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Master Thesis in Mathematics and Physics

The equivalence between second-order, Ostrogradsky-free (Galileon) Lagrangians and regular first-order theories.

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Abstract

Galileon theories are a class of scalar theories, where the second-order derivative appears in the Lagrangian. Traditionally, such second-order theories have been deemed unstable, plagued by the so-called Ostrogradsky-Ghost. Recent work however suggests that Galileon theories are stable, and in fact might be equivalent to ordinary first-order Lagrangians. In this thesis, I examine these Galileon theories, as well as other second order theories. I then use the Cartan Equivalence Algorithm to test the equivalence of second order theories to first order theoeries. From this test, I conclude that pure second order, mechanical theories are equivalent to first-order theories.

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1 Introduction

In 1850, Mikhail Ostrogradsky published a paper in which he generalised Hamilton's construction of Hamiltonians to theories containing higher-order (time) derivatives. As a consequence of this generalization, Ostrogradsky showed that such higher-order theories suffer from a linear instability, making them unsuitable for physics [1].

In 2016, two groups released papers that showed that certain second order Lagrangian theories do not suffer from this Ostrogradsky instability, or ghost [2, 3]. From this work, the question arose if these Ostrogradsky stable Lagrangians are perhaps first order theories written in a poorly chosen coordinate system, i.e. if such Lagrangians are equivalent. If they are equivalent, then these higher-order theories have nothing new to tell us. However, if there are some second-order, Ostrogradsky stable Lagrangians that are *not* equivalent to a first order theory, then these theories open up a whole new class of theories for physics to explore.

Ostrogradsky stable theories include the class of Galileon theories, a class of second-order Lagrangians that only have second-order equations of motion [4]. Galileon theories are an interesting class of theories, used in General Relativistic theories to modify gravity on large scales. They also exhibit some other properties that make them interesting to study on their own, regardless of applications [5].

For this master thesis, I was asked to study the types of Ostrogradsky stable theories, and to investigate the possible equivalence between these stable theories and ordinary first order theories. This yields the following research question: "Do the conditions that free a higher order Lagrangian of Ostrogradsky ghosts force such a Lagrangian to be divergence equivalent to a first-order Lagrangian?", and the following sub questions:

Physics: What types of Ostrogradsky stable theories are there?

Physics: What are the properties of these theories?

Math: What are the conditions to call two Lagrangians divergence equivalent?

Math: How can you determine such equivalence?

In this thesis, I first discuss the physics behind the relevant theories. In chapter 2 we discuss several approaches to classical mechanics, in chapter 3 the Ostrogradsky instability is discussed with regards to classical mechanics [1]. In particular, we discuss why the instability limits the kind of theories allowed and how to avoid it. In chapter 4 we discuss field theories, and how the Ostrogradsky instability affects them. In chapter 5 we then discuss several Ostrogradsky stable field theories. In particular the class of second order theories called Galileons is discussed [4, 5], a class that is both second order and Ostrogradsky stable. After that, we expand to theories of General Relativity in chapter 6 and couple these to scalar field theories in chapter 7.

In the second part of this thesis, we discuss the mathematics of equivalence. First, in chapter 8 we set up the mathematical framework of jet bundles and give a definition of equivalence using [6, 7]. In chapter 9 we apply this definition and framework to the galileon theories of chapter, using [8]. Next, we go over the Cartan Equivalence algorithm in chapter 10 [6]. This algorithm can be used to answer the question when two Lagrangians are equivalent, and in subsection 10.5 I show how to apply Cartan's algorithm to Lagrangians [6]. In chapter 11 we apply this algorithm to the case of second order, Ostrogradsky free, mechanical Lagrangians and check if these Lagrangians are equivalent to first order theories.

Chapter 12 is the conclusion, in which we sumarize what was discussed and attempt to answer the research question.

2 Classical Mechanics

2.1 Newtonian Mechanics

Physics, like all Natural Sciences, concerns itself with describing the world around us. In Theoretical Physics the goal is to find a theory that describes the motions and interactions of particles, however large or small, fast or slow. One famous physicist who made an attempt at deriving such a theory is Isaac Newton [9, chapter 3]. His second law

$$\boldsymbol{F} = m\boldsymbol{a} \tag{1}$$

gives the equation of motion for some object of mass m subject to a force $\boldsymbol{F}$. It is a second order differential equation, and therefore two pieces of initial data are needed to find a unique solution. Generally, they are chosen as initial position and velocity, or initial and final position at a specific time [9, chapter 3]. This method however quickly becomes complicated as more objects are described, since for each object a force equation (1) has to be defined. Newton's method also doesn't hold outside of classical mechanics [9, chapter 6].

Example 1 (Hooke's Law)**.** Let us give a well know example, that of an object of mass m attached to some spring with spring constant k, like in figure 1. This object experiences a force $F = -kx$. Applying Newton's second law, we find the acceleration to be [9, chapter 4]

$$\ddot{x} = -\frac{k}{m}x. \tag{2}$$

This is the differential equation of an harmonic oscillator and has the solution

$$x\left(t\right) = A\sin\left(\omega t + \phi\right),$$

where $\omega = \sqrt{\frac{k}{m}}$, and A and ϕ are an amplitude and phase determined by the initial conditions. This equation tells us that our object will oscillate with frequency ω [9, chapter 4].

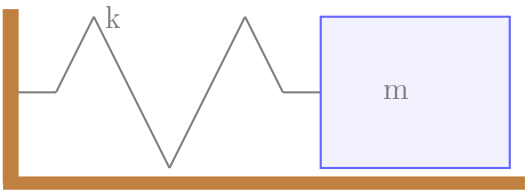

Figure 1: An object with mass m attached to a horizontal spring with spring constant k.

2.2 Lagrangian Mechanics

The Lagrangian formulation is another method to derive equations of motion. For many mechanical systems, we can define a kinetic energy T and a potential energy U [9, chapter 6], [10, chapter 13]. From these energies, we can define the *Lagrangian* $\mathcal{L} = T - V$. Next we define a functional called the *action* as the integral over the Lagrangian [9]:

$$S = \int_{t_1}^{t_2} \mathcal{L}\,\mathrm{d}t. \tag{3}$$

Let us now be a bit more general and explicit, and assume that our mechanical system has q dependent coordinates, which we label u^α. We can now extremize the action integral, and find

the Euler-Lagrange equations [10][1]:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{u}^\alpha} = \frac{\partial \mathcal{L}}{\partial u^\alpha}. \tag{4}$$

These Euler-Lagrange equations are the equations of motion for each of the u^α.

So far, for our alternate method we have defined a Lagrangian and an action out of the blue. Let us now tie them back with Newton's mechanics. Consider that in mechanics the kinetic energy is given by $T = \sum_\alpha \frac{m_\alpha}{2}\left(\dot{u}^\alpha\right)^2$ and the potential energy is just a function of position only, i.e. $U = U\left(u^\alpha\right)$. Then the Euler-Lagrange equations for this system is given by

$$m_\alpha \ddot{u}^\alpha = -\frac{\partial U}{\partial u^\alpha}. \tag{5}$$

Furthermore, the force is often conservative, which is precisely $F_\alpha = -\frac{\partial U}{\partial u^\alpha}$. Using this substitution, we observe that the Euler-Lagrange equations are an equivalent method to Newton's equations to describe physics.

The Lagrangian method can be generalized. To that end, now consider p independent coordinates labelled x^i, and also consider higher-order derivatives up to order r. The Lagrangian now does not have to be related to a kinetic and potential energy any more, but instead is an arbitrary function of the independent coordinates x^i, the dependent coordinates u^α, and the up to r^{th} order derivatives u^α_I [2]. The action then looks like:

$$S = \int \mathcal{L}\left(x^i, u^\alpha, u^{(r)}\right) \mathrm{d}_{vol} x. \tag{6}$$

We can still extremize this action integral to again obtain Euler-Lagrange equations. Setting these Euler-Lagrange equations equal to zero then determines the equations of motion of the dependent coordinates.

Theorem 1. *The Euler-Lagrange equations belonging to the action in equation* (6) *are given by [11, section 5]*

$$\sum_{\sigma=0}^{r} \sum_{|I|=\sigma} (-1)^\sigma \frac{\mathrm{d}^\sigma}{\mathrm{d}x^I}\left(\frac{\partial \mathcal{L}}{\partial u^\alpha_I}\right) = 0. \tag{7}$$

Proof. To find the Euler-Lagrange equations, we must extremize the action (6). To that end, we must find the differential of S for the functions $u^\alpha\left(x^i\right)$, similar to how we did in appendix B. Take q such functions, and vary them by adding functions $h^\alpha\left(x^i\right)$. These functions have the property that $h^\alpha\left(x^i_0\right) = h^\alpha\left(x^i_f\right) = 0$, such that $\bar{u}^\alpha = u^\alpha + h^\alpha$ agrees with u^α at the end points $\boldsymbol{x}_0$ and $\boldsymbol{x}_f$. Then the difference between the actions found from these functions is given by

$$\begin{aligned}
\Delta S = S(u+h) - S(h) &= \int_{x_0}^{x_f} \left[\mathcal{L}\left(x^i, u^\alpha + h^\alpha, u^{(r)} + h^{(r)}\right) - \mathcal{L}\left(x^i, u^\alpha, u^{(r)}\right)\right] \mathrm{d}x \\
&= \int_{x_0}^{x^f} \left[\frac{\partial \mathcal{L}}{\partial u^\alpha} h^\alpha + \frac{\partial \mathcal{L}}{\partial u^\alpha_i} h^\alpha_i + \sum_{2 \leqslant |I| \leqslant r} \frac{\partial \mathcal{L}}{\partial u^\alpha_I} h^\alpha_I\right] \mathrm{d}x + O\left(h^2\right) \\
&= F(h) + R.
\end{aligned}$$

The extremal than follows from those functions u^α for which $F(h) = 0$ for all h [10, chapter 12]. To see that this condition equals the Euler-Lagrange equations, we perform integration by

[1] See appendix B for how to extremize the action

[2] See appendix A for the definitions of this notation.

parts on $F(h)$. First, we find that

$$\int_{x_0}^{x^f} \frac{\partial \mathcal{L}}{\partial u_i^\alpha} h_i^\alpha \, \mathrm{d}x = -\int_{x_0}^{x_f} h^\alpha \frac{\mathrm{d}}{\mathrm{d}x^i} \frac{\partial \mathcal{L}}{\partial u_i^\alpha} \, \mathrm{d}x + \left[h^\alpha \frac{\partial \mathcal{L}}{\partial u_i^\alpha} \right]_{x_0^i}^{x_f^i}.$$

Due to the condition that $h^\alpha = 0$ at the boundary of the integral, the boundary term is zero. We can perform this integration by part on each term in the derivative $F(h)$, continuing on until we have a term involving h^α itself. Again, we get boundary terms which vanish due to the boundary conditions on h. The result is

$$\frac{\partial \mathcal{L}}{\partial u_I^\alpha} h_I^\alpha = (-1)^{|I|} h^\alpha \frac{\mathrm{d}^{|I|}}{\mathrm{d}x^I} \left(\frac{\partial \mathcal{L}}{\partial u_I^\alpha} \right),$$

from which we find that F is equal to

$$F(h) = \int_{x_0}^{x^f} \left[\sum_{\sigma=0}^{r} \sum_{|I|=\sigma} (-1)^\sigma \frac{\mathrm{d}^\sigma}{\mathrm{d}x^I} \left(\frac{\partial \mathcal{L}}{\partial u_I^\alpha} \right) \right] h^\alpha \, \mathrm{d}x,$$

which is zero for all h precisely when the Euler-Lagrange equations (7) are equal to zero. Thus, we conclude that the Euler-Lagrange equations (7) are precisely those conditions for the functions u^α to extremize the action S (6). □

Example 2. Let us end the introduction of the Lagrangian formulation with a look at Hooke's law in a Lagrangian setting. The potential energy for an object on a spring is given by $U = \frac{1}{2}kx^2$, and its kinetic energy is $T = \frac{1}{2}m\dot{x}^2$ [9, chapter 6]. This potential is conservative, since $-\frac{\mathrm{d}U}{\mathrm{d}x} = -kx$ is the equation for the spring force.

For this system then the Lagrangian is given by

$$\mathcal{L} = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2. \tag{8}$$

For this system (7) reduces to

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathcal{L}}{\partial \dot{x}} = 0.$$

Applied to our Lagrangian, we find the Euler-Lagrange equation

$$\begin{aligned} 0 &= \frac{\partial \mathcal{L}}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathcal{L}}{\partial \dot{x}} \\ &= -kx - \frac{\mathrm{d}}{\mathrm{d}t} (m\dot{x}) \\ &= -kx - m\ddot{x}, \\ m\ddot{x} &= -kx. \end{aligned} \tag{9}$$

This is equal to (2), as expected. The solution is again a harmonic oscilator. This explicitly shows the equivalence of Newton's and Lagrange's method with respect to Hooke's law.

Last, we shall give a quick example of a theory that can only be written in the Lagrangian formulation.

Example 3. The Dirac Lagrangian is known to be [12, page 43]

$$\mathcal{L} = \bar{\psi} \left(i\gamma^\mu \partial_\mu - m \right) \psi.$$

Unlike classical physics, in QFT there are 4 independent coordinates, 1 time-like and 3 space-like. Furthermore, each of the spinors ψ and $\bar{\psi}$ has 4 components, leading to 8 dependent coordinates. In general, since there are 8 dependent variables, there would be 8 Euler-Lagrange equations. However, due to the nature of the dependent variables (they are two spinor-fields), we need to write down only 2 equations[3]. They are

$$\begin{aligned} 0 &= \frac{\partial \mathcal{L}}{\partial \psi} - \frac{\mathrm{d}}{\mathrm{d}x^\mu} \frac{\partial \mathcal{L}}{\partial \partial_\mu \psi} \\ &= -m\bar{\psi} - \partial_\mu \left(\bar{\psi} i \gamma^\mu \right) \\ &= -i \partial_\mu \bar{\psi} \gamma^\mu - m\bar{\psi}, \end{aligned}$$

and

$$\begin{aligned} 0 &= \frac{\partial \mathcal{L}}{\partial \bar{\psi}} - \frac{\mathrm{d}}{\mathrm{d}x^\mu} \frac{\partial \mathcal{L}}{\partial \partial_\mu \bar{\psi}} \\ &= \left(i \gamma^\mu \partial_\mu - m \right) \psi - \partial_\mu 0 \\ &= \left(i \gamma^\mu \partial_\mu - m \right) \psi \end{aligned}$$

This last equation is the Dirac equation [12, page 42]. One can easily observe that neither of these equations can be written in the form of Newton's second law (1), since these equations are first-order equations and not second order.

2.3 Hamiltonian Mechanics

The third method to formulate classical mechanics is called Hamiltonian Mechanics. Where Lagrangian Mechanics uses a function called the Lagrangian, Hamiltonian Mechanics uses a Hamiltonian. The Hamiltonian is a function that depends on the position and momentum coordinates of the system (as well as potentially any independent coordinates, such as time), and can be derived from the Lagrangian as follows. Let $\mathcal{L}\left(t, u^\alpha, \dot{u}^\alpha\right)$ be a Lagrangian that depends on q dependent coordinates u^α, their time derivatives $\dot{u}^\alpha$ and perhaps even on time t directly. For each of the coordinates u^α, a momentum coordinate can be defined by $p_\alpha = \partial \mathcal{L} / \partial \dot{u}^\alpha$ [10, chapter 3]. With the momentum thus defined, we arrive at the Hamiltonian by performing a Legendre transform on the Lagrangian [10]:

$$H\left(t, u^\alpha, p^\alpha\right) = p_\alpha \dot{u}^\alpha - \mathcal{L}\left(t, u^\alpha, \dot{u}^\alpha\right). \tag{10}$$

From this Hamiltonian, equations of motion can be derived. They are given as [10, chapter 3]

$$\dot{p}_\alpha = -\frac{\partial H}{\partial u^\alpha}, \tag{11}$$

$$\dot{u}^\alpha = \frac{\partial H}{\partial p_\alpha}. \tag{12}$$

These equations can be derived from the Euler-Lagrange equations (7). First, from $p_\alpha = \partial \mathcal{L} / \partial \dot{u}^\alpha$ it follows that

$$\begin{aligned} \dot{p}_\alpha &= \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathcal{L}}{\partial \dot{u}^\alpha} \\ &= \frac{\partial \mathcal{L}}{\partial u^\alpha} \\ &= -\frac{\partial H}{\partial u^\alpha}, \end{aligned}$$

[3]Since the spinors are each (represented by) 4 component vectors, each of the 2 equations can be read as a vector equation.

where I use the E-L equations to go from the first to the second line, and the Legendre transform (10) to go from the second to the third line. To derive (12), only the Legendre transform is needed, since

$$\begin{aligned}\frac{\partial H}{\partial p_\alpha} &= \frac{\partial p_\beta \dot{u}^\beta - \mathcal{L}\left(t, u^\beta, \dot{u}^\beta\right)}{\partial p_\alpha} \\ &= \dot{u}^\alpha - \frac{\partial \mathcal{L}}{\partial p_\alpha} \\ &= \dot{u}^\alpha,\end{aligned}$$

where I have used that the Lagrangian itself does not (explicitly) depend on the momenta p_α.

The Hamiltonian also has a physical interpretation, it is the energy function. This is easy to see in classical mechanics, where the Lagrangian is given by $\mathcal{L} = T - U$, with $T = \sum_\alpha \frac{m_\alpha}{2} \left(\dot{u}^\alpha\right)^2$ and $U = U\left(u^\alpha\right)$ does not depend on the velocities $\dot{u}^\alpha$. The momenta are then given by

$$p_\alpha = \partial \mathcal{L} / \partial \dot{u}^\alpha = m_\alpha \dot{u}_\alpha,$$

and if we perform the Legendre transform, we find

$$\begin{aligned}H &= p_\alpha \dot{u}^\alpha - \mathcal{L}\left(t, u^\alpha, \dot{u}^\alpha\right) \\ &= m_\alpha \dot{u}_\alpha \dot{u}^\alpha - \sum \frac{m_\alpha}{2} \left(\dot{u}^\alpha\right)^2 + U \\ &= \sum \frac{m_\alpha}{2} \left(\dot{u}^\alpha\right)^2 + U \\ &= T + U,\end{aligned}$$

which is precisely the total energy.

Example 4. Let us again discuss Hooke's law, but now using the Hamiltonian. The potential energy remains $U = \frac{1}{2} k x^2$ [9, chapter 6], but we need to rewrite the kinetic energy term in terms of the momentum p. The Lagrangian is (8)

$$\mathcal{L} = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2,$$

from which we can calculate the momentum as

$$p = \frac{\mathrm{d}\mathcal{L}}{\mathrm{d}\dot{x}} = m \dot{x}.$$

The kinetic energy is thus given by $T = \frac{p^2}{2m}$, and the Hamiltonian for a spring is

$$H = T + U = \frac{p^2}{2m} + \frac{1}{2} k x^2.$$

The equations of motion are now given by Hamilton's equations (11)(12):

$$\begin{aligned}\dot{p} &= -\frac{\partial H}{\partial x} = -kx \\ \dot{x} &= \frac{\partial H}{\partial p} = \frac{p}{m}.\end{aligned}$$

The first of these becomes the E-L equation for a spring (9) if we back substitute the momentum, while the second equation is just the definition of the momentum.

As shown above, we can derive the equations of motion from the Hamiltonian. We can however do something more powerful, we can use the Hamiltonian to directly calculate the time-evolution of any function $F(u^\alpha, p_\alpha)$! Since in the Hamiltonian setting, all functions can only depend on the 'position' coordinates u^α and the momentum coordinates p_α, this is quite powerful. To proof this statement, we first need to define the Poisson bracket.

Definition 1. Let us work in a Hamiltonian description, with q coordinates u^α, and their q momentum coordinates p_α. The *Poisson bracket* between two functions A and B depending on these coordinates is given by [10, chapter 8]:

$$\{A, B\} = \frac{\partial A}{\partial u^\alpha}\frac{\partial B}{\partial p_\alpha} - \frac{\partial A}{\partial p_\alpha}\frac{\partial B}{\partial u^\alpha} \tag{13}$$

Theorem 2. *Let us work in a Hamiltonian description, with q coordinates u^α, and their q momentum coordinates p_α. Let A, B and C be functions depending on these coordinates, and $a, b \in \mathbb{R}$ be two real numbers. Then the Poisson bracket between these functions has the following three properties [10, chapter 8]:*

1. *The Poisson bracket is skew symmetric: $\{A, B\} = \{B, A\}$.*
2. *The Poisson bracket is linear: $\{aA + bB, C\} = a\{A, C\} + b\{B, C\}$.*
3. *The Poisson bracket admits the Jacobi Identity:*

$$\{\{A, B\}, C\} + \{\{B, C\}, A\} + \{\{C, A\}, B\} = 0.$$

The proof of this theorem we omit for brevity, but can be found in Arnol'd [10, chapter 8].

With the Poisson bracket thus defined, we can calculate the time derivatives from the next theorem.

Theorem 3. *Let us work in a Hamiltonian description, with q coordinates u^α, and their q momentum coordinates p_α. Let H be the Hamiltonian function, and let F be an arbitrary function that depends on these coordinates and possibly time. Then the time evolution of this function is given by*

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \frac{\partial F}{\partial t} + \{F, H\}. \tag{14}$$

Proof. To proof this theorem, we first consider the 'trivial' functions $F = u^\alpha$ and $F = p_\alpha$:

$$\begin{aligned}\{u^\alpha, H\} &= \frac{\partial u^\alpha}{\partial u^\beta}\frac{\partial H}{\partial p_\beta} - \frac{\partial u^\alpha}{\partial p_\beta}\frac{\partial H}{\partial u^\beta} \\ &= \frac{\partial H}{\partial p_\alpha} = \dot{u}^\alpha,\end{aligned}$$

and

$$\begin{aligned}\{p_\alpha, H\} &= \frac{\partial p_\alpha}{\partial u^\beta}\frac{\partial H}{\partial p_\beta} - \frac{\partial p_\alpha}{\partial p_\beta}\frac{\partial H}{\partial u^\beta} \\ &= -\frac{\partial H}{\partial u^\alpha} = \dot{p}_\alpha.\end{aligned}$$

These two relations are exactly the equations of motion we derived earlier. Using this, we can proof the theorem for an arbitrary function F. First, by definition, the total time derivative of F is given by

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \frac{\partial F}{\partial t} + \dot{u}^\alpha\frac{\partial F}{\partial u^\alpha} + \dot{p}_\alpha\frac{\partial F}{\partial p_\alpha},$$

and the Poisson bracket between F and H is given by

$$\begin{aligned}\{F,H\} &= \frac{\partial F}{\partial u^\alpha}\frac{\partial H}{\partial p_\alpha} - \frac{\partial F}{\partial p_\alpha}\frac{\partial H}{\partial u^\alpha} \\ &= \frac{\partial F}{\partial u^\alpha}\dot{u}^\alpha - \frac{\partial F}{\partial p_\alpha}\cdot -\dot{p}_\alpha \\ &= \frac{\partial F}{\partial u^\alpha}\dot{u}^\alpha + \frac{\partial F}{\partial p_\alpha}\dot{p}_\alpha,\end{aligned}$$

ergo

$$\frac{\partial F}{\partial t} + \{F,H\} = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial u^\alpha}\dot{u}^\alpha + \frac{\partial F}{\partial p_\alpha}\dot{p}_\alpha = \frac{\mathrm{d}F}{\mathrm{d}t}.$$

□

2.4 Constraint Analysis

Theorem 3 shows the power of using the Hamiltonian formulation. Another thing one can do in the Hamiltonian formulation is counting of degrees of freedom using contraint analysis. In principle, a physical theory has as many degrees of freedom as it has dependent coordinates. However, there could be additional degrees of freedom hiding inside the theory. These hidden degrees often pop out as ghosts, and are undesired. The Ostrogradsky ghost is one of these extra degrees of freedom, and we will discuss them in chapter 3.

The other possibility is that there are hidden constraints in the theory that reduce the number of degrees of freedom. This is beneficial, as it reduces the complexity of the problem. For example, if you are able to reduce the number of degrees of freedom to zero, then the theory is fully integrable and thus solvable [10, chapter 9], independent of initial conditions.

In order to discover the total number of degrees of freedom, one must perform a constraint analysis. A constrain analysis can be done in both the Lagrangian and the Hamiltonian formulation. In this thesis, we shall focus on the Hamiltonian analysis using [3]. A Lagrangian analysis can be found in [2,8].

Example 5. To explain this procedure, we are going to use a simple, but quite general theory of a particle in the plane. We are going to use an arbitrary Lagrangian: $\mathcal{L} = \mathcal{L}(x, y, \dot{x}, \dot{y})$. This Lagrangian is equivalent to a different Lagrangian[4]:

$$\mathcal{L}_{eq} = \mathcal{L}(x, y, X, Y) + \lambda(\dot{x} - X) + \kappa(\dot{y} - Y), \tag{15}$$

where λ and κ are Lagrange multipliers. The equations of motion for these multipliers ensure that X and Y behave as the velocities of x and y:

$$\begin{aligned}0 = EL(\mathcal{L}_{eq})_\lambda &= \frac{\partial \mathcal{L}_{eq}}{\partial \lambda} \\ &= \dot{x} - X, \\ 0 = EL(\mathcal{L}_{eq})_\kappa &= \frac{\partial \mathcal{L}_{eq}}{\partial \kappa} \\ &= \dot{y} - Y,\end{aligned}$$

[4]We also give a more strict definition of equivalence in chapter 8, that does not hold for the equivalence we have here. The Lagrangians here are called equivalent, because they lead to the same physics

while the other equations combine to give the equations of motion you would expect from a standard Lagrangian. Let us show this for the equations for x:

$$\begin{aligned} 0 = EL\left(\mathcal{L}_{eq}\right)_X &= \frac{\partial \mathcal{L}_{eq}}{\partial X} \\ &= \frac{\partial \mathcal{L}}{\partial X} - \lambda \Leftrightarrow \\ \lambda &= \frac{\partial \mathcal{L}}{\partial \dot{x}} \\ 0 = EL\left(\mathcal{L}_{eq}\right)_x &= \frac{\partial \mathcal{L}_{eq}}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}_{eq}}{\partial \dot{x}} \\ &= \frac{\partial \mathcal{L}}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\lambda \\ &= \frac{\partial \mathcal{L}}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{x}} \end{aligned}$$

A similar process will yield the equations for y.

Now knowing that these Lagrangians give the same physics, let us rewrite $\mathcal{L}_{eq}$ into the Hamiltonian formulation. First, we need to define 6 momentum variables. Using the Poisson brackets, they can be defined as precisely those variables that admit the following relations [3]

$$\{p_x, x\} = \{p_y, y\} = \{P_X, X\} = \{P_Y, Y\} = \{\rho_1, \lambda\} = \{\rho_2, \kappa\} = 1, \tag{16}$$

with all other Poisson brackets equal to zero.

Of course, we have already defined the momentum variables through another method, i.e [10, chapter 3]

$$\begin{aligned} p_x &= \frac{\partial \mathcal{L}}{\partial \dot{x}} = \lambda, & P_X &= \frac{\partial \mathcal{L}}{\partial \dot{X}} = 0, & \rho_1 &= \frac{\partial \mathcal{L}}{\partial \dot{\lambda}} = 0, \\ p_y &= \frac{\partial \mathcal{L}}{\partial \dot{y}} = \kappa, & P_Y &= \frac{\partial \mathcal{L}}{\partial \dot{Y}} = 0, & \rho_2 &= \frac{\partial \mathcal{L}}{\partial \dot{\kappa}} = 0. \end{aligned}$$

The advantage of the Poisson brackets is that they allow one to define momentum variables even in the absence of a Lagrangian. Regardless, these 6 equations produce the 6 primary constraints:

$$\begin{aligned} \Phi_x &= p_x - \lambda \approx 0, & P_x &\approx 0, & \rho_1 &\approx 0, \\ \Phi_y &= p_y - \kappa \approx 0, & P_y &\approx 0, & \rho_2 &\approx 0. \end{aligned}$$

We use an $\approx$ instead of an $=$ since these are constraints; they are not identically zero, they are zero because we constrain them to be.

Let us now explain the counting of degrees of freedom. The formula is as follows

$$DOF = \frac{1}{2}\left(\#\text{variables } - \#\text{secondary class constraints } - 2 \cdot \#\text{primary class constraints}\right). \tag{17}$$

This formula requires some explanations. First is the $\frac{1}{2}\#$variables, since for each dependent coordinate we have 2 variables, i.e. the coordinate itself and its associated momentum, we need to halve #variables in order to obtain the appropriate number of degrees of freedom.

Second are the constraints, these can be divided into two classes, the primary class and the secondary class. A constraint is of the primary class, if its Poisson bracket commutes with all other constraints and if it commutes with the Hamiltonian[5]. All other constraints are part of

[5]We say that two constraints f and g commute, if $\{f, g\} = 0$. Please note, this Poisson bracket must be identically zero, it should not be zero because we constrain it to be zero. The same holds for commutation with the Hamiltonian.

the secondary class. The constraints of primary class are stronger constraints and as a result reduce the DOF by 1 each.

To test if the 6 constraints we found thus far are of primary or secondary class, we must calculate the Poisson brackets. We find only two non-zero Poisson brackets

$$\{\Phi_x, \rho_1\} = \{\Phi_y, \rho_2\} = -1.$$

This shows that 4 constraints are already of secondary class. To check the other two, P_X and P_Y, we first need to define the Hamiltonian.

One first defines the Hamiltonian as one would for the regular Lagrangian,

$$H_0 = p_x X + p_y Y - \mathcal{L}(x, y, X, Y). \tag{18}$$

This is not the total Hamiltonian however, as it does not include the other variables that we have defined for our analysis. We arrive at the total Hamiltonian, by adding the constraints times a Lagrange multiplier [3]. We add the multipliers that enforce the primary constraints. The total Hamiltonian then becomes [3]

$$H_T = H_0 + \mu_i \Phi_i + \nu_i \rho_i + \xi_i P_i. \tag{19}$$

We now calculate the Poisson brackets of the Hamiltonian and the constraints:

$$\begin{aligned}
\dot{\rho}_i = \{\rho_i, H_t\} &= \{\rho_i, H_0\} + \mu_j \{\rho_i, \Phi_j\} \\
&= \{\rho_i, H_0\} + \mu_i = \mu_i, \\
\dot{\Phi}_i = \{\Phi_i, H_t\} &= \{\Phi_i, H_0\} + \nu_j \{\Phi_i, \rho_j\} \\
&= \{\Phi_i, H_0\} + \nu_i.
\end{aligned}$$

These constraints fix the Lagrange multipliers μ_i and ν_i, and thus can be made identically zero. The more interesting brackets are those of the constraints of unknown class:

$$\begin{aligned}
\chi_i = \dot{P}_i &= \{P_i, H_T\} \\
&= \{P_i, H_0\} + 0 \\
&= \{P_i, p_x X + P_y Y - \mathcal{L}\} \\
&= -p_i - \{P_i, \mathcal{L}\} \\
&= -p_i + \mathcal{L}_{X^i}.
\end{aligned}$$

These constraints are not identically zero. We can however constrain them to be zero, which just enforces that $p_x = \partial\mathcal{L}/\partial X = \partial\mathcal{L}/\partial\dot{x}$ and $p_y = \partial\mathcal{L}/\partial\dot{y}$. These additional constraints χ_i do not commute with the P_i, as we find out when calculating the Poisson bracket of χ_i with H_T:

$$\begin{aligned}
\dot{\chi}_i = \{\chi_i, H_t\} &= \{\chi_i, H_0\} + \nu_j \{\chi_i, \rho_j\} + \xi_j \{\chi_i, P_j\} \\
&= \{\chi_i, H_0\} + \nu_j \{\{P_i, H_t\}, \rho_j\} + \xi_j \{-p_i + \mathcal{L}_{X^i}, P_j\} \\
&= \{\chi_i, H_0\} + \xi_j \mathcal{L}_{X^i X^j}
\end{aligned} \tag{20}$$

If you identify $X = \dot{x}$ and $Y = \dot{y}$, you discover that $\mathcal{L}_{X^i X^j}$ is the kinetic matrix. If this matrix is invertible, then (20) fixes the Lagrange multipliers ξ_i. In that case, we have identified a total of 8 constraints, $\rho_i, \Phi_i, P_i, \chi_i$, all of which are second class[6], and thus we can calculate the degrees of freedom from (17) as

$$\begin{aligned}
DOF &= \frac{1}{2}(\#\text{variables} - \#\text{secondary class constraints} - 2 \cdot \#\text{primary class constraints}) \\
&= \frac{1}{2}(12 - 8 - 2 \cdot 0) \\
&= \frac{4}{2} = 2,
\end{aligned}$$

[6] ρ_i and Φ_i do not commute, and neither do P_i and χ_i, thus they are all second class.

precisely as desired.

Of course, in a general theory, there is no guarantee that $\mathcal{L}_{X^iX^j}$ is invertible. Proceeding the analysis like in [3], if $\mathcal{L}_{X^iX^j}$ has a one-dimensional kernel, i.e. only 1 linearly independent null-vector, then one obtains an additional constraint. Let the null-vector be (a scalar times) (w^x, w^y), then (20) produces as tertiary constraint

$$\Sigma = w^x \{\chi_x, H_0\} + w^y \{\chi_y, H_0\}.$$

Requiring that its time derivative is also zero, i.e. $\{\Sigma, H_t\} \approx 0$, generally gives another constraint, and brings down the degrees of freedom to 1.

Last is the case in which $\mathcal{L}_{X^iX^j}$ vanishes, in which case we have the two constraints

$$\Sigma_i = \{\chi_i, H_0\} \approx 0.$$

Depending on the time derivative of Σ_i, these may give rise to at most 2 additional constraints. Thus, we end up with either 1 or 0 degrees of freedom.

The analysis above can be generalised to cases with q dependent variables, simply by adding more additional fields X^α and Lagrange multipliers λ_α for each additional dependent variable u^α. If the generalised kinetic matrix $\mathcal{L}_{X^\alpha X^\beta} = \mathcal{L}_{\dot{u}^\alpha \dot{u}^\beta}$ is invertible, then one will find $DOF = q$. If the matrix is not invertible, one will find additional constraints, resulting in $DOF < q$.

The other generalisation is to increase the order of the derivatives appearing in the Lagrangian, i.e. add $\ddot{u}^\alpha$. We will do this analysis in chapter 3, and thus bring about the Ostrogradsky ghost.

2.5 A comparison between approaches

In this chapter, we discussed three possible methods to describe physics and derive equations of motion, the methods of Newton, Lagrange and Hamilton. It therefore would be good to end this chapter with a short discussion about each of these methods, and explain why some are more favoured in physics than others.

First is Newton's method of $F = m \cdot a$. It has the advantage of being an easy method that immediately arrives at the equations of motion for the theory. It is intuitively easy to understand: if you apply a force to an object, say you push a couch, that object moves, and how fast it moves depends on how much force you apply. However, it also quickly becomes complicated as the amount of objects and sources of forces increases. Furthermore, Newton's equations are no longer valid when one considers either the really fast ((Special) Relativity) or the really small (Quantum Mechanics).

The Lagrangian and Hamiltonian method both have the advantage of being just one formula from which the equations follow. Any changes to your theory thus require you to modify only one equation, instead of however many force equations. The disadvantages are that they are mathematically more difficult to understand, or prove their validity, and are less intuitive to understand. Hamilton's method is favoured in non-relativistic quantum mechanics, for example, it is used in the Schrödinger equation [13],

$$i\hbar \frac{\partial}{\partial t} \Phi\left(x^i, t\right) = \hat{H} \Phi\left(x^i, t\right), \tag{21}$$

which explicitly depends on the Hamiltonian.

The Lagrangian method can be used for both quantum mechanics and relativity, and where both combine in Quantum Field Theory. To explain the importance of the Lagrangian in the latter, we use the imagery used by David Morin in [9, page 225] and Paul Dirac in [14]. In

quantum field theory, particles don't take only one path between two points, they take all possible paths. However, some paths are more favoured by others. The weight of a path is called the phase of a path, and is given by a factor $e^{iS/\hbar}$, with S the action defined in (6). The phases from all possible paths have to be added up to determine the amplitude of going from a point A to a point B. The square of the absolute value of the amplitude gives the probability of a particle going from point A to point B. For non-stationary values of S, and thus for non-stationary paths, the phases for the different values of S vary wildly, due to the smal-ness of $\hbar$. Thus, one sums over essentially random vectors in the plane of complex numbers for the amplitudes, which sums to approximately zero. Thus, the probability of a path with a non-stationary S is essentially zero. However, for paths with stationary values of S, the phase is constant and no cancelling occurs. Thus it are the paths with stationary actions that we observe and thus it are the paths with stationary actions that we care about in physics. See also Dirac [14] for an argument *why* this weight factor is given by $e^{iS/\hbar}$.

It should be noted that in the rest of this thesis we are working on classical, non-quantum theories, and only do classical field theory. Thus specifics from QFT do not apply to us.

3 The Ostrogradsky instability and how to avoid it

In the previous section, we only discussed first-order theories. Physics in general also only uses first-order Lagrangians. Why is this the case? A related question might be why Newton's force equation (1) is a second order differential equation, and not a first order, or third order equation[7]? One might say that it is because these theories are what experiments agree with, and one is indeed right. However, such an answer doesn't really offer any new physical insight. Is there a deeper reason for only using first-order Lagrangians? It turns out there is, and this reason was discovered by Ostrogradsky [1]. We already hinted at in at the end of chapter 2; higher-order theories harbour an additional, un-physical degree of freedom. This extra degree of freedom is called the Ostrogradsky ghost, and it causes these higher order theories to be unstable.

In this section, we shall follow R.P Woodard's review and construction of the Ostrogradsky instability in [1], followed with a Hamiltonian analysis as in [3]. After that, we look into the works by Klein, Roest in [2] and Motohashi et al. in [3] that derive constraints to eliminate the Ostrogradsky instability in second-order classic mechanical theories.

3.1 The second-order case according to Woodard

Consider a Lagrangian that depends on second derivatives. We are primarily concerned with the $p = 1$, arbitrary q case, thus $\mathcal{L} = \mathcal{L}\left(t, u^{(2)}\right)$. For this system, the q Euler-Lagrange equations (7) are

$$\frac{\partial \mathcal{L}}{\partial u^\alpha} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{u}^\alpha} + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{\partial^2 \mathcal{L}}{\partial \ddot{u}^{\alpha 2}} = 0. \tag{22}$$

We assume non-degeneracy. For a second order system, this means that $\frac{\partial^2 \mathcal{L}}{\partial \ddot{u}^\alpha \partial \ddot{u}^\beta}$ is invertible. Because this matrix is invertible, we can rewrite (22) into a form similar to Newton's equations (1). Due to the last term in (22), this is now a fourth-order differential equation:

$$\frac{\mathrm{d}^4 u^\alpha}{\mathrm{d}t^4} = \mathcal{F}^\alpha\left(t, u^{(3)}\right).$$

Since this is a (system of) fourth-order differential equations, we need $4q$ initial data to solve the system [1]: $u_0^\alpha = u^\alpha(0)$, $\dot{u}_0^\alpha = \dot{u}^\alpha(0)$, $\ddot{u}_0^\alpha = \ddot{u}^\alpha(0)$ and $\dddot{u}_0^\alpha = \dddot{u}^\alpha(0)$. Since $4q$ initial data are needed, there are $4q$ canonical variables for a Hamiltonian analysis. Following Ostrogradsky, Woodard choose these coordinates as [1]

$$U_1^\alpha = u^\alpha, \qquad P_1^\alpha = \frac{\partial \mathcal{L}}{\partial \dot{u}^\alpha} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \ddot{u}^\alpha},$$
$$U_2^\alpha = \dot{u}^\alpha, \qquad P_2^\alpha = \frac{\partial \mathcal{L}}{\partial \ddot{u}^\alpha}.$$

Since this system is non-degenerate, in principle the system can be written in terms of the canonical variables, i.e. there exist functions $A'^\beta\left(U_1^\alpha, U_2^\alpha, P_1^\alpha, P_2^\alpha\right)$ such that $\ddot{u}^\beta = A'^\beta\left(U_1^\alpha, U_2^\alpha, P_1^\alpha, P_2^\alpha\right)$ and $\frac{\partial \mathcal{L}}{\partial \ddot{u}^\alpha}\big|_{u^\beta = U_1^\beta; \dot{u}^\beta = U_2^\beta; \ddot{u}^\beta = A'^\beta} = P_2^\alpha$ [1].

There now arise two considerations that lead us to change the formula for A'^β. The first is that the original system only had $3q$ variables: u^α, $\dot{u}^\alpha$ and $\ddot{u}^\alpha$. Thus, A^β should depend on at most $3q$ canonical variables[8]. The second can be found in the restrictions A has to meet: A does not have to be tuned such that the definition for P_1 holds. As a result, A need not depend

[7] Or any order other than 2 really.

[8] This could also be a hint to the astute reader that something is afoot: why do two equivalent descriptions have different numbers of variables?

on P_1. Thus we can instead define the acceleration as $\ddot{u}^\beta = A^\beta\left(U_1^\alpha, U_2^\alpha, P_2^\alpha\right)$ and state that it has to meet $\ddot{u}^\beta = A^\beta\left(U_1^\alpha, U_2^\alpha, P_2^\alpha\right)$ and $\frac{\partial \mathcal{L}}{\partial \ddot{u}^\alpha}|_{u^\beta = U_1^\beta; \dot{u}^\beta = U_2^\beta; \ddot{u}^\beta = A^\beta} = P_2^\alpha$.

The above thus defined, Ostrogradsky's Hamiltonian is obtained through the Legendre transformation

$$\begin{aligned} H\left(U_1^\alpha, U_2^\alpha, P_1^\alpha, P_2^\alpha\right) &= P_1^\beta \dot{u}^\beta + P_2^\beta \ddot{u}^\beta - \mathcal{L} \\ &= P_1^\beta U_2^\beta + P_2^\beta A^\beta\left(U_1^\alpha, U_2^\alpha, P_2^\alpha\right) - \mathcal{L}\left(t, U_1^\alpha, U_2^\alpha, A\left(U_1^\beta, U_2^\beta, P_2^\beta\right)\right). \end{aligned}$$

The time evolution follows from computing Hamilton's equations [1], [10, chapter 3]. As in the first-order case, the Hamiltonian is the energy function. The interesting thing to note is that this Hamiltonian only has a dependence on P_1^β in the $P_1^\beta U_2^\beta$ term, and thus only depends linearly on P_1^β. As a result, the Hamiltonian is unbounded from below and can not be stable [15]. This result is quite general, because we only made one assumption on $\mathcal{L}$, that of non-degeneracy.

Example 6. Let us consider an example, taken from Woodard [1]. We look at the regular harmonic oscillator, but with an added second order term dependent on a dimensionless parameter ϵ. The Lagrangian is given by [1]

$$\mathcal{L} = -\frac{\epsilon m}{2\omega^2}\ddot{x}^2 + \frac{m}{2}\dot{x}^2 - \frac{m\omega^2}{2}x^2. \tag{23}$$

The Euler-Lagrange equation (22) and its solution are given by

$$\begin{aligned} 0 &= -m\left[\frac{\epsilon}{\omega^2}\ddddot{x} + \ddot{x} + \omega^2 x\right], \\ x(t) &= C_+\cos(k_+ t) + S_+\sin(k_+ t) + C_-\cos(k_- t) + S_-\sin(k_- t). \end{aligned}$$

The Euler-Lagrange equation is clearly fourth-order, which is problematic. The two frequencies are given by

$$k_\pm = \omega\sqrt{\frac{1 \mp \sqrt{1-4\epsilon}}{2\epsilon}}.$$

The constants $C_\pm$ and $S_\pm$ depend on the initial data, the formulae can be found in [1] and are

$$C_+ = \frac{k_-^2 x_0 + \ddot{x}_0}{k_-^2 - k_+^2}, \qquad S_+ = \frac{k_-^2 \dot{x}_0 + \dddot{x}_0}{k_+\left(k_-^2 - k_+^2\right)},$$
$$C_- = \frac{k_+ x_0 + \ddot{x}_0}{k_+^2 - k_-^2}, \qquad S_- = \frac{k_+^2 \dot{x}_0 + \dddot{x}_0}{k_-\left(k_+^2 - k_-^2\right)}.$$

Following Ostrogradsky's choice of canonical variables, we can compute the Hamiltonian for this system. This Hamiltonian can be expressed in terms of the two frequencies and the constants $C_\pm$ and $S_\pm$. In this form, it looks like [1]

$$H = \frac{m}{2}\sqrt{1-4\epsilon}k_+^2\left(C_+^2 + S_+^2\right) - \frac{m}{2}\sqrt{1-4\epsilon}k_-^2\left(C_-^2 + S_-^2\right).$$

In this form, it is clear that C_- and S_- carry negative energy. If we are to quantize this system, C_- and S_- would correspond to negative-energy creation operators, allowing for (infinite) negative energy particles. As a result, no state can be stable, since each state can always decay into a state with same energy, just with one additional positive-energy and one additional negative-energy particle.

The regular, first order version of this system can be obtained by taking $\epsilon \to 0$. Then the Hamiltonian becomes

$$H = \frac{m}{2}\omega^2\left(C_+^2 + S_+^2\right)^2.$$

This system has no negative energy parameters, and is stable [1]. This shows that the instability indeed resides in the higher-order part of the Lagrangian.

3.2 The second-order case, using Hamiltonian constraint theory

Of course, the above analysis can also be done using Lagrange multipliers and constraint analysis, as we did on a first-order theory in chapter 2. To do so, consider a second order Lagrangian $\mathcal{L} = \mathcal{L}\left(t, \ddot{u}^\alpha, \dot{u}^\alpha, u^\alpha\right)$. To perform the analysis, define the equivalent Lagrangian

$$\mathcal{L}_{eq} = \mathcal{L}\left(t, Q^\alpha, q^\alpha, u^\alpha\right) + \lambda_\beta\left(\dot{u}^\beta - q^\beta\right) + \kappa_\beta\left(\dot{q}^\beta - Q^\beta\right). \tag{24}$$

Again, we eliminate all derivatives in the Lagrangian itself, in favour of introducing additional variables q^β and Q^β and using the Lagrange multipliers λ_α and κ_α to force these additional variables to behave as derivatives. Unlike the first order case, where we only had 1 additional variable per dependent variable, the appearance of second order derivatives requires us to use 2 additional variables per dependent variable[9]. Since this Lagrangian depends on $5q$ variables, we must define $5q$ pairs of conjugate variables. We choose them such to admit to the following Poisson brackets,

$$\{Q^\alpha, P_\beta\} = \{q^\alpha, p_\beta\} = \{u^\alpha, \pi_\beta\} = \left\{\lambda_\alpha, \rho^\beta\right\} = \left\{\kappa_\alpha, \sigma^\beta\right\} = \delta^\alpha_\beta,$$

with all other Poisson brackets equal to zero. Comparing this definition of the momenta with the usual definition gives rise to the following equations:

$$P_\alpha = \frac{\partial \mathcal{L}}{\partial \dot{Q}^\alpha} = 0, \qquad p_\alpha = \frac{\partial \mathcal{L}}{\partial \dot{q}^\alpha} = \kappa_\alpha, \qquad \pi_\alpha = \frac{\partial \mathcal{L}}{\partial \dot{u}^\alpha} = \lambda_\alpha,$$
$$\rho^\alpha = \frac{\partial \mathcal{L}}{\partial \dot{\lambda}_\alpha} = 0, \qquad \sigma^\alpha = \frac{\partial \mathcal{L}}{\partial \dot{\kappa}_\alpha} = 0.$$

These equations produce the $5q$ primary constraints:

$$P_\alpha \approx 0, \qquad \rho^\alpha \approx 0, \qquad \sigma^\alpha \approx 0,$$
$$\Phi_\alpha = p_\alpha - \kappa_\alpha \approx 0, \qquad \Psi_\alpha = \pi_\alpha - \lambda_\alpha \approx 0.$$

To compute time-evolution we define the two Hamiltonians as we did in chapter 2. First the regular Hamiltonian

$$H_0 = p_\alpha Q^\alpha + \pi_\alpha q^\alpha - \mathcal{L}\left(t, Q^\alpha, q^\alpha, u^\alpha\right), \tag{25}$$

and the total Hamiltonian

$$H_T = H_0 + \mu^\alpha \Phi_\alpha + \nu^\alpha \Psi_\alpha + \xi^\alpha P_\alpha + \zeta^1_\alpha \rho^\alpha + \zeta^2_\alpha \sigma^\alpha. \tag{26}$$

There are only 2 groups of non-zero Poisson brackets between the constraints:

$$\{\rho^\alpha, \Psi_\beta\} = \{\sigma^\alpha, \Phi_\beta\} = \delta^\alpha_\beta.$$

From this, it is immediate that the Poisson brackets of $\rho^\alpha, \Psi_\beta, \sigma^\alpha, \Phi_\beta$ with H_T fixes the multipliers $\mu^\alpha, \nu^\alpha, \zeta^1_\alpha, \zeta^2_\alpha$; in particular $\mu^\alpha = \nu^\alpha = 0$. On the contrary, the time evolution of P_α gives

$$\begin{aligned}
\chi_\alpha = \dot{P}_\alpha &= \{P_\alpha, H_T\} \\
&= \{P_\alpha, H_0\} \\
&= \left\{P_\alpha, p_\beta Q^\beta + \pi_\beta q^\beta - \mathcal{L}\left(t, Q^\beta, q^\beta, u^\beta\right)\right\} \\
&= -p_\alpha - \left\{P_\alpha, \mathcal{L}\left(t, Q^\beta, q^\beta, u^\beta\right)\right\} \\
&= -p_\alpha + \mathcal{L}_{Q^\alpha} \approx 0.
\end{aligned}$$

[9] In section II A of [3], Motohashi et al. perform a similar analysis where they only need 1 additional variable per dependent variable for second order theories. They choose not to remove all derivatives from the Lagrangian, only the second order ones. Removing all derivatives is easier to generalize, so we choose that route.

We constrain this to be zero, which gives us an additional q constraints. Like in the first-order case, these constraints do not commute with the P_α. As a result, the time derivative $\dot{\chi}_\alpha$ is given by

$$\begin{aligned}\{\chi_\alpha, H_T\} &= \{\chi_\alpha, H_0\} + \xi^\beta \{\chi_\alpha, P_\beta\} + \zeta^1_\beta \{\chi_\alpha, \rho^\beta\} + \zeta^2_\beta \{\chi_\alpha, \sigma^\beta\} \\ &= \{\chi_\alpha, H_0\} + \xi^\beta \{\mathcal{L}_{Q^\alpha} - p_\alpha, P_\beta\} \\ &= \{\chi_\alpha, H_0\} + \xi^\beta \mathcal{L}_{Q^\alpha Q^\beta}. \end{aligned} \tag{27}$$

If the matrix $\mathcal{L}_{Q^\alpha Q^\beta}$ is invertible, this equation allows us to fix the Lagrange multipliers ξ^α. With all Lagrange multipliers fixed, and all constraints thus determined, the analysis ends. None of the constraints commute with all other constraints, and thus all are of second class. The total number of degrees of freedom then is

$$\begin{aligned} DOF &= \frac{1}{2}\left(\#\text{variables } - \#\text{secondary class constraints } - 2\cdot\#\text{primary class constraints}\right) \\ &= \frac{1}{2}\left(2\cdot 5q - 6q - 2\cdot 0\right) \\ &= \frac{4q}{2} = 2q. \end{aligned}$$

We have only expected q DOF, since we only have q dependent variables. There thus are an additional q ghost DOF. This is because the matrix $\mathcal{L}_{Q^\alpha Q^\beta}$ is invertible, if it is not, it has null-vectors which lead to additional constraints, and thus fewer degrees of freedom. If we back-substitute our variables, we identify $\mathcal{L}_{Q^\alpha Q^\beta} = \mathcal{L}_{\ddot{u}^\alpha \ddot{u}^\beta}$. Ergo, the requirement of invertibility is exactly the requirement of non-degeneracy Woodard and Ostrogradsky used in their analysis [1].

We have now shown in two different methods that theories having second-order derivatives in their Lagrangian are unfavoured. The difference between the analyses is that while using Lagrange multipliers only shows that there are additional degrees of freedom, the analysis by Woodard [1] also explicitly shows why these additional degrees of freedom are bad.

3.3 The general r$^{\text{th}}$ order case

The construction by Ostrogradsky and Woodard can be extended to a general r-th order case to show that these cases are also unstable [1]. We could also extend the Hamiltonian analysis, but for brevity we shall only discuss the analysis by Woodard. Consider a Lagrangian $\mathcal{L}\left(t, u^{(r)}\right)$. We assume that this Lagrangian is non-degenerate, i.e. $\frac{\partial^2 \mathcal{L}}{\partial(\mathrm{d}^r u^\alpha/\mathrm{d}t^r)\partial\left(\mathrm{d}^r u^\beta/\mathrm{d}t^r\right)} \neq 0$. There are now $2qr$ coordinates in phase space. If we generalize Ostrogradsky's choice, the canonical coordinates are given by:

$$U^\alpha_i = \frac{\mathrm{d}^{i-1}u^\alpha}{\mathrm{d}t^{i-1}} \quad \text{and} \quad P^\alpha_i = \sum_{j=i}^{r} (-1)^{j-i} \frac{\mathrm{d}^{j-i}}{\mathrm{d}t^{j-i}} \frac{\partial \mathcal{L}}{\partial\left(\mathrm{d}^j u^\alpha/\mathrm{d}t^j\right)}.$$

Due to non-degeneracy, we should be able to solve for the $\mathrm{d}^r u^\alpha/\mathrm{d}t^r$ in terms of P^α_r and the U^α_i's. Ergo, we can find functions $\mathcal{A}^\alpha\left(U^\beta_1, \ldots, U^\beta_r, P^\beta_r\right)$ with the property that

$$\frac{\partial \mathcal{L}}{\partial \frac{\mathrm{d}^r u^\alpha}{\mathrm{d}t^r}}|_{U^\alpha_i = \mathrm{d}^{i-1}u^\alpha/\mathrm{d}t^{i-1}; U^\alpha_r = \mathcal{A}^\alpha} = P^\alpha_r.$$

With these functions, we can construct the Hamiltonian. It is [1]

$$\begin{aligned} H &= P^\alpha_i \frac{\mathrm{d}^i u^\alpha}{\mathrm{d}t^i} - \mathcal{L}, \\ &= P^\alpha_1 U^\alpha_2 + P^\alpha_2 U^\alpha_3 + \cdots + P^\alpha_{r-1} U^\alpha_r + P^\alpha_r \mathcal{A}^\alpha - \mathcal{L}\left(t, U^\beta_1, \ldots, U^\beta_r, \mathcal{A}^\beta\right), \end{aligned} \tag{28}$$

where we sum over α. Again, Hamilton's equations will give the time evolution of the system.

From (28), one can see that the Hamiltonian is linear in all coordinates P_i^α, for all α and for all $i < r$. Ergo, the energy is unbounded from below for most momentum coordinates. Adding more higher derivatives just increases the fraction of unstable coordinates ($\frac{r-1}{2r}$) to $\frac{1}{2}$ ($\lim_{r\to\infty} \frac{r-1}{2r} = \frac{1}{2}$), and thus does not fix the problem, it rather seems to make it worse [1].

Woodard states 6 principle reasons why this instability is bad. They are summarized below. For a longer explanation, we refer the reader to Woodard's article [1]

1. The Ostrogradsky instability drives the dynamical variables (u^α) to a special kind of time dependence, not towards a special numerical value. This is because the Ostrogradsky instability is a problem with the kinetic energy and not with the potential energy. In the latter, more familiar case, energy is released as the dynamical variable approaches some special value. However, in this case the energy is released as the dynamical variable approaches a special time dependence.

2. The Ostrogradsky dynamical variable carries both positive and negative energy creation and annihilation operators.

3. If a system with this instability interacts, the vacuum decays into a collection of positive and negative energy excitations. Because there are creators and annihilators for both positive and negative energy excitations, both will appear. Since more particles equals more entropy, any particle will thus immediately decay into positive and negative energy particles.

4. If the theory is furthermore a continuum field theory, then the vast entropy of having many particles makes this decay instantaneous.

5. Furthermore, this decay does not decouple from low energy physics. I.e., even at low energy the above decay happens and can not be relegated to a high energy sector, higher than we have observed so far. Therefore, we can not state that our unstable theory is merely an Effective Field Theory, valid (and stable) up to a UV-cutoff scale.

6. A single, global constraint on the energy function does not fix the Ostrogradsky instability.

3.4 Ostrogradsky-free Mechanics

In the previous section, we saw how and why many higher-order Lagrangians are unstable. However, not all higher-order Lagrangians are unstable. In 2016 and 2017 two groups, one by Crisostomi, Klein and Roest [2] based in Groningen, and Motohashi, Noui, Suyama, Yamaguchi and Langlois [3] based in Tokyo and Paris, published papers in which they independently derived constraints on second-order Lagrangians to remove the Ostrogradsky ghost.

Let us start this subsection with an example for such an Ostrogradsky stable, second-order theory.

Example 7. Let us start with the simplest of example, a second order theory with just 1 independent and just 1 dependent variable, i.e. $\mathcal{L} = \mathcal{L}(t, \ddot{u}, \dot{u}, u)$. If it is non-degenerate [1], i.e. $\frac{\partial^2 \mathcal{L}}{\partial \ddot{u}^2} \neq 0$ it must have the Ostrogradsky ghost. Thus, to avoid the ghost, we at least need degeneracy, i.e. $\frac{\partial^2 \mathcal{L}}{\partial \ddot{u}^2} = 0$. From this requirement, we know that our Lagrangian is of the form

$$\mathcal{L} = \ddot{u} f(t, \dot{u}, u) + g(t, \dot{u}, u), \tag{29}$$

with $f \neq 0$. We can calculate the Euler-Lagrangian equation (22) for this theory, and find

$$\begin{aligned}
0 &= \frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{\partial \mathcal{L}}{\partial \ddot{u}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{u}} + \frac{\partial \mathcal{L}}{\partial u} \\
&= \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(f\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\ddot{u}f_{\dot{u}} + g_{\dot{u}}\right) + \ddot{u}f_u + g_u \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(f_t + \dot{u}f_u + \ddot{u}f_{\dot{u}}\right) - \dddot{u}f_{\dot{u}} - \ddot{u}f_{t\dot{u}} - \ddot{u}\dot{u}f_{u\dot{u}} - \ddot{u}^2 f_{\dot{u}\dot{u}} - g_{t\dot{u}} - \dot{u}g_{u\dot{u}} - \ddot{u}g_{\dot{u}\dot{u}} + \ddot{u}f_u + g_u \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(f_t + \dot{u}f_u\right) + \ddot{u}f_{\dot{u}} + \ddot{u}\frac{\mathrm{d}f_{\dot{u}}}{\mathrm{d}t} - \dddot{u}f_{\dot{u}} - \ddot{u}f_{t\dot{u}} - \ddot{u}\dot{u}f_{u\dot{u}} - \ddot{u}^2 f_{\dot{u}\dot{u}} - g_{t\dot{u}} - \dot{u}g_{u\dot{u}} - \ddot{u}g_{\dot{u}\dot{u}} + \ddot{u}f_u + g_u \\
&= F\left(t,u,\dot{u}\right)\ddot{u} + G\left(t,u,\dot{u}\right).
\end{aligned}$$

This Euler-Lagrange equation is second order, and thus does not have the ghost. This can also be seen using the Hamiltonian constraint analysis. We have already done the analysis for general q, ending at (27)

$$\dot{\chi}_\alpha = \{\chi_\alpha, H_T\} = \{\chi_\alpha, H_0\} + \xi^\beta \mathcal{L}_{Q^\alpha Q^\beta},$$

which could be made identically zero by choosing the appropriate ξ^β. For our $q = 1$ case, (27) becomes

$$\dot{\chi} = \{\chi, H_0\} + \xi \mathcal{L}_{\ddot{u}\ddot{u}}.$$

Since we have degeneracy, $\mathcal{L}_{\ddot{u}\ddot{u}} = 0$ and thus we can not fix ξ. Instead, this constraint reads

$$\begin{aligned}
\Xi = \{\chi, H_T\} &= \{\chi, H_0\} \\
&= \{\mathcal{L}_Q - p, pQ + \pi q - \mathcal{L}\left(t,Q,q,u\right)\} \\
&= \{\mathcal{L}_Q, pQ + \pi q - \mathcal{L}\left(t,Q,q,u\right)\} - \{p, pQ + \pi q - \mathcal{L}\left(t,Q,q,u\right)\} \\
&= Q\mathcal{L}_{qQ} + \mathcal{L}_{uQ}q + \pi - \mathcal{L}_q \approx 0.
\end{aligned}$$

This is not identically zero, and thus produces a new constraint Ξ. To end the analysis, we need to see calculate the time evolution of this constraint; and constraint it to be zero if it is not already so:

$$\begin{aligned}
\Sigma = \dot{\Xi} &= \{\Xi, H_t\} \\
&= \{\Xi, H_0\} \\
&= \{Q\mathcal{L}_{qQ} + \mathcal{L}_{uQ}q + \pi - \mathcal{L}_q, pQ + \pi q - \mathcal{L}\left(t,Q,q,u\right)\} \\
&= Q^2\mathcal{L}_{Qqq} + 2qQ\mathcal{L}_{Qqu} + Q\mathcal{L}_{Qu} + q^2\mathcal{L}_{Qqu} + \mathcal{L}_u - Q\mathcal{L}_{qq} - q\mathcal{L}_{qu} \approx 0.
\end{aligned}$$

We thus have yet another constraint. Fortunately, this constraint does not commute with P, and thus we get

$$\begin{aligned}
\{\Sigma, H_t\} &= \{\Sigma, H_0\} + \xi\{\Sigma, P\} \\
&= \{\Sigma, H_0\} + \xi\left(q\mathcal{L}_{Qqu} + 2\mathcal{L}_{Qu} - \mathcal{L}_{qq}\right),
\end{aligned}$$

which can be tuned to zero by fixing ξ. We have thus found all constraints on the theory, we have derived 8 constraints, and the degrees of freedom of this theory are

$$\begin{aligned}
DOF &= \frac{1}{2}\left(\#\text{variables } - \#\text{secondary class constraints } - 2\cdot\#\text{primary class constraints}\right) \\
&= \frac{1}{2}\left(2\cdot 5 - 8 - 2\cdot 0\right) \\
&= 1.
\end{aligned}$$

Thus, a theory with just one dependent variable with only has a linear dependence on the second order derivative does not have the Ostrogradsky ghost.

In the example, we only had to assume degeneracy to eliminate the Ostrogradsky ghost. However, for theories with multiple dependent variables only assuming this degeneracy is not enough. To continue the analysis, like Klein et al. [2] and Motohashi [3], we will consider two cases. First is the case where all q dependent coordinates appear in second order fashion, i.e. $\partial\mathcal{L}/\partial\ddot{u}^\alpha \neq 0\ \forall\alpha$. The second case is where of the q dependent coordinates, only m appear with a second order derivative, a further $n = q - m \geqslant 1$ appear with only first order derivatives. These n first-order coordinates can be used to remove the Ostrogradsky ghost from the m second order derivatives, as we shall see. We will first discuss the all second-order case.

Our system of all second-order coordinates can be analysed in three different methods. We can look at the Euler-Lagrange equations, or apply either a Hamiltonian or Lagrangian constraint analysis. First, we look at the Euler-Lagrange equations. The Lagrangian is given by $\mathcal{L} = \mathcal{L}\left(\ddot{u}^\alpha, \dot{u}^\alpha, u^\alpha\right)$, and the q Euler-Lagrange equations (7) are given by [2]

$$\begin{aligned} E_\alpha &= \frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{\partial\mathcal{L}}{\partial\ddot{u}^\alpha} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{u}^\alpha} + \frac{\mathrm{d}\mathcal{L}}{\mathrm{d}u^\alpha} \\ &= \mathcal{L}_{\ddot{u}^\alpha\ddot{u}^\beta}\frac{\partial^4 u^\beta}{\partial t^4} + \left(\mathcal{L}_{\ddot{u}^\alpha\dot{u}^\beta} - \mathcal{L}_{\ddot{u}^\beta\dot{u}^\alpha}\right)\frac{\partial^3 u^\beta}{\partial t^3} + \text{ lower order terms.} \end{aligned}$$

From Woodard's analysis [1], we know that for the Ostrogradsky ghost to be absent, the Euler-Lagrange equations must contain at most second order time derivatives. To remove the fourth order derivatives, we need not just degeneracy, i.e. $\det\mathcal{L}_{\ddot{u}^\alpha\ddot{u}^\beta} = 0$, we need full degeneracy, i.e. $\mathcal{L}_{\ddot{u}^\alpha\ddot{u}^\beta} = 0$. If the system is not fully degenerate, then there remain(s a linear combination of) Euler-Lagrange equations that contains the fourth order time derivative. The condition

$$\frac{\partial^2\mathcal{L}}{\partial\ddot{u}^\alpha\partial\ddot{u}^\beta} = 0 \tag{30}$$

is called the *primary constraint*. However, from the E-L equations, we can also see that imposing the primary constraint is not enough, there still remain third-order terms that need to be removed. This can be done by imposing the *secondary constraint*

$$\frac{\partial^2\mathcal{L}}{\partial\ddot{u}^\alpha\partial\dot{u}^\beta} - \frac{\partial^2\mathcal{L}}{\partial\ddot{u}^\beta\partial\dot{u}^\alpha} = 0. \tag{31}$$

This secondary constraint is anti-symmetric, which explains why we did not encounter it in our example with just 1 dependent variable: it was automatically satisfied. Imposing both constraints upon a second-order Lagrangian produces second-order equations of motion, and thus removes the Ostrogradsky ghost [2].

Next we use a Hamiltonian constraint analysis, such as done by Motohashi et al. in [3]. We have already done the first part of this in section 3.2. We ended up with the time derivative of the constraint χ_α (27),

$$\dot{\chi}_\alpha = \{\chi_\alpha, H_T\} = \{\chi_\alpha, H_0\} + \xi^\beta\mathcal{L}_{Q^\alpha Q^\beta}.$$

We discussed that if $\mathcal{L}_{Q^\alpha Q^\beta}$ is invertible, the analysis ends. If not, each linearly independent null-vector of $\mathcal{L}_{Q^\alpha Q^\beta}$ gives rise to a new constraint. In order to obtain maximal constraints, we need maximal degeneracy, i.e. the primary constraint

$$\mathcal{L}_{Q^\alpha Q^\beta} = 0,$$

which is the same constraint as (30). We thus have q additional constraints

$$\begin{aligned} \Xi_\alpha = \{\chi_\alpha, H_T\} &= \{\chi_\alpha, H_0\} \\ &= \left\{\mathcal{L}_{Q^\alpha} - p_\alpha, p_\beta Q^\beta + \pi_\beta q^\beta - \mathcal{L}\left(Q, q, u\right)\right\} \\ &= Q^\beta\mathcal{L}_{Q^\alpha q^\beta} + q^\beta\mathcal{L}_{Q^\alpha u^\beta} + \pi_\alpha - \mathcal{L}_{q_\alpha} \approx 0. \end{aligned}$$

The time derivative of these constraints is given by

$$\begin{aligned}\dot{\Xi}_\alpha = \{\Xi_\alpha, H_T\} &= \{\Xi_\alpha, H_0\} + \xi^\gamma \{\Xi_\alpha, P_\gamma\} + \zeta^1_\gamma (\Xi_\alpha, \rho^\gamma)\, \zeta^2_\gamma (\Xi_\alpha, \sigma^\gamma) \\ &= \{\Xi_\alpha, H_0\} + \xi^\gamma \left(\mathcal{L}_{Q^\alpha q^\gamma} - \mathcal{L}_{Q^\gamma q^\alpha}\right).\end{aligned}$$

If the matrix $\mathcal{L}_{Q^\alpha q^\gamma} - \mathcal{L}_{Q^\gamma q^\alpha}$ now is invertible, we can fix ξ^α and end the analysis. A quick calculation then tells us that there are $\frac{3}{2}q > q$ degrees of freedom, still to many. Thus, we arrive at the secondary constraint

$$\mathcal{L}_{Q^\alpha q^\gamma} - \mathcal{L}_{Q^\gamma q^\alpha} = 0,$$

which is again the same as (31). This results in q constraints $\Sigma_\alpha = \{\Xi_\alpha, H_0\} \approx 0$. Time evolving this constraint then generally fixes the ξ^α, and the analysis ends. Our two additional sets of constraints reduce the degrees of freedom to

$$\begin{aligned}DOF &= \frac{1}{2}\left(\#\text{variables } - \#\text{secondary class constraints } - 2\cdot\#\text{primary class constraints}\right) \\ &= \frac{1}{2}\left(2\cdot 5q - 8q - 2\cdot 0\right) \\ &= q,\end{aligned}$$

as desired.

The third method involves a Lagrangian constraint analysis, and is performed by Klein and Roest in [2]. We omit it here, but it is no surprise that they find the same primary and secondary constraint as we have.

Next we will consider the case where there are m second order and n first order variables, labelled respectively ϕ^a and q^i. This case too can be analysed in three separate methods. However, for the sake of brevity, we shall only perform the Hamiltonian constraint analysis. We consider the Lagrangian $\mathcal{L} = \mathcal{L}\left(\ddot{\phi}^a, \dot{\phi}^a, \phi^a, \dot{q}^i, q^i\right)$; it is second order in ϕ^a, but only first order in q^i. Assuming no constraints, one will find that generally this Lagrangian has $2m + n$ degrees of freedom, of which m are Ostrogradsky ghosts [3]. We thus need to find m first class constraint, or $2m$ second class constraint (or a mix of both) in order to reduce the degrees of freedom to the desired $q = m + n$.

To perform the analysis and calculate which constraints we need to impose, consider the following equivalent Lagrangian

$$\mathcal{L}_{eq}\left(Q^a_2, Q^a_1, \phi^a, q^i_2, q^i_1, \lambda_a, \kappa_a, \iota_i\right) = \mathcal{L} + \lambda_a\left(\dot{\phi}^a - Q^a_1\right) + \kappa_a\left(\dot{Q_1}^a - Q^a_2\right) + \iota_i\left(\dot{q_1}^i - q^i_2\right). \quad (32)$$

We have defined a total of $5m + 3n$ 'position' coordinates, and thus we need to define $5m + 3n$ momentum coordinates. They are defined according to the following Poisson brackets

$$\begin{aligned}\{Q^a_2, P_{2b}\} = \{Q^a_1, P_{1b}\} = \{\phi^a, \phi_b\} &= \delta^a_b, \\ \left\{\lambda_a, \rho^b\right\} = \left\{\kappa_a, \sigma^b\right\} &= \delta^a_b, \\ \left\{q^i_2, p_{2j}\right\} = \left\{q^i_1, p_{1j}\right\} = \left\{\iota^i, \tau_j\right\} &= \delta^i_j.\end{aligned}$$

Comparing these to the other definition of momenta produces constraints

$$\begin{aligned}P_{2a} &= \frac{\partial\mathcal{L}}{\partial\dot{Q}^a_2} = 0, & P_{1a} &= \frac{\partial\mathcal{L}}{\partial\dot{Q}^a_1} = \kappa_a, & \pi_a &= \frac{\partial\mathcal{L}}{\partial\dot{\phi}^a} = \lambda_a, \\ \rho^a &= \frac{\partial\mathcal{L}}{\partial\dot{\lambda}_a} = 0, & \sigma^a &= \frac{\partial\mathcal{L}}{\partial\dot{\kappa}_a} = 0, & & \\ p_{2i} &= \frac{\partial\mathcal{L}}{\partial\dot{q}^i_2} = 0, & p_{1i} &= \frac{\partial\mathcal{L}}{\partial\dot{q}^i_1} = \iota_i, & \tau^i &= \frac{\partial\mathcal{L}}{\partial\dot{\iota}_i} = 0.\end{aligned}$$

These directly produce $3m+2n$ constraints, and a further $2m+n$ can be found as

$$\Phi_a = P_{1a} - \kappa_a \approx 0, \qquad \Psi_a = \pi_a - \lambda_a \approx 0, \qquad \Upsilon_i = p_{1i} - \iota_i \approx 0.$$

These constraints admit Poisson brackets, most of which are zero. The non-zero ones are

$$\left\{\Phi_a, \sigma^b\right\} = \left\{\Psi_a, \rho^b\right\} = \delta_b^a, \qquad \left\{\Upsilon_i, \tau^j\right\} = \delta_i^j.$$

Next, we define the two Hamiltonians:

$$\begin{aligned} H_0 &= P_{1a}Q_2^a + \pi_a Q_1^a + p_{1i}q_2^i - \mathcal{L}\left(Q_2^a, Q_1^a, \phi^a, q_2^i, q_1^i\right), \\ H_T &= \mu^a\Phi_a + \nu^a\Psi_a + \upsilon^i\Upsilon_i + \xi_2^a P_{2a} + \xi_1^i p_{2i} + \zeta_a^1\rho^a + \zeta_a^2\sigma^a + \eta_i\tau^i + H_0. \end{aligned}$$

Due to the non-zero Poisson brackets, the time evolution of $\Phi_a, \Psi_a, \Upsilon_i, \sigma^a, \rho_a, \tau_i$ fixes the Lagrange multipliers $\zeta_a^2, \zeta_a^1, \eta_i, \mu^a, \nu^a, \upsilon^i$. In particular, it fixes $\mu^a = \nu^a = \upsilon^i = 0$. This leaves the multipliers ξ_2^a and ξ_1^i unfixed, because their constraints P_{2a} and p_{2i} commute with all other constraints. However, they do evolve non-trivially with time. This gives two new sets of constraints, χ_{1i} and χ_{2a}:

$$\begin{aligned} \chi_{1i} = \dot{p}_{2i} &= \{p_{2i}, H_T\} \\ &= \{p_{2i}, H_0\} \\ &= \left\{p_{2i}, P_{1a}Q_2^a + \pi_a Q_1^a + p_{1j}q_2^j - \mathcal{L}\left(Q_2^a, Q_1^a, \phi^a, q_2^i, q_1^i\right)\right\} \\ &= -p_{1i} + \mathcal{L}_{q_2^i} \approx 0, \end{aligned}$$

and

$$\begin{aligned} \chi_{2a} = \dot{P}_{2a} &= \{P_{2a}, H_T\} \\ &= \{P_{2a}, H_0\} \\ &= \left\{P_{2a}, P_{1b}Q_2^b + \pi_b Q_1^b + p_{1j}q_2^j - \mathcal{L}\left(Q_2^a, Q_1^a, \phi^a, q_2^i, q_1^i\right)\right\} \\ &= -P_{1a} + \mathcal{L}_{Q_2^a} \approx 0. \end{aligned}$$

These constraints do not commute with the P_{2a} and p_{2i}. As such their time derivatives are

$$\begin{aligned} \dot{\chi}_{1i} &= \{\chi_{1i}, H_T\} \\ &= \xi_2^b\{\chi_{1i}, P_{2b}\} + \xi_1^j\{\chi_{1i}, p_{2j}\} + \{\chi_{1i}, H_0\} \\ &= \xi_2^b\mathcal{L}_{q_2^i Q_2^b} + \xi_1^j\mathcal{L}_{q_2^i q_2^j} + \{\chi_{1i}, H_0\}, \end{aligned} \tag{33}$$

and

$$\begin{aligned} \dot{\chi}_{2a} &= \{\chi_{2a}, H_T\} \\ &= \xi_2^b\{\chi_{2a}, P_{2b}\} + \xi_1^j\{\chi_{2a}, p_{2j}\} + \{\chi_{2a}, H_0\} \\ &= \xi_2^b\mathcal{L}_{Q_2^a Q_2^b} + \xi_1^j\mathcal{L}_{Q_2^a q_2^j} + \{\chi_{2a}, H_0\}. \end{aligned}$$

These derivatives can be made identically zero if the kinetic matrix

$$\mathsf{K} = \begin{pmatrix} \mathcal{L}_{Q_2^a Q_2^b} & \mathcal{L}_{Q_2^a q_2^j} \\ \mathcal{L}_{q_2^i Q_2^b} & \mathcal{L}_{q_2^i q_2^j} \end{pmatrix}$$

is invertible, since then we can solve the system

$$\begin{pmatrix} \mathcal{L}_{Q_2^a Q_2^b} & \mathcal{L}_{Q_2^a q_2^j} \\ \mathcal{L}_{q_2^i Q_2^b} & \mathcal{L}_{q_2^i q_2^j} \end{pmatrix} \begin{pmatrix} \xi_2^b \\ \xi_1^j \end{pmatrix} = -\begin{pmatrix} \{\chi_{2a}, H_0\} \\ \{\chi_{1i}, H_0\} \end{pmatrix}.$$

If we can solve the system, we have determined all Lagrange multipliers and the time evolution of all constraints, and thus we are finished. We then compute that we have

$$DOF = \frac{1}{2}\left(2\cdot\left(5m+3n\right) - \left(5m+3n\right) - m - n\right) = 2m+n$$

degrees of freedom. We thus need to pose constraints on the matrix K, it can not be invertible. We need to obtain m additional constraints from $\dot{\chi}_{1i}$ and $\dot{\chi}_{2a}$, which can be achieved if we find m null-vectors for the matrix K. These can be found by imposing the following condition on the sub-matrices of K:

$$\mathcal{L}_{Q_2^a Q_2^b} - \mathcal{L}_{Q_2^a q_2^i}\mathcal{L}^{q_2^i q_2^j}\mathcal{L}_{q_2^j Q_2^b} = 0, \tag{34}$$

where $\mathcal{L}^{ij}$ is the inverse of $\mathcal{L}_{q_2^i q_2^j}$. This condition requires that the sub-matrix $\mathcal{L}_{q_2^i q_2^j}$ is invertible. The m null-vectors are then found to be $\boldsymbol{V}_a = \left(\delta_a^b, V_a^i\right)$, where V_a^i is given by $V_a^i = -\mathcal{L}_{Q_2^a q_2^j}\mathcal{L}^{q_2^j q_2^i}$. These conditions (34) are the modified *primary conditions* from before, and they reduce to the previous primary conditions in the absence of first order variables.

Having these m null vectors, we can still fix n of the multipliers ξ. Using (33), we can fix ξ_1^i in terms of ξ_2^a by fixing $\dot{\chi_{1i}} = 0$. In doing so, we obtain

$$\xi_1^i = -\mathcal{L}^{ij}\left\{\chi_{1j}, H_0\right\} + V_a^i \xi_2^a. \tag{35}$$

From the null vectors, we derive the constraints

$$\begin{aligned}
\Xi_a &= \left\{\chi_{2a}, H_0\right\} + V_a^i\left\{\chi_{1i}, H_0\right\} \\
&= \mathcal{L}_{Q_2^a Q_1^b} Q_2^b + \mathcal{L}_{Q_2^a \phi^b} Q_1^b + \mathcal{L}_{Q_2^a q_1^j} q_2^j + \pi_a - \mathcal{L}_{Q_1^a} \\
&+ V_a^i\left(\mathcal{L}_{q_2^i Q_1^b} Q_2^b + \mathcal{L}_{q_2^i \phi^b} Q_1^b + \mathcal{L}_{q_2^i q_1^j} q_2^j - \mathcal{L}_{q_1^i}\right) \approx 0.
\end{aligned}$$

These constraints should not evolve with time as well, so we compute

$$\begin{aligned}
\dot{\Xi}_a &= \left\{\Xi_a, H_t\right\} \\
&= \xi_2^b\left\{\Xi_a, P_{2b}\right\} + \xi_1^j\left\{\Xi_a, p_{2j}\right\} + \left\{\Xi_a, H_0\right\} \\
&= \xi_2^b\left(\left\{\Xi_a, P_{2b}\right\} + V_b^j\left\{\Xi_a, p_{2j}\right\}\right) - \mathcal{L}^{ji}\left\{\chi_{1i}, H_0\right\}\left\{\Xi_a, p_{2j}\right\} + \left\{\Xi_a, H_0\right\}.
\end{aligned}$$

We can use this equation to fix the remaining multipliers ξ_2^b, and thus end the analysis. To do so requires that the matrix $\mathsf{L}_{ab} = \left\{\Xi_a, P_{2b}\right\} + V_b^j\left\{\Xi_a, p_{2j}\right\}$ is invertible. However, we are still m constraints short to remove all Ostrogradsky ghosts. We can find these additional m constraints if $\mathsf{L}_{ab} = 0$. This condition will give us our *secondary conditions*, so let us expand L_{ab}:

$$\begin{aligned}
\mathsf{L}_{ab} &= \left\{\Xi_a, P_{2b}\right\} + V_b^j\left\{\Xi_a, p_{2j}\right\} \\
&= \frac{\partial \Xi_a}{\partial Q_2^b} + V_b^j\frac{\partial \Xi_a}{\partial q_2^j} \\
&= \mathcal{L}_{Q_2^b Q_2^a Q_1^c} Q_2^c + \mathcal{L}_{Q_2^b Q_2^a \phi^c} Q_1^c + \mathcal{L}_{Q_2^b Q_2^a q_1^j} q_2^j + \mathcal{L}_{Q_2^a Q_1^b} - \mathcal{L}_{Q_2^b Q_1^a} \\
&+ V_a^i\left(\mathcal{L}_{Q_2^b q_2^i Q_1^c} Q_2^c + \mathcal{L}_{Q_2^b q_2^i \phi^c} Q_1^c + \mathcal{L}_{Q_2^b q_2^i q_1^j} q_2^j + \mathcal{L}_{q_2^i Q_1^b} - \mathcal{L}_{Q_2^b q_1^i}\right) \\
&+ \frac{\partial V_a^i}{\partial Q_2^b}\left(\mathcal{L}_{q_2^i Q_1^b} Q_2^b + \mathcal{L}_{q_2^i \phi^b} Q_1^b + \mathcal{L}_{q_2^i q_1^j} q_2^j - \mathcal{L}_{q_1^i}\right) \\
&+ \left(\mathcal{L}_{q_2^j Q_2^a Q_1^c} Q_2^c + \mathcal{L}_{q_2^j Q_2^a \phi^c} Q_1^c + \mathcal{L}_{q_2^j Q_2^a q_1^i} q_2^i + \mathcal{L}_{Q_2^a q_1^j} - \mathcal{L}_{q_2^j Q_1^a}\right) V_b^j \\
&+ V_a^i\left(\mathcal{L}_{q_2^j q_2^i Q_1^c} Q_2^c + \mathcal{L}_{q_2^j q_2^i \phi^c} Q_1^c + \mathcal{L}_{q_2^j q_2^i q_1^k} q_2^k + \mathcal{L}_{q_2^i q_1^j} - \mathcal{L}_{q_2^j q_1^i}\right) V_b^j \\
&+ \frac{\partial V_a^i}{\partial q_2^j}\left(\mathcal{L}_{q_2^i Q_1^b} Q_2^b + \mathcal{L}_{q_2^i \phi^b} Q_1^b + \mathcal{L}_{q_2^i q_1^j} q_2^j - \mathcal{L}_{q_1^i}\right) V_b^j.
\end{aligned}$$

To reduce this equation, we enforce the primary condition (34):

$$0 = \mathcal{L}_{Q_2^a Q_2^b} - \mathcal{L}_{Q_2^a q_2^i}\mathcal{L}^{ij}\mathcal{L}_{q_2^j Q_2^b} = \mathcal{L}_{Q_2^a Q_2^b} + V_a^i \mathcal{L}_{q_2^i Q_2^b},$$

The time derivatives of (34) are also identically zero, i.e.

$$\begin{aligned}
0 &= \mathcal{L}_{Q_2^a Q_2^b Q_1^c} + V_a^i \mathcal{L}_{q_2^i Q_2^b Q_1^c} + \frac{\partial V_a^i}{\partial Q_1^c}\mathcal{L}_{q_2^i Q_2^b} \\
&= \mathcal{L}_{Q_2^a Q_2^b Q_1^c} + V_a^i \mathcal{L}_{q_2^i Q_2^b Q_1^c} + \mathcal{L}_{q_2^j Q_2^a Q_1^c} V_b^j + V_a^i \mathcal{L}_{q_2^i q_2^j Q_1^c} V_b^j, \\
0 &= \mathcal{L}_{Q_2^a Q_2^b \phi^c} + V_a^i \mathcal{L}_{q_2^i Q_2^b \phi^c} + \frac{\partial V_a^i}{\partial \phi^c}\mathcal{L}_{q_2^i Q_2^b} \\
&= \mathcal{L}_{Q_2^a Q_2^b \phi^c} + V_a^i \mathcal{L}_{q_2^i Q_2^b \phi^c} + \mathcal{L}_{q_2^j Q_2^a \phi^c} V_b^j + V_a^i \mathcal{L}_{q_2^i q_2^j \phi^c} V_b^j, \\
0 &= \mathcal{L}_{Q_2^a Q_2^b q_1^j} + V_a^i \mathcal{L}_{q_2^i Q_2^b q_1^j} + \frac{\partial V_a^i}{\partial q_1^j}\mathcal{L}_{q_2^i Q_2^b} \\
&= \mathcal{L}_{Q_2^a Q_2^b q_1^k} + V_a^i \mathcal{L}_{q_2^i Q_2^b q_1^k} + \mathcal{L}_{q_2^j Q_2^a q_1^k} V_b^j + V_a^i \mathcal{L}_{q_2^i q_2^j q_1^k} V_b^j.
\end{aligned}$$

Using these constraints, L_{ab} reduces to

$$\mathsf{L}_{ab} = \mathcal{L}_{Q_2^a Q_1^b} - \mathcal{L}_{Q_2^b Q_1^a} + V_a^i \left(\mathcal{L}_{q_2^i Q_1^b} - \mathcal{L}_{Q_2^b q_1^i}\right) + \left(\mathcal{L}_{Q_2^a q_1^j} - \mathcal{L}_{q_2^j Q_1^a}\right) V_b^j + V_a^i \left(\mathcal{L}_{q_2^i q_1^j} - \mathcal{L}_{q_2^j q_1^i}\right) V_b^j.$$

We thus have our secondary condition:

$$\mathcal{L}_{Q_2^a Q_1^b} - \mathcal{L}_{Q_2^b Q_1^a} = -V_a^i \left(\mathcal{L}_{q_2^i Q_1^b} - \mathcal{L}_{Q_2^b q_1^i}\right) - \left(\mathcal{L}_{Q_2^a q_1^j} - \mathcal{L}_{q_2^j Q_1^a}\right) V_b^j - V_a^i \left(\mathcal{L}_{q_2^i q_1^j} - \mathcal{L}_{q_2^j q_1^i}\right) V_b^j. \quad (36)$$

Again, in the absence of healthy first-order variables this equation reduces to the previous secondary condition (31).

Under this condition, we can again constrain $\Theta_a = \dot{\Xi}_a = \{\Xi_a, H_0\} - \mathcal{L}^{ji}\{\Sigma_{1i}, H_0\}\{\Xi_a, p_{2j}\} \approx 0$. The time evolution of Θ_a will generally fix the Lagrange multipliers ξ_2^a, and we thus end up with $m+n$ degrees of freedom. Else, there is further degeneracy and we end up with less degrees of freedom.

In order to make the comparison with field theories in chapter 4 easier, we can rewrite the conditions (34) and (36). Let ψ_A be the collection of our dependent variables $\dot{\phi}^a$ and q^i, where the index A first runs over the $\dot{\phi}^a$ and then over the q^i[10]. The elements of the null vectors $\boldsymbol{V}_a$ can then be labelled by $v_a^A = \left(\delta_a^b, V_a^i\right)$. The primary condition can then be written as [8]

$$0 = v_a^A \frac{\partial^2 \mathcal{L}}{\partial \dot{\psi}_A \partial \dot{\psi}_B} v_b^B = P_{ab}. \quad (37)$$

Similarly, the secondary condition becomes [8]

$$0 = v_a^A \frac{\partial^2 \mathcal{L}}{\partial \dot{\psi}_{[A} \partial \psi_{B]}} v_b^B = S_{[ab]}. \quad (38)$$

Example 8. Let us end with an example of a case where the first-order sector helps remove the instability. Our example Lagrangian has one second-order variable ϕ, and one first-order variable q:

$$\mathcal{L} = \epsilon \ddot{\phi}^2 + \frac{m}{2}\dot{\phi}^2 - \frac{k}{2}\phi^2 + \frac{M}{2}\dot{q}^2 - \sqrt{2M\epsilon}\dot{q}\ddot{\phi}. \quad (39)$$

[10] I.e. $A \in \{1, \ldots, m, 1, \ldots, n\}$.

For this theory, (37) is given by

$$\frac{\partial^2 \mathcal{L}}{\partial \ddot{\phi}^2} = \frac{\partial^2 \mathcal{L}}{\partial \ddot{\phi} \partial \dot{q}} \cdot \left(\frac{\partial^2 \mathcal{L}}{\partial \dot{q}^2} \right)^{-1} \cdot \frac{\partial^2 \mathcal{L}}{\partial \ddot{\phi} \partial \dot{q}}$$

To determine if this condition holds we compute the relevant derivatives of the Lagrangian. They are

$$\begin{aligned}
\frac{\partial^2 \mathcal{L}}{\partial \ddot{\phi}^2} &= 2\epsilon, & \frac{\partial^2 \mathcal{L}}{\partial \dot{q}^2} &= M, \\
\frac{\partial^2 \mathcal{L}}{\partial \ddot{\phi} \partial \dot{q}} &= -\sqrt{2M\epsilon}, & \frac{\partial^2 \mathcal{L}}{\partial \ddot{\phi} \partial \dot{\phi}} &= 0, \\
\frac{\partial^2 \mathcal{L}}{\partial \dot{\phi} \partial \dot{q}} &= 0, & \frac{\partial^2 \mathcal{L}}{\partial \ddot{\phi} \partial q} &= 0, \\
\frac{\partial^2 \mathcal{L}}{\partial \dot{q} \partial q} &= 0, & V = -\left(\frac{\partial^2 \mathcal{L}}{\partial \dot{q}^2} \right)^{-1} \cdot \frac{\partial^2 \mathcal{L}}{\partial \ddot{\phi} \partial \dot{q}} &= -\sqrt{\frac{2\epsilon}{M}}
\end{aligned}$$

Now, the left hand side of the primary condition (37) is

$$\frac{\partial^2 \mathcal{L}}{\partial \ddot{\phi}^2} = 2\epsilon,$$

and the right hand side of (37) is

$$\frac{\partial^2 \mathcal{L}}{\partial \ddot{\phi} \partial \dot{q}} \cdot \left(\frac{\partial^2 \mathcal{L}}{\partial \dot{q}^2} \right)^{-1} \cdot \frac{\partial^2 \mathcal{L}}{\partial \ddot{\phi} \partial \dot{q}} = -\sqrt{2M\epsilon} \cdot \frac{1}{M} \cdot -\sqrt{2M\epsilon} = \frac{2M\epsilon}{M},$$

and these two are equal. Thus the primary condition is met.

The secondary condition automatically vanishes, since the condition is anti-symmetric and we have only one second-order variable.

To see that the system is indeed Ostrogradsky free, let us look at the Euler-Lagrange equation (7). They are

$$\begin{aligned}
0 = E_\phi &= \frac{\mathrm{d}^2}{\mathrm{d}t^2} \frac{\partial \mathcal{L}}{\partial \ddot{\phi}} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathcal{L}}{\partial \dot{\phi}} + \frac{\partial \mathcal{L}}{\partial \phi} \\
&= 2\epsilon \dddot{\phi} - \sqrt{2M\epsilon} \ddot{q} - m\ddot{\phi} - k\phi, \qquad (40) \\
0 = E_q &= \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} \\
&= M\ddot{q} - \sqrt{2M\epsilon} \dddot{\phi}. \qquad (41)
\end{aligned}$$

As can be seen, these equations have higher-order terms. However, we can take combinations of them that are still zero in order to remove these higher-order terms. One such combination is $\bar{E}_\phi = E_\phi + \frac{\mathrm{d}VE_q}{\mathrm{d}t}$

$$\begin{aligned}
0 = \bar{E}_\phi &= 2\epsilon \dddot{\phi} - \sqrt{2M\epsilon} \ddot{q} - m\ddot{\phi} - k\phi + \frac{\mathrm{d}\left(\sqrt{\frac{2\epsilon}{M}} \left[M\ddot{q} - \sqrt{2M\epsilon} \ddot{\phi} \right] \right)}{\mathrm{d}t} \\
&= -m\ddot{\phi} - k\phi.
\end{aligned}$$

This is the equation of motion for a harmonic oscillator. We still have a third order term in the equations for q. Choose the combination $\bar{E}_q = E_q + B\frac{\mathrm{d}\bar{E}_\phi}{\mathrm{d}t}$, for a yet to be determined constant B. Then

$$\begin{aligned} 0 = \bar{E}_q &= M\ddot{q} - \sqrt{2M\epsilon}\dddot{\phi} + B\frac{\mathrm{d}\left(-m\ddot{\phi} - k\phi\right)}{\mathrm{d}t} \\ &= M\ddot{q} - \left(\sqrt{2M\epsilon} + Bm\right)\dddot{\phi} - Bk\dot{\phi}, \end{aligned}$$

which has no third-order terms if we choose $B = -\frac{\sqrt{2M\epsilon}}{m}$. It is also in this expression that we can see the coupling between ϕ and q: the acceleration of q depends on the velocity of ϕ. These final equations of motion are up to second order and thus stable. This concludes the example.

4 Field Theories

Up until now, we have only discussed (classical) mechanics. We have discussed three separate methods to describe and solve problems in classical mechanics, and showed why the Ostrogradsky Instability, or Ghost, forbids higher order theories. However, in physics we have moved on from classical mechanics, into quantum mechanics and relativistic theories, combined into quantum field theories. In this thesis, we will not discuss the quantum field theories, but we shall discuss classical field theories. In this section, I shall introduce classical field theory and in particular introduce scalar field theories. After that, we shall discuss the Ostrogradsky instability in field theories, and the primary and secondary constraints required to remove these ghosts from second order theories. In the sections after this one we will discuss field theories that are Ostrogradsky free.

4.1 What are field theories?

Field theories, in the simplest formulation possible, are theories that describe fields. Whereas in classical mechanics, the dependent variables are often the position coordinates of the objects you're interested in (1, 2 or 3 coordinates per object, depending on the dimensions of your theory), in field theory the coordinates are promoted to independent variables. Instead, each 'object' gets its own 'field'. This field describes, for each possible position and time, 'how much' of the object is at that particular time and place. As a comparison, consider the waves on the ocean. In classical mechanics, each wave would be given two coordinates, a latitude and longitude for example, and we describe the evolution of these waves with $2q$ equations of motion, 2 equations for each of the q waves. In field theory, in contrast, there is only one wave 'field', which describes how high the water level is at each point at each time. The waves themselves are the maxima of this field at each time.

In field theory, there are multiple independent coordinates. There is generally one time coordinate t, and d space coordinates x^i, depending on the amount of space dimensions considered. In the example of waves, we only needed 2 space coordinates, however, if we would want to describe the winds in the Netherlands at different heights, we would also need a third spatial coordinate to describe the vertical. These space and time coordinates are together called *space-time* coordinates, and are together denoted by the notation x^μ. The index μ runs over $0, 1, \ldots, d$, with $x^0 = t$. For further shorthand, we use $\partial_\mu = \frac{\partial}{\partial x^\mu}$ to describe derivatives w.r.t. the independent coordinates. If we split up space and time, we shall continue to use dotted derivatives to refer to time derivatives, i.e. $\partial\phi/\partial t = \dot{\phi}$, and use ∂_i for derivatives explicitly towards the space coordinates, i.e. $\partial\phi/\partial x^i = \partial_i\phi$. In this thesis, we shall restrict ourselves to three space coordinates. There are theories that use more space or time coordinates, but those are not relevant to the work we discuss. In either case, most of the work in this section generalizes to include more spatial coordinates.

Field theories are described using a Lagrangians that depends on, in principle, the independent coordinates t and x^i, the fields ϕ and their derivatives $\partial_\mu\phi$. They could in principle also depend on higher order derivatives, e.g. $\partial_\mu\partial_\nu\phi$, but this generally brings about the Ostrogradsky Ghost, as we shall see later.

From these Lagrangians, we derive equations of motions using the Euler-Lagrange equations (7). These were derived for a very general case in chapter 2, but we restate them for field theories as

$$0 = \frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}, \tag{42}$$

for first-order Lagrangians and

$$0 = \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} + \partial_\mu \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \partial_\nu \phi \right)} \right), \tag{43}$$

for second-order Lagrangians.

Unlike classical mechanics, the field theoretical Euler-Lagrange equations produce partial differential equations, which are harder to solve [16]. Nonetheless, field theories are preferred above classical mechanics. The primary reason is that field theories treat time and space on equal footing, as independent coordinates. Because of this, field theories can be used to describe relativistic theories. Field theories that are relativistic are called Lorentz-invariant theories, for reasons that we will discuss in the next subsection.

4.2 Lorentz invariance and Scalar theories

Scalar theories are a particular class of relativistic field theories. The scalar fields they describe are invariant under Lorentz transformations, which are a particular type of transformations on space time. Space time transformations allow us to describe the same physics using a different coordinate system. A general space time transformation is given by [17]

$$\bar{x}^\mu = f^\mu \left(x^\nu \right). \tag{44}$$

Lorentz transformations are linear space-time transformations, and are precisely those transformations that observe Einstein's principle of relativity. Einstein's principle can be caught in two postulates [18, appendix A]:

1. The speed of light c is the same in all inertial frames.

2. The laws of nature are the same in all inertial frames.

A common example of the second postulate is that of an observer, you for example, placed on board of a sound-proofed train car with no windows. If you were to drop a ball in this train car, you should not be able to tell if the train is stationary or moving at constant speed from how this ball falls. However, if the train is accelerating, you would notice this since the ball would not drop straight down, but instead fall down to the back-end of the train.

The second postulate was known for a long time. It is preserved under the Galilean space-time transformations. However, the fact that the speed of light is constant only began to be known around Einstein's time. As a result of this fist postulate, the thereunto used Galilean transformations are no longer valid, and instead Lorentz' space-time transformations have to be used.

Let us set up two observers moving at constant relative velocity v w.r.t. each other In particular, this movement is according on the x-axis for both observers; and their y and z axis also point in the same direction. In this case, the Galilean transformations are given by

$$\bar{t} = t, \qquad \bar{x} = x - v \cdot t, \qquad \bar{y} = y, \bar{z} \qquad = z.$$

In contrast, the Lorentz transformation between these two observers is given by [18]

$$\bar{t} = \frac{t - \frac{vx}{c^2}}{1 - \frac{v^2}{c^2}}, \qquad \bar{x} = \frac{x - vt}{1 - \frac{v^2}{c^2}}, \qquad \bar{y} = y, \qquad \bar{z} = z,$$

where c refers to the speed of light. In $3D$-space there are 6 Lorentz transformations: there are three rotations, one for each axis; and three 'boosts'. The boosts are used to transform between

observers travelling at different velocities, and there is one boost for each space direction. A similar set-up can be used in theories with a different number of (space) dimensions, and in those theories one will find a different number of possible transformations[11]. In any number of space dimensions, the Lorentz transformations are linear, and can thus be described by the equation

$$\bar{x}^\mu = \Lambda^\mu{}_\nu x^\nu. \tag{45}$$

The matrix $\Lambda^\mu{}_\nu$ is called a Lorentz matrix.

These matrices, and thus the Lorentz transformations, can also be defined differently, as precisely those matrices that leave the Minkowski metric invariant. The Minkowski metric is given by [18]

$$\eta_{\mu\nu} = \begin{pmatrix} -c^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and the invariance requirement is given by

$$\eta_{\mu\nu} = \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu \eta_{\rho\sigma}. \tag{46}$$

Theorem 4. *The invariance condition of the Lorentz transformation* (46) *preserves distances in space-time.*

Proof. Let $\gamma(\tau)$ be a curve in space time, $\gamma = [x^\mu | x^\mu(\tau) = x^\mu, \tau_i \leqslant \tau \leqslant \tau_f]$[12]. From appendix B and [10], we know that the length of this curve is given by

$$\Phi(\gamma) = \int_{\tau_i \tau^f} \sqrt{\eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}\, \mathrm{d}\tau,$$

where this time $\dot{x}^\mu = \mathrm{d}x^\mu/\mathrm{d}\tau$. This equation can be written coordinate free, i.e. not dependend on the parametrization τ:

$$\Phi(\gamma) = \int_\gamma \sqrt{\eta_{\mu\nu}\, \mathrm{d}x^\mu\, \mathrm{d}x^\nu}.$$

Next, apply a Lorentz transformation to the coordinates x^μ. This transformation transforms the integrand as

$$\begin{aligned} \mathrm{d}\bar{x}^\mu\, \mathrm{d}\bar{x}^\nu \eta_{\mu\nu} &= \Lambda^\mu{}_\rho\, \mathrm{d}x^\rho \Lambda^\nu{}_\sigma\, \mathrm{d}x^\sigma \Lambda^\kappa{}_\mu \Lambda^\lambda{}_\nu \eta_{\kappa\lambda} \\ &= \Lambda^\kappa{}_\mu \Lambda^\mu{}_\rho\, \mathrm{d}x^\rho \Lambda^\lambda{}_\nu \Lambda^\nu{}_\sigma\, \mathrm{d}x^\sigma \eta_{\kappa\lambda} \\ &= \delta^\kappa_\rho\, \mathrm{d}x^\rho \delta^\lambda_\sigma\, \mathrm{d}x^\sigma \eta_{\kappa\lambda} \\ &= \mathrm{d}x^\rho\, \mathrm{d}x^\sigma \eta_{\rho\sigma}, \end{aligned}$$

and thus

$$\Phi(\gamma) = \int_\gamma \sqrt{\eta_{\mu\nu}\, \mathrm{d}x^\mu\, \mathrm{d}x^\nu} = \int_\gamma \sqrt{\mathrm{d}\bar{x}^\rho\, \mathrm{d}\bar{x}^\sigma \eta_{\rho\sigma}}.$$

Ergo, Lorentz invariance preserves distances. □

[11] Of course, a transformation between two observers now travelling at a velocity v' instead of v in the same direction is technically a different transformation as well. However, it is the same kind of transformation. The same applies to the rotations: rotations of 30° v 135° around the same (x) axis are also technically different, but they are of the same kind, a rotation around the x axis.

[12] We parametrize to variable τ rather than t to avoid confusion.

Since we now have defined Lorentz transformations, we can discuss Lorentz invariance or covariance. An object object is called Lorentz invariant, if it is of the same form before and after a Lorentz transformation, i.e.

$$S\left(x^{\mu}\right) = S\left(\Lambda^{\mu}_{\nu} x^{\nu}\right). \tag{47}$$

If an invariant object has no indices, then such an object is called a *Lorentz scalar* or often just *scalar*. The scalar fields we discuss are called scalar fields because they are Lorentz scalars. Another example of a Lorentz scalar is the Lagrangian of a Lorentz invariant theory, in fact such a theory is called invariant precisely because the Lagrangian is a Lorentz scalar.

An object can also be Lorentz covariant, such objects are called *Lorentz vectors* or *Lorentz tensors*, depending on the amount of indices the object has. A particular example of a Lorentz vector is the position vector x^{μ}, since this transforms as $\bar{x}^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu}$ by definition. A general covariant Lorentz vector A^{μ} transforms as [18][13]

$$\bar{A}^{\mu}\left(\bar{x}^{\mu}\right) = \Lambda^{\mu}_{\nu} A^{\nu}\left(x^{\mu}\right). \tag{48}$$

By contrast, a contravariant Lorentz tensor transforms as

$$\bar{A}_{\mu}\left(\bar{x}^{\mu}\right) = \left(\Lambda^{-1}\right)^{\nu}_{\mu} A_{\nu}\left(x^{\mu}\right). \tag{49}$$

We can generalize this into a Lorentz tensor $T^{\mu_1 \mu_2 \cdots \mu_n}$, which transforms as

$$\bar{T}^{\mu_1 \mu_2 \cdots \mu_n}\left(\bar{x}^{\mu}\right) = \left(\prod_{i=1}^{n} \Lambda^{\mu_i}{}_{\nu_i}\right) T^{\nu_1 \cdots \nu_n}\left(x^{\mu}\right). \tag{50}$$

Examples of Lorentz vectors and tensors are the derivatives of Lorentz scalars, i.e. $\partial_{\mu}\phi$ is a Lorentz vector and $\partial_{\mu}\partial_{\nu}\phi$ is a Lorentz 2-tensor.

With these preliminaries out of the way, we can discuss scalar field theories. As mentioned above, the 'scalar' in 'scalar field theory' refers to the fact that the fields ϕ are Lorentz scalars. They are not the only Lorentz scalars, the Lagrangian itself also needs to be Lorentz invariant. As a result, the following Lagrangian could be a scalar theory:

$$\mathcal{L} = \frac{1}{2}\partial^{\mu}\phi\partial_{\mu}\phi - \frac{m^2}{2}\phi^2, \tag{51}$$

while this Lagrangian is not

$$\mathcal{L} = \dot{\phi}\phi - \frac{m}{2}\partial^{i}\partial_{i}\phi.$$

Equation (51) is one of the most basic Lagrangians in field theory, and its Euler-Lagrange equation is the well-known Klein-Gordon equation [12, chapter 2]

$$\partial_{\mu}\partial^{\mu}\phi = -m^2\phi. \tag{52}$$

This theory is called a *free theory*, because the field it describes is a free field, it does not interact, not with other fields (because there aren't any), nor with itself. This can be proven using the Feynman path-integral formalism from Quantum Field Theory [12]. An explanation of this formalism would be too far astray of the goal of this thesis, but a short summary of the result is that a Lagrangian has interactions if the field ϕ and/or its derivatives appear more than quadratically.

The next step from a single-field theory is a multi-field theory, describing q scalar fields ϕ^a. We can construct a very general multi-field Lagrangian, $\mathcal{L} = A\left(\phi^a, \partial_{\mu}\phi^a\right)$. Of course, such a Lagrangian is to general to tell us anything. In order to say anything meaningfull, we need to restrain the theory. We can do this by imposing symmetries on the theory.

[13] This relation is also the definition of a covariant tensor.

Definition 2. A *symmetry* of a Lagrangian theory is a transformation Φ of the (in)dependent coordinates that leaves the Euler-Lagrange equations invariant.

A standard example of a symmetry to impose upon a theory with q scalar fields is an $SO(q)$ symmetry. Roughly speaking, this symmetry states that nothing changes upon relabelling the fields. E.g. the transformation $\phi^i \leftrightarrow \phi^j$ does not change the equations of motion. An example of this is the Klein-Gordon Lagrangian (51) generalised to q fields:

$$\mathcal{L} = \frac{1}{2}\partial^\mu \phi^a \partial_\mu \phi_a - \frac{m}{2}\phi^a \phi_a.$$

One could also impose other symmetries. For example, one could split the scalars up into two groups, and impose an $SO(m)$ symmetry on the first m fields and an $SO(q-m)$ on the remaining n. Other symmetries are shift and Galilean symmetries, which we shall discuss in chapter 5. All symmetries lead to a conserved current and conserved quantity, according to Noether's theorem [11,12]. As a result, symmetries are often used or imposed in order to reduce the theory and make it easier to solve the equations of motion.

4.3 The Ostrogradsky ghosts in field theories

As hinted at in the above, field theories are not immune to the Ostrogradsky ghost. As in mechanics, in field theory the Ostrogradsky ghost appears whenever higher order derivatives are used in the Lagrangian. In particular, it is the presence of higher order time derivatives that causes the ghost to appear, as we shall also see when discussing conditions to remove the ghost in the next subsection.

In this subsection and the next, I shall closely follow the derivation by Crisostomi, Klein and Roest in [8, appendix B]. We shall immediately discuss the most general case, where there are m second order fields ϕ^a and n first order fields q^α. The Lagrangian is then given as $\mathcal{L} = \mathcal{L}\left(\phi^a, \partial_\mu \phi^a, \partial_\mu \partial_\nu \phi^a, q^\alpha, \partial_\mu q^\alpha\right)$. Rather than using the full Hamiltonian analysis, in which every derivative is replaced by a helper field, we will only remove the second order derivatives by use of the fields A_μ. The equivalent Lagrangian is then given by

$$\mathcal{L}_{eq} = \mathcal{L}\left(\phi^a, A^a_\mu, \partial_\nu A^a_\mu, q^\alpha, \partial_\mu q^\alpha\right) + \lambda^\mu_a \left(\partial_\mu \phi^a - A^a_\mu\right). \tag{53}$$

Through the Euler-Lagrange equations for the multipliers λ^μ_a, we can find the following relations:

$$\begin{aligned} 0 &= \frac{\partial \mathcal{L}_{eq}}{\partial \lambda^\mu_a} \Rightarrow \\ \partial_\mu \phi^a &= A^a_\mu \Leftrightarrow \\ \partial_\nu \phi^a &= A^a_\nu. \end{aligned}$$

We can take the derivative of the second equation to ∂_ν and the third equation to ∂_μ and find

$$\partial_\nu A^a_\mu = \partial_\nu \partial_\mu \phi^a = \partial_\mu \partial_\nu \phi^a = \partial_\mu A^a_\nu,$$

where for the central equal sign I have used that we can change the order of integration. In particular, this equation implies $\dot{A}^a_i = \partial_i A^a_0$. We can use this relation to split the space and time components of the Lagrange multipliers and the helper fields, and obtain

$$\mathcal{L}_{eq} = \mathcal{L}\left(\phi^a, A^a_0, A^a_i, \dot{A}^a_0, \partial_i A^a_0, \partial_i A^a_j, q^\alpha, \dot{q}^\alpha, \partial_i q^\alpha\right) + \lambda^0_a \left(\dot{\phi}^a - A^a_0\right) + \lambda^i_a \left(\partial_i \phi^a - A^a_i\right). \tag{54}$$

In the next part of the analysis, we define the conjugate momenta using Poisson brackets.

$$\begin{aligned} \{\phi^a, \pi_b\} &= \delta^a_b, & \left\{A^a_\mu, P^\nu_b\right\} &= \delta^a_b \delta^\nu_\mu, \\ \left\{\lambda^\mu_a, \Lambda^b_\nu\right\} &= \delta^b_a \delta^\mu_\nu, & \{q^\alpha, p_\beta\} &= \delta^\alpha_\beta. \end{aligned}$$

Next, we compare them with the Lagrangian definition of momenta to derive our primary constraints.

$$\pi_a = \frac{\partial \mathcal{L}_{eq}}{\partial \dot{\phi}^a} = \lambda_a^0, \qquad \Lambda_\mu^a = \frac{\partial \mathcal{L}_{eq}}{\partial \dot{\lambda}_a^\mu} = 0, \qquad P_a^i = \frac{\partial \mathcal{L}_{eq}}{\partial \dot{A}_i^a} = 0,$$
$$P_a^0 = \frac{\partial \mathcal{L}_{eq}}{\partial \dot{A}_0^a} = \frac{\partial \mathcal{L}}{\partial \dot{A}_0^a}, \qquad p_\alpha = \frac{\partial \mathcal{L}_{eq}}{\partial \dot{q}^\alpha} = \frac{\partial \mathcal{L}}{\partial \dot{q}^\alpha}.$$

The top row of these constraints produce in total $m + (D+1)\cdot m + D \cdot m$ primary constraints, where D is the amount of spatial dimensions. If there is no additional degeneracy, then the bottom row allows us to express $\dot{A}_0^a$ and $\dot{q}^\alpha$ in terms of the other variables :

$$\dot{A}_0^a = f^a\left(P_b^0, \phi^b, A_0^b, A_i^b, \partial_i A_0^b, \partial_i A_j^b, q^\alpha, p_\alpha\right),$$
$$\dot{q}^\alpha = g^\alpha\left(P_b^0, \phi^b, A_0^b, A_i^b, \partial_i A_0^b, \partial_i A_j^b, q^\alpha, p_\alpha\right).$$

For the next step, we need to time evolve the constraints. This in turn requires us to define the Hamiltonian. Whereas in classical mechanics, the Hamiltonian is just the Legendre transform of the Lagrangian, field theory is more involved. Legendre transforming the Lagrangian yields a function called the *Hamiltonian density* [12]. It is the integral of this Hamiltonian density over all space that produces the Hamiltonian. Similar to chapters 2 and 3, we define two Hamiltonian densities:

$$\mathcal{H}_0 = P_a^0 f^a + p_\alpha g^\alpha - \mathcal{L},$$
$$\mathcal{H}_T = \mathcal{H}_0 + a^a\left(\pi_a - \lambda_a^0\right) + b_a^0 \Lambda_0^a + b_a^i \Lambda_i^a + c_i^a P_a^i.$$

The total Hamiltonian is then defined as

$$H_T = \int \mathcal{H}_T \, \mathrm{d}^D x.$$

The functions a^a, b_a^μ, c_i^a are the Lagrange multipliers that enforce the primary constraints. We have already split b_a^μ into time and space components, since the time evolution of the constraints Λ_μ^a produces different results for its time and space components. Indeed, the next step in our analysis is to compute this time evolution:

$$\{\Lambda_i^a, H_T\} = \partial_i \phi^a - A_i^a \approx 0, \qquad \left\{P_a^i, H_T\right\} = \frac{\partial \mathcal{L}}{\partial A_i^a} - P_b^0 \frac{\partial f^b}{\partial A_i^a} - \lambda_a^i \approx 0,$$
$$\{\Lambda_0^a, H_T\} = A_0^a - a^a = 0, \qquad \left\{\pi_a - \lambda_a^0, H_T\right\} = \frac{\partial \mathcal{L}}{\partial \dot{\phi}^a} + P_b^0 \frac{\partial f^b}{\partial \phi^a} + \partial_i \lambda_a^i - b_a^0 = 0.$$

Again the top row produces constraints, this time $2D \cdot m$ secondary constraints. The bottom row can be used to fix the multipliers a^a and b_a^0. Further time evolving our secondary constraints allows us to fix the remaining multipliers:

$$\{\partial_i \phi^a - A_i^a, H_T\} = \{\partial_i \phi^a, H_T\} - c_i^a = 0,$$
$$\left\{\frac{\partial \mathcal{L}}{\partial A_i^a} - P_b^0 \frac{\partial f^b}{\partial A_i^a} - \lambda_a^i, H_T\right\} = \left\{\frac{\partial \mathcal{L}}{\partial A_i^a} - P_b^0 \frac{\partial f^b}{\partial A_i^a}, H_T\right\} - b_a^i = 0.$$

At this point the analysis ends. All constraints are of secondary class, and thus we can calculate the degrees of freedom using (17):

$$\#DOF = \frac{1}{2}\left(2\left(m + 2\left(D+1\right)m + n\right) - 2\cdot 0 - \left(m + (D+1)m + Dm\right) - 2Dm\right) = 2m + n.$$

As we can see, we have m too many degrees of freedom, all of whom are Ostrogradsky ghosts.

4.4 Ostrogradsky-free field theories

We wish to remove the ghost degrees of freedom from our second order theory. To do so, we must impose additional constraints upon the Lagrangian. In particular, these constraints must be imposed upon the m second order fields, since it are those fields that cause the ghost. We can impose the following m constraints upon these fields:

$$\chi_a = P_a^0 - F_a\left(A_0^b, \partial_i A_0^b, q^\alpha, \partial_i q^\alpha, p_\alpha\right) \approx 0. \tag{55}$$

We can not impose this constraint on any Lagrangian. Imposing this constraint requires that the relation $P_a^0 = \frac{\partial \mathcal{L}_{eq}}{\partial \dot{A}_0^a}$ is no longer invertible for all $P0_a$. This is the case when the kinetic matrix

$$\mathsf{K} = \begin{pmatrix} \mathcal{L}_{\dot{A}^a \dot{A}^b} & \mathcal{L}_{\dot{A}^a \dot{q}^\beta} \\ \mathcal{L}_{\dot{q}^\alpha \dot{A}^b} & \mathcal{L}_{\dot{q}^\alpha \dot{q}^\beta} \end{pmatrix}$$

has m null-vectors in the $\dot{A}^b$ sector. We can use the same null-vectors as we did for classical mechanics, and obtain the null vectors $\boldsymbol{V}_a = \left(\delta_a^b, V_a^\alpha\right)$, where $V_a^\alpha = -\mathcal{L}_{\dot{A}^a \dot{q}^\beta} \mathcal{L}^{-1}_{\dot{q}^\beta \dot{q}^\alpha}$. The condition that these vectors are then actual null vectors of K is similar to the primary condition (37) from classical mechanics. Using the same notation as we did in chapter 3, let ψ^A be the collection of the dependent variables $\dot{\phi}^a$ and q^i, where the index A first runs over the $\dot{\phi}^a$ and then over the q^i. In this notation, the *primary condition* becomes [8]

$$0 = P_{(ab)} = v_a^A \mathcal{L}_{\dot{\psi}^A \dot{\psi}^B} v_b^B. \tag{56}$$

The constraints χ_a need to be added to the total Hamiltonian. Since we have already dealt with the other constraints in the previous section, we can ignore them for our continued analysis and define our total Hamiltonian as

$$H_T = H_0 + \int \xi^a \chi_a \, \mathrm{d}^D x,$$

where $H_0 = \int \mathcal{H}_0 \, \mathrm{d}^D x$. We then compute the time evolution of χ_a as

$$\left\{\chi_a(x), H_T\right\} = \left\{\chi_a(x), H_0\right\} + \left\{\chi_a(x), \int \xi^b(y) \chi_b(y) \, \mathrm{d}^D y\right\},$$

where we have made the dependence on x^μ in the integral explicit to avoid confusion. The last Poisson bracket is composed of several terms, and these are

$$\left\{P_a^0, \int \xi^b F_b \, \mathrm{d}^D y\right\} = \left(\partial_i \frac{\partial F_b}{\partial \partial_i A_0^a} - \frac{\partial F_b}{\partial A_0^a}\right) \xi^b + \frac{\partial F_b}{\partial \partial_i A_0^a} \partial_i \xi^b, \tag{57}$$

$$\left\{F_a, \int \xi^b P_b^0 \, \mathrm{d}^D y\right\} = \frac{\partial F_a}{\partial A_0^b} \xi^b + \frac{\partial F_a}{\partial \partial_i A_0^b} \partial_i \xi^b, \tag{58}$$

$$\left\{F_a, \int \xi^b F_b \, \mathrm{d}^D y\right\} = \left(\{F_a, F_b\} + \frac{\partial F_{(a}}{\partial \partial_i q^\alpha} \partial_i \frac{\partial F_{b)}}{\partial p_\alpha}\right) \xi^b + \left(\frac{\partial F_{(a}}{\partial \partial_i q^\alpha} \frac{\partial F_{b)}}{\partial p_\alpha}\right) \partial_i \xi^b. \tag{59}$$

We can group these brackets together, and find that

$$\left\{\chi_a(x), H_T\right\} = \left\{\chi_a(x), H_0\right\} + S_{[ab]} \xi^b + \left(S^i\right)_{(ab)} \partial_i \xi^b.$$

Since we need m additional constraints, we need to remove the dependency of ξ^b from this equation, else this equation can be used to fix some of the ξ^b. In that case $\left\{\chi_a(x), H_T\right\} \approx 0$ would be identically zero for certain χ_a and thus we can not make it a constraint. Thus we need all of the dependency removed and therefore we have the constraints $S_{[ab]} = 0$ and $\left(S^i\right)_{(ab)} = 0$.

To find the exact form of these constraints, we use not only the above Poisson brackets, but also the constraint (55). This constraint implies $F_a \approx P^0_a = \frac{\partial \mathcal{L}}{\partial \dot{A}^0_a}$, and we can use this to calculate the derivatives of F_a. Doing this computation, we find that the two sets of *secondary conditions* are given by [8]

$$0 = (S_i)_{(ab)} = 2v^A_a \mathcal{L}_{\dot{\psi}_{(A}\partial_i\phi_{B)}} v^B_b, \tag{60}$$

and

$$0 = S_{[ab]} = 2v^A_a \mathcal{L}_{\dot{\psi}_{[A}\psi_{B]}} v^B_b + 2v^A_{[a}\mathcal{L}_{\dot{\psi}_A\partial_i\dot{\psi}_B}\partial_i v^B_{b]} - \delta_i\left(v^A_a \mathcal{L}_{\dot{\psi}_{[A}\partial_i\psi_{B]}} v^B_b\right). \tag{61}$$

Note that these secondary conditions reduce to the conditions from classical mechanics in the absence of space coordinates. In particular, (60) vanishes, while (61) reduces to (38) in the absence of spatial derivatives.

The conditions (56), (60) and (61) apply to a general field theory. However, as stated previously, we are primarily interested in Lorentz invariant field theories. We are interested to see if requiring the constraint of Lorentz invariance causes any changes in the primary and secondary conditions. Fortunately, Crisostomi et al. also investigated this [8, section 2.3]. Let $\mathcal{L}$ be a Lorentz invariant Lagrangian, then after a Lorentz transformation we find the (general relation)

$$\bar{\mathcal{L}} = \mathcal{L} + \delta\mathcal{L},$$

where

$$\delta\mathcal{L} = \mathcal{L}_{\phi^a}\delta\phi^a + \mathcal{L}_{\partial_\mu\phi^a}\delta\partial_\mu\phi^a + \mathcal{L}_{\partial_\mu\partial_\nu\phi^a}\delta\partial_\mu\partial_\nu\phi^a + \mathcal{L}_{q^\alpha} + \delta q^\alpha + \mathcal{L}_{\partial_\mu q^\alpha}\delta\partial_\mu q^\alpha.$$

For a Lorentz invariant theory, a Lorentz transformation should not change the degeneracies. E.g., if the primary condition is satisfied pre-transformation, then it should also be satisfied after the transformation. As a result, we have

$$0 = \bar{P}_{(ab)} = P_{(ab)} + \delta P_{(ab)}.$$

If we assume that the primary condition is satisfied, then $\delta P_{(ab)}$ must also vanish. Let us now consider a particular Lorentz transformation, an arbitrary boost in the i-direction. The change in $P_{(ab)}$ is given by

$$\delta_i P_{(ab)} = \left(P_{(ab)}\right)_{\dot{\Psi^N}}\partial_i\Psi^N + \left(P_{(ab)}\right)_{\partial_i\Psi^N}\dot{\Psi}^N + (S_i)_{(ab)}, \tag{62}$$

where the 'field' Ψ^N runs over all fields, i.e. $\Psi \equiv (\phi^a, \partial_\mu\phi^a, q^\alpha)$[14]. If the primary condition is met for a Lorentz invariant theory, this equation equals zero. Further, since $P_{(ab)} = 0$, its derivatives should also be zero. As a result, (62) becomes

$$(S_i)_{(ab)} = 0.$$

Thus, for a Lorentz invariant theory, satisfying the primary condition (56) also satisfies the symmetric secondary condition (60).

Further note that, even for a Lorentz invariant theory, the constraints are not Lorentz invariant. Lorentz invariance however should not be expected, since the Ostrogradsky theorem itself is not invariant. The theorem concerns itself only with second order time derivatives, and the instability only propagates in the time derivatives, so the derivatives in the spatial direction are irrelevant. This violates Lorentz invariance, and causes non-Lorentz invariant constraints. This also manifests itself in the 'healthy' formulation of the equations of motion, where the manifestly second order formulation of the equations of motion belonging to a healthy Lagrangian $\mathcal{L}(\partial\partial\phi_m, \partial\phi_m, \phi_m, \partial q_\alpha, q_\alpha)$ are not Lorentz invariant, and vice-versa: the manifestly Lorentz-invariant equations of motion are not necessarily second-order. We have

[14] Compare to the definition of ψ^A above.

shown in example 8 that in an Ostrogradsky-free theory we can take certain combinations of the Euler-Lagrange equations in order to obtain pure second order equations. This can also be done for field theories. In particular, for a field theory, take the combination [8]

$$E_{\phi^a}+\frac{\mathrm{d}}{\mathrm{d}t}\left(V_a^\alpha E_{q^\alpha}\right)+\partial_i\left(\alpha_a^{i\alpha}E_{q^\alpha}\right)+U_a^\alpha E_{q^\alpha}=P_{(ab)}\phi_b^{(4)}+\left(S_i\right)_{(ab)}\partial_i\phi_b^{(3)}+S_{[ab]}\phi_b^{(3)}+\left(\ddot{\phi},\dot{q},\dots\right), \quad (63)$$

where

$$\alpha_a^{i\alpha}=-\left(\mathcal{L}_{\partial_i\dot{\phi}^a\dot{q}^\beta}+\mathcal{L}_{\ddot{\phi}^a\partial_i q^\beta}+2V_a^\gamma\mathcal{L}_{\dot{q}^{(\gamma}\dot{q}^{\beta)}}\right)\mathcal{L}^{-1}_{\dot{q}^\beta\dot{q}^\alpha},$$

$$U_a^\alpha=\left(\left(C_a\right)_{\dot{q}^\beta}-\alpha_a^{i\gamma}\partial_i\mathcal{L}_{\dot{q}^\gamma\dot{q}^\beta}\right)\mathcal{L}^{-1}_{\dot{q}^\beta\dot{q}^\alpha},$$

and

$$C_a=E_{\dot{\phi}^a}+V_a^\alpha E_{q^\alpha}.$$

As can be seen from the form of (63), satisfying the constraints (56), (60) and (61) causes this combination to be second order in derivatives at the highest. The Euler-Lagrange equations of a Lorentz invariant theory are also Lorentz invariant. The combination of Euler-Lagrange equations (63) can only be Lorentz invariant if $W_m^{\mu\alpha}=(V_a^\alpha,\alpha_a^{i\alpha})$ is a Lorentz vector and U_a^α is a Lorentz scalar, so that (63) becomes

$$E_{\phi^a}+\partial_\mu\left(W_a^{\mu\alpha}E_{q^\alpha}\right)+U_a^\alpha E_{q^\alpha}.$$

This equation would then be Lorentz invariant. Unfortunately, $W_m^{\mu\alpha}$ and U_a^α generally do not meet this requirement of Lorentz (co/in)variance. As a result, we have something of a trade-off: We can either have manifestly Lorentz invariant equations of motion, but that do contain higher-order derivatives, or have manifestly lower-order equations of motion, but are not Lorentz invariant [8].

Example 9. Let us discuss a simple example of a Lorentz Invariant stable theory with both one second order and one first order field. Our theory is described by the Lagrangian

$$\mathcal{L}\left(\phi,q\right)=\left(\Box\phi\right)^2+2\Box\phi\partial_\mu\phi\partial^\mu q+\partial_\mu\phi\partial_\nu\phi\partial^\mu q\partial^\nu q. \quad (64)$$

For this case, the primary constraint is

$$0=\frac{\partial^2\mathcal{L}}{\partial\ddot{\phi}^2}-\frac{\mathcal{L}^2_{\ddot{\phi}\dot{q}}}{\mathcal{L}_{\dot{q}\dot{q}}}.$$

We have

$$\frac{\partial^2\mathcal{L}}{\partial\ddot{\phi}^2}=2,\ \mathcal{L}_{\ddot{\phi}\dot{q}}=2\dot{\phi}\text{ and }\mathcal{L}_{\dot{q}\dot{q}}=2\dot{\phi}^2$$

so this condition is satisfied. On account of Lorentz Invariance, the symmetric condition is automatically satisfied. The anti-symmetric condition is also satisfied on account of there only being one second-order field.

The Euler-Lagrange equations are given by

$$\begin{aligned}0&=\partial_\mu\partial_\nu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\partial_\nu\phi}\right)-\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\right)+\frac{\partial\mathcal{L}}{\partial\phi}\\&=2\Box\left[\Box\phi\right]+4\partial_\mu\partial_\nu\phi\partial^\mu\partial^\nu q+2\partial_\mu\phi\partial^\mu\Box q-2\Box\phi\Box q-2\partial_\mu\partial_\nu\phi\partial^\mu q\partial^\nu q-2\partial_\mu\phi\Box q\partial^\mu q-2\partial_\mu\phi\partial_\nu q\partial^\mu\partial^\nu q,\end{aligned}$$

and

$$\begin{aligned} 0 &= \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu q} \right) - \frac{\partial \mathcal{L}}{\partial q} \\ &= 2\partial_\mu \Box\phi \partial^\mu \phi + 2\Box\phi\Box\phi + 2\Box\phi\partial_\mu\phi\partial^\mu q + 2\partial_\mu\phi\partial_\mu\partial_\nu\phi\partial^\nu q + \partial_\mu\phi\partial_\nu\phi\partial^\mu\partial^\nu q. \end{aligned}$$

These equations of motion are clearly Lorentz invariant, but also fourth-order. Let us try to rewrite E_ϕ into second-order form. We have

$$\begin{aligned} V &= \frac{\mathcal{L}_{\ddot{\phi}\dot{q}}}{\mathcal{L}_{\dot{q}\dot{q}}} = \frac{1}{\dot{\phi}}, \\ \alpha^i &= -\left(\mathcal{L}_{\partial_i\dot{\phi}\dot{q}} + \mathcal{L}_{\ddot{\phi}\partial_i q} + 2V\mathcal{L}_{\dot{q}\dot{q}} \right) \mathcal{L}_{\dot{q}\dot{q}}^{-1} = -\frac{\partial^i q}{\dot{\phi}} - \frac{\partial^i \phi}{\dot{\phi}^2} - \frac{2}{\dot{\phi}^2}. \end{aligned}$$

$W^\mu = (V, \alpha^i)$ is not a Lorentz vector, since $\alpha^i \neq \frac{1}{\partial_i \phi}$. As a result, the transformed Euler Lagrange equation $E_\phi + \partial_\mu (W^\mu E_q) + U E_q$ will not be Lorentz invariant, though it will be second order. Calculating the exact form of this transformed equation of motion is left as an exercise to the reader.

5 Galileons, and other second order field theories

In the previous section, we have discussed field theories and how the Ostrogradsky ghost is manifest in those. We also discuss constraints to avert the ghost. In this section, we shall discuss some of the second order theories that are free of the Ostrogradsky ghost. In particular, we shall focus on the Galilean theories, a class of second-order Lagrangian theories that have only second-order derivatives in their equations of motion, before generalising to the most general ghost-free multi-scalar theory.

5.1 One Galileon

Galileon theories are given that name because they obey a so called Galilean symmetry. This Galilean symmetry is a symmetry in the fields, and thus a Galilean theory is symmetric under the transformation

$$\bar{\phi} = \phi + b_\mu x^\mu + c. \tag{65}$$

This symmetry is a generalization of the 3D Galilean symmetry, with the field ϕ acting as a fourth spacial dimension [5], hence its name. For a theory to be symmetric under the Galilean transformation, only the equations of motion need to be invariant. Clearly, Galilean symmetry requires that the equations of motion are of strict second order, i.e. only second order terms appear[15]. Nicolis et al. [4] described the general Lagrangian that obeys the Galilean symmetry. This general Lagrangian is a linear combination of the following five Lagrangians:

$$\begin{aligned}
\mathcal{L}_1 &= \phi, \\
\mathcal{L}_2 &= -\frac{1}{2}\partial^\mu\phi\partial_\mu\phi, \\
\mathcal{L}_3 &= -\frac{1}{2}\partial_\mu\partial^\mu\phi\partial^\nu\phi\partial_\nu\phi, \\
\mathcal{L}_4 &= -\frac{1}{2}\left[(\partial^\mu\partial_\mu\phi)^2 - \partial^\mu\partial_\nu\phi\partial^\nu\partial_\mu\phi\right]\partial^\rho\phi\partial_\rho\phi \\
\mathcal{L}_5 &= -\frac{1}{2}\left[(\partial^\mu\partial_\mu\phi)^3 - 3\partial^\mu\partial_\mu\phi\partial^\nu\partial_\rho\phi\partial^\rho\partial_\nu\phi + 2\partial^\mu\partial_\nu\phi\partial^\nu\partial_\rho\phi\partial^\rho\partial_\mu\phi\right]\partial^\sigma\phi\partial_\sigma\phi.
\end{aligned} \tag{66}$$

The total Lagrangian is then given by $\mathcal{L} = \sum_{m=1}^{5}\alpha^m\mathcal{L}_m$. By computing the associated Euler-Lagrange equations (7), we can see that these Lagrangians are indeed symmetric under the Galilean symmetry (65). The Euler-Lagrange equations derived from these Lagrangians are given by [4, 5]:

$$\begin{aligned}
E(\mathcal{L}_1) &= 1, \\
E(\mathcal{L}_2) &= \Box\phi, \\
E(\mathcal{L}_3) &= (\Box\phi)^2 - \partial_\mu\partial^\nu\phi\partial_\nu\partial^\mu\phi, \\
E(\mathcal{L}_4) &= (\Box\phi)^3 - 3\Box\phi\partial_\mu\partial^\nu\phi\partial_\nu\partial^\mu\phi + 2\partial^\mu\partial_\nu\phi\partial^\nu\partial_\rho\phi\partial^\rho\partial_\mu\phi, \\
E(\mathcal{L}_5) &= (\Box\phi)^4 - 6(\Box\phi)^2\partial_\mu\partial^\nu\phi\partial_\nu\partial^\mu\phi + 8\Box\phi\partial^\mu\partial_\nu\phi\partial^\nu\partial_\rho\phi\partial^\rho\partial_\mu\phi \\
&\quad + 3(\partial_\mu\partial^\nu\phi\partial_\nu\partial^\mu\phi)^2 - 6\partial^\mu\partial_\nu\phi\partial^\nu\partial_\rho\phi\partial^\rho\partial_\sigma\phi\partial^\sigma\partial_\mu\phi.
\end{aligned}$$

$E(\mathcal{L}_1)$ is a constant, and thus clearly invariant under any transformation to the field ϕ. All other Euler-Lagrange equations are purely second order, and from (65) it is clear that $\partial_\mu\partial^\nu\bar{\phi} = \partial_\mu\partial^\nu\phi$.

[15] This symmetry allows for higher-order terms, but those are blocked by Ostrogradsky's theorem. See chapter 4 or [1].

There are only 5 Lagrangians, and the question is why not more? From the structure of the Lagrangians (66), a general formula can be derived as

$$\mathcal{L}_m = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi\delta^{\mu_3\cdots\mu_m}_{\nu_3\ldots\nu_m}\partial_{\mu_3}\partial^{\nu_3}\phi\cdots\partial_{\mu_m}\partial^{\nu_m}\phi, \tag{67}$$

for $m \geqslant 3$, and where $\delta^{\mu_3\cdots\mu_m}_{\nu_3\ldots\nu_m}$ is the generalized Dirac-delta function, see appendix A. This generalized Dirac-delta function is zero if it has more than $p = d + 1$ indices, where d is the amount of spatial dimensions. This eliminates $\mathcal{L}_m$ for $m \geqslant 6$.

There are other formulations of the Galilean Lagrangians (66), which can be used for different generalizations [5]. However, they all produce the same Euler-Lagrange equations [5], and thus we won't discuss them here.

5.2 Self duality

An interesting thing about the theory of Galileons, is that it is dual to itself. That is to say, the Galileon Lagrangians (66) are transformed to another Galilean Lagrangian under the coordinate transformation [19, 20]

$$\bar{x}^\mu = x^\mu + \partial^\mu\phi\left(x\right). \tag{68}$$

This transformation admits an inverse transformation. Impose that a dual scalar field $\bar{\pi}\left(\bar{x}\right)$ exists, such that the inverse relation is

$$x^\mu = \bar{x}^\mu - \bar{\partial}^\mu\bar{\phi}\left(\bar{x}\right). \tag{69}$$

In this equation, as well as in the rest of this section, assume that barred coordinates and functions refer to coordinates and functions in the dual system.

It is a property of this map that the derivative of ϕ transforms like a scalar, instead of the field ϕ itself, i.e. $\partial^\mu\phi = \bar{\partial}^\mu\bar{\phi}$. The transformation rules for ϕ and $\bar{\phi}$ are instead given by [19]

$$\bar{\phi}\left(\bar{x}\right) = \phi\left(x\right) + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi \tag{70}$$

$$\phi\left(x\right) = \bar{\phi}\left(\bar{x}\right) - \frac{1}{2}\bar{\partial}_\mu\bar{\phi}\partial^\mu\bar{\phi} \tag{71}$$

By comparison, it are the second derivative tensors $\Pi^\mu{}_\nu = \partial^\mu\partial_\nu\phi$ that transform in a covariant, vectorial manner: $\Pi^\mu{}_\nu = J^\mu{}_\rho\bar{\Pi}^\rho_\nu$, with $J^\mu{}_\nu$ the Jacobian of the coordinate transformation (68), $J^\mu{}_\nu = \frac{\partial\bar{x}^\mu}{\partial x^\nu} = \delta^\mu_\nu + \partial_\nu\partial^\mu\phi$.

Next to consider is how the Lagrangians transform. The total Lagrangian is a linear combination of the Lagrangians (66), given as $\mathcal{L} = \sum_{i=2}^5 c_i\mathcal{L}_i\left(\phi, \partial\phi, \partial\partial\phi\right)$[16]. After performing the duality transformation (68), the Lagrangian becomes $\mathcal{L} = \sum_{i=2}^5 \bar{c}_i\mathcal{L}_i\left(\bar{\phi}, \bar{\partial}\bar{\phi}, \bar{\partial}\bar{\partial}\bar{\phi}\right)$ for different constants $\bar{c}_i$ This shows that under this duality transformation, a galileon indeed transforms into a galileon. The relations between the coefficients c_i are given by [19]:

$$\begin{aligned}
\bar{c}_2 &= c_2, \\
\bar{c}_3 &= c_2 + c_3, \\
\bar{c}_4 &= c_2 + 2c_3 + c_4, \\
\bar{c}_5 &= c_2 + 3c_3 + 3c_4 + c_5.
\end{aligned}$$

These relations follow Newton's binomial. An interesting effect of this duality is that it maps a free scalar theory into a Galilean theory [20]. A free scalar theory is a galileon theory with Lagrangian $\mathcal{L} = c_2\mathcal{L}_2 = \frac{c_2}{2}\partial_\mu\phi\partial^\mu\phi$, and under the duality it maps to

$$\mathcal{L} = c_2\mathcal{L}_2 + c_2\mathcal{L}_3 + c_4\mathcal{L}_3 + c_2\mathcal{L}_5.$$

[16] For this analysis, $c_1 = 0$, i.e. we ignore $\mathcal{L}_0$.

The final property to examine is the Galileon transformation. The theory of Galileons is based on the symmetry under the Galilean transformation. It turns out under the duality transformation (68), the Galileon symmetry transformation becomes a mixed transformation, consisting of both a symmetry transformation and a translation [20]. In formula form, if we have the Galilean symmetry transformation

$$\phi' = \phi(x) + b_\mu x^\mu,$$

then its dual field transforms as

$$\bar{\phi}'(x) = \bar{\phi}(x - b) - b_\mu x^\mu.$$

This mixing of symmetries can be understood from the origins of Galileon theory, as a decoupled version of DGP gravity [20, 21]. In this decoupling, a galileon transformation on ϕ is equivalent to a translation of the metric coordinates $\bar{x}$ in the barred coordinate system, and vice versa. Thus, when a galileon transformation is performed on ϕ, its dual $\bar{\phi}$ undergoes not just the galileon transformation, but also the equivalent translation. We will come back to this, after we have discussed general relativity and DGP gravity in chapters 6 and further.

5.3 Multi Galileons

We can generalize galileon theory by adding additional galileon fields. The trivial (and uninteresting) is to just add additional the Lagrangians for different galileons together, e.g. $\mathcal{L} = \mathcal{L}_{gal}(\phi^1) + \mathcal{L}_{gal}(\phi^2)$. Mixing galileon fields and thus making them interact is much more interesting. Of course, the end result still must both be Galilean invariant and Ostrogradsky free. Trodden and Hinterbichler [22] and Padilla and Sivanesan [23] all looked into these multi-galileon theories. The general Lagrangian for q galileons is given by [23]

$$\mathcal{L} = \alpha_a \phi^a + \alpha_{ab} \partial_\mu \phi^a \partial^\mu \phi^b + \sum_{m=3}^{5} \alpha_{a_1 \ldots a_m} \partial_\mu \phi^{a_1} \partial^\mu \phi^{a_2} \delta^{\mu_3 \ldots \mu_m}_{\nu_3 \ldots \nu_m} \partial_{\mu_3} \partial^{\nu_3} \phi^{a_3} \cdots \partial_{\mu_m} \partial^{\nu_m} \phi^{a_m}, \tag{72}$$

where the tensor $\alpha_{a_1 \ldots a_m}$ is fully symmetric in all its indices. The $\alpha_{a_1 \ldots a_m}$ behave as Lorentz scalars, since the a_m are not Lorentz indices, i.e. they do not count over the independent variables. Thus (72) is Lorentz invariant. It is of a similar form to (67), and indeed reduces to it for $q = 1$.

The Lagrangian (72) is also ghost free. Due to the symmetry of $\alpha_{a_1 \ldots a_m}$, the higher derivative terms in the Euler-Lagrange equations (7) cancel against each other. We can show this by computing the primary (56) and secondary conditions (61)[17]. Since our theory concerns only second order variables, the conditions reduce to

$$P_{(ab)} = \mathcal{L}_{\ddot{\phi}^a \ddot{\phi}^b} \tag{73}$$

and

$$S_{[ab]} = 2\mathcal{L}_{\ddot{\phi}^{[a} \dot{\phi}^{b]}} - \partial_i \mathcal{L}_{\ddot{\phi}^{[a} \partial_i \dot{\phi}^{b]}}. \tag{74}$$

We can compute these conditions. For $m = 1, 2, 3$, $P_{(ab)} = 0$, simply because there are either no second order derivatives, or they appear only linearly. For $m = 4, 5$ we find

$$\begin{aligned} P_{(ab)} &= \mathcal{L}_{\ddot{\phi}^a \ddot{\phi}^b} \\ &= \frac{\partial}{\partial \ddot{\phi}^b} \sum_{m=4}^{5} \alpha_{a_1 a_2 a \ldots a_m} (m-2) \partial_\mu \phi^{a_1} \partial^\mu \phi^{a_2} \delta^{i_4 \ldots i_m}_{j_4 \ldots j_m} \partial_{i_4} \partial^{j_4} \phi^{a_4} \cdots \partial_{i_m} \partial^{j_m} \phi^{a_m} \\ &= 0. \end{aligned}$$

[17] Lorentz invariance takes care of condition (60).

It is due to the anti-symmetric nature of the generalized Dirac-delta matrix that no second-order time derivatives appear in the $\mathcal{L}_{\ddot{\phi}^a}$, and thus $P_{(ab)} = 0$. Similarly, we can compute (74). We shall do this in two parts:

$$\begin{aligned}
\mathcal{L}_{\ddot{\phi}^{[a}\dot{\phi}^{b]}} &= \mathcal{L}_{\ddot{\phi}^a\dot{\phi}^b} - \mathcal{L}_{\ddot{\phi}^b\dot{\phi}^a} \\
&= \frac{\partial}{\partial\dot{\phi}^b}\left(\alpha_{a_1a_2a}\partial_\mu\phi^{a_1}\partial^\mu\phi^{a_2} + 2\alpha_{a_1a_2aa_4}\partial_\mu\phi^{a_1}\partial^\mu\phi^{a_2}\partial_i\partial^i\phi^{a_4} + 3\alpha_{a_1a_2aa_4a_5}\partial_\mu\phi^{a_1}\partial^\mu\phi^{a_2}\delta^{i_4i_5}_{j_4j_5}\partial_{i_4}\partial^{j_4}\phi^{a_4}\partial_{i_5}\partial^{j_5}\phi^{a_5}\right) \\
&\quad - \frac{\partial}{\partial\dot{\phi}^a}\left(\alpha_{a_1a_2b}\partial_\mu\phi^{a_1}\partial^\mu\phi^{a_2} + 2\alpha_{a_1a_2ba_4}\partial_\mu\phi^{a_1}\partial^\mu\phi^{a_2}\partial_i\partial^i\phi^{a_4} + 3\alpha_{a_1a_2ba_4a_5}\partial_\mu\phi^{a_1}\partial^\mu\phi^{a_2}\delta^{i_4i_5}_{j_4j_5}\partial_{i_4}\partial^{j_4}\phi^{a_4}\partial_{i_5}\partial^{j_5}\phi^{a_5}\right) \\
&= \left(2\alpha_{a_1ba}\dot{\phi}^{a_1} + 42\alpha_{a_1baa_4}\dot{\phi}^{a_1}\partial_i\partial^i\phi^{a_4} + 6\alpha_{a_1baa_4a_5}\dot{\phi}^{a_1}\delta^{i_4i_5}_{j_4j_5}\partial_{i_4}\partial^{j_4}\phi^{a_4}\partial_{i_5}\partial^{j_5}\phi^{a_5}\right) \\
&\quad - \left(2\alpha_{a_1ab}\dot{\phi}^{a_1} + 42\alpha_{a_1aba_4}\dot{\phi}^{a_1}\partial_i\partial^i\phi^{a_4} + 6\alpha_{a_1aba_4a_5}\dot{\phi}^{a_1}\delta^{i_4i_5}_{j_4j_5}\partial_{i_4}\partial^{j_4}\phi^{a_4}\partial_{i_5}\partial^{j_5}\phi^{a_5}\right) \\
&= 0,
\end{aligned}$$

where the symmetry of the α tensors causes the cancellation. Last, we have

$$\begin{aligned}
\mathcal{L}_{\ddot{\phi}^a\partial_i\dot{\phi}^b} &= \frac{\partial}{\partial\partial_i\dot{\phi}^b}\left(\alpha_{a_1a_2a}\partial_\mu\phi^{a_1}\partial^\mu\phi^{a_2} + 2\alpha_{a_1a_2aa_4}\partial_\mu\phi^{a_1}\partial^\mu\phi^{a_2}\partial_i\partial^i\phi^{a_4} + 3\alpha_{a_1a_2aa_4a_5}\partial_\mu\phi^{a_1}\partial^\mu\phi^{a_2}\delta^{i_4i_5}_{j_4j_5}\partial_{i_4}\partial^{j_4}\phi^{a_4}\partial_{i_5}\partial^{j_5}\right) \\
&= 0.
\end{aligned}$$

Again, the anti-symmetric nature of the generalized Dirac-delta matrix avoids mixed derivatives in $\mathcal{L}_{\ddot{\phi}^a}$. Combining these results, we have $S_{[ab]} = 0$ for our multi-galileon Lagrangian (72), which shows that our multi-galileon theories are Ostrogradsky stable.

So far, we have only discussed Galileon theories for their mathematical properties, i.e. as Ostrogradsky-free second order theories. However, Galileon theories also have some interesting physical properties. They are used in scalar-tensor theories of gravity to modify the behaviour of gravity at large distances in order to explain the cosmological constant [4]. We will discuss these modifications after we have discussed General Relativity and its scalar, tensorial and scalar-tensorial realizations in chapters 6 and 7.

5.4 Shift Symmetry

We have previously discussed second order scalar theories that obey Galilean invariance. Of course, these are not the only second order Ostrogradsky free theories one can construct. Rather than the Galilean symmetry, we impose the shift symmetry $\bar{\phi} = \phi + c$, with c an arbitrary constant. This symmetry is less restrictive than the Galilean symmetry, clearly any theory that admits the Galilean symmetry admits the shift-symmetry. This shift symmetry is used in certain 'inflaton' inflation theories [24, 25]. The symmetry that is spontaneously broken in the Goldstone mechanism is also a shift-symmetry [12, 26].

Similar to Galileons, it are the equations of motion that should be invariant under the shift-symmetry. Clearly, any derivative is invariant, while an appearance of ϕ itself would show variance. Thus, the Euler-Lagrange equations should only depend on derivatives of ϕ, not on ϕ itself. Furthermore, we are dealing with Lorentz invariant theories. As a result, the first derivatives can only appear in pairs. We can thus introduce the variable $X = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi$. Clearly, any Lagrangian that is an arbitrary function of X will be shift-symmetric and Ostrogradsky free. There are more general Lagrangians though, in particular the Galilean invariant Lagrangians (66) are shift-symmetric and ghost free. We can generalize these Lagrangians to obtain the

following 5 shift symmetric Lagrangians:

$$\begin{aligned}
\mathcal{L}_1 &= \phi, \\
\mathcal{L}_2 &= f_2(X), \\
\mathcal{L}_3 &= \partial_\mu \partial^\mu \phi f_3(X), \\
\mathcal{L}_4 &= \left[\left(\partial^\mu \partial_\mu \phi\right)^2 - \partial^\mu \partial_\nu \phi \partial^\nu \partial_\mu \phi \right] f_4(X) \\
\mathcal{L}_5 &= \left[\left(\partial^\mu \partial_\mu \phi\right)^3 - 3\partial^\mu \partial_\mu \phi \partial^\nu \partial_\rho \phi \partial^\rho \partial_\nu \phi + 2\partial^\mu \partial_\nu \phi \partial^\nu \partial_\rho \phi \partial^\rho \partial_\mu \phi \right] f_5(X)
\end{aligned} \tag{75}$$

where the functions $f_i(X)$ are arbitrary functions of X. The total Lagrangian is given by $\mathcal{L} = \alpha \mathcal{L}_1 + \sum_{m=2}^{5} \mathcal{L}_m$[18]. All but $\mathcal{L}_1$ are shift-symmetric themselves; and due to their second derivative structure, they are all ghost free. For $\mathcal{L}_1$, we have the Euler-Lagrange equations $E_1 = 1$, which also does not depend on ϕ.

The question is whether these five are all shift-symmetric Lagrangians. We shall investigate if there could be other first-order Lagrangians. For a first-order Lagrangian, we can rewrite the Euler-Lagrange equations (7) as

$$E\left(\mathcal{L}\right) = \frac{\partial \mathcal{L}}{\partial \phi} - \Box\phi \frac{\partial \mathcal{L}}{\partial X} - \partial^\mu \phi \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial X} \right), \tag{76}$$

written in terms of X rather than $\partial_\mu \phi$. On these equations, we impose the constraint $\frac{\partial E(\mathcal{L})}{\partial \phi} = 0$, which translates into the constraint

$$\frac{\partial E\left(\mathcal{L}\right)}{\partial \phi} = \left(1 - X\right) \frac{\partial^2 \mathcal{L}}{\partial \phi^2} - \left(\Box\phi \partial_\mu \phi \partial_\nu \phi \partial^\mu \partial^\nu \phi\right) \frac{\partial^2 \mathcal{L}}{\partial \phi \partial X} \approx 0. \tag{77}$$

If a Lagrangian is linear in ϕ, this reduces to just

$$\frac{\partial \mathcal{L}}{\partial X} = 0, \tag{78}$$

i.e $\mathcal{L}$ is not allowed to depend on X. Only $\mathcal{L}_1$ and constant multiples of it satisfy this constraint. For all other Lagrangians, (77) reduces to a differential equation coupling the dependence on X to the dependence on ϕ:

$$\left(1 - X\right) \frac{\partial \mathcal{L}}{\partial \phi} = \left(\Box\phi + \partial_\mu \phi \partial_\nu \phi \partial^\mu \partial^\nu \phi\right) \frac{\partial \mathcal{L}}{\partial X} + f\left(X\right), \tag{79}$$

where $f\left(X\right)$ can be any arbitrary function of X. Of course, this derivation only holds is we are looking for a first-order Lagrangian. Adding second-order terms to the Lagrangian will of course change the condition (77) and thus the differential equation (79).

Next we generalize our shift-symmetric theory to a multi-scalar, shift-symmetric theory. We generalize the Lagrangians (75), which themselves were derived from the Lagrangians (66) for Galileons. As such, we can derive our shift-symmetric Lagrangians from the multi-Galileon Lagrangians (72) by allowing the constants $\alpha_{a_1 \ldots a_m}$ to be functions that depend on $X^{ab} = \frac{1}{2} \partial_\mu \phi^a \partial^\mu \phi^b$. As a result, we obtain the Lagrangian

$$\mathcal{L} = \alpha_a \phi^a + \alpha\left(X^{ab}\right) + \sum_{m=3}^{5} \alpha_{a_1 \ldots a_m}\left(X^{ab}\right) \delta^{\mu_3 \ldots \mu_m}_{\nu_3 \ldots \nu_m} \partial_{\mu_3} \partial^{\nu_3} \phi^{a_3} \cdots \partial_{\mu_m} \partial^{\nu_m} \phi^{a_m}. \tag{80}$$

The functions $\alpha_{a_1 \ldots a_m}\left(X^{ab}\right)$ are still required to be symmetric in all indices a_i, and α_a remains a constant to satisfy (78).

[18] The constants from the Galilean Lagrangian are now made part of the functions $f_m(X)$.

5.5 General Scalar Theory

In the previous subsections, we have discussed several second-order scalar theories that were all subject to some constraints or symmetries. In this subsection, we shall state the most general, second-order, ghost-free, multi-scalar theory. This theory possesses no constraints or symmetries, other than those required to remove the Ostrogradsky ghost. For a single scalar, this general Lagrangian is given by [23]

$$\mathcal{L} = f(\phi, X) + \sum_{m=3}^{5} f^m(\phi, X)\,\mathcal{L}_m, \tag{81}$$

where the $\mathcal{L}_m$ are the Galileon Lagrangians from (66). It is a generalization of both (66) and (75). For the Galileons, we required that c_i be constants, due to the imposed Galilean symmetry. For shift-symmetric Lagrangians, we eased this requirement since we were only imposed the less restrictive shift symmetry. Thus, the c_m were allowed to be functions of X and became $f_m(X)$. Now that we also drop the shift symmetry, we can allow the f_m to be arbitrary functions of ϕ and X. The desired absence of Ostrogradsky ghosts still forces a certain dependence on the second-order derivatives. This dependence can be ensured by the use of the generalized Dirac-delta.

We can use the same 'easing of constraints' to generalize (80) into a general multi-scalar theory. The resulting Lagrangian becomes [23]

$$\mathcal{L} = \alpha\left(\phi^a, X^{ab}\right) + \sum_{m=3}^{5} \alpha_{a_1\ldots a_m}\left(\phi^a, X^{ab}\right)\delta^{\mu_3\ldots\mu_m}_{\nu_3\ldots\nu_m}\partial_{\mu_3}\partial^{\nu_3}\phi^{a_3}\cdots\partial_{\mu_m}\partial^{\nu_m}\phi^{a_m}. \tag{82}$$

6 General Relativity

In chapters 4 and 5 we discussed field theories, and in particular relativistic field theories. These field theories do not concern themselves with gravity. Since gravity is indeed observed, we would like our theories to also be able to describe gravity. When Einstein formulated his theory of Special Relativity, it was quickly noted that Newton's theory of gravity is not relativistic. Thus, work soon began by a variety of scientists to come up with a relativistic theory of gravity. Today, we know that Einstein eventually found the 'correct' theory, in which the gravitational 'force' is mediated by the metric tensor [5, chapter 1]. However, in the early 1900's, a large number of different theories were proposed, and in those gravity was often mediated not by a tensor, but by a vector or even a scalar [27].

In this chapter, we shall discuss the principles behind general relativity and an attempt made by Nordström and others to generalize Newton's theory of gravity into a scalar theory of general relativity. This attempt was ultimately unsuccessful, and we shall thus also discuss the tensorial theory that Einstein derived instead. We will conclude with a generalization of the tensorial theory, aimed to find the most general pure tensorial Ostrogradsky free theory of general relativity, called Lovelock gravity.

6.1 The principles of GR

The idea of relativity is that irrespective of your particular frame, the laws of physics must remain the same. This was the idea behind Galilean relativity, and also behind Einstein's idea of special relativity. The difference between these two are the set of invariant laws, Einstein also required that the speed of light is invariant. The other question is the definition of *invariance* or *equivalence*. To answer this question, three principles of equivalence have been formulated [5, Chapter 1]. They are

- *Weak Equivalence Principle (WEP)*: All uncharged, feely falling test particles follow the same orbits, for equal initial positions and velocities.
- *Einstein Equivalence Principle (EEP)*: The above is valid, and locally space-time follows the laws of special relativity, i.e. is flat.
- *Strong Equivalence Principle (SEP)*: All uncharged, feely falling test particles, even massive ones, follow the same orbits, for equal initial positions and velocities and locally space-time follows the laws of special relativity, i.e. is flat.

Of course, it is to question if our universe itself obeys these equivalence principles. All three of these principles have been experimentally tested, and found to be accurate to order 10^{-13} or better. As such, valid theories of gravity have to satisfy these principles, at least to the accuracy of experiments [5, Chapter 2].

6.2 Scalar gravity and Nordström's theorem.

In this subsection, we shall investigate the particular scalar theory of general relativity proposed by Gunnar Nordström. It was one of the more conservative scalar theories, which in this context means that it was closest to Newton's theory. Nordström based his work on a dispute between Einstein and Max Abraham regarding the constancy of the speed of light c. The former proposed that c should depend on the gravitational potential, to avoid a problem in the equation for the gravitational force. Abraham used this hypothesis to derive a theory of gravitation that was no longer special relativistic [27]. Let us explain the problem.

In both Abraham and Nordström's theory, the gravitational potential is a scalar Φ that follows the field equation

$$\Box\Phi = -4\pi G\nu, \tag{83}$$

where G is the gravitational constant and ν is the rest mass density [27, page 6]. This is simply a Lorentz covariant formulation of Newton's scalar field equation $\Delta\Phi = -4\pi G\nu$. In these theories, the source of gravity is the mass density distribution. With the force defined as $F_\mu = -\partial_\mu\Phi$, the natural force equation becomes

$$-\partial_\mu\Phi = F_\mu = m\frac{\mathrm{d}U_\mu}{\mathrm{d}\tau}, \tag{84}$$

with U_μ the four-velocity and τ proper time [27, page 6]. Although a simple extension, it fails when the constancy of c is taken into account. We have $c^2 = U_\mu U^\mu$, and the constancy of c leads to the equation

$$0 = \frac{\mathrm{d}c^2}{\mathrm{d}\tau} = 2U_\mu\frac{\mathrm{d}U^\mu}{\mathrm{d}\tau}. \tag{85}$$

Inserting (84) into this equation then leads to the condition

$$0 = mU_\mu\frac{\mathrm{d}U^\mu}{\mathrm{d}\tau} = F^\mu U_\mu = -m\frac{\mathrm{d}\Phi}{\mathrm{d}\tau}, \tag{86}$$

which can only be satisfied in the narrow case that Φ is constant along the world-line of a particle. Abraham tried to solve this problem, by stating that c is not a constant. Nordström however, in his first theory of gravity, instead let the inertial mass m depend on the gravitational potential [27, page 17]. Thus, his new force law is

$$F_\mu = m\frac{\mathrm{d}U_\mu}{\mathrm{d}\tau} + U_\mu\frac{\mathrm{d}m}{\mathrm{d}\tau}. \tag{87}$$

Note the explicit dependence of m on time. It is this extra term that prevents the derivation of condition (86). Instead, the following condition is found

$$m\frac{\mathrm{d}\Phi}{\mathrm{d}\tau} = c^2\frac{\mathrm{d}m}{\mathrm{d}\tau}, \tag{88}$$

which has as solution for m

$$m = m_0\exp\left(\frac{\Phi}{c^2}\right).$$

Inserting this back into (87) yields the equation of motion of a point particle [27, page 18]

$$-\partial_\mu\Phi = \frac{\mathrm{d}U_\mu}{\mathrm{d}\tau} + \frac{U_\mu}{c^2}\frac{\mathrm{d}\Phi}{\mathrm{d}\tau}. \tag{89}$$

This equation of motion has one drawback, which was Einstein's primary reason to dismiss this theory: an object that also moves in a horizontal direction falls slower than an object that only moves in the vertical direction [27, page 21]. Einstein felt that this was a problem for all (Lorentz covariant) scalar theories of gravity.

Later, based on work by Laue on relativistic stretched bodies, Einstein reasoned that the source for gravity could not be the rest mass density ν, but instead should be the trace of the stress-energy tensor $T_{\mu\nu}$ [27, page 40]. He argued in his 1913 paper, co-written with Marcel Grossmann, that scalar-theories of gravity violate the conservation of energy. See pages 40-42 of [27] for a thought experiment that aims to prove this violation.

Nordström amended his theory, in an attempt to get around Einstein's critique. Einstein himself, in a 1913 lecture in Vienna, considered only two possible theories of gravity: his own

Entwurf theory, that would become General Relativity, and Nordström's amended theory [27, page 53].

The end of scalar gravitational theories then came soon after the publication of Einstein's theory of General Relativity in 1915. One of the important result of Einstein's theory is the prediction of the anomalous perihelion of Mercury, which matched experiments of the time. Nordström's theory also predicts a perihelion shift, however his perihelion has the wrong sign. Thus, Nordström's theory was eventually dismissed, even by its most ardent supporters, in favour of Einstein's theory of General Relativity [27, chapter 16].

6.3 Einstein's tensorial theory of General Relativity

In the previous subsection, we already gave hints to Einstein's own theory of general relativity. It is a tensorial theory, where the metric is made dynamical in order to 'bend' space time around massive objects and thus give the effect of gravitational attraction. In this subsection, we shall discuss metric geometry and why Einstein arrived at his formulation. Last, we shall cast Einstein's equation into a Lagrangian theory.

Einstein's GR treats space-time as a 4-dimensional manifold. On this manifold, there is a metric tensor called $g_{\mu\nu}$. This metric is used to measure distances in space time. Let γ be a curve in space time, parametrised by λ, i.e. $x^\mu = x^\mu(\lambda)$. The length of this curve is then given by [5, Chapter 2] [18, Chapter 1]

$$s = \int_\gamma \mathrm{d}\lambda \sqrt{g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}, \tag{90}$$

where $\dot{x}^\mu = \mathrm{d}x^\mu/\mathrm{d}\lambda$. We saw the same thing when discussing Lorentz invariant field theories in chapter 4, where the metric was the static Minkowski metric $g_{\mu\nu} = \eta_{\mu\nu}$. Furthermore, on a general manifold with non-constant metric, differentiation w.r.t. the space-time coordinates is no longer covariant. Instead, a connection $\Gamma^\mu_{\nu\rho}$ is associated with differentiation [5, Chapter 2]. In particular, the derivative of the metric is given by

$$\nabla_\mu g_{\nu\rho} = \frac{\partial g_{\nu\rho}}{\partial x^\mu} - \Gamma^\sigma_{\mu\nu} g_{\sigma\rho} - \Gamma^\sigma_{\mu\rho} g_{\nu\sigma}. \tag{91}$$

In General Relativity, the connection is chosen to be the Levi-Civita connection, which is given by [5, Chapter 2]

$$\begin{Bmatrix} \mu \\ \rho\sigma \end{Bmatrix} = \Gamma^\mu_{\rho\sigma} = \frac{1}{2} g^{\mu\nu} \left(\partial_\rho g_{\nu\sigma} + \partial_\sigma g_{\rho\nu} - \partial_\nu g_{\rho\sigma} \right).$$

As a result of this choice, the covariant derivative of the metric vanishes, $\nabla_\mu g_{\nu\rho} = 0$. This choice of connection tells us that everything on the manifold of space-time is governed by the metric tensor. Such a manifold is called a Riemannian manifold. On a Riemannian manifold, we can calculate the curvature of the manifold, i.e. how much it deviates from the 'straight plane'. Roughly speaking: a flat piece of paper has no curvature, a sphere does. This curvature is given by the so called Riemann tensor, given by [18, Chapter 3]

$$R^\mu{}_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\rho} + \Gamma^\tau_{\nu\sigma}\Gamma^\mu_{\tau\rho} - \Gamma^\tau_{\nu\rho}\Gamma^\mu_{\tau\sigma}. \tag{92}$$

With these preliminaries about the metric tensor discussed, we can discuss Einstein's theory. As mentioned in the previous section, Einstein identified the stress-energy tensor $T_{\mu\nu}$ as the source of gravity, motivated by Laue's work on relativistic stretched bodies [27, page 40]. Since this is a tensor, it makes sense to conclude that gravity itself is mediated by a tensor. Einstein then assumed that it was the metric tensor that mediates gravity, with the stress-energy tensor somehow 'bending' the metric, and thus altering the orbits followed by test particles.

The next question is how to couple the stress-energy tensor to the metric tensor. The naive method is $g_{\mu\nu} = \kappa T_{\mu\nu}$, for some coupling constant κ. However this equation does not reduce to Newton's laws in the non-relativistic case [18, section 3.5]. Since Newton's laws have been tested extensively, and found to be a good description of a non-relativistic massive system, this is a problem.

Newton's law for the gravitational potential is a second order differential equation, thus the next obvious candidate is to somehow couple the second order derivatives of $g_{\mu\nu}$ to $T_{\mu\nu}$. This could be done through the Ricci tensor, $R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu}$. The coupling would then be $R_{\mu\nu} = \kappa T_{\mu\nu}$. Einstein himself proposed this equation in 1915 [18, section 3.5]. However, it is wrong. One of the properties of $T_{\mu\nu}$ is that it is divergence free: $\nabla^{\mu} T_{\mu\nu} = 0$, like the metric tensor. The Ricci tensor is, in general, not divergence free, and thus taking the divergence of both sides of $R_{\mu\nu} = \kappa T_{\mu\nu}$ leads to an inequality. Realizing this, later in 1915 Einstein proposed the more familiar equation [18]

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu}. \tag{93}$$

The left-hand-side of this equation is divergence free. Furthermore, this equation reduces to Newton's laws in the non-relativistic case. This reduction fixes the coupling constant to $\kappa = -\frac{8\pi G}{c^4}$ [18, section 3.5]. Furthermore, (93) satisfies both the WEP and EEP, part due to the lack of divergence and in part since the manifold of space time in General Relativity is a Riemannian manifold [5, Chapter 2]. The left-hand-side of this equation is often dubbed the Einstein tensor and is defined as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R. \tag{94}$$

Einstein's equation (93) describes how the metric $g_{\mu\nu}$ evolves through time and space, in a similar method that the Euler-Lagrange equations (7) are the evolution equations of fields. Thus, one could try to formulate (93) as the Euler-Lagrange equation derived from some Lagrangian. This can be done, and the resulting action is called the Einstein-Hilbert action. It is given by [5, Chapter 2]

$$S = \int \frac{\sqrt{g} R}{16\pi G} \, \mathrm{d}^4 x + \int \mathcal{L}_m \left(g_{\mu\nu}, \psi \right) \mathrm{d}^4 x. \tag{95}$$

This action consists of the integral over two Lagrangians. First is the gravitational Lagrangian $\mathcal{L}_g = \frac{\sqrt{g} R}{16\pi G}$. Performing the principle of stationary action produces the left hand side of (93). It is this gravitational Lagrangian that describes the dynamics of gravity. The second Lagrangian, $\mathcal{L}_m$, contains the so called matter content of the theory. The field ψ is a stand in for all fields that are part of the theory, not just any scalar fields ϕ, but it also includes all other types of fields, such as vector fields A^{μ}. This matter content leads to the stress-energy tensor through the definition [5, Chapter 2]

$$T^{\mu\nu} = \frac{2}{\sqrt{g}} \frac{\partial \mathcal{L}}{\partial g_{\mu\nu}}. \tag{96}$$

Through this definition, the Euler-Lagrange equations for the metric $g_{\mu\nu}$ for the Lagrangian (95) are the Einstein equations (93) with $\kappa = -\frac{8\pi G}{c^4}$.

In considering this procedure, one will note that (95) is a second-order Lagrangian, since $R_{\mu\nu}$ contains second-order derivatives of the metric. Ostrogradsky's theorem is quite general, and does not apply to just scalars. As a result, there should be Ostrogradsky ghost. Fortunately, this is not the case. Einstein-Hilbert gravity is a degenerate theory, and the additional constraints remove the Ostrogradsky ghosts. This is already apparent since Einstein's equation (93) is only second-order, not fourth-order as a non-degenerate second order metric theory should be.

6.4 Lovelock Gravity

Einstein's theory is perhaps the most famous tensorial gravitational theory, but is not the most general tensorial theory. In 1971, David Lovelock [28] aimed to generalize Einstein's theory in order to come up with the most general tensorial theory of gravity yielding second order equations of motion, not just in our 3+1 dimensional space time, but in arbitrary D space-time dimensions.

Lovelock stated that the most general theory of general relativity involves finding the most general tensor $A_{\mu\nu}$ that satisfies the following four properties [28]

a) $A_{\mu\nu}$ is symmetric: $A_{\mu\nu} = A_{\nu\mu}$.

b) $A_{\mu\nu}$ is divergence free: $\nabla^\mu A_{\mu\nu} = 0$.

c) $A_{\mu\nu}$ is a function only of $g_{\mu\nu}$ and its first two derivatives.

d) $A_{\mu\nu}$ is at most linear in the second derivatives of $g_{\mu\nu}$.

The field equation in vacuum is given by $A_{\mu\nu} = 0$, and in general the field equation is of the form $A_{\mu\nu} = \kappa T_{\mu\nu}$. This explains the first two conditions; since $T_{\mu\nu}$ is both symmetric and divergence free, so must $A_{\mu\nu}$. Conditions c) and d) follow from the Ostrogradsky mechanism and the desire to keep the system ghost-free, see chapters 3 and 4.

According to Lovelock, the most general tensor $A_{\mu\nu}$ satisfying at least conditions a) - c) in general D dimensions is given by [28]

$$A_{\mu\nu} = \sum_{p=1}^{\lceil \frac{D}{2}-1 \rceil} a_p \left(\delta^{\rho\alpha_1\ldots\alpha_{2p}}_{\mu\beta_1\ldots\beta_{2p}} g_{\rho\nu} + \delta^{\rho\alpha_1\ldots\alpha_{2p}}_{\nu\beta_1\ldots\beta_{2p}} g_{\rho\mu} \right) \prod_{t=1}^{p} R_{\alpha_{2t-1}\alpha_{2t}}{}^{\beta_{2t-1}\beta_{2t}} + a g_{\mu\nu}, \tag{97}$$

where a and a_p are constants. Choosing the constants then ensures condition d) to be met.

In 3+1 dimensions, this equation reduces to [28]

$$A_{\mu\nu} = a g_{\mu\nu} - b G_{\mu\nu}. \tag{98}$$

Like in general relativity, the field equation $A_{\mu\nu} = \kappa T_{\mu\nu}$ is the Euler-Lagrange equation of a Lagrangian. In our general, D-dimensional case, this Lagrangian is given by [28]

$$\mathcal{L} = \sqrt{g} \sum_{p=1}^{\lceil \frac{D}{2}-1 \rceil} 2a_p \left(\delta^{\alpha_1\ldots\alpha_{2p}}_{\beta_1\ldots\beta_{2p}} \right) \prod_{t=1}^{p} R_{\alpha_{2t-1}\alpha_{2t}}{}^{\beta_{2t-1}\beta_{2t}} + 2a\sqrt{g}. \tag{99}$$

This Lagrangian is similar to the Lagrangian for galileons (72). Both use several copies of the second derivatives ($R^\mu{}_{\nu\rho\sigma}$ and $\partial_\mu\partial_\nu\phi$ respectively) of their fields ($g_{\mu\nu}$ and ϕ), multiplied together through the use of the generalized Dirac-delta function. The generalized Dirac-delta function ensures that there are no higher order derivatives present in the equations of motion. This similarity was noticed by Klein et al., who used this similarity to discuss Galileons as an analogue to general relativity [29].

In our space-time, the Lovelock Lagrangian (99) becomes

$$\mathcal{L} = \sqrt{g} 4 a_p R + 2a\sqrt{g}. \tag{100}$$

This is the Lagrangian of the Einstein-Hilbert action (95) with a cosmological constant a [5, Chapter 2]. From this, we conclude that in our regular space-time, Einstein's General Relativity is the most general, purely tensorial, theory of gravity.

7 Modern gravity: both scalars and tensors are important

In Einstein's theory of General Relativity, the gravitational 'force' is mediated by the rank-2 metric tensor. When quantized, this metric tensor becomes a massless spin-2 particle [5, Chapter 3]. General Relativity, with only the rank-2 tensor field, is generally in good agreement with experiments, however theories with more than just one rank-2 tensor field are not necessarily excluded by experiment [5]. A common extension of General Relativity then is to add an extra scalar-field, which is coupled to the metric tensor in such a manner that General Relativity is not violated at length scales where General Relativity is well-tested, i.e. Solar-system scales. This can be done in various manners, one could employ the Vainshtein mechanism [30], or have the coupling be very weak [5]. This type of theories are called *scalar-tensor theories*. The equations of motion derived from such theories are relatively simple and allows for exact solutions, which is another explanation of their popularity. Often a scalar-tensor theory is found as the dimensionally reduced, effective theory of a higher-dimensional theory. Examples of such higher dimensional theories are Kaluza-Klein, string theory [5] and DGP gravity [21].

In this section, we will first go over the Vainshtein mechanism and how it protects scalar-tensor theories from modifying gravity in the solar system. Then, we shall discuss DGP gravity and how the galileon can be derived from this theory. After having derived Galileons as a scalar-tensor theory, we will discuss Horndeski's general single-scalar-tensor theory, a generalization from the galileon-tensor theory. We will generalize his work to include an arbitrary amount of scalar fields before moving on to even more general scalar-tensor theories. These most general scalar-tensor theories are no longer second order, but still Ostrogradsky-free, and are called Beyond-Horndeski theories.

7.1 Vainshtein Mechanism

The Vainshtein mechanism is a mechanism introduced by Vainshtein in [30] to remove a discontinuity in the theory of a massive graviton, or metric tensor [39]. If furthermore can be used to shield scalar-tensor theories from modifying gravity on solar-system scales, which makes it relevant for our purposes.

In Fierz-Pauli theory, a massive graviton theory, the propagator for a massless graviton is given by [39]

$$\bar{D}^{m=0}_{\mu\nu\rho\sigma} = \frac{1}{k^2}\left(\eta_{\rho\mu}\eta_{\sigma\nu} + \eta_{\rho\nu}\eta_{\sigma\mu} - \eta_{\rho\sigma}\eta_{\mu\nu}\right), \tag{101}$$

while the propagator for a massive metric tensor is given by [39]

$$\bar{D}^{m\neq 0}_{\mu\nu\rho\sigma} = \frac{1}{k^2+m^2}\left(\eta_{\rho\mu}\eta_{\sigma\nu} + \eta_{\rho\nu}\eta_{\sigma\mu} - \frac{2}{3}\eta_{\rho\sigma}\eta_{\mu\nu} - \frac{2}{3}\eta_{\rho\sigma}\frac{k_\mu k_\nu}{m^2}\right). \tag{102}$$

The important difference lies in the third term of these equations. This difference of $\frac{1}{3}\eta_{\rho\sigma}\eta_{\mu\nu}$ is at the root of the so-called vDVZ discontinuity, and is caused by the additional degree of freedom exhibited by the massive graviton. The discontinuity causes different predictions for the same physical process, of which only the massless case is in agreement with predictions [39].

Fierz-Pauli is a linear theory, however there exist non-linear variants of Fierz-Pauli theory. For regimes very far away from a gravitational source (such as a massive object) these non-linear theories can be approximated linearly. In this linear regime, the vDVZ discontinuity appears. Vainshtein pointed out that, at some distance r_V to the source, this linear regime is no longer valid. For distances closer to the source, the theory must be non-linearly realized [30, 39]. In order to solve this non-linear regime, an expansion in e.g. the graviton mass can be used, rather than an expansion in the distance r [39]. In this non-linear regime, GR-predictions are recovered, and in those cases these extended theories are not in contradiction with experiments.

To give a rough explanation, in the non-linear regime there are strong kinetic self-couplings, that hide the extra degrees of freedom generated by a massive graviton [39]. Due to these self-couplings, the effective propagator of the graviton follows (101) and we recover predictions that match solar-scale experiments. Since these self-couplings are strong, their range is short and thus at long distances can be ignored. Thus, at long distances we enter the linear regime, where the graviton has all its degrees of freedom, and propagates as in equation (102). Therefore, for the Vainshtein mechanism to work we need a theory that is non-linear and has strong couplings.

7.2 DGP-gravity

The origin of Galileon theories lies in a pure tensorial theory. The catch is that this tensorial theory is not defined in regular 4-dimensional space-time, but in a 5-dimensional one [21]. This theory is called DGP-gravity, after its inventors Dvali, Gabadadze and Porrati [5, Chapter 5]. They claim that gravity is observed to be weak, because we live on a 3-brane [31][19], embedded in a higher dimensional world [21]. Unlike other higher-dimensional theories [5], the extra dimensions are infinite in size, and not compactified. When we then reduce to the brane, this extra fifth dimension appears as a scalar field.

As an analogue to the Brane mechanism, consider the 2-Brane of the surface of the earth in our three-dimensional world. In principle, the earth is described by three coordinates in space. However, when we reduce to the surface, we can describe coordinates using only latitude and longitude. The third coordinate, height, then acts like a scalar field on the surface, with a value for each lat- and longitude.

Let us now show how to derive the 3-brane from the 5-dimensional 'space time'. Let x^μ denote the coordinates on the brane, and y the extra coordinates. Furthermore, I use capital letters and sup- and subscripts for the 5-dimensional quantities, e.g. X^A. If we do not consider additional (matter) fields, then the DGP-action is given by [21]

$$S = M_5^3 \int \mathrm{d}^5 X \sqrt{G} R_5 + M_4^2 \int \mathrm{d}^4 x \sqrt{g} R_4, \tag{103}$$

with M_i the i-dimensional Planck mass, R_i the i-dimensional Ricci scalar, G the metric on $5-D$-space and g the metric induced on the brane. If we assume that the brane is located at $y = 0$, then the metric g is given by [21]

$$g_{\mu\nu}(x) = G_{\mu\nu}(x, y = 0).$$

With the action thus defined, Dvali et al. [21] found the following equation for the gravitational potential on the brane

$$V(r) = -\frac{1}{8\pi^2 M_4^2} \frac{1}{r} \left[\sin\left(\frac{r}{r_c}\right) \mathrm{Ci}\left(\frac{r}{r_c}\right) + \frac{1}{2} \cos\left(\frac{r}{r_c}\right) \left\{ \pi - 2\mathrm{Si}\left(\frac{r}{r_c}\right) \right\} \right], \tag{104}$$

where $Ci(z) = \gamma + \ln(z) + \int_0^z \frac{(\cos t - 1)}{t} \mathrm{d}t$, with $\gamma = 0.577$ the Euler-Masceroni constant, and $\mathrm{Si}(z) = \int_0^z \frac{\sin t}{t} \mathrm{d}t$. The constant r_c is defined as $r_c = \frac{M_4^2}{2M_5^3}$ and is the critical distance below which the theory behaves like ordinary 4-dimensional gravity and above which the theory behaves like a 5-dimensional theory of gravity [5, Chapter 5]. We can see this by taking the limit $r << r_c$, then (104) becomes [21]

$$V(r) = -\frac{1}{8\pi^2 M_4^2} \frac{1}{r} \left[\frac{1}{2}\pi + \left\{ -1 + \gamma + \ln\left(\frac{r}{r_c}\right) \right\} \left(\frac{r}{r_c}\right) + O(r^2) \right].$$

[19] A 3-brane is basically a manifold with 3 spatial dimensions.

This equation behaves like ordinary 4-dimensional Newtonian gravity, except for the logarithmic term, which is repulsive.

On the other hand, for large distances, $r >> r_c$, (104) becomes [21]

$$V(r) = -\frac{1}{8\pi^2 M_4^2}\frac{1}{r}\left[\frac{r_c}{r} + O(\frac{1}{r^2})\right].$$

This equation behaves like $\frac{1}{r^2}$, which is expected for a 5-dimensional, Newtonian theory.

The DGP gravity can be described in a strictly four-dimensional theory as mediated by the tensorial metric field and an additional scalar [5, 21, 32]. Furthermore, the DGP-model has a self-accelerating solution [5, 33]. Such a self-accelerating solution could be used to describe the accelerated expansion of the universe we observe, instead of through a cosmological constant [33]. Unfortunately, the self-accelerating solution has a ghost-instability and is superluminal at long distances [5, Chapter 5].

7.3 Galileons as scalar-tensor theory

Above, we derived DGP gravity. Nicolis et al. used the fact that DGP can be described as a 4D theory with a scalar ϕ to derive Galileons [4, 32]. As a result, the galileon theory is inherently a scalar-tensor theory. In chapter 5.1 we discussed galileons as a pure scalar theory, but now we have the tools to discuss how Galileons were derived as a scalar-tensor theory.

Nicolis et al. studied the class of scalar-tensor theories with the property that cosmological short distances, the 4-dimensional DGP theory is recovered [4, 5]. These scalar-tensor theories are derived on a flat, Minkowski background. As a result, the coupling to the metric is not covariant, i.e. ∂_μ is used instead of the covariant ∇_μ. Furthermore, we only focus on the 4-dimensional part of DGP. Taking this into account, consider the general Hilbert-Einstein gravitational action [5, Chapter 2] (95)

$$S = \int \mathrm{d}^4x \frac{1}{2} M_4^2 \sqrt{g} R_4.$$

To ensure that the modifications are long-distance modification to GR, the coupling of ϕ to the metric should be of a kinetic type. At quadratic order, the action then becomes [4]

$$S = \int \mathrm{d}^4 \frac{1}{2} M_4^2 \sqrt{g}\,(1 - 2\phi)\, R_4 + \mathcal{L}_\phi, \tag{105}$$

where $\mathcal{L}_\phi$ describes the dynamics of the galileon. We will see that $\mathcal{L}_\phi$ will be a combination of the Galileon Lagrangians in (66). Rather then using the metric itself as a dynamical variable, we will use the deviation from the flat metric as our graviton field. This deviation is given by $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$ [5, Chapter 4]. To de-mix the galileons from the metric, the Weyl Rescaling is used [4]. This rescaling is given by $h_{\mu\nu} = \hat{h}_{\mu\nu} + 2\phi\eta_{\mu\nu}$. Doing both these computations, the action 105 becomes [4]

$$S = \int \mathrm{d}^4x \left[\frac{1}{2} M_4^2 \sqrt{\hat{g}} \hat{R} + \frac{1}{2}\hat{h}_{\mu\nu} T^{\mu\nu} + \mathcal{L}_\phi + \phi T^\mu{}_\mu\right], \tag{106}$$

where $\hat{g}_{\mu\nu}$ is defined as $\hat{g}_{\mu\nu} = \eta_{\mu\nu} + \hat{h}_{\mu\nu}$, and $\hat{R}$ is the Ricci scalar derived from $\hat{g}_{\mu\nu}$.

To derive the form of $\mathcal{L}_\phi$, some conditions are to be met.

1. On cosmological scales the strength of the galileon exchange must be comparable to the gravitational strength. However, on short scales we do not want deviations from Einstein's General Relativity. This is impossible in a purely linear theory for ϕ, as per the Vainshtein mechanism. Thus, we need that the dynamics of ϕ are non-linear.

2. Galilean invariance, i.e. both $\mathcal{L}_\phi$ and the equations of motion should be invariant under transformations $\phi \to \phi + b_\mu x^\mu + c$. Since $h_{\mu\nu}$ only appears in second-order derviative in (105), $\hat{h}_{\mu\nu}$ admits the Galilean symmetry, and in general there is no condition that breaks this symmetry. Therefore, we must also allow a general Galileon shift in ϕ. [4].

3. The equations of motion are only second-order, to avoid Ostrogradsky ghosts.

It are the second and third conditions that fix $\mathcal{L}_\phi$ to be of the form (66).

With the Lagrangian given, Nicolis et al. investigated the existence of a self-accelerating solution to these equations of motion. A self-accelerating solution could explain the observed (accelerated) expansion of the universe [5, Chapter 5]. In galileon theory, such a solution can indeed exist and be stable, for a suitable combination of the Lagrangians (66), but unfortunately the speed of propagation for small fluctuations in the self-accelerating solution is faster than the speed of light [4]. However, these galileons are not coupled to the dynamical metric, and thus this is not the entire story.

In order to properly describe galileons in a gravity context, we need to couple galileon theory to a dynamical, gravitational metric. Deffayet et al. described the process of making the Lagrangians for the galileon covariant in [34]. The simple replacement $\partial_\mu \to \nabla_\mu$ leads to higher-order terms in the equations of motion. This can be solved by adding counter-terms, but regardless, Galileon invariance is lost. After adding these counter-terms, the five Lagrangians are given by [34]

$$
\begin{aligned}
\mathcal{L}_1 &= \sqrt{g}\phi, \\
\mathcal{L}_2 &= -\sqrt{g}\frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi, \\
\mathcal{L}_3 &= -\sqrt{g}\frac{1}{2}\nabla_\mu\nabla^\mu\phi\nabla_\nu\phi\nabla^\nu\phi, \\
\mathcal{L}_4 &= -\sqrt{g}\nabla_\lambda\phi\nabla^\lambda\phi\left[2\left(\Box\phi\right)^2 - 2\left(\nabla_\mu\nabla_\nu\phi\nabla^\mu\nabla^\nu\phi\right) - \frac{1}{2}\left(\nabla_\mu\phi\nabla^\mu\phi\right)R\right], \\
\mathcal{L}_5 &= -\sqrt{g}\frac{5}{2}\nabla_\lambda\phi\nabla^\lambda\phi\left[\left(\Box\phi\right)^3 - 3\Box\phi\nabla_\mu\nabla_\nu\phi\nabla^\mu\nabla^\nu\phi + 2\nabla_\mu\nabla_\nu\phi\nabla^\nu\nabla_\rho\phi\nabla^\rho\nabla^\mu\phi\right] \\
&\quad + \sqrt{g}\frac{5}{2}\nabla_\lambda\phi\nabla^\lambda\phi\left[6\nabla_\mu\phi\nabla^\mu\nabla^\nu\phi\left(R_{\nu\rho} - \frac{1}{2}g_{\nu\rho}R\right)\nabla^\rho\phi\right].
\end{aligned}
$$

This theory was generalized to the case of general n galileons by Padilla and Sivanesan [23]. They started from the non-coupled multi-galileon theory, such as derived in [22] and also described in chapter 5. Their result is:

$$
\mathcal{L} = \sqrt{g}\sum_{m=1}^{5}\alpha_{a_1\ldots a_m}\phi^{a_1}\delta^{\mu_2\ldots\mu_m}_{\nu_2\ldots\nu_m}\nabla_{\mu_2}\nabla^{\nu_2}\phi^{a_2}\cdots\nabla_{\mu_m}\nabla^{\nu_m}\phi^{a_m} + \sqrt{g}\sum_{m=3}^{5}\sum_{n=1}^{\frac{m-1}{2}}C^n_m, \tag{107}
$$

where the counter-terms are

$$
\begin{aligned}
C^m_n =& \frac{1}{(-4)^n}\frac{(m-1)!}{(m-2n-1)!\,(n!)^2}\alpha_{a_1\ldots a_m}\phi^{a_1}X^{a_2a_3}\cdots X^{a_{2n}a_{2n+1}} \\
&\cdot\delta^{\mu_{2n+2}\ldots\mu_m b_1c_1\ldots b_nc_n}_{\nu_{2n+2}\ldots\nu_m b_1c_1\ldots d_ne_n}\nabla_{\mu_{2n+2}}\nabla^{\nu_{2n+2}}\phi^{a_{2n+2}}\cdots\nabla_{\mu_m}\nabla^{\nu_m}\phi^{a_m}R^{d_1e_1}{}_{b_1c_1}\cdots R^{d_ne_n}{}_{b_nc_n},
\end{aligned} \tag{108}
$$

and $X^{ab} = \frac{1}{2}\nabla_\mu\phi^a\nabla^\mu\phi^b$ is the first-derivative variable, as in chapter 5.

As discussed above, a single galileon theory can not be self-accelerating, stable and have subluminal propagation fluctuations all at the same time. A multi-galileon theory however might have all of these properties. In particular, Padilla, Saffin and Zhou constructed a bi-galileon

theory from this general Lagrangian [35,36]. Instead of coupling both galileon fields to matter, only one of the two fields, called π, is directly coupled. The other field, called ξ, is only coupled through its coupling with π [5, Chapter 4]. The action is then given by [35]

$$S = \int \mathrm{d}^4x \left[\frac{1}{2} M_4^2 \sqrt{\hat{g}} \hat{R} + \frac{1}{2} \hat{h}_{\mu\nu} T^{\mu\nu} + \mathcal{L}_{\pi\xi} + \pi T^\mu_\mu \right], \tag{109}$$

with

$$\mathcal{L}_{\pi\xi} = \sum_{0 \leqslant m+n \leqslant 4} (\alpha_{m,n}\pi + \beta_{m,n}\xi)\, \mathcal{E}_{m,n} \left(\partial_\mu\partial_\nu\pi, \partial_\rho\partial_\sigma\xi \right),$$

where

$$\mathcal{E}_{m,n} \left(\partial_\mu\partial_\nu\pi, \partial_\rho\partial_\sigma\xi \right) = (m+n)! \delta^{\mu_1 \ldots \mu_m \rho_1 \ldots \rho_n}_{\nu_1 \ldots \nu_m \sigma_1 \ldots \sigma_n} \partial_{\mu_1}\partial^{\nu_1}\pi \ldots \partial_{\mu_m}\partial^{\nu_m}\pi \partial_{\rho_1}\partial^{\sigma_1}\xi \ldots \partial_{\rho_n}\partial^{\sigma_n}\xi.$$

The functions $\mathcal{E}_{m,n}$ are related to the equations of motion, which are [5,35]

$$\begin{aligned} 0 &= \sum_{0 \leqslant m+n \leqslant 4} a_{m,n} \mathcal{E}_{m,n} + T^\mu_\mu \\ 0 &= \sum_{0 \leqslant m+n \leqslant 4} b_{m,n} \mathcal{E}_{m,n}, \end{aligned}$$

with the constants given by

$$\begin{aligned} a_{m,n} &= (m+1) \left(\alpha_{m,n} + \beta_{m+1,n_1} \right) \\ b_{m,n} &= (n+1) \left(\beta_{m,n} + \alpha_{m-1,n+1} \right), \end{aligned}$$

with $\alpha_{-1,n} = \beta_{m,-1} = 0$.

In such a bi-galileon theory, there can exist a stable, consistent, self-accelerating solution, given the right parameters [36]. With the proper parameters, we can have a theory that satisfies [5, chapter 4]

(i) there is no $\mathcal{L} = \phi$, or tadpole term,

(ii) there is a self-accelerating vacuum,

(iii) there are no ghosts in the self-accelerating solution,

(iv) the spherical symmetric excitations of the vacuum are screened by the Vainshtein mechanism in at least the solar system, i.e. solar system gravity is unchanged,

(v) the fluctuations are never faster than the speed of light,

(vi) the fluctuations do not lead to excessive Cherenkov radiation,

(vii) the strong coupling that allows for the scalar-tensor description does not happen at low momentum scales and

(viii) the back-reaction of the galileons to the metric does alter the vacuum solution, i.e. the Minkowski metric remains a solution.

Thus a bi-galileon theory can be a suitable alternative to a cosmological constant dark energy theory [5].

Other possible candidates are the Fab-Four theories from Charmousis et al. [37,38], which might have interesting self-tuning properties [35,36].

7.4 Horndeski's general scalar-tensor theory

In 1973, Gregory Horndeski found the general second-order Lagrangian for a scalar-tensor theory that is only second-order in the equations of motion, and thus ghost-free [40]. This result is a generalization of the covariant galileon scalar-tensor theory discussed in the previous section, though without any (Galilean) symmetry imposed. Horndeski came with his theory without any consideration of a galileon symmetry, since galileons were not a thing back then.

Instead, Horndeski was interested in the most general, covariant scalar-tensor theory that had up-to second order equations of motion in both the scalar and the tensor field. His derivation relied upon the observation that the Euler-Lagrange equations for the scalar field and the tensor field must be related. If $E_{\mu\nu}\left(\mathcal{L}\right)$ describes the Euler-Lagrange equations for the metric $g_{\mu\nu}$, and $E\left(\mathcal{L}\right)$ those for the scalar ϕ, then this relation is given by [40]

$$\nabla_\nu E^{\mu\nu}\left(\mathcal{L}\right) = \frac{1}{2}\nabla^\mu E\left(\mathcal{L}\right). \tag{110}$$

Horndeski found that the general Lagrangian was given by a linear combination of Lagrangians [5, 40].

$$\mathcal{L}_1 = \delta^{\alpha\beta\gamma}_{\mu\nu\sigma}\left(K_1\nabla^\mu\nabla_\alpha\phi R_{\beta\gamma}{}^{\nu\sigma} - \frac{4}{3}K_{1,X}\nabla^\mu\nabla_\alpha\phi\nabla^\nu\nabla_\beta\phi\nabla^\sigma\nabla_\gamma\phi\right) \tag{111}$$

$$\mathcal{L}_3 = \delta^{\alpha\beta\gamma}_{\mu\nu\sigma}\left(K_3\nabla_\alpha\phi\nabla^\mu\phi R_{\beta\gamma}{}^{\nu\sigma} - 4K_{3,X}\nabla_\alpha\phi\nabla^\mu\phi\nabla^\nu\nabla_\beta\phi\nabla^\sigma\nabla_\gamma\phi\right) \tag{112}$$

$$\mathcal{L}_8 = 2\delta^{\alpha\beta}_{\mu\nu}K_8\nabla_\alpha\phi\nabla^\mu\phi\nabla^\nu\nabla_\beta\phi - 3XK_8\Box\phi \tag{113}$$

$$\mathcal{L}_9 = K_9 \tag{114}$$

$$\mathcal{L}_F = \delta^{\alpha\beta}_{\mu\nu}\left(FR_{\alpha\beta}{}^{\mu\nu} - 4F_{,X}\nabla^\mu\nabla_\alpha\phi\nabla^\nu\nabla_\beta\phi\right) - 6F_{,\phi}\Box\phi, \tag{115}$$

where the K_i are arbitrary functions of ϕ and X, as is F. However, F has the constraint $F_{,X} = K_{1,\phi} - K_3 - 2XK_{3,X}$. The strange labelling of the K_i is due to the generality of Horndeski's theory, he derived his theory for general D dimensions and found up to 10 arbitrary functions [40]. When reduced down to 4 dimensions, only the functions K_1, K_3, K_8 and K_9 remain non-zero. It is easy to see that if one chooses $F = \frac{1}{4}M_{Pl}^2$ and all other functions zero, the resulting Lagrangian becomes

$$\mathcal{L} = \Sigma_{i=1}^6\mathcal{L}_i = \mathcal{L}_F = M_{Pl}R_4, \tag{116}$$

which is the standard Einstein-Hilbert action (95) [5], without any scalar component.

Horndeski's work can be generalized into a general multi-scalar-tensor theory. This is done by making the Lagrangian for a general multi-scalar-tensor theory, (82), covariant and then add the necessary counter-terms to remove any higher-order derivatives [23]. Padilla and Sivanesan found that this general Lagrangian is given by

$$\begin{aligned}\mathcal{L} =& \sqrt{g}A\left(X^{ab},\phi^d\right) + A_c\left(X^{ab},\phi^d\right)\Box\phi^c - 2\frac{\partial B_2\left(X^{ab},\phi^d\right)}{\partial X^{c_1c_2}}\left(\Box\phi^{c_1}\Box\phi^{c_2} - \nabla_\mu\nabla_\nu\phi^{c_1}\nabla^\mu\nabla^\nu\phi^{c_2}\right) \\ &- \frac{2}{3}\frac{\partial B_{3,c_3}}{\partial X^{c_1c_2}}\epsilon^{\mu_1\mu_2\mu_3}\epsilon^{\nu_1\nu_2\nu_3}\nabla_{\mu_1}\nabla_{\nu_1}\phi^{c_1}\nabla_{\mu_2}\nabla_{\nu_2}\phi^{c_2}\nabla_{\mu_3}\nabla_{\nu_3}\phi^{c_3} + B_2R + B_{3,c}\Box\phi^cR,\end{aligned} \tag{117}$$

where the functions A, B and their derivatives w.r.t. X^{ab} are all symmetric in all of their indices [23].

7.5 Beyond Horndeski

Horndeski's theory is the most general scalar-tensor theory that leads to second order equations of motion. However, it is not the most general, Ostrogradsky-free, scalar-tensor theory. These more general theories have higher-order equations of motion, but these higher-order terms appear in such a way that they do not propagate any additional degrees of freedom. This class of most general scalar-theories is often simply called beyond-Horndeski theories [41]. For starters, when we made the galileon scalar-tensor covariant above, we added counter terms to ensure second order equations of motion. According to Chagoya and Tasinato [19] these counter-terms are not necessary to produce a ghost-free theory. The naive covariantization $\partial_\mu \to \nabla_\mu$ is enough to produce a ghost-free theory. However, the Galilean symmetry is still lost, since $\Gamma^\mu_{\nu\rho}$ is not Galilean invariant.

Earlier, in 2014, Gleyzes et al. [41] came up with a more general ghost-free scalar-tensor theory. Their Lagrangian is given by a linear combination of the following Lagrangians

$$
\begin{aligned}
\mathcal{L}_2 &= G_2(\phi, X), \\
\mathcal{L}_3 &= G_3(\phi, X)\,\Box\phi, \\
\mathcal{L}_4 &= G_4(\phi, X)\,R - 2G_{4,X}(\phi, X)\,\delta^{\mu\nu}_{\alpha\beta}\nabla^\alpha\nabla^\beta\phi\nabla_\mu\nabla_\nu\phi + F_4(\phi, X)\,\delta^{\mu\nu\rho}_{\alpha\beta\gamma}\nabla^\alpha\phi\nabla_\mu\phi\nabla^\beta\nabla_\nu\phi\nabla^\gamma\nabla_\rho\phi, \\
\mathcal{L}_5 &= G_5(\phi, X)\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\nabla^\mu\nabla^\nu\phi + \frac{1}{3}G_{5,X}(\phi, X)\,\delta^{\mu\nu\rho}_{\alpha\beta\gamma}\nabla^\alpha\nabla_\mu\phi\nabla^\beta\nabla_\nu\phi\nabla^\gamma\nabla_\rho\phi \\
&\quad + F_5(\phi, X)\,\delta^{\mu\nu\rho\sigma}_{\alpha\beta\gamma\epsilon}\nabla_\mu\phi\nabla^\alpha\phi\nabla_\nu\nabla^\beta\phi\nabla_\rho\nabla^\gamma\phi\nabla_\sigma\nabla^\epsilon\phi.
\end{aligned}
$$

These reduce to Horndeski theory for $F_4 = F_5 = 0$ and for appropriate choices of the functions G_i. In their paper, Gleyzes et al. proof this lack of ghosts in two methods, one using an Arnowitt-Deser-Misner formulation and a Hamiltonian analysis of the degrees of freedom, the other uses a covariant formulation but does not include $\mathcal{L}_5$ due to the extra complexity this term generates.

It is however not at all certain that these equations provide the most general scalar-tensor theory that is free from the Ostrogradsky ghost. The conditions found in [2, 3, 8] have not yet been generalised to a (multi) scalar-tensor system, which makes finding the most general system difficult.

8 When to call two Lagrangians equivalent

8.1 Why equivalence?

As stated before, the aim of physics is to find theories that describe the motions and interactions of particles. If there are two theories that give the same equations of motions for the same particles, are these theories then actually different? Theories with the same equations of motions, will give the same predictions for experiments. There is thus no way to distinguish between two such theories on an experimental level, and thus they are equivalent.

It is easy to create two Lagrangians with the same equations of motion. Take an arbitrary Lagrangian with p independent and q dependent coordinates $\mathcal{L}\left(x^i, u^\alpha, u_I^\alpha\right)$, with the highest derivatives appearing at order r, and add to it the (total) divergence of some p-tuple F: $\bar{\mathcal{L}} = \mathcal{L} + \nabla F$[20]. If one computes the Euler-Lagrange equations (7) for both Lagrangians, one shall see that they are the same [2, 6].

Example 10. For an example, again look at the harmonic oscillator. Its Lagrangian is given by [9]

$$\mathcal{L} = \frac{m}{2}\dot{u}^2 - \frac{k}{2}u^2.$$

We have computed its Euler Lagrange equation (7) before, it is

$$m\ddot{u} = -ku.$$

Let us take for our 1-tuple the function $f = \dot{u}^3 - 25u$, which has the total derivative

$$\nabla f = 3\dot{u}^2\ddot{u} - 25\dot{u},$$

and our modified Lagrangian $\bar{\mathcal{L}} = \mathcal{L} + \nabla F$ looks like

$$\bar{\mathcal{L}} = 3\dot{u}^2\ddot{u} + \frac{m}{2}\dot{u}^2 - 25\dot{u} - \frac{k}{2}u^2.$$

Since this Lagrangian is of second order, the Euler-Lagrange equation (7) is computed differently, it now also includes a second-order term. The Euler-Lagrange equation is

$$\begin{aligned}
0 &= \frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{\partial\bar{\mathcal{L}}}{\partial\ddot{u}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\bar{\mathcal{L}}}{\partial\dot{u}} + \frac{\partial\bar{\mathcal{L}}}{\partial u} \\
&= \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(3\dot{u}^2\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(6\dot{u}\ddot{u} + m\dot{u} - 25\right) - ku \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(6\dot{u}\ddot{u}\right) - \left(6\ddot{u}^2 + 6\dot{u}\dddot{u} + m\ddot{u}\right) - ku \\
&= \left(6\ddot{u}^2 + 6\dot{u}\dddot{u}\right) - \left(6\ddot{u}^2 + 6\dot{u}\dddot{u}\right) - m\ddot{u} - ku \\
&= -m\ddot{u} - ku.
\end{aligned}$$

This Euler-Lagrange equation equation is identical to the pervious one, as expected.

However, Lagrangians with different equations of motions can still be called equivalent. Roughly speaking, if there exists an invertible transformation Φ such that $\bar{\mathcal{L}}\left(\Phi(x^i), \Phi(u^\alpha), \Phi(u_I^\alpha)\right) = \mathcal{L}\left(x^i, u^\alpha, u_I^\alpha\right)$, then we can call $\bar{\mathcal{L}}$ and $\mathcal{L}$ equivalent. An example of this would be to rewrite a Lagrangian in a different coordinate system, for example to move between polar and Cartesian coordinates.

[20] See also appendix A for more about the notation used.

Example 11. For example, consider the system of two masses orbiting each other in free, 2-D space. In cartesian coordinates, the Lagrangian describing this system is given by [9, chapter 5]

$$\mathcal{L} = \frac{1}{2}m_1\left(\dot{x}_1^2 + \dot{y}_1^2\right) + \frac{1}{2}m_2\left(\dot{x}_2^2 + \dot{y}_2^2\right) + \frac{Gm_1m_2}{\sqrt{\left(x_1 - x_2\right)^2 + \left(y_1 - y_2\right)^2}}. \tag{118}$$

We can rewrite this system in polar coordinates around the Centre of Mass. In this system, the Lagrangian becomes [9, chapter 7]

$$\bar{\mathcal{L}} = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\phi}\right) + \frac{Gm_1m_2}{r},$$

where $\mu = \frac{m_1m_2}{m_1+m_2}$ is the reduced mass. These two Lagrangians are different, and lead to different (looking) equations of motion. Yet, they describe the same physical system, and thus we call them equivalent.

Example 10 is interesting, because the modified Lagrangian $\bar{\mathcal{L}}$ is clearly second order. It is also Ostrogradsky-free, which can be seen both in the computed Euler-Lagrange equation and also by computing the primary (30) and secondary conditions (31). This is an interesting observation. Clearly any first-order Lagrangian can be turned into a second-order Lagrangian by adding a suitable divergence. This second-order Lagrangian thus created is clearly Ostrogradsky-free, i.e. healthy, since its equations of motion are only up-to second order. The interesting question is whether this can also be done in reverse, i.e., given any healthy second-order Lagrangian $\mathcal{L}$, does there exist a p-tuple F such that $\bar{\mathcal{L}} = \mathcal{L} + \nabla F$ is a first-order Lagrangian? Or do we also need to perform a coordinate transformation? In general then, we ask: are all healthy second-order Lagrangians equivalent to a first-order Lagrangian?

Fortunately, we are not the first to ask this question on equivalence. Cartan established an algorithm to determine when/if this is the case, which Olver and Kamran [6, 17, 42] later expanded and they looked at the results of Cartan's algorithm applied to certain cases. We want to apply their theory to the case of healthy, second-order Lagrangians. Before we can apply their work, we first need a more rigorous definition of equivalence and the allowed coordinate transformations.

8.2 Jet bundles and contact forms

When discussing equivalence, we use coordinate transformations. These transformations are diffeomorphisms that map the space of coordinates to itself: $\Phi : J \to J$, where J refers to the space of coordinates. In order to better understand these allowed transformations, we first need to describe this space J. This space is called a jet-space and is a manifold that contains the independent and dependent coordinates, as well as the derivatives of the latter. My construction/explanation of jet spaces is based on that of Ivey and Landsberg in §1.9 of [7], but similar constructions can be found by Olver in chapter 4 of [6] or by Kiselev in the first chapter of [11].

Definition 3 (Equivalence of k-jets)**.** Let $f, g : \mathbb{R} \to \mathbb{R}$ be two smooth functions. They are said to have the same k-jet at a point x_0 if and only if

$$f\left(x_0\right) = g\left(x_0\right), \ \frac{\mathrm{d}f}{\mathrm{d}x}\left(x_0\right) = \frac{\mathrm{d}g}{\mathrm{d}x}\left(x_0\right), \ \frac{\mathrm{d}^2f}{\mathrm{d}x^2}\left(x_0\right) = \frac{\mathrm{d}^2g}{\mathrm{d}x^2}\left(x_0\right), \ldots, \frac{\mathrm{d}^kf}{\mathrm{d}x^k}\left(x_0\right) = \frac{\mathrm{d}^kg}{\mathrm{d}x^k}\left(x_0\right). \tag{119}$$

Further more, let $f, g : M \to N$ be smooth functions to and from manifolds of arbitrary dimensions p and q. These functions have the same k-jet at a point $x_0 \in M$ if and only if $f\left(x_0\right) = g\left(x_0\right)$ and if for any functions $u : \mathbb{R} \to M$, $v : N \to \mathbb{R}$ the compositions $v \circ f \circ u$ and $v \circ g \circ u$ have the same k-jet at $t = u^{-1}\left(x_0\right)$ [7].

The condition of equivalence of k-jets on functions between general manifolds can also be stated locally. Let x^i be local coordinates on M and u^α local coordinates on N. Then two functions f, g have the same k-jet at x_0 if and only if

$$\frac{\partial^n f^\alpha}{\partial x^I}(x_0) = \frac{\partial^n g^\alpha}{\partial x^I}(x_0),$$

for $0 \leqslant n \leqslant k$ and I a multi-index of length n [7].

This relation of equivalence between k-jet allows us to define the equivalence class $j^k_{x_0}(f)$ as the class of all functions $M \to N$ that have the same k-jet as f at the point x_0, or, in symbolic language[21]

$$j^k_{x_0}(f) = \left\{ g \in C^\infty(M,N) \;\middle|\; \frac{\partial^n f^\alpha}{\partial x^I}(x_0) = \frac{\partial^n g^\alpha}{\partial x^I}(x_0) \, \forall n \leqslant k, |I| = n \right\}.$$

With the equivalence relations and classes thus defined, we can look at the spaces on which the k-jets live. These spaces are essentially the space of all functions between M and N modulo the equivalence between k-jets. They are called the jet spaces.

Definition 4 (Jet Spaces)**.** Let $x_0 \in M$ and $u_0 \in N$ be two points. Then $J^k_{x_0,u_0}(M,N)$ denotes the space of all k-jets that map x_0 to u_0, and $J^k(M,N)$ is the space of all k-jets of all maps from M to N. It is called the k^{th}-order jet space between M and N.

The local coordinates on $J^k(M,N)$ are given by $x^i, u^\alpha, p^\alpha_i, \ldots, p^\alpha_I$. A point $j^k_{x_0}(x_0) \in J^k(M,N)$ then has coordinates $x^i_0, u^\alpha = f^\alpha(x_0), p^\alpha_I = \frac{\partial^{|I|} f^\alpha}{\partial x^I}(x_0)$ for $1 \leqslant |I| \leqslant k$ [7].

For any $l < k$, the jet space $J^l(M,N)$ is a sub-manifold of the larger space $J^k(M,N)$ and there exists the natural projection $\pi^k_l : J^k(M,N) \to J^l(M,N)$, formed by dropping the derivatives of order greater than l [6]. There exists a similar method to increase the order of a jet space. One simply appends the derivative coordinates with the higher-order derivative coordinates p^α_I necessary [11]. Of course, the thus defined map $(\pi^{-1})^k_l$ is not well-defined, since any point in $J^k(M,N)$ maps to many points in $J^{k+1}(M,N)$.

The spaces $J^k(M,N)$ each form a bundle over both M and $M \times N$, and any functions $f : M \to N$ induces a section $p \to j^k_{x_0}(f)$ of the bundle $J^k(M,N)$ over M, called the *lift* of f [7]. The lift of a function, either from $M \to N$ or from $M \to J^l(M,N)$ to some higher-order jet space $J^k(M,N)$ $(l < k)$ is also called the *prolongation* of a function [6].

Unfortunately, the allowed functions are not simply all diffeomorphisms that map a jet-space $J^k(M,N)$ onto itself. The differential coordinates p^α_I do not 'know' that they are supposed to be derivatives of coordinates u^α. As a result, a general diffeomorphism may mess up this relation. To explain, take a general function $f : M \to N$ and it's lift $j^k(f)$ on the bundle $J^k(M,N)$. The derivative coordinates of this lift are described as $p^\alpha_I = \frac{\partial^{|I|} f^\alpha}{\partial x^I}(x_0)$. Let $\Phi : J^k(M,N) \to J^k(M,N)$ be such a general diffeomorphism, and barred coordinates refer to the coordinates after this transformation: $\bar{x}^i = \Phi(x^i, u^\alpha, p^\alpha_I)$, etc. After this transformation, we may have $\bar{p}^\alpha_I \neq \frac{\partial^{|I|} \bar{f}^\alpha}{\partial \bar{x}^I}(\bar{x}_0)$, and thus $\bar{j}^k(f)$ may no longer be a lift of $\bar{f}$. This not desired, and thus we must reduce the allowed transformations. Note that it is possible that there exists a different function f' for which this corodinate transformation leaves the lift intact, i.e. $\bar{p}^\alpha_I = \frac{\partial^{|I|} \bar{f}'^\alpha}{\partial \bar{x}^I}(\bar{x}_0)$. Thus, the reduction in allowed diffeomorphisms depends on the functions that we want to be preserved.

In our case, when talking about Lagrangian equivalence, the previous discussion defined that equivalence has to do with the equations of motion that follow from the Lagrangian. The

[21] It is not strictly needed that g lives in $C^\infty(M,N)$, as long as g is smooth enough, such that the equivalence relations are well defined, i.e. $g \in C^k(M,N)$ is sufficient.

functions whose lifts need to be preserved by a transformation are therefore the solutions to these equations of motion. More general, the to be preserved functions are solutions to a system of partial differential equations. Let us therefore define differential equations on these jet spaces. Take

$$F^l\left(x^i, u^\alpha, \frac{\partial^{|I|}u^\alpha}{\partial x^I}\right) = 0, \tag{120}$$

to be a k^{th}-order system of p partial differential equations for maps $f : M \to N$. As mentioned, the coordinates on J^k do not know of differentials and do not know that they are 'supposed' to be derivatives of coordinates u^α that solve these equations. Ergo, the equations $F^l(x^i, u^\alpha, p^\alpha_I) = 0$ do not define sections in $J^k(M,N)$ (which would be lifts of solutions to the PDE), but instead they define a sub-manifold $\Sigma \subseteq J^k(M,N)$ by $\Sigma = \left\{(x^i, u^\alpha, p^\alpha_I) \in J^k(M,N) \,|F^l(x^i, u^\alpha, p^\alpha_I) = 0\right\}$.

To obtain suitable sections on this manifold, define on the jet space manifold the p-form[22] $\Omega = \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^p$ and the *differential contact ideal* $\mathcal{I}$[23]. These one-forms are called the *contact forms*, which is generated differentially by the 1-forms

$$\theta^\alpha = \mathrm{d}u^\alpha - p^\alpha_i \,\mathrm{d}x^i,$$
$$\theta^\alpha_I = \mathrm{d}p^\alpha_I - p^\alpha_{I,j}\,\mathrm{d}x^j,$$

with $|I| < k$. This p-form Ω and contact ideal $\mathcal{I}$ together define a so-called *exterior differential system with independence condition* (EDS) on the sub-manifold Σ defined by (120) [7, definition 1.9.1, page 27]. Solutions to this EDS are given according to definition 1.9.2 of [7]:

Definition 5. An *integral manifold* or *solution* of the system $(\mathcal{I}, \Omega)$ is an immersed p-fold $f : M \to \Sigma$ such that $f^*(\omega) = 0\ \forall \omega \in \mathcal{I}$ and $f^*(\Omega) \neq 0$ at each point of M.

These integral manifolds f are then exactly the lifts of proper solutions to the PDE (120). To see how this works in practise, consider the following example, taken in part from [7].

Example 12. Let $M = \mathbb{R}^2$, with coordinates (x, y), and $N = \mathbb{R}$, with coordinate u. Furthermore, we work on the first-order jet space $J^1 = J^1(\mathbb{R}^2, \mathbb{R})$. On this jet-space, define the following quite general PDE

$$\begin{aligned} u_x &= A(x,y,u), \\ u_y &= B(x,y,u). \end{aligned} \tag{121}$$

On J^1 differential coordinates don't know that they are supposed to be differentials, thus these equations define the 3-dimensional manifold $j : \Sigma \hookrightarrow J^1$. This is the manifold defined by the equations.

$$\begin{aligned} p_x &= A(x,y,u), \\ p_y &= B(x,y,u). \end{aligned}$$

The contact ideal on J^1 is generated by the one-form

$$\theta = \mathrm{d}u - p_x\,\mathrm{d}x - p_y\,\mathrm{d}y,$$

and the independence condition is given by $\Omega = \mathrm{d}x \wedge \mathrm{d}y \neq 0$.

[22] $p = \dim M$, see also appendix A

[23] For a definition of differential ideals of one-forms, I refer to §B.4 of [7]

Let $i : S \hookrightarrow \Sigma$ be a surface such that $i^*\theta = 0$ and $i^*\Omega \neq 0$. On this surface x and y are the only independent coordinates, thus the coordinates on this surface can be expressed as functions of the coordinates x and y by the Implicit Function Theorem:

$$\begin{aligned} u &= u(x, y) \\ p_x &= p_x(x, y) \\ p_y &= p_y(x, y). \end{aligned}$$

Also, since $i^*\theta = 0$, we have

$$i^* \, \mathrm{d}u = p_x \, \mathrm{d}x + p_y \, \mathrm{d}y.$$

On the other hand, since u is a function of x and y on S, we have

$$\mathrm{d}u = \mathrm{d}u \, (x, y) = u_x \, \mathrm{d}x + u_y \, \mathrm{d}y.$$

Since x and y are independent on S by the independence condition $i^*\Omega \neq 0$, the above equations then necessarily imply

$$\begin{aligned} p_x &= u_x, \\ p_y &= u_y, \end{aligned}$$

i.e. on S, the function u is indeed a solution of (121).

So the contact forms/ideal are the important tools, that tell the coordinates on a jet space $J^k \, (M, N)$ associated with derivatives (i.e. the p^α_I) to actually act like they are the derivatives of the functions u^α. Of course, this only works if the independence condition is satisfied ($i^*\Omega \neq 0$), since this ensures that the coordinates u^α actually may be expressed as functions of the coordinates x^i.

For the equivalence problem for Lagrangians, the allowed transformations are precisely those diffeomorphism that preserve the contact ideal for this Lagrangian.

8.3 Prolongated and allowed transformations

Continuing down the rabbit-hole of allowed diffeomorphism, to find the allowed diffeomorphisms we need to define their order r, and for that we need to know what the *prolongation* of a diffeomorphism is.

Definition 6 (Prolongations)**.** Let Φ be a diffeomorphsism on $J^r \, (\mathbb{R}^p, \mathbb{R}^q)$: $\Phi : \; J^r \, (\mathbb{R}^p, \mathbb{R}^q) \to J^r \, (\mathbb{R}^p, \mathbb{R}^q)$. This diffeomorphism can locally be described as $\bar{x}^i = \Phi^i \left(x^j, u^{(r)}\right)$, $\bar{u}^\alpha = \Phi^\alpha \left(x^j, u^{(r)}\right)$, $\bar{p}^\alpha_I = \Phi^\alpha_I \left(x^j, u^{(r)}\right)$, $|I| \leqslant r$. The prolongation Φ' of this diffeomorphism to a higher order jet-bundle $J^s \, (\mathbb{R}^p, \mathbb{R}^q)$, $s > r$, is locally given by $\bar{u}^\alpha = \Phi^\alpha \left(x^j, u^{(r)}\right)$, $\bar{p}^\alpha_I = \Phi^\alpha_I \left(x^j, u^{(r)}\right)$ for $|I| \leqslant r$, and $\bar{p}^\alpha_{I,J} = \frac{\partial \bar{p}^\alpha_I}{\partial \bar{x}^J}$ for $|I| = r$, $|J| \geqslant 1$. I.e., the additional higher derivative coordinates are made to transform according to the chain rule. Please note that $\bar{p}^\alpha_I$, $|I| \leqslant r$ is not made to abide any chain rule.

With the definition of a prolongation given, we can define an order for every diffeomorphism of the jet-bundle J^r. This can be used to classify diffeomorphisms, based on the lowest order jet-bundle on which they can live.

Definition 7 (Order of a diffeomorphism)**.** We define that a diffeomorphism has order r if it is the prolongation of a diffeomorphism on J^r, and can not be described as the prolongation of a diffeomorphism on any lower order jet bundle J^k, $k < r$. Thus, all diffeomorphisms on the jet bundle J^r are either prolongations of a lower order diffeomorphism, or are themselves of order r.

We want to find diffeomorphisms Φ that preserve the contact ideal, i.e. that satisfy $\Phi^*(\mathcal{I}) = \mathcal{I}$. According to Bäcklund's Theorem, only zero-th and first order diffeomorphisms can posses this property [6, Chapter 4]. Kamran and Olver categorised such functions into three classes [17].

1. *Fiber-preserving transformations* are those diffeomorphism where the new independent variables $\bar{x}^i$ only depend on the old independent variables x^i [17]. In this case, Φ has the form

$$\bar{x}^i = \phi^i\left(x^j\right), \qquad \bar{u}^\alpha = \psi^\alpha\left(x^i, u^\beta\right).$$

   The formulas for the derivatives p^α_I follow from the chain rule. This is a more restricted class of the zero-th order diffeomorphisms and does not mix dependent and independent variables.

2. General *point transformations*. Now the new independent coordinates are allowed to depend on the old dependent coordinates [17]. This is a generalisation of the fiber-preserving transformation. The diffeomorphisms have the form

$$\bar{x}^i = \phi^i\left(x^j, u^\alpha\right), \qquad \bar{u}^\alpha = \psi^\alpha\left(x^i, u^\beta\right).$$

   Again, the derivatives p^α_I follow from applying the chain rule.

3. *Contact transformations* are those transformations, where ϕ and ψ are allowed to depend on the first-derivatives of u [17]. Furthermore, they must preserve the contact ideal[24]. Thus, the diffeomorphisms have the form

$$\bar{x}^i = \phi^i\left(x^j, u^{(1)}\right), \quad \bar{u}^\alpha = \psi^\alpha\left(x^i, u^{(1)}\right), \quad \bar{p}^\alpha_i = \psi^\alpha_i\left(x^i, u^{(1)}\right)$$

   with the restriction

$$\Phi^*(\mathcal{I}) = \mathcal{I},$$

   where $\mathcal{I}$ is the contact ideal generated by the contact forms on J^r.

It is evident that the last category preserves the contact ideal. Next, I shall show that the first and second categories also preserve the contact ideal. It is given that the derivative coordinates transform according to the chain rule. Thus, these transformations are given by

$$\begin{aligned}
\bar{p}^\alpha_i &= \frac{\partial \bar{u}^\alpha}{\partial \bar{x}^i} = \frac{\partial x^j}{\partial \bar{x}^i}\frac{\partial \psi^\alpha}{\partial x^j} + \frac{\partial u^\beta}{\partial \bar{x}^i}\frac{\partial \psi^\alpha}{\partial u^\beta}, \\
\bar{p}^\alpha_{I,j} &= \frac{\partial \bar{p}^\alpha_I}{\partial \bar{x}^j} = \frac{\partial x^k}{\partial \bar{x}^j}\frac{\partial \bar{p}^\alpha_I}{\partial x^k} + \frac{\partial u^\beta}{\partial \bar{x}^j}\frac{\partial \bar{p}^\alpha_I}{\partial u^\beta} + \sum_{|J|\leqslant|I|}\frac{\partial p^\beta_J}{\partial \bar{x}^j}\frac{\partial \bar{p}^\alpha_I}{\partial p^\beta_J}.
\end{aligned}$$

To show that this ensures the preservation of the contact ideal, we prove the following theorem.

Theorem 5. *The fiber-preserving and point transformations preserve the contact ideal:*

$$\Phi^*(\mathcal{I}) = \mathcal{I}.$$

[24]Since I enforce this preservation of the contact ideal, I can consider the $p^\alpha_i = u^\alpha_i$ as actual derivatives, and not just coordinates on J^r.

Proof. Since the contact ideal is algebraically generated by the contact forms, we only have to prove that the contact forms are mapped to (linear combinations of) contact forms. To remind ourselves, we have

$$\begin{aligned} \Phi^*\left(\theta^\alpha\right) &= \bar{\theta}^\alpha = \mathrm{d}\bar{u}^\alpha - \bar{p}_i^\alpha \,\mathrm{d}\bar{x}^i, \\ \Phi^*\left(\theta_I^\alpha\right) &= \bar{\theta}_I^\alpha = \mathrm{d}\bar{p}_I^\alpha - \bar{p}_{I,j}^\alpha \,\mathrm{d}\bar{x}^j. \end{aligned} \tag{122}$$

In the *fiber preserving* case, we have

$$\begin{aligned} \mathrm{d}\bar{x}^i &= \frac{\partial \phi^i}{\partial x^j}\,\mathrm{d}x^j, \\ \mathrm{d}\bar{u}^\alpha &= \frac{\partial \psi^\alpha}{\partial x^i}\,\mathrm{d}x^i + \frac{\partial \psi^\alpha}{\partial u^\beta}\,\mathrm{d}u^\beta, \\ \bar{p}_i^\alpha &= \frac{\partial x^j}{\partial \bar{x}^i}\frac{\partial \psi^\alpha}{\partial x^j} + \frac{\partial u^\beta}{\partial \bar{x}^i}\frac{\partial \psi^\alpha}{\partial u^\beta}, \end{aligned} \tag{123}$$

and

$$\begin{aligned} \mathrm{d}\bar{p}_I^\alpha &= \frac{\partial \bar{p}_I^\alpha}{\partial x^j}\,\mathrm{d}x^j + \frac{\partial \bar{p}_I^\alpha}{\partial u^\beta}\,\mathrm{d}u^\beta + \frac{\partial \bar{p}_I^\alpha}{\partial p_J^\beta}\,\mathrm{d}p_J^\beta, \\ \bar{p}_{I,j}^\alpha &= \frac{\partial x^k}{\partial \bar{x}^j}\frac{\partial \bar{p}_I^\alpha}{\partial x^k} + \frac{\partial u^\beta}{\partial \bar{x}^j}\frac{\partial \bar{p}_I^\alpha}{\partial u^\beta} + \frac{\partial p_J^\beta}{\partial \bar{x}^j}\frac{\partial \bar{p}_I^\alpha}{\partial p_J^\beta}, \end{aligned} \tag{124}$$

where $0 < |J| \leqslant |I|$. Substituting (123) into (122) results in

$$\begin{aligned} \bar{\theta}^\alpha &= \frac{\partial \psi^\alpha}{\partial x^i}\,\mathrm{d}x^i + \frac{\partial \psi^\alpha}{\partial u^\beta}\,\mathrm{d}u^\beta - \left(\frac{\partial x^k}{\partial \bar{x}^i}\frac{\partial \psi^\alpha}{\partial x^k} + \frac{\partial u^\beta}{\partial \bar{x}^i}\frac{\partial \psi^\alpha}{\partial u^\beta}\right)\frac{\partial \phi^i}{\partial x^j}\,\mathrm{d}x^j \\ &= \frac{\partial \psi^\alpha}{\partial x^i}\,\mathrm{d}x^i - \delta_j^k \frac{\partial \psi^\alpha}{\partial x^k}\,\mathrm{d}x^j + \frac{\partial \psi^\alpha}{\partial u^\beta}\left(\mathrm{d}u^\beta - \frac{\partial u^\beta}{\partial \bar{x}^i}\frac{\partial \phi^i}{\partial x^j}\,\mathrm{d}x^j\right) \\ &= \frac{\partial \psi^\alpha}{\partial u^\beta}\theta^\beta. \end{aligned}$$

Substituting (124) into (122) results in

$$\begin{aligned} \bar{\theta}_I^\alpha &= \frac{\partial \bar{p}_I^\alpha}{\partial x^j}\,\mathrm{d}x^j + \frac{\partial \bar{p}_I^\alpha}{\partial u^\beta}\,\mathrm{d}u^\beta + \frac{\partial \bar{p}_I^\alpha}{\partial p_J^\beta}\,\mathrm{d}p_J^\beta - \left(\frac{\partial x^k}{\partial \bar{x}^j}\frac{\partial \bar{p}_I^\alpha}{\partial x^k} + \frac{\partial u^\beta}{\partial \bar{x}^j}\frac{\partial \bar{p}_I^\alpha}{\partial u^\beta} + \frac{\partial p_J^\beta}{\partial \bar{x}^j}\frac{\partial \bar{p}_I^\alpha}{\partial p_J^\beta}\right)\frac{\partial \phi^j}{\partial x^i}\,\mathrm{d}x^i \\ &= \frac{\partial \bar{p}_I^\alpha}{\partial x^j}\,\mathrm{d}x^j - \delta_i^k \frac{\partial \bar{p}_I^\alpha}{\partial x^k}\,\mathrm{d}x^k + \frac{\partial \bar{p}_I^\alpha}{\partial u^\beta}\left(\mathrm{d}u^\beta - \frac{\partial u^\beta}{\partial \bar{x}^j}\frac{\partial \phi^j}{\partial x^i}\,\mathrm{d}x^i\right) + \frac{\partial \bar{p}_I^\alpha}{\partial u_J^\beta}\left(\mathrm{d}p_J^\beta - \frac{\partial p_J^\beta}{\partial \bar{x}^j}\frac{\partial \phi^j}{\partial x^i}\,\mathrm{d}x^i\right) \\ &= \frac{\partial \bar{p}_I^\alpha}{\partial u^\beta}\left(\mathrm{d}u^\beta - p_i^\beta\,\mathrm{d}x^i\right) + \frac{\partial \bar{p}_I^\alpha}{\partial p_J^\beta}\left(\mathrm{d}p_J^\beta - p_{J,i}^\beta\,\mathrm{d}x^i\right) \\ &= \frac{\partial \bar{p}_I^\alpha}{\partial u^\beta}\theta^\beta + \sum_{|J|\leqslant|I|}\frac{\partial \bar{p}_I^\alpha}{\partial p_J^\beta}\theta_J^\beta. \end{aligned}$$

The above calculations show that fiber-preserving transformations map contact forms to (linear combinations of) contact forms, and thus they preserve the contact ideal.

For *point transformations*, only the first equation of (123) changes. It becomes

$$\mathrm{d}\bar{x}^i = \frac{\partial \phi^i}{\partial x^j}\,\mathrm{d}x^j + \frac{\partial \phi^i}{\partial u^\beta}\,\mathrm{d}u^\beta.$$

Now, substitute the above equation and the other equations of (123) into (122) to obtain

$$\begin{aligned}\bar{\theta}^\alpha &= \frac{\partial\psi^\alpha}{\partial x^i}\,\mathrm{d}x^i + \frac{\partial\psi^\alpha}{\partial u^\beta}\,\mathrm{d}u^\beta - \left(\frac{\partial x^k}{\partial\bar{x}^j}\frac{\partial\psi^\alpha}{\partial x^k} + \frac{\partial u^\beta}{\partial\bar{x}^j}\frac{\partial\psi^\alpha}{\partial u^\beta}\right)\left(\frac{\partial\phi^j}{\partial x^i}\,\mathrm{d}x^i + \frac{\partial\phi^j}{\partial u^\gamma}\,\mathrm{d}u^\gamma\right)\\ &= \left(\frac{\partial\psi^\alpha}{\partial x^i} - \frac{\partial\psi^\alpha}{\partial x^k}\frac{\partial x^k}{\partial\bar{x}^j}\frac{\partial\phi^j}{\partial x^i} - \frac{\partial\psi^\alpha}{\partial u^\beta}\frac{\partial u^\beta}{\partial\bar{x}^j}\frac{\partial\phi^j}{\partial x^i}\right)\mathrm{d}x^i + \left(\frac{\partial\psi^\alpha}{\partial u^\beta} - \frac{\partial\psi^\alpha}{\partial x^k}\frac{\partial x^k}{\partial\bar{x}^j}\frac{\partial\phi^j}{\partial u^\beta} - \frac{\partial\psi^\alpha}{\partial u^\gamma}\frac{\partial u^\gamma}{\partial\bar{x}^j}\frac{\partial\phi^j}{\partial u^\beta}\right)\mathrm{d}u^\beta\end{aligned}$$

To continue, we need to introduce some identifiers for the Kronecker-delta function.

$$\begin{aligned}\delta^i_j &= \frac{\partial x^i}{\partial x^j} = \frac{\partial x^i}{\partial\bar{x}^k}\frac{\partial\phi^k}{\partial x^j} + \frac{\partial x^i}{\partial\bar{u}^\alpha}\frac{\partial\psi^\alpha}{\partial x^j}\\ \delta^\alpha_\beta &= \frac{\partial u^\alpha}{\partial u^\beta} = \frac{\partial u^\alpha}{\partial\bar{x}^i}\frac{\partial\phi^i}{\partial u^\beta} + \frac{\partial u^\alpha}{\partial\bar{u}^\gamma}\frac{\partial\psi^\gamma}{\partial u^\beta}\end{aligned}$$

We can substitute these functions into the $\frac{\partial x^k}{\partial\bar{x}^j}\frac{\partial\phi^j}{\partial x^i}$ and $\frac{\partial u^\gamma}{\partial\bar{x}^j}\frac{\partial\phi^j}{\partial u^\beta}$ terms.

$$\begin{aligned}\bar{\theta}^\alpha &= \left(\frac{\partial\psi^\alpha}{\partial x^i} - \frac{\partial\psi^\alpha}{\partial x^k}\delta^k_i + \frac{\partial\psi^\alpha}{\partial x^k}\frac{\partial x^k}{\partial\bar{u}^\beta}\frac{\partial\psi^\beta}{\partial x^i} - \frac{\partial\psi^\alpha}{\partial u^\beta}\frac{\partial u^\beta}{\partial\bar{x}^j}\frac{\partial\phi^j}{\partial x^i}\right)\mathrm{d}x^i\\ &\quad + \left(\frac{\partial\psi^\alpha}{\partial u^\beta} - \frac{\partial\psi^\alpha}{\partial x^k}\frac{\partial x^k}{\partial\bar{x}^j}\frac{\partial\phi^j}{\partial u^\beta} - \frac{\partial\psi^\alpha}{\partial u^\gamma}\delta^\gamma_\beta + \frac{\partial\psi^\alpha}{\partial u^\gamma}\frac{\partial u^\gamma}{\partial\bar{u}^\delta}\frac{\partial\psi^\delta}{\partial u^\beta}\right)\mathrm{d}u^\beta\\ &= \left(\frac{\partial\psi^\alpha}{\partial x^k}\frac{\partial x^k}{\partial\bar{u}^\beta}\frac{\partial\psi^\beta}{\partial x^i} - \frac{\partial\psi^\alpha}{\partial u^\beta}\frac{\partial u^\beta}{\partial\bar{x}^j}\frac{\partial\phi^j}{\partial x^i}\right)\mathrm{d}x^i\\ &\quad + \left(\frac{\partial\psi^\alpha}{\partial u^\gamma}\frac{\partial u^\gamma}{\partial\bar{u}^\delta}\frac{\partial\psi^\delta}{\partial u^\beta} - \frac{\partial\psi^\alpha}{\partial x^k}\frac{\partial x^k}{\partial\bar{x}^j}\frac{\partial\phi^j}{\partial u^\beta}\right)\mathrm{d}u^\beta\end{aligned}$$

Again, use the Kronecker-delta substitutions, this time on $\frac{\partial\psi^\alpha}{\partial x^k}\frac{\partial x^k}{\partial\bar{u}^\beta}$ and $\frac{\partial\psi^\alpha}{\partial u^\gamma}\frac{\partial u^\gamma}{\partial\bar{u}^\delta}$. This results in

$$\begin{aligned}\bar{\theta}^\alpha &= \left(\delta^\alpha_\beta\frac{\partial\psi^\beta}{\partial x^i} - \frac{\partial\psi^\alpha}{\partial u^\beta}\frac{\partial u^\beta}{\partial u^\gamma}\frac{\partial\psi^\gamma}{\partial x^i} - \frac{\partial\psi^\alpha}{\partial u^\beta}\frac{\partial u^\beta}{\partial\bar{x}^j}\frac{\partial\phi^j}{\partial x^i}\right)\mathrm{d}x^i\\ &\quad + \left(\delta^\alpha_\delta\frac{\partial\psi^\delta}{\partial u^\beta} - \frac{\partial\psi^\alpha}{\partial x^k}\frac{\partial x^k}{\partial\bar{u}^\delta}\frac{\partial\psi^\delta}{\partial u^\beta} - \frac{\partial\psi^\alpha}{\partial x^k}\frac{\partial x^k}{\partial\bar{x}^j}\frac{\partial\phi^j}{\partial u^\beta}\right)\mathrm{d}u^\beta\\ &= \frac{\partial\psi^\alpha}{\partial x^i}\,\mathrm{d}x^i - \frac{\partial\psi^\alpha}{\partial u^\beta}\frac{\partial u^\beta}{\partial x^i}\,\mathrm{d}x^i + \frac{\partial\psi^\alpha}{\partial u^\beta}\,\mathrm{d}u^\beta - \frac{\partial\psi^\alpha}{\partial x^k}\frac{\partial x^k}{\partial u^\beta}\,\mathrm{d}u^\beta\\ &= \frac{\partial\psi^\alpha}{\partial u^\beta}\left(\mathrm{d}u^\beta - p^\beta_i\,\mathrm{d}x^i\right) - \frac{\partial\psi^\alpha}{\partial x^k}\left(\frac{\partial x^k}{\partial u^\beta}\,\mathrm{d}u^\beta - \delta^k_i\,\mathrm{d}x^i\right)\\ &= \frac{\partial\psi^\alpha}{\partial u^\beta}\theta^\beta - \frac{\partial\psi^\alpha}{\partial x^k}\frac{\partial x^k}{\partial u^\beta}\theta^\beta\\ &= \frac{\mathrm{d}\psi^\alpha}{\mathrm{d}u^\beta}\theta^\beta.\end{aligned}$$

A similar computation can be done on $\bar{\theta}^\alpha_I$ (the second equation of (122)), the result is

$$\bar{\theta}^\alpha_I = \frac{\mathrm{d}\psi^\alpha_I}{\mathrm{d}u^\beta}\theta^\beta + \frac{\mathrm{d}\psi^\alpha_I}{\mathrm{d}p^\beta_J}\theta^\beta_J,$$

where $0 < |J| \leqslant |I|$. Thus, I have shown that for both the fiber-preserving transformations and for the point transformation, the contact ideal is preserved.

□

Example 13. To give an example of a non-trivial contact transformation, take a look at the Legendre transformation on $J^1(\mathbb{R},\mathbb{R})$ [6, chapter 4].

$$\bar{x} = p, \quad \bar{u} = u - xp \quad \bar{p} = -x. \tag{125}$$

To cast this in a more familiar setting, let $u = \mathcal{L}$, $\bar{u} = \mathcal{H}$, $x = \dot{q}$ and $p = \frac{\partial \mathcal{L}}{\partial \dot{q}}$. Then (125) becomes

$$\mathcal{H} = \mathcal{L} - \dot{q}\frac{\partial \mathcal{L}}{\partial \dot{q}}.$$

Clearly, the transformation rule for $\bar{p}$ is not simply the chain rule, a factor of $\frac{\partial u}{\partial p}$ is missing. The Legendre transformation is thus not a prolonged point transformation, however it does preserve the contact form. To prove this, first calculate $\mathrm{d}\bar{u}$ and $\mathrm{d}\bar{x}$:

$$\begin{aligned}\mathrm{d}\bar{u} &= \mathrm{d}\left(u - xp\right) = \mathrm{d}u - x\,\mathrm{d}p - p\,\mathrm{d}x,\\ \mathrm{d}\bar{x} &= \mathrm{d}p.\end{aligned}$$

Next, substitute these into the zero$^{\text{th}}$-order contact form:

$$\begin{aligned}\bar{\theta} &= \mathrm{d}\bar{u} - \bar{p}\,\mathrm{d}\bar{x}\\ &= (\mathrm{d}u - x\,\mathrm{d}p - p\,\mathrm{d}x) - (-x)\,\mathrm{d}p\\ &= \mathrm{d}u - p\,\mathrm{d}x\\ &= \theta.\end{aligned}$$

Thus, we have found a contact-transformation on $J^1(\mathbb{R},\mathbb{R})$

So far, we have defined jet-bundles, contact forms and ideals, and diffeomorphisms on jet-bundles. From this, we have found the types of diffeomorphism that preserve the contact ideals, and called those the transformations under which two Lagrangians can be equivalent to each other. Next, we need to define 'equivalence'.

8.4 The types of equivalence

Let $\mathcal{L}$ and $\bar{\mathcal{L}}$ be two general Lagrangians defined on a jet-bundle $J^r(\mathbb{R}^p,\mathbb{R}^q)$. Kamran and Olver describe two types of equivalence between these two general Lagrangians [17, 42].

1. First, we can require that the functionals defined by $\mathcal{L}$ and $\bar{\mathcal{L}}$ agree on all possible functions $u^\alpha = f^\alpha(x^i)$. This is called the *standard equivalence problem.* Thus, these two Lagrangians are equivalent if they are related by

$$\bar{\mathcal{L}}\left(\bar{x}^i, \bar{u}^{(r)}\right) = \frac{\mathcal{L}\left(x^i, u^{(r)}\right)}{|\det J|}, \tag{126}$$

   where $(J)^i_j = \frac{\mathrm{d}\phi^i}{\mathrm{d}x^j}$, the Jacobian of the transformation of the independent variables.

2. Second, we can require that the variational problems defined by $\mathcal{L}$ and $\bar{\mathcal{L}}$ agree only on extremals. That is to say, they produce the same Euler-Lagrange equations. This is the case, if and only if the Lagrangians differ by a divergence. In formula form, this can be stated as

$$\bar{\mathcal{L}}\left(\bar{x}^i, \bar{u}^{(r)}\right) = \frac{\mathcal{L}\left(x^i, u^{(r)}\right) + \nabla F}{|\det J|}, \tag{127}$$

   with again $(J)^i_j = \frac{\mathrm{d}\phi^i}{\mathrm{d}x^j}$ and $F\left(x^i, u^{(r)}\right)$ an arbitrary p-tuple of functions $F : J^r \to \mathbb{R}^p$. This type is called the *divergence equivalence problem.* It is this type of equivalence that the Lagrangians from example 10 exhibit.

It is easily seen that any two Lagrangians that are standard equivalent, are also divergence equivalent, simply use the function $F = 0$. The divergence equivalence problem is thus less restrictive and still meets our goal: preservation of the equations of motion, as mentioned in the introduction to this section. Therefore, I shall focus on this type of equivalence in the rest of this thesis. However, the Cartan Equivalence Algorithm introduced in section 10 can also be applied to the standard equivalence problem.

9 Equivalence of Galileons

In this section we use the results of Crisostomi et al. [8] and the previous sections to see if Galileon theories are equivalent to first order theories. We are still interested in the widest class of equivalence, divergence equivalent under contact transformations, but we shall see that that there is still equivalence in the absence of coordinate transformations.

We first recap the Ostrogradsky-free conditions, and then go into the classes of Ostrogradsky-stable Lagrangians found by Crisostomi et al. [8] and how these classes are divergence equivalent to first-order theories. We then apply these results to ever more general Galileon theories, from just one Galileon without additional fields, till arbitrary Galileons with arbitrary additional helper fields.

9.1 Equivalence conditions

As a recap the conditions found by Crisostomi et al. for Ostrogradsky stability for Lorentz Invariant theories are [8] (56)(61)

$$\begin{aligned} 0 &= P_{(ab)} = v_a^A \mathcal{L}_{\dot\psi^A \dot\psi^B} v_b^B, \\ 0 &= S_{[ab]} = 2v_a^A \mathcal{L}_{\dot\psi^{[A} \psi^{B]}} v_b^B + 2v_{[a}^A \mathcal{L}_{\dot\psi^A \partial_i \dot\psi^B} \partial_i v_{b]}^B - \delta_i \left(v_a^A \mathcal{L}_{\dot\psi^{[A} \partial_i \psi^{B]}} v_b^B \right). \end{aligned}$$

They also found certain conditions under which certain Ostrogradsky free theories are divergence equivalent to first order theories. We shall explain the procedure. Let us divide the stable Lagrangians into three classes, based on their null-vector $V_a^\alpha = -\mathcal{L}_{\ddot\phi^a \dot q^\beta} \mathcal{L}^{-1}_{\dot q^\beta \dot q^\alpha}$.

$$\text{Class I} : V_a^\alpha = 0 \tag{128}$$

$$\text{Class II} : V_a^\alpha = V_a^\alpha \left(\phi^b, \partial_\mu \phi^b, q^\beta \right) \tag{129}$$

$$\text{Class III} : V_a^\alpha = V_a^\alpha \left(\phi^b, \partial_\mu \phi^b, q^\beta, \partial_\mu \partial_\nu \phi^b, \partial_\mu q^\beta \right) \tag{130}$$

Class I constraints

Both mechanical and Lorentz invariant Class I Lagrangians are divergence equivalent to first-order theories. In fact, no coordinate transformation is required. We shall proof this statement in two theories, one for classical mechanics and one for field theories. For both proofs, we shall follow Crisostomi et al. [8].

Theorem 6. *Let $\mathcal{L}$ be a mechanical, second-order, Class I Lagrangian. $\mathcal{L}$ is divergence equivalent to a first order Lagrangian $\bar{\mathcal{L}}$ under the relation*

$$\mathcal{L} = \bar{\mathcal{L}} + \frac{\mathrm{d}F}{\mathrm{d}t}, \tag{131}$$

where $F = F\left(t, \phi^a, \dot\phi^a\right)$.

Proof. For a second-order, Ostrogradsky free Lagrangian, $\mathcal{L}\left(\ddot\phi_a, \dot\phi_a, \phi_a, \dot q_\alpha, \dot q_\alpha\right)$, the Class I condition (128) reduces the Ostrogradsky constraints (37) and (38) to

$$\begin{aligned} 0 &= P_{(ab)} = \mathcal{L}_{\ddot\phi^a \ddot\phi^b}, \\ 0 &= S_{[ab]} = \mathcal{L}_{\ddot\phi^{[a} \dot\phi^{b]}}. \end{aligned}$$

The primary condition implies linearity in the $\ddot\phi^a$, and thus the Lagrangian can be written as

$$\mathcal{L} = \ddot\phi^a f_a \left(\dot\phi^b, \phi^b, q^\alpha \right) + g \left(\dot\phi^b, \phi^b, \dot q^\alpha, q^\alpha \right). \tag{132}$$

The secondary condition then implies $\partial f_a/\partial\dot{\phi}^b = \partial f_b/\partial\dot{\phi}^a$. To remove the second order terms then, we need to find a function F such that $F_{\dot{\phi}^a} = f_a$. We can define this function as an integral on $J^2(\mathbb{R},\mathbb{R}^q)$:

$$F = \int f_a \,\mathrm{d}\dot{\phi}^a.$$

This function matches the criteria, since

$$\begin{aligned}\frac{\partial F}{\partial\dot{\phi}^a} &= \int \frac{\partial f_b}{\partial\dot{\phi}^a}\,\mathrm{d}\dot{\phi}^b \\ &= \int \frac{\partial f_a}{\partial\dot{\phi}^b}\,\mathrm{d}\dot{\phi}^b \\ &= f_a,\end{aligned}$$

where we have used the secondary condition $\partial f_a/\partial\dot{\phi}^b = \partial f_b/\partial\dot{\phi}^a$. As a result, the mechanical Class I Lagrangians can all be written as

$$\begin{aligned}\mathcal{L} &= \bar{\mathcal{L}} + \frac{\mathrm{d}F}{\mathrm{d}t} \\ &= \bar{\mathcal{L}} + \frac{\partial F}{\partial t} + \frac{\partial F}{\partial\phi^a}\dot{\phi}^a + \frac{\partial F}{\partial\dot{\phi}^a}\ddot{\phi}^a \\ &= \bar{\mathcal{L}} + \frac{\partial F}{\partial t} + \frac{\partial F}{\partial\phi^a}\dot{\phi}^a + f_a\ddot{\phi}^a.\end{aligned}$$

Comparing with (132) shows us that $g = \bar{\mathcal{L}} + \frac{\partial F}{\partial t} + \frac{\partial F}{\partial\phi^a}\dot{\phi}^a$, which is a purely first order function. Thus $\bar{\mathcal{L}}\left(\dot{\phi}_a, \phi_a, \dot{q}_\alpha, \dot{q}_\alpha\right)$ is a first order Lagrangian that is divergence equivalent to $\mathcal{L}$. We conclude that all mechanical Class I Lagrangians are divergence equivalent to a first order Lagrangian. □

There is a similar theorem and proof for Lorentz Invariant theories:

Theorem 7. *Let $\mathcal{L}$ be a second-order, Lorentz Invariant, Class I Lagrangian. $\mathcal{L}$ is divergence equivalent to a first order Lagrangian $\bar{\mathcal{L}}$ under the relation*

$$\mathcal{L} = \bar{\mathcal{L}} + \partial_\mu F^\mu, \tag{133}$$

where F^μ is 4-tuple of functions, and $\partial_\mu F^\mu$ its total derivative.

Proof. For field theories, the Class I condition (128) reduces the constraints (56), (60) and (61) to

$$\begin{aligned}0 &= P_{(ab)} = \mathcal{L}_{\ddot{\phi}^a\ddot{\phi}^b}, \\ 0 &= (S_i)_{(ab)} = \mathcal{L}_{\ddot{\phi}^{(a}\partial_i\dot{\phi}^{b)}} \\ 0 &= S_{[ab]} = \mathcal{L}_{\ddot{\phi}^{[a}\dot{\phi}^{b]}}.\end{aligned}$$

The primary constraint again implies linearity, i.e.

$$\mathcal{L} = \ddot{\phi}^b f_b\left(\partial_i\partial_\mu\phi^a, \partial_\mu\phi^a, \phi^a, \partial_i q^\alpha, q^\alpha\right) + g\left(\partial_i\partial_\mu\phi^a, \partial_\mu\phi^a, \phi^a, \partial_i q^\alpha, q^\alpha\right), \tag{134}$$

and the secondary constraints become

$$\begin{aligned}0 &= \frac{\partial f_{(a}}{\partial\partial_i\dot{\phi}^{b)}}, \\ 0 &= \frac{\partial f_{[a}}{\partial\dot{\phi}^{b]}} - \frac{1}{2}\partial_i\frac{\partial f_{[a}}{\partial\partial_i\dot{\phi}^{[b}}.\end{aligned}$$

These constraints are expanded from the mechanical case, and we are now looking for a function F^0 such that $F^0_{\dot{\phi}^a} - \partial_i F^0_{\partial_i \dot{\phi}^a} = f^a$. This function F does not generically exist for a general field theory, so we turn to the requirement of Lorentz Invariance. Due to Lorentz invariance, we have $\frac{\partial f_a}{\partial \partial_i \dot{\phi}^b} = 0$. This causes the symmetric secondary constraint (60) to vanish, and the anti-symmetric constraint (61) reduces to

$$\frac{\partial f_a}{\partial \dot{\phi}^b} - \frac{\partial f_b}{\partial \dot{\phi}^a} = 0,$$

similar to the mechanics case. We thus again have that there exists a function F^0 such that $F^0_{\dot{\phi}^a} = f_a$, and thus we have that

$$\mathcal{L} = \bar{\mathcal{L}} + \partial_\mu F^\mu,$$

where the F^i are arbitrary functions $F^i\left(\partial_\mu \phi^a, \phi^a, q^\alpha\right)$, e.g. $F^i = 0$. The form of the F^i, ensures that $\bar{\mathcal{L}}$ is first order. However, there is no guarantee that the tuple F^μ is a Lorentz vector, even for certain choices of the F^i. Therefore, we have only proven that all Lorentz invariant, Class I Lagrangians are divergence equivalent to a first order Lagrangian, but not necessarily a Lorentz invariant first order Lagrangian. □

Non-Lorentz invariant theories might still satisfy $\frac{\partial f_a}{\partial \partial_i \dot{\phi}^b} = 0$. By the above proof, these theories are also equivalent to a first order Lagrangian. In particular, theories with only one dependent variable satisfy $\frac{\partial f}{\partial \partial_i \dot{\phi}} = 0$. However, Crisostomi et al. did not find a proof that a function F always exists for the cases $\frac{\partial f_a}{\partial \partial_i \dot{\phi}^b} \neq 0$ [8]. Thus, there is no general theorem for a field theoretical Class I Lagrangian.

Class II constraints

Crisostomi et al. proved that all Class II theories are standard equivalent to a Class I Lagrangian under a contact transformation on the helper fields q^α. Since this Class I Lagrangian need not be Lorentz Invariant, we then do not have that in general all Class II Lagrangians are first order.

Theorem 8. *All second-order, Class II Lagrangians $\mathcal{L}_{II}$ are standard equivalent to a Class I Lagrangian $\bar{\mathcal{L}}_I$ under the contact transformation*

$$\bar{x}^\mu = x^\mu \qquad \bar{\phi}^a = \phi^a \qquad \bar{q}^\alpha = \bar{q}^\alpha\left(q^\beta, \phi^b, \partial_\mu \phi^b\right),$$

Proof. We shall closely follow the proof from Crisostomi et al. [8]. In particular we shall proof that the specific dependence of the null-vector $V^\alpha_a = V^\alpha_a\left(\phi^b, \partial_\mu \phi^b, q^\beta\right)$, i.e. the Class II condition (129), requires the existence of this contact transformation and vice-versa, that the existence of such a transformation for a Class II Lagrangian requires that the transformed Lagrangian is of Class I.

First, assume that there exists this contact transformation between two Lagrangians:

$$\mathcal{L}_{II}\left(\partial_i \partial_j \phi^a, \partial_i \phi^a, \phi^a, \partial_i q^\alpha, q^\alpha\right) = \bar{\mathcal{L}}_I\left(\partial_i \partial_j \phi^a, \partial_i \phi^a, \phi^a, \partial_i \bar{q}^\alpha, \bar{q}^\alpha\right),$$

where $\mathcal{L}_{II}$ belongs to Class II and $\bar{\mathcal{L}}_I$ belongs to Class I. The null-vectors of both Lagrangians are related by [8]

$$\bar{V}^\alpha_a = \frac{\partial \bar{q}^\alpha}{\partial \dot{\phi}^a} + V^\beta_a \frac{\partial \bar{q}^\alpha}{\partial q^\beta}. \tag{135}$$

Since $\bar{\mathcal{L}}_I$ belongs to Class I, $\bar{V}_a^\alpha = 0$. Futhermore, $\bar{q}^\alpha = \bar{q}^\alpha\left(q^\beta, \phi^B, \partial_i\phi^b\right)$ is invertible. We can combine these two statements to rewrite (135) as

$$V_a^\beta = \frac{\partial \bar{q}^\alpha}{\partial \dot{\phi}^a}\left(\frac{\partial \bar{q}^\alpha}{\partial q^\beta}\right)^{-1},$$

and thus $V_a^\beta = V_a^\beta\left(\phi^b, \partial_\mu\phi^b, q^\beta\right)$.

In the other direction, we assume that we have a Class II Lagrangian $\mathcal{L}_{II}$, with $V_a^\alpha = V_a^\alpha\left(\phi^b, \partial_\mu\phi^b, q^\beta\right)$. Consider the partial differential equations

$$\frac{\partial u}{\partial \dot{\phi}^a} + V_a^\beta\left(\phi^b, \partial_\mu\phi^b, q^\beta\right)\frac{\partial u}{\partial q^\beta} = 0. \tag{136}$$

By Frobenius' theorem [16], these equations have n independent solutions $\bar{q}^\alpha$ if and only if the integrability conditions

$$0 = \frac{\partial V_{[a}^\alpha}{\partial \dot{\phi}^{b]}} + V_{[a}^\beta\frac{\partial V_{b]}^\alpha}{\partial q^\beta} \equiv \mathcal{F}_{ab}^\alpha \tag{137}$$

are satisfied. Using the definition of $V_a^\alpha = -\mathcal{L}_{\ddot{\phi}^a\dot{q}^\beta}\mathcal{L}^{-1}_{\dot{q}^\beta\dot{q}^\alpha}$ and the fact that $\mathcal{L}_{II}$ obeys the primary condition (56) to calculate these conditions, leads to the reduction

$$\mathcal{F}_{ab}^\alpha = \mathcal{L}^{-1}_{\dot{q}^\beta\dot{q}^\alpha}\frac{\partial}{\partial \dot{q}^\beta}S_{[ab]},$$

which vanishes by construction, since $\mathcal{L}_{II}$ also obeys the secondary conditions (61).

Thus, we have our n independent solutions $\bar{q}^\alpha$. Furthermore, in the derivation of the conditions (56) and (61) we have assumed that the q^α are non degenerate, i.e. $\mathcal{L}^{-1}_{\dot{q}^\beta\dot{q}^\alpha}$ exists. Therefore, the q^α are also independent. Thus, we can conclude that $\frac{\partial \bar{q}^\alpha}{\partial q^\beta}$ is invertible [43] and we have the invertible contact transformation

$$\bar{x}^i = x^i \qquad \bar{\phi}^a = \phi^a \qquad \bar{q}^\alpha = \bar{q}^\alpha\left(q^\beta, \phi^B, \partial_i\phi^b\right).$$

Define the transformed Lagrangian $\bar{\mathcal{L}}\left(\partial_\mu\partial_\nu\phi^a, \partial_\mu\phi^a, \phi^a, \partial_\mu\bar{q}^\alpha, \bar{q}^\alpha\right) = \mathcal{L}_{II}\left(\partial_\mu\partial_\nu\phi^a, \partial_\mu\phi^a, \phi^a, \partial_\mu q^\alpha, q^\alpha\right)$. The null-vectors transform as in (135)

$$\bar{V}_a^\alpha = \frac{\partial \bar{q}^\alpha}{\partial \dot{\phi}^a} + V_a^\beta\frac{\partial \bar{q}^\alpha}{\partial q^\beta},$$

which is just (136), of which the $\bar{q}^\alpha$ were defined to be solutions. As a result, we have $\bar{V}_a^\alpha = 0$ and thus $\bar{\mathcal{L}}$ belongs to Class I, conluding this direction of the proof.

With both the 'if' and 'only if' proven, we can thus conclude that all second-order, Class II Lagrangians are standard equivalent to a Class I Lagrangian. □

This theorem leads directly to the following corollary.

Corollary 1. *All mechanical Class II Lagrangians are divergent equivalent to a regular, first-order Lagrangian.*

Proof. Theorem 8 states that all Class II Lagrangians are standard equivalent to a Class I Lagrangian. Since standard equivalence is a subset of divergence equivalence, we thus have that all Class II Lagrangians are also divergence equivalent to a Class I Lagrangian. From theorem 6 we know that all mechanical Class I Lagrangians are divergence equivalent to a first-order Lagrangian. Combining these two statements then leads to the conclusion that all mechanical Class II Lagrangians are divergent equivalent to a regular, first-order Lagrangian. □

There does not exist a similar corollary for Lorentz invariant Lagrangians, since the Class I Lagrangian equivalent to the Class II Lorentz invariant Lagrangian need not be Lorentz invariant.

Class III constraints

The Class III theories can not always be made divergence equivalent to a first order Lagrangian. Although for some Class III Lagrangians there exists a contact transformation that transform them into a Class I Lagrangian, Crisostomi et al. were not able to find a proof that such a transformation is possible for all Class III theories [8]. As we shall not deal with Class III Lagrangians much in the rest of this chapter, we refer the reader to [8] for the Lagrangians that are equivalent.

9.2 One Galileon

In this subsection, we shall proof that all single-Galileon Lagrangians are explictly divergence equivalent to a first-order Lagrangian. In particular, we shall compute the functions F such that $\bar{\mathcal{L}} = \mathcal{L} + \frac{\mathrm{d}F}{\mathrm{d}t}$ is first-order. As described in section 5.1, the Lagrangian for a single galileon is a linear combination of (66)

$$\begin{aligned}
\mathcal{L}_1 &= \phi, \\
\mathcal{L}_2 &= -\frac{1}{2}\partial^\mu\phi\partial_\mu\phi, \\
\mathcal{L}_3 &= -\frac{1}{2}\partial_\mu\partial^\mu\phi\partial^\nu\phi\partial_\nu\phi, \\
\mathcal{L}_4 &= -\frac{1}{2}\left[(\partial^\mu\partial_\mu\phi)^2 - \partial^\mu\partial_\nu\phi\partial^\nu\partial_\mu\phi\right]\partial^\rho\phi\partial_\rho\phi \\
\mathcal{L}_5 &= -\frac{1}{2}\left[(\partial^\mu\partial_\mu\phi)^3 - 3\partial^\mu\partial_\mu\phi\partial^\nu\partial_\rho\phi\partial^\rho\partial_\nu\phi + 2\partial^\mu\partial_\nu\phi\partial^\nu\partial_\rho\phi\partial^\rho\partial_\mu\phi\right]\partial^\sigma\phi\partial_\sigma\phi.
\end{aligned}$$

$\mathcal{L}_1$ and $\mathcal{L}_2$ are already in first order form, and thus we need not discuss them. The other three Lagrangians all have no secondary fields, thus for all $V_m^\alpha = 0$ and they belong in Class I. Let us rewrite them in the form of (134):

$$\begin{aligned}
\mathcal{L}_3 &= \ddot{\phi}\left(-\frac{1}{2}\partial_\nu\phi\partial^\nu\phi\right) - \frac{1}{2}\partial_i\partial^i\phi\partial_\nu\phi\partial^\nu\phi, \\
\mathcal{L}_4 &= \ddot{\phi}\left(-\partial_i\phi\partial^i\partial_\rho\phi\partial^\rho\phi\right) - \frac{1}{2}\left(\partial_i\partial^i\phi\partial_j\partial^j\phi - 2\partial_i\dot{\phi}\partial^i\dot{\phi} - \partial_i\partial_j\phi\partial^i\partial^j\phi\right), \\
\mathcal{L}_5 &= \ddot{\phi}\left(-\frac{3}{2}\partial_i\partial^i\phi\partial_j\partial^j\phi + \frac{3}{2}\partial_i\partial_j\phi\partial^i\partial^j\phi\right)\partial_\sigma\phi\partial^\sigma\phi + 3\partial_i\dot{\phi}\partial_j\dot{\phi}\partial^i\partial^j\phi\partial_\sigma\phi\partial^\sigma\phi.
\end{aligned}$$

From these equations, we find the following functions 'f':

$$\begin{aligned}
f_3 &= -\frac{1}{2}\partial_\nu\phi\partial^\nu\phi, \\
f_4 &= -\partial_i\phi\partial^i\phi\partial_\rho\phi\partial^\rho\phi, \\
f_5 &= \frac{3}{2}\left(-\partial_i\partial^i\phi\partial_j\partial^j\phi + \partial_i\partial_j\phi\partial^i\partial^j\phi\right)\partial_\sigma\phi\partial^\sigma\phi.
\end{aligned}$$

Each of these satisfies $\frac{\partial f_k}{\partial\partial_i\dot{\phi}} = 0$, with $k = 3, 4, 5$, and thus each of these Lagrangians can be turned into a first order Lagrangian. The relevant functions are

$$\begin{aligned}
F_3 &= -\frac{1}{6}\dot{\phi}^3 - \frac{1}{2}\partial_i\phi\partial^i\phi\dot{\phi}, \\
F_4 &= -\partial_i\phi\partial^i\phi\left(\frac{1}{3}\dot{\phi}^3 + \partial_j\phi\partial^j\phi\dot{\phi}\right), \\
F_5 &= \frac{3}{2}\left(-\partial_i\partial^i\phi\partial_j\partial^j\phi + \partial_i\partial_j\phi\partial^i\partial^j\phi\right)\left(\frac{1}{3}\dot{\phi}^3 + \partial_j\phi\partial^j\phi\dot{\phi}\right).
\end{aligned}$$

Furthermore, the equivalent first order Lagrangians can be calculated. They are

$$\begin{aligned}
\bar{\mathcal{L}}_3 &= \mathcal{L}_3 - \frac{\mathrm{d}F_3}{\mathrm{d}t} \\
&= -\frac{1}{2}\partial_i\partial^i\phi\partial_\nu\phi\partial^\nu\phi + \dot{\phi}\partial_i\dot{\phi}\partial^i\phi, \\
\bar{\mathcal{L}}_4 &= \mathcal{L}_4 - \frac{\mathrm{d}F_4}{\mathrm{d}t} \\
&= -\frac{1}{2}\left(\partial_i\partial^i\phi\partial_j\partial^j\phi - 2\partial_i\dot{\phi}\partial^i\dot{\phi} - \partial_i\partial_j\phi\partial^i\partial^j\phi\right) - \partial_i\dot{\phi}\left(\frac{2}{3}\partial^i\phi\dot{\phi}^3 + 4\partial^i\phi\partial_j\phi\partial^j\phi\dot{\phi}\right) \\
\bar{\mathcal{L}}_5 &= \mathcal{L}_5 - \frac{\mathrm{d}F_5}{\mathrm{d}t} \\
&= 3\partial_i\dot{\phi}\partial_j\dot{\phi}\partial^i\partial^j\phi\partial_\sigma\phi\partial^\sigma\phi - 3\left(\partial_i\partial_j\phi\partial^i\partial^j\phi - \partial_i\partial^i\phi\partial_j\partial^j\phi\right)\partial_k\phi\partial^k\dot{\phi}\dot{\phi} \\
&+ 3\left(\partial_i\partial_j\phi\partial^i\partial^j\dot{\phi} - \partial_i\partial^i\phi\partial_j\partial^j\dot{\phi}\right)\left(\frac{1}{3}\dot{\phi}^3 + \partial_k\phi\partial^k\phi\dot{\phi}\right)
\end{aligned}$$

Clearly, we lose Lorentz invariance, but the resulting Lagrangians are all first order in time derivatives, and thus ghost free.

9.3 One Galileon, many scalars

A step up from considering only one galileon, is to consider this galileon together with some additional scalars. These additional scalars are all healthy, that is, the scalars will only be first-order in derivatives. Unlike the previous subsection, our Lagrangian can now be in any of the three classes. Thus, we can not show for all of these cases that the Lagrangians are divergence equivalent to first-order theories.

The simplest theory with one galileon and many first-order scalars, is a theory were these two types of fields are not coupled:

$$\mathcal{L}\left(\phi, \partial\phi, \partial\partial\phi, q^\alpha, \partial q^\alpha\right) = \mathcal{L}_{gal}\left(\phi, \partial\phi, \partial\partial\phi\right) + \mathcal{L}_{scal}\left(q^\alpha, \partial q^\alpha\right) \tag{138}$$

Such a theory is not very interesting, and, due to the decoupled nature of the theory, can in fact be treated as in the previous subsection.

Instead, we want the galileon to interact with the galileons. The total theory still has to be ghost-free, and thus satisfies (56):

$$0 = P_{11} = \mathcal{L}_{\ddot{\phi}\ddot{\phi}} - \mathcal{L}_{\ddot{\phi}\dot{q}^\alpha}\mathcal{L}^{-1}_{\ddot{\phi}\dot{q}^\beta}\mathcal{L}^{-1}_{\dot{q}^\beta\dot{q}^\alpha}.$$

The symmetric secondary condition (60) is satisfied, since we are working with Lorentz invariant theories. The anti-symmetric condition (61) is automatically satisfied, since there is only one second-order field.

Next, we need to consider which class our Lagrangian is in. Class I has the requirement that $V_1^\alpha = \mathcal{L}_{\ddot{\phi}\dot{q}^\beta}\mathcal{L}^{-1}_{\dot{q}^\beta\dot{q}^\alpha} = 0$, under which the ghost-free condition becomes

$$P_{11} = \mathcal{L}_{\ddot{\phi}\ddot{\phi}}.$$

The Class I condition can be met by ensuring that there is no coupling between the second derivative of the unhealthy field with any first derivative of the healthy field. Furthermore, the primary constraint states that the Class I Lagrangians are all linear in the unhealthy field. Therefore, these Lagrangians are of the form

$$\mathcal{L} = \ddot{\phi}f\left(\phi, \partial_i\phi, \dot{\phi}, \partial_i\dot{\phi}, \partial_i\partial_j\phi, q^\alpha, \partial_i q^\alpha\right) + g\left(\phi, \partial_i\phi, \dot{\phi}, \partial_i\dot{\phi}, \partial_i\partial_j\phi, q^\alpha, \partial_i q^\alpha, \dot{q}^\alpha\right), \tag{139}$$

The functions f and g are almost arbitrary, the only requirement we place upon them is that they make the Lagrangian Lorentz invariant.

Since we are calling our unhealthy field a Galileon, let us give an example of such a Lagrangian that obeys the Galilean symmetry:

$$\mathcal{L} = \mathcal{L}_{GAL} f\left(q^{\alpha}\right) + g\left(q^{\alpha}, \partial q^{\alpha}\right), \tag{140}$$

with f and g arbitrary functions and $\mathcal{L}_{GAL}$ the Lagrangian for a single galileon (66). This Lagrangian is equivalent to a first order Lagrangian by theorem 7. The relevant function F^0 is given by $F^0 = F_i \cdot f$, with the F_i the total derivative functions of the one galileon case.

Of course, there are other Lorentz invariant Lagrangians that are in Class I, but those do not obey the Galilean symmetry in ϕ.

Next, we generalize to Class II theories. The null-vector now is non-zero, but does not depend on all derivatives. For one second-order field, this can be written as

$$\mathcal{L}_{\ddot{\phi}\dot{q}^{\beta}} \mathcal{L}^{-1}_{\dot{q}^{\beta}\dot{q}^{\alpha}} = V_1^{\alpha}\left(\phi, \partial_{\mu}\phi, q^{\beta}\right). \tag{141}$$

The easiest way to accomplish this is to enforce that all components on the left-hand side do not depend on the higher-order derivatives, i.e.

$$\begin{aligned}
\mathcal{L}_{\ddot{\phi}\dot{q}^{\beta}} &= \mathcal{L}_{\ddot{\phi}\dot{q}^{\beta}}\left(\phi, \partial_{\mu}\phi, q^{\gamma}\right), \\
\mathcal{L}_{\dot{q}^{\beta}\dot{q}^{\alpha}} &= \mathcal{L}_{\dot{q}^{\beta}\dot{q}^{\alpha}}\left(\phi, \partial_{\mu}\phi, q^{\gamma}\right).
\end{aligned}$$

We can draw several conclusions from these lines:

- Any quadratic appearance of the second-order time derivative of the galileon is not coupled to the time derivative of the scalar.
- The second-order time derivative of the galileon is only linearly coupled to the first-order time derivative of the scalars.
- The Lagrangians depends at most quadratically on the first order derivatives of the scalar.

We are however interested in not just the form of these Class II theories, but we want to check their equivalence to a first order theory. According to Crisostomi et al. [8], all Class II theories can be mapped to a Class I theory via a field redefinition of the q^{α}. The transformed fields $\bar{q}^{\alpha}$ are the solutions to the system of partial differential equations [8]

$$\frac{\partial u}{\partial \dot{\phi}} + V_1^{\beta}\left(\partial\phi, \phi, q^{\alpha}\right) \frac{\partial u}{\partial q^{\beta}} = 0, \tag{142}$$

and the transformed Lagrangian is given by

$$\mathcal{L}\left(\partial_{\mu}\partial_{\nu}\phi, \partial_{\mu}\phi, \phi, \partial_{\mu}q^{\alpha}, q^{\alpha}\right) = \bar{\mathcal{L}}_I\left(\partial_{\mu}\partial_{\nu}\phi, \partial_{\mu}\phi, \phi, \partial_{\mu}\bar{q}^{\alpha}, \bar{q}^{\alpha}\right). \tag{143}$$

Lorentz invariant Class I theories are always a total divergence first-order theory, but the Class I theory we receive due to the field redefinition need not be Lorentz invariant. We can however still end up with a fist-order theory. Let $\bar{\mathcal{L}}$ be the thus gained Class I theory. It can be rewritten into the form of (139),

$$\bar{\mathcal{L}}_I = \ddot{\phi} f\left(\partial_i\partial_{\mu}\phi, \partial_{\mu}\phi, \phi, \partial_i q_{\alpha}, q_{\alpha}\right) + g\left(\partial_i\partial_{\mu}\phi, \partial_{\mu}\phi, \phi, \partial_i q_{\alpha}, q_{\alpha}\right).$$

Although this Lagrangian need not be Lorentz invariant, there is only one second order variable, thus the condition $\frac{\partial f_a}{\partial \partial_i \dot{\phi}^b} = 0$ still holds. Since we only have one second order variable, we

do not have the anti-symmetric requirement $\frac{\partial f_a}{\partial \dot{\phi}^b} = \frac{\partial f_b}{\partial \dot{\phi}^a}$. Thus there exists an F^0 such that $F^0_{\dot{\phi}} = f$ and $\bar{\mathcal{L}} + \frac{\mathrm{d}}{\mathrm{d}t}F^0$ is first-order, and thus we conclude that all Class II One-Galileon, many scalar Lagrangians are divergence equivalent to a regular first-order Lagrangian, though this Lagrangian need not be Lorentz Invariant.

This leaves us with the Class III theories. For these theories, Crisostomi et al. [8] did not find any relation between Class III theories and first-order Lagrangians. As a result, our analysis of one-galileon theories stops at this point. The existence of a divergence equivalent first-order Lagrangian can be tested using Cartan's equivalence algortihm discussed in chapters 10 and 10.5.

9.4 Many Galileons

The next step is to promote the additional scalar fields to galileon fields. In other words, we are going to investigate multi-galileon theories. From section 5.1, and the work by Padilla and Sivanesan [23] referenced therein, the Lagrangian for such a general Lagrangian is given by (72):

$$\mathcal{L} = \sum_{m=1}^{5} \alpha_{a_1 \dots a_m} \phi^{a_1} \delta^{\mu_2 \dots \mu_m}_{\nu_2 \dots \nu_m} \partial_{\mu_2} \partial^{\nu_2} \phi^{a_2} \cdots \partial_{\mu_m} \partial^{\nu_m} \phi^{a_m},$$

where the indexes a_j count over the fields. Since there are no first order scalar fields present, these theories are all in Class I. Since they are Lorentz invariant, we know that they are divergence equivalent to first-order Lagrangians by theorem 7. To arrive at the required total derivative function F, we shall split up the sum in (72) into its component parts, and then rewrite each term in the form of (128) individually. Linearity of differentials then allows us to consider each of these components separately.

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_5 \\
\mathcal{L}_1 &= \alpha_a \phi^a \\
\mathcal{L}_2 &= \alpha_{ab} \phi^a \partial^\mu \partial_\mu \phi^b \\
&= \alpha_{ab} \phi^a \ddot{\phi}^b + \alpha_{mn} \phi^a \partial^i \partial_i \phi^b \\
\mathcal{L}_3 &= \alpha_{a_1 a_2 a_3} \phi^{a_1} \delta^{\mu_2 \mu_3}_{\nu_2 \nu_3} \partial_{\mu_2} \partial^{\nu_2} \phi^{a_2} \partial_{\mu_3} \partial^{\nu_3} \phi^{a_3} \\
&= \alpha_{a_1 a_2 a_3} \phi^{a_1} \left(\ddot{\phi}^{a_2} \partial_j \partial^j \phi^{a_3} + \partial_j \partial^j \phi^{a_2} \ddot{\phi}^{a_3} + \partial_j \partial^j \phi^{a_2} \partial_k \partial^k \phi^{a_3} - 2 \partial_j \dot{\phi}^{a_2} \partial^j \dot{\phi}^{a_3} - \partial_j \partial_k \phi^{a_2} \partial^j \partial^k \phi^{a_3} \right) \\
&= \ddot{\phi}^b \left(2 \alpha_{a_1 b a_3} \phi^{a_1} \partial_j \partial^j \phi^{a_3} \right) + g_3 \\
\mathcal{L}_4 &= \alpha_{a_1 a_2 a_3 a_4} \phi^{a_1} \delta^{\mu_2 \mu_3 \mu_4}_{\nu_2 \nu_3 \nu_4} \partial_{\mu_2} \partial^{\nu_2} \phi^{a_2} \partial_{\mu_3} \partial^{\nu_3} \phi^{a_3} \partial_{\mu_4} \partial^{\nu_4} \phi^{a_4} \\
&= \alpha_{a_1 a_2 a_3 a_4} \phi^{a_1} \left(3 \partial_j \partial^j \phi^{a_2} \partial^k \partial_k \phi^{a_3} \ddot{\phi}^{a_4} + 6 \partial_j \dot{\phi}^{a_2} \partial^j \partial_k \phi^{a_3} \partial^k \dot{\phi}^{a_4} - 6 \partial_j \partial^j \phi^{a_2} \partial_k \dot{\phi}^{a_3} \partial^k \dot{\phi}^{a_4} - 3 \partial_j \partial_k \phi^{a_2} \partial^j \partial^k \phi^{a_3} \ddot{\phi}^{a_4} \right) \\
&= 3 \alpha_{a_1 a_2 a_3 a_4} \phi^{a_1} \left(\partial_j \partial^j \phi^{a_2} \partial^k \partial_k \phi^{a_3} - \partial_j \partial_k \phi^{a_2} \partial^j \partial^k \phi^{a_3} \right) \ddot{\phi}^{a_4} \\
&\quad + 6 \alpha_{a_1 a_2 a_3 a_4} \phi^{a_1} \left(\partial_j \dot{\phi}^{a_2} \partial^j \partial_k \phi^{a_3} - \partial_k \dot{\phi}^{a_2} \partial_j \partial^j \phi^{a_3} \right) \partial^k \dot{\phi}^{a_4} \\
\mathcal{L}_5 &= \alpha_{a_1 a_2 a_3 a_4 a_5} \phi^{a_1} \delta^{\mu_2 \mu_3 \mu_4 \mu_5}_{\nu_2 \nu_3 \nu_4 \nu_5} \partial_{\mu_2} \partial^{\nu_2} \phi^{a_2} \partial_{\mu_3} \partial^{\nu_3} \phi^{a_3} \partial_{\mu_4} \partial^{\nu_4} \phi^{a_4} \partial_{\mu_5} \partial^{\nu_5} \phi^{a_5} \\
&= 4 \alpha_{a_1 a_2 a_3 a_4 a_5} \phi^{a_1} \delta^{j_2 j_3 j_4}_{k_2 k_3 k_4} \partial_{j_2} \partial^{k_2} \phi^{a_2} \partial_{j_3} \partial^{k_3} \phi^{a_3} \partial_{j_4} \partial^{k_4} \phi^{a_4} \ddot{\phi}^{a_5} + g \left(\partial_i \partial_\mu \phi^a, \partial_\mu \phi^a, \phi^a \right)
\end{aligned}$$

In this split, we have that $\mathcal{L}_1$ is zero-th order, and thus naturally ghost free. For the more interesting cases have $f_{2,a} = \alpha_{ab} \phi_b$ and $f_{3,b} = 2 \alpha_{a_1 a_2 b} \phi^{a_1} \partial_j \partial^j \phi^{a_2}$. For the fourth Lagrangian, we have $f_{4,b} = 3 \alpha_{a_1 a_2 a_3 b} \phi^{a_1} \left(\partial_j \partial^j \phi^{a_2} \partial^k \partial_k \phi^{a_3} - \partial_j \partial_k \phi^{a_2} \partial^j \partial^k \phi^{a_3} \right)$, and last for the fifth Lagrangian $f_{5,b} = 4 \alpha_{a_1 a_2 a_3 a_4 b} \phi^{a_1} \delta^{j_2 j_3 j_4}_{k_2 k_3 k_4} \partial_{j_2} \partial^{k_2} \phi^{a_2} \partial_{j_3} \partial^{k_3} \phi^{a_3} \partial_{j_4} \partial^{k_4} \phi^{a_4}$. We could calculate which functions F_i produce the functions f_i, and thus calculate the first order Lagrangians. This is left as an exercise for the reader.

Finally, we note that the Galilean invariance is not responsible for the equivalence. All Ostrogradsky free Lagrangians with only second order variables are in Class I, and thus are divergence equivalent to a first order theory by them 7, although this may not be a Lorentz invariant theory.

9.5 Many Galileons, and many scalars

Essentially, this is the most general case. Our ghost free theories could fall in any of the three classes. We discuss the theories by class.

- The Class I theories are all equivalent to a first-order theory, by theorem 7.
- The Class II theories are all equivalent to a Class I theory, by theorem 8. Unfortunately, the resulting Class I Lagrangian need not be Lorentz invariant, and neither is it generically true that $\frac{\partial f_a}{\partial \partial_i \dot{\phi}^b} = 0$. As a result, we can not at this point state whether all Class II theories are equivalent to a first-order theory.
- Unfortunately, Crisostomi et al. [8] found no general transformation that turns a Class III theory into a Class I theory, and from there into a first order one. As a result, we can not state anything about them right now.

The Class II and Class III theories can be looked into, using the Cartan algortihm discussed in chapters 10 and 10.5.

9.6 Covariant Galileons

In the previous part of this section, only non-covariant galileons, and galileon like Lagrangians, were considered, i.e. theories on a flat background $g_{\mu\nu} = \eta_{\mu\nu}$. The reason for this is that, as far as I'm aware, no one has yet formulated general criteria for second order tensor theories to be Ostrogradsky ghost free. Therefore is no class of theories to apply any equivalence calculations or consideration to.

However, as Chogoya and Tasinato [19] found that several Galileon theories remain ghost-free upon minimal-coupling-covariantization, it is not illogical to assume that the theories from this section remain ghost-free upon covariantization.

10 An explanation of Cartan's equivalence algorithm

Elie Cartan formulated an algorithm to solve equivalence problems on differential forms. He also formulated a procedure to cast several other equivalence problems in differential form. This allows one to use his algorithm to solve these problems [6, 17]. In this section, I shall closely follow the book by Olver [6, chapters 8 to 10] to explain Cartan's algorithm, and use his paper together with Kamran [17] to show how to rewrite the Lagrangian equivalence problem in differential form.

First, we have the strict version of Cartan's algorithm. It answers the question: Given two coframes $\bar{\theta}^i$ an θ^i, when does there exist a diffeomorphism Φ such that $\Phi^*\left(\bar{\theta}^i\right) = \theta^i$? Next, we concern ourselves with a more general question: Given two coframes $\bar{\theta}^i$ and θ^i, when does there exist a diffeomorphism Φ and a matrix A^i_j (subject to some constraints) such that $\Phi^*\left(\bar{\theta}^i\right) = A^i_j\theta^j$? We solve this more general question by reducing to the strict problem.

10.1 Strict equivalence of coframes

To answer the strict question, we first need to define coframes and the equivalence problem on these coframes.

Definition 8 (Frames and coframes, [6])**.** Let M be a smooth, n-dimensional manifold. A *frame* on M is a set of vector fields $V = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ with the property that they form a basis on the tangent space T_xM at each point $x \in M$. Dually, a *coframe* on M is an ordered set of one-forms $\boldsymbol{\theta} = \{\theta^1, \ldots, \theta^n\}$ which form a basis on the cotangent space T^*_xM at each point $x \in M$.

Given a coframe $\boldsymbol{\theta}$, its *dual* frame of vector fields is the frame of vector fields $\{\partial/\partial\theta^1, \ldots, \partial/\partial\theta^n\}$ that satisfies

$$\left\langle \theta^i; \frac{\partial}{\partial\theta^j} \right\rangle = \delta^i_j.$$

Definition 9 (Equivalence of coframes [6])**.** Let $\boldsymbol{\theta} = \{\theta^1, \ldots, \theta^n\}$ be a coframe on a manifold M and $\bar{\boldsymbol{\theta}} = \left\{\bar{\theta}^1, \ldots, \bar{\theta}^n\right\}$ a coframe on a potentially different manifold $\bar{M}$, with $\dim M = \dim \bar{M}$. We say that these two coframes are equivalent if and only if there exists a diffeomorphism $\Phi : M \to \bar{M}$ such that

$$\Phi^*\bar{\boldsymbol{\theta}}^i = \boldsymbol{\theta}^i. \tag{144}$$

The key observation made by Cartan in developing his algorithm was that the exterior derivative operator d is invariant under diffeomorphisms [6]. Thus, for any two equivalent coframes, we also necessarily have

$$\Phi^*\,\mathrm{d}\bar{\boldsymbol{\theta}}^i = \mathrm{d}\boldsymbol{\theta}^i \tag{145}$$

Since $\boldsymbol{\theta}$ is a coframe, we can write the two-forms $\mathrm{d}\theta^i$ as a linear combination of wedge products of the θ^i. This gives the fundamental *structure equations*

$$\mathrm{d}\theta^i = T^i_{jk}\theta^j \wedge \theta^k. \tag{146}$$

We can also compute these for $\bar{\boldsymbol{\theta}}$, this yields

$$\mathrm{d}\bar{\theta}^i = \bar{T}^i_{jk}\bar{\theta}^j \wedge \bar{\theta}^k.$$

Substituting these two equations into (145) gives

$$
\begin{aligned}
& T^i_{jk}(x)\,\theta^j \wedge \theta^k \\
=& \,\mathrm{d}\theta^i \\
=& \Phi^*\,\mathrm{d}\bar{\theta}^i \\
=& \Phi^*\left(\bar{T}^i_{jk}\bar{\theta}^j \wedge \bar{\theta}^k\right) \\
=& \bar{T}^i_{jk}(\Phi(x))\,\Phi^*\bar{\theta}^j \wedge \Phi^*\bar{\theta}^k \\
=& \bar{T}^i_{jk}(\bar{x})\,\theta^j \wedge \theta^k,
\end{aligned}
$$

where in the last line the equivalence of the coframes (144) is used to write $\Phi^*\bar{\theta}^j \wedge \Phi^*\bar{\theta}^k = \theta^j \wedge \theta^k$. The above calculation implies the invariance of the structure equations:

$$
\bar{T}^i_{jk}(\bar{x}) = T^i_{jk}(x), \text{ when } \bar{x} = \Phi(x). \tag{147}
$$

This invariance already gives some necessary conditions for equivalence, for if T^i_{jk} for $\boldsymbol{\theta}$ is a constant for some indices i, j, k, then for any equivalent coframe $\bar{\boldsymbol{\theta}}$, $\bar{T}^i_{jk}$ must assume the same constant value for the same indices i, j, k [6].

These structure equations appear not just in the derivatives of the coframe, if we compute the commutation relations for the dual-frame to our coframe, we find the following:

$$
\left[\frac{\partial}{\partial\theta^j}, \frac{\partial}{\partial\theta^k}\right] = -T^i_{jk}\frac{\partial}{\partial\theta^i}. \tag{148}
$$

These equations will later allow us to rewrite the order of derivation of the structure equations using lower order derivatives.

We now have a set of $\frac{1}{2}n^2(n-1)$ invariant functions. However, we can easily compute more. Let $I(x)$ be any scalar invariant (T^i_{jk} is an example), which is mapped to $\bar{I}(\bar{x})$ under the action of Φ, then the differentials of these invariants must also agree: $\Phi^*\,\mathrm{d}\bar{I} = \mathrm{d}I$. We can re-express these differentials in terms of the coframes, using the dual frames. Thus, we find

$$
\frac{\partial I(x)}{\partial\theta^j}\theta^j = \mathrm{d}I(x) = \Phi^*\,\mathrm{d}\bar{I}(\bar{x}) = \frac{\partial\bar{I}(\Phi(x))}{\partial\bar{\theta}^j}\theta^j, \tag{149}
$$

or, since the θ^i are linearly independent,

$$
\frac{\partial\bar{I}}{\partial\bar{\theta}^i}(\bar{x}) = \frac{\partial I}{\partial\theta^i}(x). \tag{150}
$$

We can do this ad infinitum (or at least ad nauseam), giving us an infinite number of invariants. These invariants may not be all independent, there may be functional relations between them. These relations are also invariant. For example, assume that we have 3 invariants, I_1, I_2 and I_3, with the relation $I_3 = H(I_1, I_2)$, for some function $H(x_1, x_2)$. Then any equivalent coframe must have this same functional relation: $\bar{I}_3 = H\left(\bar{I}_1, \bar{I}_2\right)$ for the *same* function $H(x_1, x_2)$.

We can introduce these dependence relations by permuting the derivatives. The Lie brackets (148) allow us to do this, at the expense of introducing lower order derivatives, using the relation:

$$
\frac{\partial^2 I}{\partial\theta^k\partial\theta^j} - \frac{\partial^2 I}{\partial\theta^j\partial\theta^k} = T^i_{jk}\frac{\partial I}{\partial\theta i} \tag{151}
$$

Before continuing, let us introduce some notation for the derivatives of the structure equations:

$$
T_\sigma = \frac{\partial^s T^i_{jk}}{\partial\theta^{l_s}\partial\theta^{l_{s-1}}\cdot\ldots\cdot\partial\theta^{l_1}}, \text{ where } \sigma = (i, j, k, l_1, \ldots, l_s).
$$

This is the most general *structure invariant* associated with the coframe [6]. σ is an *ordered* multi-index, since the order of derivation matters. The integer $s = \text{ order } \sigma = |\sigma| - 3$ is called the order of the derived invariant. According to the above calculations, for two equivalent coframes, their structure equations and all their derivatives must agree, i.e.

$$\bar{T}_\sigma(\bar{x}) = T_\sigma(x), \text{ with } \bar{x} = \Phi(x), \ \forall \sigma. \tag{152}$$

Through constant derivation, we have an infinite cascade of necessary conditions for equivalence, but we would like some sufficient conditions. In order to reach these conditions, we need to introduce the classifying spaces and manifolds.

Definition 10 (Classifying spaces [6])**.** The s^{th} order *classifying space* $\mathbb{K}^{(s)} = \mathbb{K}^{(s)}(n)$ associated with an n-dimensional manifold M is the Euclidean space of dimension $q_s(n) = \frac{1}{2}n^2(n-1)\binom{n+s}{n}$, which has local coordinates given by $z^{(s)} = (\dots, z_\sigma, \dots)$. The entries of $z^{(s)}$ are labelled by non-decreasing multi-indices $\sigma = (i, j, k, l_1, \dots, l_r)$, with $j < k$, $1 \leqslant l_1 \leqslant l_2 \leqslant \dots \leqslant l_r \leqslant n$, and $0 \leqslant r \leqslant s$.

Definition 11 (Structure map [6])**.** The s^{th} order *structure map* associated with a coframe $\boldsymbol{\theta}$ on M is the map $\boldsymbol{T}^{(s)} : M \to \mathbb{K}^{(s)}$ whose components are the structure constants: $z_\sigma = T_\sigma(x)$, for order $(\sigma) \leqslant s$.

Thus, the structure map is the set of all structure equations and their derivatives, and the classifying space is where the values of the components of the structure map lie. These definitions already take into account that we are allowed to permute the order of differentiation in our invariants, thus only the $q_s(n)$ indices with non-decreasing indices are needed.

If two coframes $\boldsymbol{\theta}$ and $\bar{\boldsymbol{\theta}}$ are equivalent, then the invariance equations (152) tell us that their structure maps must have the same image. We can give this image a name: the classifying set:

Definition 12 (Classifying set)**.** The s^{th} order *classifying set* $\mathcal{C}^{(s)} = \mathcal{C}^{(s)}(\boldsymbol{\theta}, U)$ associated with a coframe $\boldsymbol{\theta}$ on an open subset $U \subset M$ is defined as the image of the structure map $\boldsymbol{T}^{(s)}$:

$$\mathcal{C}^{(s)}(\boldsymbol{\theta}, U) = \left\{ \boldsymbol{T}^{(s)}(x) \,|\, x \in U \right\} \subset \mathbb{K}^{(s)}.$$

This definition and (152) then immediately give rise to the following proposition

Proposition 1. *Suppose $\boldsymbol{\theta}$ and $\bar{\boldsymbol{\theta}}$ are equivalent coframes under $\Phi : M \to \bar{M}$. Then for each $s \geqslant 0$ the s^{th} order classifying sets are the same, e.g. $\mathcal{C}^{(s)}\left(\bar{\boldsymbol{\theta}}, \bar{U}\right) = \mathcal{C}^{(s)}(\boldsymbol{\theta}, U)$, where $U \subset M$ is the domain and $\bar{U} \subset \bar{M}$ is the range of the local equivalence map Φ.*

Ergo, two coframes are equivalent if and only if all of their classifying sets are the same [6].

From the definition of the classifying set, it should be clear that if $\mathcal{C}^{(s)}\left(\bar{\boldsymbol{\theta}}, \bar{U}\right) = \mathcal{C}^{(s)}(\boldsymbol{\theta}, U)$ for a certain s, then also $\mathcal{C}^{(r)}\left(\bar{\boldsymbol{\theta}}, \bar{U}\right) = \mathcal{C}^{(r)}(\boldsymbol{\theta}, U)$ for each $r \leqslant s$.

This proposition is not just a necessary condition, it is also sufficient. However, it does require us to compute infinitely many classifying sets. The next paragraph aims to reduce the amount of classifying sets that need to be computed to determine equivalence.

Let $\rho_s = \operatorname{rank}\left(\boldsymbol{T}^{(s)}\right)$ be the rank of the structure map. As long as the structure map is regular, $\mathcal{C}^{(s)}$ is a ρ_s dimensional sub-manifold of $\mathbb{K}^{(s)}$. This rank ρ_s is equal to the number of functionally independent structure invariants among the T_σ, and all structure invariants in $\mathcal{C}^{(s)}$ can be expressed as a function of a suitably chosen set of ρ_s invariants.

It should be clear from the construction of the classifying set that its dimension is non-decreasing. Furthermore, it is bounded by n, the dimension of M. Therefore, it must converge to a constant value. These considerations lead to the following definition.

Definition 13. Let $\boldsymbol{\theta}$ be a coframe and let ρ_s denote the rank of the s^{th} order structure map $\boldsymbol{T}^{(s)}$. The smallest s for which $\rho_s = \rho_{s+1}$ is called the order of the coframe. We also have

$$0 \leqslant \rho_o < \rho_1 < \cdots < \rho_s = \rho_{s+1} = \rho_{s+2} = \cdots = r \leqslant n.$$

This stabilizing rank r is called the *rank* of the coframe.

This proposition gives an upper bound to the order of $\mathcal{C}^{(s)}$, and thus an upper bound to the size of the classifying sets. Thus we can sumarize this section in the following theorem:

Theorem 9 (Fundamental Theorem of Equivalence of Coframes [6])**.** *Let $\boldsymbol{\theta}$ and $\bar{\boldsymbol{\theta}}$ be smooth, regular coframes defined, respectively, on n-dimensional manifolds M and $\bar{M}$. There exists a local diffeomorphism $\Phi : M \to \bar{M}$ mapping the coframes to each other, $\Phi^* \bar{\boldsymbol{\theta}} = \boldsymbol{\theta}$, if and only if they have the same order, $\bar{s} = s$, the same rank, $\bar{r} = r$, and their $(s+1)^{th}$ order classifying sets $\mathcal{C}^{(s+1)}\left(\bar{\boldsymbol{\theta}}\right)$ and $\mathcal{C}^{(s+1)}\left(\boldsymbol{\theta}\right)$ overlap.*

Example 14. Let us discuss an example of the computation. Let $M = \mathbb{R}^2 \backslash \{x = 0\}$, on which we define the coframe

$$\theta^1 = \mathrm{d}x, \qquad \theta^2 = x\,\mathrm{d}y.$$

Its dual frame is given by

$$\frac{\partial}{\partial \theta^1} = \frac{\partial}{\partial x}, \qquad \frac{\partial}{\partial \theta^2} = \frac{1}{x}\frac{\partial}{\partial y}.$$

Compute the differentials:

$$\begin{aligned}
\mathrm{d}\theta^1 &= \mathrm{d}\,\mathrm{d}x = 0, \\
\mathrm{d}\theta^2 &= \mathrm{d}\left(x\,\mathrm{d}y\right) \\
&= \mathrm{d}x \wedge \mathrm{d}y \\
&= \frac{x}{x}\,\mathrm{d}x \wedge \mathrm{d}y \\
&= \frac{1}{x}\theta^1 \wedge \theta^2.
\end{aligned}$$

Thus $T^1_{12} = 0$ and $T^2_{12} = \frac{1}{x}$. Our rank-0 structure map thus only has 1 independent structure constant. To see if this is enough, compute the rank-1 structure map.

$$\begin{aligned}
T_{\{2,1,2,1\}} = \frac{\partial T^2_{12}}{\partial \theta^1} &= \frac{\partial \frac{1}{x}}{\partial x} \\
&= -\frac{1}{x^2} \\
T_{\{2,1,2,2\}} = \frac{\partial T^2_{12}}{\partial \theta^2} &= \frac{1}{x}\frac{\partial \frac{1}{x}}{\partial y} \\
&= 0
\end{aligned}$$

$T_{\{2,1,2,1\}}$ is a function of T^2_{12}: $T_{\{2,1,2,1\}} = -\left(T^2_{12}\right)^2$ and is thus not independent. Therefore, our coframe has rank 1 and order 0. Its 1^{th} order classifying set is given by

$$\mathcal{C}^1\left(\mathrm{d}x, x\,\mathrm{d}y, \mathbb{R}^2 \backslash \{0\}\right) = \left\{T^1_{12}, T^2_{12}, T_{1,1,2,1}, T_{1,1,2,2}, T_{2,1,2,1}, T_{2,1,2,2}\right\} = \left\{0, \frac{1}{x}, 0, 0, \frac{-1}{x^2}, 0\right\}.$$

For an equivalent coframe, consider

$$\bar{\theta}^1 = \mathrm{d}\bar{y}, \qquad \bar{\theta}^2 = \bar{y}\,\mathrm{d}\bar{x}. \tag{153}$$

Clearly, it is equivalent through the diffeomorphism Φ given as

$$\begin{aligned}\bar{x} &= \Phi^1(x,y) = y,\\ \bar{y} &= \Phi^2(x,y) = x.\end{aligned}$$

To prove the equivalence, calculate the pull-backs:

$$\begin{aligned}\Phi^*\left(\bar{\theta}^1\right) &= \Phi^*\left(\mathrm{d}\bar{y}\right)\\ &= \mathrm{d}x\\ &= \theta^1,\\ \Phi^*\left(\bar{\theta}^2\right) &= \Phi^*\left(\bar{y}\,\mathrm{d}\bar{x}\right)\\ &= x\,\mathrm{d}y\\ &= \theta^2.\end{aligned}$$

However, we can also compute the rank, order and classifying sets of $\bar{\boldsymbol{\theta}}$. First, the dual frame is now given by

$$\frac{\partial}{\partial\bar{\theta}^1} = \frac{\partial}{\partial\bar{y}}, \qquad \frac{\partial}{\partial\theta^2} = \frac{1}{\bar{y}}\frac{\partial}{\partial\bar{x}}. \tag{154}$$

The structure equations are

$$\begin{aligned}T^2_{12} &= \frac{1}{\bar{y}},\\ T_{2,1,2,1} &= \frac{-1}{\bar{y}^2},\end{aligned}$$

with all others equal to zero. The first-order classifying set is now given as

$$\mathcal{C}^1\left(\mathrm{d}y, y\,\mathrm{d}x, \mathbb{R}^2\backslash\{0\}\right) = \left\{T^1_{12}, T^2_{12}, T_{1,1,2,1}, T_{1,1,2,2}, T_{2,1,2,1}, T_{2,1,2,2}\right\} = \left\{0, \frac{1}{\bar{y}}, 0, 0, \frac{-1}{\bar{y}^2}, 0\right\},$$

which is identical, concluding this example.

10.2 Equivalence with structure-groups

In the previous section, the equivalence problem for coframes was solved. However, for the Lagrangians we do not require the strict equivalence. For example, we only require that the contact ideal $\mathcal{I}$ as a whole is preserved. If θ^α_I represents a contact form, then the relation $\Phi^*\bar{\theta}^\alpha_I = \theta^\alpha_I$ certainly conserves the contact ideal, but so does $\Phi^*\bar{\theta}^\alpha_I = \left(a^\alpha_\beta\right)^J_I\theta^\beta_J$, i.e. a linear transformation mixing the contact forms themselves. Thus, we need to somehow include these allowed transformations in the formulation of the equivalence problem. We do this by the use of structure groups, which are the groups of allowed transformations.

Definition 14 (G-equivalence problem [6])**.** Let $G \subseteq GL(n)$ be a Lie group. Let ω and $\bar{\omega}$ be coframes defined respectively on the n-dimensional manifolds M and $\bar{M}$. The *G-valued equivalence problem* for these coframes is to determine if there exists a diffeomorphism $\Phi : M \to \bar{M}$ and a G-valued function $g : M \to G$ with the property that

$$\Phi^*\bar{\omega} = g\,(x)\,\omega. \tag{155}$$

The group G is called the structure group of the problem.

We use ω instead of $\boldsymbol{\theta}$ to denote the coframes to differentiate between the regular equivalence problem and the G-valued equivalence problem.

There are two special cases to look at. The first is if $G = \{e\}$. In that case, (155) reduces to (144) and we have reduced the problem to the case described above, in section 10.1. The other case is if $G = GL(n)$. In that case, any two coframes are equivalent and the problem is trivial. Thus the interesting cases are those when $\{e\} \subset G \subset GL(n)$. We want to reduce this group G, through a series of invariant operations (to be defined later), to the trivial group $\{e\}$, which allows us to use the already developed machinery from subsection 10.1.

First, we make (155) more symmetric. Instead of just one G-valued function, we now search for two functions, $g : M \to G$ and $\bar{g} : \bar{M} \to G$ such that (155) becomes

$$\Phi^* \left[\bar{g}\left(\bar{x}\right)\bar{\omega}\right] = g\left(x\right)\omega. \tag{156}$$

If this can be done, then the two coframes defined by $\bar{\boldsymbol{\theta}} = \bar{g}\left(\bar{x}\right)\bar{\omega}$ and $\boldsymbol{\theta} = g\left(x\right)\omega$ are obviously invariant: $\Phi^*\bar{\boldsymbol{\theta}} = \boldsymbol{\theta}$. So, how to find these functions? The key lies in finding functions that depend on the group parameters g^i_j and maybe the coordinates x^k that are invariant or unchanged by the coordinate transformations. We can then use such a function to fix one of the group parameters[25]. More precisely, we are looking for scalar functions $H(g, \omega)$ that satisfy

$$H\left(\bar{g}\left(\bar{x}\right), \bar{\omega}|_{\bar{x}}\right) = H\left(g\left(x\right), \omega|_x\right), \tag{157}$$

whenever $\bar{x} = \Phi\left(x\right)$ and $\Phi^*\left(\bar{g}\left(\bar{x}\right)\bar{\omega}\right) = g\left(x\right)\omega$.

For each such a function H, we can then normalize one of the group parameters and set it equal to some constant or combination of the other group parameters. Of course, in doing so, the restrictions imposed by the group must be taken into account. E.g, if a group parameter must be non-zero, we can not normalize this parameter to zero. This reduces the diameter of the group G by one. If this is done often enough, the problem is reduced to the $\{e\}$-equivalence problem that was discussed in section 10.1 [6, Chapter 10].

The invariant combinations used are often the structure functions computed by taking the derivative of $\boldsymbol{\theta} = g\left(x\right)\omega$:

$$\begin{aligned}
\theta^i &= g^i_j\left(x\right)\omega^j \\
\mathrm{d}\theta^i &= \mathrm{d}\left(g^i_j\left(x\right)\omega^j\right) \\
&= \mathrm{d}\left(g^i_j\left(x\right)\right) \wedge \omega^j + g^i_j\left(x\right)\mathrm{d}\omega^j.
\end{aligned}$$

The two forms $\mathrm{d}\omega^j$ can be written as the wedge product of the one forms ω^i, since ω forms a coframe on M by construction. Furthermore, we can rewrite these one forms ω^i in terms of the one forms θ^i, resulting in

$$\mathrm{d}\theta^i = \gamma^i_j\theta^j + T^i_{jk}\theta^j \wedge \theta^k. \tag{158}$$

The functions T^i_{jk} are in general *not* invariants of the problem. In order to get to actual invariants, the process of *absorption* is used.

First, one notes that the functions γ^i_j can be written in the Maurer-Cartan basis [6, chapter 2] of the group G: $\gamma^i_j = A^i_{j\kappa}\alpha^\kappa$. Thus, (158) becomes

$$\mathrm{d}\theta^i = A^i_{j\kappa}\alpha^\kappa \wedge \theta^j + T^i_{jk}\theta^j \wedge \theta^k, \tag{159}$$

and for the equivalent frame:

$$\mathrm{d}\bar{\theta}^i = A^i_{j\kappa}\bar{\alpha}^\kappa \wedge \bar{\theta}^j + \bar{T}^i_{jk}\bar{\theta}^j \wedge \bar{\theta}^k. \tag{160}$$

[25]This fixing of parameters is also called normalization, because often the structure constants are normalized to zero or one.

Not all of the terms $A^i_{j\kappa}$ are non-zero. We also do not care about the exact form of the α^κ forms. Thus, we can choose to *absorb* other terms into the α^κ forms. The precise mathematical foundations can be found in [6, pages 307-310]. In particular, for a given α^κ wedging θ^j (i.e. $A^i_{j\kappa}$ is non-zero) we can redefine it as

$$\pi^\kappa = \alpha^\kappa - z^\kappa_i \theta^i, \tag{161}$$

with the z^κ_i satisfying

$$A^i_{j\kappa} z^\kappa_k - A^i_{k\kappa} z^\kappa_j = -T^i_{jk}. \tag{162}$$

With this substitution, (159) becomes

$$\mathrm{d}\theta^i = A^i_{j\kappa} \pi^\kappa \wedge \theta^j + U^i_{jk} \theta^j \wedge \theta^k. \tag{163}$$

The remaining non-zero terms U^i_{jk} are the *essential torsion*. These essential torsion coefficients may depend explicitly on the group-parameters. These are then normalized to convenient constant values (usually $\pm 1, 0$) by solving for the group-parameters. After thus fixing the group parameters, one substitutes the thus obtained formulae back into the formula for the coframe and recompute. This continues until either all group parameters are normalized and we have an explicit, invariant coframe, or we are left with some undetermined group parameters but no essential torsion to fix them. We shall deal with those cases in the next subsections.

To summarise, the procedure is as follows. One is given a coframe $\boldsymbol{\omega}$ and an structure group G. The invariant coframe is then given by $\boldsymbol{\theta} = g\boldsymbol{\omega}$ for a certain group element $g \in G$. Perform the process of normalization and absorption as described above to obtain the essential torsion elements and fix the group-parameters to fix g. Thus, the invariant coframe $\boldsymbol{\theta}$ is now fully determined, and one can compute the order, rank and classifying sets for this invariant frame.

Example 15. As an example, consider $M = \mathbb{R}^2 \setminus \{x = 0 \text{ or } y = 0\}$. As coframe, take

$$\omega^1 = y\,\mathrm{d}x, \qquad \omega^2 = x\,\mathrm{d}y.$$

For the structure group, use

$$g = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix},$$

for some constant $a \neq 0$. Then $(\theta^1, \theta^2)^T = g\,(\omega^1, \omega^2)^T$. Next, compute the differentials. First for θ^1:

$$\begin{aligned} \mathrm{d}\theta^1 &= \mathrm{d}\,(y\,\mathrm{d}x) \\ &= \mathrm{d}y \wedge \mathrm{d}x \\ &= \frac{\omega^2 \wedge \omega^1}{xy} \\ &= -\frac{1}{xya}\theta^1 \wedge \theta^2, \end{aligned}$$

and then for θ^2:

$$\begin{aligned} \mathrm{d}\theta^2 &= \mathrm{d}\,(ax\,\mathrm{d}y) \\ &= x\,\mathrm{d}a \wedge \mathrm{d}y + a\,\mathrm{d}x \wedge \mathrm{d}y \\ &= \mathrm{d}a \wedge \omega^2 + \frac{a}{xy}\omega^1 \wedge \omega^2 \\ &= \frac{\mathrm{d}a}{a} \wedge \theta^2 + \frac{1}{xy}\theta^1 \wedge \theta^2 \\ &= \alpha \wedge \theta^2. \end{aligned}$$

In the last line, I have absorbed $\frac{\theta^1}{xy}$ into $\alpha = \frac{\mathrm{d}a}{a} + \frac{\theta^1}{xy}$.

We are thus left with one essential torsion component $U^1_{12} = \frac{-1}{xya}$. This can be normalized to -1 by setting $a = \frac{1}{xy}$. After normalization, we repeat the procedure of computing the differentials. Again, first we look at θ^1:

$$\begin{aligned}\mathrm{d}\theta^1 &= \mathrm{d}\left(y\,\mathrm{d}x\right)\\ &= \mathrm{d}y \wedge \mathrm{d}x\\ &= \frac{\omega^2 \wedge \omega^1}{xy}\\ &= -\theta^1 \wedge \theta^2.\end{aligned}$$

The torsion component did not change, as expected. Next, look at θ^2:

$$\begin{aligned}\mathrm{d}\theta^2 &= \mathrm{d}\left(\frac{1}{y}\,\mathrm{d}y\right)\\ &= -\frac{1}{y^2}\,\mathrm{d}y \wedge \mathrm{d}y\\ &= 0,\end{aligned}$$

which thus yields no additional torsion components.

Thus, the final invariant frame is given by

$$\begin{aligned}\theta^1 &= y\,\mathrm{d}x\\ \theta^2 &= \frac{1}{y}\,\mathrm{d}y.\end{aligned}$$

With this invariant frame, we go through the strict algorithm. However, we have already done the first step, the differentials of θ^i. The torsion components are $T^1_{12} = -1$ and $T^2_{12} = 0$. Since both of these are constants, their derivatives are zero and we are left with a rank 0, order 1 system and classifying set

$$\mathcal{C}^1\left(\theta^1, \theta^2, M\right) = \left\{T^1_{12}, T^2_{12}, T_{1,1,2,1}, T_{1,1,2,2}, T_{2,1,2,1}, T_{2,1,2,2}\right\} = \{-1, 0, 0, 0, 0, 0\}$$

10.3 Involutive systems

It is possible that after a number of loops through the algorithm we have eliminated all essential torsion, but have not fixed all group parameters. The reason for this is the existence of a symmetry group of the coframe, and the dimension of this symmetry group changes the solution space [6, Chapter 11]. There are two cases to distinguish, as the dimension of the symmetry group is either finite or infinite, and each of these cases is dealt with through a different method.

First, we need to determine which case we're in. For this, we will use *Cartan's test for involutivity*. This test determines whether a system is in involution, which precisely means that the system has an infinite dimensional symmetry group [6].

Before we can head to the test, we have to define some new concepts.

Definition 15 (Degree of indeterminacy)**.** The absorption equations (162) may not be completely solvable, i.e. some of the functions z^κ_i are not fixed by solving these equations. The dimension of the solution space of (162) is the degree of indeterminacy of the system, and is labelled by $r^{(1)}$ [6].

$$A^i_{j\kappa} z^\kappa_k - A^i_{k\kappa} z^\kappa_j = -T^i_{jk}. \quad (162)$$

Next are the Cartan Characters of the system.

Definition 16 (Cartan Characters)**.** Let $\boldsymbol{v} \in \mathbb{R}^n$, with n the dimension of the manifold M. Then define $\mathsf{L}[\boldsymbol{v}] \in \mathbb{R}^{n\times r}$ as the matrix with elements

$$\mathsf{L}^i_\kappa[\boldsymbol{v}] = A^i_{j\kappa} v^j.$$

From this matrix, we can determine the n *reduced characters*. The first one is given by

$$s'_1 = \max\left\{\operatorname{rank}(\mathsf{L}[\boldsymbol{v}]) \mid \boldsymbol{v} \in \mathbb{R}^m\right\}.$$

The next ones, up to and including the $(m-1)^{\text{th}}$ one, are then subsequently given by

$$s'_1 + s'_2 + \cdots + s'_k = \max\left\{ \operatorname{rank}\begin{pmatrix} \mathsf{L}[\boldsymbol{v}_1] \\ \mathsf{L}[\boldsymbol{v}_2] \\ \vdots \\ \mathsf{L}[\boldsymbol{v}_k] \end{pmatrix} \;\middle|\; \boldsymbol{v}_1, \ldots \boldsymbol{v}_k \in \mathbb{R}^m \right\},$$

and the last one is defined by $s'_1 + s'_2 + \cdots + s'_n = r$, with r the dimension of the original, un-reduced structure group [6].

With these two concepts defined, we can now use Cartan's test for involution. According to him, a system of coframes $\boldsymbol{\theta}$ is involutive if [6]

$$s'_1 + 2s'_2 + \cdots + ns'_n = r^{(1)}. \tag{164}$$

If the system at hand is both involutive, and has only constant essential torsion remaining, then it is called transitive. Furthermore, the following theorem about equivalence is true [6]:

Theorem 10 (Transitive Equivalence)**.** *Let $\boldsymbol{\theta}$ and $\bar{\boldsymbol{\theta}}$ both be coframes on M and $\bar{M}$ respectively, with the same structure group G. If both $\boldsymbol{\theta}$ and $\bar{\boldsymbol{\theta}}$ are in involution, have only constant essential torsion, and the torsion coefficients are equal, then $\boldsymbol{\theta}$ and $\bar{\boldsymbol{\theta}}$ are equivalent.*

If the essential torsion is not constant, then we have the intransitive case. The next step is then to look at the derivatives of the essential torsion coefficients. It is possible that these *do* depend on some group parameters. In this case, use this dependence to fix the group parameters and normalize these derivative essential torsion and go through the algorithm again. Eventually, either all group parameters will be fixed or all essential torsion derivatives will be independent of the group parameters. For the latter case, we have the following theorem [6]

Theorem 11. *Let $\boldsymbol{\theta}$ and $\bar{\boldsymbol{\theta}}$ both be lifted coframes on M and $\bar{M}$ respectively, with the same structure group G. Furthermore, both $\boldsymbol{\theta}$ and $\bar{\boldsymbol{\theta}}$ are involutive and neither the essential torsion, nor its derivatives, depend on the group parameters. Then $\boldsymbol{\theta}$ and $\bar{\boldsymbol{\theta}}$ are equivalent only if they have the same order $s = \bar{s}$ and their $(s+1)^{th}$ order classifying manifolds $\mathcal{C}^{(s+1)}\left(\bar{\boldsymbol{\theta}}\right)$ and $\mathcal{C}^{(s+1)}\left(\boldsymbol{\theta}\right)$ overlap.*

10.4 Prolonged systems

In this subsection we will treat the non-involutive systems. These are systems with a finite symmetry group with dimension greater than 0. These systems then are defined on a too small base manifold to handle this additional symmetry group. The solution to this is to prolong the system onto a larger base manifold, and try to solve the system on this larger manifold.

The prolongation takes us from the base manifold M to the manifold $M \times G$, i.e. the base manifold appended with the *reduced* structure group. On this new, larger manifold, the coframe

will be formed by formed by the coframe $\boldsymbol{\theta}$ on M, appended by the modified Maurer-Cartan forms π^κ on G. These modified Maurer-Cartan forms are given by

$$\pi^\kappa = \alpha^\kappa - z_i^\kappa \theta^i,$$

where α^κ are the regular Maurer-Cartan forms on G, and the z_j^κ are the solutions to (162). If the degree of indeterminacy $r^{(1)} = 0$, the z_i^κ are uniquely determined and we have a fully defined coframe on $M \times G$. Thus, we can try to solve the system on $M \times G$.

However, maybe $r^{(1)} > 0$, which is called the *indeterminate* case. Now, we have $r^{(1)}$ free variables, which I denote by w^u, on which the z_i^κ depend. In this case, equivalence implies the existence that there exists both a $g = g(x) \in G$ and $w = w(x) \in \mathbb{R}^{r^{(1)}}$ under which both $\boldsymbol{\theta}$ and $\boldsymbol{\pi}$ are invariant. The appended coframe now comprises of both $\boldsymbol{\theta} = g \cdot \boldsymbol{\omega}$ and $\pi^\kappa = \alpha^\kappa + \mathsf{S}_j^\kappa \theta^j$, where S_j^κ is the particular solution to (162) with $w^u = 0$. In order to determine the functions $w^u(x)$ that are part of the equivalence procedure, we have an additional structure group $G^{(1)}$, with elements given by

$$G^{(1)} = \left\{ \begin{pmatrix} I_n & 0 \\ \mathsf{K}[w] & I_r \end{pmatrix} \;\middle|\; w \in \mathbb{R}^{r^{(1)}} \right\}, \tag{165}$$

with $\mathsf{K}[w]$ the homogeneous solution to (162), i.e. a solution to

$$A^i_{j\kappa} z^\kappa_k - A^i_{k\kappa} z^\kappa_j = 0. \tag{166}$$

The final coframe is then given by

$$\boldsymbol{\Theta} = g^{(1)} \cdot \begin{pmatrix} g \cdot \boldsymbol{\omega} \\ \boldsymbol{\pi} \end{pmatrix}. \tag{167}$$

Next, use the absorption and normalization procedure to determine the values for $w^u(x)$ and determine the remaining group parameters of $g(x)$.

Should one again terminate to a solution where not all group parameters are determined, the procedure is to see if the new, reduced system, is in involution or not. If not, again prolong the system by adding another copy of G. Keep doing this, until all group parameters are determined. This procedure will terminate (for non-involutive systems)[26] [6].

Thus, all possible branches of the equivalence problem have now been solved. Next, we cast the Lagrangian equivalence problem into differential form.

10.5 Application of Cartan's algorithm to Lagrangians

Cartan's algorithm allows us to check if two coframes attached to a manifold M are equivalent. We want to apply his algorithm to the problem of equivalence between Lagrangians. To do that, we need to rewrite (127) into a differential form, and set up the rest of the coframe to use in our equivalence problem. Last, we need to define the structure group to use.

As a reminder, we call two Lagrangians divergence equivalent if there exists a map $\Phi : J^k(\mathbb{R}^p, \mathbb{R}^q) \to J^k(\mathbb{R}^p, \mathbb{R}^q)$ and a function $F : \mathbb{R}^p \to \mathbb{R}$ such that (127)

$$\bar{\mathcal{L}}\left(\bar{x}^i, \bar{u}^{(r)}\right) = \frac{\mathcal{L}\left(x^i, u^{(r)}\right) + \nabla F}{|\det J|},$$

where J^i_j is the Jacobian associated with Φ. This relation opperates on the jet-bundle $J^k(\mathbb{R}^p, \mathbb{R}^q)$, and thus this is the manifold on which we need to define our coframe.

[26] This non-termination is why involutive systems are treated differently

The first elements of our coframe we find by considering the variational problem described by $\mathcal{L}$. It is given by the functional $S = \int \mathcal{L}\,\mathrm{d}\boldsymbol{x}$, with $\mathrm{d}\boldsymbol{x}$ the volume form. It is the integrand that is to be preserved, modulo a total divergence. From [42], the pull back of $\bar{\mathcal{L}}$ is

$$\Phi^*\left(\bar{\mathcal{L}}\,\mathrm{d}\bar{\boldsymbol{x}}\right) = \bar{\mathcal{L}}\det J\,\mathrm{d}\boldsymbol{x} + \bar{\mathcal{L}}\frac{\mathrm{d}x^i}{\mathrm{d}\bar{u}^\alpha}\theta^\alpha \wedge \mathrm{d}x^{\hat{i}},$$

where $\mathrm{d}x^{\hat{i}} = \prod_{j\in\{1,\ldots,p\}\setminus\{i\}} \wedge\, \mathrm{d}x^j$. Since we want divergence equivalence, we can use (127) to substitute $\bar{\mathcal{L}}\det J$. We also consider the contact ideal $\mathcal{I}$. The allowed transformations preserve this ideal. Furthermore, solutions to the variational problem live on the submanifold on which the contact forms vanish. Thus, we can remove $\frac{\mathrm{d}x^i}{\mathrm{d}\bar{u}^\alpha}\theta^\alpha \wedge \mathrm{d}x^{\hat{i}}$ by working $\mod \mathcal{I}$.

$$\Phi^*\left(\bar{\mathcal{L}}\,\mathrm{d}\bar{\boldsymbol{x}}\right) = \mathcal{L}\,\mathrm{d}\boldsymbol{x} + \nabla F\,\mathrm{d}\boldsymbol{x} \mod \mathcal{I}. \tag{168}$$

Define the one-forms $\omega^i = \sqrt[p]{\mathcal{L}}\,\mathrm{d}x^i$. These obey $\omega_1 \wedge \cdots \wedge \omega_p = \mathcal{L}\,\mathrm{d}\boldsymbol{x}$. We are interested in how these one-forms transform under fiber-preserving, point and contact transformations. Since we want (168) to hold for the wedge product of ω^i, a general transformation can be written as [42]

$$\begin{aligned}\Phi^*\left(\bar{\omega}^i\right) &= \left(B_0\right)^i_\alpha\,\theta^\alpha + \left(B_1\right)^{i,j}_\alpha\,\theta^\alpha_j + \left(B_2\right)^{i,J}_\alpha\,\theta^\alpha_J + J^i_j\omega^j \\ &= J^i_j\omega^j \mod \mathcal{I}.\end{aligned}$$

If Φ is a point transformation, we have $B_2 = 0$, and if Φ is a fiber-preserving transformation, we have $B_2 = 0$ and $B_1 = 0$, as a consequence of Bäcklund's theorem [42].

The transformation laws for the contact forms were already given in the proof of theorem 5, they are

$$\begin{aligned}\Phi^*\left(\bar{\theta}^\alpha\right) &= \frac{\mathrm{d}\psi^\alpha}{\mathrm{d}u^\beta}\theta^\beta, \\ \Phi^*\left(\bar{\theta}^\alpha_I\right) &= \frac{\mathrm{d}\psi^\alpha_I}{\mathrm{d}u^\beta}\theta^\beta + \sum_{|J|\leqslant|I|}\frac{\mathrm{d}\psi^\alpha_I}{\mathrm{d}p^\beta_J}\theta^\beta_J.\end{aligned} \tag{169}$$

These transformations are all linear transformations, and thus we can follow Kamran and Olver [42]; we rescale the contact forms and place them into a column vector. Define

$$\boldsymbol{\theta} = \left(\theta_0, \mathcal{L}^{-1/p}\theta_1, \mathcal{L}^{-2/p}\theta_2, \ldots, \mathcal{L}^{-(r-1)/p}\theta_{r-1}\right),$$

so that (169) becomes

$$\bar{\boldsymbol{\theta}} = \mathsf{A}\boldsymbol{\theta}, \tag{170}$$

where A is a block-lower triangular matrix with the diagonal blocks satisfying $A_k = A_0 \otimes \odot^k J^{-T}$ [42]. With these two equations (168) and (170), we have given part of our coframe and defined the structure group for this coframe. These one-forms however do not span a full co-frame on the jet-space. We are lacking coframe elements corresponding to the r^{th}-order derivatives. Thus, append the coframe by adding elements [42]

$$\pi^\alpha_I = \mathcal{L}^{-r/p}\,\mathrm{d}p^\alpha_I,$$

with $|I| = r$. The transformation law for these elements are governed by the fact that $\bar{p}^\alpha_I$ has to obey the chain rule.

Grouping this all together, we now have the following transformation matrix group of one-forms

$$\begin{pmatrix}\bar{\boldsymbol{\theta}} \\ \bar{\boldsymbol{\omega}} \\ \bar{\boldsymbol{\pi}}\end{pmatrix} = \begin{pmatrix}A & 0 & 0 \\ B & J & 0 \\ C & D\cdot J & E\end{pmatrix}\begin{pmatrix}\boldsymbol{\theta} \\ \boldsymbol{\omega} \\ \boldsymbol{\pi}\end{pmatrix}.$$

The restrictions on these block-matrices can be found in [42, page 378]. We now need to insert the divergence into this equation. We do this through considering the Jacobian J^i_j. Introduce p additional coordinates w^i, which transform according to $\bar{w}^i = w^i + F^i$. We are thus no longer working on J^r, but on $J^r \times \mathbb{R}^p$ and we need to extend our coframe accordingly. Like Kamran and Olver [42], append the co-frame with the one forms $v^i = \mathcal{L}^{1/p-1}\,\mathrm{d}w^i$. With this addition, the transformation law becomes

$$\begin{pmatrix} \bar{\boldsymbol{\theta}} \\ \bar{\boldsymbol{\omega}} \\ \bar{\boldsymbol{\pi}} \\ \bar{\boldsymbol{v}} \end{pmatrix} = \begin{pmatrix} A & 0 & 0 & 0 \\ B & J & 0 & 0 \\ C & D\cdot J & E & 0 \\ Q & T\cdot R & T\cdot M & T \end{pmatrix} \begin{pmatrix} \boldsymbol{\theta} \\ \boldsymbol{\omega} \\ \boldsymbol{\pi} \\ \boldsymbol{v} \end{pmatrix}. \tag{171}$$

The exact restrictions on the block-matrices can again be found in [42, page 378], but the important one is that we restrict $\det J = 1 + \mathrm{trace}R$.

Next, we show that this transformation law indeed transforms the Lagrangian as in (127). Let $\left(b^i_\mu\right)$ be the entries of B, J^i_j the entries of J and $\mathcal{I}$ the contact ideal generated by the contact forms $\boldsymbol{\theta}$[27]. Under the transformation (171), the Lagrangian $\mathcal{L}$ becomes

$$\begin{aligned}
\bar{\mathcal{L}}\,\mathrm{d}\bar{x} &= \bar{\omega}^1 \wedge \bar{\omega}^2 \wedge \cdots \wedge \bar{\omega}^p \\
&= \left(b^1_\mu\theta^\mu + J^1_i\omega^i\right) \wedge \left(b^2_\mu\theta^\mu + J^2_i\omega^i\right) \wedge \cdots \wedge \left(b^p_\mu\theta^\mu + J^p_i\omega^i\right) \\
&= \det J \cdot \omega^1 \wedge \omega^2 \wedge \cdots \wedge \omega^p \mod \mathcal{I} \\
&= (1 + \mathrm{trace}R)\cdot\omega^1 \wedge \omega^2 \wedge \cdots \wedge \omega^p \mod \mathcal{I} \\
&= \mathcal{L}\,\mathrm{d}x + \mathrm{trace}R\cdot\mathcal{L}\,\mathrm{d}x \mod \mathcal{I},
\end{aligned}$$

To find the trace of R we look at the transformation law for $\boldsymbol{v}$ from two perspectives. First, using the transformation (171);

$$\begin{aligned}
\bar{v}^i &= q^i_\mu\theta^\mu + \frac{1}{\det J}j^i_k r^k_j\omega^j + \frac{1}{\det J}j^i_k m^{kI}_\alpha \pi^\alpha_I + \frac{1}{\det J}j^i_j v^j \\
&= q^i_\mu\theta^\mu + \frac{1}{\det J}j^i_k r^k_j\omega^j + \frac{1}{\det J}j^i_k m^{kI}_\alpha \pi^\alpha_I + \frac{1}{\det J}j^i_j \mathcal{L}^{(1/p)-1}\,\mathrm{d}w^j,
\end{aligned}$$

and next, using the definition of v:

$$\begin{aligned}
\bar{v}^i &= \bar{\mathcal{L}}^{(1/p)-1}\,\mathrm{d}\bar{w}^i \\
&= \bar{\mathcal{L}}^{(1/p)-1}\,\mathrm{d}\left(w^i + F^i\right) \\
&= \bar{\mathcal{L}}^{(1/p)-1}\left(\mathrm{d}w^i + F^i_{x^j}\,\mathrm{d}x^j + F_{\theta^\mu}\theta^\mu\right).
\end{aligned}$$

We compare these two transformation laws. Using the linear indepence of the one-forms $\mathrm{d}w^i$ and ω^i, we see:

$$\begin{aligned}
\bar{\mathcal{L}}^{(1/p)-1}\,\mathrm{d}w^m &= \frac{1}{\det J}j^m_n\mathcal{L}^{(1/p)-1}\,\mathrm{d}w^n, \\
\bar{\mathcal{L}}^{(1/p)-1}F^i_{x^j}\mathcal{L}^{-(1/p)}\omega^j &= \frac{1}{\det J}j^i_k r^k_j\omega^j.
\end{aligned} \tag{172}$$

To distil the trace of R out of this, first rewrite the second line,

$$\delta^m_j\bar{\mathcal{L}}^{(1/p)-1} = \frac{\mathcal{L}^{(1/p)}}{\det J}\left(F^{-1}_x\right)^m_i\, j^i_k r^k_j$$

[27] Here θ^μ runs over the entire $\boldsymbol{\theta}$. The index μ thus incorporates both the index α and the multi-indices I.

and then substitute this into (172)

$$\begin{aligned}
\frac{\mathcal{L}^{(1/p)}}{\det J}\left(F_x^{-1}\right)_i^m j_k^i r_j^k \,\mathrm{d}w^j &= \frac{1}{\det J} j_j^m \mathcal{L}^{(1/p)-1}\,\mathrm{d}w^j \\
F_{x^m}^l \left(F_x^{-1}\right)_i^m j_k^i r_j^k &= \mathcal{L}^{-1} F_{x^m}^l j_j^m \\
\left(j^{-1}\right)_l^h j_k^l r_j^k &= \mathcal{L}^{-1}\left(j^{-1}\right)_l^h F_{x^m}^l j_j^m \\
r_j^h = \mathcal{L}^{-1}\left(j^{-1}\right)_l^h F_{x^m}^l j_j^m &
\end{aligned}$$

Thus the trace of R is given by

$$\begin{aligned}
r_h^h &= \mathcal{L}^{-1}\left(j^{-1}\right)_l^h F_{x^m}^l j_h^m \\
&= \mathcal{L}^{-1} F_{x^m}^l \delta_l^m \\
&= \mathcal{L}^{-1}\mathrm{div}F
\end{aligned}$$

and the transformation for $\mathcal{L}\,\mathrm{d}x$ becomes

$$\begin{aligned}
\bar{\mathcal{L}}\,\mathrm{d}\bar{x} &= \mathcal{L}\,\mathrm{d}x + \mathrm{trace}R\cdot\mathcal{L}\,\mathrm{d}x \mod \mathcal{I} \\
&= \mathcal{L}\,\mathrm{d}x + \mathrm{div}F\,\mathrm{d}x \mod \mathcal{I},
\end{aligned}$$

as desired.

To conclude, in this section we have written the Lagrangian equivalence problem (127) into a form to which we can apply Cartan's equivalence algorithm. In the next section, we will apply the algorithm to the specific problem of determining if there is an equivalence between second-order Ostrogradsky free and regular first-order Lagrangians for the case of mechanics, i.e. only 1 independent variable.

11 The case for 1 independent, and q dependent variables, all second order

In this section, we use the Cartan equivalence algorithm to determine whether second-order, Ostrogradsky-free Lagrangians are divergent equivalent to regular first-order theories. As the title says, in this section we will deal with Lagrangians that are second order in all of their dependent variables, i.e. $\frac{\partial \mathcal{L}}{\partial \ddot{u}^\alpha} \neq 0$. Of course, we do want the Lagrangian as a whole to be healthy, i.e. we enforce that (30) and (31) hold.

Let us set-up the coframe needed. We are working on $J^2(\mathbb{R}, \mathbb{R}^q)$, on which we define the coframe according to [42]:

$$\begin{aligned}
\theta^\alpha &= \mathrm{d}u^\alpha - \dot{u}^\alpha \,\mathrm{d}x \\
\theta^\alpha_x &= \mathrm{d}\dot{u}^\alpha - \ddot{u}^\alpha \,\mathrm{d}x \\
\omega &= \mathcal{L}\,\mathrm{d}x \\
\pi^\alpha &= \mathcal{L}^{-2}\,\mathrm{d}\ddot{u}^\alpha \\
v &= \mathrm{d}w
\end{aligned}$$

Including normalization on the contact forms, the coframe vector is $\Sigma = (\boldsymbol{\theta}, \mathcal{L}^{-1}\boldsymbol{\theta}_{\boldsymbol{x}}, \omega, \boldsymbol{\pi}, v)$. Since $q > 1$, according to Bäcklund's Theorem all contact transformations are prolongations of a point transformation [42]. Therefore, we the most general transformation we use in this case are point transformations. In this case then, the structure group is given by the $(q+q+1+q+1)^2$-block matrix

$$\mathsf{H} = \begin{pmatrix}
(A_1)^\alpha_\beta & 0 & 0 & 0 & 0 \\
(A_2)^\alpha_\beta & \frac{1}{j}(A_1)^\alpha_\beta & 0 & 0 & 0 \\
(b_1)_\beta & 0 & j & 0 & 0 \\
(C_1)^\alpha_\beta & (C_2)^\alpha_\beta & (d_1)^\alpha & \frac{1}{j^2}(A_1)^\alpha_\beta & 0 \\
(q_1)_\beta & (q_2)_\beta & j-1 & 0 & 1
\end{pmatrix}. \tag{173}$$

However, we can be more restrictive than just that. In [8] it was already shown that $p = 1$, $r = 2$, $q =$ arbitrary, all second-order, Ostrogradsky-free Lagrangians are divergence equivalent to first order fields using only a total derivative, i.e. without a transformation to the (in)dependent coordinates. We have discussed this in section 9 using theorem 6. Since we want to check this result using Cartan's algorithm, we can restrict the structure group to not include any transformation on the dependent coordinates. A positive result in this restricted case, is still valid for the unrestricted case. In our restriction, the only allowed transformations are transformations purely on the independent coordinate and the addition of a total derivative. After this restriction, the structure group matrix is given by

$$\mathsf{H} = \begin{pmatrix}
\delta^\alpha_\beta & 0 & 0 & 0 & 0 \\
0 & \frac{\delta^\alpha_\beta}{j} & 0 & 0 & 0 \\
0 & 0 & j & 0 & 0 \\
0 & 0 & 0 & \frac{\delta^\alpha_\beta}{j^2} & 0 \\
(q_1)_\beta & (q_2)_\beta & j-1 & 0 & 1
\end{pmatrix}. \tag{174}$$

The inverse transformation is also required to calculate the differentials in the proper co-frame. It is given by

$$\mathsf{H}^{-1} = \begin{pmatrix}
\delta^\alpha_\beta & 0 & 0 & 0 & 0 \\
0 & j\delta^\alpha_\beta & 0 & 0 & 0 \\
0 & 0 & \frac{1}{j} & 0 & 0 \\
0 & 0 & 0 & j^2\delta^\alpha_\beta & 0 \\
-(q_1)_\beta & -j(q_2)_\beta & \frac{j-1}{j} & 0 & 1
\end{pmatrix}.$$

Next we need the differentials. The total differential of a function F is given by

$$\mathrm{d}F\left(x,u^{\alpha},\dot{u}^{\alpha},\ddot{u}^{\alpha}\right)=F_{u^{\alpha}}\bar{\theta}^{\alpha}+j\mathcal{L}F_{\dot{u}^{\alpha}}\bar{\theta}_{x}^{\alpha}+\frac{D_{x}F}{j\mathcal{L}}\bar{\omega}+j^{2}\mathcal{L}^{2}F_{\ddot{u}^{\alpha}}\bar{\pi}^{\alpha}$$

and the differential $\mathrm{d}\Sigma$ is given by

$$\begin{aligned}
\mathrm{d}\theta^{\alpha} &= -\bar{\theta}_{x}^{\alpha}\wedge\bar{\omega}\\
\frac{1}{\mathcal{L}}\,\mathrm{d}\theta_{x}^{\alpha} &= j\bar{\omega}\wedge\bar{\pi}^{\alpha}\\
\mathrm{d}\omega &= \frac{\mathrm{d}\mathcal{L}}{\mathcal{L}}\wedge\omega\\
&= \frac{\mathrm{d}\mathcal{L}}{j\mathcal{L}}\wedge\bar{\omega}\\
&= \frac{\mathcal{L}_{u^{\alpha}}}{j\mathcal{L}}\bar{\theta}^{\alpha}\wedge\bar{\omega}+\mathcal{L}_{\dot{u}^{\alpha}}\bar{\theta}_{x}^{\alpha}\wedge\bar{\omega}-j\mathcal{L}\mathcal{L}_{\ddot{u}^{\alpha}}\bar{\omega}\wedge\bar{\pi}^{\alpha}\\
\mathrm{d}\pi^{\alpha} &= -2j^{2}\frac{\mathcal{L}_{u^{\beta}}}{\mathcal{L}}\bar{\theta}^{\beta}\wedge\bar{\pi}^{\alpha}-2j^{3}\mathcal{L}_{\dot{u}^{\beta}}\bar{\theta}_{x}^{\beta}\wedge\bar{\pi}^{\alpha}-2j\frac{D_{x}\mathcal{L}}{\mathcal{L}^{2}}\bar{\omega}\wedge\bar{\pi}^{\alpha}-2j^{4}\mathcal{L}\mathcal{L}_{\ddot{u}^{\beta}}\bar{\pi}^{\beta}\wedge\bar{\pi}^{\alpha}\\
&= -2j^{3}\frac{\mathrm{d}\mathcal{L}}{\mathcal{L}}\wedge\bar{\pi}^{\alpha}
\end{aligned}$$

The first part of $\mathrm{d}\bar{\Sigma}$ is relatively easy to compute:

$$\begin{aligned}
\mathrm{d}\bar{\theta}^{\alpha} &= \delta_{\beta}^{\alpha}\,\mathrm{d}\theta^{\beta}=-\bar{\theta}_{x}^{\alpha}\wedge\bar{\omega}\\
\mathrm{d}\bar{\theta}_{x}^{\alpha} &= \delta_{\beta}^{\alpha}\,\mathrm{d}\frac{1}{j\mathcal{L}}\theta_{x}^{\beta}\\
&= -\frac{1}{j}\,\mathrm{d}j\wedge\bar{\theta}_{x}^{\alpha}-\frac{1}{\mathcal{L}}\,\mathrm{d}\mathcal{L}\wedge\bar{\theta}_{x}^{\alpha}+\bar{\omega}\wedge\bar{\pi}^{\alpha}\\
&= -\iota\wedge\bar{\theta}_{x}^{\alpha}+\bar{\omega}\wedge\bar{\pi}^{\alpha}\\
\mathrm{d}\bar{\omega} &= \mathrm{d}j\omega\\
&= \mathrm{d}j\wedge\omega+j\,\mathrm{d}\omega\\
&= \iota\wedge\bar{\omega}\\
\mathrm{d}\bar{\pi}^{\alpha} &= \delta_{\beta}^{\alpha}\,\mathrm{d}\frac{1}{j^{2}}\pi^{\beta}\\
&= -\frac{2}{j^{3}}\,\mathrm{d}j\wedge\pi^{\alpha}+\frac{1}{j^{2}}\,\mathrm{d}\pi^{\alpha}\\
&= -2\iota\wedge\bar{\pi}^{\alpha},
\end{aligned}$$

where we use the substitution $\iota=\frac{\mathrm{d}j}{j}+\frac{\mathrm{d}\mathcal{L}}{\mathcal{L}}$. The last differential is given by

$$\begin{aligned}
\mathrm{d}\bar{v} &= \mathrm{d}\left(\left(q_{1}\right)_{\beta}\theta^{\beta}\right)+\mathrm{d}\left(\left(q_{2}\right)_{\beta}\frac{1}{\mathcal{L}}\theta_{x}^{\beta}\right)+\mathrm{d}\left(\left(j-1\right)\omega\right)+\mathrm{d}v\\
&= \mathrm{d}\left(q_{1}\right)_{\beta}\wedge\theta^{\beta}+\left(q_{1}\right)_{\beta}\mathrm{d}\theta^{\beta}+\mathrm{d}\left(q_{2}\right)_{\beta}\wedge\frac{1}{\mathcal{L}}\theta_{x}^{\beta}+\left(q_{2}\right)_{\beta}\mathrm{d}\left(\frac{1}{\mathcal{L}}\theta_{x}^{\beta}\right)+\mathrm{d}j\wedge\omega+\left(j-1\right)\mathrm{d}\omega\\
&= \left(\xi_{1}\right)_{\alpha}\wedge\bar{\theta}^{\alpha}+\left(\xi_{2}\right)_{\alpha}\wedge\bar{\theta}_{x}^{\alpha}+\iota\wedge\bar{\omega}+j\left[\left(q_{2}\right)_{\alpha}+\mathcal{L}\mathcal{L}_{\ddot{u}^{\alpha}}\right]\bar{\omega}\wedge\bar{\pi}^{\alpha}
\end{aligned}$$

These equations give 1 set of essential torsion coefficients to be normalized: $j\left[\left(q_{2}\right)_{\alpha}+\mathcal{L}\mathcal{L}_{\ddot{u}^{\alpha}}\right]\bar{\omega}\wedge\bar{\pi}^{\alpha}$. We normalize to zero by fixing $\left(q_{2}\right)_{\alpha}=-\mathcal{L}\mathcal{L}_{\ddot{u}^{\alpha}}$

With this normalization, we enter the second round. Since this choice of group parameters does not affect the first 4 groups of coframe elements, we only need to consider the last coefficient. Again, we have

$$\begin{aligned}
\mathrm{d}\bar{v} &= \mathrm{d}\left(\left(q_1\right)_\beta \theta^\beta\right) - \mathrm{d}\left(\mathcal{L}_{\ddot{u}^\beta}\theta_x^\beta\right) + \mathrm{d}\left(\left(j-1\right)\omega\right) + \mathrm{d}v \\
&= \mathrm{d}\left(q_1\right)_\beta \wedge \theta^\beta + \left(q_1\right)_\beta \mathrm{d}\theta^\beta - \mathrm{d}\mathcal{L}_{\ddot{u}^\beta} \wedge \theta_x^\beta - \mathcal{L}_{\ddot{u}^\beta}\,\mathrm{d}\theta_x^\beta + \mathrm{d}j \wedge \omega + j\,\mathrm{d}\omega - \mathrm{d}\omega \\
&= \left(\xi_1\right)_\alpha \wedge \bar{\theta}^\alpha + \iota \wedge \omega - \left[\left(q_1\right)_\alpha + \mathcal{L}_{\dot{u}^\alpha} - D_x\mathcal{L}_{\ddot{u}^\alpha}\right]\bar{\theta}_x^\alpha \wedge \bar{\omega}
\end{aligned}$$

Again, there is only one set of coefficients to normalize, $-\left[\left(q_1\right)_\alpha + \mathcal{L}_{\dot{u}^\alpha} - D_x\mathcal{L}_{\ddot{u}^\alpha}\right]\bar{\theta}_x^\alpha \wedge \bar{\omega}$. This can be normalized to zero by choosing $\left(q_1\right)_\beta = D_x\mathcal{L}_{\ddot{u}^\beta} - \mathcal{L}_{\dot{u}^\beta}$

We now need to compute $\mathrm{d}\bar{v}$ for the third time.

$$\begin{aligned}
\mathrm{d}\bar{v} &= \mathrm{d}\left(\left(D_x\mathcal{L}_{\ddot{u}^\beta} - \mathcal{L}_{\dot{u}^\beta}\right)\theta^\beta\right) - \mathrm{d}\left(\mathcal{L}_{\ddot{u}^\beta}\theta_x^\beta\right) + \mathrm{d}\left(\left(j-1\right)\omega\right) + \mathrm{d}v \\
&= \mathrm{d}D_x\mathcal{L}_{\ddot{u}^\beta} \wedge \theta^\beta - \mathrm{d}\mathcal{L}_{\dot{u}^\beta} \wedge \bar{\theta}^\beta + \left(D_x\mathcal{L}_{\ddot{u}^\beta} - \mathcal{L}_{\dot{u}^\beta}\right)\mathrm{d}\theta^\beta - \mathrm{d}\mathcal{L}_{\ddot{u}^\beta} \wedge \theta_x^\beta - \mathcal{L}_{\ddot{u}^\beta}\,\mathrm{d}\theta_x^\beta + \mathrm{d}j \wedge \omega + j\,\mathrm{d}\omega - \mathrm{d}\omega \\
&= \iota \wedge \bar{\omega} + \left(D_x\mathcal{L}_{\ddot{u}^{[\beta}u^{\alpha]}} - \mathcal{L}_{\dot{u}^{[\beta}u^{\alpha]}}\right)\bar{\theta}^\alpha \wedge \bar{\theta}^\beta + j\mathcal{L}\left(\mathcal{L}_{\dot{u}^\alpha\dot{u}^\beta} - \mathcal{L}_{\ddot{u}^\beta u^\alpha} - D_x\mathcal{L}_{\ddot{u}^\alpha\dot{u}^\beta} - \mathcal{L}_{\ddot{u}^\alpha u^\beta}\right)\bar{\theta}^\alpha \wedge \bar{\theta}_x^\beta \\
&- \frac{D_xD_x\mathcal{L}_{\ddot{u}^\beta} - D_x\mathcal{L}_{\dot{u}^\beta} + \mathcal{L}_{u^\beta}}{j\mathcal{L}}\bar{\theta}^\beta \wedge \bar{\omega}
\end{aligned}$$

This calculation leaves us with three sets of essential torsion coefficients. $\left(D_x\mathcal{L}_{\ddot{u}^{[\beta}u^{\alpha]}} - \mathcal{L}_{\dot{u}^{[\beta}u^{\alpha]}}\right)\bar{\theta}^\alpha \wedge \bar{\theta}^\beta$ can not be normalized, since it does not contain any group parameters. As such, it is a fundamental invariant of the system. $j\mathcal{L}\left(\mathcal{L}_{\dot{u}^\alpha\dot{u}^\beta} - \mathcal{L}_{\ddot{u}^\beta u^\alpha} - D_x\mathcal{L}_{\ddot{u}^\alpha\dot{u}^\beta} - \mathcal{L}_{\ddot{u}^\alpha u^\beta}\right)\bar{\theta}^\alpha \wedge \bar{\theta}_x^\beta$ can be normalized. We can not normalize to 0, since $j = 0$ is not allowed. Normalization to δ_β^α is also not possible, since we have only 1 parameter to fix. Thus, instead normalize to $\left(\mathcal{L}_{\dot{u}^\alpha\dot{u}^\beta} - \mathcal{L}_{\ddot{u}^\beta u^\alpha} - D_x\mathcal{L}_{\ddot{u}^\alpha\dot{u}^\beta} - \mathcal{L}_{\ddot{u}^\alpha u^\beta}\right)\bar{\theta}^\alpha \wedge \bar{\theta}_x^\beta$ by fixing $j = \mathcal{L}^{-1}$. The last torsion coefficient is $\frac{D_xD_x\mathcal{L}_{\ddot{u}^\beta} - D_x\mathcal{L}_{\dot{u}^\beta} + \mathcal{L}_{u^\beta}}{j\mathcal{L}}\bar{\theta}^\beta \wedge \bar{\omega} = EL\left(\mathcal{L}\right)_{u^\beta}\bar{\theta}^\beta \wedge \bar{\omega}$[28], i.e. the Euler-Lagrange equations for this system. The Euler-Lagrange equations can thus also not be normalized, and thus are also fundamental invariants of the system.

With j thus chosen, all group parameters are fixed and the final structure group matrix looks like

$$\mathsf{H} = \begin{pmatrix}
\delta_\beta^\alpha & 0 & 0 & 0 & 0 \\
0 & \mathcal{L}\delta_\beta^\alpha & 0 & 0 & 0 \\
0 & 0 & \mathcal{L}^{-1} & 0 & 0 \\
0 & 0 & 0 & \mathcal{L}^2\delta_\beta^\alpha & 0 \\
D_x\mathcal{L}_{\ddot{u}^\beta} - \mathcal{L}_{\dot{u}^\beta} & -\mathcal{L}\mathcal{L}_{\ddot{u}^\alpha} & \mathcal{L}^{-1} - 1 & 0 & 1
\end{pmatrix}. \tag{175}$$

Let us compute the differentials of our invariant coframe.

$$\begin{aligned}
\mathrm{d}\bar{\theta}^\alpha &= \mathrm{d}\theta^\alpha = -\bar{\theta}_x^\alpha \wedge \bar{\omega}, \\
\mathrm{d}\bar{\theta}_x^\alpha &= \mathrm{d}\left(\mathcal{L}\frac{1}{\mathcal{L}}\theta_x^\alpha\right) = \mathrm{d}\theta_x^\alpha = \bar{\omega} \wedge \bar{\pi}^\alpha, \\
\mathrm{d}\bar{\omega} &= \mathrm{d}\left(\mathcal{L}^{-1}\omega\right) = \mathrm{d}\left(\mathcal{L}^{-1}\mathcal{L}\,\mathrm{d}x\right) = 0, \\
\mathrm{d}\bar{\pi}^\alpha &= \mathrm{d}\left(\mathcal{L}^2\pi^\alpha\right) = \mathrm{d}\left(\mathcal{L}^2\mathcal{L}^{-2}\,\mathrm{d}\ddot{u}^\alpha\right) = 0,
\end{aligned}$$

[28] In this step, I have both used the normalization $j = \mathcal{L}^{-1}$, and that for the E-L equations the truncated differential and the regular total differential match-up.

and the last differential is

$$\begin{aligned}
\mathrm{d}\bar{v} &= \mathrm{d}\left(\left(D_x\mathcal{L}_{\ddot{u}^\beta} - \mathcal{L}_{\dot{u}^\beta}\right)\theta^\beta\right) - \mathrm{d}\left(\mathcal{L}_{\ddot{u}^\beta}\theta_x^\beta\right) + \mathrm{d}\left(\left(\mathcal{L}^{-1} - 1\right)\omega\right) + \mathrm{d}v \\
&= \left(D_x\mathcal{L}_{\ddot{u}^\beta u^\alpha} - \mathcal{L}_{\dot{u}^\beta u^\alpha}\right)\bar{\theta}^\alpha \wedge \bar{\theta}^\beta + \left(\mathcal{L}_{\dot{u}^\alpha\dot{u}^\beta} - \mathcal{L}_{\ddot{u}^\beta u^\alpha} - D_x\mathcal{L}_{\ddot{u}^\alpha\dot{u}^\beta} - \mathcal{L}_{\ddot{u}^\alpha u^\beta}\right)\bar{\theta}^\alpha \wedge \bar{\theta}_x^\beta \\
&\quad - \left(D_xD_x\mathcal{L}_{\ddot{u}^\beta} - D_x\mathcal{L}_{\dot{u}^\beta} + \mathcal{L}_{u^\beta}\right)\bar{\theta}^\beta \wedge \bar{\omega}
\end{aligned}$$

We thus have two constant torsion coefficients; $T^{\theta^\alpha}_{\theta^\alpha_x\omega} = -1$ and $T^{\theta^\alpha_x}_{\omega\pi^\alpha} = 1$; and three sets of non-constant torsion coefficients; $T^v_{\theta^\alpha\theta^\beta} = \left(D_x\mathcal{L}_{\ddot{u}^\beta u^\alpha} - \mathcal{L}_{\dot{u}^\beta u^\alpha}\right)$, $T^v_{\theta^\alpha\theta^\beta_x} = \left(\mathcal{L}_{\dot{u}^\alpha\dot{u}^\beta} - \mathcal{L}_{\ddot{u}^\beta u^\alpha} - D_x\mathcal{L}_{\ddot{u}^\alpha\dot{u}^\beta} - \mathcal{L}_{\ddot{u}^\alpha u^\beta}\right)$ and $T^v_{\theta^\alpha\omega} = -\left(D_xD_x\mathcal{L}_{\ddot{u}^\alpha} - D_x\mathcal{L}_{\dot{u}^\alpha} + \mathcal{L}_{u^\alpha}\right)$; on our invariant coframe. Since the wedge product anti-commutes, the $T^v_{\theta^\alpha\theta^\beta}$ only yield $\frac{1}{2}q\left(q-1\right)$ independent coefficients. Similarly, due to the symmetric nature of $T^v_{\theta^\alpha\theta^\beta_x}$, this coefficient only gives us another $\frac{1}{2}q\left(q+1\right)$ coefficients. The Euler-Lagrange equations contribute another q invariants. Together, the coefficients give q^2+q+1 independent coefficients, and therefore the zero$^{\text{th}}$ order classifying set has an order of (at most) q^2+q+1. Thus, the zero$^{\text{th}}$ order classifying manifold is given by

$$\begin{aligned}
&\left\{T^{\theta^\alpha}_{\theta^\alpha_x\omega}, T^v_{\theta^\alpha\theta^\beta}, T^v_{\theta^\alpha\theta^\beta_x}, T^v_{\theta^\alpha\omega}\right\} = \\
&\left\{-1, \left(D_x\mathcal{L}_{\ddot{u}^\beta u^\alpha} - \mathcal{L}_{\dot{u}^\beta u^\alpha}\right), \left(\mathcal{L}_{\dot{u}^\alpha\dot{u}^\beta} - \mathcal{L}_{\ddot{u}^\beta u^\alpha} - D_x\mathcal{L}_{\ddot{u}^\alpha\dot{u}^\beta} - \mathcal{L}_{\ddot{u}^\alpha u^\beta}\right), -\left(D_xD_x\mathcal{L}_{\ddot{u}^\alpha} - D_x\mathcal{L}_{\dot{u}^\alpha} + \mathcal{L}_{u^\alpha}\right)\right\}
\end{aligned}$$

Since we work on $J^2\left(\mathbb{R},\mathbb{R}^q\right)$, there can be no more than $\dim J^2\left(\mathbb{R},\mathbb{R}^q\right) = 3q+2$ independent coefficients. Therefore, for $q \geqslant 3$, the zero$^{\text{th}}$ classifying set is already 'large enough' to have enough invariants. Still, we need to calculate the derivatives of our coframe elements and find the first order classifying set. To calculate this set, first we again need the total differential:

$$\mathrm{d}F\left(x, u^\alpha, \dot{u}^\alpha, \ddot{u}^\alpha\right) = F_{u^\alpha}\bar{\theta}^\alpha + F_{\dot{u}^\alpha}\bar{\theta}^\alpha_x + D_xF\bar{\omega} + F_{\ddot{u}^\alpha}\bar{\pi}^\alpha$$

Using this differential, the derivatives of $T^v_{\theta^\alpha\theta^\beta}$ can be found to be

$$\begin{aligned}
\mathrm{d}T^v_{\theta^\alpha\theta^\beta} =& \mathrm{d}\left(D_x\mathcal{L}_{\ddot{u}^\beta u^\alpha} - \mathcal{L}_{\dot{u}^\beta u^\alpha}\right) \\
=& \left(D_x\mathcal{L}_{\ddot{u}^\beta u^\alpha} - \mathcal{L}_{\dot{u}^\beta u^\alpha}\right)_{u^\gamma}\bar{\theta}^\gamma + \left(D_x\mathcal{L}_{\ddot{u}^\beta u^\alpha} - \mathcal{L}_{\dot{u}^\beta u^\alpha}\right)_{\dot{u}^\gamma}\bar{\theta}^\gamma_x \\
&+ D_x\left(D_x\mathcal{L}_{\ddot{u}^\beta u^\alpha} - \mathcal{L}_{\dot{u}^\beta u^\alpha}\right)\bar{\omega} + \left(D_x\mathcal{L}_{\ddot{u}^\beta u^\alpha} - \mathcal{L}_{\dot{u}^\beta u^\alpha}\right)_{\ddot{u}^\gamma}\bar{\pi}^\gamma \\
=& \left(D_x\mathcal{L}_{\ddot{u}^\beta u^\alpha u^\gamma} - \mathcal{L}_{\dot{u}^\beta u^\alpha u^\gamma}\right)\bar{\theta}^\gamma + \left(D_x\mathcal{L}_{\ddot{u}^\beta\dot{u}^\gamma u^\alpha} + \mathcal{L}_{\ddot{u}^\beta u^\gamma u^\alpha} - \mathcal{L}_{\dot{u}^\beta\dot{u}^\gamma u^\alpha}\right)\bar{\theta}^\gamma_x \\
&+ \left(D_xD_x\mathcal{L}_{\ddot{u}^\beta u^\alpha} - D_x\mathcal{L}_{\dot{u}^\beta u^\alpha}\right)\bar{\omega} + \left(\mathcal{L}_{\ddot{u}^\beta\dot{u}^\gamma u^\alpha} - \mathcal{L}_{\ddot{u}^\gamma\dot{u}^\beta u^\alpha}\right)\bar{\pi}^\gamma \\
=& \left(D_x\mathcal{L}_{\ddot{u}^\beta u^\alpha u^\gamma} - \mathcal{L}_{\dot{u}^\beta u^\alpha u^\gamma}\right)\bar{\theta}^\gamma + \left(D_x\mathcal{L}_{\ddot{u}^\beta\dot{u}^\gamma u^\alpha} + \mathcal{L}_{\ddot{u}^\beta u^\gamma u^\alpha} - \mathcal{L}_{\dot{u}^\beta\dot{u}^\gamma u^\alpha}\right)\bar{\theta}^\gamma_x + \left(D_xD_x\mathcal{L}_{\ddot{u}^\beta u^\alpha} - D_x\mathcal{L}_{\dot{u}^\beta u^\alpha}\right)\bar{\omega} \\
=& T^v_{\theta^\alpha\theta^\beta;\theta^\gamma}\bar{\theta}^\gamma + T^v_{\theta^\alpha\theta^\beta;\theta^\gamma_x}\bar{\theta}^\gamma_x + T^v_{\theta^\alpha\theta^\beta;\omega}\bar{\omega} + T^v_{\theta^\alpha\theta^\beta;\pi^\gamma}\bar{\pi}^\gamma + T^v_{\theta^\alpha\theta^\beta;v}\bar{v}
\end{aligned}$$

Similarly, the derivatives of $T^v_{\theta^\alpha\theta^\beta_x}$ can be computed as

$$\begin{aligned}
\mathrm{d}T^v_{\theta^\alpha\theta^\beta_x} =& \mathrm{d}\left(\mathcal{L}_{\dot{u}^\alpha\dot{u}^\beta} - D_x\mathcal{L}_{\ddot{u}^\alpha\dot{u}^\beta} - \mathcal{L}_{\ddot{u}^{(\alpha}u^{\beta)}}\right) \\
=& \left(\mathcal{L}_{\dot{u}^\alpha\dot{u}^\beta} - \mathcal{L}_{\ddot{u}^\beta u^\alpha} - D_x\mathcal{L}_{\ddot{u}^\alpha\dot{u}^\beta} - \mathcal{L}_{\ddot{u}^\alpha u^\beta}\right)_{u^\gamma}\bar{\theta}^\gamma + \left(\mathcal{L}_{\dot{u}^\alpha\dot{u}^\beta} - \mathcal{L}_{\ddot{u}^\beta u^\alpha} - D_x\mathcal{L}_{\ddot{u}^\alpha\dot{u}^\beta} - \mathcal{L}_{\ddot{u}^\alpha u^\beta}\right)_{\dot{u}^\gamma}\bar{\theta}^\gamma_x \\
&+ D_x\left(\mathcal{L}_{\dot{u}^\alpha\dot{u}^\beta} - \mathcal{L}_{\ddot{u}^\beta u^\alpha} - D_x\mathcal{L}_{\ddot{u}^\alpha\dot{u}^\beta} - \mathcal{L}_{\ddot{u}^\alpha u^\beta}\right)\bar{\omega} + \left(\mathcal{L}_{\dot{u}^\alpha\dot{u}^\beta} - \mathcal{L}_{\ddot{u}^\beta u^\alpha} - D_x\mathcal{L}_{\ddot{u}^\alpha\dot{u}^\beta} - \mathcal{L}_{\ddot{u}^\alpha u^\beta}\right)_{\ddot{u}^\gamma}\bar{\pi}^\gamma \\
=& \left(\mathcal{L}_{\dot{u}^\alpha\dot{u}^\beta u^\gamma} - \mathcal{L}_{\ddot{u}^\beta u^\alpha u^\gamma} - D_x\mathcal{L}_{\ddot{u}^\alpha\dot{u}^\beta u^\gamma} - \mathcal{L}_{\ddot{u}^\alpha u^\beta u^\gamma}\right)\bar{\theta}^\gamma \\
&+ \left(\mathcal{L}_{\dot{u}^\alpha\dot{u}^\beta\dot{u}^\gamma} - \mathcal{L}_{\ddot{u}^\beta\dot{u}^\gamma u^\alpha} - D_x\mathcal{L}_{\ddot{u}^\alpha\dot{u}^\beta\dot{u}^\gamma} - \mathcal{L}_{\ddot{u}^\alpha\dot{u}^\beta u^\gamma} - \mathcal{L}_{\ddot{u}^\alpha\dot{u}^\gamma u^\beta}\right)\bar{\theta}^\gamma_x \\
&+ \left(D_x\mathcal{L}_{\dot{u}^\alpha\dot{u}^\beta} - D_x\mathcal{L}_{\ddot{u}^\beta u^\alpha} - D_xD_x\mathcal{L}_{\ddot{u}^\alpha\dot{u}^\beta} - D_x\mathcal{L}_{\ddot{u}^\alpha u^\beta}\right)\bar{\omega} + \left(\mathcal{L}_{\ddot{u}^\gamma\dot{u}^\alpha\dot{u}^\beta} - \mathcal{L}_{\ddot{u}^\alpha\dot{u}^\beta\dot{u}^\gamma}\right)\bar{\pi}^\gamma \\
=& \left(\mathcal{L}_{\dot{u}^\alpha\dot{u}^\beta u^\gamma} - \mathcal{L}_{\ddot{u}^\beta u^\alpha u^\gamma} - D_x\mathcal{L}_{\ddot{u}^\alpha\dot{u}^\beta u^\gamma} - \mathcal{L}_{\ddot{u}^\alpha u^\beta u^\gamma}\right)\bar{\theta}^\gamma \\
&+ \left(\mathcal{L}_{\dot{u}^\alpha\dot{u}^\beta\dot{u}^\gamma} - D_x\mathcal{L}_{\ddot{u}^\alpha\dot{u}^\beta\dot{u}^\gamma} - \mathcal{L}_{\ddot{u}^\alpha\dot{u}^\beta u^\gamma} - \mathcal{L}_{\ddot{u}^\beta\dot{u}^\gamma u^\alpha} - \mathcal{L}_{\ddot{u}^\gamma\dot{u}^\alpha u^\beta}\right)\bar{\theta}^\gamma_x \\
&+ \left(D_x\mathcal{L}_{\dot{u}^\alpha\dot{u}^\beta} - D_x\mathcal{L}_{\ddot{u}^\beta u^\alpha} - D_xD_x\mathcal{L}_{\ddot{u}^\alpha\dot{u}^\beta} - D_x\mathcal{L}_{\ddot{u}^\alpha u^\beta}\right)\bar{\omega} \\
=& T^v_{\theta^\alpha\theta^\beta_x;\theta^\gamma}\bar{\theta}^\gamma + T^v_{\theta^\alpha\theta^\beta_x;\theta^\gamma_x}\bar{\theta}^\gamma_x + T^v_{\theta^\alpha\theta^\beta_x;\omega}\bar{\omega} + T^v_{\theta^\alpha\theta^\beta_x;\pi^\gamma}\bar{\pi}^\gamma + T^v_{\theta^\alpha\theta^\beta_x;v}\bar{v}
\end{aligned}$$

Last is the differential of the Euler-Lagrange equations,

$$\begin{aligned}
\mathrm{d}T^v_{\theta^{\alpha}\omega} =& -\mathrm{d}\left(D_xD_x\mathcal{L}_{\ddot{u}^\alpha} - D_x\mathcal{L}_{\dot{u}^\alpha} + \mathcal{L}_{u^\alpha}\right)\\
=& -\left(D_xD_x\mathcal{L}_{\ddot{u}^\alpha} - D_x\mathcal{L}_{\dot{u}^\alpha} + \mathcal{L}_{u^\alpha}\right)_{u^\gamma}\bar{\theta}^\gamma - \left(D_xD_x\mathcal{L}_{\ddot{u}^\alpha} - D_x\mathcal{L}_{\dot{u}^\alpha} + \mathcal{L}_{u^\alpha}\right)_{\dot{u}^\gamma}\bar{\theta}^\gamma_x\\
& - D_x\left(D_xD_x\mathcal{L}_{\ddot{u}^\alpha} - D_x\mathcal{L}_{\dot{u}^\alpha} + \mathcal{L}_{u^\alpha}\right)\bar{\omega} - \left(D_xD_x\mathcal{L}_{\ddot{u}^\alpha} - D_x\mathcal{L}_{\dot{u}^\alpha} + \mathcal{L}_{u^\alpha}\right)_{\ddot{u}^\gamma}\bar{\pi}^\gamma\\
=& -\left(D_xD_x\mathcal{L}_{\ddot{u}^\alpha u^\gamma} - D_x\mathcal{L}_{\dot{u}^\alpha u^\gamma} + \mathcal{L}_{u^\alpha u^\gamma}\right)\bar{\theta}^\gamma - \left(D_xD_x\mathcal{L}_{\ddot{u}^\alpha\dot{u}^\gamma} + 2D_x\mathcal{L}_{\ddot{u}^\alpha u^\gamma} - D_x\mathcal{L}_{\dot{u}^\alpha\dot{u}^\gamma} - \mathcal{L}_{\dot{u}^{[\alpha}u^{\gamma]}}\right)\bar{\theta}^\gamma_x\\
& - \left(D_xD_xD_x\mathcal{L}_{\ddot{u}^\alpha} - D_xD_x\mathcal{L}_{\dot{u}^\alpha} + D_x\mathcal{L}_{u^\alpha}\right)\bar{\omega} - \left(D_x\mathcal{L}_{\ddot{u}^\alpha\dot{u}^\gamma} - \mathcal{L}_{\dot{u}^\alpha\dot{u}^\gamma} + \mathcal{L}_{\ddot{u}^{(\alpha}u^{\gamma)}}\right)\bar{\pi}^\gamma\\
=& T^v_{\theta^\alpha\omega;\theta^\gamma}\bar{\theta}^\gamma + T^v_{\theta^\alpha\omega;\theta^\gamma_x}\bar{\theta}^\gamma_x + T^v_{\theta^\alpha\omega;\omega}\bar{\omega} + T^v_{\theta^\alpha\omega;\pi^\gamma}\bar{\pi}^\gamma + T^v_{\theta^\alpha\omega;v}\bar{v}
\end{aligned}$$

The total number of non-zero first-order torsion coefficients is $q\left(2q^2+4q+1\right)$. Combined with the zero$^{\text{th}}$ order invariants, this is larger than $3q+2$. Therefore, there must be functional dependency between these coefficients. Since the functional dependence of the invariants is also invariant [6, Chapter 8], any equivalent Lagrangian must have the same functional dependence between invariants.

First, we can see that $T^v_{\theta^\alpha\theta^\beta_x} = T^v_{\theta^\alpha\omega;\pi^\beta}$. A second relation is

$$\begin{aligned}
T^v_{\theta^\beta\theta^\alpha_x;\omega} + T^v_{\theta^{[\alpha}\theta^{\beta]}} - T^v_{\theta^\alpha\omega;\theta^\beta_x} =& \left(D_x\mathcal{L}_{\dot{u}^\beta\dot{u}^\alpha} - D_x\mathcal{L}_{\ddot{u}^\alpha u^\beta} - D_xD_x\mathcal{L}_{\ddot{u}^\beta\dot{u}^\alpha} - D_x\mathcal{L}_{\ddot{u}^\beta u^\alpha}\right) + \left(D_x\mathcal{L}_{\ddot{u}^\beta u^\alpha} - \mathcal{L}_{\dot{u}^\beta u^\alpha}\right)\\
& - \left(D_x\mathcal{L}_{\ddot{u}^\alpha u^\beta} - \mathcal{L}_{\dot{u}^\alpha u^\beta}\right) + \left(D_xD_x\mathcal{L}_{\ddot{u}^\alpha\dot{u}^\beta} + 2D_x\mathcal{L}_{\dot{u}^\alpha u^\beta} - D_x\mathcal{L}_{\dot{u}^\alpha\dot{u}^\beta} - \mathcal{L}_{\dot{u}^{[\alpha}u^{\beta]}}\right)\\
=& D_x\mathcal{L}_{\dot{u}^\beta\dot{u}^\alpha} - D_x\mathcal{L}_{\ddot{u}^\alpha u^\beta} - D_xD_x\mathcal{L}_{\ddot{u}^\beta\dot{u}^\alpha} - D_x\mathcal{L}_{\ddot{u}^\beta u^\alpha} + D_x\mathcal{L}_{\ddot{u}^\beta u^\alpha} - \mathcal{L}_{\dot{u}^\beta u^\alpha}\\
& - D_x\mathcal{L}_{\ddot{u}^\alpha u^\beta} + \mathcal{L}_{\dot{u}^\alpha u^\beta} + D_xD_x\mathcal{L}_{\ddot{u}^\alpha\dot{u}^\beta} + 2D_x\mathcal{L}_{\dot{u}^\alpha u^\beta} - D_x\mathcal{L}_{\dot{u}^\alpha\dot{u}^\beta} - \mathcal{L}_{\dot{u}^{[\alpha}u^{\beta]}}\\
=& 0
\end{aligned}$$

The third relation is $T^v_{\theta^\alpha\theta^\beta;\theta^\gamma_x} = T^v_{\theta^\beta\theta^\gamma_x;\theta^\alpha}$.

From this we have the first order classifying manifold

$$\begin{aligned}
&\left\{T^{\theta^\alpha}_{\theta^\alpha_x\omega}, T^v_{\theta^\alpha\theta^\beta}, T^v_{\theta^\alpha\theta^\beta_x}, T^v_{\theta^\alpha\omega}, T^v_{\theta^\alpha\theta^\beta;\theta^\gamma}, T^v_{\theta^\alpha\theta^\beta;\theta^\gamma_x}, T^v_{\theta^\alpha\theta^\beta;\omega}, T^v_{\theta^\alpha\theta^\beta_x;\theta^\gamma_x}, T^v_{\theta^\alpha\theta^\beta_x;\omega}, T^v_{\theta^\alpha\omega;\theta^\gamma}, T^v_{\theta^\alpha\omega;\omega}\right\}\\
=&\{-1, \left(D_x\mathcal{L}_{\ddot{u}^\beta u^\alpha} - \mathcal{L}_{\dot{u}^\beta u^\alpha}\right), \left(\mathcal{L}_{\dot{u}^\alpha\dot{u}^\beta} - \mathcal{L}_{\ddot{u}^\beta u^\alpha} - D_x\mathcal{L}_{\ddot{u}^\alpha\dot{u}^\beta} - \mathcal{L}_{\ddot{u}^\alpha u^\beta}\right), -\left(D_xD_x\mathcal{L}_{\ddot{u}^\alpha} - D_x\mathcal{L}_{\dot{u}^\alpha} + \mathcal{L}_{u^\alpha}\right),\\
&\left(D_x\mathcal{L}_{\ddot{u}^\beta u^\alpha u^\gamma} - \mathcal{L}_{\dot{u}^\beta u^\alpha u^\gamma}\right), \left(D_x\mathcal{L}_{\ddot{u}^\beta\dot{u}^\gamma u^\alpha} + \mathcal{L}_{\ddot{u}^\beta u^\gamma u^\alpha} - \mathcal{L}_{\dot{u}^\beta\dot{u}^\gamma u^\alpha}\right), \left(D_xD_x\mathcal{L}_{\ddot{u}^\beta u^\alpha} - D_x\mathcal{L}_{\dot{u}^\beta u^\alpha}\right),\\
&\left(\mathcal{L}_{\dot{u}^\alpha\dot{u}^\beta\dot{u}^\gamma} - D_x\mathcal{L}_{\ddot{u}^\alpha\dot{u}^\beta\dot{u}^\gamma} - \mathcal{L}_{\ddot{u}^\alpha\dot{u}^\beta u^\gamma} - \mathcal{L}_{\ddot{u}^\beta\dot{u}^\gamma u^\alpha} - \mathcal{L}_{\ddot{u}^\gamma\dot{u}^\alpha u^\beta}\right), \left(D_x\mathcal{L}_{\dot{u}^\alpha\dot{u}^\beta} - D_x\mathcal{L}_{\ddot{u}^\beta u^\alpha} - D_xD_x\mathcal{L}_{\ddot{u}^\alpha\dot{u}^\beta} - D_x\mathcal{L}_{\ddot{u}^\alpha u^\beta}\right)\\
&- \left(D_xD_x\mathcal{L}_{\ddot{u}^\alpha u^\gamma} - D_x\mathcal{L}_{\dot{u}^\alpha u^\gamma} + \mathcal{L}_{u^\alpha u^\gamma}\right), -\left(D_xD_xD_x\mathcal{L}_{\ddot{u}^\alpha} - D_xD_x\mathcal{L}_{\dot{u}^\alpha} + D_x\mathcal{L}_{u^\alpha}\right)\}.
\end{aligned}$$

Of course, just having the invariant coframe and its invariants is not enough, we need to compare it to the invariants that belong to a first order Lagrangian. We set up the same coframe and start with the same structure group.

$$\mathsf{H} = \begin{pmatrix} \delta^\alpha_\beta & 0 & 0 & 0 & 0\\ 0 & \frac{\delta^\alpha_\beta}{j} & 0 & 0 & 0\\ 0 & 0 & j & 0 & 0\\ 0 & 0 & 0 & \frac{\delta^\alpha_\beta}{j^2} & 0\\ (q_1)_\beta & (q_2)_\beta & j-1 & 0 & 1 \end{pmatrix}. \tag{176}$$

The total differential of a function F *not* depending on $\ddot{u}^\alpha$ is given by

$$\mathrm{d}F\left(x, u^\alpha, \dot{u}^\alpha\right) = F_{u^\alpha}\bar{\theta}^\alpha + j\mathcal{L}F_{\dot{u}^\alpha}\bar{\theta}^\alpha_x + \frac{D_xF}{j\mathcal{L}}\bar{\omega}$$

and the differential $\mathrm{d}\Sigma$ is given by

$$\begin{aligned}
\mathrm{d}\theta^\alpha &= -\bar{\theta}^\alpha_x \wedge \bar{\omega} \\
\frac{1}{\mathcal{L}}\,\mathrm{d}\theta^\alpha_x &= j\bar{\omega} \wedge \bar{\pi}^\alpha \\
\mathrm{d}\omega &= \frac{\mathrm{d}\mathcal{L}}{\mathcal{L}} \wedge \omega \\
&= \frac{\mathrm{d}\mathcal{L}}{j\mathcal{L}} \wedge \bar{\omega} \\
&= \frac{\mathcal{L}_{u^\alpha}}{j\mathcal{L}}\bar{\theta}^\alpha \wedge \bar{\omega} + \mathcal{L}_{\dot{u}^\alpha}\bar{\theta}^\alpha_x \wedge \bar{\omega} \\
\mathrm{d}\pi^\alpha &= -2\frac{\mathrm{d}\mathcal{L}}{\mathcal{L}}\pi^\alpha \\
&= -2j^2\frac{\mathrm{d}\mathcal{L}}{\mathcal{L}}\bar{\pi}^\alpha \\
&= -2j^2\frac{\mathcal{L}_{u^\beta}}{\mathcal{L}}\bar{\theta}^\beta \wedge \bar{\pi}^\alpha - 2j^3\mathcal{L}_{\dot{u}^\beta}\bar{\theta}^\beta_x \wedge \bar{\pi}^\alpha - 2j\frac{D_x\mathcal{L}}{\mathcal{L}^2}\bar{\omega} \wedge \bar{\pi}^\alpha
\end{aligned}$$

The first set of differentials in $\mathrm{d}\bar{\Sigma}$ remains unchanged:

$$\begin{aligned}
\mathrm{d}\bar{\theta}^\alpha &= \delta^\alpha_\beta\,\mathrm{d}\theta^\beta = -\bar{\theta}^\alpha_x \wedge \bar{\omega} \\
\mathrm{d}\bar{\theta}^\alpha_x &= \delta^\alpha_\beta\,\mathrm{d}\frac{1}{j\mathcal{L}}\theta^\beta_x \\
&= -\frac{1}{j}\,\mathrm{d}j \wedge \bar{\theta}^\alpha_x - \frac{1}{\mathcal{L}}\,\mathrm{d}\mathcal{L} \wedge \bar{\theta}^\alpha_x + \bar{\omega} \wedge \bar{\pi}^\alpha \\
&= -\iota \wedge \bar{\theta}^\alpha_x + \bar{\omega} \wedge \bar{\pi}^\alpha \\
\mathrm{d}\bar{\omega} &= \mathrm{d}j\omega \\
&= \mathrm{d}j \wedge \omega + j\,\mathrm{d}\omega \\
&= \iota \wedge \bar{\omega} \\
\mathrm{d}\bar{\pi}^\alpha &= \delta^\alpha_\beta\,\mathrm{d}\frac{1}{j^2}\pi^\beta \\
&= -\frac{2}{j^3}\,\mathrm{d}j \wedge \pi^\alpha + \frac{1}{j^2}\,\mathrm{d}\pi^\alpha \\
&= -2\iota \wedge \bar{\pi}^\alpha
\end{aligned}$$

The last differential is a different though:

$$\begin{aligned}
\mathrm{d}\bar{v} &= \mathrm{d}\left(\left(q_1\right)_\beta\theta^\beta\right) + \mathrm{d}\left(\left(q_2\right)_\beta\frac{1}{\mathcal{L}}\theta^\beta_x\right) + \mathrm{d}\left(\left(j-1\right)\omega\right) + \mathrm{d}v \\
&= \mathrm{d}\left(q_1\right)_\beta \wedge \theta^\beta + \left(q_1\right)_\beta\mathrm{d}\theta^\beta + \mathrm{d}\left(q_2\right)_\beta \wedge \frac{1}{\mathcal{L}}\theta^\beta_x - \left(q_2\right)_\beta\frac{\mathrm{d}\mathcal{L}}{\mathcal{L}^2} \wedge \theta^\beta_x + \left(q_2\right)_\beta\frac{1}{\mathcal{L}}\,\mathrm{d}\theta^\beta_x + \mathrm{d}j \wedge \omega + j\,\mathrm{d}\omega - \mathrm{d}\omega \\
&= \left(\xi_1\right)_\alpha \wedge \bar{\theta}^\alpha + \left(\xi_2\right)_\alpha \wedge \bar{\theta}^\alpha_x + \iota \wedge \bar{\omega} + j\left(q_2\right)_\alpha\bar{\omega} \wedge \bar{\pi}^\alpha
\end{aligned}$$

The non-zero, non-constant torsion coefficient $T^v_{\omega\pi^\beta}$ can be normalized to zero by choosing $\left(q_2\right)_\beta = 0$.

Since fixing $\left(q_2\right)_\beta$ does not affect the other differentials, we only need to recompute $\mathrm{d}\bar{v}$.

$$\begin{aligned}
\mathrm{d}\bar{v} &= \mathrm{d}\left(\left(q_1\right)_\beta\theta^\beta\right) + \mathrm{d}\left(\left(j-1\right)\omega\right) + \mathrm{d}v \\
&= \mathrm{d}\left(q_1\right)_\beta \wedge \theta^\beta + \left(q_1\right)_\beta\mathrm{d}\theta^\beta + \mathrm{d}j \wedge \omega + j\,\mathrm{d}\omega - \mathrm{d}\omega \\
&= \left(\xi_1\right)_\alpha \wedge \bar{\theta}^\alpha + \iota \wedge \bar{\omega} - \left(\left(q_1\right)_\alpha + \mathcal{L}_{\dot{u}^\alpha}\right)\bar{\theta}^\alpha_x \wedge \bar{\omega}
\end{aligned}$$

We again have 1 non-constant torsion coefficient, $T^v_{\theta^\beta_x \omega}$, and this is normalized to zero by choosing $(q_1)_\beta = -\mathcal{L}_{\dot{u}^\beta}$.

Again, we only need to recompute $\mathrm{d}\bar{v}$:

$$\begin{aligned}
\mathrm{d}\bar{v} &= -\mathrm{d}\left(\mathcal{L}_{\dot{u}^\beta}\theta^\beta\right) + \mathrm{d}\left((j-1)\,\omega\right) + \mathrm{d}v \\
&= -\mathrm{d}\mathcal{L}_{\dot{u}^\beta} \wedge \theta^\beta - \mathcal{L}_{\dot{u}^\beta}\,\mathrm{d}\theta^\beta + \mathrm{d}j \wedge \omega + j\,\mathrm{d}\omega - \mathrm{d}\omega \\
&= \iota \wedge \bar{\omega} - \mathcal{L}_{\dot{u}^\beta u^\alpha}\bar{\theta}^\alpha \wedge \bar{\theta}^\beta + j\mathcal{L}\mathcal{L}_{\dot{u}^\beta \dot{u}^\alpha}\bar{\theta}^\beta \wedge \bar{\theta}^\alpha_x + \frac{D_x\mathcal{L}_{\dot{u}^\alpha} - \mathcal{L}_{u^\alpha}}{j\mathcal{L}}\bar{\theta}^\alpha \wedge \bar{\omega}
\end{aligned}$$

We are in a similar situation as with the second-order Lagrangians. From these torsion coefficients, we choose the same normalization for j as we did for the second-order Lagrangians, i.e. $j = \mathcal{L}^{-1}$.

With this, we find the following structure group

$$\mathsf{H} = \begin{pmatrix} \delta^\alpha_\beta & 0 & 0 & 0 & 0 \\ 0 & \mathcal{L}\delta^\alpha_\beta & 0 & 0 & 0 \\ 0 & 0 & \mathcal{L}^{-1} & 0 & 0 \\ 0 & 0 & 0 & \mathcal{L}^2\delta^\alpha_\beta & 0 \\ -\mathcal{L}_{\dot{u}^\beta} & 0 & \mathcal{L}^{-1} - 1 & 0 & 1 \end{pmatrix}. \tag{177}$$

Let us compute the differentials of our invariant coframe.

$$\begin{aligned}
\mathrm{d}\bar{\theta}^\alpha &= \mathrm{d}\theta^\alpha = -\bar{\theta}^\alpha_x \wedge \bar{\omega}, \\
\mathrm{d}\bar{\theta}^\alpha_x &= \mathrm{d}\left(\mathcal{L}\frac{1}{\mathcal{L}}\theta^\alpha_x\right) = \mathrm{d}\theta^\alpha_x = \bar{\omega} \wedge \bar{\pi}^\alpha, \\
\mathrm{d}\bar{\omega} &= \mathrm{d}\left(\mathcal{L}^{-1}\omega\right) = \mathrm{d}\left(\mathcal{L}^{-1}\mathcal{L}\,\mathrm{d}x\right) = 0, \\
\mathrm{d}\bar{\pi}^\alpha &= \mathrm{d}\left(\mathcal{L}^2\pi^\alpha\right) = \mathrm{d}\left(\mathcal{L}^2\mathcal{L}^{-2}\,\mathrm{d}\ddot{u}^\alpha\right) = 0,
\end{aligned}$$

and the last differential is

$$\begin{aligned}
\mathrm{d}\bar{v} &= -\mathrm{d}\left(\mathcal{L}_{\dot{u}^\beta}\theta^\beta\right) + \mathrm{d}\left(\left(\mathcal{L}^{-1} - 1\right)\omega\right) + \mathrm{d}v \\
&= -\mathrm{d}\mathcal{L}_{\dot{u}^\beta} \wedge \theta^\beta - \mathcal{L}_{\dot{u}^\beta}\,\mathrm{d}\theta^\beta - \mathcal{L}^{-2}\,\mathrm{d}\mathcal{L} \wedge \omega + \mathcal{L}^{-1}\,\mathrm{d}\omega - \mathrm{d}\omega \\
&= -\mathcal{L}_{\dot{u}^\beta u^\alpha}\bar{\theta}^\alpha \wedge \bar{\theta}^\beta + \mathcal{L}_{\dot{u}^\beta \dot{u}^\alpha}\bar{\theta}^\beta \wedge \bar{\theta}^\alpha_x + \left(D_x\mathcal{L}_{\dot{u}^\alpha} - \mathcal{L}_{u^\alpha}\right)\bar{\theta}^\alpha \wedge \bar{\omega}
\end{aligned}$$

In total, this gives us the same number of invariants as the second-order case did, from the same sources. The zero$^{\text{th}}$ order classifying manifold is thus given by:

$$\left\{T^{\theta^\alpha}_{\theta^\alpha_x \omega}, T^v_{\theta^\alpha \theta^\beta}, T^v_{\theta^\alpha \theta^\beta_x}, T^v_{\theta^\alpha \omega}\right\} = \left\{-1, -\mathcal{L}_{\dot{u}^\beta u^\alpha}, \mathcal{L}_{\dot{u}^\alpha \dot{u}^\beta}, D_x\mathcal{L}_{\dot{u}^\alpha} - \mathcal{L}_{u^\alpha}\right\}$$

To compare these first-order with the second order Lagrangians, we also need to compute the first order classifying manifold. For that, we first need the differential:

$$\mathrm{d}F\left(x, u^\alpha, \dot{u}^\alpha, \ddot{u}^\alpha\right) = F_{u^\alpha}\bar{\theta}^\alpha + F_{\dot{u}^\alpha}\bar{\theta}^\alpha_x + D_xF\bar{\omega} + F_{\ddot{u}^\alpha}\bar{\pi}^\alpha$$

Using this differential, the derivatives of $T^v_{\theta^\alpha\theta^\beta}$ can be found to be

$$\begin{aligned}
\mathrm{d}T^v_{\theta^\alpha\theta^\beta} &= -\mathrm{d}\mathcal{L}_{\dot{u}^\beta u^\alpha} \\
&= -\left(\mathcal{L}_{\dot{u}^\beta u^\alpha}\right)_{u^\gamma}\bar{\theta}^\gamma - \left(\mathcal{L}_{\dot{u}^\beta u^\alpha}\right)_{\dot{u}^\gamma}\bar{\theta}^\gamma_x - D_x\mathcal{L}_{\dot{u}^\beta u^\alpha}\bar{\omega} - \left(\mathcal{L}_{\dot{u}^\beta u^\alpha}\right)_{\ddot{u}^\gamma}\bar{\pi}^\gamma \\
&= -\mathcal{L}_{\dot{u}^\beta u^\alpha u^\gamma}\bar{\theta}^\gamma - \mathcal{L}_{\dot{u}^\beta \dot{u}^\gamma u^\alpha}\bar{\theta}^\gamma_x - D_x\mathcal{L}_{\dot{u}^\beta u^\alpha}\bar{\omega} \\
&= T^v_{\theta^\alpha\theta^\beta;\theta^\gamma}\bar{\theta}^\gamma + T^v_{\theta^\alpha\theta^\beta;\theta^\gamma_x}\bar{\theta}^\gamma_x + T^v_{\theta^\alpha\theta^\beta;\omega}\bar{\omega} + T^v_{\theta^\alpha\theta^\beta;\pi^\gamma}\bar{\pi}^\gamma + T^v_{\theta^\alpha\theta^\beta;v}\bar{v}
\end{aligned}$$

Similarly, the derivatives of $T^v_{\theta^\alpha \theta^\beta_x}$ can be computed as

$$\begin{aligned}
\mathrm{d}T^v_{\theta^\alpha \theta^\beta_x} &= \mathrm{d}\mathcal{L}_{\dot{u}^\alpha \dot{u}^\beta} \\
&= \left(\mathcal{L}_{\dot{u}^\alpha \dot{u}^\beta}\right)_{u^\gamma} \bar{\theta}^\gamma + \left(\mathcal{L}_{\dot{u}^\alpha \dot{u}^\beta}\right)_{\dot{u}^\gamma} \bar{\theta}^\gamma_x + D_x \mathcal{L}_{\dot{u}^\alpha \dot{u}^\beta} \bar{\omega} + \left(\mathcal{L}_{\dot{u}^\alpha \dot{u}^\beta}\right)_{\ddot{u}^\gamma} \bar{\pi}^\gamma \\
&= \mathcal{L}_{\dot{u}^\alpha \dot{u}^\beta u^\gamma} \bar{\theta}^\gamma + \mathcal{L}_{\dot{u}^\alpha \dot{u}^\beta \dot{u}^\gamma} \bar{\theta}^\gamma_x + D_x \mathcal{L}_{\dot{u}^\alpha \dot{u}^\beta} \bar{\omega} \\
&= T^v_{\theta^\alpha \theta^\beta_x ; \theta^\gamma} \bar{\theta}^\gamma + T^v_{\theta^\alpha \theta^\beta_x ; \theta^\gamma_x} \bar{\theta}^\gamma_x + T^v_{\theta^\alpha \theta^\beta_x ; \omega} \bar{\omega} + T^v_{\theta^\alpha \theta^\beta_x ; \pi^\gamma} \bar{\pi}^\gamma + T^v_{\theta^\alpha \theta^\beta_x ; v} \bar{v}
\end{aligned}$$

Last is the differential of the Euler-Lagrange equations,

$$\begin{aligned}
\mathrm{d}T^v_{\theta^\alpha \omega} &= \mathrm{d}\left(D_x \mathcal{L}_{\dot{u}^\alpha} - \mathcal{L}_{u^\alpha}\right) \\
&= \left(D_x \mathcal{L}_{\dot{u}^\alpha} - \mathcal{L}_{u^\alpha}\right)_{u^\gamma} \bar{\theta}^\gamma + \left(D_x \mathcal{L}_{\dot{u}^\alpha} - \mathcal{L}_{u^\alpha}\right)_{\dot{u}^\gamma} \bar{\theta}^\gamma_x + D_x\left(D_x \mathcal{L}_{\dot{u}^\alpha} - \mathcal{L}_{u^\alpha}\right) \bar{\omega} + \left(D_x \mathcal{L}_{\dot{u}^\alpha} - \mathcal{L}_{u^\alpha}\right)_{\ddot{u}^\gamma} \bar{\pi}^\gamma \\
&= \left(D_x \mathcal{L}_{\dot{u}^\alpha u^\gamma} - \mathcal{L}_{u^\alpha u^\gamma}\right) \bar{\theta}^\gamma + \left(D_x \mathcal{L}_{\dot{u}^\alpha \dot{u}^\gamma} + \mathcal{L}_{\dot{u}^{[\alpha} u^{\gamma]}}\right) \bar{\theta}^\gamma_x + \left(D_x D_x \mathcal{L}_{\dot{u}^\alpha} - D_x \mathcal{L}_{u^\alpha}\right) \bar{\omega} + \mathcal{L}_{\dot{u}^\alpha \dot{u}^\gamma} \bar{\pi}^\gamma \\
&= T^v_{\theta^\alpha \omega ; \theta^\gamma} \bar{\theta}^\gamma + T^v_{\theta^\alpha \omega ; \theta^\gamma_x} \bar{\theta}^\gamma_x + T^v_{\theta^\alpha \omega ; \omega} \bar{\omega} + T^v_{\theta^\alpha \omega ; \pi^\gamma} \bar{\pi}^\gamma + T^v_{\theta^\alpha \omega ; v} \bar{v}
\end{aligned}$$

Part of comparing these differentials with those from the second order system, is to try and find the same functional relations between them. Again, we have $T^v_{\theta^{\alpha_\omega ; \pi^\beta}} = T^v_{\theta^\alpha \theta^\beta_x}$, and also $T^v_{\theta^\beta \theta^\alpha_x ; \omega} + T^v_{\theta^{[\alpha} \theta^{\beta]}} - T^v_{\theta^\alpha \omega ; \theta^\beta_x} = 0$. The third relation is again $T^v_{\theta^\alpha \theta^\beta ; \theta^\gamma_x} = T^v_{\theta^\beta \theta^\gamma_x ; \theta^\alpha}$, and these are all relations.

Again, we have the first-order classifying manifold.

$$\begin{aligned}
&\left\{T^{\theta^\alpha}_{\theta^\alpha_x \omega}, T^v_{\theta^\alpha \theta^\beta}, T^v_{\theta^\alpha \theta^\beta_x}, T^v_{\theta^\alpha \omega}, T^v_{\theta^\alpha \theta^\beta ; \theta^\gamma}, T^v_{\theta^\alpha \theta^\beta ; \theta^\gamma_x}, T^v_{\theta^\alpha \theta^\beta ; \omega}, T^v_{\theta^\alpha \theta^\beta_x ; \theta^\gamma_x}, T^v_{\theta^\alpha \theta^\beta_x ; \omega}, T^v_{\theta^\alpha \omega ; \theta^\gamma}, T^v_{\theta^\alpha \omega ; \omega}\right\} \\
=&\{-1, -\mathcal{L}_{\dot{u}^\beta u^\alpha}, \mathcal{L}_{\dot{u}^\alpha \dot{u}^\beta}, D_x \mathcal{L}_{\dot{u}^\alpha} - \mathcal{L}_{u^\alpha}, -\mathcal{L}_{\dot{u}^\beta u^\alpha u^\gamma}, -\mathcal{L}_{\dot{u}^\beta \dot{u}^\gamma u^\alpha}, \\
&- D_x \mathcal{L}_{\dot{u}^\beta u^\alpha}, \mathcal{L}_{\dot{u}^\alpha \dot{u}^\beta \dot{u}^\gamma}, D_x \mathcal{L}_{\dot{u}^\alpha \dot{u}^\beta}, \left(D_x \mathcal{L}_{\dot{u}^\alpha u^\gamma} - \mathcal{L}_{u^\alpha u^\gamma}\right), \left(D_x D_x \mathcal{L}_{\dot{u}^\alpha} - D_x \mathcal{L}_{u^\alpha}\right)\}
\end{aligned}$$

We can start comparing these coefficients to those generated from the second order Lagrangians. Referring to the first-order Lagrangians with $\bar{\mathcal{L}}$ from now on, we have the following conditions for equivalence:

$$\begin{aligned}
T^v_{\theta^\alpha \theta^\beta} = \bar{T}^v_{\theta^\alpha \theta^\beta} &\Leftrightarrow \\
D_x \mathcal{L}_{\ddot{u}^\beta u^\alpha} - \mathcal{L}_{\dot{u}^\beta u^\alpha} &= -\bar{\mathcal{L}}_{\dot{u}^\beta u^\alpha},
\end{aligned} \tag{178}$$

$$\begin{aligned}
T^v_{\theta^\alpha \theta^\beta_x} = \bar{T}^v_{\theta^\alpha \theta^\beta_x} &\Leftrightarrow \\
\mathcal{L}_{\dot{u}^\alpha \dot{u}^\beta} - D_x \mathcal{L}_{\ddot{u}^\alpha \dot{u}^\beta} - \mathcal{L}_{\ddot{u}^{(\alpha} u^{\beta)}} &= \bar{\mathcal{L}}_{\dot{u}^\beta \dot{u}^\alpha},
\end{aligned} \tag{179}$$

and

$$\begin{aligned}
T^v_{\theta^\alpha_x \omega} = \bar{T}^v_{\theta^\alpha_x \omega} &\Leftrightarrow \\
D_x D_x \mathcal{L}_{\ddot{u}^\alpha} - D_x \mathcal{L}_{\dot{u}^\alpha} + \mathcal{L}_{u^\alpha} &= -D_x \bar{\mathcal{L}}_{\dot{u}^\alpha} + \bar{\mathcal{L}}_{u^\alpha}.
\end{aligned} \tag{180}$$

This last conditions states that the Euler-Lagrange equations must overlap, as we expected and required from the start.

We want to see if a second-order, Ostrogradsky-free Lagrangian is divergent equivalent to a regular first order Lagrangian. To check this, we calculated the invariants listed above, and now we need to see if both theories have the same invariants. Assume that we have such a second order Lagrangian $\mathcal{L}$ and a first order Lagrangian $\bar{\mathcal{L}}$ that are equivalent, i.e. $\exists\ F = F\left(x, u^\alpha, \dot{u}^\alpha\right)$ s.t. $\mathcal{L} = \bar{\mathcal{L}} + D_x F$[29]. Compute the left half of (178),

[29]Since F does not depend on $\ddot{u}^\alpha$, the truncated and full total derivatives are equivalent.

$$
\begin{aligned}
T^v_{\theta^\alpha \theta^\beta} &= D_x \mathcal{L}_{\ddot{u}^\beta u^\alpha} - \mathcal{L}_{\dot{u}^\beta u^\alpha} \\
&= D_x \left(\bar{\mathcal{L}} + D_x F\right)_{\ddot{u}^\beta u^\alpha} - \left(\bar{\mathcal{L}} + D_x F\right)_{\dot{u}^\beta u^\alpha} \\
&= D_x \left(D_x F\right)_{\ddot{u}^\beta u^\alpha} - \bar{\mathcal{L}}_{\dot{u}^\beta u^\alpha} - \left(D_x F\right)_{\dot{u}^\beta u^\alpha} \\
&= D_x \left(F_{\dot{u}^\beta u^\alpha}\right) - \left(D_x F_{\dot{u}^\beta} + F_{u^\beta}\right)_{u^\alpha} - \bar{\mathcal{L}}_{\dot{u}^\beta u^\alpha} \\
&= -F_{u^\beta u^\alpha} - \bar{\mathcal{L}}_{\dot{u}^\beta u^\alpha} \\
&= -F_{u^\beta u^\alpha} + \bar{T}^v_{\theta^\alpha \theta^\beta}.
\end{aligned}
$$

This shows that the function F must satisfy $F_{u^\beta u^\alpha} = 0$ for equivalence.

Next, we compare (179),

$$
\begin{aligned}
T^v_{\theta^\alpha \theta^\beta_x} &= \mathcal{L}_{\dot{u}^\alpha \dot{u}^\beta} - D_x \mathcal{L}_{\ddot{u}^\alpha \dot{u}^\beta} - \mathcal{L}_{\ddot{u}^{(\alpha} u^{\beta)}} \\
&= \left(\bar{\mathcal{L}} + D_x F\right)_{\dot{u}^\alpha \dot{u}^\beta} - D_x \left(\bar{\mathcal{L}} + D_x F\right)_{\ddot{u}^\alpha \dot{u}^\beta} - \left(\bar{\mathcal{L}} + D_x F\right)_{\ddot{u}^{(\alpha} u^{\beta)}} \\
&= \bar{\mathcal{L}}_{\dot{u}^\alpha \dot{u}^\beta} + \left(D_x F\right)_{\dot{u}^\alpha \dot{u}^\beta} - D_x \left(D_x F\right)_{\ddot{u}^\alpha \dot{u}^\beta} - \left(D_x F\right)_{\ddot{u}^{(\alpha} u^{\beta)}} \\
&= \bar{\mathcal{L}}_{\dot{u}^\alpha \dot{u}^\beta} + \left(D_x F_{\dot{u}^\alpha} + F_{u^\alpha}\right)_{\dot{u}^\beta} - D_x \left(F_{\dot{u}^\alpha}\right)_{\dot{u}^\beta} - \left(F_{\dot{u}^{(\alpha}}\right)_{u^{\beta)}} \\
&= \bar{\mathcal{L}}_{\dot{u}^\alpha \dot{u}^\beta} + D_x F_{\dot{u}^\alpha \dot{u}^\beta} + F_{\dot{u}^\alpha u^\beta} + F_{\dot{u}^\beta u^\alpha} - D_x F_{\dot{u}^\alpha \dot{u}^\beta} - F_{\dot{u}^{(\alpha} u^{\beta)}} \\
&= \bar{\mathcal{L}}_{\dot{u}^\alpha \dot{u}^\beta} \\
&= \bar{T}^v_{\theta^\alpha \theta^\beta_x}
\end{aligned}
$$

This calculation yields no new conditions on F.

Last, we need to compare the Euler-Lagrange equations, i.e. equations (180):

$$
\begin{aligned}
T^v_{\theta^\alpha_x \omega} &= D_x D_x \mathcal{L}_{\ddot{u}^\alpha} - D_x \mathcal{L}_{\dot{u}^\alpha} + \mathcal{L}_{u^\alpha} \\
&= D_x D_x \left(\bar{\mathcal{L}} + D_x F\right)_{\ddot{u}^\alpha} - D_x \left(\bar{\mathcal{L}} + D_x F\right)_{\dot{u}^\alpha} + \left(\bar{\mathcal{L}} + D_x F\right)_{u^\alpha} \\
&= D_x D_x F_{\dot{u}^\alpha} - D_x \bar{\mathcal{L}}_{\dot{u}^\alpha} - D_x \left(D_x F_{\dot{u}^\alpha} + F_{u^\alpha}\right) + \bar{\mathcal{L}}_{u^\alpha} + D_x F_{u^\alpha} \\
&= -D_x \bar{\mathcal{L}}_{\dot{u}^\alpha} + \bar{\mathcal{L}}_{u^\alpha} \\
&= \bar{T}^v_{\theta^\alpha_x \omega},
\end{aligned}
$$

as expected.

Since these invariants already match-up everywhere, there is no need to also compare the derivatives of these invariants and we can instead conclude that all second-order, Ostrogradsky-free Lagrangians are equivalent to a first-order Lagrangian plus a total derivative.

12 Conclusion

This goal of this thesis was to answer the question: "Do the conditions that free a higher order Lagrangian of Ostrogradsky ghosts force such a Lagrangian to be divergence equivalent to a first-order Lagrangian?" In trying to answer this question, we also discussed the subquestions

- What types of Ostrogradsky stable theories are there?
- What are the properties of these theories?
- What are the conditions to call two Lagrangians divergence equivalent?
- How can you determine such equivalence?

We found that there are seval types of Ostrogradsky stable theories. First, we can split them up into classical mechanical and field theories. Second, the field theories can be split up, depending on Lorentz (in)variance. Last, all theories can be split up depending on the presence of first-order helper fields.

Within the Lorentz invariant field theories, Galileons are the most interesting theory. Especially when combined with gravity, they allow for interesting modifications to gravity. We discussed these couplings in chapter 7. A more general coupling of an Ostrogradsky-free second-order scalar to gravity leads to Beyond-Horndeski theories, which is an attempt to find the most general Ostrogradsky-free tensor-scalar theory. Tensorial General Relativity itself is also an Ostrogradsky free theory, as we found in section 6.3.

After having discussed these types of Ostrogradsky free problems, we defined the conditions under which two Lagrangians are divergence equivalent. To be able to define this equivalence, we discussed the theory of Jet spaces and contact transformations in chapter 8. In the end, we used the definition in section 8.4, which states that two Lagrangians are divergence equivalent if and only if there exists a contact transformation $\Phi : J^2\left(\mathbb{R}^p, \mathbb{R}^q\right) \to J^2\left(\mathbb{R}^p, \mathbb{R}^q\right)$ and an p-tuple $F : J^2\left(\mathbb{R}^p, \mathbb{R}^q\right) \to \mathbb{R}^p$ that relates the two Lagrangian under

$$\bar{\mathcal{L}}\left(\bar{x}^i, \bar{u}^{(r)}\right) = \frac{\mathcal{L}\left(x^i, u^{(r)}\right) + \nabla F}{|\det J|}, \tag{127}$$

where $(J)^i_j = \frac{\mathrm{d}\phi^i}{\mathrm{d}x^j}$.

We applied this definition to several Galileon theories in section 9, using the work by Crisostomi et al. [8], before we expaned to try and find a more general answer. To this end, we found and explained the Cartan Equivalence algorithm in chapter 10, which can be used to answer the main research question. In chapter 11 we applied Cartan's algorithm to classical mechanics, with an arbitrary number of second-order variables. I was unable to apply Cartan's algorithm to classical mechanics with helper fields, and field theories altogether due to time constraints and the additional complexity. As a result, the main research question can not be answered in full generality. Using the results from chapters 11 and 9 we can formulate the following conclusion:

Question "Do the conditions that free a higher order Lagrangian of Ostrogradsky ghosts force such a Lagrangian to be divergence equivalent to a first-order Lagrangian?"

Answer For theories with only second-order variables, the conditions that remove the Ostrogradsky ghosts also force such a Lagrangian to be divergence equivalent to a first-order Lagrangian. Field theories with only one second-order variable are also always equivalent, independent on the presence of first-order fields. For theories with general number of second-order and first-order fields, we were unable to draw any conclusions.

As I was unable to fully answer the research question, there still is some further work tp determine equivalence for the unanswered cases. Furthermore, the conditions for Ostrogradsky-free theories have only been formulated for scalar fields. These conditions can be generalized to spinor, vector and tensor fields. With these conditions then known, we can then consider the equivalence of these Ostrogradsky-free theories to first-order theories, be it through Cartan's algorithm or through some other method.

A Notation

Throughout this document, several methods of notation are used. Below, I have tried to list them all to give an overview.

A.1 Summation conventions

Throughouth this thesis, we use the Einstein summation convention, and thus summation over repeated indices is implied:

$$a_i b^i = \sum_i a_i b^i.$$

Futhermore, we use $[i, j]$ to refer to the anti-symmetric sum over the indices i and j, while (i, j) refers to the symmetric sum. for example

$$S_{[i,j]} = S_{i,j} - S_{j,i}$$

and

$$S_{(i,j)} = S_{i,j} + S_{j,i}.$$

A.2 Jet Bundles

The notation in the table below refers to notation used in reference to jet bundles, and contact forms thereon. The notation is primarily taken from Ivey and Landsberg in [7] and Olver in [6].

x^i	The independent coordinates. In general, there are p of them [42, page 371]. When discussing field theories, the notation x^μ is also used.
∂_i	The partial derivative w.r.t. x^i.
$\Box = \partial_i \partial^i$	The second order derivative.
u^α	The dependent coordinates. In general, there are q of them [42, page 371]. In the physics chapters 2-9 also referred to as ϕ^a or q^α.
$u^\alpha_i = \frac{\partial u^\alpha}{\partial x^i}$	The first derivatives of the dependent coordinates to the independent coordinates.
$I = \{i_1, i_2, \ldots, i_s\}$	A multi index of order s. Possibly un-ordered.
$u^\alpha_I = \frac{\partial^s u^\alpha}{\partial x^{i_1} \cdot \partial x^{i_2} \cdot \ldots \cdot \partial x^{i_s}}$	The $\lvert I\rvert = s^{\text{th}}$ order derivative of u^α w.r.t. $x^I = \{x^{i_1}, x^{i_2}, \ldots, x^{i_s}\}$ [42, page 371].
u_s	The set of all derivatives of all the functions u^α of order exactly s.
$u^{(s)}$	The set of all derivatives of all the functions u^α of order up to and including s, i.e. $u^{(s)} = \{u_0, u_1, \ldots, u_s\}$.
$\Upsilon(X)$	The space of all curves over the domain X.
$J^k(M, N)$	The k^{th} jet-bundle from M to N. Sometimes shortened to J^k if it is obvious what M and N are.
$\mathcal{I}$	The contact ideal on a jet space $J^k(M, N)$. Its exact form depends on k, M and N [7, Appendix §B.4].
$[\omega] = \omega + \mathcal{J}$	The equivalence class of a differential form ω modulo some differential ideal $\mathcal{J}$.

A.3 Geometry

The notation below refers to notation used in the sections dealing with gravity and geometry, and is the same as used by Horndeski in [40], unless otherwise specified.

$\delta^{i_1 \dots i_h}_{j_1 \dots j_h} = \det \begin{vmatrix} \delta^{i_1}_{j_1} & \dots & \delta^{i_1}_{j_h} \\ \vdots & \ddots & \vdots \\ \delta^{i_h}_{j_1} & \dots & \delta^{i_h}_{j_h} \end{vmatrix}$	The generalised Kronecker delta.
$g_{\mu\nu}$	The metric tensor in regular (3+1)-dimensional space.
$g = \lvert \det g_{\mu\nu} \rvert$	The determinant of the metric.
$\eta_{\mu\nu}$	The Minkowski metric in (3+1)-dimensions, I use a mostly-positive signature.
G_{AB}	The metric tensor in $(3 + N + 1)$ dimensions.
$G = \lvert \det G_{AB} \rvert$	The determinant of this metric.
η_{AB}	The Minkowski metric in $(3 + N + 1)$ dimensions.
$\Gamma^i_{jk} = \frac{1}{2} g^{ih} \left(\partial_k g_{jh} + \partial_j g_{kh} - \partial_h g_{jk} \right)$	The Christoffel symbols for a general metric.
∇_i	The covariant derivative w.r.t x^i and the Christoffel symbols defined above.
$R_{\mu\ \rho\sigma}^{\ \nu}$	The Riemann tensor.
$R_{\mu\nu} = R_{\mu\ \nu\rho}^{\ \rho}$	The Ricci-tensor, derived from the Riemann tensor.
$R = R^\mu_\mu$	The Ricci-scalar, the trace of the Ricci-tensor.
$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$	The Einstein Tensor.

B Variational Problems

The calculus of variations is concerned with the extremals of functions, whose domain is the infinite-dimensional space of curves. Such functions are called *functionals* [10]. To explain this some more, let us consider $\mathbb{R}^p$. Then let $\Upsilon(\mathbb{R}^p)$ denote the space of all curves over $\mathbb{R}^p$. A functional Φ is a map $\Phi : \Upsilon(\mathbb{R}^p) \to \mathbb{R}$. Note that the value of Φ only depends on the entire curve $f \in \Upsilon(\mathbb{R}^p)$, it does *not* depend on any local coordinates $x \in \mathbb{R}^p$.

Example 16. One example of a functional is the length of a curve in the Euclidean plane. Let $\gamma = \{(t, x, y) \,|x(t) = x, y(t) = y, t_i < t < t_f\}$ describe a curve[30]. Furthermore, let $\dot{x} = \mathrm{d}x/\mathrm{d}t$ and $\dot{y} = \mathrm{d}y/\mathrm{d}t$. Then the functional that describes the length of this curve is given by

$$\Phi(\gamma) = \int_{t_i}^{t_f} \sqrt{\dot{x}^2 + \dot{y}^2}\,\mathrm{d}t.$$

To consider extremes of a functional, like we do when we consider extremes of a regular function, we need to know what happens to the value of the functional if we make a small change to the input arguments. Let us consider an 'approximation' γ' to γ, $\gamma' = \{(t, x_i) : x_i = x_i(t) + h_i(t)\}$. Let us call it $\gamma' = \gamma + h$, with the requirement that $h(t_i) = h(t_f) = 0$. Consider the change in Φ, $\Delta\Phi = \Phi(\gamma + h) - \Phi(\gamma)$. We want this change to meet certain conditions, if we want so speak of extremes of a functional. To be exact, we look at Arnol'd [10]

Definition 17. A functional Φ is called *differentiable* if

$$\Delta\Phi = \Phi(\gamma + h) - \Phi(\gamma) = F + R,$$

where F depends linearly on h and $R(h, \gamma) = O(h^2)$ [10, page 56].

The function F can be called the differential of Φ at γ, in the same way one defines the derivative of a regular function.

Next, we can define what it means for a functional to have an extreme.

Definition 18. An *extremal* of a differentiable functional $\Phi(\gamma)$ is a curve γ such that $F(h) = 0$ for all functions h [10, page 57].

Again, this is similar to the definition of the extreme of a function, where we say a function obtains an extremal point at those points where $\frac{\mathrm{d}f}{\mathrm{d}x} = 0$.

For a functional of the form $\Phi(\gamma) = \int L(x, \dot{x}, t)\,\mathrm{d}t$, the derivative is given by

$$F(h) = \int \left[\frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}}\right] h\,\mathrm{d}t + \left(\frac{\partial L}{\partial \dot{x}} h\right),$$

and an extremal is reached ($F(h) = 0 \forall h$) when [10, page 57]

$$\frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} = 0.$$

This is exactly the Euler-Lagrange equation (7) for the case $p = q = r = 1$. For derivation of the Euler-Lagrange equations for higher-order Lagrangians ($r > 1$), I refer to [11], or to chapter 2 for a less rigorous derivation.

[30] This curve technically does not exist in the plane, but in 3-space. The curve in the xy-plane is the projection of the above curve to the xy-plane. If γ behaves nice enough, the projection γ_2 is itself a curve that can be described as $\gamma_2 = \{(x, y) \,|y(x) = y, x_i < x < x_f\}$ and then $\Phi_2(\gamma_2) = \int_{x_i}^{x_f} \sqrt{1 + (\mathrm{d}y/\mathrm{d}x)^2}\,\mathrm{d}x = \Phi(\gamma)$.

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