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A search for the regular tessellations of closed hyperbolic surfaces

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Abstract

In this thesis we study regular tessellations of closed orientable surfaces of genus 2 and higher. We differentiate between a purely topological setting and a metric setting. In the topological setting we will describe an algorithm that finds all possible regular tessellations. We also provide the output of this algorithm for genera 2 up to and including 10. In the metric setting we will prove that all topological regular tessellations can be realized metrically. Our method provides an alternative to that of Edmonds, Ewing and Kulkarni.

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1 Introduction

In this thesis we will show how to find all possible regular tessellations of a genus-g surface, where $g \geq 2$ (genus-2^+ surfaces for short). Precise definitions of (regular) tessellations will be given in Section 2.6. We will differentiate between topological regular tessellations—basically embedded regular graphs with regular dual—and metric regular tessellations.

We begin with a short discussion regarding the preliminary knowledge that will be needed. We will define what tessellations are and will recall the basics of hyperbolic geometry. Then we will define, side pairings and fundamental domains. This will lead to a brief discussion of Poincaré's (polygon) Theorem. We will then be able to give a quick summary of the relation between the hyperbolic plane and genus-2^+ surfaces.

In Section 3 we will start looking for tessellations that can fit on a genus-2^+ surface, in a strictly topological sense. The first part of this section will focus on how we can represent a genus-2^+ surface by a polygon, using a specific kind of tessellation. In the last part we will derive an algorithm to find all regular tessellations that a genus-2^+ surface admits. The whole section is an elaborate discussion on necessary and sufficient conditions to find if a regular tessellation is possible.

Once we know that a genus-2^+ surface admits a certain tessellation in the topological sense, we want to know if we can fit it onto such a surface metrically. In Section 4 we will explore how we can find this out. We will give our own constructive proof to serve as an alternative to the proofs available in the current literature.

Specifically we will focus on the following questions:

- For what values of p and q is there a closed genus-2^+ Riemann surface with a topological regular tessellation by p-gons, such that q of them meet in each vertex?

- Can we construct a closed genus-2^+ Riemann surface with a metric regular tessellation by p-gons, such that q of them meet in each vertex, whenever this is topologically possible?

2 Preliminaries

In this section we will present an overview of some definitions used in this thesis and of preliminary knowledge.

2.1 Hyperbolic geometry

Hyperbolic geometry studies spaces of constant negative curvature. In this thesis we choose to use the Poincaré disk model to represent the hyperbolic plane. We will denote this by $\mathbb{D}$. One of the remarkable characteristics of the hyperbolic plane is that the angle sum of any triangle is strictly less than π.

$\mathbb{D}$ is an representation of the hyperbolic plane in the Euclidean plane. $\mathbb{C}$ is usually used for the Euclidean plane. $\mathbb{D}$ is then the open unit disk equipped with the metric

$$\mathrm{d}s^2 = \frac{4\mathrm{d}z^2}{(1-|z|^2)^2} = \frac{4\left(\mathrm{d}x^2+\mathrm{d}y^2\right)}{\left(1-(x^2+y^2)\right)^2},$$

which we call $d_{\mathbb{D}}$. One can now calculate that $\mathbb{D}$ has constant Gaußian curvature $K=-1$.

In $\mathbb{D}$, lines appear either as Euclidean diameters of the unit disk or as Euclidean circular arcs which meet $\partial\mathbb{D}$ orthogonally. In both cases lines are geodesic curves in $\mathbb{D}$ (a geodesic is a path that is locally the shortest between two points). Moreover, $\mathbb{D}$ is a conformal model, meaning that angles are not distorted with respect to the Euclidean plane. Distance on the other hand is heavily distorted.

Just like in the Euclidean plane, an equilateral equiangular polygon is called a regular polygon. If $\frac{2\pi}{q}$ is the size of the angles of a regular p-gon in the hyperbolic plane, then we call it a q-regular p-gon and we must have $(p-2)(q-2) > 4$. This can be derived from the fact that the angle sum of triangles is less than π.

The orientation preserving isometries of $\mathbb{D}$ are exactly the Möbius maps

$$f(z) = \frac{az+b}{\bar{b}z+\bar{a}}, \text{ with } |a|^2 - |b|^2 = 1.$$

They form a group under composition called the automorphism group $\mathrm{Aut}(\mathbb{D})$. If $f(z) \in \mathrm{Aut}(\mathbb{D})$, then $f(\bar{z})$ is an orientation reversing isometry. We write $\mathrm{Isom}(\mathbb{D})$ for the set of all isometries of $\mathbb{D}$.

In the hyperbolic plane, all triangles having the same angles can be mapped onto each other by isometries. This means that there is no such thing as *similarity* in the hyperbolic plane, only *congruence*.

2.2 Fundamental domains

Definition 2.1. Take Γ a group of homeomorphisms acting on $\mathbb{D}$, i.e. every $g \in \Gamma$ is a homeomorphism $g : \mathbb{D} \to \mathbb{D}$. We say that Γ acts ***properly discontinuously*** on $\mathbb{D}$ if, for every compact subset K of $\mathbb{D}$, there are only finitely many $g \in \Gamma$ for which $g(K) \cap K \neq \emptyset$.

Definition 2.2. If a group of isometries Γ acts properly discontinuously on $\mathbb{D}$, then $X \subset \mathbb{D}$ is a ***fundamental domain*** for Γ if

- $X \cap g(X) \neq \emptyset \implies g = id$,
- $\bigcup_{g\in\Gamma} g(\overline{X}) = \mathbb{D}$.

If a fundamental domain is a polygon, we will speak of a ***fundamental polygon*** for Γ. We use analogous terminology for other shapes.

One could say that the images of X under Γ form a tessellation of $\mathbb{D}$, where each $g(X)$ corresponds to a tile. Strictly speaking this is an abuse of the definition of tessellation (which we give in Section 2.6.1), but thinking of it in this way helps to provide an intuitive understanding of the interaction between Γ and X.

We will assume here that each compact subset K of $\mathbb{D}$ meets only finitely many $g(X)$. A fundamental domain with this property is called ***locally finite***. Note that this is in general not part of the definition of a fundamental domain. For a rigorous discussion on fundamental domains, their existence and what it means to be locally finite we refer to [1].

A fundamental domain is never unique. We can easily see this by taking a fundamental domain X for Γ and a $g \neq id$. Then gX is another fundamental domain and so is $(X \backslash A) \cup g(A)$, for any $A \subset X$.

2.3 Side pairings

Throughout this section we will assume that X is a fundamental finite sided polygon for some group $\Gamma \subset \mathrm{Isom}(\mathbb{D})$ acting properly discontinuously on $\mathbb{D}$ (this is allowed: [1]).

Definition 2.3 (Copied from [1])**.** A ***side*** of X is a geodesic segment of the form $\overline{X} \cap g(\overline{X})$, for some $g \in \Gamma\backslash\{id\}$. A ***vertex*** of X is a single point of the form $\overline{X} \cap g(\overline{X}) \cap h(\overline{X})$, for some distinct $g, h \in \Gamma\backslash\{id\}$.

Definition 2.4. A ***side pairing (transformation)*** is an element $g \in \Gamma$ for which $\overline{X} \cap g\left(\overline{X}\right)$ is a side of X.

Because X is a fundamental domain, there is a unique $g_s \in \Gamma$ for each side s of X, that maps s to some other side t of X. For if $s = \overline{X} \cap g(\overline{X})$ is a side of X, then so is $t \coloneqq g^{-1}(s) = \overline{X} \cap g^{-1}(\overline{X})$. We set $g_s = g^{-1}$ and call it the ***side pairing associated to*** s. It is easily seen that g_s^{-1} is the side pairing associated to t. They essentially *pair* s and t.

Lemma 2.5. *The side pairings of X generate Γ.*

Proof. Because X is a fundamental domain for Γ, there is a bijection between the elements of Γ and the copies of X. So to prove the lemma we need only show that we can map X to any $g(X)$, using only side pairings.

Pick any $g \in \Gamma$ and consider a path of finite length from a point in X to a point in $g(X)$ that does not pass through any vertices. We can label the copies of X crossed by this path as $X_0 = X, \ldots, X_n$ and let $X_i = g_i(X)$. We do this in such a way that two consecutive copies are adjacent. This means that they share a side, or equivalently that $g_i(\overline{X}) \cap g_{i+1}(\overline{X}) \neq \emptyset$. This also means that we set $g_0 = id$ and $g_n = g$.

Now we note that

$$g_i(\overline{X}) \cap g_{i+1}(\overline{X}) \neq \emptyset \implies \overline{X} \cap g_i^{-1} g_{i+1}(\overline{X}) \neq \emptyset,$$

meaning that $g_i^{-1} g_{i+1}$ is a side pairing of X. This means that $g = \prod_{i=0}^{n-1} g_i^{-1} g_{i+1}$ is a decomposition of g into side pairings of X. ℛ

The technique used in the proof above can also be applied to a very specific case. We choose a path from X to itself that makes a little circle around some vertex v of X. We choose it so that this path only passes through copies of X containing v, in a consecutive way. Then by the same reasoning as before, $id = \prod_{i=0}^{n-1} g_i^{-1} g_{i+1}$, were each $g_i^{-1} g_{i+1}$ is a side pairing. This is known as a ***cycle relation***, or ***vertex relation***.

Definition 2.6. Let $g_0 = id, g_1, \ldots, g_{\ell-1}, g_\ell = id$ be side pairings of X such that $g_1(X), \ldots, g_\ell(X)$ are consecutive copies of X, when walking around some vertex v_0 of X exactly once (Figure 1a). Set $h_i \coloneqq g_{i-1} g_i^{-1}$. Note that, by the same reasoning as before, all h_i's are side pairings of X. If we have a cycle

$$v_0 \xmapsto{h_1} v_1 \xmapsto{h_2} v_2, \xmapsto{h_3}, \ldots, \xmapsto{h_k} v_k = v_0$$

in which v_0 only occurs at the beginning and end, then we call this a ***vertex cycle*** of v_0 (Figure 1b). If we complete our walk around v_0, then the cycle

$$v_0 \xmapsto{h_1} v_1 \xmapsto{h_2} v_2, \xmapsto{h_3}, \ldots, \xmapsto{h_\ell} v_\ell = v_0$$

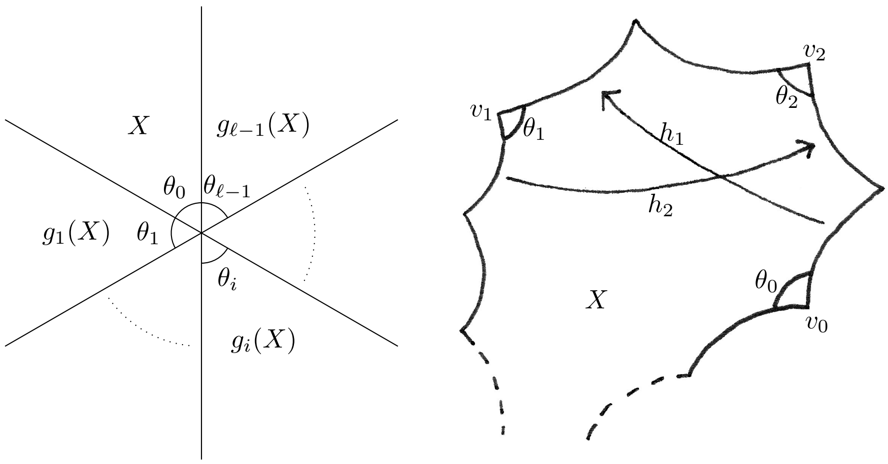
(in which v_0 might also occur in the middle somewhere) is called the ***full vertex cycle*** of v_0. Note that only for a full vertex cycle, $\prod_{i=1}^{\ell} h_i = id$.

Definition 2.7. Take X, v_i, g_i and h_i as above, such that $v_0, v_1, \ldots, v_k$ is a vertex cycle. Let θ_i be the angle at v_i in X. Note that this is also the size of the angle at v_0 in $g_i(X)$. It is a standard result (see [1] and [16]) that either of the following holds (see also Figure 1):

- $\displaystyle\sum_{i=0}^{\ell-1} \theta_i = 2\pi$
- $\displaystyle\sum_{i=1}^{k} \theta_i = \frac{2\pi}{\kappa}, \quad \kappa \in \mathbb{N}_{>1}.$

We call $\sum \theta_i$ the ***angle sum*** of the vertex cycle of v_0. If we feel like abusing notation, we will simply call it the *angle sum* of v_0.

If the angle sum of v_0 equals 2π, then its vertex cycle is a full vertex cycle. We refer to this case by saying that v_0 has a ***full angle sum***.



(a) *Copies of X around the vertex v_0.* **(b)** *The first three vertices in the vertex cycle of v_0.*

Figure 1: *Construing-aid for the definition of a (full) vertex cycle.*

2.4 A brief discussion on Poincaré's Theorem

In the previous section we started with a fundamental finite sided convex polygon X for some group Γ acting properly discontinuously on $\mathbb{D}$, such that its copies under Γ cover $\mathbb{D}$ in a locally finite manner. From this, we derived that Γ is generated by the side pairings of X. We also saw that the angle sums of the vertices of X are equal to $\frac{2\pi}{\kappa}, \kappa \in \mathbb{N}_{\geq 1}$.

It turns out that we can also move in the other way. That is, we can start with some nice polygon; choose nice side pairings and end up with a group Γ that acts properly discontinuously on $\mathbb{D}$. Poincaré's Theorem below, makes this precise.

Poincaré's Theorem. *Take X a finite sided polygon in $\mathbb{D}$, whose sides are paired by the side pairings $g_1, \ldots, g_n$. The side pairings are isometries. Further assume that each vertex of X has a an angle sum of $\frac{2\pi}{\kappa}$, $\kappa \in \mathbb{N}_{\geq 0}$. Then*

- *$g_1, \ldots, g_n$ generate a group Γ, that acts properly discontinuously on $\mathbb{D}$.*
- *We have that $\Gamma = \langle g_1, \ldots, g_n \mid \text{vertex relations} \rangle$.*
- *X is a fundamental domain for Γ and its copies under Γ are a locally finite cover of $\mathbb{D}$.*

We will not give a proof of the theorem here, but we would like to discuss some delicacies regarding it. Firstly, we would like to note that we left out the cases where X has a vertex that lies on $\partial\mathbb{D}$. This is simply because we will not concern ourselves with these cases here.

Secondly, it is very common that authors assume that the polygon X is convex, moreover. This is however not strictly necessary. In Section 4 we will explicitly need that Poincaré's Theorem also applies to non-convex polygons.

For a proof of the theorem we refer the reader to §9.8 of [1] and [13]. Both give essentially the same proof, which does not rely on X being convex.

2.5 Riemann Surfaces

We will now introduce Riemann surfaces and list some well known topological properties of these surfaces. In particular, the topological invariants "genus" and "Euler characteristic" will be defined here. We refer the reader to either [10] or [17] for proofs of all the claims we make below and for more elaborate definitions than those we give here.

2.5.1 Definitions and characteristics

A ***Riemann surface*** is a one-dimensional connected complex manifold. Two-dimensional real manifolds are Riemann surfaces precisely when they are orientable and metrizable. Riemann surfaces carry a structure that allows one to measure angles on the surface. A (Riemann) surface is called ***closed*** when it is compact and without boundary.

Definition 2.8 (Copied from [17])**.** Suppose that a collection Δ of triangles is defined on a Riemann surface Σ such that each point $x \in \Sigma$ belongs to at least one triangle in Δ and that

- if x belongs to a triangle t of Δ but is not on an edge of t, then t is the only triangle containing x and t is a neighbourhood of x;
- if x belongs to an edge e of a triangle t_1 in Δ and x is not a vertex of t_1, then there is exactly one other triangle t_2 in Δ such that t_1 and t_2 are the only triangles containing x, $t_1 \cap t_2 = e$ and $t_1 \cup t_2$ is a neighbourhood of x;
- if x is a vertex of t_1, there is a finite number of triangles $t_1, t_2, \ldots, t_\ell$, each having x as a vertex, such that each successive pair of triangles t_i, t_{i+1} or t_1, t_ℓ have only one edge in common, whilst $t_1, \ldots, t_\ell$ are the only triangles containing x and $t_1 \cup \ldots \cup t_\ell$ forms a neighbourhood of x.

When Δ satisfies the conditions above, then we call it a ***triangulation***.

Every compact Riemann surface can be triangulated. If the triangulation of some Riemann surface has V vertices, E edges and F faces, then we can define the ***Euler characteristic*** χ of that surface as $\chi = V - E + F$. The Euler characteristic is a topological invariant. For closed Riemann surfaces we define the ***genus*** g of that surface as $g = \frac{2-\chi}{2}$. Intuitively this is the number of holes of a surface.

2.5.2 Coverings

Definition 2.9 (Copied from [10])**.** The manifold M^* is said to be a ***(branched) covering manifold*** of the manifold M if there is a continuous surjective map $f : M^* \to M$ with the following property: For each $x^* \in M^*$ there exists a local coordinate z^* on M^* vanishing at x^*, a local coordinate z on M vanishing at $f(x)$ and an integer $n > 0$ such that f is given by $z = (z^*)^n$ in terms of these local coordinates. Here the integer n depends only on the point $x^* \in M^*$. If $n > 1$, then x^* is called a ***branch point of order*** $n-1$ or ***ramification point of order*** n. If $n = 1$ for all points $x^* \in M^*$, then the cover is called a ***smooth cover***. The map f is called a ***(branched) covering map***.

We call $x = f(x^*)$ the ***projection*** of x^* onto M. We say that x^* ***lies over*** x.

We call M^* an ***unlimited*** covering manifold of M provided that for every curve $c \subseteq M$ and every point $x^* \in M^*$ with $f(x^*) = c(0)$, there exists a curve $c^* \subseteq M^*$ with initial point P^* and $f(c^*) = c$. We call c^* a ***lift of*** c.

Definition 2.10 (Copied from [17])**.** Suppose we have two points x and y of a Riemann surface Σ, two curves c_1 and c_2 on Σ with initial point x and terminal point y and a continuous map $h : [0,1]^2 \to \Sigma$ such that

$$\begin{aligned} h(t,0) &= c_1(t), \\ h(t,1) &= c_2(t), \\ h(0,u) &= x, \\ h(1,u) &= y, \end{aligned}$$

for all $t, u \in [0,1]$. Then we say that c_1 and c_2 are ***homotopic*** and we write $c_1 \overset{h}{\sim} c_2$.

For any point $x \in \Sigma$ consider all curves on M with x as initial and terminal point, i.e. closed curves through x. Two such curves are ***equivalent*** whenever they are homotopic. The set $\pi_1(\Sigma, x)$ of equivalence classes of all closed curves through x forms a group. This group is called the ***fundamental group of*** Σ ***based at*** x. For Riemann surfaces $\Sigma \ni x, y$ the groups $\pi(\Sigma, x)$ and $\pi(\Sigma, y)$ are naturally equivalent. We can therefore drop the dependency on the base point from our notation and just speak of the ***fundamental group of*** Σ**,** $\pi_1(\Sigma)$. We refer to [17] for a rigorous discussion of the group structure mentioned above.

If M^* is a covering manifold of M with covering map f, then a homeomorphism $h : M^* \to M^*$ with the property that $f \circ h = f$ is called a ***covering transformation of*** M^*. The set of covering transformations forms a group. This group is called transitive if it acts transitively on any set $f^{-1}(x), x \in M$.

A smooth unlimited covering M^* has a transitive group of covering transformations if and only if $\pi_1(M^*)$ is isomorphic to a normal subgroup of $\pi_1(M)$. In this case the group of covering transformations of M^* is isomorphic to $\pi_1(M)/\pi_1(M^*)$.

To this thesis, the most important consequence of the theory above is this: Every Riemann surface Σ has a ***universal covering*** $\tilde{\Sigma}$. This is the smooth unlimited covering whose fundamental group $\pi_1(\Sigma)$ is isomorphic to $\{0\}$. $\tilde{\Sigma}$ is also a Riemann surface.

We call two Riemann surfaces ***conformally equivalent*** if there is a bijective analytic function between them. The *Uniformization Theorem* states that every simply-connected Riemann surface is conformally equivalent to either $\mathbb{C}$, $\hat{\mathbb{C}}(=\mathbb{C}\cup\{\infty\})$ or $\mathbb{D}$. We refer to [10] and [17] for a proof of the Uniformization Theorem and of the following.

Proposition 2.11. *If $\tilde{\Sigma}$ is the universal covering of a Riemann surface Σ, then*

- *$\tilde{\Sigma}$ is simply connected,*
- *the group of covering transformations of $\tilde{\Sigma}$ is isomorphic to $\pi_1(\Sigma)$,*
- *Σ is conformally equivalent to $\tilde{\Sigma}/\pi_1(\Sigma)$.*

2.5.3 Hyperbolic surfaces

If a surface is a hyperbolic 2-manifold—a 2-manifold whose charts are subsets of $\mathbb{D}$, then we simply call it a ***hyperbolic surface***. Every closed genus-g Riemann surface is conformally equivalent to a closed 2-manifold of constant curvature. This curvature is negative, precisely when $g \geq 2$. This means that genus-2^+ surfaces are equivalent to a quotient $\mathbb{D}/\Gamma$ of $\mathbb{D}$, where Γ is a subgroup of $\mathrm{Aut}(\mathbb{D})$ whose action on $\mathbb{D}$ is free (i.e. $g(x) = x \implies g = id$) and properly discontinuous. In particular, this implies that all genus-2^+ surfaces are hyperbolic surfaces. This is all a consequence of the *Uniformization Theorem* which we mentioned above. Again, for a more scrutinous discussion of the Uniformization Theorem and its consequences we refer to [10] and [17]. It thus makes sense that we exploit the connection between the hyperbolic plane and genus-2^+ surfaces.

It also makes sense to give Poincaré's Theorem another look. It provides us with a method to construct a group Γ acting properly discontinuously on $\mathbb{D}$, which is not necessarily free. If Γ were also free, then $\mathbb{D}/\Gamma$ would be a genus-2^+ surface, according to the discussion above. It turns out that the group Γ, that we get from Poincaré's Theorem, acts freely on $\mathbb{D}$ if all vertices of the polygon X have a full angle sum. This is very useful information, because it allows us to construct hyperbolic surfaces with relative ease.

Lastly, suppose that Γ is a group of isometries acting freely and properly discontinuously on $\mathbb{D}$. Say that $\psi : \mathbb{D} \to \mathbb{D}/\Gamma$ is the quotient map from $\mathbb{D}$ to $\mathbb{D}/\Gamma$. Then the ***induced metric*** d on the surface $\mathbb{D}/\Gamma$ is defined as

$$d(x, y) := \min\{d_{\mathbb{D}}(s, t) \mid s \in \psi^{-1}(x), t \in \psi^{-1}(y)\}. \tag{2.1}$$

Usually, surfaces will not be described as $\mathbb{D}/\Gamma$, but as $\overline{X}/\Gamma$, where X is a fundamental domain for Γ. This is, for all intents and purposes, the same surface. In this case we must alter the definition of the induced metric slightly.

Definition 2.12. We say two points $x, y \in \overline{X}$ are equivalent ($x \sim y$) if they lie in the same Γ-orbit. We write $[x]$ for the equivalence class containing $x \in \overline{X}$. Then the ***induced metric*** d on the surface $\overline{X}/\Gamma$ is defined as

$$d\big([x],[y]\big) = \inf \left\{ \sum_{i=0}^{n} d_{\mathbb{D}}(\xi_i, \xi_i') \right\},$$

where the infimum is taken over all $\xi_0 \in [x]$, $\xi_n' \in [y]$, $\xi_i' \sim \xi_{i+1}$ for $i = 0, \ldots, n-1$, and $n > 0$.

This definition of induced metric agrees with (2.1) on $\overline{X}$.

2.6 Tessellations

2.6.1 Definitions and notations

The following definitions will depend on the use and knowledge of elementary graph theory and elementary theory on graph embeddings. The reader who is not sufficiently skilled in these subjects is referred to [2] and [12]. Just to be on the safe side: A graph is allowed to have loops and multiple edges between vertices.

Definition 2.13. Let Σ be a topological surface without boundary. A ***tessellation*** T on Σ is a non-empty, connected, locally finite graph G embedded in Σ, such that each component of $\Sigma \backslash G$ is homeomorphic to a disk. G will be referred to as the graph of the tessellation.

The components of $\Sigma \backslash G$ are called ***open faces***; the closure of an open face is called a ***closed face***. Note that closed faces need not be simply-connected. When this happens, the difference between open and closed faces is vital. When this doesn't happen, it is rarely essential to emphasise if a face is open or closed. In these cases we will generally just speak of ***faces***. It is also quite common to use the word ***tile*** in stead of face. These can and will be used interchangeably. We write V_T, E_T, F_T for respectively the sets of vertices, edges and faces of T.

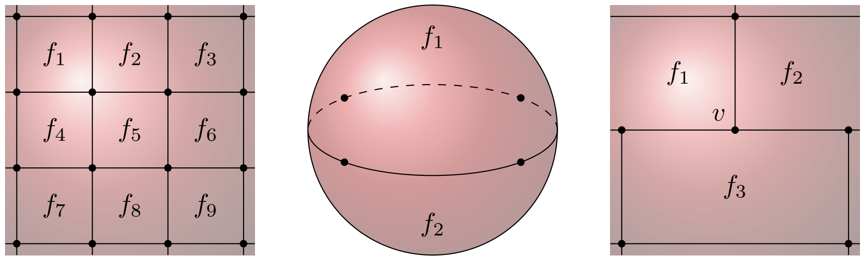
We define a polygon as a closed disk, whose boundary is divided into p segments (called edges) by p vertices. Every closed face f of a tessellation is either itself a p-gon, or can be obtained from a p-gon by making identifications on its sides. In either case we say that the face f has ***edge number*** p. To find the edge number we count the edges of a closed face with multiplicity. An edge of a closed face f has multiplicity 2 if it lies only in f. It has multiplicity 1 if it lies in f and in some distinct other closed face f^*.

We have defined tessellations in such a way that they have the following general properties:

- The interiors of two distinct tiles never intersect.
- An intersection point of two distinct tiles is either a common vertex or lies in a common edge.

In Figure 2 we try to make the above notion a bit more intuitive.

If our tessellation looks like Figure 3a or Figure 3b anywhere, then we say it has a ***trivial vertex***. In a tessellation without trivial vertices, every edge is



(a) *Tessellation* **(b)** *Tessellation* **(c)** *Tessellation only if v is considered a vertex of f_1, f_2 and f_3, meaning f_3 is a pentagon.*

Figure 2: *Examples of what we consider tessellations and what not.*

essential to the definition of a certain face. This is not the case in Figure 3. In Figure 3a we could just as well delete the vertex v and the edge e. In Figure 3b we could just as well consider $a \cup v \cup b$ as one long edge. You could say that the vertex v is *trivial* in these cases. Removing it only changes the edge number of some face. It keeps the number of faces the same.

(a) **(b)**

Figure 3: *Tessellating in a trivial way.*

Definition 2.14. If T is a tessellation of a topological surface then it is called a ***topological regular tessellation*** or ***topologically regular*** if each face has edge number p and each vertex has valence q, for some p, q. We also call this a ***topological*** $\{p, q\}$ ***tessellation***.

Note that a regular tessellation cannot be trivial, unless we are tessellating a sphere and $q = 2$ (Figure 2b for instance).

In this thesis the only metric spaces we will be studying are Riemann surfaces. Lengths and angles are well-defined on these surfaces. This allows us to define the metric equivalent of the definition above.

Definition 2.15. If T is a tessellation of a Riemann surface then it is called a ***metric regular tessellation*** or ***metrically regular*** if it is a topological regular tessellation for which

- every edge is a geodesic segment of fixed length,
- every angle between two distinct edges is of equal size,

- every face is convex, meaning any two points in it can be connected by a geodesic that lies entirely within the face.

We also call this a ***metric*** $\{p, q\}$ ***tessellation***.

When it is clear if we are in a topological setting or in a metric one, then we simply say that we have ***regular tessellation*** or $\{p, q\}$ tessellation.

2.6.2 Dual tessellations

For any tessellation T of some surface Σ, there is a ***dual tessellation*** T^*. This dual is constructed as follows:

- Start with the tessellation T of Σ and its graph G.
- Pick a point in the interior of each face f_i of T. These are the dual vertices f_i^*.
- If f_i and f_j are adjacent—i.e. share an edge e_k—then draw an edge e_k^* between the dual vertices f_i^* and f_j^*. The dual edge e_k^* should intersect only e_k and no other (dual) vertices or (dual) edges, its endpoints excepted. N.B.: If e_k has multiplicity 2 then $f_i = f_j$, $f_i^* = f_j^*$ and e_k^* is a loop.
- The dual vertices and the dual edges now form a dual graph G^*. The dual tessellation T^* is the tessellation formed by embedding G^* into Σ (this is possible by construction). Each face v_i^* of T^* contains/corresponds to a vertex v_i of T.

Recall that a graph isomorphism is a bijection on its vertex-set and its edge-set, preserving incidence. We write $G \cong H$, if two graphs are isomorphic. Now note that in general G^* is unique, only up to a graph isomorphism. We always have $(G^*)^* \cong G$. In a purely topological setting we have that

$$T \text{ is a } \{p, q\} \text{ tessellation} \iff T^* \text{ is a } \{q, p\} \text{ tessellation.} \tag{2.2}$$

We choose to lay some restrictions on the definition of the dual of a *metric* regular tessellation. In this case we demand that f_i^* is the incentre of f_i and that the dual edges are geodesic segments. It is not hard to see that (2.2) still holds with this definition for the dual of a metric regular tessellation. Moreover, in this case we can say that $(G^*)^*$ and G are not only isomorphic, but in fact equal. This equality now also holds for $(T^*)^*$ and T.

2.6.3 Euler's formula

With any tessellation of a closed surface Σ, we canonically associate the numbers V, E, and F. Here V is the number of vertices of the tessellation, E is the number of edges (not counted with multiplicity) and F is the number of faces. These numbers are used in the well known identity

$$V - E + F = \chi = 2 - 2g. \tag{2.3}$$

Here χ is the Euler characteristic of the surface and g its genus.

When we write $(V, E, F)_T(\Sigma) = (x, y, z)$, we mean that the tessellation T of Σ has x vertices, y egdes and z faces. When it is clear what Σ is then we just write $(V, E, F)_T = (x, y, z)$. If T happens to be a regular $\{p, q\}$ tessellation, we will even write $(V, E, F)_{\{p,q\}} = (x, y, z)$.

2.6.4 Regular maps

A tessellation of a closed surface is sometimes also called a ***map***. An ***automorphism*** of a map is defined as a permutation of its faces, preserving properties of incidence and adjacency. The automorphisms of a map form a group, referred to as its ***automorphism group.***

Usually when authors use the word "map" instead of "tessellation" they are interested primarily in *regular maps.* A map is called a ***regular map*** if it has two specific automorphisms:

- One that cyclically permutes the edges of a particular face;
- One that cyclically permutes the edges of a vertex belonging to that same face.

These two automorphisms then generate the automorphism group of the map.

Regular maps are be completely determined by their automorphism groups. This correspondence has been used to produce exhaustive lists of regular maps for genera up to 301 [3, 4, 5, 6].

Regular maps are also regular tessellations, but regular tessellations might not be regular maps. This is because the definition of a regular tessellation does not rely on the existence of certain automorphisms. Many contemporary research focuses on regular maps. See for instance [7, 14, 15, 18, 19]. In this thesis we will consider regular tessellations and we do not limit our scope to regular maps. Specifically we will find a way to describe regular tessellations.

2.6.5 Necessary and sufficient conditions for the existence of regular tessellations

Suppose we have a $\{p, q\}$ tessellation of an orientable surface without boundary. Then we have Euler characteristic

$$V - E + F = \chi = 2 - 2g, \tag{2.4}$$

where g is the genus of the surface. Along the edge of any tile, two tiles meet. Every face has p edges, so we must have $E = \frac{pF}{2}$. Likewise we must have $V = \frac{pF}{q}$. This is equivalent to saying that

$$qV = 2E = pF. \tag{2.5}$$

From (2.4) and (2.5) we can derive that

$$\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -\frac{p}{2} \\ 1 & 0 & -\frac{p}{q} \end{pmatrix} \begin{pmatrix} V \\ E \\ F \end{pmatrix} = \begin{pmatrix} 2-2g \\ 0 \\ 0 \end{pmatrix}.$$

The 3×3 matrix has determinant $\frac{2p-pq+2q}{-2q}$. This equals zero, if and only if, $\{p, q\}$ is a Euclidean tessellation. Since we are only interested in hyperbolic tessellations, we are free to write

$$\begin{pmatrix} V \\ E \\ F \end{pmatrix} = \frac{1}{2p - pq + 2q} \begin{pmatrix} 2p & 2p & 2q - pq \\ pq & 2p + 2q & -pq \\ 2q & 2q & -2q \end{pmatrix} \begin{pmatrix} 2-2g \\ 0 \\ 0 \end{pmatrix}. \tag{2.6}$$

From this we can derive that

$$V(p,q) = \frac{2p(2-2g)}{2p-pq+2q}, \tag{2.7a}$$

$$E(p,q) = \frac{pq(2-2g)}{2p-pq+2q}, \tag{2.7b}$$

$$F(p,q) = \frac{2q(2-2g)}{2p-pq+2q}. \tag{2.7c}$$

For convenience, we define these formulas for $(p,q) \in \mathbb{R}^2_{\geq 3}$.

We can now use this to derive some necessary conditions for the existence of a topological $\{p,q\}$ tessellation on an orientable genus-g surface. If we are given p, q and g then we can use (2.7) to check if $V, E, F \in \mathbb{Z}_{\geq 1}$. If not, then a topological $\{p,q\}$ tessellation is not possible.

It turns out that these conditions are also sufficient. So for given p, q and g, there is always a topological $\{p,q\}$ tessellation on an orientable genus-g surface if (2.7) implies that V, E and F are positive integers.

Example 2.16. Suppose we want to know if a genus-2 surface can be tessellated by a topological $\{8,8\}$ tessellation. Using (2.7) we find that this is possible and that we must have $(V,E,F) = (1,4,1)$. Because $F = 1$, this tessellation will have only one tile. Likewise, we can find that $\{3,10\}$ is also possible. It gives $(V,E,F) = (3,15,10)$.

Proposition 2.17. *For given $p, q \in \mathbb{Z}_{\geq 3}$ and $g \in \mathbb{Z}_{\geq 0}$ there exists a topological $\{p,q\}$ tessellation on some genus-g surface if and only if $V, E, F \in \mathbb{Z}_{\geq 1}$ according to* (2.7).

A proof of the sufficiency part of Proposition 2.17[1] can be found in [8]. The authors (Edmonds, Ewing and Kulkarni) essentially look for branched coverings of tessellations whose existence is easily verified. The lifts of these tessellations are then exactly the tessellation they sought for. In Example 2.18 we illustrate this for a particular tessellation.

Example 2.18. We take $\{p,q\} = \{7,3\}$. In [8], Edmonds, Ewing and Kulkarni use heuristics to derive the existence of the tessellation shown in Figure 4 (see the article for the details of those heuristics). They show in Lemma 3.2 of their paper that this tessellation can be lifted to a branched cover, such that v_5 and v_6 are lifted to ramification points of order 3. All other points are lifted to ramification points of order 1.

After lifting, we have a tessellation with 6 vertices (each of valence 3), 21 edges, and 6 faces (each with edge number 7). This is a $\{7,3\}$ tessellation on a non-orientable closed surface with Euler characteristic $\tilde{\chi} = -1$. This surface has an orientable double cover (i.e. each point lifts to two distinct points) of Euler characteristic $\chi = -2$. This implies that it is a surface with genus 2, on which we have a $\{7,3\}$ tessellation with 28 vertices, 42 edges and 12 faces. This shows that a topological $\{7,3\}$ tessellation exists on a closed genus-2 surface (and thus also its dual: the topological $\{3,7\}$ tessellation).

[1]Actually the article is not even limited to just orientable hyperbolic surfaces. It explicitly considers regular tessellations of general closed surfaces.

[2]To create a cross cap, remove the interior of a circle from the surface. On the circular border that this creates, we identify every point with its antipodal point. We denote this by $\otimes$.

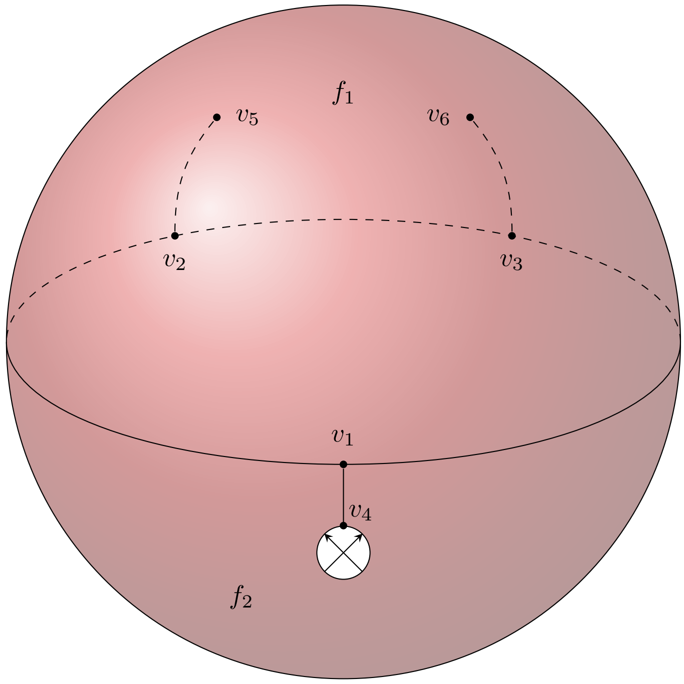


Figure 4: *A sphere with a topological $\{3, 2\}$ tessellation to which three vertices (and edges) and a cross cap*[2] *are added.*

3 Tessellations of a closed orientable genus-2^+ surface

In this section we study tessellations of closed genus-g surfaces, for $g \geq 2$. To keep things brief, we will simply say genus-2^+ surfaces. We take on a topological point of view. This means that there is exactly one closed orientable genus-g surface for each $g \in \mathbb{Z}_{\geq 2}$. We denote this by Σ_g from now on. Section 3.1 will focus on tessellations of genus-2^+ surfaces that have only one tile. In Section 3.2 we will show how exactly this tile can be used to represent the surface Σ_g. In Section 3.3 we show how to find all regular tessellations of Σ_g. Note that until Section 3.3 we do not assume all tessellations to be regular.

3.1 Tessellations of a closed orientable genus-2^+ surface consisting of one tile

Let's look at a general orientable genus-g surface, Σ_g, for $g \geq 2$. We are going to look for for a tessellation of Σ_g having only one tile with edge number p. We will require that this tessellation has no trivial vertices. This tile can be obtained from a p-gon by making identifications on its sides. We call this p-gon P_p. We will slightly abuse notation from now on and say that P_p tessellates or tiles Σ_g in this case.

Lemma 3.1. *We can tessellate Σ_g without trivial vertices by exactly one polygon P_p if and only if $p \in \{4g, 4g + 2, \ldots, 12g - 6\}$.*

Proof. Let's look at the tessellation of Σ_g by the single tile P_p. If we draw this tessellation on Σ_g, then we get a graph G. By construction, every edge of G corresponds to exactly one pair of edges of P_p. This makes sense, because every edge has multiplicity 2 in a tessellation with only one tile. So we have $E_{P_p} = 2E_G$. Every vertex of G must correspond to at least 3 vertices of P_p, or else we would have a vertex that is trivial. Using (2.3) we find that $E_G = V_G + 2g - 1$. Lastly we note that $E_{P_p} = V_{P_p} = p$.

To sum up, so far we know that:

$$\begin{aligned} p = E_{P_p} &= V_{P_p}, \\ E_G &= V_G + 2g - 1, \\ E_{P_p} &= 2E_G, \\ V_{P_p} &\geq 3V_G. \end{aligned} \tag{3.1}$$

From this we can deduce that

$$2E_G \geq V_G \implies 2E_G \geq 3(E_G + 1 - 2g) \implies E_G \leq 3(2g - 1).$$

And since $p = E_{P_p} = 2E_G$, we can see that $p \leq 12g - 6$. We can also deduce that

$$p = E_{P_p} = 2E_G = 2(V_G + 2g - 1) \geq 4g.$$

We will see in Section 3.3 (Claim 3.4) that Σ_g always has the tessellations $\{4g, 4g\}$ and $\{12g - 6, 3\}$. Both of these consist of 1 tile. The first has only 1 vertex, of valence $4g$. The latter has $4g - 2$ vertices, all of valence 3. This means that the lowest and highest possible values for p always occur. These are regular, so they do not have any trivial vertices. And since p must be even, this leaves only the claimed possibilities.

To see that all possibilities indeed occur, we generalise the techniques used in [11] (sections 5 and 6 in particular). We start with the $\{12g - 6, 3\}$ tessellation. We then perform the following steps:

S(1) Pick an edge of the tessellation that connects two distinct vertices.

S(2) ***Collapse*** this edge. (See Figure 5.)

The result is a tessellation by exactly one $(12g - 8)$-gon. By repeating S(1) and S(2) we will eventually arrive at the $\{4g, 4g\}$ tessellation. In getting there we will have seen all desired tessellations occur.

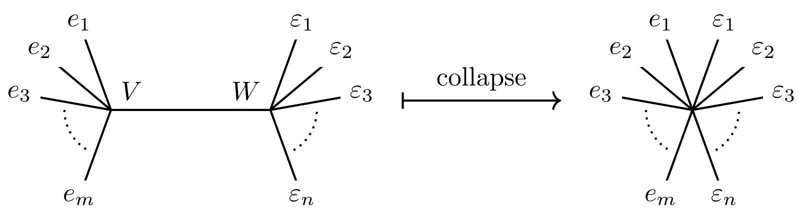


Figure 5: *The collapsing of an edge. V and W are two distinct vertices, with the same image under the* collapse*. The e_i's and ε_i's are edges incident to V and W respectively. Note that not all these edges need to be distinct.*

3.2 How to represent a closed orientable genus-2^+ surface as a polygon.

3.2.1 How can we determine the representation?

We now know all p-gons that can tessellate a certain Σ_g. We said that we can formally obtain our only tile from this p-gon, by *making identifications on its sides.* To be a bit more precise, this means that there is a natural map $\pi : P_p \to \Sigma_g$. This map is an embedding of the interior of P_p onto the (only) open face of the tessellation. It maps ∂P to the graph G of the tessellation as dictated by the identifications. In this section we will find how we can determine what these identifications should be.

Lemma 3.2. *Fix Σ_g and choose p such that the p-gon P_p can tessellate Σ_g (we explained what we mean by this at the very beginning of Section 3.1). Say G is the graph of this tessellation.*

Walking around the boundary of P_p exactly once, now corresponds to a closed walk on G with the following conditions:

Walk Conditions:

(W1) Each edge is traversed exactly two times.

(W2) Each edge is traversed exactly once in each direction.

(W3) For each edge, the two times that edge is being traversed are not consecutive, nor are they the beginning and end of the walk.

Proof.
(W1): Trivial.
(W2): Because we want M to be orientable.
(W3): If (W3) were false then we would have a trivial vertex, like in Figure 3a. We chose not to consider tessellations with trivial vertices.

Every walk that satisfies the *Walk Conditions* corresponds to a set of side pairings of P_p and vice versa. So if we manage to describe all walks of G satisfying the Walk Conditions, then we know all possible sets of side pairings for P_p that meet our needs. Determining all of these walks is no standardised task. The quickest method to determine them all will be different every time. An example for the case $p = 18$ and $g = 2$ can be found in Sections 2–3 of [11].

Example 3.3. The technique discussed above can be applied to any closed surface, not just the genus-2^+ ones. So we will use the torus ($g = 1$) in this example. Consider the standard way to cut the torus so that it becomes a square. In Figure 6 we have drawn the corresponding graph, both on the torus and as a plane graph.

If we travel along a in the "right" direction, we write a. If we travel along a in the "wrong" direction, we write $\bar{a}$. We do the same for b.

It is quite obvious that, up to cyclic permutation, the only walks satisfying the Walk Conditions are $ab\bar{a}\bar{b}$ and $a\bar{b}\bar{a}b$. So the identification schemes shown in Figure 7 follow. These are, as expected, the standard identification patterns for the torus (modulo mirror images). This all becomes more complex if the graph has more vertices and the surface has higher genus.

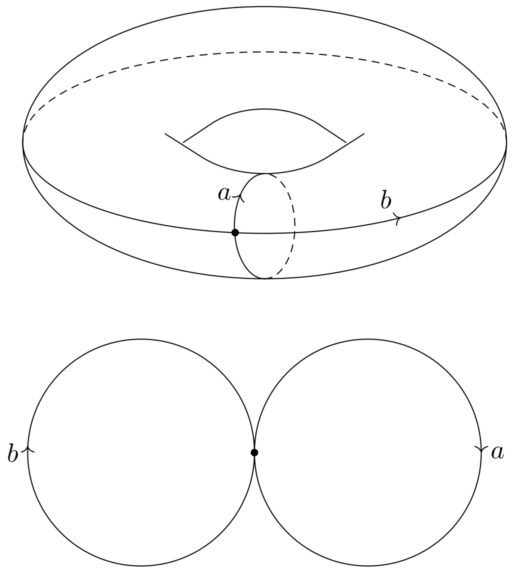


Figure 6: *A standard graph to cut along to make a square out of a torus.*

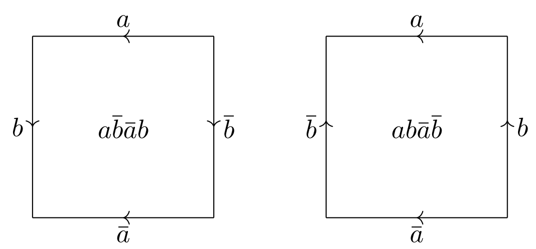


Figure 7: *Two identification schemes to turn a square into a torus.*

3.3 Find all regular tessellations of a closed orientable genus-2^+ surface

In the previous sections, we explored tessellations consisting of one tile, and showed how we can use these to represent Σ_g as a polygon. Note that not all of these tessellations were regular. For instance, according to Lemma 3.1 we can tessellate Σ_2 by a single 16-gon. This tessellation, however, is not topologically regular.

In this section we will show how to find all possible topological $\{p, q\}$ tessellations of Σ_g, for $g \geq 2$. We will show that p and q are bounded for fixed g. Since p and q are integers, this means that we can determine all possible values algorithmically. This is exactly what we will do. In Appendix A we included an Octave[3] script that can print what all possible $\{p, q\}$ are into a file, using this algorithm.

[3] A scientific programming language that is mostly compatible with Matlab. See `https://www.gnu.org/software/octave/` for more information.

Assume throughout this section that $g \geq 2$ is fixed. We may also assume that $p, q \geq 3$, since we are in hyperbolic space. So we already have lower bounds for p and q. The next step is to look for upper bounds. For this we will use that

$$V(p,q) = \frac{2p(2-2g)}{2p - pq + 2q}, \tag{2.7a}$$

$$E(p,q) = \frac{pq(2-2g)}{2p - pq + 2q}, \tag{2.7b}$$

$$F(p,q) = \frac{2q(2-2g)}{2p - pq + 2q}. \tag{2.7c}$$

These equations will be of great help to derive the upper bounds.

We will start with an upper bound for q. For this we use (2.7a). It tells us that $V(p,q)$ is continuous and descending in q. It makes no sense for V to be smaller than one. So we fix p and derive

$$\frac{2p(2-2g)}{2p - pq + 2q} = 1 \implies q = \frac{2p(2-2g) - 2p}{2-p}.$$

This means that $\frac{2p(2-2g)-2p}{2-p}$ is an upper bound for q.

To find an upper bound for p, we use (2.7c). From it we can derive that $F(p,q)$ is continuous and descending in both p and q. We also revisit the proof of Lemma 3.1, where we claimed that P_{4g} and P_{12g-6} can always tile Σ_g. We will now prove this claim, by showing that they always give rise to topological regular tessellations with exactly one face.

Claim 3.4. *We can always tessellate Σ_g with a topological $\{4g, 4g\}$ tessellation and a topological $\{12g-6, 3\}$ tessellation.*

Proof. We use Equation 2.6 with $(p,q) = (4g, 4g)$ and find that $\{4g, 4g\}$ is possible with $(V, E, F) = (1, 2g, 1)$. Likewise we find that $\{12g-6, 3\}$ is possible with $(V, E, F) = (4g-2, 6g-3, 1)$.

So we know that $\{12g-6, 3\}$ is always a possible topological regular tessellation for Σ_g, with $F(12g-6, 3) = 1$. Because $F(p,q)$ is continuous and descending in both p and q, we can only choose $p > 12g-6$ if $q < 3$. This is not allowed, so $12g-6$ is an upper bound for p.

So for Σ_g we can determine all possible $\{p, q\}$ by the following algorithm.

```
For p = 3 To 12g − 6;
  For q = 3 To (2p(2 − 2g) − 2p)/(2 − p);
    Calculate V, E and F;
    If (V, E, F) ∈ Z^3_{>0};
      Show "Yes";
      Else Show "No"; % Can also be left out
    End;
  End;
End;
```

Listing 1: *An algorithm to determine all possible topological $\{p, q\}$ tessellations for some Σ_g.*

We used this algorithm to produce the tables in Appendix B.

4 Making metric regular tessellations out of topological regular tessellations

4.1 Exploring the possibilities

Suppose we found a topological regular tessellation of a genus-2^+ surface. We would then like to know if we can also fit this tessellation metrically.

Proposition 4.1. *Every topological regular $\{p, q\}$ tessellation T of a genus-2^+ surface Σ_g can also be realized as a metric regular $\{p, q\}$ tessellation S of some genus-2^+ surface.*

4.1.1 What does the literature say

Information regarding Proposition 4.1 appears to be very scarce. Its truthfulness mostly seems to be taken for granted. The only papers we found that treated this claim are [8] and [9], both by A. Edmons, J. Ewing and S. Kulkarni.

Edmonds, Ewing and Kulkarni discuss Proposition 4.1 in the first part of §8 of [8] and in Proposition 4.3 of [9]. In both cases the proof relies on the fact that we can somehow "insert" some metrically regular polygon into a face of the tessellation at hand. It is then claimed that this can be done for all faces. They claim that this naturally turns our topological regular tessellation into a metrically regular one.

It is the author's opinion that this proof is not as accessible as it could be. A priori, the surface Σ_g has no metric structure defined on it. So to say that we insert a regular polygon into some face of T seems ill-defined. Secondly, we wonder if the "insertion method" can be used as a global argument. For example: Suppose that we do have a clear way to insert a metrically regular polygon into a face of T and suppose that we have done this for all but one face. Can our metrically regular polygon then be inserted into the last face, without deforming the other faces? We therefore have decided to take another look at Proposition 4.1 and will write down our own alternative proof in Section 4.2.

4.1.2 A heuristic method

Let's start with the first entry from Table 1 in Appendix B: $\{3, 7\}$. According to Conder [6] this is not a regular map. But, we can tessellate a genus-2 surface with it in a metrically regular way. To see this is true we can use Figure 8. Here we have taken 28 tiles from the $\{3, 7\}$ tessellation of $\mathbb{D}$, such that they form a convex 18-gon. We took exactly 28 because $(V, E, F)_{\{3,7\}} = (12, 42, 28)$ for a genus-2 surface.

According to Poincaré's Theorem, the side pairings shown in Figure 8 generate a group Γ acting properly discontinuously on $\mathbb{D}$. Dividing by the orbits of this group turns our 18-gon into a closed orientable surface Σ. We know that it is orientable because the side pairing isometries in Figure 8 are orientation preserving. We can deduce that this surface must have genus 2, by observing that we have $V - E + F = -2$. Now note that each vertex cycle has a full angle sum. This means we do not get any cusps or cone points. The metric of $\mathbb{D}$ induces a metric on $\mathbb{D}/\Gamma$. If we endow our surface with the induced metric from $\mathbb{D}$, then we have described how $\{3, 7\}$ can be a metric regular tessellation of a genus-2 surface.

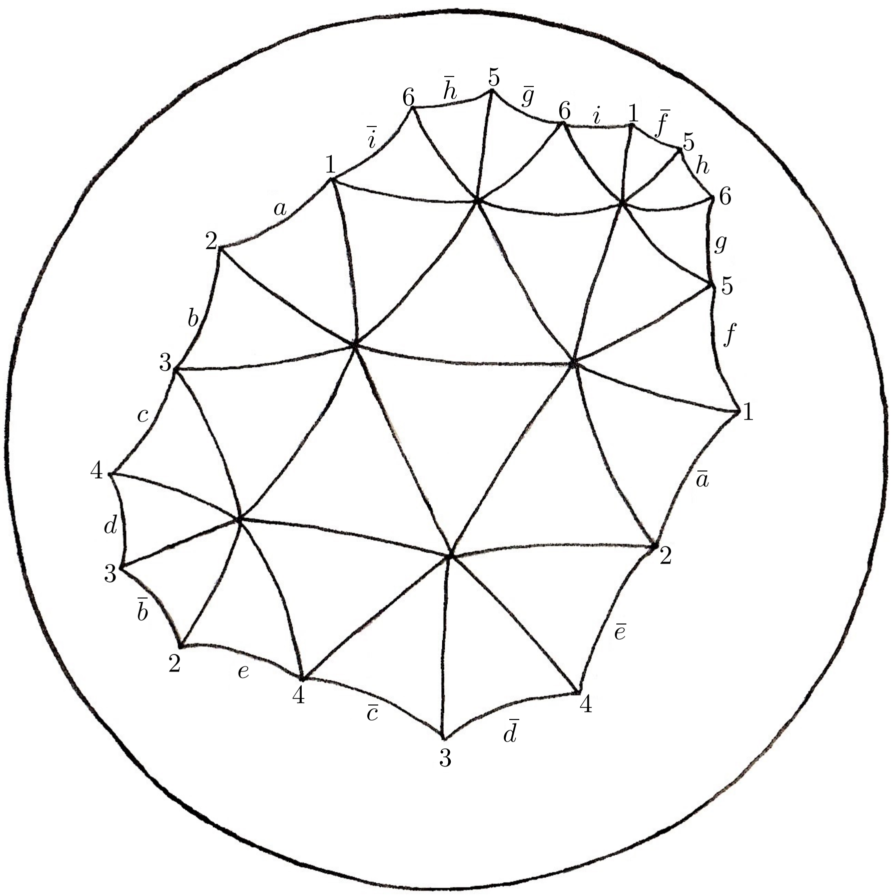


Figure 8: 28 *congruent triangles of a* $\{3, 7\}$ *tessellation of* $\mathbb{D}$*, forming an* 18*-gon. This* 18*-gon is equipped with side pairings that transform it into an orientable genus-*2 *surface. Sides are given letters, vertices on the border are numbered.*

The question now arises if we can use the method we described above to come to a proof of Proposition 4.1. It turns out that we can. We will give an algorithm that shows that a polygon like the one in Figure 8 can be constructed, starting from a topological tessellation. To put it briefly, *we are going to construct a spanning tree of the dual of the graph of the tessellation. At the same time we will construct an abstract graph in $\mathbb{D}$, isomorphic to this spanning tree. Its vertices will be regular polygons and the edges will be adjacency relations. This will form a polygon $S \subset \mathbb{D}$. The edges of the tessellation's dual graph that are not in the spanning tree will be associated with side pairings of F.*

This formulation is probably more brief than comprehensible. Therefore, we will give a thorough and lengthy explanation of the algorithm in the next section.

4.2 Going from topologically regular to metrically regular

In what follows, we will be needing the definition of a demi-edge of a graph.

Definition 4.2. A ***demi-edge*** is an edge of a graph G that is only associated with one vertex. If the vertices and edges of a graph are respectively points and arcs on some surface, then we can turn an edge into two demi-edges by removing one point from its interior. Note that this means that an embedded demi-edge is a half open subset of the original edge. See also Figure 9.

Figure 9: *An edge (left) and a demi-edge (right).*

Assume that we have a topological regular $\{p, q\}$ tessellation of some surface Σ_g, with V vertices, E edges and F faces. Call the tessellation T and call the graph of this tessellation G. We are going to construct a polygon $S \subset \mathbb{D}$ that we can identify with Σ_g made of F (metrically) regular p-gons, by an algorithm. We call this algorithm the *Topological-Tessellation to Metric-Tessellation Algorithm*, TT2MT in abbreviated form.

TT2MT Algorithm

Tessellate $\mathbb{D}$—Regularly tessellate $\mathbb{D}$ with a (metric) $\{p, q\}$ tessellation. Call this tessellation $\tilde{T}$.

Dual of T—Take the graph of a dual tessellation T^* of T. Call it G^*.

Dual of $\tilde{T}$—In $\mathbb{D}$, draw the vertices $\tilde{f}_i^*$ of the dual $\tilde{T}^*$ of $\tilde{T}$. Each of these vertices is the incentre of some face f_i of $\tilde{T}$.

Choose initial vertices—Choose an initial vertex $f_0^* \in V_{G^*}$. Also choose an initial vertex $\tilde{f}_0^* \in V_{\tilde{T}^*}$. Now map $f_0^* \mapsto \tilde{f}_0^*$.

Make spanning tree—Create a spanning tree τ^* of G^*. Make f_0^* the ***tree root*** (i.e. start at f_0^*). An edge $e_i^* \in E_{G^*} \backslash E_{\tau^*}$ is called a ***non-tree edge*** (NTE abbreviated).

Map spanning tree—Map the tree τ^* to the tree $\tilde{\tau}^* \subset \mathbb{D}$ by a map called φ that is defined by the following rules:

- $f_0^* \overset{\varphi}{\mapsto} \tilde{f}_0^*$, i.e. $\tilde{f}_0^*$ is the tree root of $\tilde{\tau}$.
- $E_{\tau^*} \overset{\varphi}{\to} E_{\tilde{T}^*}$, i.e. edges of τ^* are mapped to edges of $\tilde{T}^*$.
- Properties of incidence are preserved by φ, i.e. vertices and/or edges are incident, if and only if, their images are incident as well.

Note that $\varphi : \tau^* \to \tilde{\tau}^*$ is a graph isomorphism.

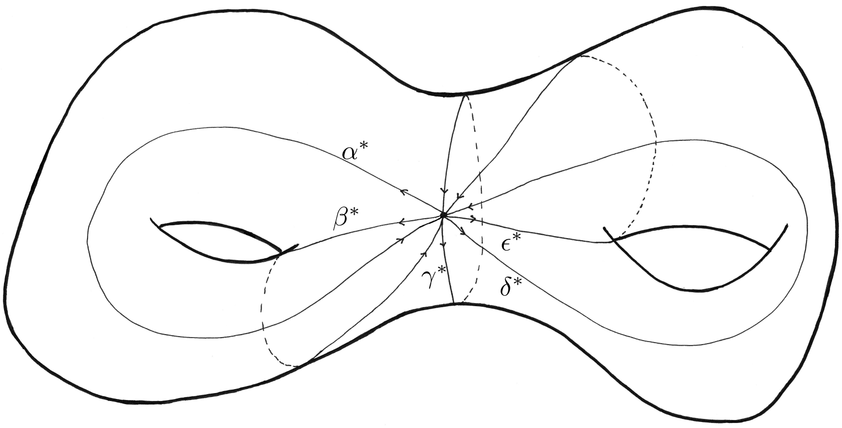
Make polygon—Each vertex of $\tilde{\tau}^*$ corresponds to exactly one face of $\tilde{T}$ and each edge to an adjacency relation between two faces. The union of all these faces form a polygon S in $\mathbb{D}$. This polygon is made of exactly F regular p-gons. Each side of S is a side of some tile of $\tilde{T}$. Be aware of the fact that some consecutive sides of S might meet at an angle π or greater.

Side pairings—Derive the side pairings from the NTEs. Do this as follows:

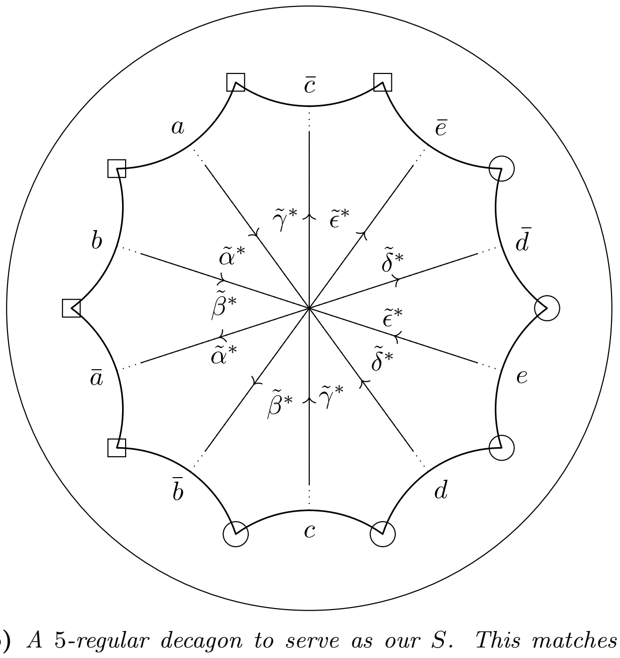
- Pick a vertex $f_i^* \in V_{G^*}$ that is incident to an NTE. Consecutively label its edges $e_1^*, \ldots, e_q^*$, such that e_1^* is not an NTE.
- Pick an NTE out of the e_i^*'s, were i is as small as possible. Call it n. It is incident to f_i^* and f_j^*, not necessarily distinct. Consecutively label the edges of f_j^* as $\varepsilon_1^*, \ldots, \varepsilon_q^*$, such that ε_1^* is not an NTE.
- These vertices are mapped, under φ, to $\tilde{f}_i^*$ and $\tilde{f}_j^*$ respectively. Say $\tilde{f}_i^* \in \tilde{f}_i$ and $\tilde{f}_j^* \in \tilde{f}_j$, where $\tilde{f}_i$ and $\tilde{f}_j$ are faces of $\tilde{T}$. Note that they lie in S.
- Take all edges of $\tilde{T}^*$ that are incident to $\tilde{f}_i^*$ and $\tilde{f}_j^*$. Consecutively label these $\tilde{e}_1^*, \ldots, \tilde{e}_q^*$ for $\tilde{f}_i^*$ and $\tilde{\varepsilon}_1^*, \ldots, \tilde{\varepsilon}_q^*$ for $\tilde{f}_j^*$, such that $\varphi(e_i^*) = \tilde{e}_i^*$ and $\varphi(\varepsilon_i^*) = \tilde{\varepsilon}_i^*$ for all non-NTEs. Note that the $\tilde{e}_i^*$'s are not necessarily distinct from the $\tilde{\varepsilon}_i^*$'s.
- Remove the point from all NTE's where they intersect with an edge of G. This creates a bunch of demi-edges. In particular, the NTE that we called n is now divided into two demi-edges n_i and n_j, incident to f_i^* and f_j^* respectively. We can now extend φ to all these new demi-edges, by requiring that it preserves the cyclic order of incident (demi-)edges around a vertex. Also, the length of the image of a demi-edge in $\tilde{T}^*$ under φ should be exactly half of that of the other edges in $\tilde{T}^*$.
- $\varphi(n_i)$ and $\varphi(n_j)$ now have points arbitrarily close to distinct sides of S. These sides are to be paired. This can be done by an isometry.

The result is a polygon S equipped with side pairings, which we call $g_1, \ldots, g_\ell$. Say $\Gamma = \langle g_1, \ldots, g_\ell \rangle$ and say $\psi : S \to S/\Gamma$. We define the metric on S/Γ to be the induced metric. Then we claim that $\psi(S \cap T)$ is a metric regular tessellation on S/Γ, which is a closed genus-g surface.

Before we prove our claim we would like to note that the last step we described above (Side pairings), is not necessary to prove the existence of the metrically regular tessellation. It only serves to aid in the description of such a tessellation.



(a) *The dual of a topological* $\{10, 5\}$ *tessellation on an orientable genus-2 surface.*



(b) *A* 5*-regular decagon to serve as our* S. *This matches with Figure 10a.*

Figure 10: *An example of parts of the TT2MT algorithm for* $\{p, q\} = \{10, 5\}$ *and genus* $g = 2$.

Example 4.3. Suppose we are on Σ_2 and we have a topological $\{10, 5\}$ tessellation T. We start by drawing its dual T^*, i.e. the topological $\{5, 10\}$ tessellation (Figure 10a) and the metric $\{10, 5\}$ tessellation $\tilde{T}$ in $\mathbb{D}$. The spanning tree τ^* is exactly the only vertex of T^*. It is mapped to a single vertex of $\tilde{T}^*$, corresponding to a 5-regular decagon (i.e. an equilateral decagon with angle sizes $\frac{2\pi}{5}$). This decagon is drawn in Figure 10b.

Now the edges $\alpha^*, \beta^*, \gamma^*, \delta^*, \epsilon^*$ are turned into 10 demi-edges. These are mapped to the demi-edges $\tilde{\alpha}^*, \tilde{\beta}^*, \tilde{\gamma}^*, \tilde{\delta}^*, \tilde{\epsilon}^*$ of Figure 10b respectively. Their cyclic order must be preserved during the process. The sides of the decagon must be paired as is now dictated by these demi-edges.

In Figure 10b we try to illustrate how the demi-edges determine the side pairings. Here we show how the demi-edges of G^* are mapped into S. It illustrates that we pair two sides if they are approached by corresponding demi-edges. We can also see that we have $V - E + F = 2 - 5 + 1 = -2$. We now have a description of a metric $\{10, 5\}$ tessellation on what must be a closed genus-2 Riemann surface.

Claim 4.4. *The TT2MT Algorithm proves the existence of a metric regular tessellation for every topological regular tessellation.*

Proof. It is not hard to see that we can always *cut open* Σ_g such that we are left with a simply-connected shape S'. Just cut along the edges of G that do not intersect τ^*.

Because S' was made by cutting along edges of G, we can naturally divide the border of S' into sides; each side corresponds to an edge of G. Moreover, the cutting dictates how we should identify pairs of these sides in order to get Σ_g back.

By construction, the (cyclic) order in which the sides of S are paired is the same as the order in which the sides of S' are paired (the former is derived from the latter). So we can safely say the the side pairings we have equipped S with, at least turn it into a topological surface of the right orientation and genus.

Also by construction, the sides of S are all of equal length (S is a union of congruent polygons). So we are free to choose orientation preserving isometries as side pairings.

All that's left to check are the angle sums. Each set of identified vertices is incident to q edges, where paired edges are counted as one. Each two neighbouring edges make an angle of size $\frac{2\pi}{q}$ by construction. There are q pairs of neighbouring edges. This means that our vertex has a full angle sum, as desired. See Figure 12 for an example.

We may now use Poincaré's Theorem on S and its side pairings. It is now trivial to see that we have described a metrically regular tessellation of a closed genus-g surface. ℛ

Example 4.5. Consider the case of a $\{3, 7\}$ tessellation on a genus-2 surface. In Figure 11a we have drawn a topological $\{3, 7\}$ tessellation T and a spanning tree τ^* of the graph of its dual tessellation T^*. In Figure 11a:

- The topological $\{3.7\}$ tessellation T is drawn with solid lines.
- We use colours to indicate how the triangles "wrap around" the surface (same colour=same triangle). If no confusion is possible, triangles are left white.

- The dual vertices are dots.
- The spanning tree τ^* of the graph of the dual tessellation T^* is drawn with dashed lines.
- We have drawn two NTEs with double dashed lines.

In Figure 11b we have drawn the tree $\tilde{\tau}^*$ in a convenient way. Each of the dual vertices lies within some face of an equilateral triangle, with all angles $\frac{2\pi}{7}$. A priori there is only adjacency when there is a tree-edge connecting two triangles, so one might say that this picture is misleading. We will shortly see that this will not be a problem.

We have indicated three dual vertices f_i^* f_j^* and f_k^* in Figure 11a. Their images under φ—$\tilde{f}_i^*$, $\tilde{f}_j^*$ and $\tilde{f}_k^*$ respectively—are indicated in Figure 11b.

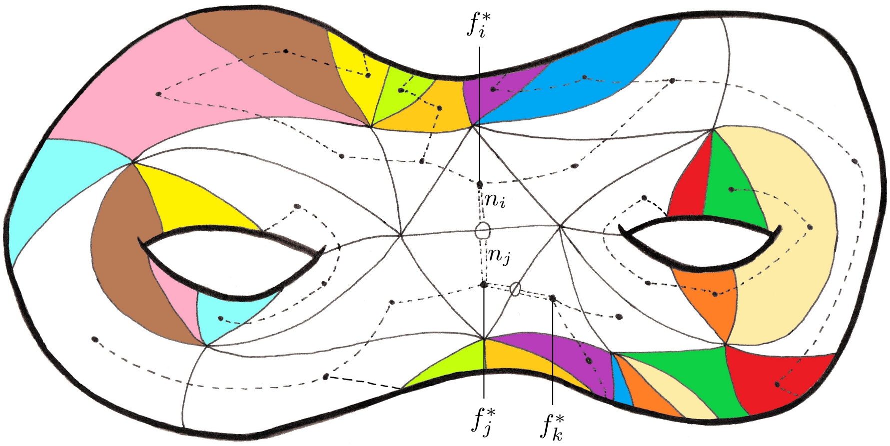
In Figure 11a we can see the NTE between f_i^* and f_j^*. It is divided into two demi-edges n_i and n_j by removing the pont indicated by a circle. The map φ can be extended to these two half edges in a way that preserves the cyclic order around the vertices f_i^* and f_j^*. Their images are drawn Figure 11b. Now we know that the sides we (prematurely) labelled a and $\bar{a}$ must be identified.

The NTE between f_j^* and f_k^* analogously induces an identification. The same thing must of course also be done for all other NTEs. The sides they identify are all, by construction, line segments in $\mathbb{D}$ of equal length. Thus, we can choose orientation preserving side pairings to make the identifications.

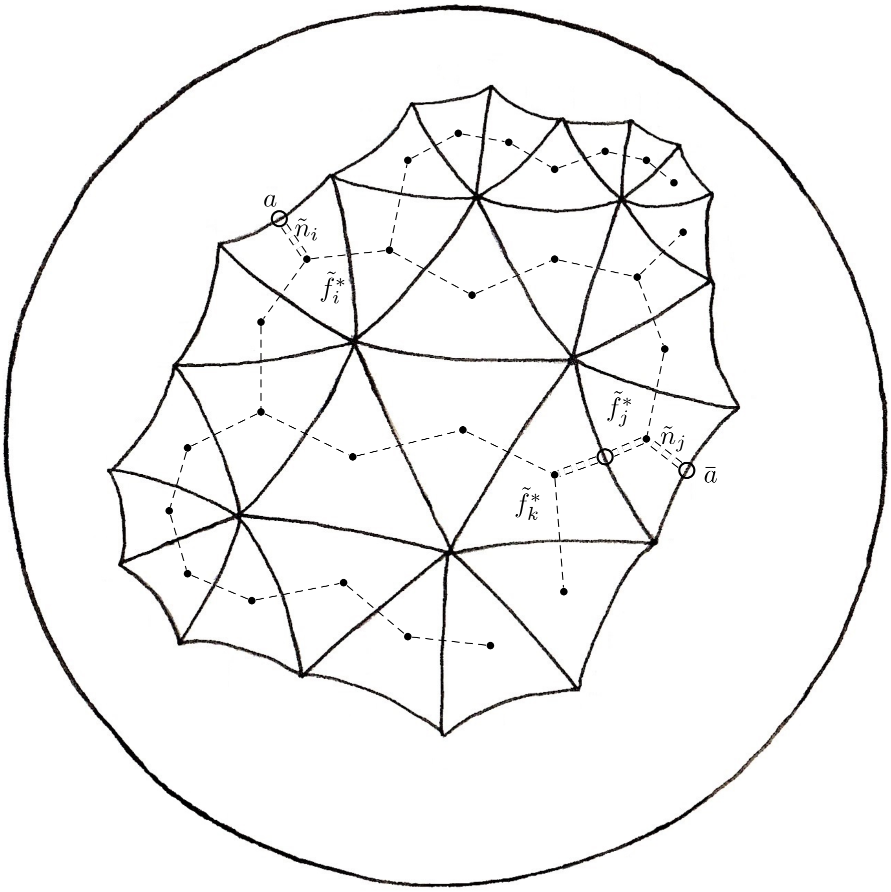
It is now obvious why we called our positioning of the tree $\tilde{\tau}^*$ "convenient": The triangles of Figure 11b that appear adjacent are in fact "made adjacent" by choosing side pairings. Many of these side parings will be the identity in Figure 11b.

After we made the identifications, we are left with Figure 12. Vertex v_3 is now incident, in cyclic order, to the seven edges $b, j, k, c = \bar{c}, m, \bar{d} = d, \ell, \bar{b} = b$. The angle between each consecutive pair is $\frac{2\pi}{7}$ and there are seven angles formed. So, v_3 has a full angle sum. Analogously, we show that every vertex on the edge of our shape (for instance v_7) has a full angle sum.

Using Poincaré's Theorem we can now deduce that we have found a metric $\{3, 7\}$ tessellation of a closed genus-2 Riemann surface.



(a)



(b)

Figure 11: *Deriving a description for a metric* $\{3, 7\}$ *tessellation on a closed genus-2 surface.*

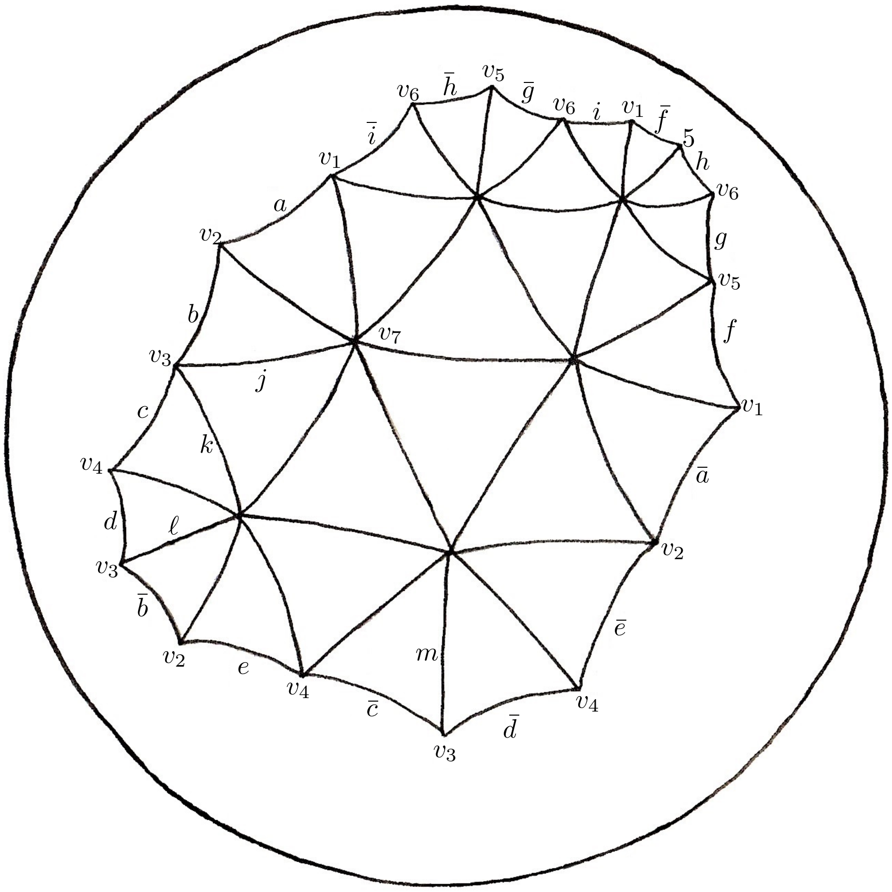


Figure 12: *v_3 has full angle sum.*

A An Octave script that prints what all possible $\{p, q\}$ tessellations for some closed orientable genus-g surface are into a file

In this appendix we show an Octave[4] script that will allow us to see what all possible $\{p, q\}$ tessellations are for a closed oreintable genus-g surface, for a certain g. The script is given in Listing 2. It is an implementation of the algorithm shown in Listing 1.

The script first checks if `manualinput=1`. This means that `manualinput` must be defined before running the script. If `manualinput=1`, then the script will ask to input a value for `g`. If `manualinput`$\neq$`1`, then it will not set a value for `g`. So in this case `g` must be defined before using the script.

Next it opens the files `Tessellations-of-a-genus-`g`-surface.txt` and `tableforgenus`g`.tex`. If those files don't exist, then they are craeated at the spot. If either or both of those files do exist, then their contents are deleted. Immediately thereafter it prints the first lines of those files.

The file `Tessellations-of-a-genus-`g`-surface.txt` is an ordinary .txt file. If the script finds that $\{p, q\}$ is possible, then it will say so in that file. At the end of the file it will say "There are no more possibilities.", emphasising that the list is exhaustive. It also prints the number of tessellations that were found.

The file `tableforgenus`g`.tex` creates a .tex file to be used by LaTeX. It contains a longtable-environment that will display all possible $\{p, q\}$ in a table. The tables in Appendix B are created with this method. Note that this requires the use of the LaTeX package "longtable".

After the first lines have been printed, we enter a double `for`-loop. It runs over all values of `p` and `q` described in Section 3.3. The script then calculates `V`, `E` and `F`, if possible. Else it sets `V=-1`.

Once this is done, the script checks if `V`, `E` and `F` all have positive integer values. The checking uses the `&&` operator. This acts like an ordinary logical `AND`, only it does not bother to check any more arguments, once it encounters a false one. So if `V`, `E` and `F` could not be calculated (and thus it set `V=-1`), we don't get an error message.

Once the script has found that a certain tessellation is possible, it prints corresponding lines into the .txt and .tex files. If not, then it does nothing. When the double `for`-loop is completed, it prints the final lines of the files and then closes them. Listing 3 and Listing 4 show the two files that are created by the script, when `g=2`.

```
% This is an Octave script that creates two files in which it prints what all possible topological regular {p,q} tessellations are for a closed orientable genus-g surface (g at least 2).
if manualinput==1 % Must be defined before using the script.
  g=input('Give genus: ');
end
```

[4] A scientific programming language that is mostly compatible with Matlab. See `https://www.gnu.org/software/octave/` for more information.

```
g % g stands for genus and must be defined before using the scipt, or through manual input (if manualinput=1).

% Create filenames that depend on g.
textfilename=sprintf('Tessellations-of-a-genus-%d-surface.txt',g);
tablefilename=sprintf('tableforgenus%d.tex',g);

% Open/Create files (delete all that is currently present).
textfile=fopen(textfilename,'w');
tablefile=fopen(tablefilename,'w');

% Print the first rules of the files.
fprintf(textfile,'This file lists all possible topological regular {p,q} tessellations for a closed orientable genus-%d surface.\r\nIt also gives the number of vertices (V), edges (E) and faces (F) of those tessellations.\r\n\r\n',g)
fprintf(tablefile,'\\begin{longtable}{|c|c|c}\r\n    \\caption{All possible topological regular tesselations of a closed orientable genus-$%d$ surface. A {\\normalfont *} denotes a tessellation with exactly one tile.\\label{table-for-genus-%d}}\\\\\r\n\\cline{1-2}\r\n    $\\mathbf{\\{p,q\\}}$&$\\mathbf{(V,E,F)}$&~\\endfirsthead\r\n    \\caption{genus-$%d$ continued\\ldots}\\\\\r\n\\cline{1-2}\r\n    $\\mathbf{\\{p,q\\}}$&$\\mathbf{(V,E,F)}$&~\\\\\\cline{1-2}\\endhead\r\n',g,g,g)
numberOfTessellations=0;

% Loop through all reasonable p and q.
for p = 3: 12*g-6;
    for q =3: (2*p*(2-2*g)-2*p)/(2-p);
      if 2*p-p*q+2*q~=0;
        V=(2*p*(2-2*g))/(2*p-p*q+2*q);
        E=(p*q*(2-2*g))/(2*p-p*q+2*q);
        F=(2*q*(2-2*g))/(2*p-p*q+2*q);
        else V=-1;
      end
      % Check if {p,q} is a possible tessellation. If so, write lines to the files.
      if V>0&&V==floor(V)&&E>0&&E==floor(E)&&F>0&&F==floor(F);
          fprintf(textfile,'(%d,%d) is a possibility.\r\n    It gives (V,E,F)=(%d,%d,%d).\r\n',p,q,V,E,F);
          numberOfTessellations=numberOfTessellations+1;
          if F==1;
```

```
                fprintf(textfile,'    This means that the %d-regular %d-gon can serve as a fundamental polygon.\r\n\r\n',q,p)
                fprintf(tablefile,'\\cline{1-2}\r\n$\\{%d,%d\\}$&$(%d,%d,%d)$&\\multicolumn{1}{@{}l}{*}\\\\\r\n',p,q,V,E,F)
              else fprintf(tablefile,'\\cline{1-2}\r\n$\\{%d,%d\\}$&$(%d,%d,%d)$&\\\\\r\n',p,q,V,E,F)
              end
        end
      end
end

% Print end of files.
fprintf(textfile,'There are no more possibilities.\r\nThere are %d tessellations in total',numberOfTessellations);
fprintf(tablefile,'\\cline{1-2}\r\n\\end{longtable}')

% Close files.
fclose(textfile);
fclose(tablefile);
```

Listing 2: *Octave script that finds all possible tessellations $\{p, q\}$ of a genus-g surface. The results are printed into a .txt file and into a .tex file. The source file has a .m extension.*

```
This file lists all possible topological regular {p,q} tessellations for a closed orientable genus-2 surface.
It also gives the number of vertices (V), edges (E) and faces (F) of those tessellations.

(3,7) is a possibility.
    It gives (V,E,F)=(12,42,28).
(3,8) is a possibility.
    It gives (V,E,F)=(6,24,16).
(3,9) is a possibility.
    It gives (V,E,F)=(4,18,12).
(3,10) is a possibility.
    It gives (V,E,F)=(3,15,10).
(3,12) is a possibility.
    It gives (V,E,F)=(2,12,8).
(3,18) is a possibility.
    It gives (V,E,F)=(1,9,6).
(4,5) is a possibility.
    It gives (V,E,F)=(8,20,10).
(4,6) is a possibility.
    It gives (V,E,F)=(4,12,6).
(4,8) is a possibility.
```

```
    It gives (V,E,F)=(2,8,4).
(4,12) is a possibility.
    It gives (V,E,F)=(1,6,3).
(5,4) is a possibility.
    It gives (V,E,F)=(10,20,8).
(5,5) is a possibility.
    It gives (V,E,F)=(4,10,4).
(5,10) is a possibility.
    It gives (V,E,F)=(1,5,2).
(6,4) is a possibility.
    It gives (V,E,F)=(6,12,4).
(6,6) is a possibility.
    It gives (V,E,F)=(2,6,2).
(7,3) is a possibility.
    It gives (V,E,F)=(28,42,12).
(8,3) is a possibility.
    It gives (V,E,F)=(16,24,6).
(8,4) is a possibility.
    It gives (V,E,F)=(4,8,2).
(8,8) is a possibility.
    It gives (V,E,F)=(1,4,1).
    This means that the 8-regular 8-gon can serve as a fundamental polygon.

(9,3) is a possibility.
    It gives (V,E,F)=(12,18,4).
(10,3) is a possibility.
    It gives (V,E,F)=(10,15,3).
(10,5) is a possibility.
    It gives (V,E,F)=(2,5,1).
    This means that the 5-regular 10-gon can serve as a fundamental polygon.

(12,3) is a possibility.
    It gives (V,E,F)=(8,12,2).
(12,4) is a possibility.
    It gives (V,E,F)=(3,6,1).
    This means that the 4-regular 12-gon can serve as a fundamental polygon.

(18,3) is a possibility.
    It gives (V,E,F)=(6,9,1).
    This means that the 3-regular 18-gon can serve as a fundamental polygon.

There are no more possibilities.
There are 25 tessellations in total
```

Listing 3: *The contents of the file `Tessellations-of-a-genus-2-surface.txt` that the script in Listing 2 creates when `g=2`.*

```latex
\begin{longtable}{|c|c|c}
    \caption{All possible topological regular tesselations of a closed orientable genus-$2$ surface. A {\normalfont *} denotes a tessellation with exactly one tile.\label{table-for-genus-2}}\\
\cline{1-2}
    $\mathbf{\{p,q\}}$&$\mathbf{(V,E,F)}$&~\endfirsthead
    \caption{genus-$2$ continued\ldots}\\
\cline{1-2}
    $\mathbf{\{p,q\}}$&$\mathbf{(V,E,F)}$&~\\\cline{1-2}\endhead
\cline{1-2}
$\{3,7\}$&$(12,42,28)$&\\
\cline{1-2}
$\{3,8\}$&$(6,24,16)$&\\
\cline{1-2}
$\{3,9\}$&$(4,18,12)$&\\
\cline{1-2}
$\{3,10\}$&$(3,15,10)$&\\
\cline{1-2}
$\{3,12\}$&$(2,12,8)$&\\
\cline{1-2}
$\{3,18\}$&$(1,9,6)$&\\
\cline{1-2}
$\{4,5\}$&$(8,20,10)$&\\
\cline{1-2}
$\{4,6\}$&$(4,12,6)$&\\
\cline{1-2}
$\{4,8\}$&$(2,8,4)$&\\
\cline{1-2}
$\{4,12\}$&$(1,6,3)$&\\
\cline{1-2}
$\{5,4\}$&$(10,20,8)$&\\
\cline{1-2}
$\{5,5\}$&$(4,10,4)$&\\
\cline{1-2}
$\{5,10\}$&$(1,5,2)$&\\
\cline{1-2}
$\{6,4\}$&$(6,12,4)$&\\
\cline{1-2}
$\{6,6\}$&$(2,6,2)$&\\
\cline{1-2}
$\{7,3\}$&$(28,42,12)$&\\
\cline{1-2}
$\{8,3\}$&$(16,24,6)$&\\
\cline{1-2}
$\{8,4\}$&$(4,8,2)$&\\
\cline{1-2}
$\{8,8\}$&$(1,4,1)$&\multicolumn{1}{@{}l}{*}\\
```

```
\cline{1-2}
$\{9,3\}$&$(12,18,4)$&\\
\cline{1-2}
$\{10,3\}$&$(10,15,3)$&\\
\cline{1-2}
$\{10,5\}$&$(2,5,1)$&\multicolumn{1}{@{}l}{*}\\
\cline{1-2}
$\{12,3\}$&$(8,12,2)$&\\
\cline{1-2}
$\{12,4\}$&$(3,6,1)$&\multicolumn{1}{@{}l}{*}\\
\cline{1-2}
$\{18,3\}$&$(6,9,1)$&\multicolumn{1}{@{}l}{*}\\
\cline{1-2}
\end{longtable}
```

Listing 4: *The contents of the file `tableforgenus2.tex` that the script in Listing 2 creates when `g=2`. pdfLaTeX makes Table 1 out of these lines.*

B All regular tessellations of closed orientable surfaces of genus 2 to 10

Table 1: *All possible topological regular tesselations of a closed orientable genus-2 surface. A * denotes a tessellation with exactly one tile.*

$\{\mathbf{p},\mathbf{q}\}$	$(\mathbf{V},\mathbf{E},\mathbf{F})$
$\{3,7\}$	$(12,42,28)$
$\{3,8\}$	$(6,24,16)$
$\{3,9\}$	$(4,18,12)$
$\{3,10\}$	$(3,15,10)$
$\{3,12\}$	$(2,12,8)$
$\{3,18\}$	$(1,9,6)$
$\{4,5\}$	$(8,20,10)$
$\{4,6\}$	$(4,12,6)$
$\{4,8\}$	$(2,8,4)$
$\{4,12\}$	$(1,6,3)$
$\{5,4\}$	$(10,20,8)$
$\{5,5\}$	$(4,10,4)$
$\{5,10\}$	$(1,5,2)$

Table 1: *genus-2 continued...*

$\{\mathbf{p},\mathbf{q}\}$	$(\mathbf{V},\mathbf{E},\mathbf{F})$	
$\{6,4\}$	$(6,12,4)$	
$\{6,6\}$	$(2,6,2)$	
$\{7,3\}$	$(28,42,12)$	
$\{8,3\}$	$(16,24,6)$	
$\{8,4\}$	$(4,8,2)$	
$\{8,8\}$	$(1,4,1)$	*
$\{9,3\}$	$(12,18,4)$	
$\{10,3\}$	$(10,15,3)$	
$\{10,5\}$	$(2,5,1)$	*
$\{12,3\}$	$(8,12,2)$	
$\{12,4\}$	$(3,6,1)$	*
$\{18,3\}$	$(6,9,1)$	*

Table 2: *All possible topological regular tesselations of a closed orientable genus-3 surface. A * denotes a tessellation with exactly one tile.*

$\{\mathbf{p},\mathbf{q}\}$	$(\mathbf{V},\mathbf{E},\mathbf{F})$
$\{3,7\}$	$(24,84,56)$
$\{3,8\}$	$(12,48,32)$
$\{3,9\}$	$(8,36,24)$
$\{3,10\}$	$(6,30,20)$
$\{3,12\}$	$(4,24,16)$
$\{3,14\}$	$(3,21,14)$
$\{3,18\}$	$(2,18,12)$
$\{3,30\}$	$(1,15,10)$
$\{4,5\}$	$(16,40,20)$
$\{4,6\}$	$(8,24,12)$
$\{4,8\}$	$(4,16,8)$
$\{4,12\}$	$(2,12,6)$
$\{4,20\}$	$(1,10,5)$
$\{5,4\}$	$(20,40,16)$

Table 2: *genus-3 continued…*

$\{\mathbf{p}, \mathbf{q}\}$	$(\mathbf{V}, \mathbf{E}, \mathbf{F})$	
$\{5, 5\}$	$(8, 20, 8)$	
$\{5, 6\}$	$(5, 15, 6)$	
$\{5, 10\}$	$(2, 10, 4)$	
$\{6, 4\}$	$(12, 24, 8)$	
$\{6, 5\}$	$(6, 15, 5)$	
$\{6, 6\}$	$(4, 12, 4)$	
$\{6, 9\}$	$(2, 9, 3)$	
$\{7, 3\}$	$(56, 84, 24)$	
$\{7, 14\}$	$(1, 7, 2)$	
$\{8, 3\}$	$(32, 48, 12)$	
$\{8, 4\}$	$(8, 16, 4)$	
$\{8, 8\}$	$(2, 8, 2)$	
$\{9, 3\}$	$(24, 36, 8)$	
$\{9, 6\}$	$(3, 9, 2)$	
$\{10, 3\}$	$(20, 30, 6)$	
$\{10, 5\}$	$(4, 10, 2)$	
$\{12, 3\}$	$(16, 24, 4)$	
$\{12, 4\}$	$(6, 12, 2)$	
$\{12, 12\}$	$(1, 6, 1)$	*
$\{14, 3\}$	$(14, 21, 3)$	
$\{14, 7\}$	$(2, 7, 1)$	*
$\{18, 3\}$	$(12, 18, 2)$	
$\{20, 4\}$	$(5, 10, 1)$	*
$\{30, 3\}$	$(10, 15, 1)$	*

Table 3: *All possible topological regular tesselations of a closed orientable genus-4 surface. A * denotes a tessellation with exactly one tile.*

$\{\mathbf{p}, \mathbf{q}\}$	$(\mathbf{V}, \mathbf{E}, \mathbf{F})$
$\{3, 7\}$	$(36, 126, 84)$
$\{3, 8\}$	$(18, 72, 48)$

Table 3: *genus*-4 *continued...*

$\{\mathbf{p}, \mathbf{q}\}$	$(\mathbf{V}, \mathbf{E}, \mathbf{F})$
$\{3, 9\}$	$(12, 54, 36)$
$\{3, 10\}$	$(9, 45, 30)$
$\{3, 12\}$	$(6, 36, 24)$
$\{3, 15\}$	$(4, 30, 20)$
$\{3, 18\}$	$(3, 27, 18)$
$\{3, 24\}$	$(2, 24, 16)$
$\{3, 42\}$	$(1, 21, 14)$
$\{4, 5\}$	$(24, 60, 30)$
$\{4, 6\}$	$(12, 36, 18)$
$\{4, 7\}$	$(8, 28, 14)$
$\{4, 8\}$	$(6, 24, 12)$
$\{4, 10\}$	$(4, 20, 10)$
$\{4, 12\}$	$(3, 18, 9)$
$\{4, 16\}$	$(2, 16, 8)$
$\{4, 28\}$	$(1, 14, 7)$
$\{5, 4\}$	$(30, 60, 24)$
$\{5, 5\}$	$(12, 30, 12)$
$\{5, 10\}$	$(3, 15, 6)$
$\{6, 4\}$	$(18, 36, 12)$
$\{6, 6\}$	$(6, 18, 6)$
$\{6, 12\}$	$(2, 12, 4)$
$\{7, 3\}$	$(84, 126, 36)$
$\{7, 4\}$	$(14, 28, 8)$
$\{7, 7\}$	$(4, 14, 4)$
$\{8, 3\}$	$(48, 72, 18)$
$\{8, 4\}$	$(12, 24, 6)$
$\{8, 8\}$	$(3, 12, 3)$
$\{9, 3\}$	$(36, 54, 12)$
$\{9, 18\}$	$(1, 9, 2)$
$\{10, 3\}$	$(30, 45, 9)$
$\{10, 4\}$	$(10, 20, 4)$

Table 3: *genus-4 continued...*

$\{\mathbf{p}, \mathbf{q}\}$	$(\mathbf{V}, \mathbf{E}, \mathbf{F})$	
$\{10, 5\}$	$(6, 15, 3)$	
$\{10, 10\}$	$(2, 10, 2)$	
$\{12, 3\}$	$(24, 36, 6)$	
$\{12, 4\}$	$(9, 18, 3)$	
$\{12, 6\}$	$(4, 12, 2)$	
$\{15, 3\}$	$(20, 30, 4)$	
$\{16, 4\}$	$(8, 16, 2)$	
$\{16, 16\}$	$(1, 8, 1)$	*
$\{18, 3\}$	$(18, 27, 3)$	
$\{18, 9\}$	$(2, 9, 1)$	*
$\{24, 3\}$	$(16, 24, 2)$	
$\{28, 4\}$	$(7, 14, 1)$	*
$\{42, 3\}$	$(14, 21, 1)$	*

Table 4: *All possible topological regular tesselations of a closed orientable genus-5 surface. A * denotes a tessellation with exactly one tile.*

$\{\mathbf{p}, \mathbf{q}\}$	$(\mathbf{V}, \mathbf{E}, \mathbf{F})$
$\{3, 7\}$	$(48, 168, 112)$
$\{3, 8\}$	$(24, 96, 64)$
$\{3, 9\}$	$(16, 72, 48)$
$\{3, 10\}$	$(12, 60, 40)$
$\{3, 12\}$	$(8, 48, 32)$
$\{3, 14\}$	$(6, 42, 28)$
$\{3, 18\}$	$(4, 36, 24)$
$\{3, 22\}$	$(3, 33, 22)$
$\{3, 30\}$	$(2, 30, 20)$
$\{3, 54\}$	$(1, 27, 18)$
$\{4, 5\}$	$(32, 80, 40)$
$\{4, 6\}$	$(16, 48, 24)$
$\{4, 8\}$	$(8, 32, 16)$

Table 4: *genus-5 continued...*

$\{\mathbf{p}, \mathbf{q}\}$	$(\mathbf{V}, \mathbf{E}, \mathbf{F})$
$\{4, 12\}$	$(4, 24, 12)$
$\{4, 20\}$	$(2, 20, 10)$
$\{4, 36\}$	$(1, 18, 9)$
$\{5, 4\}$	$(40, 80, 32)$
$\{5, 5\}$	$(16, 40, 16)$
$\{5, 6\}$	$(10, 30, 12)$
$\{5, 10\}$	$(4, 20, 8)$
$\{5, 30\}$	$(1, 15, 6)$
$\{6, 4\}$	$(24, 48, 16)$
$\{6, 5\}$	$(12, 30, 10)$
$\{6, 6\}$	$(8, 24, 8)$
$\{6, 7\}$	$(6, 21, 7)$
$\{6, 9\}$	$(4, 18, 6)$
$\{6, 15\}$	$(2, 15, 5)$
$\{7, 3\}$	$(112, 168, 48)$
$\{7, 6\}$	$(7, 21, 6)$
$\{7, 14\}$	$(2, 14, 4)$
$\{8, 3\}$	$(64, 96, 24)$
$\{8, 4\}$	$(16, 32, 8)$
$\{8, 8\}$	$(4, 16, 4)$
$\{8, 24\}$	$(1, 12, 3)$
$\{9, 3\}$	$(48, 72, 16)$
$\{9, 6\}$	$(6, 18, 4)$
$\{10, 3\}$	$(40, 60, 12)$
$\{10, 5\}$	$(8, 20, 4)$
$\{11, 22\}$	$(1, 11, 2)$
$\{12, 3\}$	$(32, 48, 8)$
$\{12, 4\}$	$(12, 24, 4)$
$\{12, 12\}$	$(2, 12, 2)$
$\{14, 3\}$	$(28, 42, 6)$
$\{14, 7\}$	$(4, 14, 2)$

Table 4: *genus-5 continued...*

$\{\mathbf{p}, \mathbf{q}\}$	$(\mathbf{V}, \mathbf{E}, \mathbf{F})$	
$\{15, 6\}$	$(5, 15, 2)$	
$\{18, 3\}$	$(24, 36, 4)$	
$\{20, 4\}$	$(10, 20, 2)$	
$\{20, 20\}$	$(1, 10, 1)$	*
$\{22, 3\}$	$(22, 33, 3)$	
$\{22, 11\}$	$(2, 11, 1)$	*
$\{24, 8\}$	$(3, 12, 1)$	*
$\{30, 3\}$	$(20, 30, 2)$	
$\{30, 5\}$	$(6, 15, 1)$	*
$\{36, 4\}$	$(9, 18, 1)$	*
$\{54, 3\}$	$(18, 27, 1)$	*

Table 5: *All possible topological regular tesselations of a closed orientable genus-6 surface. A * denotes a tessellation with exactly one tile.*

$\{\mathbf{p}, \mathbf{q}\}$	$(\mathbf{V}, \mathbf{E}, \mathbf{F})$
$\{3, 7\}$	$(60, 210, 140)$
$\{3, 8\}$	$(30, 120, 80)$
$\{3, 9\}$	$(20, 90, 60)$
$\{3, 10\}$	$(15, 75, 50)$
$\{3, 11\}$	$(12, 66, 44)$
$\{3, 12\}$	$(10, 60, 40)$
$\{3, 16\}$	$(6, 48, 32)$
$\{3, 18\}$	$(5, 45, 30)$
$\{3, 21\}$	$(4, 42, 28)$
$\{3, 26\}$	$(3, 39, 26)$
$\{3, 36\}$	$(2, 36, 24)$
$\{3, 66\}$	$(1, 33, 22)$
$\{4, 5\}$	$(40, 100, 50)$
$\{4, 6\}$	$(20, 60, 30)$
$\{4, 8\}$	$(10, 40, 20)$

Table 5: *genus-6 continued...*

$\{\mathbf{p},\mathbf{q}\}$	$(\mathbf{V},\mathbf{E},\mathbf{F})$
$\{4,9\}$	$(8,36,18)$
$\{4,12\}$	$(5,30,15)$
$\{4,14\}$	$(4,28,14)$
$\{4,24\}$	$(2,24,12)$
$\{4,44\}$	$(1,22,11)$
$\{5,4\}$	$(50,100,40)$
$\{5,5\}$	$(20,50,20)$
$\{5,10\}$	$(5,25,10)$
$\{5,20\}$	$(2,20,8)$
$\{6,4\}$	$(30,60,20)$
$\{6,6\}$	$(10,30,10)$
$\{6,8\}$	$(6,24,8)$
$\{6,18\}$	$(2,18,6)$
$\{7,3\}$	$(140,210,60)$
$\{8,3\}$	$(80,120,30)$
$\{8,4\}$	$(20,40,10)$
$\{8,6\}$	$(8,24,6)$
$\{8,8\}$	$(5,20,5)$
$\{8,16\}$	$(2,16,4)$
$\{9,3\}$	$(60,90,20)$
$\{9,4\}$	$(18,36,8)$
$\{9,9\}$	$(4,18,4)$
$\{10,3\}$	$(50,75,15)$
$\{10,5\}$	$(10,25,5)$
$\{10,15\}$	$(2,15,3)$
$\{11,3\}$	$(44,66,12)$
$\{12,3\}$	$(40,60,10)$
$\{12,4\}$	$(15,30,5)$
$\{13,26\}$	$(1,13,2)$
$\{14,4\}$	$(14,28,4)$
$\{14,14\}$	$(2,14,2)$

Table 5: *genus-6 continued...*

$\{\mathbf{p},\mathbf{q}\}$	$(\mathbf{V},\mathbf{E},\mathbf{F})$	
$\{15,10\}$	$(3,15,2)$	
$\{16,3\}$	$(32,48,6)$	
$\{16,8\}$	$(4,16,2)$	
$\{18,3\}$	$(30,45,5)$	
$\{18,6\}$	$(6,18,2)$	
$\{20,5\}$	$(8,20,2)$	
$\{21,3\}$	$(28,42,4)$	
$\{24,4\}$	$(12,24,2)$	
$\{24,24\}$	$(1,12,1)$	*
$\{26,3\}$	$(26,39,3)$	
$\{26,13\}$	$(2,13,1)$	*
$\{36,3\}$	$(24,36,2)$	
$\{44,4\}$	$(11,22,1)$	*
$\{66,3\}$	$(22,33,1)$	*

Table 6: *All possible topological regular tesselations of a closed orientable genus-7 surface. A * denotes a tessellation with exactly one tile.*

$\{\mathbf{p},\mathbf{q}\}$	$(\mathbf{V},\mathbf{E},\mathbf{F})$
$\{3,7\}$	$(72,252,168)$
$\{3,8\}$	$(36,144,96)$
$\{3,9\}$	$(24,108,72)$
$\{3,10\}$	$(18,90,60)$
$\{3,12\}$	$(12,72,48)$
$\{3,14\}$	$(9,63,42)$
$\{3,15\}$	$(8,60,40)$
$\{3,18\}$	$(6,54,36)$
$\{3,24\}$	$(4,48,32)$
$\{3,30\}$	$(3,45,30)$
$\{3,42\}$	$(2,42,28)$
$\{3,78\}$	$(1,39,26)$

Table 6: *genus-7 continued...*

$\{\mathbf{p},\mathbf{q}\}$	$(\mathbf{V},\mathbf{E},\mathbf{F})$
$\{4,5\}$	$(48,120,60)$
$\{4,6\}$	$(24,72,36)$
$\{4,7\}$	$(16,56,28)$
$\{4,8\}$	$(12,48,24)$
$\{4,10\}$	$(8,40,20)$
$\{4,12\}$	$(6,36,18)$
$\{4,16\}$	$(4,32,16)$
$\{4,20\}$	$(3,30,15)$
$\{4,28\}$	$(2,28,14)$
$\{4,52\}$	$(1,26,13)$
$\{5,4\}$	$(60,120,48)$
$\{5,5\}$	$(24,60,24)$
$\{5,6\}$	$(15,45,18)$
$\{5,10\}$	$(6,30,12)$
$\{6,4\}$	$(36,72,24)$
$\{6,5\}$	$(18,45,15)$
$\{6,6\}$	$(12,36,12)$
$\{6,9\}$	$(6,27,9)$
$\{6,12\}$	$(4,24,8)$
$\{6,21\}$	$(2,21,7)$
$\{7,3\}$	$(168,252,72)$
$\{7,4\}$	$(28,56,16)$
$\{7,7\}$	$(8,28,8)$
$\{7,14\}$	$(3,21,6)$
$\{8,3\}$	$(96,144,36)$
$\{8,4\}$	$(24,48,12)$
$\{8,8\}$	$(6,24,6)$
$\{9,3\}$	$(72,108,24)$
$\{9,6\}$	$(9,27,6)$
$\{9,18\}$	$(2,18,4)$
$\{10,3\}$	$(60,90,18)$

Table 6: *genus-7 continued...*

$\{\mathbf{p}, \mathbf{q}\}$	$(\mathbf{V}, \mathbf{E}, \mathbf{F})$	
$\{10, 4\}$	$(20, 40, 8)$	
$\{10, 5\}$	$(12, 30, 6)$	
$\{10, 10\}$	$(4, 20, 4)$	
$\{12, 3\}$	$(48, 72, 12)$	
$\{12, 4\}$	$(18, 36, 6)$	
$\{12, 6\}$	$(8, 24, 4)$	
$\{12, 12\}$	$(3, 18, 3)$	
$\{14, 3\}$	$(42, 63, 9)$	
$\{14, 7\}$	$(6, 21, 3)$	
$\{15, 3\}$	$(40, 60, 8)$	
$\{15, 30\}$	$(1, 15, 2)$	
$\{16, 4\}$	$(16, 32, 4)$	
$\{16, 16\}$	$(2, 16, 2)$	
$\{18, 3\}$	$(36, 54, 6)$	
$\{18, 9\}$	$(4, 18, 2)$	
$\{20, 4\}$	$(15, 30, 3)$	
$\{21, 6\}$	$(7, 21, 2)$	
$\{24, 3\}$	$(32, 48, 4)$	
$\{28, 4\}$	$(14, 28, 2)$	
$\{28, 28\}$	$(1, 14, 1)$	*
$\{30, 3\}$	$(30, 45, 3)$	
$\{30, 15\}$	$(2, 15, 1)$	*
$\{42, 3\}$	$(28, 42, 2)$	
$\{52, 4\}$	$(13, 26, 1)$	*
$\{78, 3\}$	$(26, 39, 1)$	*

Table 7: *All possible topological regular tesselations of a closed orientable genus-8 surface. A * denotes a tessellation with exactly one tile.*

$\{\mathbf{p}, \mathbf{q}\}$	$(\mathbf{V}, \mathbf{E}, \mathbf{F})$
$\{3, 7\}$	$(84, 294, 196)$

Table 7: *genus-8 continued...*

$\{\mathbf{p},\mathbf{q}\}$	$(\mathbf{V},\mathbf{E},\mathbf{F})$
$\{3,8\}$	$(42,168,112)$
$\{3,9\}$	$(28,126,84)$
$\{3,10\}$	$(21,105,70)$
$\{3,12\}$	$(14,84,56)$
$\{3,13\}$	$(12,78,52)$
$\{3,18\}$	$(7,63,42)$
$\{3,20\}$	$(6,60,40)$
$\{3,27\}$	$(4,54,36)$
$\{3,34\}$	$(3,51,34)$
$\{3,48\}$	$(2,48,32)$
$\{3,90\}$	$(1,45,30)$
$\{4,5\}$	$(56,140,70)$
$\{4,6\}$	$(28,84,42)$
$\{4,8\}$	$(14,56,28)$
$\{4,11\}$	$(8,44,22)$
$\{4,12\}$	$(7,42,21)$
$\{4,18\}$	$(4,36,18)$
$\{4,32\}$	$(2,32,16)$
$\{4,60\}$	$(1,30,15)$
$\{5,4\}$	$(70,140,56)$
$\{5,5\}$	$(28,70,28)$
$\{5,8\}$	$(10,40,16)$
$\{5,10\}$	$(7,35,14)$
$\{5,15\}$	$(4,30,12)$
$\{5,50\}$	$(1,25,10)$
$\{6,4\}$	$(42,84,28)$
$\{6,6\}$	$(14,42,14)$
$\{6,10\}$	$(6,30,10)$
$\{6,24\}$	$(2,24,8)$
$\{7,3\}$	$(196,294,84)$
$\{7,42\}$	$(1,21,6)$

Table 7: *genus*-8 *continued...*

$\{\mathbf{p},\mathbf{q}\}$	$(\mathbf{V},\mathbf{E},\mathbf{F})$	
$\{8,3\}$	$(112,168,42)$	
$\{8,4\}$	$(28,56,14)$	
$\{8,5\}$	$(16,40,10)$	
$\{8,8\}$	$(7,28,7)$	
$\{8,12\}$	$(4,24,6)$	
$\{8,40\}$	$(1,20,5)$	
$\{9,3\}$	$(84,126,28)$	
$\{10,3\}$	$(70,105,21)$	
$\{10,5\}$	$(14,35,7)$	
$\{10,6\}$	$(10,30,6)$	
$\{10,20\}$	$(2,20,4)$	
$\{11,4\}$	$(22,44,8)$	
$\{11,11\}$	$(4,22,4)$	
$\{12,3\}$	$(56,84,14)$	
$\{12,4\}$	$(21,42,7)$	
$\{12,8\}$	$(6,24,4)$	
$\{12,36\}$	$(1,18,3)$	
$\{13,3\}$	$(52,78,12)$	
$\{15,5\}$	$(12,30,4)$	
$\{17,34\}$	$(1,17,2)$	
$\{18,3\}$	$(42,63,7)$	
$\{18,4\}$	$(18,36,4)$	
$\{18,18\}$	$(2,18,2)$	
$\{20,3\}$	$(40,60,6)$	
$\{20,10\}$	$(4,20,2)$	
$\{24,6\}$	$(8,24,2)$	
$\{27,3\}$	$(36,54,4)$	
$\{32,4\}$	$(16,32,2)$	
$\{32,32\}$	$(1,16,1)$	*
$\{34,3\}$	$(34,51,3)$	
$\{34,17\}$	$(2,17,1)$	*

Table 7: *genus-8 continued...*

$\{\mathbf{p}, \mathbf{q}\}$	$(\mathbf{V}, \mathbf{E}, \mathbf{F})$	
$\{36, 12\}$	$(3, 18, 1)$	*
$\{40, 8\}$	$(5, 20, 1)$	*
$\{42, 7\}$	$(6, 21, 1)$	*
$\{48, 3\}$	$(32, 48, 2)$	
$\{50, 5\}$	$(10, 25, 1)$	*
$\{60, 4\}$	$(15, 30, 1)$	*
$\{90, 3\}$	$(30, 45, 1)$	*

Table 8: *All possible topological regular tesselations of a closed orientable genus-9 surface. A * denotes a tessellation with exactly one tile.*

$\{\mathbf{p}, \mathbf{q}\}$	$(\mathbf{V}, \mathbf{E}, \mathbf{F})$
$\{3, 7\}$	$(96, 336, 224)$
$\{3, 8\}$	$(48, 192, 128)$
$\{3, 9\}$	$(32, 144, 96)$
$\{3, 10\}$	$(24, 120, 80)$
$\{3, 12\}$	$(16, 96, 64)$
$\{3, 14\}$	$(12, 84, 56)$
$\{3, 18\}$	$(8, 72, 48)$
$\{3, 22\}$	$(6, 66, 44)$
$\{3, 30\}$	$(4, 60, 40)$
$\{3, 38\}$	$(3, 57, 38)$
$\{3, 54\}$	$(2, 54, 36)$
$\{3, 102\}$	$(1, 51, 34)$
$\{4, 5\}$	$(64, 160, 80)$
$\{4, 6\}$	$(32, 96, 48)$
$\{4, 8\}$	$(16, 64, 32)$
$\{4, 12\}$	$(8, 48, 24)$
$\{4, 20\}$	$(4, 40, 20)$
$\{4, 36\}$	$(2, 36, 18)$
$\{4, 68\}$	$(1, 34, 17)$

Table 8: *genus*-9 *continued...*

$\{\mathbf{p},\mathbf{q}\}$	$(\mathbf{V},\mathbf{E},\mathbf{F})$
$\{5,4\}$	$(80,160,64)$
$\{5,5\}$	$(32,80,32)$
$\{5,6\}$	$(20,60,24)$
$\{5,10\}$	$(8,40,16)$
$\{5,14\}$	$(5,35,14)$
$\{5,30\}$	$(2,30,12)$
$\{6,4\}$	$(48,96,32)$
$\{6,5\}$	$(24,60,20)$
$\{6,6\}$	$(16,48,16)$
$\{6,7\}$	$(12,42,14)$
$\{6,9\}$	$(8,36,12)$
$\{6,11\}$	$(6,33,11)$
$\{6,15\}$	$(4,30,10)$
$\{6,27\}$	$(2,27,9)$
$\{7,3\}$	$(224,336,96)$
$\{7,6\}$	$(14,42,12)$
$\{7,14\}$	$(4,28,8)$
$\{8,3\}$	$(128,192,48)$
$\{8,4\}$	$(32,64,16)$
$\{8,8\}$	$(8,32,8)$
$\{8,24\}$	$(2,24,6)$
$\{9,3\}$	$(96,144,32)$
$\{9,6\}$	$(12,36,8)$
$\{10,3\}$	$(80,120,24)$
$\{10,5\}$	$(16,40,8)$
$\{11,6\}$	$(11,33,6)$
$\{11,22\}$	$(2,22,4)$
$\{12,3\}$	$(64,96,16)$
$\{12,4\}$	$(24,48,8)$
$\{12,12\}$	$(4,24,4)$
$\{14,3\}$	$(56,84,12)$

Table 8: *genus-9 continued...*

$\{\mathbf{p},\mathbf{q}\}$	$(\mathbf{V},\mathbf{E},\mathbf{F})$	
$\{14,5\}$	$(14,35,5)$	
$\{14,7\}$	$(8,28,4)$	
$\{14,21\}$	$(2,21,3)$	
$\{15,6\}$	$(10,30,4)$	
$\{18,3\}$	$(48,72,8)$	
$\{19,38\}$	$(1,19,2)$	
$\{20,4\}$	$(20,40,4)$	
$\{20,20\}$	$(2,20,2)$	
$\{21,14\}$	$(3,21,2)$	
$\{22,3\}$	$(44,66,6)$	
$\{22,11\}$	$(4,22,2)$	
$\{24,8\}$	$(6,24,2)$	
$\{27,6\}$	$(9,27,2)$	
$\{30,3\}$	$(40,60,4)$	
$\{30,5\}$	$(12,30,2)$	
$\{36,4\}$	$(18,36,2)$	
$\{36,36\}$	$(1,18,1)$	*
$\{38,3\}$	$(38,57,3)$	
$\{38,19\}$	$(2,19,1)$	*
$\{54,3\}$	$(36,54,2)$	
$\{68,4\}$	$(17,34,1)$	*
$\{102,3\}$	$(34,51,1)$	*

Table 9: *All possible topological regular tesselations of a closed orientable genus-*10 *surface. A * denotes a tessellation with exactly one tile.*

$\{\mathbf{p},\mathbf{q}\}$	$(\mathbf{V},\mathbf{E},\mathbf{F})$
$\{3,7\}$	$(108,378,252)$
$\{3,8\}$	$(54,216,144)$
$\{3,9\}$	$(36,162,108)$
$\{3,10\}$	$(27,135,90)$

Table 9: *genus*-10 *continued...*

$\{\mathbf{p}, \mathbf{q}\}$	$(\mathbf{V}, \mathbf{E}, \mathbf{F})$
$\{3, 12\}$	$(18, 108, 72)$
$\{3, 15\}$	$(12, 90, 60)$
$\{3, 18\}$	$(9, 81, 54)$
$\{3, 24\}$	$(6, 72, 48)$
$\{3, 33\}$	$(4, 66, 44)$
$\{3, 42\}$	$(3, 63, 42)$
$\{3, 60\}$	$(2, 60, 40)$
$\{3, 114\}$	$(1, 57, 38)$
$\{4, 5\}$	$(72, 180, 90)$
$\{4, 6\}$	$(36, 108, 54)$
$\{4, 7\}$	$(24, 84, 42)$
$\{4, 8\}$	$(18, 72, 36)$
$\{4, 10\}$	$(12, 60, 30)$
$\{4, 12\}$	$(9, 54, 27)$
$\{4, 13\}$	$(8, 52, 26)$
$\{4, 16\}$	$(6, 48, 24)$
$\{4, 22\}$	$(4, 44, 22)$
$\{4, 28\}$	$(3, 42, 21)$
$\{4, 40\}$	$(2, 40, 20)$
$\{4, 76\}$	$(1, 38, 19)$
$\{5, 4\}$	$(90, 180, 72)$
$\{5, 5\}$	$(36, 90, 36)$
$\{5, 10\}$	$(9, 45, 18)$
$\{6, 4\}$	$(54, 108, 36)$
$\{6, 6\}$	$(18, 54, 18)$
$\{6, 12\}$	$(6, 36, 12)$
$\{6, 30\}$	$(2, 30, 10)$
$\{7, 3\}$	$(252, 378, 108)$
$\{7, 4\}$	$(42, 84, 24)$
$\{7, 7\}$	$(12, 42, 12)$
$\{7, 10\}$	$(7, 35, 10)$

Table 9: *genus*-10 *continued...*

$\{\mathbf{p},\mathbf{q}\}$	$(\mathbf{V},\mathbf{E},\mathbf{F})$
$\{7,28\}$	$(2,28,8)$
$\{8,3\}$	$(144,216,54)$
$\{8,4\}$	$(36,72,18)$
$\{8,8\}$	$(9,36,9)$
$\{9,3\}$	$(108,162,36)$
$\{9,18\}$	$(3,27,6)$
$\{10,3\}$	$(90,135,27)$
$\{10,4\}$	$(30,60,12)$
$\{10,5\}$	$(18,45,9)$
$\{10,7\}$	$(10,35,7)$
$\{10,10\}$	$(6,30,6)$
$\{10,25\}$	$(2,25,5)$
$\{12,3\}$	$(72,108,18)$
$\{12,4\}$	$(27,54,9)$
$\{12,6\}$	$(12,36,6)$
$\{12,24\}$	$(2,24,4)$
$\{13,4\}$	$(26,52,8)$
$\{13,13\}$	$(4,26,4)$
$\{15,3\}$	$(60,90,12)$
$\{16,4\}$	$(24,48,6)$
$\{16,16\}$	$(3,24,3)$
$\{18,3\}$	$(54,81,9)$
$\{18,9\}$	$(6,27,3)$
$\{21,42\}$	$(1,21,2)$
$\{22,4\}$	$(22,44,4)$
$\{22,22\}$	$(2,22,2)$
$\{24,3\}$	$(48,72,6)$
$\{24,12\}$	$(4,24,2)$
$\{25,10\}$	$(5,25,2)$
$\{28,4\}$	$(21,42,3)$
$\{28,7\}$	$(8,28,2)$

Table 9: *genus*-10 *continued...*

$\{\mathbf{p}, \mathbf{q}\}$	$(\mathbf{V}, \mathbf{E}, \mathbf{F})$	
$\{30, 6\}$	$(10, 30, 2)$	
$\{33, 3\}$	$(44, 66, 4)$	
$\{40, 4\}$	$(20, 40, 2)$	
$\{40, 40\}$	$(1, 20, 1)$	*
$\{42, 3\}$	$(42, 63, 3)$	
$\{42, 21\}$	$(2, 21, 1)$	*
$\{60, 3\}$	$(40, 60, 2)$	
$\{76, 4\}$	$(19, 38, 1)$	*
$\{114, 3\}$	$(38, 57, 1)$	*

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