{O}^i_M$ is d_i, ρ_i is dimensionless.

Finding the CZ equation we look again at the Green's function with l_i insertions of the operator $\mathcal{O}^i_M$:

$$G^{(n)}=\langle\phi_1,\cdots\phi_n(\rho_i M^{d-d_i}\mathcal{O}^i_M)^l\rangle \tag{A.9}$$

The fields transform under an infinitesimal change as previously defined. And:

$$\rho_i\to\rho_i+\delta\rho_i \tag{A.10a}$$

$$\mathcal{O}^i_M\to\mathcal{O}^i_M(1+\delta\eta_i) \tag{A.10b}$$

Results in:

$$dG^{(n)}=\left(n\delta\eta+l_i\delta\eta_i+l_i\frac{\delta\rho_i}{\rho_i}+(d-d_i)l\frac{\delta M}{M}\right)G^{(n)} \tag{A.11}$$

and also:

$$dG^{(n)} = \frac{\partial G}{\partial M}\delta M + \frac{\partial G}{\partial \rho_i}\delta\rho_i \tag{A.12}$$

Combining results in:

$$\left(M\frac{\partial}{\partial M} + n\gamma + \left[\rho_i\gamma_i - (d-d_i)\rho_i\right]\frac{\partial}{\partial\rho_i}\right)G^{(n)} = 0 \tag{A.13}$$

With:

$$\gamma = -M\frac{\delta\eta}{\delta M},\ \ \gamma_i = -M\frac{\delta\eta_i}{\delta M} \tag{A.14}$$

We define:

$$\beta_i = \rho_i\gamma_i - (d-d_i)\rho_i \tag{A.15}$$

In a field theory with dimensionful coupling constant λ, the following general law can be stated for the β function of a dimensionless coupling constant g ($\lambda = gM^x$), in d dimensions:

$$\beta^{(d)}(g) = \beta^{d_M}(g) - x\cdot g + \mathcal{O}(x^2g^2) \tag{A.16}$$

with $x =$ powers of mass needed to make g dimensionless and $d_M =$ dimension in which λ is dimensionless.

The previous can for example be applied to a scalar field theory in d dimensions of the form:

$$\mathcal{L} = \mathcal{L}_0 - \frac{1}{2}\rho_m M^2\phi_M^2 - \frac{1}{4}\lambda M^{4-d}\phi_M^4, \tag{A.17a}$$

with all the coupling constants dimensionless. This results in the following CZ equation:

$$\left(M\frac{\partial}{\partial M} + n\gamma + \beta_m\frac{\partial}{\partial\rho_m} + \beta\frac{\partial}{\partial\lambda}\right)G^{(n)} = 0 \tag{A.17b}$$

with

$$\beta_m = \rho_m\gamma_m - 2\rho_m \tag{A.17c}$$

$$\beta^{(d)} = \lambda\gamma_\lambda - (4-d)\lambda \equiv \beta^{(4)}(\lambda) + (d-4)\lambda \tag{A.17d}$$

A.3 Standard Model

In this section the β functions of the SM parameters will be given. Subsequently the initial values of these parameters will be determined. These β functions and initial values are used to plot the running of the SM coupling constants in figure 3.1.

β functions
The one-loop β functions of the SM couplings can be found in for example [19] and [23]. For completeness they are also stated below.

$$\begin{aligned}
\beta_{g_1} &= \frac{1}{16\pi^2}\frac{41}{6}g_1^3 \\
\beta_{g_2} &= -\frac{1}{16\pi^2}\frac{19}{6}g_2^3 \\
\beta_{g_3} &= -\frac{7}{16\pi^2}g_3^3 \\
\beta_{y_t} &= \frac{1}{16\pi^2}\left(-\frac{17}{12}g_1^2 y_t - \frac{9}{4}g_2^2 y_t - 8g_3^2 y_t + \frac{9}{2}y_t^3\right) \\
\beta_\lambda &= \frac{1}{16\pi^2}\left[24\lambda^2 + 6(2\lambda - y_t^2)y_t^2 - 3\lambda(g_1^2 + 3g_2^2) + \frac{3}{8}(g_1^4 + 3g_2^4 + 2g_1^2 g_2^2)\right]
\end{aligned}$$

Standard Model masses
To determine the running of the couplings constants an initial value is needed. The initial values can be calculated once the particle masses are known. The table below summarizes all relevant masses (from [1] and [5]).

Particle	Theory (SM)	Experiment
t	$m_t = \frac{1}{\sqrt{2}}y_t\langle\phi_0\rangle$	$m_t = 173.21$ GeV
$W^\pm$	$M_W^2 = \frac{1}{4}g^2\langle\phi_0\rangle^2$	$M_W = 80.385$ GeV
Z	$M_Z^2 = \frac{1}{4}(g^2 + g'^2)\langle\phi_0\rangle^2$	$M_Z = 91.188$ GeV
γ	$M_\gamma^2 = 0$	$M_\gamma = 0$
H	$M_H^2 = -2\mu^2 = 2\lambda\langle\phi_0\rangle^2$	$M_H = 125$ GeV

TABLE A.1: Theoretical and experimental value of some SM particles

In this table g is the $SU(2)$ coupling constant (also called g_2) and g' the hypercharge $U(1)$ coupling constant (also called g_1). y_t is the Yukawa coupling of the top quark.

Electroweak VEV
$v = \langle\phi_0\rangle$ is determined by comparing the amplitude of muon decay in the Fermi theory with the amplitude of muon decay in the SM:

$$\frac{1}{\sqrt{2}}G_F = \frac{g^2}{8M_W^2}$$

Using the SM result that $M_W^2 = \frac{1}{4}g^2v^2$ the VEV is obtained:

$$v = (\sqrt{2}G_F)^{-1/2}$$

From muon lifetime experiments the Fermi coupling is determined to be $G_F = 1.166 \cdot 10^{-5}$ GeV^{-2}. Thus, $v = 246$ GeV.

Initial values
By combining the value of the VEV with the previous table the following indirect values for the VEV and the coupling constants can be determined.

Parameter	Value
$v = \langle \phi_0 \rangle$	246 GeV
g'	0.349
g	0.654
y_t	0.996
λ	0.129

TABLE A.2: Indirectly determined values for couplings and VEV

These are taken as the initial values of the coupling constants at the scale $\mu = 173$ GeV. Additionally the initial value of g_3 is taken to be 1.26 at this scale.

Appendix B

Renormalization

The theory of renormalization can also be found in most textbook, e.g. [1] and [29]. The basic principles of renormalization will be shown by means of an example. For simplicity the theory describing an interacting massive scalar field is used.

The bare Lagrangian describing an interacting massive scalar field is given by:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_0)^2 - \frac{1}{2}m_0^2\phi_0^2 - \frac{\lambda_0}{4!}\phi_0^4 \tag{B.1}$$

The fields and constants in this equation are the bare quantities. When calculating loop diagrams using this Lagrangian infinities are encountered. These infinities are dealt with by proposing that the parameters in the Lagrangian are not measurable, the quantities we do measure are the physical parameters which are in general energy dependent.

The Lagrangian can be written in these physical parameters by the following changes:

$$\begin{aligned}\phi_0 &= Z^{1/2}\phi \\ \delta_Z &= Z - 1 \\ \delta_m &= m_0^2 Z - m^2 \\ \delta_\lambda &= \lambda_0 Z^2 - \lambda\end{aligned}$$

This results in a Lagrangian of the form:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 + \frac{1}{2}\delta_Z(\partial_\mu \phi)^2 - \frac{1}{2}\delta_m\phi^2 - \frac{\delta_\lambda}{4!}\phi^4 \tag{B.2}$$

All constants with subscript 0, like λ_0, describe the bare constants. All constants without this subscript describe the renormalized quantities.

The infinities are absorbed by the renormalization terms, making the physical parameters finite. For every loop order the infinities can be determined and absorbed in the renormalization constants, using specific renormalization conditions.

The renormalization terms can be expanded in loop order like:
$\delta_m = \delta_m^{(1)} + \delta_m^{(2)} + \delta_m^{(3)} + \cdots$

At tree level the renormalization terms are zero, by definition.

In the thesis several different renormalization conditions are used. Usually they boil down to setting the mass and vertices terms equal to the renormalized values at some specific renormalization scale (M).

To deal with the infinities popping up in calculations one also needs a systematic way to calculate them. There are several different ways to calculate infinities, some more useful than others. In this thesis only the "UV cut-off" is used.

Note: The one and two loop β functions are independent of the choice of regularization.

UV cut-off

The first method does exactly what we would expect from its name; instead of taking the integral to infinity the integral is taken to Λ. Some integrals can be calculated very simply using this approach, especially integrals in massless theories. However the cut off destroys gauge invariance and is therefore not always applicable. Since the theories in this thesis are all effective field theory the UV cut-off can be applied.

An example of an integral which is easy to calculate using a UV cut-off is:

$$\int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2} \tag{B.3}$$

To avoid the poles this integral is Wick rotated into the complex plane ($p^0 \to ip_E^0$). This reduces the integral to a simple spherical integral in four (Euclidean) dimension. The area of a sphere in four dimensions is given by $\int d\Omega = 2\pi^2$.

Below a step by step calculation of the integral above:

$$\begin{aligned}
\int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2} &= \int \frac{id^4p_E}{(2\pi)^4}\frac{i}{-p_E^2} \\
&= \int d\Omega \int \frac{idp_E}{(2\pi)^4}\frac{ip_E^3}{-p_E^2} \\
&= \frac{1}{8\pi^2}\int dp_E p_E \\
&= \frac{1}{16\pi^2}\Lambda^2
\end{aligned}$$

Appendix C

Mathematics

This Appendix summarizes some of the mathematics used in this thesis.

Geometric series

$$\sum_{n=1}^{\infty} \frac{1}{n} x^n = -\log(1-x)$$

Gamma matrix identities

- $\text{Tr}(\gamma^\nu, \gamma^\mu \cdots \gamma^\rho) = 0$ for uneven number of gamma matrices
- $(\gamma^5)^2 = 1$
- $\{\gamma^\mu \gamma^\nu\} \equiv \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\nu = 2\eta^{\mu\nu}\mathrm{I}$

Polygon Integral

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{i}{2} \int \frac{d^4k}{(2\pi)^4} \frac{1}{n} \left(\frac{1/2 \cdot \lambda \phi_c^2}{k^2} \right)^n &= -\frac{i}{2} \int \frac{d^4k}{(2\pi)^4} \log \left(1 - \frac{\lambda \phi_c^2}{2k^2} \right) \\
&= \frac{1}{2} \int \frac{d^4k_E}{(2\pi)^4} \log \left(1 + \frac{\lambda \phi_c^2}{2k_E^2} \right) \\
&= \frac{1}{2^4 \pi^2} \int dk_E k_E^3 \log \left(1 + \frac{\lambda \phi_c^2}{2k_E^2} \right) \\
&= \frac{1}{2^6 \pi^2} \left[\lambda \phi_c^2 \Lambda^2 + \frac{\lambda^2 \phi_c^4}{4} \left(\log \left(\frac{\lambda \phi_c^2}{2\Lambda^2} \right) - \frac{1}{2} \right) \right] \qquad \text{(C.1)}
\end{aligned}$$

To calculate the integral of the first line one has to rotate from Minkowski spacetime (x^0, x^1, x^2, x^3) to Euclidean space $(x_E^0, x_E^1, x_E^2, x_E^3)$. This is achieved through a Wick rotation which is defined as:

$$x^0 = -ix_E^0$$

$$k^0 = ik_E^0$$

such that

$$x^2 = x_\mu x^\mu = (x^0)^2 - (x^i)^2 2 = -(x_E)^2$$

$$k^2 = k_\mu k^\mu = (k^0)^2 - (k^i)^2 = -(k_E)^2.$$

The integral of the third line is calculated using a UV cut-off and the standard integral:

$$\int_0^\Lambda dx x^3 \log(1 + a/x^2) = \frac{1}{4}\left[a(\Lambda^2 - a\log(a + \Lambda^2)) + \Lambda^4 \log(a/\Lambda^2 + 1)\right] + \frac{1}{4}a^2 \log(a)$$

which reduces to the following when Λ is big:

$$\begin{aligned}
\int_0^\Lambda x^3 \log(1 + a/x^2) &\approx \frac{1}{4}\left[a\Lambda^2 - a^2\log(\Lambda^2) + \Lambda^4\left(\frac{a}{\Lambda^2} - \frac{a^2}{2\Lambda^4}\right) + a^2\log(a)\right] \\
&= \frac{1}{4}\left[2a\Lambda^2 - \frac{a^2}{2} + a^2\log(a/\Lambda^2)\right] \\
&= \frac{1}{4}\left[2a\Lambda^2 + a^2\left(\log(a/\Lambda^2) - \frac{1}{2}\right)\right]
\end{aligned}$$

Bibliography

[1] M. Peskin and D. Schroeder. *An introduction to quantum field theory*. 1995.

[2] T.P. Cheng and L.F. Li. *Gauge theory of elementary particle physics*. Clarendon press Oxford, 1984.

[3] D. Griffiths. *Introduction to Elementary Particles*. Physics textbook. Wiley, 2008. ISBN: 9783527406012.

[4] ATLAS Collaboration. *Standard Model Total Production Cross section Measurements summary plot*. https://atlas.web.cern.ch/Atlas/ .

[5] C. Patrignani et al. "Review of Particle Physics". In: *Chin. Phys.* C40.10 (2016), p. 100001. DOI: 10.1088/1674-1137/40/10/100001.

[6] Joan Elias-Miro et al. "Higgs mass implications on the stability of the electroweak vacuum". In: *Phys. Lett.* B709 (2012), pp. 222–228. DOI: 10.1016/j.physletb.2012.02.013. arXiv: 1112.3022 [hep-ph].

[7] O. Lebedev and A. Westphal. "Metastable Electroweak Vacuum: Implications for Inflation". In: *Phys. Lett.* B719 (2013), pp. 415–418. DOI: 10.1016/j.physletb.2012.12.069. arXiv: 1210.6987 [hep-ph].

[8] J. Elias-Miro et al. "Stabilization of the Electroweak Vacuum by a Scalar Threshold Effect". In: *JHEP* 06 (2012), p. 031. DOI: 10.1007/JHEP06(2012)031. arXiv: 1203.0237 [hep-ph].

[9] D. Buttazzo et al. "Investigating the near-criticality of the Higgs boson". In: *JHEP* 12 (2013), p. 089. DOI: 10.1007/JHEP12(2013)089. arXiv: 1307.3536 [hep-ph].

[10] F. Brümmer. *Lecture notes on Beyond the Standard Model Physics, given at the DESY Summer Student Programme*. https://summerstudents.desy.de/hamburg/e224192/e225266/Beyond-the-Standard-Model-1.pdf. 2016.

[11] G. F. Giudice. "Naturalness after LHC8". In: *PoS* EPS-HEP2013 (2013), p. 163. arXiv: 1307.7879 [hep-ph].

[12] W.A. Bardeen. "On naturalness in the standard model". In: *preprint FERMILAB-CONF-95-391-T*. 1995.

[13] G. 't Hooft. "Naturalness, chiral symmetry, and spontaneous chiral symmetry breaking". In: *Recent developments in gauge theories. Proceedings of the NATO Advanced Study Institute on recent developments in gauge theories, held in Cargèse, Corsica, August 26-September 8, 1979* (1980), pp. 135–157.

[14] C. Englert et al. "Emergence of the Electroweak Scale through the Higgs Portal". In: *JHEP* 04 (2013), p. 060. DOI: 10.1007/JHEP04(2013)060. arXiv: 1301.4224 [hep-ph].

[15] T. Behnke et al. "The International Linear Collider Technical Design Report - Volume 1: Executive Summary". In: (2013). arXiv: 1306.6327.

[16] S.R. Coleman and E.J. Weinberg. "Radiative Corrections as the Origin of Spontaneous Symmetry Breaking". In: *Phys. Rev.* D7 (1973), pp. 1888–1910. DOI: 10.1103/PhysRevD.7.1888.

[17] S. Weinberg. *The quantum theory of fields*. Vol. 2. Cambridge university press, 1996.

[18] E.J. Weinberg. "Radiative corrections as the origin of spontaneous symmetry breaking". PhD thesis. Harvard U., 1973. arXiv: hep-th/0507214 [hep-th].

[19] A.J. Helmboldt et al. "Minimal Conformal Extensions of the Higgs Sector". In: (2016). arXiv: 1603.03603 [hep-ph].

[20] E. Gildener and S. Weinberg. "Symmetry Breaking and Scalar Bosons". In: *Phys. Rev.* D13 (1976), p. 3333. DOI: 10.1103/PhysRevD.13.3333.

[21] R. Foot, A. Kobakhidze, and R.R. Volkas. "Electroweak Higgs as a pseudo-Goldstone boson of broken scale invariance". In: *Phys. Lett.* B655 (2007), pp. 156–161. DOI: 10.1016/j.physletb.2007.06.084. arXiv: 0704.1165 [hep-ph].

[22] H. Hendriksen. "A minutious introduction to the Minimal Scale Invariant Standard Model". Master Thesis. Van Swinderen Institute, University of Groningen, 2016.

[23] M. Shaposhnikov and C. Wetterich. "Asymptotic safety of gravity and the Higgs boson mass". In: *Phys. Lett.* B683 (2010), pp. 196–200. DOI: 10.1016/j.physletb.2009.12.022. arXiv: 0912.0208 [hep-th].

[24] S. Weinberg. "Ultraviolet Divergences in Quantum Theories of Gravitation". In: *General Relativity: An Einstein Centenary Survey*. 1980, pp. 790–831.

[25] H.J. Warringa. "Thermodynamics of QCD-inspired theories". 2006.

[26] Alessandro Codello and Roberto Percacci. "Fixed Points of Nonlinear Sigma Models in d>2". In: *Phys. Lett.* B672 (2009), pp. 280–283. DOI: 10.1016/j.physletb.2009.01.032. arXiv: 0810.0715 [hep-th].

[27] F. Bazzocchi et al. "Fermions and Goldstone bosons in an asymptotically safe model". In: *Phys. Lett.* B705 (2011), pp. 388–392. DOI: 10.1016/j.physletb.2011.10.029. arXiv: 1105.1968 [hep-ph].

[28] G. M. Pelaggi et al. "Naturalness of asymptotically safe Higgs". In: (2017). arXiv: 1701.01453 [hep-ph].

[29] A. Zee. *Quantum field theory in a nutshell*. Princeton university press, 2010.