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A Distributed Observer for a Linear Time-Invariant System

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Abstract

This paper deals with the construction of a distributed observer. A distributed observer is a network of m observers, where each observer uses only local information to approximate the state of a given system. Sufficient conditions for the existence of a distributed observer are found. In particular, if the graph underlying the observer network is strongly connected, and the system is globally observable, it is shown that there exists a distributed observer of dimension $mn + m - 1$, where n is the dimension of the observed system. Furthermore, it is shown that the convergence rates of the estimates can be preassigned at arbitrary values. The proofs provide ideas for a construction strategy, which is described in the end of the paper.

Contents

1 Introduction

In many control problems it is required to know the state of the system to be controlled. In practical applications, however, it is often the case that the state is not available for measurement, hence it has to be approximated by an observer. An observer is a linear system that yields an asymptotically correct estimate of the state of a given linear system by using the available output measurement. For the system $\dot{x} = Ax$, with output $y = Cx$, it is well known that a full-state observer exists, provided that the matrix pair (C, A) is detectable. Methods for constructing an observer of least dimension are also known [1].

The classic observer requires the output measurement to be available at a single location, but in practice it might happen that the measurement is acquired by multiple agents. This motivates the generalization of the observer problem to a network of m observers, each of which can observe part of some global output measurement. We require that each observer arrives at an asymptotically correct estimate of the state x, while using information only from the observers in its reception range. We call such an observer a distributed observer. The goal is to provide sufficient conditions for the existence of a distributed observer together with construction techniques. We begin by formally defining the problem and establishing concrete conditions for the existence of an observer. These conditions come in the form of equations with certain restrictions stemming from the network topology and the definition of an observer. The equations are then simplified and related to the well studied topics of decentralized stabilization and decentralized control. Finally, using the work in [2], [3], [4] and [5], we arrive at the final result of this paper.

2 Notation and Preliminaries

This section is meant to introduce some of the notation and preliminary concepts that will be used throughout this paper. To begin with, let I_n and 0_n denote the identity and zero matrix, respectively, of size $n \times n$, and let $\mathbb{1}_m$ denote the m-dimensional vector with ones as its elements, that is $\mathbb{1}_m = \begin{pmatrix} 1 & \cdots & 1 \end{pmatrix}^T$. For matrices $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{k \times \ell}$, the Kronecker product $A \otimes B$ is the $mk \times nl$ matrix defined by

$$A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix},$$

where a_{ij} is the element in row i and column j of A. The Kronecker product has the following properties:

1. $(A + B) \otimes C = A \otimes C + B \otimes C$, if the addition is defined.

2. $A \otimes (B + C) = A \otimes B + A \otimes C$, if the addition is defined.

3. $(A \otimes B)(C \otimes D) = AC \otimes BD$, if the multiplication is defined.

4. $\operatorname{rank}(A \otimes B) = \operatorname{rank} A \operatorname{rank} B$.

The discussion in this paper requires some basic knowledge of graph theory. A graph is defined as an ordered pair of sets $(\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is the set of vertices and $\mathcal{E}$ is the set of edges. The set of vertices is just a labelling set, while the set of edges consists of ordered pairs of vertices, indicating the vertices the edge connects as well as the direction of the edge. For example, if i and j are vertices, then $(j, i) \in \mathcal{E}$ if and only if there is an edge from j to i. Often we will not explicitly define the set of edges, but only describe the condition under which there is an edge from j to i. For notational convenience, we will denote the graph with a symbol, like $\mathbb{G}$, and write ij instead of (j, i) for the edge from j to i. A graph is said to be *strongly connected* if there is a path between any two vertices in the graph. A path from j to i is a sequence of vertices $(v_1, \ldots, v_p)$, where $v_1 = j$ and $v_p = i$, such that there is an edge from v_k to v_{k+1} for all $k \in \{1, \ldots, p-1\}$. Similarly, a graph is said to be *weakly connected* if there is a weak path between any two nodes, where a weak path is a path in which the direction of the edges connecting consecutive vertices is irrelevant. The *incidence matrix* of a graph is an $m \times n$ matrix M, where m and n are the number of vertices and edges, respectively. Each edge corresponds to a column in M and the column corresponding to the edge from j to i is given by $e_j - e_i$, where $e_i \in \mathbb{R}^m$ is the ith standard basis vector. The rank of the incidence matrix depends on the connectivity properties of the graph. In particular, Theorem 8.3.1 in [2] states that $\operatorname{rank} M = m - d$ where d is the number of weakly connected components of the graph.

Finally, we will need some basic concepts from systems theory. Consider the linear system

$$\begin{aligned} \dot{x} &= Ax + Bu, \quad && x \in \mathbb{R}^n, \ u \in \mathbb{R}^q, \\ y &= Cx, && y \in \mathbb{R}^p. \end{aligned}$$

which is completely determined by the matrices A, B and C, and will be denoted by (C, A, B). The pair (A, B) is called *controllable* if

$$\operatorname{rank} \begin{pmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{pmatrix} = n,$$

and it has *controllability index* m if m is the smallest integer such that

$$\operatorname{rank} \begin{pmatrix} B & AB & A^2B & \cdots & A^{m-1}B \end{pmatrix} = n.$$

The definition of controllability index makes sense only for a controllable pair, and in such a case the controllability index is not greater than n and not smaller than $\frac{n}{q}$, where q is the number of columns in B. On the other hand, the pair (C, A) is called *observable* if the pair (A^T, C^T) is controllable, and it has *observability index* m if (A^T, C^T) has controllability index m. The system (C, A, B) is called *minimal* if (C, A) is observable and (A, B) is controllable. Throughout this paper, we will consider the system without input, given by

$$\begin{aligned} \dot{x} &= Ax, \quad && x \in \mathbb{R}^n, \\ y &= Cx, && y \in \mathbb{R}^p, \end{aligned}$$

where the pair (C, A) is assumed to be observable. A linear system of the form

$$\dot{z} = Hz + Ky, \quad z \in \mathbb{R}^n, \tag{1}$$

is called an *observer* if the state z is an asymptotically correct estimate of the state x. In other words, if we define the estimation error as $\varepsilon = z - x$, then the linear system in (1) is an observer if $\varepsilon \to 0$ as $t \to \infty$. The error dynamics, given by

$$\dot{\varepsilon} = Hz + Ky - Ax = H\varepsilon + (H + KC - A)x,$$

suggest that if the equation $H = A - KC$ is satisfied and $\sigma(H) \in \mathbb{C}^-$, then

$$\dot{\varepsilon} = H\varepsilon \quad \Rightarrow \quad \varepsilon \to 0 \text{ as } t \to \infty.$$

Given that (C, A) is observable, there exists a matrix $K \in \mathbb{R}^{n\times p}$ such that $\sigma(A - KC) \in \mathbb{C}^-$, hence finding such a K and setting $H = A - KC$ makes the linear system in (1) an observer. The observer design equation in the following chapter will be derived in a very similar way.

3 Observer Design Equation

In this paper we are interested in constructing m observers, each of which has access to only part of the output of the system

$$\begin{aligned} \dot{x} &= Ax, \quad x \in \mathbb{R}^n, \\ y &= Cx, \quad y \in \mathbb{R}^p. \end{aligned}$$

In particular, let $C = \begin{pmatrix} C_1^T & \dots & C_m^T \end{pmatrix}^T$ and $y = \begin{pmatrix} y_1^T & \cdots & y_m^T \end{pmatrix}^T$, and suppose that observer i has access to

$$y_i = C_i x, \quad y_i \in \mathbb{R}^{p_i}.$$

We assume that the pair (C, A) is observable, but making this assumption for any of the pairs (C_i, A) would be too restrictive. Instead, we allow the observers to communicate in order to estimate the state of the system, thus creating a communication network. If we let $\mathcal{N}_i$ be the set of observers in the range of observer i, together with i itself, then the network can be formally described by a graph $\mathbb{Y}$ with vertices in $\mathcal{M} = \{1, \dots, m\}$ and set of edges defined by $\mathcal{E} = \{ij : \ i \in \mathcal{M}, \ j \in \mathcal{N}_i\}$.

We are looking to construct a family of m linear systems of the form

$$\dot{z}_i = \sum_{j\in\mathcal{N}_i} H_{ij} z_j + K_{ij} y_j, \quad z_i \in \mathbb{R}^n, \ i \in \mathcal{M}, \tag{2}$$

such that every state z_i, $i \in \mathcal{M}$, is an asymptotically correct estimate of the state x. If we define the estimation error of observer i as $\varepsilon_i = z_i - x$, then the family of linear systems in (2) is called a *distributed observer* if $\varepsilon_i \to 0$ as $t \to \infty$ for all $i \in \mathcal{M}$. To construct a distributed observer, we need to find matrices H_{ij} and K_{ij}, for all $ij \in \mathcal{E}$, such that all estimation errors converge to zero.

To begin with, we will establish more concrete conditions that need to be satisfied in order for the family of linear systems (2) to be a distributed observer. Let $H = (H_{ij})$ denote the $mn \times mn$ matrix defined by

$$H = \begin{pmatrix} H_{11} & \cdots & H_{1m} \\ \vdots & \ddots & \vdots \\ H_{m1} & \cdots & H_{mn} \end{pmatrix},$$

where $H_{ij} = 0_n$ if $ij \notin \mathcal{E}$. Similarly, let $K = (K_{ij})$ and $K_{ij} = 0$ if $ij \notin \mathcal{E}$, so that the joint observer equations can be expressed more compactly as

$$\dot{z} = Hz + Ky, \quad z = \begin{pmatrix} z_1 \\ \vdots \\ z_m \end{pmatrix} \in \mathbb{R}^{mn}, \ y = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} \in \mathbb{R}^p.$$

The estimation errors can be collected in a vector $\varepsilon \in \mathbb{R}^{mn}$, where

$$\varepsilon = \begin{pmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_m \end{pmatrix} = \begin{pmatrix} z_1 \\ \vdots \\ z_m \end{pmatrix} - \begin{pmatrix} x \\ \vdots \\ x \end{pmatrix} = z - \mathbb{1}_m \otimes x,$$

and the requirement for a distributed observer is that $\varepsilon \to 0$ as $t \to \infty$. Just like in the single observer case, we look at the error dynamics given by

$$\begin{aligned} \dot{\varepsilon} &= Hz + Ky - \mathbb{1}_m \otimes Ax \\ &= H\varepsilon + (H(\mathbb{1}_m \otimes I_n) + KC - \mathbb{1}_m \otimes A)x. \end{aligned}$$

The latter suggests that if the matrices H and K are such that the equation

$$H(\mathbb{1}_m \otimes I_n) = \mathbb{1}_m \otimes A - KC \tag{3}$$

is satisfied, then $\dot{\varepsilon} = H\varepsilon$, hence $\varepsilon \to 0$, provided that $\sigma(H) \in \mathbb{C}^-$. We will refer to (3) as the *design equation* for our distributed observer. To emphasize the fact that H and K need to have a specific zero pattern due to communication restrictions, we can write out (3) as the set of equations

$$\sum_{j \in \mathcal{N}_i} H_{ij} = A - \sum_{j \in \mathcal{N}_i} K_{ij} C_j, \quad i \in \mathcal{M}.$$

Although these equations are not necessary for the existence of an observer, they will prove to be sufficiently general for the purpose of this paper.

4 Distributed Observer

In this section we propose a method for the construction of a distributed observer in the case when $K_{ij} = 0$ for $i \neq j$. This means that each observer is allowed to use information from its own output only and the matrix K is block diagonal. The approach we are taking involves relating condition from the design equation (3) to the well studied topic of decentralized stabilization [3]. To achieve this, we need to express H as the dynamics matrix obtained after stabilization by output feedback, which should not be surprising given that H has to be stable.

In our new set up, the design equation (3) can be expanded to m equations that read

$$\sum_{j=1}^{m} H_{ij} = A - K_{ii} C_i, \quad i \in \mathcal{M},$$

and we can rearrange terms to get

$$H_{ii} = A - K_{ii} C_i - \sum_{\substack{j=1 \\ j \neq i}}^{m} H_{ij}, \quad i \in \mathcal{M}.$$

Using these equations, we can obtain an expression for H such that the design equation (3) is implicitly satisfied. With this in mind, note that substituting H_{ii} from the equation above in the ith block row of H yields

$$\begin{aligned}\begin{pmatrix}H_{i1} & \cdots & H_{ii} & \cdots & H_{im}\end{pmatrix} = A\begin{pmatrix}0 & \cdots & I & \cdots & 0\end{pmatrix} + H_{i1}\begin{pmatrix}I & \cdots & -I & \cdots & 0\end{pmatrix} + \cdots \\ \cdots - K_{ii}\begin{pmatrix}0 & \cdots & C_i & \cdots & 0\end{pmatrix} + \cdots + H_{im}\begin{pmatrix}0 & \cdots & -I & \cdots & I\end{pmatrix}.\end{aligned}$$

To arrive at something more comprehensible, let e_i denote the ith standard basis vector in $\mathbb{R}^m$ and consider the matrices C_{ij} and $F_{ij},\ i \in \mathcal{M},\ j \in \mathcal{M}$, defined by

$$C_{ij} = \begin{cases} e_i^T \otimes C_i, & \text{if } i = j, \\ (e_j^T - e_i^T) \otimes I_n, & \text{otherwise} \end{cases}, \qquad F_{ij} = \begin{cases} -K_{ii}, & \text{if } i = j, \\ H_{ij}, & \text{otherwise.} \end{cases}$$

Now we can write the ith block row of H like

$$\begin{pmatrix}H_{i1} & \cdots & H_{ii} & \cdots & H_{im}\end{pmatrix} = A\begin{pmatrix}0 & \cdots & I & \cdots & 0\end{pmatrix} + \sum_{j=1}^{m} F_{ij}C_{ij}.$$

If, in addition, we define $B_i = e_i \otimes I_n$, then H can be written out like

$$\begin{aligned} H &= \sum_{i=1}^{m} B_i \begin{pmatrix}H_{i1} & \cdots & H_{im}\end{pmatrix} \\ &= I_m \otimes A + \sum_{i=1}^{m}\sum_{j=1}^{m} B_i F_{ij} C_{ij}. \end{aligned}$$

Taking into account the restriction that $F_{ij} = 0$ if $(i,j) \notin \mathcal{E}$, results in

$$H = I_m \otimes A + \sum_{ij \in \mathcal{E}} B_i F_{ij} C_{ij}, \tag{4}$$

and in this form it should be clear that H has the required zero pattern. Additionally, given the definition of the matrices $F_{ij},\ ij \in \mathcal{E}$, using (4) to define H and setting $K = \text{blockdiag}(F_{ii} : i \in \mathcal{M})$, implies that the design equation (3) is implicitly satisfied. Therefore, the problem is reduced to finding $F_{ij},\ ij \in \mathcal{E}$, such that $\sigma(H) \in \mathbb{C}^-$, where H is defined by (4). Note that H can be seen as the dynamics matrix obtained after applying feedback laws $u_{ij} = F_{ij}y_{ij}$ to the system

$$\dot{\varepsilon} = (I_m \otimes A)\varepsilon + \sum_{ij \in \mathcal{E}} B_i u_{ij}, \qquad y_{ij} = C_{ij}\varepsilon,\ ij \in \mathcal{E}, \tag{5}$$

hence the problem is translated to stabilization of the error dynamics. In this context, we can turn to methods from decentralized stabilization [3] and decentralized control [4] to arrive at the final result of this paper. At this point we make a small remark about the definition of C_{ij} for $ij \in \mathcal{E}$ and $i \neq j$. The vector c_{ij}, defined as the row in the transpose of the incidence matrix of the graph $\mathbb{Y}$ corresponding to the edge from j to i, can be written as $c_{ij} = e_j^T - e_i^T$, thus we have $C_{ij} = c_{ij} \otimes I_n,\ ij \in \mathcal{E},\ i \neq j$. The fact that the matrices $C_{ij},\ ij \in \mathcal{E}$, are related to the incidence matrix of $\mathbb{Y}$ will be used in the proofs to come later in this paper.

4.1 Decentralized stabilization

As already mentioned, the problem of constructing a distributed observer is reduced to finding appropriate matrices F_{ij}, $ij \in \mathcal{E}$, such that the system in (5) is stabilized. One way to do this is to first find matrices F_{ij}, $ij \in \mathcal{E}$, such that the system $(C_{k\ell}, H, B_k)$ is minimal for some $k\ell \in \mathcal{E}$. Having accomplished this, we can use standard centralized techniques to construct a stabilizing controller for the system $(C_{k\ell}, H, B_k)$. With this in mind, consider the following theorem:

Theorem 1. *If the graph $\mathbb{Y}$ is strongly connected, then there exist matrices F_{ij}, $ij \in \mathcal{E}$, such that for all $k\ell \in \mathcal{E}$ the system $(C_{k\ell}, H, B_k)$ is minimal with controllability index m, where m is the number of vertices of $\mathbb{Y}$.*

The proof of this theorem is fairly involved and will be presented in the next subsection but, in the meantime, we will write down the equations for a distributed observer by using this result. Firstly, note that although the theorem makes a statement for all $k\ell \in \mathcal{E}$, we need to choose a particular $k\ell$ to use for stabilization. It will be shown that the resulting observer equations differ depending on whether $k = \ell$ or $k \neq \ell$, but in both cases the total dimension of the distributed observer will be the same.

In any case, we can fix a pair $k\ell \in \mathcal{E}$ and consider the system $(C_{k\ell}, H, B_k)$ given by

$$\begin{aligned} \varepsilon &= H\varepsilon + B_k u_{k\ell}, \\ y_{k\ell} &= C_{k\ell}\varepsilon \end{aligned}$$

where H is defined by (4). This is the system that results after applying feedback laws $u_{ij} = F_{ij} y_{ij}$ to the system in (5) and neglecting the unnecessary input and output channels. Using the techniques described in [5], we can construct a stabilizing controller of the form

$$\begin{aligned} \dot{\bar{z}} &= \bar{A}\bar{z} + \bar{B}y_{k\ell}, \\ u_{k\ell} &= \bar{C}\bar{z} + \bar{D}y_{k\ell} \end{aligned}$$

where $\bar{z} \in \mathbb{R}^{m-1}$ due to the fact that m is the controllability index of the pair $(H, B_{k\ell})$. The resulting stable closed loop system is given by

$$\begin{pmatrix} \dot{\varepsilon} \\ \dot{\bar{z}} \end{pmatrix} = \begin{pmatrix} H + B_k\bar{D}C_{k\ell} & B_k\bar{C} \\ \bar{B}C_{k\ell} & \bar{A} \end{pmatrix} \begin{pmatrix} \varepsilon \\ \bar{z} \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} \varepsilon \\ \bar{z} \end{pmatrix} \to 0 \quad \text{as} \quad t \to \infty,$$

hence, in particular, ε converges to zero as $t \to \infty$. It is not yet clear how we can use this in the equations for our distributed observer. Unlike in the beginning, where we had $\dot{\varepsilon} = H\varepsilon$, we now have

$$\dot{\varepsilon} = (H + B_k\bar{D}C_{k\ell})\varepsilon + B_k\bar{C}\bar{z},$$

together with the dynamics of the stabilizing controller

$$\dot{\bar{z}} = A\bar{z} + \bar{B}C_{k\ell}\varepsilon.$$

Nevertheless, if we impose that $\varepsilon = z - \mathbb{1}_m \otimes x$ and use this to derive the observer equations, then the resulting estimation error will be given by ε, hence it will converge to zero as $t \to \infty$.

With this in mind, let $K = \text{blockdiag}(F_{ii} : \ i \in \mathcal{M})$, so that the design equation (3) is implicitly satisfied. The observer equations would have to be such that

$$\begin{aligned}\dot{z} = \dot{\varepsilon} + \mathbb{1} \otimes \dot{x} &= (H + B_k\bar{D}C_{k\ell})\varepsilon + B_k\bar{C}\bar{z} + (\mathbb{1}_m \otimes A)x \\ &= Hz + B_k\bar{D}C_{k\ell}\varepsilon + B_k\bar{C}\bar{z} + (\mathbb{1}_m \otimes A - H(\mathbb{1}_m \otimes I_n))x \\ &= Hz + B_k\bar{D}C_{k\ell}\varepsilon + B_k\bar{C}\bar{z} + KCx \\ &= Hz + Ky + B_k\bar{C}\bar{z} + B_k\bar{D}C_{k\ell}\varepsilon.\end{aligned}$$

The difference between the case when $k = \ell$ and $k \neq \ell$ is only in the very last term, for which we have

$$C_{k\ell}\varepsilon = \begin{cases} z_\ell - z_k, & \text{if } k = \ell \\ C_k z_k - y_k, & \text{if } k \neq \ell. \end{cases}$$

Therefore, in the case when $k = \ell$, we can write

$$\begin{aligned}\dot{z} &= Hz + Ky + B_k\bar{C}\bar{z} + B_k\bar{D}(z_\ell - z_k) \\ \dot{\bar{z}} &= \bar{A}\bar{z} + \bar{B}(z_\ell - z_k).\end{aligned}$$

To show that the communication restriction is not violated, note that $\ell \in \mathcal{N}_k$ and we can write out the observer equations in detail, like

$$\begin{aligned}\dot{z}_i &= \sum_{j \in \mathcal{N}_i} H_{ij} z_j + K_{ii} y_i, \qquad i \in \mathcal{M} \setminus \{k\}, \\ \dot{z}_k &= \sum_{j \in \mathcal{N}_k} H_{kj} z_j + K_{kk} y_k + \bar{D}(z_\ell - z_k) + \bar{C}\bar{z}, \\ \dot{\bar{z}} &= \bar{A}\bar{z} + \bar{B}(z_\ell - z_k).\end{aligned}$$

It is clear that for $m - 1$ of the observers the equations are unchanged, while for the kth observer the dynamics differ slightly and the state has been extended by the new state component $\bar{z}$. In other words, the stabilizing controller can be seen as a controller of the kth observer. In this way we have constructed a distributed observer where $m-1$ of the observers have dimension n and one observer has dimension $n + m - 1$.

Quite similarly, in the case when $k = \ell$, we obtain the observer equations

$$\begin{aligned}\dot{z}_i &= \sum_{j \in \mathcal{N}_i} H_{ij} z_j + K_{ii} y_i, \qquad i \in \mathcal{M} \setminus \{k\}, \\ \dot{z}_k &= \sum_{j \in \mathcal{N}_k} H_{kj} z_j + K_{kk} y_k + \bar{D}(C_k z_k - y_k) + \bar{C}\bar{z}, \\ \dot{\bar{z}} &= \bar{A}\bar{z} + \bar{B}(C_k z_k - y_k),\end{aligned}$$

where, just like before, we have $m-1$ observers of dimension n and one observer of dimension $n+m-1$. There does not seem to be any advantage to picking either $k = \ell$ or $k \neq \ell$, at least not in terms of dimension and complexity of the equations.

The pole placement method described in [5] allows us to assign an arbitrary spectrum to the closed loop matrix

$$\begin{pmatrix} H + B_k\bar{D}C_{k\ell} & B_k\bar{C} \\ \bar{B}C_{k\ell} & \bar{A} \end{pmatrix}.$$

This implies that, in principle, we can preassign arbitrary convergence rate of the distributed observer approximation.

5 Analysis

In this section we aim to provide a proof of Theorem 1, which will be done in two parts. In the context of the proof, we can treat the set of matrices $F_{ij},\ ij \in \mathcal{E}$, as a point in some finite dimensional space $\mathscr{F}$. Therefore a point in $\mathscr{F}$ corresponds to a matrix H through the equation in (4) and we can talk about the set of matrices H as a set of points in $\mathscr{F}$. We will first show that the set of points H for which all pairs (H, B_i), $i \in \mathcal{M}$, have controllability index m is open and dense in $\mathscr{F}$. Then we will show that the set of points H for which all systems (C_{ij}, H, B_i), $ij \in \mathcal{E}$, are minimal is open and dense in $\mathscr{F}$, thus the intersection of those two sets must also be open and dense. This would imply that the intersection is non-empty, or that there exists a point in $\mathscr{F}$ that corresponds to an H that satisfies both conditions. In other words, we would have shown that there exists a set of matrices F_{ij}, $ij \in \mathcal{E}$, such that all systems (C_{ij}, H, B_i), $ij \in \mathcal{E}$, are minimal with controllability index m, thus proving Theorem 1.

5.1 Decentralized control

This subsection is meant to provide a summary of the relevant results from decentralized control [4] which will be used throughout the proof of Theorem 1. The notation used here is restricted to this section only and should not be confused with notation used up to now.

Let $\mathcal{I}$ be some indexing set and consider the linear system system

$$\begin{aligned} \dot{x} &= Ax + \sum_{i\in\mathcal{I}} B_i u_i, \quad x \in \mathbb{R}^n, \\ y_i &= C_i x, \qquad i \in \mathcal{I}, \end{aligned}$$

which we will denote by $(\mathbf{C}, A, \mathbf{B})$, where $\mathbf{C} = \text{blockcol}(C_i :\ i \in \mathcal{I})$ and $\mathbf{B} = \text{blockrow}(B_i : i \in \mathcal{I})$. Consider feedback laws of the form $u_i = F_i y_i$ and let A_F denote the matrix that gives the resulting dynamics, that is $A_F = A + \sum_{i\in\mathcal{I}} B_i F_i C_i$. The following is an interpretation of Theorem 3 and Corollary 1 from [4].

Theorem 2 (Corollary 1 in [4]). *There exist matrices F_i, $i \in \mathcal{I}$, such that the triple (C_j, A_F, B_j) is minimal for all $j \in \mathcal{I}$ if and only if the system $(\mathbf{C}, A, \mathbf{B})$ is minimal and complete.*

To define completeness of a system we need the notion of a complementary subsystem. Let $\mathcal{C} \subset \mathcal{I}$ be a non-empty proper subset and define $\mathbf{B}(\mathcal{C}) = \text{blockrow}(B_i :\ i \in \mathcal{C})$ and $\mathbf{C}(\mathcal{C}) = \text{blockcol}(C_i :\ i \in \mathcal{I} \setminus \mathcal{C})$. Then $(\mathbf{C}(\mathcal{C}), A, \mathbf{B}(\mathcal{C}))$ is called a *complementary subsystem* of $(\mathbf{C}, A, \mathbf{B})$, which can be written out as

$$\dot{x} = Ax + \sum_{i\in\mathcal{C}} B_i u_i, \qquad y_i = C_i x,\ i \in \mathcal{I} \setminus \mathcal{C}.$$

Although it appears that the definition of a complementary subsystem depends on the ordering, the properties that characterize completeness do not.

Definition 3. *The system $(\mathbf{C}, A, \mathbf{B})$ is complete if for each proper subset $\mathcal{C} \subset \mathcal{I}$ the resulting complementary subsystem $(\mathbf{C}(\mathcal{C}), A, \mathbf{B}(\mathcal{C})$ has a non-zero transfer matrix $\mathbf{C}(\mathcal{C})(sI - A)^{-1}\mathbf{B}(\mathcal{C})$ and the matrix*

$$\pi(\mathcal{C}) = \begin{pmatrix} \lambda I - A & \mathbf{B}(\mathcal{C}) \\ \mathbf{C}(\mathcal{C}) & 0 \end{pmatrix}$$

has rank no less than n for all $\lambda \in \mathbb{C}$.

The condition that all complementary subsystems of $(\mathbf{C}, A, \mathbf{B})$ have a non-zero transfer matrix can be described in terms of the graph of $(\mathbf{C}, A, \mathbf{B})$. The graph of $(\mathbf{C}, A, \mathbf{B})$ is a graph with vertices in $\mathcal{I}$ and set of edges given by $\{ij : \ C_i(sI - A)^{-1}B_j \neq 0\}$. Lemma 8 from [4] states the following.

Lemma 4 (Lemma 8 in [4])**.** *The transfer matrices of all complementary subsystems of $(\mathbf{C}, A, \mathbf{B})$ are non-zero if and only if the graph of $(\mathbf{C}, A, \mathbf{B})$ is strongly connected.*

Finally, Theorem 1 from [4] is a statement for a fixed $i \in \mathcal{I}$ which we can extend to a statement for all $i \in \mathcal{I}$ by using the definition of completeness of a system.

Theorem 5 (Theorem 1 in [4])**.** *There exist matrices $F_i, \ i \in \mathcal{I}$, such that the pair (A_F, B_j) is controllable for all $j \in \mathcal{I}$ if and only if the system $(\mathbf{C}, A, \mathbf{B})$ is controllable and complete.*

5.2 Controllability index

In this subsection we will show that the set of points for which all pairs (H, B_k), $k \in \mathcal{M}$, have controllability index m is open and dense in $\mathscr{F}$. Let $k \in \mathcal{M}$ and note that the pair (H, B_k) has a controllability index m if

$$\operatorname{rank} \begin{pmatrix} B_k & HB_k & \cdots & H^{m-1}B_k \end{pmatrix} = mn. \tag{6}$$

Since B_k has nm rows and n columns, the matrix on the left-hand side of (6) is square, hence m is the smallest possible controllability index the pair (H, B_k) can attain. Moreover, the rank condition (6) is equivalent to

$$\det \begin{pmatrix} B_k & HB_k & \cdots & H^{m-1}B_k \end{pmatrix} \neq 0.$$

The matrix H can be written in terms of the unknown matrices F_{ij}, $ij \in \mathcal{E}$, and the determinant in the last equation can be seen as a multivariable polynomial, where the variables are the unknown elements of all matrices $F_{ij}, \ ij \in \mathcal{E}$. Provided that the polynomial is not the zero polynomial, its zero set is a proper algebraic variety, the complement of which is open and dense in $\mathscr{F}$. This complement is precisely the set of points for which the rank condition (6) holds. If we can show that there exists at least one point for which the rank condition (6) holds, then we guarantee that the polynomial is not the zero polynomial, hence the set of points for which (H, B_k) has controllability index m is open and dense in $\mathscr{F}$. As the intersection of open and dense sets is open and dense, showing there exists a point for which (6) holds for all pairs $(H, B_k), \ k \in \mathcal{M}$, implies that the set of points for which all pairs $(H, B_k), \ k \in \mathcal{M}$, have controllability index m is open and dense in $\mathscr{F}$.

To show such a point exists, we will consider the case when $F_{ii} = 0, \ i \in \mathcal{M}$. Furthermore, let $\bar{\mathcal{E}} = \mathcal{E} \setminus \{ii : \ i \in \mathcal{M}\}$ and consider matrices of the form $F_{ij} = f_{ij}I_n, \ ij \in \bar{\mathcal{E}}$, where f_{ij} are scalars. The equation for H in (4) is reduced to

$$\begin{aligned} H &= I_m \otimes A + \sum_{ij \in \bar{\mathcal{E}}} B_i (f_{ij} I) C_{ij} \\ &= I_m \otimes A + \sum_{ij \in \bar{\mathcal{E}}} f_{ij} \left(e_i \otimes I_n\right) \left(c_{ij} \otimes I_n\right) \\ &= I_m \otimes A + \sum_{ij \in \bar{\mathcal{E}}} \left(f_{ij} e_i c_{ij}\right) \otimes I_n. \end{aligned}$$

This can be simplified by defining

$$F = \sum_{ij \in \bar{\mathcal{E}}} f_{ij} e_i c_{ij}, \tag{7}$$

so that the equation for H can be written as

$$H = I_m \otimes A + F \otimes I_n. \tag{8}$$

In this form, we can relate the rank condition in (6) to a similar one in lower dimensions using the following lemma.

Lemma 6. *There exists an $mn \times mn$ nonsingular matrix T_k such that*

$$\begin{pmatrix} B_k & HB_k & \cdots & H^{m-1}B_k \end{pmatrix} = \begin{pmatrix} e_k \otimes I_n & Fe_k \otimes I_n & \cdots & F^{m-1}e_k \otimes I_n \end{pmatrix} T_k$$

where $H = I_m \otimes A + F \otimes I_n$ and $B_k = e_k \otimes I_n$.

Proof. Writing out $H^{j-1}B_k$ yields

$$\begin{aligned} H^{j-1}B_k &= (I_m \otimes A + F \otimes I_n)^{j-1} (e_k \otimes I_n) \\ &= \sum_{i=1}^{j} \binom{j-1}{i-1} (I_m \otimes A)^{j-i} (F \otimes I_n)^{i-1} (e_k \otimes I_n). \end{aligned}$$

Using the properties of the Kronecker product, we obtain

$$\begin{aligned} H^{j-1}B_k &= \sum_{i=1}^{j} \binom{j-1}{i-1} \left(I_m \otimes A^{j-i}\right) \left(F^{i-1} \otimes I_n\right) (e_k \otimes I_n) \\ &= \sum_{i=1}^{j} \binom{j-1}{i-1} F^{i-1}e_k \otimes A^{j-i}, \end{aligned}$$

and since $F^{i-1}e_k$ is a column vector for all i, we can write

$$\begin{aligned} H^{j-1}B_k &= \sum_{i=1}^{j} \binom{j-1}{i-1} \left(F^{i-1}e_k \otimes I_n\right) \left(1 \otimes A^{j-i}\right) \\ &= \sum_{i=1}^{j} \left(F^{i-1}e_k \otimes I_n\right) \cdot \binom{j-1}{i-1} A^{j-i}. \end{aligned}$$

Let the matrices $T_{ij} \in \mathbb{R}^{n \times n},\ i \in \mathcal{M},\ j \in \mathcal{M}$, be defined by

$$T_{ij} = \begin{cases} \binom{j-1}{i-1} A^{j-i} & \text{, if } i \leqslant j, \\ 0_n & \text{, otherwise,} \end{cases}$$

which then implies that

$$H^{j-1}B_k = \sum_{i=1}^{m} \left(F^{i-1}e_k \otimes I_n\right) T_{ij}.$$

The last expression looks like the result of multiplying a block row vector with a block column vector and, in particular, if we define the matrix T_k by $T_k = (T_{ij})$, then

$$\begin{pmatrix} B_k & \cdots & H^{m-1}B_k \end{pmatrix} = \begin{pmatrix} e_k \otimes I_n & Fe_k \otimes I_n & \cdots & F^{m-1}e_k \otimes I_n \end{pmatrix} T_k.$$

To show that T_k is nonsingular, note that T_k is block upper triangular because the blocks T_{ij} are zero for $i > j$. On the block diagonal we have $T_{ii} = I_n$, hence T_k is upper triangular with ones on the diagonal and as such it is nonsingular. □

The fact that T_k is nonsigular implies that the rank condition in (6) is equivalent to

$$\operatorname{rank} \begin{pmatrix} e_k \otimes I_n & Fe_k \otimes I_n & \cdots & F^{m-1}e_k \otimes I_n \end{pmatrix} = nm,$$

which can be written as

$$\operatorname{rank} \left(\begin{pmatrix} e_k & Fe_k & \cdots & F^{m-1}e_k \end{pmatrix} \otimes I_n \right) = nm.$$

Given that $\operatorname{rank} I_n = n$, the latter is equivalent to

$$\operatorname{rank} \begin{pmatrix} e_k & Fe_k & \cdots & F^{m-1}e_k \end{pmatrix} = m,$$

and because $F \in \mathbb{R}^{m \times m}$, we can recognize this as the condition for which the pair (F, e_k) is controllable. This way, the problem is reduced to finding scalars $f_{ij},\ ij \in \bar{\mathcal{E}}$, such that the pair (F, e_k) is controllable. We can relate the definition of F in (7) to the outcome of applying feedback laws $v_{ij} = f_{ij}w_{ij},\ ij \in \bar{\mathcal{E}}$, to the system

$$\dot{\varphi} = \sum_{ij \in \bar{\mathcal{E}}} e_i u_{ij}, \qquad w_{ij} = c_{ij}\varphi,\ ij \in \bar{\mathcal{E}}. \tag{9}$$

With this in mind, we are looking for feedback laws that make the system in (9) controllable through a specific input channel. The discussion in this subsection so far has been for an arbitrary $k \in \mathcal{M}$, hence if we show that there exist scalars $f_{ij},\ ij \in \bar{\mathcal{E}}$, such that the pair (F, e_k) is controllable for all $k \in \mathcal{M}$, then we would have shown that there exist matrices $F_{ij},\ ij \in \mathcal{E}$, (where $F_{ii} = 0$ and $F_{ij} = f_{ij}I_n$) such that the pair (H, B_k) has controllability index m for all $k \in \mathcal{M}$. To do that, we will show that the system in (9) is controllable and complete. Let $\mathbf{b} = \operatorname{blockrow}(b_{ij} :\ b_{ij} = e_i,\ ij \in \bar{\mathcal{E}})$ and $\mathbf{c} = \operatorname{blockcol}(c_{ij} :\ ij \in \bar{\mathcal{E}})$, so that the system in (9) can be denoted by $(\mathbf{c}, 0_m, \mathbf{b})$.

Lemma 7. *If $\mathbb{Y}$ is strongly connected then the system $(\mathbf{c}, 0_m, \mathbf{b})$ is controllable.*

Proof. The fact that $\mathbb{Y}$ is strongly connected implies that all basis vectors $e_i,\ i \in \mathcal{M}$, are included as columns in $\mathbf{b}$. Therefore $\operatorname{rank} \mathbf{b} = m$ and the system $(\mathbf{c}, 0_m, \mathbf{b})$ is controllable. □

Let $\mathbb{G}$ denote the graph of $(\mathbf{c}, 0_m, \mathbf{b})$ with vertices in $\bar{\mathcal{E}}$ and and edge from $k\ell \in \bar{\mathcal{E}}$ to $ij \in \bar{\mathcal{E}}$ if $c_{ij}(sI)^{-1}e_k \neq 0$. Note that this definition implies that if $c_{ij}(sI)^{-1}e_k \neq 0$, then there is an edge from $k\ell$ to ij for all $\ell \in \bar{\mathcal{N}}_k$, where $\bar{\mathcal{N}}_k = \mathcal{N}_k \setminus \{k\}$. The following two lemmas will be enough to show that the system $(\mathbf{c}, 0_m, \mathbf{b})$ is complete.

Lemma 8. *If $\mathbb{Y}$ is strongly connected then $\mathbb{G}$ is strongly connected.*

Proof. Let $k \in \mathcal{M}$ and recall that $c_{ki} = e_i^T - e_k^T$ for all $i \in \bar{\mathcal{N}}_k$, which implies that $c_{ki}(sI)^{-1}e_k = -\frac{1}{s} \neq 0$ for all $i \in \bar{\mathcal{N}}_k$. Therefore, there is an edge from kj to ki for all $j \in \bar{\mathcal{N}}_k$ and all $i \in \bar{\mathcal{N}}_k$. Since this is true for all $k \in \mathcal{M}$, the subgraph $\mathbb{G}_k$ with vertices in the set $\{ki : i \in \bar{\mathcal{N}}_k\}$ must be complete for all $k \in \mathcal{M}$.

Let $\mathbb{Q}$ be the graph with vertices in $\mathcal{M}$ and an edge from $\ell \in \mathcal{M}$ to $k \in \mathcal{M}$ if there is an edge from $\mathbb{G}_\ell$ to $\mathbb{G}_k$. Given that $\mathbb{G}_k$ is complete for all $k \in \mathcal{M}$, showing that $\mathbb{Q}$ is strongly connected would imply that $\mathbb{G}$ is strongly connected too. To this end, note that there is an edge from $\mathbb{G}_\ell$ to $\mathbb{G}_k$ if and only if there is an edge from ℓj to ki for some $i \in \bar{\mathcal{N}}_k$ and $j \in \bar{\mathcal{N}}_\ell$, equivalently, if $c_{ki}(sI)^{-1}e_\ell \neq 0$ for some $i \in \bar{\mathcal{N}}_k$. Considering the definition of c_{ki}, the latter is true if and only if $\ell = k$ or $\ell \in \bar{\mathcal{N}}_k$. Therefore, there is and edge $\mathbb{G}_\ell$ to $\mathbb{G}_k$ if and only if $\ell \in \mathcal{N}_k$, which suggests that $\mathbb{Q} = \mathbb{Y}$, hence $\mathbb{Q}$ is strongly connected and so is $\mathbb{G}$. □

Lemma 9. *If $\mathbb{Y}$ is strongly connected, then for any non-empty proper subset $\mathcal{C} \subset \bar{\mathcal{E}}$, the matrix*

$$\pi(\mathcal{C}) = \begin{pmatrix} \lambda I & \mathbf{b}(\mathcal{C}) \\ \mathbf{c}(\mathcal{C}) & 0 \end{pmatrix}$$

has rank no less than m for all $\lambda \in \mathbb{C}$, where $\mathbf{b}(\mathcal{C}) = \text{blockrow}(b_{ij} : \; b_{ij} = e_i, \; ij \in \mathcal{C})$ and $\mathbf{c}(\mathcal{C}) = \text{blockcol}(C_i : \; ij \in \bar{\mathcal{E}} \setminus \mathcal{C})$.

Proof. Let $\mathcal{C} \subset \bar{\mathcal{E}}$ be a non-empty proper subset and note that the structure of $\pi(\mathcal{C})$ implies that $\operatorname{rank} \pi(\mathcal{C}) \geqslant \operatorname{rank} \mathbf{b}(\mathcal{C}) + \operatorname{rank} \mathbf{c}(\mathcal{C})$ for all $\lambda \in \mathbb{C}$, so showing that $\operatorname{rank} \mathbf{b}(\mathcal{C}) + \operatorname{rank} \mathbf{c}(\mathcal{C}) \geqslant m$ is sufficient. If d denotes the number of distinct integers $k \in \mathcal{M}$ such that $kj \in \mathcal{C}$ for some $j \in \bar{\mathcal{N}}_i$, then there are d distinct basis vectors among the columns of $\mathbf{b}(\mathcal{C})$, hence $\operatorname{rank} \mathbf{b}(\mathcal{C}) = d$. Therefore it is enough to show that $\operatorname{rank} \mathbf{c}(\mathcal{C}) \geqslant m - d$. Let $\mathbf{c}^*(\mathcal{C})$ be the matrix obtained by removing the rows c_{ki} from $\mathbf{c}(\mathcal{C})$ for which there is some $j \in \bar{\mathcal{N}}_k$ such that $kj \in \mathcal{C}$. Note that $\mathbf{c}$ is the transpose of the incidence matrix of $\mathbb{Y}$, hence $\mathbf{c}^*(\mathcal{C})$ is the transpose of the incidence matrix of $\mathbb{Y}^*$, the graph obtained after removing all incoming edges of d distinct vertices of $\mathbb{Y}$. If we group every one of these distinct vertices with the vertices they have a path to, then the leftover vertices cannot have an incoming edge from any of these d groups. But no incoming edges of the leftover vertices have been removed and $\mathbb{Y}$ is strongly connected, which is a contradiction, therefore these d groups contain all vertices of $\mathbb{Y}^*$. This implies that $\mathbb{Y}^*$ has at most d weakly connected components and from Theorem 8.3.1 in [2] we know that the incidence matrix of $\mathbb{Y}^*$ has rank not less than $m - d$. The same holds for the transpose of the incidence matrix, hence $\operatorname{rank} \mathbf{c}^*(\mathcal{C}) \geqslant m - d$ and the proof is complete. □

The only thing left is to trace the steps back and arrive at the result of this subsection, stated below.

Proposition 10. *If the graph $\mathbb{Y}$ is strongly connected, then there exist matrices $F_{ij}, \; ij \in \mathcal{E}$, such that the pairs (H, B_k) have controllability index m for all $k \in \mathcal{M}$.*

Proof. From Lemma 8 we know that the graph of $(\mathbf{c}, 0_m, \mathbf{b})$ is strongly connected, and with Lemma 4 this implies that the transfer matrices of all complementary subsystems of $(\mathbf{c}, 0_m, \mathbf{b})$ are non-zero. Moreover, with the result from Lemma 9 and Lemma 7 we can conclude that $(\mathbf{c}, 0_m, \mathbf{b})$ is controllable and complete, hence Theorem 5 implies that there exist scalars $f_{ij}, \; ij \in \bar{\mathcal{E}}$, such that the all pairs $(F, e_k), \; k \in \mathcal{M}$ are controllable, where F is defined by (7). Fix those $f_{ij}, \; ij \in \bar{\mathcal{E}}$, and let $F_{ii} = 0, \; i \in \mathcal{M}$, and $F_{ij} = f_{ij} I_n, \; ij \in \bar{\mathcal{E}},$, so that

$H = I_m \otimes A + F \otimes I_n$. From Lemma 6 we know that for all $k \in \mathcal{M}$ there exist nonsingular matrices T_k such that

$$\begin{pmatrix} B_k & HB_k & \cdots & H^{m-1}B_k \end{pmatrix} = \begin{pmatrix} e_k \otimes I_n & Fe_k \otimes I_n & \cdots & F^{m-1}e_k \otimes I_n \end{pmatrix} T_k,$$

and as a consequence, we obtain

$$\operatorname{rank} \begin{pmatrix} B_k & HB_k & \cdots & H^{m-1}B_k \end{pmatrix} = nm, \quad \text{for all } k \in \mathcal{M}.$$

This shows that all pairs $(H, B_k),\ k \in \mathcal{M}$ have controllability index m, as desired. $\square$

5.3 Minimality

In this subsection we will show that the set of points for which all systems $(C_{k\ell}, H, B_k),\ k\ell \in \mathcal{E}$, are minimal is open and dense in $\mathscr{F}$. Given that the condition for observability is dual to the condition for controllability, it should be clear that the set of points for which all systems $(C_{k\ell}, H, B_k),\ k\ell \in \mathcal{E}$, are minimal is either empty or open and dense in $\mathscr{F}$. Therefore, it is enough to show that the set is non-empty, or that there exists a set of matrices $F_{ij},\ ij \in \mathcal{E}$, such that all systems $(C_{k\ell}, H, B_k),\ k\ell \in \mathcal{E}$, are minimal.

Let $\mathbf{B} = \{B_{ij} : \ B_{ij} = B_i,\ ij \in \mathcal{E}\}$ and $\mathbf{C} = \{C_{ij} : \ ij \in \mathcal{E}\}$, so that the system in (5) can be denoted by $(\mathbf{C}, I_m \otimes A, \mathbf{B})$. Note that Proposition 10 and Theorem 5 imply that the system $(\mathbf{C}, I_m \otimes A, \mathbf{B})$ is controllable and complete, hence it is enough to show that the system is observable in order to use Theorem 2.

Lemma 11. *The system $(\mathbf{C}, I_m \otimes A, \mathbf{B})$ is observable.*

Proof. We will show that all eigenvalues of $I_m \otimes A$ are observable. To this end, suppose there exists a non-zero vector $v \in \mathbb{R}^{nm}$ such that $C_{ij}v = 0,\ ij \in \mathcal{E}$, and $(I_m \otimes A)v = \lambda v$. If we let $v = \begin{pmatrix} v_1^T & \cdots & v_m^T \end{pmatrix}^T$, where $v_i \in \mathbb{R}^n,\ i \in \mathcal{M}$, then we can write

$$\begin{aligned} Av_i &= \lambda v_i, \quad i \in \mathcal{M}, \\ C_i v_i &= 0, \quad i \in \mathcal{M}, \\ v_j - v_i &= 0, \quad ij \in \bar{\mathcal{E}}, \end{aligned}$$

where we used the definitions $C_{ii} = e_i^T \otimes C_i,\ i \in \mathcal{M}$, and $C_{ij} = (e_j^T - e_i^T) \otimes I_n,\ ij \in \bar{\mathcal{E}}$. Note that $v_j = v_i,\ ij \in \bar{\mathcal{E}}$, implies that $v_i = v_j$ for any i and j connected with a path in $\mathbb{Y}$, and because $\mathbb{Y}$ is strongly connected this also implies that $v_i = v_1,\ i \in \mathcal{M}$. Now we have that $Av_1 = \lambda v_1$ and $C_i v_1 = 0$ for all $i \in \mathcal{M}$, equivalently $Cv_1 = 0$, and given that (C, A) is observable, this implies that $v_1 = 0$, hence $v = 0$. We conclude that all eigenvalues of $I_m \otimes A$ are observable, or that the system $(\mathbf{C}, I_m \otimes A, \mathbf{B})$ is observable. $\square$

We can finally provide a proof for Theorem 1:

Proof of Theorem 1. Proposition 10 implies that the set of points for which all pairs (H, B_k), $k \in \mathcal{M}$, have controllability index m is non-empty, hence the set must be open and dense in $\mathscr{F}$. This and Theorem 5 suggest that the system $(\mathbf{C}, I_m \otimes A, \mathbf{B})$ is controllable and complete, which, together with Lemma 11 and Theorem 2, implies that there exists a set of matrices $F_{ij},\ ij \in \mathcal{E}$, such that all systems $(C_{k\ell}, H, B_k),\ k\ell \in \mathcal{E}$, are minimal. Again, the set of points for which this holds must be open and dense in $\mathscr{F}$ because it is non-empty, hence the intersection of those two sets must also be open and dense. More importantly, it is non-empty, so there exists a set of matrices $F_{ij},\ ij \in \mathcal{E}$, such that all systems $(C_{k\ell}, I_m \otimes A, B_k),\ ij \in \mathcal{E}$, are minimal with controllability index m. $\square$

Remark 12. *The fact that the set of points for which all systems $(C_{k\ell}, I_m \otimes A, B_k)$, $ij \in \mathcal{E}$, are minimal with controllability index m is open and dense in $\mathscr{F}$, suggests that picking F_{ij}, $ij \in \mathcal{E}$, at random will almost certainly result in a matrix H that satisfies the property described in Theorem 1. Even if this fails, the proofs also suggest that a small perturbation in almost any direction will eventually lead to an appropriate H. This can be seen as a construction strategy.*

In view of this remark, we can write down a step-by-step algorithm for the construction of a distributed observer, assuming that the graph $\mathbb{Y}$ is strongly connected and (C, A) is an observable pair. The steps are as follows:

1. Pick random matrices F_{ij}, $ij \in \mathcal{E}$, where $F_{ij} \in \mathbb{R}^{n\times n}$ if $i \neq j$, and $F_{ij} \in \mathbb{R}^{n\times p_i}$ if $i = j$.

2. Define
$$H = I_m \otimes A + \sum_{ij\in\mathcal{E}} B_i F_{ij} C_{ij},$$
and fix an edge $k\ell \in \mathcal{E}$. Because of Theorem 1 and Remark 12, we will almost certainly have that the system $(C_{k\ell}, H, B_k)$ is minimal with controllability index m.

3. Using the techniques described in [5], find matrices $\bar{A}$, $\bar{B}$, $\bar{C}$ and $\bar{D}$ such that
$$\sigma \begin{pmatrix} H + B_k \bar{D} C_{k\ell} & B_k \bar{C} \\ \bar{B} C_{k\ell} & \bar{A} \end{pmatrix} \subset \mathbb{C}^-,$$
where $\bar{A} \in \mathbb{R}^{(m-1)\times(m-1)}$ and the rest have corresponding dimensions.

4. Let $K_{ii} = F_{ii}$, $i \in \mathcal{M}$, and
$$H_{ij} = \begin{cases} A - F_{ii} C_i - \sum_{r\in\mathcal{N}_i\setminus\{i\}} F_{ir}, & \text{if } i = j, \\ F_{ij}, & \text{otherwise.} \end{cases}$$
In the case when $k \neq \ell$, the equations for the distributed observer are given by
$$\begin{aligned} \dot{z}_i &= \sum_{j\in\mathcal{N}_i} H_{ij} z_j + K_{ii} y_i, \qquad i \in \mathcal{M} \setminus \{k\}, \\ \dot{z}_k &= \sum_{j\in\mathcal{N}_k} H_{kj} z_j + K_{kk} y_k + \bar{D}(z_\ell - z_k) + \bar{C}\bar{z}, \\ \dot{\bar{z}} &= \bar{A}\bar{z} + \bar{B}(z_\ell - z_k), \end{aligned}$$
while in the case when $k = \ell$, they are given by
$$\begin{aligned} \dot{z}_i &= \sum_{j\in\mathcal{N}_i} H_{ij} z_j + K_{ii} y_i, \qquad i \in \mathcal{M} \setminus \{k\}, \\ \dot{z}_k &= \sum_{j\in\mathcal{N}_k} H_{kj} z_j + K_{kk} y_k + \bar{D}(C_k z_k - y_k) + \bar{C}\bar{z}, \\ \dot{\bar{z}} &= \bar{A}\bar{z} + \bar{B}(C_k z_k - y_k). \end{aligned}$$

6 Conclusion

In this paper we have shown sufficient conditions and techniques for the construction of a distributed observer consisting of m observers. Assuming that the graph describing the distributed observer network is strongly connected and the observed system is observable, it has been shown that we can construct a distributed observer consisting of $m-1$ observers of dimension n, the dimension of the state of the observed system, and one observer of dimension $n+m-1$. Furthermore, the construction method allows for arbitrary convergence rate of the state approximations. The proof of Theorem 1 also suggest that the observers are stable in the sense that small perturbation in the matrices that determine the observer dynamics will almost never substantially affect the convergence properties of the distributed observer.

It would be interesting to investigate the construction of a distributed observer with smaller dimension. The output corresponding to each observer already gives some information about the state, hence each observer is required to approximate only the leftovers. However, this would require a more general form of the observer equations to include an output for each observer, which will serve as the approximation of the state of the observed system. Moreover, this will lead to more general observer design equations, and the observers would almost certainly have to use the output measurements of neighbouring observers.

On the other hand, applications in control problems might require the construction of a distributed observer for the system

$$\dot{x} = Ax + \sum_{i=1}^{m} B_i u_i,$$

where observer i can measure the input signal u_i. In this setting, we can think of decentralized control with the use of a distributed observer. Another problem might be to regard the inputs u_i as disturbances that have to be ignored by the potential observer. However, in the problems suggested here, the connection with decentralized stabilization does not seem to be as straightforward as in the problem solved in this paper. Instead, a new approach to the solution might be necessary in order to find a way to expand on the results.

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