Violations of the Born rule in the Black Hole

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Abstract

In this master thesis we explore the possibility of violations of the Born rule postulate in the vicinity of the black hole. We introduce the black hole information paradox and review the recent development in the firewall discussion. In the second part we introduce a thought experiment in the framework of AdS/CFT. We formulate a conflict between quantum entanglement and typicality. An exploration of this conflict by D. Marolf and J. Polchinski in 2015 [15] has resulted into the conclusion that 'non-excited' black hole states could not be dual to typical CFT states without violating the Born rule. In this thesis we try to get more insight into the matter and explicitly quantify the statements made. By computing correlation function on the eternal BTZ black hole in 2+1 dimensions, we find violations of the Born rule to be immeasurable, indicating there would be a possibility for non-excited black holes to be dual to typical CFT states.

Contents

1 Introduction

In 1972 Jacob Bekenstein connected the quantity of thermodynamic entropy to the area of black holes. The famous area law is given by the *Beckenstein-Hawking formula*[1]:

$$S = \frac{A}{4G_N} \tag{1}$$

This law relates the area of the black hole to the entropy. This statement made a very big impact. Elaborating on this calculation Stephen Hawking formulated his theory of black hole radiation and showed that black holes could evaporate [2]. The result caused a situation. A notorious situation known in the business as the "black hole information paradox". A black hole starting in a pure state would radiate away into a mixed state consisting of thermal radiation, causing to lose information of it's purity. The principle of conservation of information or quantum unitarity was a sacred one, and never seen violated before. At the black hole, where quantum mechanics and general relativity met in concrete way, the situation looked very frightening. Either the process did not obey quantum unitarity, or something different happened which was at that moment, and for many years to come not understood.

A second interesting feature of *Bekenstein-Hawking* formula is the relation between the entropy and the area of the black hole. It suggested the idea of quantum holography. Originally formulated by Gerardus 't Hooft [3]. The idea that the information inside a volume could be alternatively and fully described by the surface enclosing it. In 1998 J.C. Maldacena conjectured a concrete example of such a holographic principle. With string theory being the only true proposal as a theory for quantum gravity, he found a duality between a $N = 4$ supersymmetric Yang-Mills gauge theory without gravity, and a type IIB string theory compactified on $AdS_5 \times S^5$ with gravity [4]. The boundary of the Anti-de Sitter space was found to be exactly equal to a quantum field theory with conformal symmetry, a conformal field theory or CFT. The AdS/CFT proposal was the first example of a gauge/gravity duality. An excellent tool to describe ill defined processes in one theory via the well understood dual theory.

The birth of quantum holography shed new light on the black hole information problem. Defining the process in AdS/CFT ruled out the possibility of information loss. However the solution to the paradox continued to be unknown. Quite recently in 2009 S. Mathur, and later in 2013 by a group of researchers going by the name of AMPS (Almheiri, Marolf, Polchinski Sully), argued that the black hole information paradox redefined in AdS/CFT was a question of whether or not the interior region of the black hole existed at all[5,6]. In other words; If someone would try to cross the horizon, 'something' should destroy that special someone. AMPS called this 'something' a 'firewall'. Something quite dramatic seen in the light of general relativity, which according to the equivalence principle predicts the horizon to be a place like anywhere else.

The existence of the interior turned out to depend on whether or not the interior region of the black hole was entangled with the exterior region in a very specific way. This rather surprising, but very fundamental, fact gave birth to a formulation of a number of new paradoxes [7,8]. All originating from the newly explicification of the information problem in terms of quantum entanglement.

A black hole in AdS was shown to be dual to a very high energy state described by a thermal density matrix in the CFT [9,23]. Here one can connect the different micro-states of the ensemble in the CFT to the entropy of the black hole, all being entangled in different ways. In 2013 J. Polchinski and D. Marolf sharpened the paradox by stating that only very a-typical microstates, characterized by a highly specific entanglement, would resemble black holes without a firewall, so called 'smooth' horizon states [10]. They formulated a conflict between typicality and entanglement. Questions like, what kind of black hole then is the dual to a typical micro-state in CFT? Is there even one?
This conflict was honed even more by S. Shenker and D. Stanford. They brought in the phenomenon of quantum chaos [11,12]. With black holes known to be highly chaotic objects, the smallest perturbation was shown to have dramatic effects on the entanglement of the system. This highly specific entanglement which was needed for black holes to have smooth horizons turned from a very unusual unlikely case to a near impossible puzzle. Even so proposals for constructing a black hole interior were developed [13,14], trying to go around and solving all the paradoxes that were lying out there. However in 2015 J. Polchinski and D. Marolf came back and stated that it in principle wasn't possible to construct a AdS black hole with a smooth interior being dual to a typical CFT state within the laws of quantum mechanics. They posed that any construction of such kind would result into violations of the born rule postulate of quantum mechanics [15].

Now the Born rule has never shown to be violated anywhere in nature, neither has it been been proven to exist. The Born rule however has been verified many times by experiment and is not considered as controversial. As being a postulate of quantum mechanics, it is a foundation on which the theory is build. Taken to be true by assumption. If the born rule would be broken in the vicinity of a black hole, quantum mechanics would need a modification. Something very important to find out if searching for the real theory of quantum gravity.

In this thesis we research this statement of Marolf and Polchinski; If one constructs an AdS black hole with a smooth horizon dual to a typical state in the CFT, do we observe violations of the Born rule. In section I, a broad introduction into the background theory is supplied. First, via a derivation of Rindler space, the semi-classical form of the information paradox is explained. With this, the concept of Hawking radiation and the matter of entanglement between the inner and outer region of the black hole is analyzed. Continuing we briefly review AdS/CFT, the information problem in AdS/CFT, black holes in AdS/CFT, and the conflict between typicality and entanglement. We construct a black hole by applying quantum field theory on a black hole background in AdS/CFT. These results we can further use in our thought-experiment.
In the continuing section, II, we formulate a thought experiment to test the statement of Marolf and Polchinski. We define our black hole to have a smooth horizon and monitor the effect on the black hole state by acting on it with a unitary operator. This unitary operation will perturb the black hole state very mildly, however drastically change it's entanglement configuration. We will try to interpret the modified state, and see what has happened both in the CFT and AdS picture. The question to answer now is, if the two pictures, AdS and CFT, show similar physical situations or do they differ. To that, if they differ, do they do so within the laws of quantum mechanics? We will derive a bound within quantum mechanics to test the born rule. We will present our results by computing correlation functions and the effect on the energy of the system. We conclude with a discussion, the implications of this research, and a look towards future research on the matter. The entire thesis uses $c = \hbar = G = k = 1$

Section I

In section I is a theoretical background.. The review starts with the concept of Rindler space. Rindler space is flat space, however it functions as an excellent tool to understand what is going on at the black hole. As one will see very shortly, it shows many similarities with black holes, and can be used to grasp the concept of Hawking radiation and the entanglement issue related to black holes.

Secondly the basics of AdS/CFT will be outlined. The AdS/CFT correspondence is the framework where computations will be made in. It is therefore key to understand how one can describe the process of black hole evaporation. After this we can state the information paradox in it's holographic form. The firewall arguments will be briefly summed, as the connection without quantum chaos. When having done so, The conflict between entanglement and typicality can be best understood.

2 Rindler Space

Rindler space is as mentioned above just ordinary Minkowski space. The key feature is presented by observing the spacetime symmetries of the metric. Every spacetime symmetry is generated by a 'Killing vector', named after Wilhelm Killing. Minkowski spacetime has 10 different Killing isometries. 1 + 3 translations, 3 rotations and 3 boosts. In ordinary (t, x) coordinates time translation is generated by the Killing vector ∂/∂_t A Lorentz boost in the $x-$ direction is subsequently generated by $x\partial/\partial_t + t\partial/\partial_x$. However, if one changes coordinates to a boosted coordinate frame, time translation is now generated by $x\partial/\partial_t + t\partial/\partial_x$. Both sets of coordinates will have a different notion of time, and therefore a different energy ground state or vacuum. This is called the "Unruh effect" [21]. Relating both sets is done by a Bogoliubov transformation. This will be explained below in detail, and gives rise to interesting phenomena in quantum field theory. A good full review on the subject is given by [16]. Below the same motivation for the Rindler construction is followed.

2.1 Two sets of coordinates

The first set of coordinates will just be regular Minkowski coordinates. The Minkowski metric is given by:

$$ds^2 = dt^2 - dx^2 \tag{2}$$

With

$$\mathbf{x} = (x, y, z) \tag{3}$$

From now on we just consider x instead of $\mathbf{x}$.
Consider now the following boost operation with boost parameter β:

$$t \to t \cosh \beta + x \sinh \beta \tag{4}$$

$$x \to t \sinh \beta + x \cosh \beta \tag{5}$$

One can observe that plugging the new coordinates back into the metric equation, the metric stays the same. In other words the metric is invariant under this transformation. This isometry is generated by the boost Killing vector:

$$K_\mu = x \frac{\partial}{\partial_t} + t \frac{\partial}{\partial_x} \tag{6}$$

One could see that this symmetry suggests the following coordinate transformation: Consider now the following transformation motivated by the boost Killing vector:

$$t = \chi \sinh \eta \tag{7}$$

$$x = \chi \cosh \eta \tag{8}$$

Resulting in the following diagram:

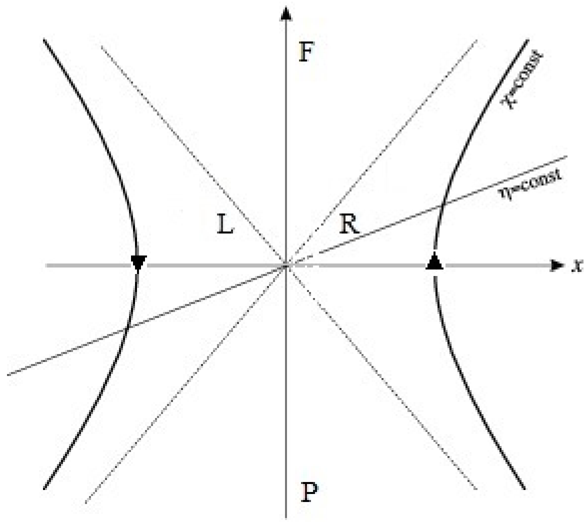

Figure 1: Rindler space

As one can observe, the trajectory runs down in the quadrant II. This due to the fact that the boost operator works consequently on negative x. Secondly, the fact that the entire Rindler plane consists out of four wedges. The *left*, *right*, *future* and *past* wedge. For now we focus on the left and right wedge and later continue to the future and past regions. Both the left and the right Rindler wedge are can be related to the Minkowski plane.

$$t = \frac{e^{\bar{\xi}a} \sinh a\bar{\eta}}{a} \tag{9}$$

$$x = \frac{e^{\bar{\xi}a} \cosh a\bar{\eta}}{a}. \tag{10}$$

and for the left wedge:

$$t = \frac{e^{\bar{\xi}a} \sinh a\bar{\eta}}{a} \tag{11}$$

$$x = -\frac{e^{\bar{\xi}a} \cosh a\bar{\eta}}{a}. \tag{12}$$

only differing by a minus sign for the spatial component.These new coordinates will uniformly accelerate the old coordinates assymptoting to the speed of light. The spacetime has become asymptotic and is now characterized by the two Rindler horizons on the lightcone at $U = V = 0$.

The metric now looks like:

$$ds^2 = e^{2\xi a}(d\eta^2 - d\xi^2). \tag{13}$$

The metric is independent of η. The Killing vector describing time translation is now given by ∂/∂_η. To describe our spacetime in quantum field theory we go first to lightcone coordinates: Minkowski coordinates can be related to lightcone coordinates in the usual way:

$$u = t - x \tag{14}$$

$$v = t + x \tag{15}$$

Also the Rindler coordinates (η, ξ) can be taken together in UV coordinates:

$$U = \eta - \xi \tag{16}$$

and,

$$V = \eta + \xi \tag{17}$$

We can relate the uv Minkowski coordinates to the UV Rindler ones as follows: For the right wedge:

$$u = t - x = \frac{e^{\xi a} \sinh a\eta}{a} - \frac{e^{\xi a} \cosh a\eta}{a} = -\frac{1}{a} e^{-aU} \tag{18}$$

Where $U = \eta - \xi$. The same for v and V:

$$v = t + x = \frac{e^{\xi a} \sinh a\eta}{a} + \frac{e^{\xi a} \cosh a\eta}{a} = \frac{1}{a} e^{-aV} \tag{19}$$

For the left wedge:

$$u = t - x = \frac{e^{a\bar{\xi}} \sinh a\bar{\eta}}{a} + \frac{e^{\bar{\xi} a} \cosh a\bar{\eta}}{a} = \frac{1}{a} e^{-a\bar{U}} \tag{20}$$

Where $U = \eta - \xi$. The same for v & V:

$$v = t + x = \frac{e^{\bar{\xi} a} \sinh a\bar{\eta}}{a} - \frac{e^{\bar{\xi} a} \cosh a\bar{\eta}}{a} = -\frac{1}{a} e^{-a\bar{V}} \tag{21}$$

We called the Rindler coordinates for the left wedge, $\bar{U}$ and $\bar{V}$. In this case these coordinates are related to η and ξ like this:

$$\bar{U} = \bar{\eta} + \bar{\xi} \tag{22}$$

and,

$$\bar{V} = \bar{\eta} - \bar{\xi} \tag{23}$$

2.2 The Boguliubov transformation

2.2.1 Expansion of Rindler modes

In the following section, we continue the story to add some quantum fields to the metric. We are going to do this according standard quantum field theory. When doing so we have two notions of time. In the Minkowski coordinates time translation is generated by the usual Killing vector ∂/∂_t, however in Rindler coordinates this is done by the above mentioned boost Killing vector $x\partial/\partial_t + t\partial/\partial_x$. Now usually the positive frequency modes are defined to correspond to the annihilation operator in the mode expansion. These therefore define the notion of the vacuum state. However now that there are two notions of time-translation, two sets of annihilation operators with different sets of positive frequency wavefunctions, we have two different vacua. What happens below is the relation between these two sets of modes. This is done by a Boguliubov transformation and is possible since both are complete sets of energy-momentum eigenstates.
The expansion of the scalar field (eq. 9) can be written in two dimensions, 1 time and 1 spatial, as follows:

$$\phi(x,t) = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{4\pi k}} \left[a(\vec{k}) e^{-\imath k^{\mu}(t-x)} + a^{\dagger}(\vec{k}) e^{\imath k^{\mu}(t-x)} \right] \tag{24}$$

or

$$\phi(x,t) = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{4\pi k}} \left[a(\vec{k}) f_M(t,x) + a^{\dagger}(\vec{k}) f_M^{\star}(t,x) \right] \tag{25}$$

with $k^{\mu} = (\omega, \vec{k})$

We can subdivide the integral in a negative and a positive part:

$$\begin{aligned}\phi(x,t) = \int_{0}^{\infty} \frac{dk}{\sqrt{4\pi k}} \Big[& a^{-}(k) e^{-\imath k^{\mu}(t-x)} + a^{-\dagger}(k) e^{\imath k^{\mu}(t-x)} \\ & + a^{+}(k) e^{-\imath k^{\mu}(t+x)} + a^{+\dagger}(k) e^{\imath k^{\mu}(t+x)} \Big]\end{aligned} \tag{26}$$

We can clean this up a bit by making use of the uv-coordinate change. Recall: $u = t - x$ and $v = t + x$

$$\begin{aligned}\phi(u,v) = \int_{0}^{\infty} \frac{dk}{\sqrt{4\pi k}} \Big[& \\ & a^{-}(k) e^{-\imath k^{\mu} u} + a^{-\dagger}(k) e^{\imath k^{\mu} u} + a^{+}(k) e^{-\imath k^{\mu} v} + a^{+\dagger}(k) e^{\imath k^{\mu} v} \Big]\end{aligned} \tag{27}$$

We can grab the u parts and v parts together and write the function like this:

$$\phi(u,v) = \phi_{-}(u) + \phi_{+}(v) \tag{28}$$

We made a distinction between the positive and negative frequencies corresponding to left and right moving waves.

The next step is to derive the same expression for fields expanded in Rindler coordinates. The KG equation in terms of Rindler coordinates (η, ξ), needs to be solved and these solutions have to be expanded in a similar way. Remember we work with a massless scalar field.

$$(\partial^t \partial_t + \partial^x \partial_x)\phi = 0 \tag{29}$$

now becomes:

$$(\partial^\eta \partial_\eta + \partial^\xi \partial_\xi)\phi = 0 \tag{30}$$

Since the Lagrangian density of the field is invariant under this transformation the quantization procedure is exactly the same and we can express the field in terms of these new coordinates like this:

$$\psi_R(\xi, \eta) = \int_{-\infty}^{\infty} \frac{d\omega'}{\sqrt{4\pi\omega}} \left[b^R(\omega) e^{-\imath\omega^\mu(\eta+\xi)} + b^{R\dagger}(\omega) e^{\imath\omega^\mu(\eta-\xi)} \right] \tag{31}$$

or

$$\psi_R(\xi, \eta) = \int_{-\infty}^{\infty} \frac{d\omega}{\sqrt{4\pi\omega}} \left[b^R(\omega) g_R(\eta, \xi) + b^{R\dagger}(\omega) g_R^\star(\eta, \xi) \right] \tag{32}$$

Again similar commutation hold for these creation and annihilation operators of Rindler modes:

$$[b(\omega), b^\dagger(\omega)] = i\delta^3(\vec{x} - \vec{y}] \tag{33}$$

$$[b^\dagger(\omega), b^\dagger(\omega'))] = 0 \tag{34}$$

$$[b(\omega), b(\omega')] = 0 \tag{35}$$

Similarly we can write the expansion in in a positive and negative part. Making use of the coordinates $U = \eta - \xi$ and $V = \eta + \xi$ we end up with:

$$\begin{aligned} \psi_R(U, V) = \int_0^{\infty} \frac{d\omega}{\sqrt{4\pi\omega}} \Bigg[\\ b_R^-(\omega) e^{-\imath\omega^\mu U} + b_R^{-\dagger}(k) e^{\imath\omega^\mu U} + b_R^+(\omega) e^{-\imath\omega^\mu V} + b_R^{+\dagger}(\omega) e^{\imath\omega^\mu V} \Bigg] \end{aligned} \tag{36}$$

And in short splitted up into a positive and negative part for k:

$$\psi_R(U, V) = \psi_{-R}(U) + \psi_{R+}(V) \tag{37}$$

Now this expansion is only for fields in the right wedge, since these coordinates map to this wedge only. This is indicated by the R in the superscript of the creation and annihilation operators. For the left wedge the expansion looks the same only with coordinates $(\bar{\eta}, \bar{\xi})$. The Rindler vacuum is defined as : $b^R(\omega) |0_R\rangle = b^L |0_R\rangle$ respectively for the right and left wedge.

Moreover due to this transformation symmetry in the Lagrangian density we can demand that the fields are equal to each other. We are going to work with the right wedge coordinates.

$$\phi(u, v) = \xi_R(U, V) \tag{38}$$

or

$$\phi_-(u) + \phi_+(v) = \psi_{-R}(U) + \psi_{R+}(V) \tag{39}$$

We consider only the positive parts, $\phi_+(v)$ and $\psi_+(V)$. We can split the left and right movers. These two can be taken separately, since they don't interact with one another. Continuing:

$$\phi_+(v) = \psi_{R+}(V) \tag{40}$$

or:

$$\begin{aligned} &\int_0^\infty \frac{dk}{\sqrt{4\pi k}} \left[a^+(k) e^{-\imath k^\mu v} + a^{+\dagger}(k) e^{\imath k^\mu v} \right] = \\ &\int_0^\infty \frac{d\omega}{\sqrt{4\pi\omega}} \left[b_R^+(\omega) e^{-\imath\omega^\mu V} + b_R^{+\dagger}(\omega) e^{\imath\omega^\mu V} \right] \end{aligned} \tag{41}$$

To solve this equation we take a Fourier transform to V on both sides. Or in other words: we multiply with $\int_{-\infty}^{\infty} \frac{dV'}{\sqrt{2\pi}} e^{(ikV')}$. Let us first look at the right hand side of equation 48.

Right hand side:

$$\begin{aligned} &\int_{-\infty}^{\infty} \frac{dV}{\sqrt{2\pi}} e^{(i\omega' V)} \int_0^\infty \frac{d\omega}{\sqrt{4\pi\omega}} \left[b_R^+(\omega) e^{-\imath\omega V} + b_R^{+\dagger}(\omega) e^{\imath\omega V} \right] \\ &= \int_0^\infty \frac{d\omega}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{dV'}{\sqrt{4\pi\omega}} \left[b_R^+(\omega) e^{-\imath(\omega-\omega')V} + b_R^{+\dagger}(\omega) e^{\imath(\omega+\omega')V} \right] \end{aligned} \tag{42}$$

By taking the Fourier transform to v we cancel to Fourier transform to k and end up with:

$$= \frac{1}{\sqrt{2|\omega|}} \begin{cases} b_R^+(\omega) & \text{for} \quad \omega > 0 \\ b_R^{+\dagger}(\omega) & \text{for} \quad \omega < 0 \end{cases} \tag{43}$$

Now the left hand side of eq. 48 is unfortunately less trivial. We start with the fourier transform to V.

Left hand side:

$$\begin{aligned} &\int_{-\infty}^{\infty} \frac{dV}{\sqrt{2\pi}} e^{(i\omega' V)} \int_0^\infty \frac{dk}{\sqrt{4\pi k}} \left[a^+(k) e^{-\imath k^\mu v} + a^{+\dagger}(k) e^{\imath k^\mu v} \right] \\ &= \int_0^\infty \frac{d}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{dV}{\sqrt{4\pi k}} \left[a^+(k) e^{\imath\omega' V - \imath k^\mu v} + a^{+\dagger}(k) e^{\imath\omega' V + \imath k^\mu v} \right] \end{aligned} \tag{44}$$

Because V is a Rindler coordinate and v a Minkowski one, we can try helping our cause by writing V in terms of v. Recall: $v = \frac{1}{a} e^{-aV}$, and write:

Left hand side:

$$= \int_0^\infty \frac{dk}{\sqrt{2k}} \left[a^+(k) F(k, \omega') + a^{+\dagger}(k) F(-k, \omega') \right] \tag{45}$$

where,

$$F(k,\omega') = \int_{-\infty}^{\infty} \frac{dV}{2\pi} \exp\left[i\omega' V + i\frac{k}{a}e^{-aV}\right] \tag{46}$$

We now set both sides equal to each other and find:

$$b_R^+(\omega) = \int_0^{\infty} dk \left[\alpha_{\omega K}^R a^+(k) + \beta_{\omega K}^R a^{+\dagger}(k)\right] \tag{47}$$

The coefficients $\alpha_{\omega k}^R$ and $\beta_{\omega K}^R$ are called the *Bogulubov Coefficients*, and are given by:

$$\alpha_{\omega k}^R = \sqrt{\frac{\omega'}{k}} F(k,\omega') = \sqrt{\frac{\omega'}{k}} \int_{-\infty}^{\infty} \frac{dV}{2\pi} \exp\left[i\omega' V + i\frac{k}{a}e^{-aV}\right] \tag{48}$$

and,

$$\beta_{\omega k}^R = \sqrt{\frac{\omega'}{k}} F(-k,\omega') = \sqrt{\frac{\omega'}{k}} \int_{-\infty}^{\infty} \frac{dV}{2\pi} \exp\left[i\omega' V - i\frac{k}{a}e^{-aV}\right] \tag{49}$$

By solving these we find the relation between the Rindler operator b_r^+ , and the Minkowski operators a^+ and $a^{+\dagger}$.
We find a similar expression for b_R^+ when we hermitian conjugate eq. 56:

$$b_R^{+\dagger}(\omega) = \int_0^{\infty} dk \left[\alpha_{\omega K}^{\star R} a^{+\dagger}(k) + \beta_{\omega K}^{\star R} a^+(k)\right] \tag{50}$$

Likewise we can find expressions for the Left Rindler wedge when substitute all R's for L's and use the transformation: $v = -\frac{1}{a}e^{-a\bar{V}}$.

Eq. 60 and Eq. 63 relate the operators between Minkowski and Rindler. It is also conveniant to have an expression that relate the modes. We can write down the following:

$$g_R(\eta,\xi) = \int_0^{\infty} dk \left[\alpha_{\omega K}^R f_M(t,x) + \beta_{\omega K}^{\star R} f_M^\star(t,x)\right] \tag{51}$$

and:

$$g_R^\star(\eta,\xi) = \int_0^{\infty} dk \left[\alpha_{\omega K}^{\star R} f_M^\star(t,x) + \beta_{\omega K}^R f_M(t,x)\right] \tag{52}$$

Another argument for this to be true is the completeness of both sets of modes.

2.2.2 Solving the Boguliubov Coefficients

The next step is solving eq. (48), here we follow [17]:

$$\alpha_{\omega k}^R = \sqrt{\frac{\omega'}{k}} F(k,\omega') = \sqrt{\frac{\omega'}{k}} \int_{-\infty}^{\infty} \frac{dV}{2\pi} \exp\left[i\omega' V + i\frac{k}{a}e^{-aV}\right] \tag{53}$$

We can start by making a couple of substitutions. We take: $x = e^{-aV}$, with $dx = -ae^{-aV}dV$, $s = -\frac{i\omega}{a}$ and $b = \frac{-ik}{a}$

$$F(k, \omega') = \frac{1}{2\pi a}\int_0^\infty dx x^{s-1}e^{-bx} \tag{54}$$

Integration boundaries change to [0,∞), since x is a positive everywhere. $\alpha^R_{\omega k}$ now becomes:

$$\alpha^R_{\omega k} = \sqrt{\frac{\omega'}{k}}\frac{1}{2\pi a}\int_0^\infty dx x^{s-1}e^{-bx} \tag{55}$$

The expression we have now, looks a lot like a gamma function. We use the identity: $e^{-s\log(b)}\Gamma(s) = \int_0^\infty dx x^{t-1}e^{-bx}$, where the logarithm is defined as: $\log(A+iB) = \log|A+iB| + i\,\mathrm{sgn}(B)\tan^{-1}(\frac{|B|}{A})$. Where $'\mathrm{sgn}'$ is the sign function, which gives the sign of a certain function. The sign function is 1 when the argument is positive, and negative when the argument is -1. So $k > 0$ gives a positive result etc. Now this is exactly what we need.
Proceding:

$$\begin{aligned}\alpha^R_{\omega k} &= \sqrt{\frac{\omega'}{k}}\frac{1}{2\pi a}\int_0^\infty dx x^{s-1}e^{-bx} \\ &= \sqrt{\frac{\omega'}{k}}\frac{1}{2\pi a}e^{-s\log(b)}\Gamma(s) \\ &= \sqrt{\frac{\omega'}{k}}\frac{1}{2\pi a}e^{\frac{i\omega'}{a}\log(\frac{-ik}{a})}\Gamma(-\frac{i\omega}{a})\end{aligned} \tag{56}$$

Using the logarithm identity and some more algebra, we end up with:

$$\alpha^R_{\omega k} = \sqrt{\frac{\omega'}{k}}\frac{1}{2\pi a}\left(\frac{a}{k}\right)^{\frac{i\omega}{a}}e^{\frac{\omega\pi}{2a}}\Gamma(-\frac{i\omega'}{a}) \tag{57}$$

Here we have used in the exponent that the argument of the logarithm goes to $\frac{\pi}{2}$, when the angle goes to infinity. It does so since the real part of the logarithm is zero.
Continuing by making use of another identity of the gamma function: $x\Gamma(x) = \Gamma(1+x)$, with $x = \frac{-i\omega'}{a}$ we get:

$$\alpha^R_{\omega k} = \sqrt{\frac{\omega'}{k}}\frac{1}{2\pi a}\left(\frac{a}{-i\omega'}\right)\left(\frac{a}{k}\right)^{\frac{i\omega'}{a}}e^{\frac{\omega\pi}{2a}}\Gamma\left(1 - \frac{i\omega}{a}\right) \tag{58}$$

It is time to clean up. We end up with:

$$\alpha^R_{\omega k} = \frac{ie^{\frac{\omega\pi}{2a}}}{\sqrt{k\omega}2\pi}\left(\frac{a}{k}\right)^{-\frac{i\omega}{a}}\Gamma\left(1 - \frac{i\omega}{a}\right) \tag{59}$$

Which is our final result for $\alpha^R_{\omega k}$. The ω' is changed for ω, since it was only a dummy variable.

Computing the other Bogulubov coefficients goes on the exact same way: We find for:

$$\beta^R_{\omega k} = -\frac{ie^{\frac{-\omega\pi}{2a}}}{\sqrt{k\omega}2\pi}\left(\frac{a}{k}\right)^{-\frac{i\omega}{a}}\Gamma\left(1-\frac{i\omega}{a}\right) \tag{60}$$

Only the sign function changes sign since, we now have dealt with: $F(-k,\omega)$ instead of $F(k,\omega)$. We can find the following relation between the two coefficients:

$$\alpha^R_{\omega k} = -e^{\frac{\pi\omega}{a}}\beta^R_{\omega k} \tag{61}$$

For the left wedge we deal we again take the positive ($\bar{V}$) part of the wavefunction. By complex conjugating the right wedge we end up in the left wedge. For the rest we substitute L's for the R's and find:

$$\alpha^L_{\omega k} = -\frac{ie^{\frac{\omega\pi}{2a}}}{\sqrt{k\omega}2\pi}\left(\frac{a}{k}\right)^{\frac{i\omega}{a}}\Gamma\left(1+\frac{i\omega}{a}\right) \tag{62}$$

and,

$$\beta^L_{\omega k} = \frac{ie^{\frac{-\omega\pi}{2a}}}{\sqrt{k\omega}2\pi}\left(\frac{a}{k}\right)^{\frac{i\omega}{a}}\Gamma\left(1+\frac{i\omega}{a}\right) \tag{63}$$

We can now relate these coefficients with each other and find the following relations between left and right.

$$\beta^R_{\omega k} = -e^{-\frac{\pi\omega}{a}}\alpha^{\star L}_{\omega k}, \qquad \beta^L_{\omega k} = -e^{-\frac{\pi\omega}{a}}\alpha^{\star R}_{\omega k}, \tag{64}$$

2.3 Unruh Effect

The first result we can derive is the following: The number operator is defined for Minkowski Operators by:

$$N_k^M = a_k^\dagger a_k \tag{65}$$

To give a trivial example on it's significance, let us find the number of particles in the Minkowski vacuum or equivalently compute the expectation value of N in $|0_M\rangle$

$$\langle 0_M| N^M |0_M\rangle = \langle 0_M| a_k^\dagger a_k |0_M\rangle = 0 \tag{66}$$

By definition, or eq. 13, we end up with 0. a_k annihilates the ket-state, as does $a_k^\dagger$ on the bra-state.
Now the same holds for the Rindler vacuum. Take the right wedge. We let the Rindler creation and annilihation operators work on the state and find:

$$\langle 0_R| N^R |0_R\rangle = \langle 0_R| b_\omega^\dagger b_\omega |0_R\rangle = 0 \tag{67}$$

The question is however what happens when we let the Rindler Operators work on the Minkowski vacuum. Right now with the Bogulubov coefficients we can make sense out of this question by relating b_R and $b_R^\dagger$ in terms of Minkowski operators and Bogulubov coefficients.

$$\begin{aligned}
\langle 0_M| N^R |0_M\rangle &= \langle 0_M| \int dk \left[\alpha_{\omega K}^{\star R} a^\dagger(k) + \beta_{\omega K}^{\star R} a^R(k)\right] \\
&\quad \times \int dk \left[\alpha_{\omega K}^{R} a^(k') + \beta_{\omega K}^{R} a^{+\dagger}(k')\right] |0_M\rangle \\
&= \int dk |\beta_{\omega k}|^2 \langle 0_M| a_k a_k'^\dagger |0_M\rangle \\
&= \int dk |\beta_{\omega k}|^2 \delta_{kk'} \\
&= \int dk |\beta_{\omega k}|^2
\end{aligned} \tag{68}$$

We use our expression for $\beta_{\omega k}$ and end up with after some algebra:

$$\langle 0_M| N^R |0_M\rangle = \frac{1}{e^{\frac{2\pi\omega}{a}} - 1} \delta(\omega - \omega') \tag{69}$$

This result is called the Unruh effect. It may seem surprising at first. Because what we see here is exactly the same expectation value as a Bose-Einstein particle in a thermal bath of Temperature $T = a/2\pi$. This result might seem strange at first sight. If you consider the vacuum to be a state with zero energy then, yes indeed this result is strange. But this definition is not correct. The vacuum is the state with the lowest energy, and for Rindler space, as being accelerated spacetime, it is not so weird that the lowest energy state is different then the vacuum state of Minkowski space. Crucial to understand is the fact that only a Rindler observer is observing particles. Since this observer is accelerating.

A Minkowski observer just freefalling through spacetime does not observe any particles, as the spacetime is still just flat Minkowski geometry.
To describe the path of a Minkowski observer in terms of Rindler modes one has to bring together the left and right wedge. This should be possible since both sets are complete and describe the entire spacetime. We make us the equation's between the operators of the different expansions (44) and (47).

$$b_R^+(\omega) = \int_0^\infty dk \left[\alpha_{\omega K}^R a^+(k) + \beta_{\omega K}^R a^{+\dagger}(k) \right] \tag{70}$$

and,

$$b_R^{+\dagger}(\omega) = \int_0^\infty dk \left[\alpha_{\omega K}^{\star R} a^{+\dagger}(k) + \beta_{\omega K}^{\star R} a^+(k) \right] \tag{71}$$

Likewise for the left Rindler modes:

$$b_L^+(\omega) = \int_0^\infty dk \left[\alpha_{\omega K}^L a^+(k) + \beta_{\omega K}^L a^{+\dagger}(k) \right] \tag{72}$$

and,

$$b_L^{+\dagger}(\omega) = \int_0^\infty dk \left[\alpha_{\omega K}^{\star L} a^{+\dagger}(k) + \beta_{\omega K}^{\star L} a^+(k) \right] \tag{73}$$

We can now use the relations between the Bogoliubov coefficients, equation (61), to connect left and right ones. Substituting these in we obtain:

$$b_R^+(\omega) = \int_0^\infty dk \left[\alpha_{\omega K}^R a^+(k) - e^{-\frac{\pi\omega}{a}} \alpha_{\omega K}^{\star L} a^{+\dagger}(k) \right] \tag{74}$$

,

$$b_R^{+\dagger}(\omega) = \int_0^\infty dk \left[\alpha_{\omega K}^{\star R} a^{+\dagger}(k) - \alpha_{\omega K}^L a^+(k) \right] \tag{75}$$

and for the left:

$$b_L^+(\omega) = \int_0^\infty dk \left[\alpha_{\omega K}^L a^+(k) - e^{-\frac{\pi\omega}{a}} \alpha_{\omega K}^{\star R} a^{+\dagger}(k) \right] \tag{76}$$

,

$$b_L^{+\dagger}(\omega) = \int_0^\infty dk \left[\alpha_{\omega K}^{\star L} a^{+\dagger}(k) - \alpha_{\omega K}^R a^+(k) \right] \tag{77}$$

They are all written down explicitly since we will need a specific combination of them to let them describe the full Minkowski vacuum. We continue by looking at the expansion of the field in terms of Minkowski modes:

$$\phi(v) = \int_0^\infty \frac{dk}{\sqrt{4\pi k}} \left[a(k) f_M(v) + a^\dagger(k) f_M^\star(v) \right] \tag{78}$$

We want to write $a^+(k)$ in terms of Rindler operators to see which combination of them annihilations the vacuum, or correspond to positive frequency modes. In order to do so, we have to invert the previous equations and isolote all the parts that correspond to the $a^+(k)$ resp $a^{+\dagger}(k)$. The way to find all operators being proportional to $a^+(k)$ is to take the following combination:

$$a^+(k) \propto b_R^+(\omega) - e^{-\frac{\pi\omega}{a}} b_L^{+\dagger}(\omega)$$

This looks like this:

$$a^+(k) = \int_0^\infty d\omega \frac{C}{\alpha_{\omega K}^R} \left[b_R^+(\omega) - e^{-\frac{\pi\omega}{a}} b_L^{+\dagger}(\omega) \right] \tag{79}$$

where C is a constant given by: $C = \frac{1}{1-e^{-\frac{2\pi\omega}{a}}}$. Furthermore $\alpha^{\star L} = -\alpha^R$ is used to relate the Bogoliubov coefficients. Of course, a similar relation is when L and R are interchanged:

$$a^+(k) = \int_0^\infty d\omega \frac{C}{\alpha_{\omega K}^L} \left[b_L^+(\omega) - e^{-\frac{\pi\omega}{a}} b_R^{+\dagger}(\omega) \right] \tag{80}$$

This relation can be written likewise for all negative frequency modes correspond to $a^{+\dagger}(k)$ only switching to the hermitian conjugate of (76) and (77). Now we can derive a very important result from here. If we plug (76) and (77) into equation (75) we find that the following combination of Rindler operators should annihilate the Minkowski vacuum:

$$b^R - e^{-\frac{\pi\omega}{a}} b^{L\dagger} |0_m\rangle = 0 \tag{81}$$

$$b^L - e^{-\frac{\pi\omega}{a}} b^{R\dagger} |0_m\rangle = 0 \tag{82}$$

These relations imply the following:

$$b^{R\dagger} b^R - b^{L\dagger} b^L = |0_R\rangle \tag{83}$$

This relation tells us that the number of Rindler particles in the left wedge is the same as in the right wedge. Right now we can write the following.

$$|0_m\rangle = \prod_i \sum_{n_i=0}^{\infty} \frac{K_n}{n_i!} (b^{R\dagger} b^{L\dagger})^{n_i} |0_R\rangle \tag{84}$$

Here we follow [1]. We use a discrete sum instead of the integral to find the K_n. The physics don't change by this. Continuing to find this parameter we use relations (eq. 85, 86) and find:

$$K_{n+1} - e^{-\frac{\pi\omega_i}{a}} K_{n_i} = 0 \tag{85}$$

Solving:

$$K_{n_i} = e^{-\frac{n_i \pi \omega_i}{a}} K_0 \tag{86}$$

Plugging back in:

$$|O_m\rangle = \prod_i \sum_{n_i=0}^{\infty} \frac{e^{-\frac{n_i \pi \omega_i}{a}} K_0}{n_i!} (b^{R\dagger} b^{L\dagger})^{n_i} |0_R\rangle \tag{87}$$

We now define the state as follows: Every state with n_i particles has n_i particles with energy ω_i in each wedge, left and right. Defining as in [1]

$$\frac{1}{n_i!}(b^{R\dagger} b^{L\dagger})^{n_i} |0_R\rangle \equiv |n_i, R\rangle \otimes |n_i, L\rangle \tag{88}$$

and the state becomes per frequency(the product is left out):

$$\boxed{|O_m\rangle = C_i \sum_{n_i=0}^{\infty} e^{-\frac{n_i \pi \omega_i}{a}} |n_i, R\rangle \otimes |n_i, L\rangle} \tag{89}$$

With normalization factor $C_i = \sqrt{1 - exp(-\frac{2\pi\omega_i}{a}})$. This is our main result from Rindler space. We see that the state of the Minkowski vacuum is an entangled state between the left and right Rindler wedge. This is a fundamental result and very important one for the continuing story. To underline the importance, when one wants to cross the Rindler horizon as a Minkowski observer the two wedges need to be in this exact entangled state. First, we can see what happens if one would put the system into a different state. For instance by perturbing the system and putting the system in a mixed state. The Minkowski vacuum now is described by the density matrix ρ. Lets see what we find: The density matrix of the system is now given by:

$$\begin{aligned}\rho = |\psi\rangle\langle\psi| = |0_M\rangle\langle 0_M i| = \\ (C_i)^2 \sum_{n_i=0}^{\infty} e^{-\frac{2n_i \pi \omega_i}{a}} \Big(|n_i, R\rangle \otimes |n_i, L\rangle \langle n_i, R| \otimes \langle n_i, L| \Big)\end{aligned} \tag{90}$$

We can write in terms of a left and right part by taking the partial trace:

$$\rho_R \otimes \rho_L \tag{91}$$

The density matrix for the right wedge, ρ_R, is reached by taking the partial trace over the left eigenstates: in matrix form:

$$\rho_R = \sum \langle n_i, L| \rho |n_i, L\rangle \tag{92}$$

and we end up with:

$$\rho_R = (C_i)^2 \sum_{n_i=0}^{\infty} e^{-\frac{2n_i \pi \omega_i}{a}} |n_i, R\rangle\langle n_i, R| \tag{93}$$

What we see is a thermal state and correlation with the left wedge is lost. To make the situation even more concrete, we could see what happens with the stress energy tensor $T_{\mu\nu}$ if the system is not in the entangled state (77). It is shown in appendix A that this quantity, $T_{\mu\nu}$, will be non-zero computed at the horizon. This means that there is energy sitting there! A first connection can be made with the concept of a 'firewall'.

2.4 Hawking radiation and connection to black holes

The connection with Black holes is not very hard to see. In the case of Rindler space the acceleration has to be supplied by a rocket booster of some sorts, however in the black hole case the gravitational field will do the trick. An immediate connection can be made describing an infalling observer in a Schwarzschild metric in terms of coordinate time and proper time. Probably known to the reader is the fact that the time needed to describe the trajectory in terms of coordinate time is infinite, while the elapsed time for an infaller to reach the horizon in terms of proper time is finite and well defined. The observer freely falling in terms of proper time is not accelerating away, however the one far away from the black hole trying to describe the process from outside of the black hole, has to accelerate away from the black hole to prevent him/herself from falling in. Expanding the field in terms of modes for both observers gives the analogy to Rindler. Relating the asymptotic modes, often called outgoing modes, to the infalling modes in similar fashion as equation (69). We observe with now $|\psi\rangle$ being the black hole vacuum for asymptotic modes:

$$\langle\psi|\, b^{\dagger}_{\omega} b_{\omega} \,|\psi\rangle = \frac{1}{e^{\frac{\omega}{T}} - 1}\delta(\omega - \omega') \tag{94}$$

The Hawking flux or black body radiation spectrum for a black hole with temperature T, which is related to the acceleration by $T = \frac{a}{2\pi}$ [18]. By explicitly doing the calculation for a quantum field in d dimensions, one finds that this quantity is reduced by a so called grey-body factor. The fact is that modes can scatter of the gravitational field of the black hole. This causes a certain probability to exist for most to be reflected back towards to horizon, limiting the chance to escape completely.

$$\langle\psi|\, b^{\dagger}_{\omega} b_{\omega} \,|\psi\rangle = \frac{\Gamma_{s\omega lm}}{e^{\frac{2\pi\omega}{a}} - 1}\delta(\omega - \omega') \tag{95}$$

This gray-body factor can be seen as a transmission coefficient, and depends on the angular momentum of the mode [19]. The Hawking flux causes the black hole to evaporate since the energy of the Hawking photons is negative. The energy of these modes can be seen as the conserved charge corresponding to the time-translation Killing vector, which is time-like outside of the horizon. However this Killing vector becomes space-like inside the horizon. The conserved charge now becomes a momentum and can have a negative signature. It turns out, due to the mixing of positive and negative frequency modes in the interior, a negative sign is needed [18]. The description of the interior of the black hole is actually a highly relevant problem, which will be looked at in the AdS/CFT section, when describing the firewall argument.

2.5 Information Paradox

With the mechanism of Hawking radiation explained, the original formulation of the information problem can be stated. Since the black hole loses it's energy in the form of thermal radiation, the black hole will eventually end up in a complete mixed state of thermal radiation. However, the trouble arises if the black hole started out in a pure state. This process of black hole evaporation would cause a pure state to transit into a mixed state, and this cannot be described by a standard S-matrix process.

$$|\psi_{BH}\rangle \to \rho_{thermal} \tag{96}$$

To keep the process unitary, information has to travel outside of the horizon into the hawking radiation, which is forbidden by causality. As a consequence we end up with two possibilities. Either the information should be lost, or the Hawking radiation should in some way contain the information about the purity of the state. This would mean that by computing all correlation functions between the Hawking photons, one would find the final state still to be pure. As is the case for the process of burning up a pure state encyclopedia.
The situation can be viewed at graphically by looking at the von Neumann entropy of the system. [1] Now describing the two scenarios once more. What Hawking proposed was a linear increase in entanglement entropy. The black hole starting out in a pure state will slowly increase it's entropy by the evaporation process and do so until there was nothing but thermal radiation left. The information preserving alternative needs the entanglement entropy to go to zero at the end of it's lifetime. In other words, there has to be some tipping point where the entropy would start decreasing. This moment in time is called the *Page time* [41]. We sketch both cases below.

[1]The von Neumann or entanglement entropy is a measure to quantify the entanglement and is given by: $S = -\operatorname{Tr}\rho\log\rho$. A pure state will have zero entropy, while a mixed state will have maximal von Neumann entropy.

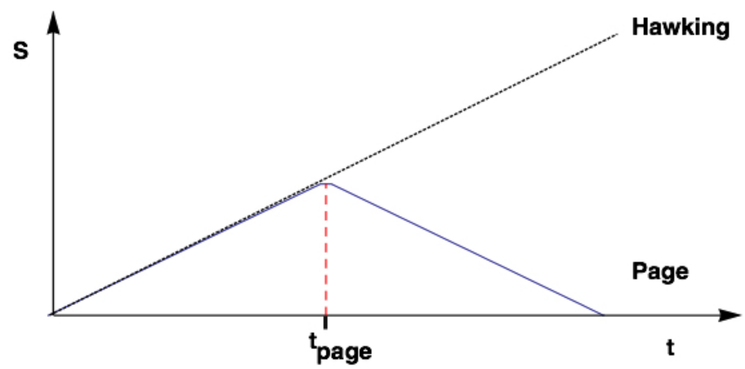

Figure 2: The von Neumann entropy versus time for an evaporating black hole according to Hawking and Page [3]

A unitary black hole evaporation process follows the Page curve. Questions like, what triggers the entropy to decrease, or how does the full S-matrix of the black hole look like, are unanswerable at the moment. A full description of quantum gravity has to give insight in these puzzles, which is a wish for many.

For completeness of the review, we mention a third option for the evaporation process. The black hole could decrease to a remnant. A Planckian size object with a very high entropy. The entanglement entropy of the object would be so large, it would exceed the Bekenstein entropy and therefore violate the fact that the number of microstates is given by the Bekenstein entropy [19]. This last option seems very implausible, however was considered by some and has to be mentioned when discussing the information paradox.

The paradox seemed/seems very solid, and indeed nobody could crack the code for over twenty years. However new hope glared on the horizon, when the AdS/CFT correspondence came into our world.

3 AdS/CFT

The black hole information paradox has gotten new light since the AdS/CFT duality was developed in 1998. In this review the basics of AdS, CFT and the connection between them is explained. After this we go back to black holes, talking about how one can desribe them in AdS/CFT. By applying quantum field theory to a BTZ-AdS metric, we will be able to construct a set up, on which we can compute correlation functions, and perform calculations on the black hole.
Secondly, the black hole information paradox in AdS/CFT is reviewed. We make the connection between typicality and entanglement. Furthermore the connection with quantum chaos is underlined, as it is an important factor in the conflict.

3.1 Introduction

A hologram is a very thin film, describing a three dimensional object. The two dimensional film containing all the information of the object, only in an alternative description. The original idea of 't Hooft [3] to suggest this might be relevant in physics was based on the work of Bekenstein and Hawking [1] [2]. Every bit of information fallen into the black hole was described on the surface of the object. With entropy usually being thought of as a quantity related to the volume, this was an extraordinary idea. 't Hooft proposed now to look at any closed surface area. He showed that the degrees of freedom inside were maximized if the area consisted of one big black hole. In the search for a theory of quantum gravity, black holes were considered to be a natural physical cut-off for quantum field theory to break down at higher energies. According to 't Hooft it was therefore logical to describe any volume less energetic then a black hole, like ordinary quantum field theories, with a description lying on the surface enclosing that volume.
Susskind then proposed the holographic principle could be realized inside string theory [22]. Five years later the development of AdS/CFT in 1998 was the first realization of such a holographic principle. The original statement of the duality is given by[4]:

$$D = 4\ ,\ \mathscr{N} = 4\ U(N)\ \text{Super Yang Mills} = \text{IIB string theory}\ on\ AdS_5 \times S^5 \tag{97}$$

where N is the rank of the field theory and $\mathscr{N}$ is the number of supersymmetries. A supersymmetric Yang Mills theory in four dimensions is found dual to a string theory on a five dimensional Anti-de Sitter space times a five dimensional sphere. The gauge theory is subject to a conformal symmetry. Together with Poincaré symmetry it is invariant under scale-transformations/dilations and special-conformal transformations. This high degree of symmetry field theory was found to be equivalent to the boundary of AdS space, which is highly symmetric manifold itself. It is characterized by a negative curvature.
It is illuminating to look at the couplings on both sides, following the review of [21]. The gauge theory is charactarized by the coupling, given by g_{YM} and the rank of the fields, N. Again 't Hooft showed that in the limit when N is large, one can perturbatively expand in terms of $1/N$ and g_{YM}^2. The amplitude now has the following form:

$$Z = \sum_{g\geq 0} N^{2-2g} \sum_{n=0} C_{g,n}\lambda^n \tag{98}$$

where $\lambda = g_{YM}^2$ the 't Hooft coupling.

This can be identified with the loop expansion from string theory. The loop expansion in Riemann surfaces for a closed string theory has a similar form as equation (98) [9].

$$Z = \sum_{g \geq 0} g_s^{2g-2} Z_g \tag{99}$$

with string coupling g_s.
If we now want to relate both sides, we have to relate the parameters describing them. Next to the string coupling g_s, which indicates the importance of quantum corrections, the string theory is defined by the curvature length L_{AdS} and the string length l_s. The ratio of the two, L_{AdS}/l_s, is a measure of how big the radius of AdS is in string lengths. If your curvature length is comparable to your string length, stringy/planckian effects are important.
For the duality to hold we find the following relations between the two sides:

$$g_s = g_{YM}^2 \sim \frac{\lambda}{N}, \qquad \left(\frac{L_{AdS}}{l_s}\right)^4 = 4\pi g_{YM}^2 N \sim \lambda \tag{100}$$

We can observe the fact that the parameters can be tuned in the that is desirable. Let us find the regime for classical gravity. For the stringy effects to be negligible, we need $L_{AdS}/l_s >> 1$. This means that we find $\lambda >> 1$. Secondly, we want quantum corrections to be small, therefore we want g_s to be small, which means we need to take next to λ also $N >> 1$. We observe a very important property of the duality. When evaluating the gravity side in the weak coupling regime, the gauge side turns out to be in the strong coupling regime. Similar in the opposite case. The AdS/CFT correspondence is what is called a weak/strong coupling duality. One of the two sides is evaluated in the weak coupling regime, when the other is evaluated in the strong regime. This is what makes the duality so valuable. When on one side perturbation theory breaks down, one can observe what is happening on the other side.

There is a lot of evidence to be found for the correspondence [27]. First of all, the symmetry groups on both sides agree. The isometry group of AdS^5 is $SO(4,2)$, which matches with the conformal group in 4 dimensions[4] After the discovery, Edward Witten and others elaborated on Maldacena's work and started constructing a map between bulk fields and boundary operators [23,24]. This was the beginning of the so called *"dictionary"*. A vademecum to relate quantities in the bulk to the boundary. The dictionary will be reviewed in section 3 of the AdS/CFT paragraph. With this, things like correlation functions [23], and causality [25,26], were tested, always with success. However, quantities like correlation functions can be hard to compute in the strongly coupled regime of the CFT. This causes a limitation on what is testable. To emphasize once more, not once the correspondence seemed to be violated. The ground on which the duality seemed to stand looks pretty solid.
After the first example from Maldacena, many different dualities were developed in all kinds of dimensions. In 2009 the general formulation of the duality was stated by Polchinski et al. [31]. The volume/bulk theory is always described with one extra dimension, with on the boundary lying the conformal field theory. The general correspondence is usually formulated as AdS^{d+1}/CFT_d. The S is a trivial part of the duality, and is therefore left out of the formulation.

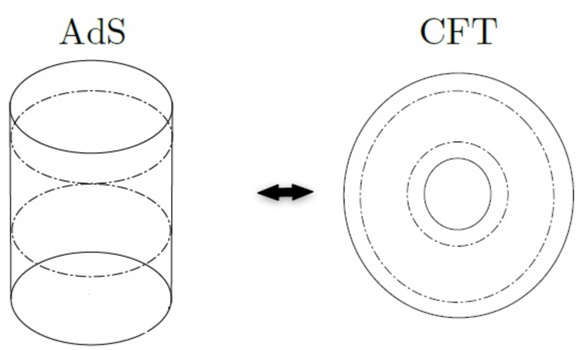

Figure 3: The conformal field theory lies on the boundary of the AdS cylinder. Different cross sections in AdS correspond to different circles on the CFT

[4]For $p > 1, q > 1, CO(\mathbb{R}^{p,q}) \cong SO(p+1, q+1)$, which for $p = 1, q = 3$ is equal to $SO(4,2)$ [27]

3.2 AdS

The bulk is described by an Anti-de Sitter space. AdS is different from Minkowski space. It is a solution of the Einstein equation with a negative cosmological constant. The space is contracting. It is maximally symmetric. The isometry group of AdS_5 has 15 elements. AdS_d is a d dimensional hyperbolic manifold, and can be described by a hyperboloid in $d+1$ dimensional flat space. A set of points $(X^1, X^2, ..., X^{d+1})$ obeys the following equation:

$$-(X^1)^2 - (X^2)^2 + \ ... \ + (X^{d-1})^2 + (X^d)^2 + (X^{d+1})^2 = -L_{AdS}^2 \tag{101}$$

with L again the curvature length/radius of AdS. These points are embedded in an $d+1$ dimensional space with metric:

$$ds^2 = -(dX^1)^2 - (dX^2)^2 + \ ... \ + (dX^{d-1})^2 + (dX^d)^2 + (dX^{d+1})^2 \tag{102}$$

We can do several coordinate transformations to get more grip on how the space looks like. Often AdS is described by the Poincaré patch. We follow the following coordinate transformations:

$$X^1 = \frac{1}{2z}(z^2 + L_{AdS}^2 + \sum_{i=3}^{d}(x^i)^2 - t^2) \tag{103}$$

$$X^2 = \frac{L_{AdS}t}{z} \tag{104}$$

$$X^i = \frac{L_{AdS}x^i}{z} \tag{105}$$

$$X^{d+1} = \frac{1}{2z}(z^2 + L_{AdS}^2 - \sum_{i=3}^{d}(x^i)^2 - t^2) \tag{106}$$

The Poincaré patch is given by:

$$ds^2 = \frac{L_{AdS}^2}{z^2}\Big[- dt^2 + d\bar{x}^2 + dz^2 \Big] \tag{107}$$

with $\bar{x} = \sum_3^d x^i$. The patch only describes part of the entire spacetime, since it is now singular at $z = 0$, therefore it only describes values for $z \neq 0$. The patch consists of Minkowski space slices "warped" in the z-direction. After a conformal rescaling we can find for $z \to 0$:

$$ds_{CFT}^2 = -dt^2 + d\bar{x}^2 \tag{108}$$

which is the expected Minkowski metric if considered in 4 dimensions.

Another very useful coordinate transformation is the global patch. We have to transform equation (102) with the following coordinate changes. We switch to spherical coordinates for the $d-2$ sphere. We introduce angles α_3 until α_d

$$z = \frac{L_{AdS}^2}{\sqrt{L_{AdS}^2 + r^2}} \tag{109}$$

$$x^3 = \frac{1}{L_{AdS}^2} \cos\alpha_1 \tag{110}$$

$$x^i = \frac{1}{L_{AdS}^2} \sin\alpha_3 ... \sin\alpha_{i-2} \sin\alpha_{i-1} \tag{111}$$

$$x^{i-1} = \frac{1}{L_{AdS}^2} \sin\alpha_3 ... \sin\alpha_{i-2} \cos\alpha_{i-1} \tag{112}$$

We apply this to the Poincaré patch, and obtain AdS in d in global coordinates:

$$ds^2 = -(1 + \frac{r^2}{L_{AdS}^2})dt^2 + \frac{1}{1 + \frac{r^2}{L_{AdS}^2}} dr^2 + r^2 d\Omega_{d-2}^2 \tag{113}$$

The time coordinate t is now the proper time in the center of the cylinder at $r = 0$ and runs from $(-\infty, \infty)$. The radial coordinate is zero at the center of the cylinder and runs to infinity at the boundary. The radius of AdS is related to the coupling λ.
Global coordinates are 'static' coordinates. This means the spacetime now stays in one place. However, since AdS has a negative curvature, there is a potential towards the center. Because of this reason, AdS is sometimes thought of as 'gravity in a box'.
As one can observe the metric is of similar form of for example the Schwarzschild metric:

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega_{d-1}^2 \tag{114}$$

Often another transformation to *tortoise* coordinates is made. Taking $\rho = \tan r$ running from $[0, 2\pi]$, we write the metric in the following form:

$$ds^2 = \frac{1}{\cos^2\rho}\left(-dt^2 + d\rho^2\right) + \sin^2\rho d\Omega_{d-1}^2 \tag{115}$$

Where we have taken $l_{AdS} = 1$. This metric is convenient when making calculations on the AdS metric, and will be used later on when expanding quantum fields on it.

We continue by looking at the Penrose diagram of global AdS-space. Here we use r, t coordinates again. [5]

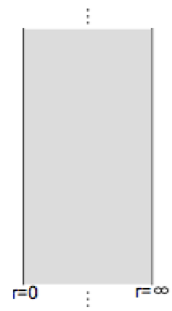

Figure 4: The Penrose diagram from global AdS space

Now one can observe several facts from this diagram. First of all, it is possible for null-geodesics to reach the boundary in finite time. This means one needs to impose boundary conditions at the boundary. By observing the diagram one notices massive particles to show a similar type of trajectory as the light rays, only they will never reach the boundary. It is nice to see how the Penrose diagram represents the potential mentioned above.
Secondly, time runs up in the diagram all the way from past infinity to future infinity. The diagram cannot be written in a more compact form, since lightlike particles keep bouncing of the boundaries forever. The boundary of AdS is timelike, and it's topological structure $\mathbb{R} \times \mathbb{S}^{d-1}$ is exactly that of a Minkowski spacetime.

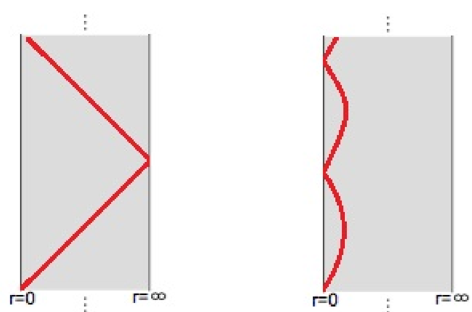

Figure 5: Trajectories of Particles in AdS. Null geodesics on the left, Massive particles on the right.

[5]Picture from [19], page 69

3.3 CFT

To get a feel of what a conformal field theory is, and what makes it interesting, it is informative to start with a quick sidestep to the concept of the renormalization group flow. This will only be for introductory purposes, therefore the tone is kept rather qualitative.
In quantum field theory the running coupling constant tells you how the theory behaves at different length scales. This quantity is governed by the renormalization group (RG) flow. The RG-flow is determined by the *Beta*-function [28]. For quantum field theories with only one coupling, like QCD and QED this is given by:

$$\beta = \mu \frac{\partial}{\partial_\mu} \alpha \tag{116}$$

We see it consisting of the slope of the coupling α to the energy scale μ. The beta function can have different values. When β is negative, the theory is called to be *"asymptotically free"*. The coupling drops to zero at large energy scales and becomes calculable. This is what happens for QCD for example. The anti-screening of the gluons causes the theory to be strongly coupled at low energies. When positive, the coupling of a quantum field theory increases at higher energy. An example of such a theory is QED. Actually, QED increases asymptotically towards the Landau pole, causing the coupling to become infinite. A similar theory is called *"infrared red free"*. Now the theory is perturbative in the low energy limit, and is an effective description of a complete theory in the UV.
Now, the beta function can become zero too. This can be the case for a theory for all length scales, on an interval, or for a theory at a 'fixed point'. The theory has become scale invariant field. Often these theories obey an even bigger symmetry group. The conformal group. The conformal group consists of:

1. **Poincaré symmetry**, Lorentz symmetry plus translations in all directions,

$$x^\mu \to \Lambda^\mu_\nu x^{nu} + a^\mu \tag{117}$$

2. **Scaling invariance / Dilatations** D^μ,

$$x \to \lambda x^\mu \tag{118}$$

3. **Special conformal transformations** K^μ

$$x^\mu \to \frac{x^\mu - b^\mu x^2}{1 - 2b \cdot x + b^2 x^2} \tag{119}$$

The main consequence one has draw from the introduction, is the fact that, due to the scale invariance of the CFT, the stress energy tensor $T_{\mu\nu}$ is traceless, $T^\mu_\mu = 0$. Put differently, the theory does not have a mass gap. The energy spectrum is continuous, causing fields not to have asymptotic states. Usually, these states are considered to be non-interacting, and used in the S-matrix

formalism to describe a scattering process. However, now due to this lack of mass gap, the entire concept of an S-matrix is ill-defined. Because of this reason, the basic observables are correlation functions of local operators $\mathcal{O}$ [29].
These local operators are often also referred to as *primary* operators. They transform under conformal rescaling in a simple way[47].

$$\mathcal{O}'(x') \to \lambda^{-\Delta}\mathcal{O}(x) \tag{120}$$

and for general conformal transformations as

$$\mathcal{O}'(x') \to \left(\frac{\partial x'}{\partial x}\right)^{-\Delta}\mathcal{O}(x). \tag{121}$$

Here Δ is the *conformal dimension* of the theory. This parameter makes sure the action of a certain quantum field theory is scale invariant. The primary fields transform under conformal transformations in general by In CFT's one is interested in computing correlation functions between primary operators. The two point function of a CFT with one scaling dimensions Δ is given by [30]:

$$\langle\mathcal{O}(x)\mathcal{O}(x')\rangle \propto \frac{1}{|x-x'|^{2\Delta}} \tag{122}$$

3.4 Relating the two sides

The two sides of the correspondence are related in the following section. AdS/CFT implies a 1-1 mapping. This includes the Hilbert spaces to be equivalent, the Hamiltonian to be the same, and, as we saw earlier, the symmetry groups to be corresponding. Next to this, we need to be able to relate any quantity in the AdS to one in the CFT. In the general case of the duality, we have seen a CFT in d dimensions, as $\mathbb{R} \times \mathbb{S}^{d-1}$, to be positioned on the boundary of the AdS^{d+1}. On the CFT side, one is interested in finding all correlation functions between local operators. Together these quantities can be encoded into a generating functional form:

$$\mathcal{Z}_{CFT}[\phi_i] = \langle e^{\int d^d x \sum_i \phi_i(x_i)\mathcal{O}(x_i)} \rangle \tag{123}$$

The correlation functions are then obtained by taking functional derivatives with respect to $\phi_i(x)$, and setting the $\phi_i(x_i)'s$ to zero. The generating functional on the CFT side can be taken exactly equivalent to the string partition function, when the boundary value of the bulk field $\phi(x)$ is given by $\phi_i(x)$. The mapping between the two sides is therefore represented by an equivalence of the partition functions.

$$\mathcal{Z}_{string}[\phi_i(x)] = \mathcal{Z}_{CFT}[\phi_i(x)] \tag{124}$$

where $\phi_i(x_i)$ corresponds to value the bulk field $\phi(x_i)$ extrapolated to the boundary at $z \to 0$.
Equation (136) is referred to by J. Polchinski as Maldacena's equation and considered to be the greatest equation of all time.

As mentioned above, one can take functional derivatives to obtain:

$$\left.\frac{\partial^n \mathcal{Z}_{CFT}}{\partial\phi_i(x_1)\partial\phi_i(x_2)...\partial\phi_i(x_n)}\right|_{\phi_i(x_i)=0} \sim \langle \mathcal{O}(x_1)...\mathcal{O}(x_n)\rangle \tag{125}$$

which works similarly on the bulk side resulting in:

$$\langle \mathcal{O}(x_1)...\mathcal{O}(x_n)\rangle = \lim_{r\to\infty} r^{-n\Delta} \langle \phi_1(t,r,\Omega)\phi_2(t,r,\Omega)...\phi_n(t,r,\Omega)\rangle \tag{126}$$

where we re-encounter the scaling dimension Δ. It is related to the mass of the field by:

$$\Delta = \frac{d}{2} + \frac{1}{2}\sqrt{d^2 + 4m^2} \tag{127}$$

Equation (125) is a statement of the correspondence more useful, when the user is interested in computing correlation functions in the bulk/boundary. In our case we will be working with scalar fields, however this statement holds equally well for higher spin fields.

3.5 Black holes in AdS/CFT

3.5.1 Thermal Field theory

Before we dive into black holes in AdS, let us take an informative detour continuing with path integrals. Our bulk path integral or the path integral for curved spacetimes in general is given by:

$$\mathcal{Z} = \int \mathcal{D}[g, \phi] e^{i \int d^d x \mathcal{L}[\phi]} \tag{128}$$

Where one does not only integrate over different field paths, but also over different metric configurations. If one considers asymptotic AdS space, this operation is not very significant, however we write it down for completeness. To evaluate this quantity, one introduces a Wick rotation to let the functional converge. Setting $\tau = it$

$$\mathcal{Z} = \int \mathcal{D}[g, \phi] e^{-\int d\tau \int d^{d-1} x \mathcal{L}_E[\phi]} \tag{129}$$

where the Lagrangian density is now in Euclidean form. We can write this differently in the canonical quantization form in terms of the Hamiltonian.

$$\int \mathcal{D}[g, \phi] e^{-\int d\tau \int d^{d-1} x \mathcal{L}_E[\phi]} = \langle \phi_f(x) | \, e^{-\tau H} \, | \phi_i(x) \rangle \tag{130}$$

Taking $\phi_f(x) = \phi_i(x)$, and integrating over ϕ we get:

$$\int D[\phi] \, \langle \phi(x) | \, e^{-\tau H} \, | \phi(x) \rangle = \operatorname{Tr} e^{-\tau H} \tag{131}$$

Identifying τ with the inverse temperature, β, we are looking at the partition function of a thermal ensemble, $\rho_{th} = \frac{e^{-\beta H}}{Z}$, at finite β.

$$\mathcal{Z} = \operatorname{Tr} e^{-\beta H} \tag{132}$$

A well known connection is made between quantum field theory at finite temperature and quantum statistical mechanics. The imaginary time can be identified with the inverse temperature. Even more so, the imaginary time τ turns out to be periodic, which can be shown when looking at expectation values of operators in a thermal state.

$$\langle O(t_1) \rangle = \operatorname{Tr}(\rho_{th} A) \tag{133}$$

Taking a product of two operators:

$$\langle O_1(t_1) O_2(t_2) \rangle = \operatorname{Tr}\big(e^{-\beta H} O_1(t_1) O_2(t_2)\big) \tag{134}$$

Working in the Heisenberg picture, and using the cyclicity of the trace:

$$\begin{aligned}
\langle O_1(t_1) O_2(t_2) \rangle &= \operatorname{Tr}\big(e^{-\beta H} O_1(t_1) O_2(t_2)\big) \\
&= \operatorname{Tr}\big(O_2(t_2) e^{-\beta H} O_1(t_1)\big) \\
&= \operatorname{Tr}\big(e^{-\beta H} O_2(t_2) e^{-\beta H} O_1(t_1) e^{\beta H}\big) \\
&= \operatorname{Tr}\big(e^{-\beta H} O_2(t_2) O_1(t_1 + i\beta)\big) \\
&= \langle O_2(t_1) O_1(t_2 + i\beta) \rangle
\end{aligned} \tag{135}$$

This result is known as the *KMS* condition. It shows that the time, t, to be periodic with $i\beta$, or τ to be periodic with β. The *KMS* condition is the condition satisfied by every *KMS* state. A system in thermal equilibrium with it's environment. A second note on the KMS condition, it tells us that Lorentz invariance is broken. Since only the time component has become periodic in Euclidean signature, we lose the fact that all spacetime directions transform in similar fashion.

With this result we can retrieve some important relations. Take for $O_1(t_1)$, and, $O_2(t_2)$, your field ϕ creation/annihilation operators $b^\dagger_\omega$, and, $b^\dagger_\omega$, and use the KMS condition.

$$\begin{aligned}\langle b^\dagger_\omega b_{\omega'}\rangle &= \mathrm{Tr}\big(e^{-\beta H} b^\dagger_\omega b_{\omega'}\big)\\ &= \mathrm{Tr}\big(e^{-\beta H} b_{\omega'} e^{-\beta H} b^\dagger_\omega e^{\beta H}\big)\end{aligned} \tag{136}$$

Using the following identity:

$$e^{\hat{A}}\hat{B}e^{-\hat{A}} = \hat{B} + [\hat{A},\hat{B}] + \frac{1}{2}[\hat{A},[\hat{A},\hat{B}]] \tag{137}$$

we write $e^{-\beta H} b^\dagger_\omega e^{\beta H}$ to first order as:

$$\begin{aligned}e^{-\beta H} b^\dagger_\omega e^{\beta H} &= b^\dagger_\omega - \beta[H, b^\dagger_\omega]\\ &= b^\dagger_\omega(1-\beta\omega)\\ &= e^{-\beta\omega} b^\dagger_\omega\end{aligned} \tag{138}$$

Where we made use of canonical commutation relation $[H, b^\dagger_\omega] = \omega b^\dagger_\omega$. Plugging this back into relation (149) and using $[b_{\omega'}, b^\dagger_\omega] = \delta_{\omega\omega'}$ we find

$$\begin{aligned}\langle b^\dagger_\omega b_{\omega'}\rangle &= e^{-\beta\omega}\langle b_\omega b^\dagger_{\omega'}\rangle\\ &= \frac{1}{e^{\beta\omega}-1}\delta_{\omega\omega'}\end{aligned} \tag{139}$$

The Bose Einstein distribution for in a thermal bath.

3.5.2 Black Holes

We have made the identification of the imaginary time to the temperature. As seen above, the black hole is a thermodynamic object with a temperature, giving of black body radiation. When trying to describe such an object in AdS/CFT this suggests an affiliation with quantum field theory at finite temperature. Within the CFT one can construct any energetic state, by acting on the vacuum with local operators. In the bulk this looks like throwing in perturbations/particles from the boundary [37]. Recall the AdS vacuum metric:

$$ds^2 = -(1+r^2)dt^2 + \frac{1}{(1+r^2)}dr^2 + r^2 d\Omega_2^2 \tag{140}$$

and the Schwarzschild metric on a Minkowski manifold.

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + \frac{1}{(1 - \frac{2M}{r})}dr^2 + r^2 d\Omega_2^2 \tag{141}$$

This suggests the following construction for a Schwarzschild black hole in AdS.

$$ds^2 = -(1 - \frac{2M}{r} + r^2)dt^2 + \frac{1}{(1 - \frac{2M}{r} + r^2)}dr^2 + r^2 d\Omega_2^2 \tag{142}$$

We can derive the temperature of the black hole by observing what happens at the horizon [46]. The horizon is at the root of $V(r) = (1 - \frac{2M}{r} + r^2)$. Rewrite this by: $r = r_h + \rho^2$.

$$\begin{aligned} 1 - \frac{2M}{r} + r^2 &= 1 - \frac{2M}{r_h + \rho^2} + (r_h + \rho^2)^2 \\ &= \frac{1}{r_h + \rho^2}\left[r_h - 2M + r_h^3 + \rho^2(1 + 3r_H^2) + 3r_h^2\rho^4 + \rho^6\right] \end{aligned} \tag{143}$$

Now looking at what happens close to the horizon, we drop the constants and look to leading order in ρ what happens for $\rho \ll 1$. The higher order terms go to zero much faster, and can be dropped too.

$$V(\rho) = \frac{1}{r_h + \rho^2}\left[r_h - 2M + r_h^3 + \rho^2(1 + 3r_H^2) + 3r_h^2\rho^4 + \rho^6\right] \to \frac{1 + 3r_h}{r_h^2}\rho^2 \tag{144}$$

As we saw earlier, the temperature is related to the euclidean time τ. Wick rotating the metric and plugging back $V(\rho)$ we find for the metric:

$$ds^2 = \frac{4r_h}{1 + 3r_h^2}\left[\left(\frac{1 + 3r_h^2}{2r_h}\right)^2 \rho^2 d\tau^2 + d\rho^2\right] + r_h^2 d\Omega_2^2 \tag{145}$$

Now there is a nice trick here. Observe the term between brackets being similar to the euclidean plane in terms of polar coordinates is given by

$$ds^2 = r^2 d\theta^2 + dr^2 \tag{146}$$

In these coordinates there is what is called a 'conical singularity' at $r = 0$. The coordinate θ is not well defined at $r = 0$. Imagine walking over the pole on the earth at $r = 0$. Your polar coordinate θ will discontinuously make a 180 degree turn. However due to the fact that θ is periodic with 2π we do not experience any problems when computing smooth curves on and over $r = 0$. In other words, the singularity is coordinate dependent now.

For the black hole we have to do he same thing. By making τ periodic with $\beta = 2\pi \times \frac{2r_h}{1+3r_h^2}$, the metric becomes smooth at $r = r_h$. For the temperature we find:

$$T = \frac{1 + 2r_h^2}{4\pi r_h} \tag{147}$$

Plotting this gives us,

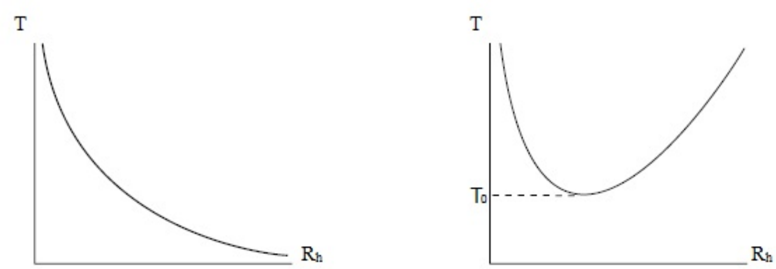

Figure 6: Black hole in Schwarschild geometry vs. Black holes in AdS

One observes black holes to have positive specific heat in AdS after critical temperature T_0. Physically, this corresponds to the Hawking radiation bouncing of the boundary in AdS and returning into the black hole. This has the effect for the black hole to become stable and in thermal equilibrium with it's environment. Black holes are named 'Big', when they are stable in AdS. Big black holes do not evaporate and are therefore eternal. The information problem remains to exist, as a question of how one consistently constructs the interior of the black hole (see chapter 3.7).

3.5.3 Two sided black hole in AdS/CFT

In 2001 J. Maldacena conjectured another very important result in AdS/CFT [35]. He proposed a duality for the maximally extended AdS-Schwarzschild geometry. In his paper he showed the geometry to be dual to two copies of a conformal field theory in a highly specifically entangled state, called the *Thermofield double state*. This state has it's origin a while ago when thermal field theory was developed. If one tries to describe a thermal system, characterized by a thermal density matrix $\rho_{th} = e^{-\beta H}/\mathcal{Z}$, by a pure state, one is not possible to do so with one Hilbert space $\mathcal{H}_1$. By doubling the degrees of freedom, forming a second copy of the Hilbert space $\mathcal{H}_2$, one constructs an entangled pure state with $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ in the following way:

$$|TFD\rangle = \sum_n \frac{e^{-\frac{\beta E_n}{2}}}{\sqrt{\mathcal{Z}}} |E_n\rangle_1 \otimes |E_n\rangle_2 \tag{148}$$

where $|E_n\rangle$ are the set of orthonormal eigenstates of the Hamiltonian H. Taking the partial trace over the full density matrix ρ , $\mathrm{Tr}_2 |TFD\rangle \langle TFD|$, over one Hilbert space to find:

$$\rho_1 = \sum_n \frac{e^{-\beta E_n}}{\mathcal{Z}} |E_n\rangle_1 \langle E_n|_1 = \frac{e^{-\beta H}}{\mathcal{Z}} \tag{149}$$

The thermal density matrix. Each of the two sides correspond exactly to a thermal state. Correlation functions within the thermofield state are given by their thermal expectation value:

$$\langle TFD| \mathcal{O}_1\mathcal{O}_2.....\mathcal{O}_n |TFD\rangle = \frac{1}{\mathcal{Z}} \mathrm{Tr}\big[e^{-\beta H}\mathcal{O}_1\mathcal{O}_2.....\mathcal{O}_n\big] \tag{150}$$

Maldacena showed this state to have a holographic interpretation. With both systems corresponding to a CFT dual to an asymptotic region of AdS connected via a wormhole. The two CFT's are together in the entangled state given by (151), and are non-interacting. The remarkable fact of the proposal is the geometric connection that forms purely from the entanglement between the two sides. It is in principle possible for two observers in both asymptotic regions to meet in the interior of the black hole.

The metric of AdS-Schwarzschild in d dimensions is given by:

$$ds^2 = -(r^2 - r_h^2)dt^2 + \frac{1}{(r^2 - r_h^2)}dr^2 + r^2 d\Omega_{d-2}^2 \tag{151}$$

extending this with kruskal transformations:

Region I	$U = -e^{-r_h(t-r_*)}$	$V = e^{r_h(t+r_*)}$
Region II	$U = e^{-r_h(t-r_*)}$	$V = e^{r_h(t+r_*)}$
Region III	$U = e^{-r_h(t-r_*)}$	$V = -e^{r_h(t+r_*)}$
Region IV	$U = -e^{-r_h(t-r_*)}$	$V = -e^{r_h(t+r_*)}$

the metric becomes:

$$ds^2 = \frac{(r^2 - r_h^2)}{r_h^2 UV} dU dV + r^2 d\Omega_{d-2}^2 \tag{152}$$

With asymptotic region I defined for $U < 0$ and $V > 0$, region III for $U > 0$ and $V < 0$ and the interior for $U > 0$ and $V > 0$. The singularity is at $U = V = 0$.

The penrose diagram is depicted below.

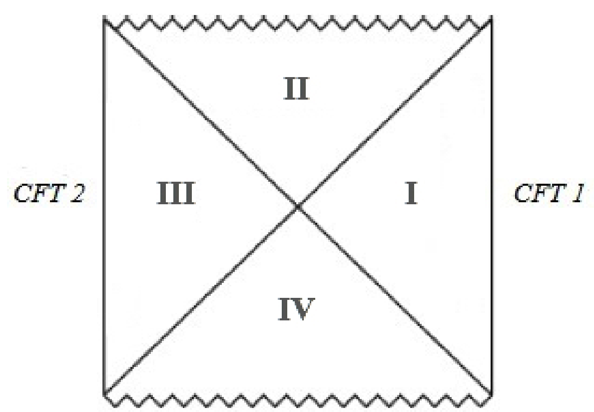

Figure 7: The Thermofield double state

A very important point is the Hamiltonian of the system. Due to the killing isometry ∂_t, time runs up in CFT 1 and down in the second CFT. Similarly in the usual extended Schwarzshild geometry. The Hamiltonian in this case is defined as:

$$H = H_1 - H_2 \tag{153}$$

The state is time-independent.

$$e^{-iHt} |TFD\rangle = \frac{1}{\sqrt{\mathcal{Z}}} \sum_n e^{-i(H_1 - H_2)t} e^{-\frac{\beta E_n}{2}} |E_n\rangle_1 \otimes |E_n\rangle_2 = |TFD\rangle \tag{154}$$

But, in this case there is **no** connection possible between the two asymptotic regions. This state just resembles a black hole in thermal equilibrium.

If one wants to be able to meet observers from the other region via the wormhole, one has to apply a CPT conjugation on one of the two sides. Time now runs up on both sides, and observers from both asymptotic regions can in principle meet in the interior of the black hole. The Hamiltonian now is given by:

$$\tilde{H} = H_1 + H_2 \tag{155}$$

and the thermofield state is in that case

$$|\tilde{TFD}\rangle = \sum_n \frac{e^{-\frac{\beta E_n}{2}}}{\sqrt{\mathcal{Z}}} |\tilde{E_n}\rangle_1 \otimes |E_n\rangle_2 \tag{156}$$

where $|\tilde{E_n}\rangle_1$ is the CPT conjugation of $|E_n\rangle_1$.
Both wedges still are described by ρ_{th}. We find for operators $\mathcal{O}$ in the thermal CFT state similar relations as (152). We can write down expectation values for two point correlators by using thermal field theory.

$$\begin{aligned}\langle TFD| \mathcal{O}^{\dagger}_{\omega}\mathcal{O}_{\omega'} |TFD\rangle &= \frac{1}{\mathcal{Z}} \operatorname{Tr}\left[e^{-\beta H}\mathcal{O}^{\dagger}_{\omega}\mathcal{O}_{\omega'}\right] \\ &= \frac{1}{e^{\beta\omega}-1}\delta_{\omega\omega'}\end{aligned} \tag{157}$$

The holographic interpretation relates the primary CFT operators in a thermal state, to creation and annihilation operators in the bulk [13].

$$\mathcal{O}_{\omega} \Leftrightarrow b_{\omega} \tag{158}$$

This very important relation shows us the identifcation between the bulk and the boundary operators. In fact the thermal state on the boundary can be interpreted as a quark gluon plasma[13].
The state is considered to be an example of a black hole with a smooth horizon. In 2013 Hartman and Maldacena provided some evidence in terms of computing correlators between the two sides, and showing the evolution of the mutual information between the two CFT's to evolve exactly like the growth of the interior of the black hole [36]. There is agreement on the smoothness of the state, which is why the thermofield state is often used as a reference point.
The characteristic entanglement of the state causing it the have a smooth horizon, is key for understanding what is needed to solve the information paradox. The astonishing fact that two non-interacting CFT's entangled in the thermofield state generate a geometric connection via a wormhole remains until today somewhat of a mystery. It suggests a very deep meaning of quantum entanglement. Van Raamsdonk and others continued investigating the idea that entanglement could generate geometry in a more general way. He formulated by the means of the Ryu Takayanagi proposal[6] a way to extract the Einstein equations purely coming from computing entanglement entropies. [38]

[6]The RT proposal relates the entanglement entropy of a region A in a CFT in a state $|\psi\rangle$ to the extremal surface area of a plane with codimension 2 whos boundary is given by the CFT [39]

3.6 Fields in AdS

To perform computations, one has to lay down quantum field theory on an AdS background. In usual quantum field theory the fields are expanded in Minkowski spacetime, with it's 'flat' spacetime signature. In our case we need to quantize the fields in a curved space-time. As stated in the quantum field theory section this procedure is different then the normal/flat case. The general second-quantization procedure of fields in curved space time is described in [32]. As for our goal the theory is used as a computational tool, we do not review this section in this thesis completely.
The easiest way to do so is to look at scalar fields. We will solve the Klein-Gordon equation for the 3 dimensional case. One can of course solve the equation in higher dimensions, however this will not do anything different with the relevant physics. The Klein Gordon equation is given by:

$$\Box\phi = m^2\phi \tag{159}$$

We work in 'mostly plus'-signature. The thing that has changed now is that the D'alembertian now is given by:

$$\Box = \frac{1}{\sqrt{|g|}}\partial_\nu\sqrt{|g|}g^{\mu\nu}\partial_\mu \tag{160}$$

here $g^{\mu\nu}$is the inverse of the metric, and g is the determinant of the metric. For a diagonal matrix this is given by the product of it's components:

$$\det(A) = \prod_i A_{ii} \tag{161}$$

Empty AdS-space-time in 2+1-dimensions in global coordinates can be written down by equation (113):

$$ds^2 = -(1+r^2)dt^2 + \frac{1}{(1+r^2)}dr^2 + r^2 d\Omega_1^2 \tag{162}$$

Where $d\Omega_1$ represent the spherically symmetric angular part in a 1-sphere. Continuing with this metric to the Klein-Gordon equation we observe that we have two Killing vector fields, one timelike and one spacelike given by: $K_1 = (-(1+r^2), 0, 0,)$, and $K_2 = (0, 0, 0, r^2\sin^2(\theta))$. This means that the time, and angular component of the Klein-Gordon equation will give trivial solutions. We see that the these components pass trough equation (117) smoothly and give straightforward results. We write the Klein-Gordon equation as a result following equation (117) for each component:

$$-\frac{1}{(1+r^2)}\frac{\partial^2\phi}{\partial t^2} + \frac{1}{r}\partial_r r(1+r^2)\partial_r\phi + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\Omega^2} - m^2\phi = 0 \tag{163}$$

where $\sqrt{|g|} = r$. To solve this equation we need a trial solution. As mentioned above, the time part as well as the angular part need trivial solutions, leaving the radial part to be solved. The solution is of a complete set of modes with corresponding creation and annihilation operators.

$$\phi(t,r,\Omega) = \sum_{\omega,l,m} [a_{\omega,lm} f_{\omega,lm}(t,r,\Omega) + a^{\dagger}_{\omega,lm} f^{*}_{\omega',l'm'}(t,r,\Omega)] \tag{164}$$

Where $f_{\omega,lm}(t,r,\Omega)$ is given by:

$$f_{\omega,lm}(t,r,\Omega) = e^{-i\omega t + ik\Omega}(\Omega)\frac{U(r)}{r} \tag{165}$$

If we plug this into the differential equation we retrieve:

$$\frac{\omega^2}{(1+r^2)}\frac{U(r)}{r} + \frac{1}{r}\partial_r r(1+r^2)\partial_r \frac{U(r)}{r} - \frac{k^2}{r^2}\frac{U(r)}{r} - m^2\frac{U(r)}{r} = 0 \tag{166}$$

We observe that we only have a radial dependence left. This is the equation to solve. Working out the algebra of the middle part, obtained from the D'alembertian and putting this back in equation (7) we obtain:

$$\begin{aligned}\frac{1}{r}(1+r^2)\frac{\partial^2 U(r)}{\partial r^2} + \frac{(r^2-1)}{r^2}\frac{\partial U(r)}{\partial r} + \frac{\omega^2}{(1+r^2)}\frac{U(r)}{r} + \frac{1}{r}(\frac{1}{r^2} - 1)U(r) \\ - \frac{k^2}{r^2}\frac{U(r)}{r} - m^2\frac{U(r)}{r} = 0\end{aligned} \tag{167}$$

Some more algebra, multiplying with r and rearranging gives us:

$$(1+r^2)^2\frac{\partial^2 U(r)}{\partial r^2} + \frac{(r^2-1)(r^2+1)}{r}\frac{\partial U(r)}{\partial r} + V(r)U(r) = -\omega^2 U(r) \tag{168}$$

with

$$V(r) = \frac{(1+r^2)}{r^2}[1 - k^2 - (m^2+1)r^2] \tag{169}$$

Now this differential equation is solvable with hypergeometric functions. The solutions are given by:

$$\begin{aligned}U(r) \rightarrow \left(r^2\right)^{\frac{k+1}{2}}\left(r^2+1\right)^{w/2}\Bigg(&C_1 * \\ &{}_2F_1\left[\frac{1}{2}(k+w-\Delta), \frac{1}{2}(k+w+\Delta); k+1; -r^2\right] \\ &+ C_2\left(-1-r^2\right)^{-k} * \\ &{}_2F_1\left[\frac{1}{2}(-k+w-\Delta), \frac{1}{2}(-k+w+\Delta); 1-k; -r^2\right]\Bigg)\end{aligned} \tag{170}$$

where $\Delta = 1 + \sqrt{m^2+1}$. The constants can be found using boundary conditions.

To observe the asymptotic behavior at $r \to \infty$, we can approximate the radial solution by a power law:

$$\phi(t, r, \Omega) = e^{-i\omega t} e^{ik\Omega} r^a \tag{171}$$

Substituting into differential equation (120) and simplifying trough the same trivial steps as before

$$-\frac{1}{r^2}\omega^2 r^a + \frac{1}{r}\partial_r r^3 \partial_r r^a + \frac{k^2}{r^2} r^a - m^2 r^a = 0 \tag{172}$$

Working out the radial part.

$$-\omega^2 r^{a-2} + a(a+2) r^a + k^2 r^{a-2} - m^2 r^a = 0 \tag{173}$$

We are interested in the limit of $r \to$, therefore only considering dominant terms, r^a,

$$a(a+2) r^a = m^2 r^a \tag{174}$$

Which gives a second order equation for a:

$$a^2 + 2a - m^2 = 0 \tag{175}$$

where the roots are given by

$$a = -1 \pm \sqrt{1 + m^2} \tag{176}$$

The negative root is exactly the conformal dimension, Δ, of the boundary field. In other words we see the how the bulk is connected to the CFT boundary .

3.6.1 Fields in eternal BTZ

In this thesis the BTZ black hole is used to do calculations on. The next step is to do the same thing we did in empty AdS, only now for a metric containing a black hole. This metric for the eternal BTZ black hole in $2+1$ dimensions is given by:

$$ds^2 = -(r^2 - r_h^2)dt^2 + \frac{1}{(r^2 - r_h^2)}dr^2 + r^2 d\Omega_1^2 \tag{177}$$

Again we solve the Klein Gordon equation:

$$\Box\phi = m^2\phi \tag{178}$$

We have a complete set of solutions:

$$\phi(t, r, \Omega) = \int_0^\infty \frac{d\omega dk}{2\pi\sqrt{2\omega}}[b_{\omega,k} f_{\omega,k}(t, r, \Omega) + b^\dagger_{\omega,k} f^*_{\omega,k}(t, r, \Omega)] \tag{179}$$

The operators are related in the usual way:

$$[b_{\omega,k}, b^\dagger_{\omega',k}] = \delta(\omega - \omega')\delta(k - k') \tag{180}$$

We have for the wavefunctions

$$f_{\omega,k}(t, r, \Omega) = e^{-i\omega t + ik\Omega}\Psi(r)_{\omega,k}. \tag{181}$$

In similar fashian as for empty AdS, we end up with a differential equation for the radial solutions:

$$-(r^2 - r_h^2)^2 \frac{\partial^2\Psi(r)}{\partial r^2} - 2r(r^2 - r_h^2)\frac{\partial\Psi(r)}{\partial r} + V(r)\Psi(r) = \omega^2\Psi(r) \tag{182}$$

with

$$V(r) = (r^2 - r_h^2)\left[\frac{k^2 + \frac{r_h^2}{4}}{r^2} + m^2 + \frac{3}{4}\right] \tag{183}$$

This is another differential equation solved by hypergeometric functions.

$$\begin{aligned}\Psi(r) \to & \left(r^2\right)^{\frac{r_h - ik}{2r_h}} \left(r^2 - r_h^2\right)^{-\frac{iw}{2r_h}} \left(c_2 e^{-\frac{\pi k}{r_h}} \left(\frac{r^2}{r_h^2}\right)^{\frac{ik}{r_h}} \right. \\ & {}_2F_1\left(\frac{r_h^2 + ikr_h - iwr_h - \sqrt{(m^2+1)\, r_h^4}}{2r_h^2}, \right. \\ & \left. \frac{r_h^2 + ikr_h - iwr_h + \sqrt{(m^2+1)\, r_h^4}}{2r_h^2}; \frac{ik}{r_h} + 1; \frac{r^2}{r_h^2}\right) + \\ & c_1\, {}_2F_1\left(\frac{r_h^2 - ikr_h - iwr_h + \sqrt{(m^2+1)\, r_h^4}}{2r_h^2}, \right. \\ & \left.\left. -\frac{-r_h^2 + ikr_h + iwr_h + \sqrt{(m^2+1)\, r_h^4}}{2r_h^2}; 1 - \frac{ik}{r_h}; \frac{r^2}{r_h^2}\right)\right)\end{aligned} \tag{184}$$

We can make several substitutions:

$$a = \frac{ik}{2r_h} \;,\; b = \frac{i\omega}{2r_h} \;,\; \Delta = 1 + \sqrt{m^2+1} \tag{185}$$

This gives us:

$$\begin{aligned}\Psi(r) \to & \left(r^2\right)^{1-a} \left(r^2 - r_h^2\right)^{-b} \left(C_2 e^{-\frac{\pi k}{r_h}} \left(\frac{r^2}{r_h^2}\right)^{2a} \right. \\ & {}_2F_1\left(1 - a - b - \frac{\Delta}{2}, a - b + \frac{\Delta}{2}; 2a+1; \frac{r^2}{r_h^2}\right) + C_1, \\ & \left. {}_2F_1\left(\frac{\Delta}{2} - a - b, \frac{\Delta}{2} - 1 + a + b; 1 - 2a; \frac{r^2}{r_h^2}\right)\right)\end{aligned} \tag{186}$$

The solution is of the form of:

$$\Psi(r) = r^2(r^2 - r_h^2)^{-b}\Big[(r^2)^{-a} C_1 P(r) + (r^2)^a C_2 Q(r)\Big] \tag{187}$$

With

$$\begin{aligned} P(r) &= {}_2F_1\left(\frac{\Delta}{2} - a - b, \frac{\Delta}{2} - 1 + a + b; 1 - 2a; \frac{r^2}{r_h^2}\right) \\ Q(r) &= (\frac{1}{r_h^2})^a\, {}_2F_1\left(1 - a - b - \frac{\Delta}{2}, a - b + \frac{\Delta}{2}; 2a+1; \frac{r^2}{r_h^2}\right)\end{aligned} \tag{188}$$

The solution should be subject to two boundary conditions at the boundary and the horizon. In the *near horizon* limit when $r \to r_h$. As ${}_2F_1(a,b,c,1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$, the hypergeometric functions contribute by a constant. It is convenient to transform to tortoise coordinate r_*

$$r_* = \frac{1}{r_h}\log\left(\frac{r-r_h}{r+r_h}\right) \tag{189}$$

The solution is given in the near horizon limit, $r_* \to -\infty$ by:

$$\Psi(r) \sim \mathcal{C}\left[e^{-i\delta_{\omega k}}e^{-i\omega r_*} + e^{i\delta_{\omega k}}e^{i\omega r_*}\right] \tag{190}$$

Where δ is a real constant in terms of gamma functions given by:

$$e^{i\delta} = 4^b\sqrt{\frac{\Gamma(-2b)\Gamma(-a+b+\Delta/2)\Gamma(-a+b+\Delta/2)}{\Gamma(2b)\Gamma(-a-b+\Delta/2)\Gamma(a-b+\Delta/2)}}. \tag{191}$$

The second boundary condition is when $r \to \infty$ at the boundary of AdS. In this limit the wavefunction should be related to the conformal dimension of the dual field theory Δ. By using the following hypergeometric identity[43]:

$$\begin{aligned}{}_2F_1(a,b,c,z) \propto\ & \frac{\Gamma(-a+b)\Gamma(c)}{\Gamma(b)\Gamma(c-a)} \\ & (-z)^{-a}\left(1+O(\frac{1}{z})\right)+\frac{\Gamma(a-b)\Gamma(c)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b}+O(\frac{1}{z})\left(1+O(\frac{1}{z})\right)\end{aligned} \tag{192}$$

We can expand around infinity and find that only a particular combination of $P(r)$ and $Q(r)$ is appropriate as the other one goes like r^{Δ} which blows up when $r \to \infty$. We normalize the solution by demanding:

$$\Psi(r) \to \frac{1}{\Upsilon(\omega,k)}r^{-\Delta} \tag{193}$$

at the boundary. Where Υ is given by [34]

$$\Upsilon = (2\pi)^{\frac{d}{2}}\sqrt{2\frac{\Gamma(\Delta-\frac{d}{2}+1)\pi^{\frac{d}{2}}}{\Gamma(\Delta)}} \underset{\text{for } d=2}{\Rightarrow} (2\pi)^{\frac{3}{2}}. \tag{194}$$

One finds for the full normalized mode:

$$\begin{aligned}&\Psi(r) = \\ &\frac{1}{\Gamma(\Delta)}\mathcal{D}_1\left(\frac{r^2}{r_h^2}\right)^a\left(\frac{r^2}{r_h^2}-1\right)^{-a-\frac{\Delta}{2}}{}_2F_1\left(a-b+\frac{\delta}{2}, a+b+\frac{\Delta}{2}, \Delta, \frac{r_h^2}{r_h^2-r^2}\right)\end{aligned} \tag{195}$$

with

$$\mathcal{D}_1 = \sqrt{\frac{\Gamma(a+b+\frac{\Delta}{2}\Gamma(a-b+\frac{\Delta}{2}\Gamma(-a+b+\frac{\Delta}{2}\Gamma(-a-b+\frac{\Delta}{2}}{\Gamma(2b)\Gamma(-2b)}} \tag{196}$$

With both limits, horizon and boundary, we can describe the full region of spacetime in region I. With a similar expansion in region III, we can continue both modes from both wedges to fully construct the interior of the black hole, region II. The expansion behind the horizon in terms of both wedges is given by

$$\begin{aligned}\phi(t,r,\Omega)_{II} = &\int_0^\infty \frac{d\omega' dk}{2\pi\sqrt{2\omega'}} \\ &b_R g^{(1)}_{\omega' k}(t,r,\Omega) + b_L^\dagger g^{*(2)}_{\omega' k}(t,r,\Omega) + b_R^\dagger g^{*(1)}_{\omega' k}(t,r,\Omega) + b_L g^{(2)}_{\omega' k}(t,r,\Omega),\end{aligned} \tag{197}$$

which in the near horizon approximation looks like for a spherically symmetric setting

$$g^{(1)}_{\omega'}(t_2,r_2) = r_h^{-\frac{1}{2}} e^{-i\delta_{\omega k}} e^{-i\omega(t+r_*)} \text{ and } g^{(2)}_{\omega'}(t_2,r_2) = r_h^{-\frac{1}{2}} e^{-i\delta_{\omega k}} e^{-i\omega(t-r_*)}. \tag{198}$$

obtaining:

$$\begin{aligned}\phi(t,r,\Omega)_{II} \underset{r\to r_h}{\to} \\ \frac{1}{\sqrt{r_h}} \int_0^\infty \frac{d\omega' dk}{2\pi\sqrt{2\omega'}} \left[b_R e^{-i\delta_{\omega k}} e^{-i\omega(t+r_*)} + b_L^\dagger e^{i\delta_{\omega k}} e^{i\omega(t-r_*)} + \text{h.c.} \right]\end{aligned} \tag{199}$$

When correlating different wedges, it is necessary to write this in Kruskal-Szekeres form, since coordinate patch describes the entire region of all four wedges together. Recall, in region II we have: $U = e^{-r_h(t-r_*)}$ and $V = e^{-r_h(t+r_*)}$:

$$\begin{aligned}\phi(t,r,\Omega)_{II} \underset{r\to r_h}{\to} \\ \frac{1}{\sqrt{r_h}} \int_0^\infty \frac{d\omega' dk}{2\pi\sqrt{2\omega'}} \left[b_R e^{-i\delta_{\omega k}} V^{-ia\omega'} + b_L^\dagger e^{i\delta_{\omega k}} U^{ia\omega'} + \text{h.c.} \right]\end{aligned} \tag{200}$$

where $a = \beta/2\pi$, and β is related to the horizon radius by $\beta = 2\pi/r_h$. We now have a full set of solutions for a scalar field in $2+1$ dimensions for the eternal BTZ-black hole. The setting can be used to actually compute correlation functions or other desired quantities.

To do so, one needs to be able to relate the left and the right operators. Here we can use the relations found from the Rindler decomposition. Since both subsystems left and right are thermal systems they are related in the same way as the left and right Rindler wedges, as they were thermal systems too. Using equation (81) and (82)

$$b_{L,\omega_1} |TFD\rangle = e^{-\frac{\beta\omega}{2}} b^\dagger_{R,\omega_2} |TFD\rangle \tag{201}$$

and

$$b^\dagger_{L,\omega_1} |TFD\rangle = e^{\frac{\beta\omega}{2}} b_{R,\omega_2} |TFD\rangle \tag{202}$$

3.7 Information problem in AdS/CFT

3.7.1 Construction of the interior

With the knowledge on how to describe black holes in AdS, we can study the information problem in the correspondence. The problem arises when describing quantum field theory in the interior of the black hole. Below there is the standard Penrose diagram[7] of a black hole formed from collapse. There are three types of quantum modes present. The red and green modes are interior modes, respectively left and right moving ones. The purple mode is a right moving exterior mode.

By definition the interior lies in the causal future of the exterior of the black

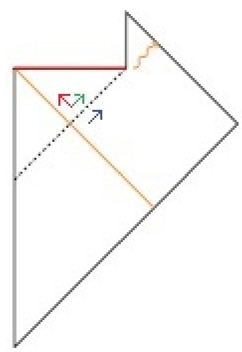

Figure 8: Black hole formed from gravitational collapse. The orange line represents a collapsing photon shell, the dotted line the event horizon, and the red line is the singularity. On the top right corner, one can see the Hawking radiation coming out. The green and blue arrows represents right moving interior/exterior modes.[9]

hole. One therefore expects to construct the interior by continuing the exterior modes forward in time. The left movers, the red arrow, can be continued without any difficulty. However, a problem arises with the right moving modes. One can observe two colored arrows in the picture. A resp. interior(green) and exterior (blue) right moving mode. Both modes extrapolated back in time coast along the horizon, and therefore will be highly blue-shifted. This blue-shifting can go up so high, one obtains center of mass energies higher then the Planck scale. This problem is referred to as the *Trans-Planckian problem.* The interior modes need to be traced back trough the shell, reflected of the boundary at $r = 0$, and back into space.

[7]For an explanation of Penrose diagrams, see appendix B

To somehow facilitate the issue here, one can construct the interior modes in a related way. The usual way to do it is to look at the eternal black hole or Hartle-Hawking state. This two-sided geometry offers the possibility for the right moving modes to be evolved from the left asymptotic region, often referred to as region III. Let's look at the this in detail: One can trace the right moving

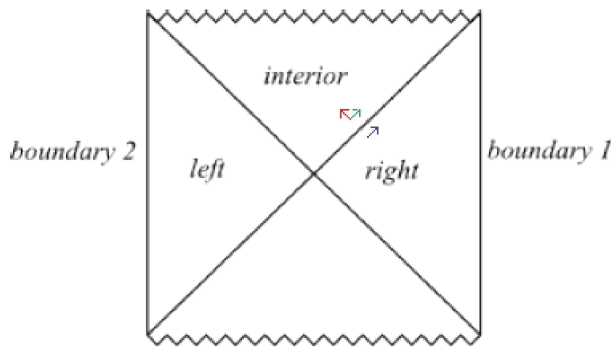

Figure 9: The Penrose diagram of the eternal AdS black hole.

modes back from the interior, region II, to the left region. In this way one can construct the interior of the black hole more naturally. What is very obvious now, is the connection with Rindler space. The main result of Rindler was the right wedge entanglement with the left wedge. This means for the black hole that modes from the interior need to be entangled with modes from exterior in a very specific way!

3.7.2 Quantum Cloning

With this prescription for the interior of the black hole, we can continue to see what more AdS/CFT has in store. If one assumes the duality to hold in whatever case, it tells you the evaporation process should be describable within the gauge theory, excluding the possibility of non-unitarity and information loss. This means that any information that falls in should be contained inside the Hawking radiation in some-kind of way. However if one does so, new troubles arise. Let us look at the evaporation of a black hole formed from collapse.

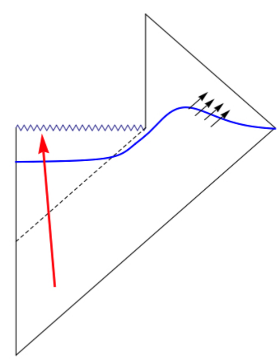

Figure 10: One sided black hole fromed from collapse. The right arrow represents one observer. The black arrows represent that observer coming out in Hawking radiation to keep the process unitary [11]

There are two arrows in figure 10. The red arrow represents an infalling observer. The black arrows represent that infalling observer coming out in Hawking radiation. One can draw a so called *nice slice* trough both copies. A *nice slice* is often called a Cauchy slice. It is a spacelike surface, $\mathcal{S}$, on a manifold, $\mathcal{M}$, intersected by every non-spacelike (causal curve) only once. For example, in Minkowski spacetime a slice at $t = a$ resembles a Cauchy slice. This blue nice slice shows us that at a certain instant in time, two copies of the same information are present. This is not possible within quantum mechanics. In fact, it violates the principle of quantum cloning. A statement which can be proven in two lines.

Take two quantum states, $|\phi\rangle \,\&\, |\psi\rangle$. Call $\hat{Q}$ the cloning operator. If we assume $\hat{Q}$ to be a linear operator, we can observe its effect by letting it act on $|\phi\rangle \,\&\, |\psi\rangle$.

$$\hat{Q}\,|\phi\rangle + \hat{Q}\,|\psi\rangle = |\phi\rangle\,|\phi\rangle + |\psi\rangle\,|\psi\rangle \tag{203}$$

However we find a contradiction if we put $|\phi\rangle \,\&\, |\psi\rangle$ in a superpositioned state:

$$\hat{Q}(|\phi\rangle + |\psi\rangle) = |\phi\rangle\,|\phi\rangle + |\psi\rangle\,|\psi\rangle + 2\,|\phi\rangle\,|\psi\rangle \tag{204}$$

Quantum superposition tells you (202) and (203) should of course be equal, which they aren't.

3.7.3 Black Hole Complementarity

A proposal for a solution to this problem came from Susskind and collaborators [32]. They came forward with a somewhat abstract proposal. Although two copies of the same information were present on one nice slice, no single observer could observe both copies. Susskind et. al. argued that to describe the process of black hole evaporation, a full formulation of quantum gravity was needed. It would therefore be very well possible, for the lower energy effective theory not to desribe the physics accurately. They presumed quantities, like the nice slice, to be ill defined in the effective description of the theory, and one should restrict the Hilbert space to the causal diamond of a single observer.
The idea seems radical at first glance, however becomes more conceivable after a second. In a way this is what happens with AdS/CFT. There are two descriptions of the same quantity. Both cannot be observed by a single observer.

3.7.4 Strong Subadditivity and Firewalls

Black hole complementarity seemed to be saving the situation, however new additional problems were formulated by S. Mathur and the AMPS group [5][6]. They were able to show inconsistencies for the construction of the interior even within the causal region of one observer. They formulated the information paradox for the infalling observer in terms of entanglement entropies.
Let us start with a black hole formed from gravitational collapse in a pure state. Next to a necessary entanglement between interior and exterior modes, we demand our final state of Hawking radiation to be pure again. This insists a high degree of entanglement between the early Hawking radiation/exterior modes and the late radiation. We run into a conflict of entanglement. As always a picture says more then a thousand words:

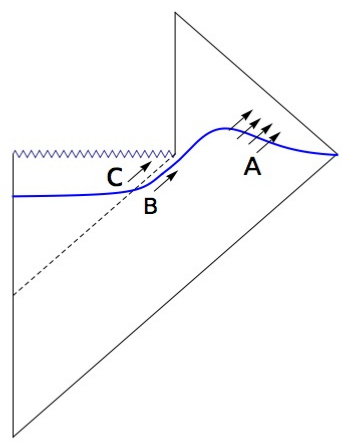

Figure 11: Black hole formed from collapse. Three sets of modes. Modes B are entangled with interior modes C. However, modes A need to be entangled with modes B as well for purity of the final state

The statements can be made precise by looking at the strong subadditivity relation for von Neumann entropies. Strong subadditivity inequality states:

$$S_{BC} + S_{BA} \geq S_B + S_{ABC} \tag{205}$$

Since Hawking modes B and C are together in a pure state, we have $S_{BC} = 0$. This tells you immediately that $S_{ABC} = S_A$.

$$S_{BA} \geq S_B + S_A \tag{206}$$

This cannot be, since S_{BA} should be lower then S_B, for the black hole to end up in a pure state.

AMPS argue that there is another time scale relevant for the evaporation process, the *scrambling time*. A time scale for infalling information to be 'scrambled' into purely thermalized bits. From that moment on the entropy of the infallen information and the black hole is maximized and should decrease after. The black hole is one of the fastest scramblers out there, causing the scrambling time to be much smaller then the Page time. This leaves the concept of an old black hole highly irrelevant, and sharpens the situation dramatically. The conclusion of the paper is that since both demands of entanglement cannot be fullfilled, the entanglement between modes B and C should not be there in the first place. They introduce the concept of a *Firewall*. Something at the horizon, which breaks up the entanglement between the interior and exterior region, to solve the paradox. This statement is highly disputable to say the least. According to the equivalence principle in general relativity, the horizon should not be a special place for the infalling observer. The firewall proposal would violate this fundamental feature dramatically.
AMPS argue that the 3 following *postulates* can not coexist together:

1. **Unitarity**. The process is describable by an S-matrix. If violated, infalling information about the purity of a state will be lost forever, and absent in the outcoming Hawking radiation.
2. **Effective Field Theory**. One can describe the region outside the black hole by gravity in terms of an effective field theory.
3. **No Drama**. Nothing happens at the horizon for the infalling observer.

Often the relation between the number of microstates and the Bekenstein entropy of the black hole is stated as a fourth postulate. This however comes down to the same issue of whether or not the entropy goes to zero at the end of the page curve. As therefore can be taken under the same postulate as postulate 1.

3.8 Entanglement vs Typicality

The period after the AMPS paper the discussion shifted to a debate of entanglement versus typicality. What exactly is meant by typicality will become clear in a moment. As we have seen, black holes in AdS thermalize under their own Hawking radiation. Which makes them, after some time, very well approximated by the two-sided black hole in AdS. In this case the dual field theory on the boundary of the AdS space is represented by a thermal density matrix ρ_{th} consisting of e^S states. Where S is the Bekenstein entropy of the black hole.[12]. Now coming back to typicality, a typical state of this ensemble is picked at random by the Haar measure.[13] For a highly energetic thermal system this would suggest random phases for all matrix elements, and absolutely not the highly specific entanglement structure of the thermofield state. The thermofield state was until that day the only state of which there was consensus on the smoothness of the horizon. However, it was just one state of the entire ensemble in the CFT. The question phrased by Marolf and Polchinski was, if typical states with generic entanglement structures were dual to smooth horizons too. Something not too much to ask, if you realize how unlikely, $\sim e^{-S}$, it is for the black hole to be exactly in the thermofield double state.

In [37] they answered the question themselves, by using the Eigenstate Thermalization Hypothesis (ETH) [42]. They approximated expectation values of operators in the dual CFT as how they would behave in a quantum chaotic system. In a thermal system you expect operators to take their thermal expectation value after some time t, the scrambling time.

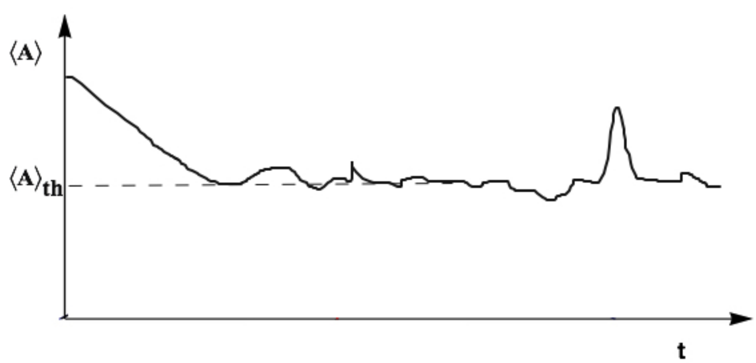

Figure 12: Thermalization of operators in a thermal system.

[12]Here we explicitly see the statistical relation between the number of microstates and the black hole entropy The AdS/CFT duality proves Bekenstein to be right after 30 years

[13]The Haar measure, named after Alfred Haar, corresponds to the measure keeping the subspace of all unitary transformations of the entire vector space rotationaly invariant. Using this measure one is possible to pick a state completely at random [48].

The ETH tells you how to do so in a consistent way with linear evolution of quantum mechanics. The ETH states:

$$\langle\alpha| A |\beta\rangle = A_{\alpha\beta} = \mathcal{A}(\bar{E}^{\alpha\beta})\delta_{\alpha\beta} + e^{-S(\bar{E}^{\alpha\beta})/2} f^A(E_\alpha, E_\beta) R^A_{\alpha\beta} \tag{207}$$

where $\mathcal{A}, f^A$ are smooth functions, and $\bar{E}^{\alpha\beta} = E^\alpha + E^\beta/2$. The smooth function $\mathcal{A}$ essentially represents the thermal expectation value for operator A. The off-diagonal contributions are characterized by $R^A_{\alpha\beta}$ and suppressed by a factor of $e^{-S/2}$. The $R^A_{\alpha\beta}$'s vary erratically. Marolf and Polchinksi showed that using this evaluation for operators in a strongly coupled CFT, two point functions between the left and the right CFT were exponentially suppressed with e^{-S} for generic states. With generic states they considered equally entangled states, however with configurations different then the thermofield state. Showing these states to obey this kind of behavior they argued it to be a strong argument in favour for typical states to possess firewalls.

3.9 Quantum Chaos

Another dimension which has to be added to the story about the conflict between entanglement and typicality is the addition of quantum chaos. Polchinski and Marolf already used the ETH for quantum chaotic systems, which was further elaborated on by Shenker and Stanford. They showed that the slightest perturbation could completely destroy the ER bridge between the two CFT's[11]. By injecting a few particles in one of the two CFT's early enough back in time, one could cause a shockwave by a blue-shifting of the energy of the particles and deform the geometry of the metric completely.

$$E_p \sim \frac{E}{r_*} e^{r_* t_i} \tag{208}$$

This resulted in an exponential decay of mutual information between the two sides. The authors found the effect already to be significant when t_i was of the order of the scrambling time.

The main idea to extract from this result, is the chaotic character of the black hole. The sensitivity to initial conditions resembles erratic behaviour. The authors called this the 'butterfly effect'. If one would interpreted the CFT side as a very hot plasma, this might seem not so far fetched. However doing so, would form a strong indication towards hot horizons. The chaotic character of the black hole made the case for black holes to possess the necessary entanglement needed for a smooth interior looked highly unlikely.

3.10 Violations of the Born Rule

In 2015 Marolf and Polchinski came to an even stronger conclusion [15]. They stated that a typical CFT state dual to a smooth horizon black hole would **violate** the Born rule of quantum mechanics. In other words, any construction that does seem to overcome all the challenges, objections and hardship; that somehow managed to dual a smooth horizon interior to a typical CFT micro state, would break a postulate of quantum mechanics, ruling out any possible solution in principle.

This conclusion would strongly stand in the way of any proposal that tries to solve the paradox in AdS/CFT. The goal of this thesis is therefore to argue the grounds of the statement of Marolf and Polchinski. In the following section the argument of [15] will be reviewed in detail. We will present a thought experiment to quantify the statements made, and try to break the born rule by all means necessary to see if we find the same conclusion.

Section II

In this second section we will first discuss the Born rule regarding Black holes. With a quick review of the argument of Marolf & Polchinski from [15], we outline the different aspects of the problem. In the second part we introduce a thought experiment. Here we test the Born rule for typical black holes states by perturbing one. We will explain the methodology and quantify our statements by computations of correlation functions in the last section. We will conclude with a discussion and a look towards further research.

4 Violations of the Born Rule

4.1 The Born Rule

The Born rule is one of the postulates of quantum mechanics, first stated by Max Born in 1926 [44]. The rule is of great importance since it bridges the mathematical framework to the physical experiment. For a state $|\psi\rangle$ we start with it's formal definition. Say we have Hermitian operator $\hat{A}$, with a corresponding complete set of orthonormal eigenvectors e_i and eigenvalues λ_i. A measurement of $\hat{A}$ yields collapse into one of it's eigenstates with corresponding eigenvalues. The probability to find a given eigenvalue is given by the Born rule.

The Born Rule: When a system is in state $|\psi\rangle$, the probability of finding eigenvalue λ_i is given by:

$$P(\hat{A} = \lambda_i|\psi) = |(e_i, \psi)|^2. \tag{209}$$

The overlap of state $|\psi\rangle$ with e_i is what gives the associated probability. We can therefore pose the following statement from this rule.

Measurement of operator $\hat{A}$ on nearly parallel states in Hilbert space H yields similar expectation values

4.2 Violations of the Born Rule for cool horizons

Marolf and Polchinski (MP) use exactly this statement in the context of the AdS/CFT to present violations of the Born rule for typical states dual to smooth horizons. Their argument is subtle. It consists of all the ingredients we reviewed. The idea makes use of the AdS/CFT duality in a fundamental way. We will first describe their argument in words together with a toy model. After that we explicitly show how to set up a thought experiment to quantify earlier statements made. The argument can be schematically represented:

(i) Start with a typical pure CFT state $|\phi\rangle$ picked randomly from the ensemble dual to a BTZ black hole in $2+1$ dimensions. Assume the black hole to have a smooth interior.

(ii) Rotate the phases of all modes in state $|\phi\rangle$. This very slightly rotates the state vector in the Hilbert space of the CFT, however drastically changes the entanglement configuration of the state.

(iii) Observe the effect in both the CFT and AdS picture. The CFT state now looks very similar to a typical state of the Hilbert space.

(iv) The AdS picture however changed notably. The smoothness of the horizon depends on the exact right entanglement structure. The new configuration would seem to cause the horizon to resemble a firewall, since the modes are not correctly entangled with the interior any more.

(v) There is a contradiction if one assumes typical state to have smooth horizons. We have two nearly *parallel* states in the CFT, the perturbed state and a typical state, with two physically *orthogonal* interpretations in the bulk.

(vi) MP conclude that the major premise was wrong in the first place. *Typical states are dual to firewall states, else one encounters strong violations of the Born rule*[15]

Things might not be completely clear after this first enumeration. The argument requires a bit more insight in the exact course of events. The main point to keep in mind is the fact that by perturbing the entanglement of a pure state, one acquires different physical interpretations in the bulk, while the CFT state remains nearly parallel to a typical state. If typical states are assumed to be "smooth", we encounter a contradiction.
The eminent question to ask is how to perform this kind of operation. The following unitary operator $\hat{U}$ fits the bill [15].

$$\hat{U} = e^{i\theta_i N_\omega} \qquad \text{with} \qquad N_\omega = \sum_i b_i^\dagger b_i \tag{210}$$

It rotates all the phases of the number operator for all frequencies with the amount of θ. To observe what the effect of this operation $\hat{U}$ is, we look at toy model construction of our situation.

Toy model

In a very simplistic picture, we represent the modes in pure state $|TFD\rangle$ on the left wedge to be entangled with the right wedge. If we act with the operator $\hat{U}$ on one of the two sides, we rotate the coefficients of the phases of all the modes on one side only, recall $\hat{U} = e^{i\theta_i b_i^\dagger b_i}$. We can see what happens to a single mode. Represent the single mode state as a fermionic mode for simplicity in this toy model with ground-state $|0\rangle$ and excited state $|1\rangle$. Say we put mode b_1^L with b_1^R into the following state:

$$|\psi\rangle_1 = \frac{|0_L, 0_R\rangle + |1_L, 1_R\rangle}{\sqrt{2}} \tag{211}$$

Now acting with $\hat{U}$ on, say the left wedge, we observe what happens: For θ small,

$$\begin{aligned} \hat{U}\,|\psi\rangle_1 &= e^{i\theta b_i^{\dagger L} b_i^L} \frac{|0_L, 0_R\rangle + |1_L, 1_R\rangle}{\sqrt{2}} \\ &\sim (1 + i\theta b_i^{\dagger L} b_i^L)(|0_L, 0_R\rangle + |1_L, 1_R\rangle) \\ &= |0_L, 0_R\rangle + (1 + i\theta)\,|1_L, 1_R\rangle \end{aligned} \tag{212}$$

The ladder operators act trivially on the ground-state and on the right wedge. One can observe the phase shift given to the left modes. In other words, the entanglement configuration has been modified a small amount. However, the total amount of entanglement remains the same. This can be quantified by looking at the von-Neumann entropy or entanglement entropy.
Recall $S = -\operatorname{Tr}\rho\log\rho$, and write for the reduced density matrix of the left mode *before* and *after* the operation

$$\begin{aligned} \text{Before} \rightarrow \rho_L &= \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \Rightarrow S = \log 2 \\ \text{After} \rightarrow \rho_L &= \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1-\theta^2}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \Rightarrow S = \log 2 \end{aligned} \tag{213}$$

Where we dropped the θ^2 assuming θ being small. We see $\hat{U}$ rotate the phases of mode with a definite frequency, therefore changing the entanglement configuration. As we know from Rindler space, for the horizon to be smooth one needs a very specific entanglement and a small deviation results into excitations on the horizon. In other words, due to the modification by $\hat{U}$, a single mode state now has a non-zero probability to be in an excited state. Even more so, when we act with $\hat{U}$ on an entire state, say $|TFD\rangle$, we sum over all frequencies. The small perturbance for all frequencies adds up to a full firewall at the horizon. For the purpose however of violating the Born rule it is sufficient to rotate only a very small amount of modes. There is no need to create a firewall. Only a few excited particles will be sufficient.

On the other hand $\hat{U}$ commutes with the CFT Hamiltonian H, since the number operator commutes with the Hamiltonian. The ensemble in the CFT stays untouched. Typical states remain typical states, however we now saw the physical picture, of what is happening at the horizon, looking dramatically different.

4.3 Discrete modes vs Wave-packets

There is one very important point which we should address now. The perspective of the infalling observer. The infalling observer has to *measure* the excitations at the horizon. In the toy model above we have used discrete sets of modes in frequency space. In other words, they are represented by a delta spike in Fourier space. The problem arises when we Fourier transform this to regular spacetime. The perturbation is distributed over the entire region of spacetime,

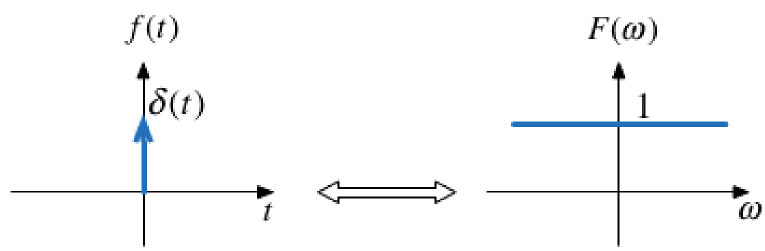

Figure 13: Fourier transform of a delta spike. $\Rightarrow \int_{-\infty}^{\infty} dte^{-2\pi i\omega t}\delta(t) = 1$

and therefore immeasurable for any observer. An observer always has to measure differences in energy, and in this case she won't measure anything.
This problem is a usual one in quantum field theory and is resolved by using wavepackets instead of delta spikes. The mode b_i now looks like this

$$b_i = \int d\omega g_i(\omega) b_\omega. \tag{214}$$

Where $g_i(\omega)$ is a smooth function, that represents the 'smearing' of the mode. The consequence of this operation is however very critical. **The commutator $[\hat{U}, H]$ is now non-zero.** This is caused by the number operator now having off-diagonal components, due to the smearing. The immediate consequence is the fact that the energy no longer is conserved for an operation with $\hat{U}$. The increase in energy for state $\hat{U}\,|\psi\rangle$ means an exponential increase in number of states available in the ensemble of the CFT. All these new states are smooth horizon states by definition. This opens up the question if the new perturbed state, $U\,|\psi\rangle$, is still a typical state. This vital point will be explained in detail below, when setting up the thought experiment. For now we can summarize this paragraph by stating the importance of the smearing of the wavepackets once more. The operator $\hat{U}$ knows two competing effects. Next to mapping smooth typical states to excited states, it increases the energy of the ensemble. Is the increase in energy small enough for the state $U|\psi\rangle$ still to be typical? A vast majority of excited states after the operation with $\hat{U}$, would emanate the contradiction between typicality and smooth horizons.

5 Bounds of the Born Rule

In this section we formulate a concrete way of how to test if there are violations of the Born rule for typical CFT states dual to smooth horizon bulk states. However, formulating a framework to find violations of the Born rule would be applicable in more quantum mechanical cases applicable then for the black hole. The formalism derived below is therefore valid for general quantum mechanical cases. Mainly, the point of interest is to get insight in measures of comparison between states. With these measures we can derive a condition, which, if not respected, will result into a violation of the Born rule.

1. Our first measure is the expectation value of a certain operator in two different quantum states, say $|\psi_1\rangle$ and $|\psi_2\rangle$. As we will see, the difference is bounded from above by the spectrum of the operator. The quantity of interest is given by:

$$|\langle\psi_2|\,\hat{A}\,|\psi_2\rangle - \langle\psi_1|\,\hat{A}\,|\psi_1\rangle\,| \tag{215}$$

2. The second measure is given by the inner-product between two states. This is linked to the *fidelity* in quantum information theory. It is a measure of the amount of overlap between vectors. It can be defined in the following way

$$|\langle\psi_2|\psi_1\rangle\,| = 1 - \frac{\epsilon^2}{2} \tag{216}$$

3. Both obtained measures will be related in order to formulate a condition to test the Born rule.

5.0.1 A First Measure

We look at the change of the expectation value of a certain operator $\hat{A}$ between two normalized pure states. The quantity we're interested in is given by

$$|\langle\psi_2|\,\hat{A}\,|\psi_2\rangle - \langle\psi_1|\,\hat{A}\,|\psi_1\rangle\,| \leq \ ? \tag{217}$$

The hypothesis is, when two states do not differ much, the change in expectation value should also be limited. The two state vectors are both normalized, and are shifted by:

$$|\psi_2\rangle - |\psi_1\rangle = |\delta\psi\rangle \tag{218}$$

Define: $|||\delta\psi\rangle\,|| = \delta$. We can work out quantity (216). We substitute equation (217) in:

$$\begin{aligned}&|\langle\psi_2|\,A\,|\psi_2\rangle - \langle\psi_1|\,A\,|\psi_1\rangle\,| \\ &= |\langle\psi_2|\,A\,|\psi_2\rangle + \langle\delta\psi|\,A\,|\psi_1\rangle + \langle\psi_2|\,A\,|\delta\psi\rangle + \langle\delta\psi|\,A\,|\delta\psi\rangle - \langle\psi_2|\,A\,|\psi_2\rangle\,| = \\ &|\langle\delta\psi|\,A\,|\psi_1\rangle + \langle\psi_2|\,A\,|\delta\psi\rangle + \langle\delta\psi|\,A\,|\delta\psi\rangle\,|\end{aligned} \tag{219}$$

We can use the triangle identity to obtain:

$$\begin{aligned} &|\langle\delta\psi|\, A\,|\psi_1\rangle + \langle\psi_2|\, A\,|\delta\psi\rangle + \langle\delta\psi|\, A\,|\delta\psi\rangle\,| \leq \\ &\qquad |\langle\delta\psi|\, A\,|\psi_1\rangle\,| + |\langle\psi_2|\, A\,|\delta\psi\rangle\,| + |\langle\delta\psi|\, A\,|\delta\psi\rangle\,| \end{aligned} \tag{220}$$

We can now make use of the Schwarz inequality to continue. For the first term, this is given by:

$$|\langle\delta\psi|\, A\,|\psi_1\rangle\,| \leq |\langle\delta\psi|\,||A|||\psi_1\rangle\,| = \delta||A|| \leq \delta \max_j |\lambda_j^a| \tag{221}$$

where $\max_j|\lambda_j^a|$ is the maximum eigenvalue of A The last inequality we can obtain if A is a complete continuous self adjoint or Hermitian operator. For the entire expression we obtain:

$$|\langle\psi_2|\, A\,|\psi_2\rangle - \langle\psi_1|\, A\,|\psi_1\rangle\,| \leq \max_j |\lambda_j^a|(2\delta + \delta^2) \tag{222}$$

Assuming δ to be small, we can take this up to first order

$$|\langle\psi_2|\, A\,|\psi_2\rangle - \langle\psi_1|\, A\,|\psi_1\rangle\,| \leq \max_j |\lambda_j^a| 2\delta. \tag{223}$$

We have found our first bound on the Born Rule.

5.0.2 A Second Measure

We will calculate the projection between two state vectors, $|\psi_1\rangle$ and $|\psi_2\rangle$,

$$|\langle\psi_2|\psi_1\rangle\,| \tag{224}$$

The quantity is a measure of overlap between the two states. For two states we can formulate a general condition.

$$|\langle\psi_2|\psi_1\rangle\,| = 1 - \frac{\epsilon^2}{2} \tag{225}$$

If the states are very similar, or parallel. ϵ will go to zero. A second measure is obtained.

5.1 Relating measures

To sum up, we have for the difference in expectation value:

$$|\langle\psi_2|\, A \,|\psi_2\rangle - \langle\psi_1|\, A \,|\psi_1\rangle\, |\leq \max_j |\lambda_j^a|(2\delta + \delta^2). \tag{226}$$

The overlap of both states.

$$|\langle\psi_2|\psi_1\rangle\, |= \alpha = 1 - \frac{\epsilon^2}{2} \tag{227}$$

Relate δ to ϵ:

$$\delta = \sqrt{||\,|\psi_1\rangle - |\psi_2\rangle\, ||^2} = \sqrt{||\,|\psi_1\rangle\, || + ||\,|\psi_2\rangle\, || - 2|\langle\psi_2|\psi_1\rangle\, |}. \tag{228}$$

The first two terms underneath the square root are products of normalized state vectors, and the last term is related to ϵ via equation (20). We obtain:

$$\delta = \sqrt{2 - 2 - \epsilon^2} = \epsilon \tag{229}$$

We find that $\delta = \epsilon$. We now plug equation (20) into equation (19) with the relation between δ and ϵ we just found. This gives:

$$|\langle\psi_2|\, A \,|\psi_2\rangle - \langle\psi_1|\, A \,|\psi_1\rangle\, |\leq \max_j |\lambda_j^a| 2\epsilon \tag{230}$$

or

$$\boxed{|\langle\psi_2|\, A \,|\psi_2\rangle - \langle\psi_1|\, A \,|\psi_1\rangle\, |\leq 8 \max_j |\lambda_j^a|(1 - |\langle\psi_2|\psi_1\rangle\, |)^2} \tag{231}$$

Where we have found a condition for operators and states to satisfy. On one side, the difference between an expectation value of an operator in two different states, compared to the projection of these two states scaled with the maximum eigenvalue of this operator on the other side. **If** this inequality is broken, the Born rule is violated. In the next section we will apply this inequality to black holes in AdS/CFT. We will do so, by acting on the thermofield state with $\hat{U}_{MP} = e^{-i\theta N_\omega}$.

6 Thought Experiment

Initial Conditions.
The commencing setting is a holographic eternal AdS BTZ black hole in $2+1$ dimensions with $2-d$ CFT Hilbert space $\mathcal{H}(E)$, where the CFT is represented by the microcanonical ensemble peaked around energy E. As described in the theoretical formulation of the argument of MP (section 4.2), we start of by picking a random typical pure state $|\psi\rangle$ from the ensemble ρ_{CFT}. We assume the state to have a non-excited horizon. In other words, we assume the vast majority of states in the ensemble to be smooth (equilibrium states are dual to smooth horizons). Our initial state can be resembled by the thermofield double state $|TFD\rangle$. Even though the thermofield state is a highly a-typical state of the ensemble, the characteristics of local operators and therefore correlation functions are similar. This frees us from specifying a construction to obtain smooth horizon states dual to typical CFT states, and giving us the possibility to do explicit calculations.

6.1 A Thought Experiment

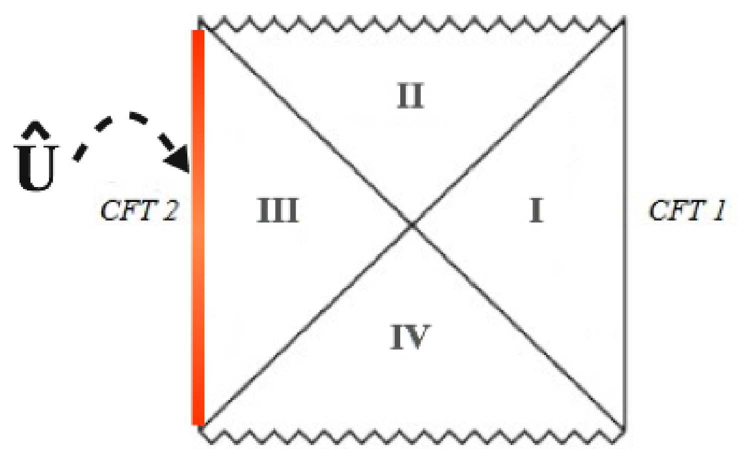

Figure 14: We will act with $\hat{U}$ on the *left* wedge of the thermofield double state

The left wedge of the thermofield state is acted on with $\hat{U}$. As we have seen, $\hat{U}$ maps all equilibrium states of the ensemble to excited states, however raise the energy of the ensemble by a finite amount δE. These **two** effects need to be captured to make statements about the violations of the Born rule.

- In order to see whether $\hat{U}$ had the desired 'excitation' effect on the thermofield state, we will compute two point functions between the interior and exterior region of the black hole in a typical state $|\psi_1\rangle$, and in the perturbed state $|\psi_2\rangle$ or equivalently $\hat{U}\,|\psi\rangle$. The change in correlator $\delta\langle\phi_1\phi_2\rangle$ can then be defined by the *left hand side* of equation (27):

$$\delta\langle\phi_1\phi_2\rangle \equiv |\langle\psi_2|\,\phi(t_1,r_1)\phi(t_2,r_2)\,|\psi_2\rangle - \langle\psi_1|\,\phi(t_1,r_1)\phi(t_2,r_2)\,|\psi_1\rangle\,| \quad (232)$$

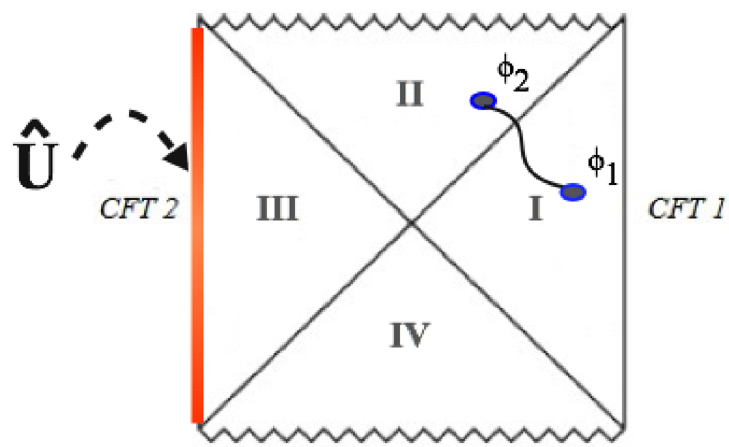

Figure 15: A two point function between a point inside and outside of the horizon

The perspective of the infalling observer is caught in picking this quantity. The infaller measures correlation functions in order to determine in what state he's in.

- The second effect of the operation is a raise in energy of the ensemble by δE, due to the smearing of the wavepackets. Recall, the effect is due to $[\hat{U}, H] \neq 0$. The increase in energy, increases the dimension of the Hilbert space. Let's examine what happens.

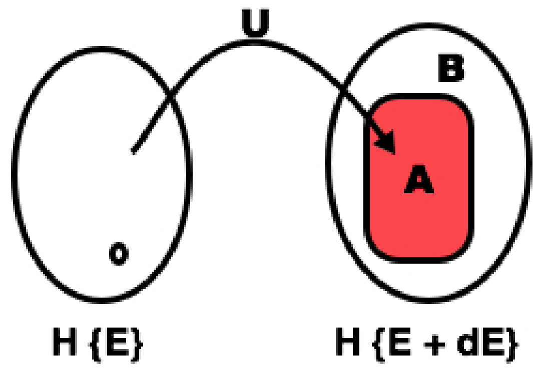

Figure 16: The effect of $\hat{U}$ on $\mathcal{H}(E)$.

Almost all of states of the ensemble in $\mathcal{H}(E)$ are "smooth", except for some excited non-equilibrium states. We can observing the effect of $\hat{U}$. $\hat{U}$ maps all typical 'smooth' states of $\mathcal{H}(E)$ to 'excited' states into subspace A.
However a second effect of this operation is the expansion of the Hilbert space into $\mathcal{H}(E+\delta E)$. This generates a new portion of states represented by subspace B.
Continuing: A newly perturbed state from subspace A or $\hat{U}\,|TFD\rangle$, call this state to $|\psi_2\rangle$, will have an excited horizon, just like all states coming from $\mathcal{H}(E)$. Now the question is whether or not a typical state from the **new** Hilbert space $\mathcal{H}(E+\delta E)$ has an excited horizon. Call a typical state from subspace A: $|\psi_1\rangle$. The question to answer is the following one:

How big is subspace A, or $\hat{U}\mathcal{H}(E)$, compared to the entire Hilbert space $\mathcal{H}(E+\delta E)$?

Answering this, we randomly pick a state vector from $\mathcal{H}(E+\delta E)$. Call this vector $|\psi\rangle$. This vector can be spanned up by two orthonormal vectors. From region A, and it's orthogonal component. As follows:

$$|\psi\rangle = \alpha\,|\psi_A\rangle + \beta\,|\psi_B\rangle \tag{233}$$

We are interested in the quantity α. To obtain this α we need to know how much the energy increased with the operation of U. The number of states in $\mathscr{H}\{E \ + \delta E\}$ we can call N. Likewise the number of states in region A we call n. This is approximately the same as the number of states in the original Hilbert space $\mathscr{H}\{E\}$. This change in dimension is related to a change in energy by the second law of thermodynamics:

$$dS = \frac{dE}{T} \tag{234}$$

The entropy is related to the number of states in a logarithmic way.

$$\begin{aligned} dS &= \log(\dim[\mathscr{H}\{E \ + \delta E\}]) - \log(\dim[\mathscr{H}\{E\}]) \\ &= \log(N) - \log(n) \\ &= \frac{dE}{T} \end{aligned} \tag{235}$$

We end up with:

$$N = ne^{\frac{dE}{T}} \tag{236}$$

To relate this to α we need to know what the projection is of the entire vector space with dimensionality N onto the subspace with dimensionality n. $||\psi_A||$ is given by:

$$||\psi_A||= \sqrt{\frac{n}{N}}. \tag{237}$$

If we look at equation (29) we can identify $||\psi_A||$ with α. We find α in terms of the change in energy using equation (234):

$$\alpha = \sqrt{\frac{n}{N}} = e^{-\frac{dE}{2T}} \tag{238}$$

Next we can identify $|\psi_A\rangle$ with $|\psi_2\rangle$, our perturbed states, and $|\psi\rangle$ with $|\psi_1\rangle$. A typical state from the new ensemble $\mathcal{H}(E + \delta E)$

$$|\psi_1\rangle = \alpha\,|\psi_2\rangle + \beta\,|\psi_B\rangle \tag{239}$$

The projection of the excited states onto the full Hilbert space can be identified with the second measure derived in section 5.0.2.

This was exactly what we've derived for two general states. The inner-product between $|\psi_1\rangle$, a typical state from $\mathcal{H}(E+\delta E)$ and $|\psi_2\rangle$, the perturbed state from $\hat{U}\mathcal{H}(E)$, is bounded by

$$|\langle\psi_2|\psi_1\rangle\,|= 1 - \frac{\epsilon^2}{2} \Rightarrow \alpha|\langle\psi_2|\,|||\psi_1\rangle\,|= \alpha = 1 - \frac{\epsilon^2}{2} \tag{240}$$

α is characterized by the change in energy and the temperature. The change in energy is our second objective. The quantity is given by:

$$\delta E = \langle\psi_2|\,H\,|\psi_2\rangle - \langle\psi_1|\,H\,|\psi_1\rangle \tag{241}$$

where $H = H_L + H_R$. The second bound represents the *right hand side* of our Born rule inequality. Together with the *left hand side* determined by the computed change in correlator, we can investigate the Born rule.

- The computed quantities will be taken together into the Born rule condition . Putting

$$\alpha = 1 - \frac{\epsilon^2}{2}$$

and

$$\delta\langle\phi_1\phi_2\rangle = |\langle\psi_2|\,\phi(t_1,r_1)\phi(t_2,r_2)\,|\psi_2\rangle - \langle\psi_1|\,\phi(t_1,r_1)\phi(t_2,r_2)\,|\psi_1\rangle\,|$$

into the Born rule inequality (26)

$$|\langle\psi_2|\,A\,|\psi_2\rangle - \langle\psi_1|\,A\,|\psi_1\rangle\,| \leq \max_j|\lambda_j^a|2\epsilon \tag{242}$$

we obtain:

$$\boxed{\delta\langle\phi_1\phi_2\rangle \leq 2\max_j|\lambda_j^{\phi_1\phi_2}|\sqrt{\frac{\delta E}{T}}} \tag{243}$$

where $\epsilon = \sqrt{2}\sqrt{1 - e^{\frac{\delta E}{2T}}}$. We have applied the derived bounds on the Born rule on the eternal black hole, and established a concrete way to find violations of this rule. **If inequality is broken, we find a violation of the Born rule** .

On the left hand side we see the change in two-point correlation function between a point inside the horizon and outside. On the right we have the change in energy over the temperature together with the maximum eigenvalue of the two point correlator. This quantity is a tricky point. We have to specify the maximum eigenvalue of the two point correlation function. Since the expansion of the field has been done in terms of scalars, this value is not definite. The issue deserves some thought. We have decided to approximate the maximum eigenvalue, $\max_j|\lambda_j^{\phi_1\phi_2}|$, as being of order of the original value of the **unperturbed correlator**. Since one would expect the correlation between a point inside and outside the horizon to be maximal when the horizon is smooth, this seems the right approximation. The value of this quantity will then depend on where the two points are situated. For every regime, we will compute the unperturbed correlator between both points, so that we have all the ingredients to apply the Born rule inequality above. In the following section all these quantities will be computed, and a judgment will be given on whether it is possible to break the Born rule for typical smooth horizon black holes.

We compare on one hand the change on a AdS bulk two-point correlation function, which is a way to describe the experience of the infalling observer. And on the other hand, the effect on the ensemble in the CFT. Which gives us a measure on how typical the new modified state is. *We expect it will be very hard to break the Born rule.* The increase in dimension is large, and the time the infaller can measure is limited. We will try by all means to break the Born rule searching for limits where the left hand side of condition (37) is large, while the right hand side is small.

Just as an interesting remark, one could ask if there are other operations on the thermofield state that would generate the same entanglement mixing effect without perturbing the state out of equilibrium. We answer this question in Appendix A.

7 Computations

7.1 Change in Energy

Below a calculation is shown of the effect on the expectation value of the energy by the operator U acting on the Thermofield state. Defining U as follows similar as in [1]:

$$U = e^{i\theta_i N_\omega} \tag{244}$$

U will rotate the phases of the creation and annihilation operators. U is a unitary operator and by letting it act on one of the two sides of the thermofield state we can modify the entanglement. This can have an effect on the smoothness of the horizon for the infalling observer.

We can calculate the effect of this operator on $|TFD\rangle$ by looking at the expectation value of the energy. In this case we rotate the phases in the left subsystem. Defining U acting on the left subsystem as U_L:

$$\langle TFD| U_L^\dagger H U_L |TFD\rangle = \langle TFD| e^{-i\theta_i N_{\omega,L}} H e^{i\theta_i N_{\omega,L}} |TFD\rangle \tag{245}$$

Here the Hamiltonian of the entire system is given by:

$$H = H_L + H_R \tag{246}$$

Because the Hamiltonian is a linear operator we can split the expectation value in a right and left part, and we observe the following:

$$\langle TFD| e^{-i\theta_i N_{\omega,L}} (H_L + H_R) e^{i\theta_i N_{\omega,L}} |TFD\rangle = \tag{247}$$

$$\langle TFD| e^{-i\theta_i N_{\omega,L}} H_L e^{i\theta_i N_{\omega,L}} |TFD\rangle + \langle TFD| H_R |TFD\rangle \tag{248}$$

Since the operator only acts on the left side the effect on the right side is zero. If we are only interested in the effect of the operator we can neglect the term on the right side, since it will drop out if we calculate the difference in energy caused by U in the following way:

$$\delta E = \langle TFD| U_L^\dagger (H_L + H_R) U_L |TFD\rangle - \langle TFD| H_L + H_R |TFD\rangle \tag{249}$$

Equation 41 will be the main goal of the underlying section.

Focusing on:

$$\langle TFD| \, e^{-i\theta_i N_{\omega,L}} H_L e^{i\theta_i N_{\omega,L}} \, |TFD\rangle \tag{250}$$

and looking at a small change of the phase we can expand the exponents in the following way:

$$\Rightarrow e^{-i\theta_i N_{\omega,L}} H_L e^{i\theta_i N_{\omega,L}} = [1 - i\theta N_{\omega,L} - \theta^2 N_{\omega,L}^2] H_L [1 + i\theta N_{\omega,L} - \theta^2 N_{\omega,L}^2] \tag{251}$$

We can substitute $X = \theta N_{\omega,L}$ and write expression 8 out:

$$\Rightarrow H_L + H_L iX - iXH_L - \frac{X^2}{2} H_L - H_L \frac{X^2}{2} + XH_L X \tag{252}$$

By looking at the terms we can see that this can be simplified using commutation relations. As in [1] we use the commutative terms and write the series up to second order as follows:

$$e^{-i\theta_i N_{\omega,L}} H_L e^{i\theta_i N_{\omega,L}} \Rightarrow H_L + i[H_L, X] - \frac{1}{2}[[H, X], X] + O(X^3) \tag{253}$$

The first order term is trivial to see and the second order term can be seen showing $[[H, X], X] = (HX - XH)X - X(HX - XH) = HXX - 2XHX - XXH$ where we see the terms in equation (9) up to a factor of 2.

We have:

$$X = \theta_i N_{\omega,L} = \sum_i \theta_i b_i^\dagger b_i \tag{254}$$

The sum over i will be omitted in following equations. The next thing to do is to introduce the creation and annihilation operators as wave packets. The wavepackets are defined in the following way:

$$b_i = \int d\omega g_i(\omega) b_\omega \tag{255}$$

The factor g_i is again a smooth scaling factor, which we will define later on. Writing X:

$$X = \theta_i \int d^2\omega g_i^*(\omega_1) b_{\omega_1}^\dagger g_i(\omega_2) b_{\omega_2} \tag{256}$$

.

Substituting this back into equation (10), we can look at the first order term, $i[H_L, X]$, first:

$$i[H_L, X] = i\theta_i \left(H_L \int d^2\omega g_i^*(\omega_1) b_i^\dagger(\omega_1) g_i(\omega_2) b_i(\omega_2) - \int d^2\omega g_i^*(\omega_1) b_i^\dagger(\omega_1) g_i(\omega_2) b_i(\omega_2) H_L \right) \tag{257}$$

Important to note is that the only elements in the expression of X that contribute to the commutator $i[H_L, X]$ are the creation and annihilation operators.

$$= i\theta_i \int d^2\omega g_i^*(\omega_1) g_i(\omega_2)[H, b_i^\dagger(\omega_2) b_i(\omega_1)] \tag{258}$$

$$= i\theta_i \int d^2\omega g_i^*(\omega_1) g_i(\omega_2)[H_L, b_i^\dagger(\omega_1)] b_i(\omega_2) + b_i^\dagger(\omega_1)[H_L, b_i(\omega_2)] \tag{259}$$

$$= i\theta_i \int d^2\omega (\omega_1 - \omega_2) g_i^*(\omega_1) g_i(\omega_2) b_i^\dagger(\omega_1) b_i(\omega_2) \tag{260}$$

In the last step we used the following commutation relations between the creation/annihilation operators and the Hamiltonian.

$$[b_i^\dagger, H] = -\omega_1 b_i^\dagger \text{ and } [b_i, H] = \omega_2 b_i \tag{261}$$

Equation 17 is the first order contribution.

We continue by looking at the second order term. The second order contribution is given by: $[H, X], X]$. We work the commutator out in the following steps. The commutator [H,X] is derived above. This contribution is commutated with another value of X.

$$[[H, X], X] = \sum_i \theta_i \theta_j g_i^*(\omega_1) g_i(\omega_2) g_i^*(\omega_3) g_i(\omega_4)(\omega_1 - \omega_2)[b_i^\dagger(\omega_1) b_i(\omega_2), b_i^\dagger(\omega_3) b_i(\omega_4)] \tag{262}$$

The sum is written here for completeness. Again we see that the only terms that actually are present in the commutator are the creation/annihilation operators coming from 2 times a contribution of X. Let's work out the commutator:

$$\begin{aligned}[b_i^\dagger(\omega_1) b_i(\omega_2), b_i^\dagger(\omega_3) b_i(\omega_4)] = {} & [b_i^\dagger(\omega_1), b_i^\dagger(\omega_3)] b_i(\omega_2) b_i(\omega_4) + \\ & b_i^\dagger(\omega_3)[b_i^\dagger(\omega_1), b_i(\omega_4)] b_i(\omega_2) + \\ & b_i^\dagger(\omega_1)[b_i(\omega_2), b_i^\dagger(\omega_3)] b_i(\omega_4) + \\ & b_i^\dagger(\omega_1) b_i^\dagger(\omega_3)[b_i(\omega_2), b_i(\omega_4)]\end{aligned} \tag{263}$$

We can now make use of the usual commutation relations between the creation and annihilation operators:$[b_i, b_j] = [b_i^\dagger, b_j^\dagger] = 0$ and$[b_i, b_j^\dagger] = \delta_{\omega\omega'}$.
After dropping the first and the last term due to these relations we end up with:

$$[b_i^\dagger(\omega_1)b_i(\omega_2), b_i^\dagger(\omega_3)b_i(\omega_4)] = b_i^\dagger(\omega_1)b_i(\omega_4)\delta_{23} - b_i(\omega_2), b_i^\dagger(\omega_3)\delta_{14} \tag{264}$$

Plugging this back into equation (19) and we put the delta functions at work we arrive at:

$$\begin{aligned}[H,X],X] = \sum_i \theta_i\theta_j \\ g_i^*(\omega_1)g_i(\omega_2)g_i^*(\omega_3)g_i(\omega_4)(\omega_1-\omega_2)(b_i^\dagger(\omega_1)b_i(\omega_4)\delta_{23} - b_i(\omega_2), b_i^\dagger(\omega_3)\delta_{14})\end{aligned} \tag{265}$$

$$= \sum_i \theta_i\theta_j |g_i(\omega_1)|^2 |g_i(\omega_3)|^2 (\omega_1-\omega_3)(b_i^\dagger(\omega_1)b_i(\omega_1) - b_i^\dagger(\omega_3)b_i(\omega_3)) \tag{266}$$

7.1.1 First order contribution

The next step is now to calculate the expectation value of the several contributions to:

$$\langle TFD| e^{-i\theta_i N_{\omega,L}} H_L e^{i\theta_i N_{\omega,L}} |TFD\rangle \tag{267}$$

In the series expansion this was given by:

$$\begin{aligned}&= \langle TFD| H_L |TFD\rangle + \langle TFD| i[H_L, X] |TFD\rangle - \\ &\langle TFD| [H_L, X], X] |TFD\rangle \\ &\quad + O(X^3)\end{aligned} \tag{268}$$

The first order term

$$\langle TFD| i\theta_i [H_L, X] |TFD\rangle \tag{269}$$

is done first. The first order contribution given by equation (17) is plugged into equation (26) above:

$$\langle TFD| i\theta_i \int d^2\omega(\omega_1-\omega_2) g_i^*(\omega_1)g_i(\omega_2) b_i^\dagger(\omega_1)b_i(\omega_2) |TFD\rangle \tag{270}$$

To compute equation (27) we need the expectation value of the number operator in the thermofield state. This is given by equation (..)

$$\langle TFD| b^\dagger(\omega)b(\omega') |TFD\rangle = \frac{1}{\sqrt{Z}} \mathrm{Tr}[e^{-\beta H} b^\dagger(\omega)b(\omega')] = \frac{1}{e^{\beta\omega}-1}\delta_{\omega\omega'} \tag{271}$$

Where $Z = \mathrm{Tr}[e^{-\beta H}]$. Right now we can calculate equation (27) and we end up with:

$$i\theta_i \int d^2\omega(\omega_1-\omega_2) g_i^*(\omega_1)g_i(\omega_2) \frac{1}{e^{\beta\omega}-1}\delta_{\omega_1\omega_2} \tag{272}$$

By looking at the delta function we see that the only contribution that will hold is when $\omega_1 = \omega_2$. This contribution however will be zero due to $(\omega_1 - \omega_2)$ term, and we can conclude that the first order contribution vanishes.

7.1.2 Second order contribution

Continuing to the second order term we have to calculate:

$$\langle TFD| \, [[H, X], X] \, |TFD\rangle \tag{273}$$

We plug in equation (23) as end result of the second order contribution and get the following:

$$\begin{gathered}\langle TFD| \sum_i \theta_i \theta_j \int d\omega_1 d\omega_3 g_i^*(\omega_1) g_j(\omega_1) g_i^*(\omega_3) g_j(\omega_3)(\omega_1 - \omega_3) \\ (b_i^\dagger(\omega_1) b_i(\omega_1) - b_i^\dagger(\omega_3) b_i(\omega_3)) \, |TFD\rangle\end{gathered} \tag{274}$$

Making use of equation (30) we end up with:

$$-\sum_i \theta_i^2 \int d\omega_1 d\omega_3 |g_i(\omega_1)|^2 |g_i(\omega_3)|^2 (\omega_1 - \omega_3) \left(\frac{1}{e^{\beta\omega_1} - 1} - \frac{1}{e^{\beta\omega_3} - 1} \right) \tag{275}$$

Where we can evaluate the dubbel sum for $i = j$. This makes sense since a product of two $g_i(\omega)'s$ falls off rapidly, when $i \neq j$. If we now calculate equation (6) we see that equation 32 is the only standing term:

$$\delta E = \langle TFD| \, U_L^\dagger (H_L + H_R) U_L \, |TFD\rangle - \langle TFD| \, H_L + H_R \, |TFD\rangle$$

$$=$$

$$\delta E = \sum_i \theta_i^2 \int_{\delta\omega} d\omega_1 d\omega_3 |g_i(\omega_1)|^2 |g_i(\omega_3)|^2 (\omega_1 - \omega_3) \left(\frac{1}{e^{\beta\omega_1} - 1} - \frac{1}{e^{\beta\omega_3} - 1} \right) \tag{276}$$

To evaluate this we have to look at the Boltzmann factors. We can write $\omega_1 = \omega_0 + \delta\omega_1$, and $\omega_3 = \omega_0 + \delta\omega_3$.

$$\left(\frac{1}{e^{\beta(\omega_0+\delta\omega_1)} - 1} - \frac{1}{e^{\beta(\omega_0+\delta\omega_3)} - 1} \right) \tag{277}$$

Since we are interested in the limit where $\delta\omega \Rightarrow 0$ we can Taylor-expand both terms in resp: $\delta\omega_1$ and $\delta\omega_3$. This gives up to first order:

$$\begin{aligned} &\left(\frac{1}{e^{\beta(\omega_0+\delta\omega_1)} -} 1 - \frac{1}{(e^{\beta\omega_0+\delta\omega_3} - 1)^2} \right) \Rightarrow \\ &\qquad \beta \frac{e^{\beta\omega_0}}{(e^{\beta\omega_0} - 1)^2} \delta\omega_3 - \beta \frac{e^{\beta\omega_0}}{(e^{\beta\omega_0} - 1)^2} \delta\omega_1 \end{aligned} \tag{278}$$

Plugging back the relations $\omega_1 = \omega_0 + \delta\omega_1$,and $\omega_3 = \omega_0 + \delta\omega_3$ we obtain:

$$(\omega_3 - \omega_1) \frac{\beta e^{\beta\omega_0}}{(e^{\beta\omega_0} - 1)^2} \tag{279}$$

This is put back into equation (36):

$$\boxed{\delta E = \theta_i^2 \int_{\delta\omega} d\omega_1 d\omega_3 \frac{\beta e^{\beta\omega_0}}{(e^{\beta\omega_0} - 1)^2} |g_i(\omega_1)|^2 |g_i(\omega_3)|^2 (\omega_1 - \omega_3)^2} \tag{280}$$

We have found our result for the change in energy.

7.2 Correlation Functions

The second objective is the correlator between the exterior and interior of the black hole. We would like to quantify the effect of the operation $\hat{U}$. In other words, how much did the correlation between points across the horizon change. This correlation is exactly observed in all n-point function between fields of the interior and exterior, and when one wants to know exactly what happened after an operation, one should calculate all of them. However, not all of them are computable. The two-point function for the eternal BTZ black hole in d = 2 + 1 is explicitly calculable. As we saw, we can solve the Klein Gordon equation explicitly. We therefore work with a *massless* scalar field in three dimensions. The derivation for massive fields of higher spin is more complicated mathematically, but based on the same grounds.

$$\begin{aligned}&\delta\langle\phi_1\phi_2\rangle = \\ &\langle TFD|\, U_L^\dagger \phi(t_1, r_1)\phi(t_2, r_2)U_L\, |TFD\rangle - \langle TFD|\, \phi(t_1, r_1)\phi(t_2, r_2)\, |TFD\rangle\end{aligned} \tag{281}$$

We substract from here the unperturbed correlator we find the difference or the change due to the operation on the state with $\hat{U}$.
The points can be situated anywhere, and to see if there are violations of the born rule one has to check all of them. We will check different regimes for this correlator, approximating the radial wavefunctions for several limits.

1. **Horizon Approximation**: We will take both points on either side in the vicinity of the horizon.

2. **Boundary Approximation** The first point will be close to the boundary, while the second one inside the horizon stays in the horizon limit.

Since the point 1 is in the right wedge, we have an expansion as in equation (70), with creation and annihilation strictly from the right wedge:

$$\phi(t_1, r_1, \Omega_1) = \int \frac{d\omega dk}{(2\pi)^2}\frac{1}{\sqrt{2\omega}}\left[b_{\omega,k} f_{\omega,k}(t_1, r_1, \Omega) + b^\dagger_{\omega,k} f^*_{\omega,k}(t_1, r_1, \Omega)\right] \tag{282}$$

For point two we need modes from both the left and the right wedge. We assumed angular symmetry here for both points in the field. The expansion of the field in this region looks like this:

$$\begin{aligned}\phi(t_2, r_2, \Omega_2) = \int \frac{d\omega dk}{(2\pi)^2}\frac{1}{\sqrt{2\omega}}&\Big[b_R g^{(1)}_{\omega,k}(t_2, r_2, \Omega_2) + b^\dagger_L g^{*(2)}_{\omega,k}(t_2, r_2, \Omega_2) \\ &+ b^\dagger_R g^{*(1)}_{\omega,k}(t_2, r_2, \Omega_2) + b_L g^{(2)}_{\omega,k}(t_2, r_2, \Omega_2)\Big]\end{aligned} \tag{283}$$

The Hamiltonian is given by: $H = H_1 + H_2$. Therefore the first two operators pair up with $e^{-i\omega t}$.

7.2.1 Two-point Function

We can start by analyzing the quantity we need. Given by equation (69). Lets name the operators on the right wedge $\hat{a}$ and left wedge $\hat{b}$ instead of $a_{\omega,k,R/L}$ for more clarity. We also left out all the time and radial indications to keep the bookkeeping process cleaner. The product between the two fields is given by the following equation.

$$\begin{aligned}\phi(t_1,r_1)\phi(t_2,r_2) = \int \frac{d\omega dk}{(2\pi)^2}\frac{d\omega' dk'}{(2\pi)^2}\frac{1}{\sqrt{2\omega}}\frac{1}{\sqrt{2\omega'}} \\ (\hat{a}_1 f + \hat{a}_2^\dagger f^*)[\hat{a}_3 g^{(1)} + \hat{b}_6^\dagger g^{*(2)} + \hat{a}_4^\dagger g^{*(1)} + \hat{b}_5 g^{(2)}]\end{aligned} \tag{284}$$

From this we get 8 terms:

$$\begin{aligned}&\Rightarrow f * g^{(1)}\hat{a}_1\hat{a}_3 + f * g^{*(2)}\hat{a}_1\hat{b}_6^\dagger + f * g^{*(1)}\hat{a}_1\hat{a}_4^\dagger + f * g^{(2)}\hat{a}_1\hat{b}_5 \\ &\quad + f^* * g^{(1)}\hat{a}_2^\dagger\hat{a}_3 + f^* * g^{*(2)}\hat{a}_2^\dagger\hat{b}_6^\dagger + f^* * g^{*(1)}\hat{a}_2^\dagger\hat{a}_4^\dagger + f^* * g^{(2)}\hat{a}_2^\dagger\hat{b}_5\end{aligned} \tag{285}$$

Now we can start looking at the entire quantity of equation (69). We expand as we did for the energy, $\Rightarrow \phi_1\phi_2 + i[\phi_1\phi_2, X] - \frac{1}{2}[[\phi_1\phi_2, X], X] + O(X^3)$, and look at what is happening to first order:

$$i\theta[\phi(t_1,r_1)\phi(t_2,r_2), \hat{b}_{\omega_7}^\dagger\hat{b}_{\omega_8}] \tag{286}$$

Each term from quantity (..) can be analyzed separately, and can be added later on. We can see that only terms 2,4,6 and 8 will give contributions, since the other terms commute with the number operator. From these four only terms 4 and 6 give a contribution after calculating the expectation value in the thermal state. Before we calculate anything, we put in the spread of the frequency:

$$b_i = \int d\omega g_i(\omega) b_\omega \tag{287}$$

We see that from equation (69) we have for term 4 and 6 three integrals defining three different frequency spreads. Term 4 is done explicitly below:

$$\text{Term 4} \Rightarrow i\theta f * g^{(2)}[\hat{a}_\omega\hat{b}_{\omega'}, \hat{b}_{\omega_7}^\dagger\hat{b}_{\omega_8}] \tag{288}$$

Only the operators in the left wedge are treated as smeared out. We calculate the commutator:

$$[\hat{a}_\omega\hat{b}_{\omega'}, \hat{b}_{\omega_7}^\dagger\hat{b}_{\omega_8}] = \hat{a}_\omega\hat{b}_{\omega_8}\delta_{\omega'\omega_7} \tag{289}$$

and end up with:

$$\int d\omega_7 d\omega_8 |g_7(\omega_7)||g_8(\omega_8)|\hat{a}_\omega\hat{b}_{\omega_8}\delta_{\omega'\omega_7} \tag{290}$$

We continue by calculating the expectation value:

$$\int d\omega_7 d\omega_8 g_7(\omega_7))||g_8(\omega_8)|\langle TFD|\,\hat{a}_\omega\hat{b}_{\omega_8}\,|TFD\rangle\,\delta_{\omega'\omega_7} \tag{291}$$

By using commutation relations and transformation rules between operators on the left and the right we get:

$$\Rightarrow \int d\omega_7 d\omega_8 |g_7(\omega_7)||g_8(\omega_8)| e^{\frac{-\beta\omega}{2}} \langle TFD| \hat{b}_{\omega_8} \hat{b}^{\dagger}_{\omega} |TFD\rangle \, \delta_{\omega'\omega_7} \tag{292}$$

Now the operator $\hat{b}_{\omega}$ is an operator on the left wedge. Using the thermal expectation values, using the following result:

$$\langle TFD| \hat{b}_{\omega_8} \hat{b}^{\dagger}_{\omega_7} |TFD\rangle = -\frac{1}{e^{-\beta\omega} - 1} \delta_{\omega\omega_8} \tag{293}$$

And we end up with for term 4:

$$\hat{a}_{\omega} \hat{b}_{\omega'} \Rightarrow -\int d\omega_7 d\omega_8 |g_7(\omega_7)||g_(\omega_8)| \frac{e^{-\frac{\beta\omega}{2}}}{e^{-\beta\omega} - 1} \delta_{\omega\omega_8} \delta_{\omega'\omega_7} \tag{294}$$

Likewise for term 6 we get:

$$\hat{a}^{\dagger}_{\omega} \hat{b}^{\dagger}_{\omega'} \Rightarrow -\int d\omega_7 d\omega_8 |g_7(\omega_7)||g_8(\omega_8)| \frac{e^{\frac{\beta\omega'}{2}}}{e^{\beta\omega'} - 1} \delta_{\omega\omega_7} \delta_{\omega'\omega_8} \tag{295}$$

We plug these two expressions back, and end up for the total change of the correlator to first order with:

$$\begin{aligned} &\Rightarrow \int \frac{d\omega dk}{(2\pi)^2} \frac{d\omega' dk'}{(2\pi)^2} \frac{1}{\sqrt{2\omega}} \frac{1}{\sqrt{2\omega'}} \Bigg[\\ &f * g^{(2)} \int d\omega_7 d\omega_8 |g_7(\omega_7)||g_(\omega_8)| \frac{e^{\frac{\beta\omega}{2}}}{e^{\beta\omega} - 1} \delta_{\omega\omega_8} \delta_{\omega'\omega_7} - \\ &f^* g^{*(2)} \int d\omega_7 d\omega_8 |g_7(\omega_7)||g_8(\omega_8)| \frac{e^{\frac{\beta\omega'}{2}}}{e^{\beta\omega'} - 1} \delta_{\omega\omega_7} \delta_{\omega'\omega_8} \Bigg] \end{aligned} \tag{296}$$

With the delta functions we kill the two integrals over ω_7 ω_8. The final expression now becomes:

$$\begin{aligned} &\delta\langle \phi(t_1, r_1) \phi(t_2, r_2) \rangle \Rightarrow \\ &\theta \int \frac{d\omega dk}{(2\pi)^2} \frac{d\omega' dk'}{(2\pi)^2} \frac{1}{\sqrt{2\omega}} \frac{1}{\sqrt{2\omega'}} |g(\omega)||g(\omega')| \left[\frac{f_{\omega} g^{(2)}_{\omega'} e^{\frac{\beta\omega}{2}}}{e^{\beta\omega} - 1} - \frac{f^*_{\omega} g^{*(2)}_{\omega'} e^{\frac{\beta\omega'}{2}}}{e^{\beta\omega'} - 1} \right] \end{aligned} \tag{297}$$

7.2.2 Pulse-approximation

We are ready to observe the smearing of the operators in more detail. Since the operators need to be spread out over a small frequency range, instead of being treated like a delta function, a good approximation would be a Gaussian. However mathematically this complicates the matter. As one can observe, the integrand already contains quite a number terms depending on ω.
Secondly, we are interested in the limit where the width of the wavepacket is small. In other words, the Gaussian would look like a Lorentzian. In this case, a good approximation would be a block pulse. The pulse will have a constant height, G_0, for a small frequency window, $\delta\omega$, and will be zero everywhere else. One can vary the width of the pulse to a desired value. The situation will look like this: In formula form this looks like:

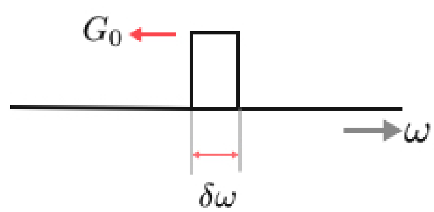

Figure 17: $g_i(\omega)$ is given by G_0 for a frequency window of $\delta\omega$ between a minimum frequency, ω_{min} and a maximum frequency, ω_{max}, and will be zero otherwise

$$g_i(\omega) \Rightarrow G_i \quad \text{for} \quad \omega_{min} \leq \omega \leq \omega_{max}. \tag{298}$$

By looking at the smearing integrals over $d\omega_i$, we see that these now simplify to the following constant values only defined between ω_{max} and ω_{min}. In other words:

$$\int d\omega_i g_i(\omega) b_{\omega_i} \Rightarrow G_i \int_{\omega_{min}}^{\omega_{max}} d\omega_i b_{\omega_i} \tag{299}$$

We look at equation (84) and focus on the ω integral. We change the integral via the pulse-approximation:

$$\boxed{\delta\langle\phi(t_1, r_1)\phi(t_2, r_2)\rangle \Rightarrow \theta G_0^2 \int_{\omega_{min}}^{\omega_{max}} \frac{d\omega d\omega'}{\sqrt{2\omega}\sqrt{2\omega'}} \left[\frac{f_\omega g_{\omega'}^{(2)} e^{\frac{\beta\omega}{2}}}{e^{\beta\omega} - 1} - \frac{f_\omega^* g_{\omega'}^{*(2)} e^{\frac{\beta\omega'}{2}}}{e^{\beta\omega'} - 1} \right]} \tag{300}$$

The k integral is left out, since it can be done trivially when keeping the angular part symmetric.

Also for the result for the change in energy equation (34) we can plug in the pulse approximation and obtain:

$$\delta E = \theta_i^2 G_0^4 \frac{\beta e^{\beta\omega_0}}{(e^{\beta\omega_0}-1)^2} \int_{\delta\omega} d\omega_1 d\omega_3 (\omega_1 - \omega_3)^2 \tag{301}$$

We identify ω_0 as ω_{min}, and evaluate the integrals:

$$\delta E \sim \theta_i^2 G_0^4 \frac{\beta e^{\beta\omega_{min}}}{(e^{\beta\omega_{min}}-1)^2} \int_{\omega_{min}}^{\omega_{\max}} d\omega_1 d\omega_3 (\omega_1 - \omega_3)^2 \tag{302}$$

Performing the ω_1 integral:

$$\theta_i^2 \frac{1}{3} G_0^4 \frac{\beta e^{\beta\omega_{min}}}{(e^{\beta\omega_{min}}-1)^2} \int_{\omega_{min}}^{\omega_{\max}} d\omega_3 (\omega_{max} - \omega_3)^3 - (\omega_{min} - \omega_3)^3 \tag{303}$$

And the ω_3 integral:

$$\theta_i^2 \frac{1}{12} G_0^4 \frac{\beta e^{\beta\omega_{min}}}{(e^{\beta\omega_{min}}-1)^2} [(\omega_{min} - \omega_{max})^4 + (\omega_{max} - \omega_{min})^4] \tag{304}$$

We find

$$\boxed{\delta E = \frac{\theta_i^2 G_0^4 \delta\omega^4}{12} \frac{\beta e^{\beta\omega_{min}}}{(e^{\beta\omega_{min}}-1)^2}} \tag{305}$$

We see the following relation:

$$\delta E \sim G_0^4 \delta\omega^4 \tag{306}$$

7.3 Horizon approximation

Before we evaluate our correlation change in this first approximation we need to specify the unperturbed correlator, which functions as the maximum eigenvalue in the Born rule inequality.

Maximum Eigenvalue We have defined $\max_j|\lambda_j^{\phi_1\phi_2}|$ to be the unperturbed correlator between both points. In the near horizon regime, this can be taken from [34]: We have:

$$\langle\phi_1\phi_2\rangle_{CFT} \sim \frac{1}{((U_1-U_2)(V_1-V_2))^{\frac{1}{2}}}$$

There is however an inconvenience with the massless scalar field in the limit. For $U_1 = U_2$ or $V_1 = V_2$ the two-point correlator blows up [34]. The issue is a well-known one, and is solved by considering derivatives of the field instead of the original scalars. The original two-point function for a massless scalar is then given by [34]:

$$\lim_{(V_1-V_2)\to 0}\langle TFD|\,\partial_{U_1}\partial_{U_2}\phi_1\phi_2\,|TFD\rangle = \frac{\beta}{4\pi^2(U_1-U_2)^2} \tag{307}$$

This quantity will be used in inequality (107) to test the Born rule as the maximum eigenvalue of the correlation function between $\phi_1\&\phi_2$.

$$\max_j|\lambda_j^{\phi_1\phi_2}|\sim \frac{\beta}{4\pi^2(U_1-U_2)^2}. \tag{308}$$

With this quantity defined, we can now focus our attention on the effect of $\hat{U}$ on the two-point correlation function.

The variation of the correlator depends on where the wavefunctions $f_\omega \& g^{(2)}_{\omega'}$ are located.

$$\delta\langle\phi(t_1,r_1)\phi(t_2,r_2)\rangle \Rightarrow \theta G_0^2 \int_{\omega_{min}}^{\omega_{max}} \frac{d\omega d\omega'}{\sqrt{2\omega}\sqrt{2\omega'}}\left[\frac{f_\omega g^{(2)}_{\omega'} e^{\frac{\beta\omega}{2}}}{e^{\beta\omega}-1} - \frac{f^*_\omega g^{*(2)}_{\omega'} e^{\frac{\beta\omega'}{2}}}{e^{\beta\omega'}-1}\right] \tag{309}$$

We evaluate both first in the near horizon approximation. Meaning we take each point on both sides very close to the horizon. We are integrating over a

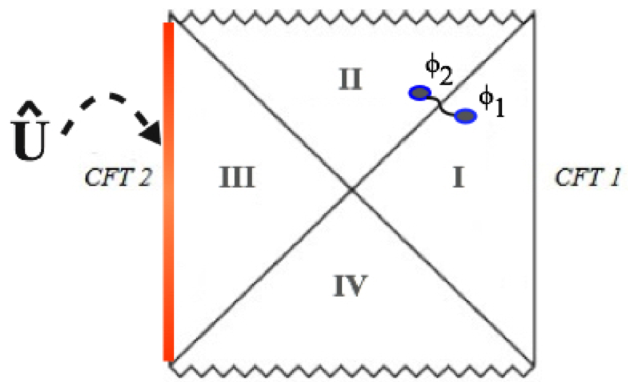

Figure 18: A two point function between a point inside and outside both in the *near horizon* limit

window, we do not have to worry about the pole at $\omega = 0$, however the integral is still quite problematic, due to the number of terms. As our domain is when $\delta\omega$ is small, we can take out the smooth terms and evaluate them, around a single frequency ω_0.

$$\delta\langle\phi_1\phi_2\rangle = \frac{\theta G_0^2}{4\pi^2\omega_0}\frac{e^{\frac{\beta\omega_0}{2}}}{e^{\beta\omega_0}-1}\int_{\omega_0}^{\omega_0+\delta\omega} d\omega d\omega'\left[f_\omega g^{(2)}_{\omega'} - f^*_\omega g^{*(2)}_{\omega'}\right] \tag{310}$$

As we have seen (section 3.6), a spherically symmetric wave function in the near horizon approximation *outside* of the horizon is given by:

$$f_\omega = r_h^{-\frac{1}{2}}(e^{i\delta_{\omega,k}}(-U_1)^{ia\omega} + e^{-i\delta_{\omega,k}}(V_1)^{-ia\omega}). \tag{311}$$

now written in terms of Kruskal coordinates. Likewise we have for *inside* the horizon wavefunctions $g^{(1)}$ and $g^{(2)}$ (recall equation (196). We only need $g^{(2)}$:

$$g^{(2)}_{\omega'}(t_2,r_2) = r_h^{-\frac{1}{2}}e^{-i\delta}(U_2)^{-ia\omega} \tag{312}$$

with

$$a = \frac{\beta}{2\pi} \tag{313}$$

Continuing to look at the product of wave functions in $\delta\langle\phi_1\phi_2\rangle$:

$$\begin{aligned}&\left[f_\omega g^{(2)}_{\omega'} - f^*_\omega g^{*(2)}_{\omega'}\right] = \\ &\quad \frac{1}{r_h}\left[(-U_1)^{ia\omega}(U_2)^{-ia\omega'} + e^{-2i\delta}(V_1)^{-ia\omega}(U_2)^{-ia\omega'} - \text{h.c.}\right]\end{aligned} \tag{314}$$

Without any loss of generality we can look at the change in correlation between the derivatives to U_1/U_2 of the fields [14]

$$\partial_{U_1}\partial_{U_2}(\delta\phi_1\phi_2) \tag{315}$$

Applying this to equation 8 we only are only left with the terms depending on U_1 and U_2. After differentiating we obtain:

$$\partial_{U_1}\partial_{U_2}\left[f_\omega g^{(2)}_{\omega'} - f^*_\omega g^{*(2)}_{\omega'}\right] = \frac{\beta^2\omega\omega'}{4\pi^2 r_h}(-U_1)^{ia\omega-1}(U_2)^{-ia\omega'-1} - \text{h.c.} \tag{316}$$

The quantity to calculate is:

$$\begin{aligned}&\partial_{U_1}\partial_{U_2}\delta\phi_1\phi_2 = \\ &\frac{\omega_0\theta G_0^2\beta^2}{16\pi^4 r_h}\frac{e^{\frac{\beta\omega_0}{2}}}{e^{\beta\omega_0}-1}\int_{\omega_0}^{\omega_0+\delta\omega} d\omega d\omega'\left[(-U_1)^{ia\omega-1}(U_2)^{-ia\omega'-1} - \text{h.c.}\right]\end{aligned} \tag{317}$$

The integral over ω is given by:

$$\int_{\omega_0}^{\omega_0+\delta\omega} d\omega(-U_1)^{ia\omega-1} = \frac{(-1+(-U_1)^{ia\delta\omega})(-U_1)^{ia\omega_0-1}}{ia\log(-U_1)} \tag{318}$$

likewise the integral over ω':

$$\int_{\omega_0}^{\omega_0+\delta\omega} d\omega'(U_2)^{-ia\omega-1} = -\frac{(-1+(U_2)^{-ia\delta\omega})(U_2)^{-ia\omega_0-1}}{ia\log(U_2)} \tag{319}$$

And the product is given by:

$$\begin{aligned}&\int_{\omega_0}^{\omega_0+\delta\omega} d\omega d\omega'(-U_1)^{ia\omega-1}(U_2)^{-ia\omega'-1} \\ &= \frac{1}{a^2}\frac{(-1+(-U_1)^{ia\delta\omega})(-U_1)^{ia\omega_0-1}(-1+(U_2)^{-ia\delta\omega})(U_2)^{-ia\omega_0-1}}{\log(-U_1)\log(U_2)} \\ &= \frac{1}{a^2U_1U_2}\left(-\frac{U_1}{U_2}\right)^{ia\omega_0}\frac{(-1+(-U_1)^{ia\delta\omega})(-1+(U_2)^{-ia\delta\omega})}{\log(-U_1)\log(U_2)} \\ &= \frac{1}{a^2U_1U_2}\left(-\frac{U_1}{U_2}\right)^{ia\omega_0}\frac{(1-(-U_1)^{ia\delta\omega}-(U_2)^{-ia\delta\omega}+\left(-\frac{U_1}{U_2}\right)^{ia\delta\omega})}{\log(-U_1)\log(U_2)}\end{aligned} \tag{320}$$

[14] The reason for taking the derivative is due to the well known problem with the lightlike massless scalar field correlation function. One has $\langle\phi_1\phi_2\rangle \sim \frac{1}{((U_1-U_2)(V_1-V_2))^{\frac{1}{2}}}$ which blows up for $U_1 = U_2$

Now the full quantity for the change of the correlator to first derivative in U_1/U_2 is given by:

$$\partial_{U_1}\partial_{U_2}\delta\phi_1\phi_2 \Rightarrow \frac{\omega_0\theta G_0^2}{4\pi^2 r_h}\frac{e^{\frac{\beta\omega_0}{2}}}{e^{\beta\omega_0}-1}\frac{1}{U_1U_2}\frac{1}{\log(-U_1)\log(U_2)}$$
$$\left[\left(-\frac{U_1}{U_2}\right)^{ia\omega_0}(1-(-U_1)^{ia\delta\omega}-(U_2)^{-ia\delta\omega}+\left(-\frac{U_1}{U_2}\right)^{ia\delta\omega})-\text{h.c.}\right] \tag{321}$$

7.4 Boundary approximation

The second regime for the variation of the two-point correlation function is one point close to the boundary outside of the black hole, with the second point inside still close to the horizon.

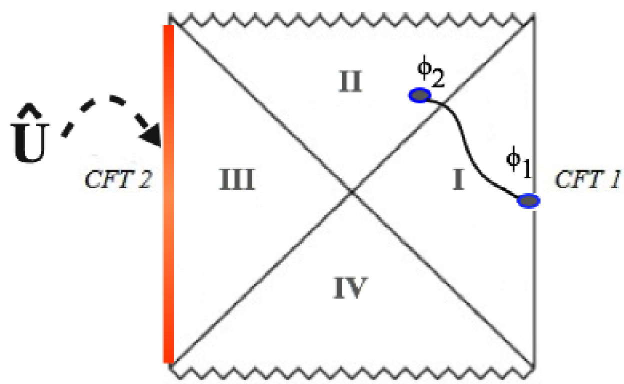

Figure 19: A two point function between a point in the *near horizon* limit and one in at the boundary

Recall the boundary limit for the full wave function for the eternal black hole

$$f_{\omega,k}(t,r,\Omega) \underset{r\to\infty}{\rightarrow} \frac{1}{\Upsilon}\frac{1}{\sqrt{r_h}} r^{-\Delta} e^{-i\omega t+ik\Omega} \tag{322}$$

with $\Delta = 1+\sqrt{1+m^2} = 2$. From now on, we assume spherical symmetry again, leaving out the angular dependence.

Maximum Eigenvalue For the new regime, the maximum eigenvalue $\max_j |\lambda_j^{\phi_1\phi_2}|$ should be specified, before computing the new correlation change. We once more define the new unperturbed two-point function between both points:

$$\langle \partial_{U_1}\partial_{U_2}\phi_{boundary}(t_1r_1)\phi_{horizon}(t_2r_2)\rangle \tag{323}$$

As we saw in the derivation for the change in correlation, in section 4.2, we have a product between the creation/annihilation operators of the exterior with the interior.

$$\begin{aligned} &\langle\phi(t_1r_1)\phi(t_2r_2)\rangle \Rightarrow \\ &f * g^{(1)}\hat{a}_\omega\hat{a}_{\omega'} + f * g^{*(2)}\hat{a}_\omega\hat{b}^\dagger_{\omega'} + f * g^{*(1)}\hat{a}_\omega\hat{a}^\dagger_{\omega'} + f * g^{(2)}\hat{a}_\omega\hat{b}_{\omega'} \\ +&f^* * g^{(1)}\hat{a}^\dagger_\omega\hat{a}_{\omega'} + f^* * g^{*(2)}\hat{a}^\dagger_\omega\hat{b}^\dagger_{\omega'} + f^* * g^{*(1)}\hat{a}^\dagger_\omega\hat{a}^\dagger_{\omega'} + f^* * g^{(2)}\hat{a}^\dagger_\omega\hat{b}_{\omega'} \end{aligned} \tag{324}$$

From the thermal expectation values, we know only terms $3, 4, 5$ and6 to give a non-trivial value. From equation (..) and (..) we can write down

- $$\hat{a}_\omega \hat{a}^\dagger_{\omega'} \rightarrow \frac{e^{\beta\omega}}{e^{\beta\omega}-1}\delta_{\omega\omega'} \tag{325}$$
- $$\hat{a}_\omega \hat{b}_{\omega'} \rightarrow \frac{e^{\frac{\beta\omega}{2}}}{e^{\beta\omega}-1}\delta_{\omega\omega'} \tag{326}$$
- $$\hat{a}^\dagger_{\omega'} \hat{a}_\omega \rightarrow \frac{1}{e^{\beta\omega}-1}\delta_{\omega\omega'} \tag{327}$$
- $$\hat{a}_\omega \hat{b}^\dagger_{\omega'} \rightarrow \frac{e^{\frac{\beta\omega}{2}}}{e^{\beta\omega}-1}\delta_{\omega\omega'} \tag{328}$$

If we then pair up these terms with their corresponding wave functions, we end up with four terms:

$$\begin{gathered}\langle\phi_{boundary}(t_1 r_1)\phi_{horizon}(t_2 r_2)\rangle \sim \frac{1}{\Upsilon}\frac{1}{r_h} r_1^{-2}\int_0^\infty \frac{d\omega}{\omega} * \\ \left[\frac{e^{-i\delta_\omega}}{e^{\beta\omega}-1}\left(e^{\frac{\beta\omega}{2}} e^{-i\omega t_1} U_2^{-ia\omega} + e^{i\omega t_1} V_2^{-ia\omega}\right) + \right. \\ \left. \frac{e^{i\delta_\omega}}{e^{\beta\omega}-1}\left(e^{\frac{\beta\omega}{2}} e^{i\omega t_1} U_2^{ia\omega} + e^{-i\omega t_1} V_2^{ia\omega}\right)\right]\end{gathered} \tag{329}$$

Before continuing with this integral, we drop the term *before* the integral. The constants Υ, r_h and the radial component at the boundary r_1 will drop out of the Born rule inequality, when we divide by the change in correlator $\delta\langle\phi_1\phi_2\rangle$.

$$\begin{aligned}\delta\langle\phi_1\phi_2\rangle \leq 2\max_j|\lambda_j^{\phi_1\phi_2}|\sqrt{\frac{\delta E}{T}} &\Rightarrow \frac{\delta\langle\phi_1\phi_2\rangle}{\max_j|\lambda_j^{\phi_1\phi_2}|} \\ &\Rightarrow \frac{\delta\langle\phi_1\phi_2\rangle}{\langle\phi_{boundary}(t_1 r_1)\phi_{horizon}(t_2 r_2)\rangle} = \sqrt{\frac{\delta E}{T}}\end{aligned}$$

Continuing by writing everything in Kruskal coordinates and taking derivatives to U_1 and U_2, both V_2 terms drop, and we are left with two terms. The quantity to compute is I:

$$\begin{aligned}I \sim \frac{a^2}{2U_1U_2}&\int_0^\infty d\omega\omega * \\ &\frac{e^{\frac{\beta\omega}{2}}}{e^{\beta\omega}-1}\left[e^{-i\delta_\omega} U_2^{-ia\omega}\left(-\frac{U_1}{V_1}\right)^{\frac{ia\omega}{2}} + e^{i\delta_\omega} U_2^{ia\omega}\left(-\frac{U_1}{V_1}\right)^{\frac{-ia\omega}{2}}\right]\end{aligned} \tag{330}$$

These terms can be taken together as:

$$\int_{-\infty}^{\infty} d\omega\omega \frac{e^{\frac{\beta\omega}{2}}}{e^{\beta\omega}-1}\left(\frac{1}{U_2}\sqrt{-\frac{U_1}{V_1}}\right)^{ia\omega}$$

The integral has n poles at $\omega = \frac{2\pi i}{\beta} n$, and is evaluated in the upper half plane when $\left|\frac{1}{U_2}\sqrt{-\frac{U_1}{V_1}}\right| < 1$. The large semi-circle at $R \to \infty$ goes to zero by Jordan's Lemma. We find:

$$\begin{aligned}\int_{-\infty}^{\infty} d\omega\omega \frac{e^{\frac{\beta\omega}{2}}}{e^{\beta\omega}-1}\left(\frac{1}{U_2}\sqrt{-\frac{U_1}{V_1}}\right)^{ia\omega} &= 2\pi i \sum \mathrm{Res}(f(\omega),\omega_0) \\ &= \lim_{\omega\to\omega_0}(\omega-\omega_0)\frac{\omega e^{\frac{\beta\omega}{2}}}{e^{\beta\omega}-1}\left(\frac{1}{U_2}\sqrt{-\frac{U_1}{V_1}}\right)^{ia\omega}\end{aligned}$$

With *L'Hoptital's* rule we find

$$\lim_{\omega\to\omega_0}(\omega-\omega_0)\frac{\omega e^{\frac{\beta\omega}{2}}}{e^{\beta\omega}-1}\left(\frac{1}{U_2}\sqrt{-\frac{U_1}{V_1}}\right)^{ia\omega} = -\frac{4\pi^2}{\beta^2}\sum_0^{\infty} n\left(U_2\sqrt{-\frac{V_1}{U_1}}\right)^n \tag{331}$$

The infinite sum has a limit for $\left|U_2\sqrt{-\frac{V_1}{U_1}}\right| < 1$, since it is exactly of the form: $\sum nx^n = \frac{x}{x^2-1}$. With some algebra we write

$$-\frac{4\pi^2}{\beta^2}\sum_0^{\infty} n\left(U_2\sqrt{-\frac{V_1}{U_1}}\right)^n = -\frac{4\pi^2}{\beta^2}\frac{U_2\sqrt{-\frac{V_1}{U_1}}}{(U_2\sqrt{V_1}-\sqrt{U_1})^2} \tag{332}$$

We conclude for quantity I :

$$I \sim \frac{\sqrt{V_1}}{U_1\sqrt{-U_1}(U_2\sqrt{V_1}-\sqrt{-U_1})^2}. \tag{333}$$

Remember this is not the full correlation function between a point at the horizon and the boundary. We have divided out the radial part r_1^{Δ} of ϕ_1, since this will be divided out when evaluating the Born rule. This is the quantity we need in our Born rule inequality to scale the change in correlation.

7.4.1 Change in Correlation

The change of the two-point horizon-boundary correlator is given by:

$$\langle \delta\phi_{bound}(t_1r_1)\phi_{hor}(t_2r_2)\rangle \sim \frac{\omega_0 a^2\theta G_0^2}{4\pi^2 U_1U_2}\frac{e^{\frac{\beta\omega_0}{2}}}{e^{\beta\omega_0}-1}\int_{\omega_0}^{\omega_0+\delta\omega} d\omega d\omega' * \\ \left[e^{-i\delta_\omega}U_2^{-ia\omega}\left(-\frac{U_1}{V_1}\right)^{\frac{ia\omega}{2}} - e^{i\delta_\omega}U_2^{ia\omega}\left(-\frac{U_1}{V_1}\right)^{\frac{-ia\omega}{2}}\right] \tag{334}$$

We could use the derivatives of the wavefunctions to U_1, U_2 from the original correlator above. The function between square brackets consists of one term and it's hermitian conjugate.
As for the horizon approximation, we take out the smooth parts of the function and evaluate the the effect of $\hat{U}$ on the wavefunctions over a small window. Next up, the integral over ω and ω'. They are straightforward and the answer can be given immediately.

$$\int_{\omega_0}^{\omega_0+\delta\omega} d\omega d\omega' U_2^{-ia\omega'}\left(-\frac{U_1}{V_1}\right)^{\frac{ia\omega}{2}} \\ \Rightarrow \frac{2\left(U_2^{-ia(\omega_0+\delta\omega)} - U_2^{ia\omega_0}\right)\left[\left(-\frac{U_1}{V_1}\right)^{\frac{ia\omega_0+\delta\omega}{2}} - \left(-\frac{U_1}{V_1}\right)^{\frac{ia\omega_0}{2}}\right]}{a^2U_1U_2\log U_2\log -\frac{U_1}{V_1})} \tag{335}$$

The total change in correlator is then given by:

$$\langle \delta\phi_{bound}(t_1r_1)\phi_{hor}(t_2r_2)\rangle \sim \frac{\omega_0\theta G_0^2}{4\pi^2U_1U_2\log U_2\log\left(-\frac{U_1}{V_1}\right)}\frac{e^{\frac{\beta\omega_0}{2}}}{e^{\beta\omega_0}-1} * \\ \left[2\left(U_2^{-\kappa_+} - U_2^{\kappa_0}\right)\left[\left(-\frac{U_1}{V_1}\right)^{\frac{\kappa_+}{2}} - \left(-\frac{U_1}{V_1}\right)^{\frac{\kappa_0}{2}}\right] - \text{H.C.}\right] \tag{336}$$

Where $\kappa_+ = ia\omega_0 + \delta\omega$ and $\kappa_0 = ia\omega_0$.

8 Analysis

Our final result for the Born Rule inequality is given by:

$$\boxed{\delta\langle\phi_1\phi_2\rangle \leq 2\max_j|\lambda_j^{\phi_1\phi_2}|\sqrt{\frac{\delta E}{T}}} \tag{337}$$

where the change in energy was given by:

$$\delta E = \frac{\theta_i^2 G_0^4 \delta\omega^4}{12}\frac{\beta e^{\beta\omega_0}}{(e^{\beta\omega_0}-1)^2} \tag{338}$$

We have calculated two different regimes for the correlation function $\phi_1\phi_2$. All the ingredients are on the table to test the Born rule.

8.1 Horizon Approximation

To test the Born rule, we substitute all necessary quantities into inequality (135). Recall our result for the change in correlation function in the near horizon approximation:

$$\begin{aligned}\partial_{U_1}\partial_{U_2}\delta\phi_1\phi_2 \Rightarrow \frac{\omega_0\theta G_0^2}{4\pi^2 r_h}\frac{e^{\frac{\beta\omega_0}{2}}}{e^{\beta\omega_0}-1}\frac{1}{U_1U_2}\frac{1}{\log(-U_1)\log(U_2)}\\ \left[\left(-\frac{U_1}{U_2}\right)^{ia\omega_0}(1-(-U_1)^{ia\delta\omega}-(U_2)^{-ia\delta\omega}+\left(-\frac{U_1}{U_2}\right)^{ia\delta\omega})-\text{h.c.}\right]\end{aligned} \tag{339}$$

For this regime we have set the maximum eigenvalue $\max_j|\lambda_j^{\phi_1\phi_2}|$ equal to

$$\max_j|\lambda_j^{\phi_1\phi_2}|\sim \langle TFD|\,\partial_{U_1}\partial_{U_2}\phi_1\phi_2\,|TFD\rangle \sim \frac{\beta}{4\pi^2(U_1-U_2)^2} \tag{340}$$

We put them all in (135) and obtain:

$$\boxed{\begin{aligned}&\frac{\omega_0(U_1-U_2)^2}{U_1U_2}\frac{1}{\log(-U_1)\log(U_2)}\left[\left(-\frac{U_1}{U_2}\right)^{ia\omega_0} *\right.\\ &\left.\left(1-(-U_1)^{ia\delta\omega}-(U_2)^{-ia\delta\omega}+\left(-\frac{U_1}{U_2}\right)^{ia\delta\omega}\right)-\text{h.c.}\right] \leq \frac{4\pi\beta}{\sqrt{3}}\delta\omega^2\end{aligned}} \tag{341}$$

Analysis

When trying to violate the Born rule, one wants to let the left hand side become large, while the right should go to zero. One can immediately observe to take ω_0 **to be large**, causing the left side to substantial. Next, there are several cases that need to be checked:

1. $\delta\omega \to 0$, $\beta \to 0$. For $\delta\omega = 0$, or $\beta = 0$, the left hand side and the right hand side are **both** zero. Recall $a = \beta/2\pi$. One needs to take $\delta\omega$ to be small but non-zero.

2. $\delta\omega, \beta$ **small**. In this case we can expand the left hand side. The case for β is similar to the one of $\delta\omega$. The only terms consisting of $\delta\omega$ or β are between the big square brackets. We expand the terms:

$$\begin{aligned}&\rightarrow\Bigg[1-1-\log\left(-\frac{1}{U_1}\right)ia\delta\omega- \\ &\quad 1+\log(U_2)ia\delta\omega+1+\log\left(-\frac{U_1}{U_2}\right)ia\delta\omega-\text{h.c.}\Bigg] \\ &=\log\left(-\frac{U_2}{U_1}\right)ia\delta\omega-\log\left(-\frac{U_2}{U_1}\right)ia\delta\omega=0\end{aligned} \tag{342}$$

This tells us $\delta\omega$ has to be sufficient, for the left hand side to be finite. Likewise for β

3. $\delta\omega$, β **finite**. Continuing holding $\delta\omega$, β **finite**, we vary the spacetime coordinates to several limits.
The case for $U_1 = -U_2$ gives a zero left hand side. One can relate both coordinates by taking $U_2 = -mU_1$. Where m is a number. In this case the inequality becomes:

$$\begin{aligned}&\frac{\omega_0(1+m)^2}{\omega_0 m}\frac{1}{\log(-U_1)\log(-mU_1)} \\ &\Bigg[\left(\frac{1}{m}\right)^{ia}\Bigg(1-(-U_1)^{ia\delta\omega}-(-mU_1)^{-ia\delta\omega} \\ &+\left(\frac{1}{m}\right)^{ia\delta\omega}\Bigg)-\left(\frac{1}{m}\right)^{-ia\omega_0}\Bigg(1-(-U_1)^{-ia\delta\omega} \\ &-(-mU_1)^{ia\delta\omega+}\left(\frac{1}{m}\right)^{-ia\delta\omega}\Bigg)\Bigg]\leq\frac{4\pi\beta}{\sqrt{3}}\delta\omega^2\end{aligned} \tag{343}$$

Right now it is dubious when we look at the $m \to 0$ limit. What we observe is, the left hand side consisting of a first term blowing up, the second, logarithmic term going to zero logarithmically, and several rapidly oscillating terms. In this limit the left hand side behaves divergent.

$$\lim_{m\to 0}\frac{\omega_0}{m}[\left(\frac{1}{m}\right)^{ia\omega_0}-\left(\frac{1}{m}\right)^{-ia\omega_0}]\to\infty \tag{344}$$

However the infalling observer measures for a finite spread in spacetime. In other words, it takes some time to measure something. In this sense one should integrate on a very small region over m, say ϵ.

$$\omega_0 \int_\epsilon dm m^{-ia\omega_0 - 1} \sim m^{-ia\omega_0}\Big|_{-\epsilon/2}^{\epsilon/2} \tag{345}$$

The same for the Hermitian conjugate term. What we observe for both terms is for ϵ being small but finite, the result behaves regular. Oscillating and being finite. Put differently, looking at the inequality for the limit $m \to 0$. One can explain the regular result by recognizing the applicability of the *Riemann-Lebesgue lemma* [15]. With ω_0 large rapidly oscillating terms on the left hand side like $m^{-ia\omega_0}$, the lemma tells us that these terms are suppressed in $O(\frac{1}{\omega_0})$

The left hand side will be of order 1 for a very specific choice of m and U_1. One does not know if this choice is physically possible, however one does know that for the arbitrary m and U_1 this possibility is extremely small, if not going to zero.

[15] The Riemann-Lebesgue lemma is a very important theorem from Fourier analysis. It states that for a continuous function $f(x)$, the fourier integral between a and b for $k \to \infty$ goes to zero when $k \to \infty$: $\lim_{k\to\infty} \int_a^b dx e^{ikx} f(x) \to O(\frac{1}{k}) \to 0$.[45]

8.2 Boundary Approximation

For clarity we once more state the Born rule inequality

$$\delta\langle\phi_1\phi_2\rangle \leq 2\max_j|\lambda_j^{\phi_1\phi_2}|\sqrt{\frac{\delta E}{T}} \tag{346}$$

The effect on the correlation for the horizon-boundary correlator is given by:

$$\begin{aligned}\langle\delta\phi_{bound}(t_1r_1)\phi_{hor}(t_2r_2)\rangle \sim \frac{\omega_0\theta G_0^2}{2\pi^2U_1U_2\log U_2\log\left(-\frac{U_1}{V_1}\right)}\frac{e^{\frac{\beta\omega_0}{2}}}{e^{\beta\omega_0}-1}*\\ \left[\left(U_2^{-\kappa_+}-U_2^{\kappa_0}\right)\left[\left(-\frac{U_1}{V_1}\right)^{\frac{\kappa_+}{2}}-\left(-\frac{U_1}{V_1}\right)^{\frac{\kappa_0}{2}}\right]-\text{H.C.}\right]\end{aligned} \tag{347}$$

We found the maximum eigenvalue of the two-point horizon-boundary correlator to scale with:

$$\max_j|\lambda_j^{\phi_1\phi_2}|\sim\frac{\sqrt{V_1}}{U_1\sqrt{-U_1}(U_2\sqrt{V_1}-\sqrt{-U_1})^2} \tag{348}$$

Since the right hand side, the change in energy, is similar to the horizon approximation one should focus on the left hand side. One can immediately identify the similarities for $\delta\omega$, $\beta\to 0$ with the horizon approximation. For the continuing analysis, we take $\delta\omega$, β small.
For the analysis of different values for U/V, one puts together $\delta\langle|\phi_1\phi_2\rangle$ and $\max_j|\lambda_j^{\phi_1\phi_2}|$.

$$\begin{aligned}\frac{\langle\delta\phi_{bound}(t_1r_1)\phi_{hor}(t_2r_2)\rangle}{\max_j|\lambda_j^{\phi_1\phi_2}|}\Rightarrow\frac{\sqrt{-U_1}(U_2\sqrt{V_1}-\sqrt{-U_1})^2}{U_2\sqrt{V_1}\log U_2\log\left(-\frac{U_1}{V_1}\right)}*\\ \left[\left(U_2^{-\kappa_+}-U_2^{\kappa_0}\right)\left[\left(-\frac{U_1}{V_1}\right)^{\frac{\kappa_+}{2}}-\left(-\frac{U_1}{V_1}\right)^{\frac{\kappa_0}{2}}\right]-\text{H.C.}\right]\end{aligned} \tag{349}$$

The fraction in front of the big square brackets is the one of interest. The term between square brackets consists of oscillating terms of order one. Observing the fraction in front:

$$\frac{\sqrt{-U_1}(U_2\sqrt{V_1}-\sqrt{-U_1})^2}{U_2\sqrt{V_1}\log U_2\log\left(-\frac{U_1}{V_1}\right)}$$

For this term to blow up, there are two different cases.

1. $U_2 \to 0$. The second point is located nearly on the horizon. This is very plausible. In this limit, taking U_1/V_1 constant C and ω_0 large again, one observes the fraction to behave in the following way:

$$\lim_{U_2 \to 0} \frac{\omega_0 C}{U_2 \log(U_2)} = \lim_{U_2 \to 0} \frac{\omega_0}{U_2} \tag{350}$$

   The term is, as before, regularized by integrating U_2 over a measuring distance ϵ and applying the *Riemann Lebesgue* lemma. The fraction is multiplied with the rapidly oscillating part between square brackets. All terms will behave like:

$$\lim_{U_2 \to 0} \frac{\omega_0}{U_2} * U_2^{\kappa_0} \to \lim_{U_2 \to 0} \omega_0 U_2^{\kappa - 1} \Rightarrow \omega_0 \int_\epsilon U_2^{\kappa_0 - 1} \sim U_2^{\kappa_0} \Big|^\epsilon \tag{351}$$

2. $V_1 \to 0$ In addition to the above, one can take $V_1 \to 0$. Again the limit is diverging, but can be normalized in the same way. Taking the other terms constant in the form of P:

$$\lim_{V_1 \to 0} \frac{\omega_0 P}{\sqrt{V_1} \log\left(\frac{P}{V_1}\right)} \to \lim_{V_1 \to 0} \frac{\omega_0}{\sqrt{V_1}} \Rightarrow \omega_0 \int_\epsilon \frac{1}{\sqrt{V_1}} \sim \sqrt{V_1} \Big|^\epsilon \tag{352}$$

Both limits together in this sense give a finite result for the left hand side, and do not break the Born rule.

9 Discussion

As the result of this thesis contradicts the predictions made by [15], we shed some light on why this is the case. The authors in [15] discuss that there is an interplay between the increase in energy caused by the smearing of the wavepackets, and the observation of Born rule violations. A comparison between our result for the change in energy with the result of Marolf and Polchinski shows a similar result, when we take $\frac{e^{\frac{\beta\omega}{2}}}{e^{\beta\omega}-1} \sim 1$, or $\beta \sim 1/\omega$. However, by computing correlators between the inside and outside the horizon, we have tried to explicitly display the experience of the infalling observer. This results into a different conclusion outcome. Our observation shows Born rule violations to be very hard to detect if not immeasurable by the infaller.

Although two-point functions give a good indication whether or not the horizon consists of an excited nature, one should check all n-point correlators to gain full clarity on the matter. Not only in two dimensions, but for higher dimensions as well. The construction of these quantities does however consist of some mathematical difficulties. Higher point functions become very complicated. Likewise computing two-point functions for scalar fields in higher dimensions, one runs into difficulties finding explicit solutions for the Klein-Gordon equation.

An improvement on the construction in this thesis might be approximating the wave packets by a Gaussian instead of a pulse. This seems a more natural representation of a wave-packet. In this case, the difference with the pulse approximation is most visible at the tails of the Gaussian. However, one is interested in the limit where the width of the Gaussian is small. In this limit the surface area of the tails is minimal, therefore the pulse approximation works quite well.

A second point of interest might be the approximation of the maximum eigenvalue of the two-point correlator. For bosons this quantity is not well defined. An interesting analysis would be to check the case for fermionic fields. In this way the correlator has a definite maximum eigenvalue, and results are more exact.

For a more complete view it is good to raise the question what would be the consequence of Born rule violations if they would turn out to occur. Even though violations of this kind have never been observed before, this does not necessarily imply a disastrous repercussion. As finding the right construction for the interior of the black hole consistent with quantum mechanics and general relativity is part of the quest for finding a UV-complete theory of quantum gravity, it could very well be an ingredient of this yet to be developed theory. Of course speculations of this kind are precarious, however it is clarifying to have a full overview of potential scenarios with corresponding features.

With the result of this thesis the endeavor remains to find a complete construction for the interior of a black hole consistent with quantum mechanics. As proposals like [13] show promising results, the hope is there so that maybe one day we all could live in a world where we are able to dual typical CFT microstates to non excited AdS black holes.

10 Conclusion

In this master thesis we have researched the question whether or not typical black hole micro states with non-excited dual horizons could exist without violating the Born rule. The matter is investigated by the construction of a quantitative representation of the Born identity in the form of a condition. The condition derived in in this thesis relates the expectation value of a certain Hermitian operator $\hat{A}$ in two different quantum states, to the overlap between these states. As the overlap between the eigenvectors of $\hat{A}$ and the quantum state in question, is related by the Born rule to the expected probability of collapse into that eigenstate, one expects parallel quantum states to yield similar expectation values for a certain operator. Therefore, by applying this derived Born rule condition to black holes within the context of holographic duality AdS/CFT, one can analyze if both sides produce similar physical interpretations when typical CFT micro-states are taken dual to smooth horizon AdS black hole states.

We have constructed a thought experiment, where a smooth horizon black hole state dual to a typical CFT state is perturbed by a unitary operator $\hat{U}$. By monitoring the perturbation on such a state, one is able to analyze the hypothesis. We have related the effect of the pertubation on the energy of the state, to the effect on two-point correlators between the inside and outside region of the black hole. These quantities have been put into our derived Born rule condition. We found the modification of these quantities to lie **within** the bounds of the Born rule for several regimes of the two-point correlation functions. We can therefore conclude that violations of the Born rule do not explicitly occur for typical black hole states.

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12 Appendix A

12.1 Other operations?

As we have seen above, the wave-packet construction causes the situation to be delicate. The increase in energy brings several issues along with it. It would be highly convenient if we could in some other way change the entanglement of the state, without changing the energy of the state. The relevant question is, if there are operations on the thermofield state $|TFD\rangle$, that leave both partial density ρ_L, ρ_R unperturbed, however modify the entanglement of the entire state.

Phase. One can add a phase-shift to one wedge. This is done by [40], by applying time-evolution to one side of the $|TFD\rangle$. The authors actually find a very interesting result. Since they produce a whole new set of new "phase-shifted" states, the question arises whether or not these states are too connected via a wormhole as the thermofield state is. They find this only to be possible if the interior is build up from so called *state dependent* operators [13]. State dependency is actually one solution for finding a smooth interior dual to typical states in the CFT. As the contents of this thesis is broader applicable, then only *state dependency* we did not review it here. The reader interested in this proposal can look at [13,34,49].

12.2 Schmidt Decomposition

So we are looking for an operation with the following properties:

$$\hat{A}\,|TFD\rangle = |TFD\rangle' \tag{353}$$

However we need

$$\rho_{L/R} = \rho^{'}_{L/R} \tag{354}$$

It turns out there are **no** other operations then a phase which have the desired effect. To show this we make use of the Schmidt decomposition. The Schmidt decomposition states the following:

Theorem 1.1: For every pure state $|\psi\rangle$ of composite system AB there exists a basis of orthonormal states $|i_A\rangle$ and $|i_B\rangle$ such that,

$$|\psi\rangle = \sum_i \lambda_i\,|i_A\rangle\,|i_B\rangle \tag{355}$$

The λ_i are called the *Schmidt coefficients*. They are non-negative eigenvalues satisfying $\sum_i \lambda^2 = 1$. An important feauture from the Schmidt Decomposition is for instance that the eigenvalues of ρ_A and ρ_B are the same since they are both given by resp: $\rho_A = \sum_i \lambda_i^2\,|i_A\rangle\,\langle i_A|$ and $\rho_B = \sum_i \lambda_i^2\,|i_B\rangle\,\langle i_B|$. One can see that both the eigenvalues are given by: λ_i^2.

Now the fact is that we can both express ρ_L and $\rho_L^{'}$ in the Schmidt-basis giving:

$$\rho_L = \sum_i \lambda_i^2 \left| i_A \right\rangle \left\langle i_A \right| \tag{356}$$

and,

$$\rho_L^{'} = \sum_i {\lambda'}_i^2 \left| i'_A \right\rangle \left\langle i'_A \right| \tag{357}$$

The fact is that if we demand

$$\rho_L = \rho_L^{'}, \tag{358}$$

one can state that the eigenfunctions and eigenvalues of ρ_1 can only differ from the eigenfunctions/values of $\rho_1^{'}$ with a phase.
This case is a classic one and can be shown by looking at 2 spin $\frac{1}{2}$ particles. The density matrix of 2-spin particles in the following state in the regular z-basis is given by:

$$\left|\psi\right\rangle = \frac{1}{2}(\left|\uparrow_z\right\rangle \left\langle\uparrow_z\right| + \left|\downarrow_z\right\rangle \left\langle\downarrow_z\right|) \to \rho_z = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \tag{359}$$

If we now write this state in terms of the x-basis and compute the density matrix again we end up with the following:

$$\begin{aligned} \left|\psi\right\rangle &= \frac{1}{2}(\left|\uparrow_x\right\rangle \left\langle\uparrow_x\right| + \left|\downarrow_x\right\rangle \left\langle\downarrow_x\right| \\ &= \frac{1}{2}(\frac{1}{\sqrt{2}} \left|\uparrow_z + \downarrow_z\right\rangle \left\langle\uparrow_z + \downarrow_z\right| + \frac{1}{\sqrt{2}} \left|\uparrow_z - \downarrow_z\right\rangle \left\langle\uparrow_z - \downarrow_z\right|) \\ &= \frac{1}{4}(\left|\uparrow_z\right\rangle \left\langle\uparrow_z\right| + \left|\uparrow_z\right\rangle \left\langle\downarrow_z\right| + \left|\downarrow_z\right\rangle \left\langle\uparrow_z\right| + \left|\downarrow_z\right\rangle \left\langle\downarrow_z\right| + \left|\uparrow_z\right\rangle \left\langle\uparrow_z\right| - \left|\uparrow_z\right\rangle \left\langle\downarrow_z\right| \\ &\qquad - \left|\downarrow_z\right\rangle \left\langle\uparrow_z\right| + \left|\downarrow_z\right\rangle \left\langle\downarrow_z\right|) \\ &= \frac{1}{2}(\left|\uparrow_z\right\rangle \left\langle\uparrow_z\right| + \left|\downarrow_z\right\rangle \left\langle\downarrow_z\right|) \end{aligned} \tag{360}$$

Which gives the exact same density matrix as found in the z-basis. The result therefore tells us that operator A can only be a phase. In this case again the powerful properties of the Schimdt Decomposition are shown.

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