# Unoriented and oriented Kontsevich graph cocycles 

 Finding infinitesimal deformations of Poisson structuresBachelor's Project Mathematics
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# INFINITESIMAL DEFORMATIONS OF ALGEBRAS AND THE KONTSEVICH UNORIENTED AND ORIENTED GRAPH COMPLEXES 

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This bachelor thesis is concerned with finding infinitesimal deformations of Poisson structures, by using the unoriented and unoriented graph complex introduced by M. Kontsevich. We shall first give a short historical introduction to deformation theory and its development in general. Even though in the thesis we are mainly concerned with infinitesimal deformations of Poisson structures, we start with a detailed introduction of deformations and infinitesimal deformations of associatvive algebras. This way we do not have to introduce both deformation theory and Poisson algebras at the same time. Moreover, by starting with deformations of associative algebras and then passing to deformations of Poisson algebras, the concept of deformation might become more clear. After having introduced deformations, we give a brief historical introduction to the recent development concerning infinitesimal deformations of Poisson structures that can be found using the oriented and unoriented graph complexes. We shall conclude this chapter with an overview of the content of the bachelor thesis.

## 1. Introduction to deformation theory

A deformation of a mathematical object can be seen as a family of the same type of objects. This family should depend on its parameters "continuously". Deformation theory aims to describe a certain type of objects in such continuous families, sometimes in order to find more of them, sometimes in order to relate the objects to each other. Actually some very well known mathematical objects are constructed by using a deformation: in the definition of the Riemann integral of a function one takes a limit over a continuous family of step-functions, i.e. a deformed step-function, that approximate the function. (This particular example can be rephrased so that it concerns a universal deformation, since Riemann-integrals apply to all Riemann-integrable functions.)

The first mathematician using this idea of continuous families of objects was probably B. Riemann. For Riemann surfaces with genus $g>0$, he described in 1857 in [25] the complex, continuous (almost everywhere analytic) family of isomorphism classes. We call this an example of analytic deformation theory since the object that was deformed here was a complex manifold with an analytic structure. One could approach both, analytic and algebraic deformation problems with infinitesimal methods. Again, this was first done for the (analytic) deformation of Riemann surfaces, by O . Teichmüller in [26] (1944). He was killed in 1943, when fighting for the Nazis in World War II, before it was published, and the work still lacked precision. In 1957, A. Froehlicher and A. Nijenhuis defined, in [13] (1958), infinitesimal deformations with much more precision
in a more general context, namely for arbitrary complex manifolds. K. Kodaira and D.C. Spencer developed this even further in [21].

Short after that, analytic deformation theory with its infinitesimal methods was extended to objects other than complex manifolds, algebras in particular. This extension was introduced first for associative algebras by M. Gerstenhaber in 1963 (see [14]) and for Lie algebras by A. Nijenhuis and R.W. Richardson in 1966 (see [24]).

## 2. Deformations of associative algebras

Intuitively, an infinitesimal deformation of an algebra $A$ consists of a continuous family of algebras $\tilde{A}$, where for each non-zero value of the real parameter $t$, the initial multiplication operation of the algebra $A$ has slightly changed. (At $t=0$ the multiplication stays untouched, therefore $A_{0}=A$.) In order to define a deformation of an algebra we proceed with the definition and remark below.
Definition 1 ([5]). Let $k$ be a field. A $k$-algebra (or just algebra if the field is not specified) $A$ is a $k$-vector with a bilinear multiplication $m: A \times A \rightarrow A$ defined on it. This multiplication is required to be distributive, i.e. for $\alpha, \beta, \gamma \in A$,

$$
m(\alpha, \beta+\gamma)=m(\alpha, \beta)+m(\alpha, \gamma) \quad \text { and } \quad m(\alpha+\beta, \gamma)=m(\alpha, \gamma)+m(\beta, \gamma)
$$

We call $A$ an associative algebra if additionally the following holds for all $\alpha, \beta, \gamma \in A$

$$
m(\alpha, m(\beta, \gamma))=m(m(\alpha, \beta), \gamma) .
$$

Example 1. Let $M$ be a smooth manifold. An example of an associative algebra is the $\mathbb{R}$-vector space $C^{\infty}(M)$ of smooth functions over $M$, with pointwise multiplication.

We define a formal deformation of an algebra on page 3. Before that, to motivate the formal definition, we give an (perhaps more intuitive) definition, Definition 2, for the concept of deformation: a "deformation family" of an algebra. In [17] it is pointed out that for any approach to deformations of mathematical objects holds that the defining properties of the deformed object are preserved under a deformation. Both Definitions 2 and 5 illustrate this. Namely, a deformation family of an associative algebra is a family of algebras wherein each algebra is associative. A formal deformation of an associative algebra is a new (and larger) associative algebra.
Definition 2 (A similar concept is described in [17]). Let $A$ be an normed associative $\mathbb{R}$-algebra with addition $a: A \times A \rightarrow A$ and multiplication $m_{0}: A \times A \rightarrow A$. Let $I \subset \mathbb{R}$ be an open interval containing zero. Then a deformation family of $A$ is a set $\mathcal{A}:=$ $\left\{A_{s}: s \in \mathcal{I}\right\}$, where by definition $A_{s}$ is an associative $\mathbb{R}$-algebra for each $s \in \mathcal{I}$ with the following properties:
(1) $A_{s}=A$ as sets.
(2) Addition in $A_{s}$ is given by $a$, the addition in $A$.
(3) Multiplication in $A_{s}$ is given by a $\mathbb{R}$-bilinear map $\hat{m}_{s}$ of the form

$$
\hat{m}_{s}=\sum_{i=0}^{\infty} s^{i} m_{i}=m_{0}+s m_{1}+s^{2} m_{2}+\cdots,
$$

where $\left\{m_{n}\right\}_{n \in \mathbb{N}_{\geq 0}}$ is a sequence of $\mathbb{R}$-bilinear maps $A \times A \rightarrow A$. Here the first element in the sequences is given by $m_{0}$, the multiplication of $A$.
Remark 1. For every deformation family $\mathcal{A}$ of associative algebra $A$ we have $A_{0}=A$ as algebras at $s=0$.
Remark 2. Since in Definition 2 we require $A_{s}$ to be an algebra for all $s \in \mathcal{I}$, the space $A_{s}$ is, by assumption, closed under the respective multiplication $\hat{m}_{s}$, i.e. the infinite sum $\hat{m}_{s}(\alpha, \beta)$ must converge under the given norm for $A$ for all $\alpha, \beta \in A$, at every $s \in I$.

To approach the notion of formal deformations of algebras (independent of possible norms and convergence) we need the definitions of a formal power series and a power series ring.

Definition 3. Let $R$ be a ring. A formal power series in formal parameter $t$ with coefficients in $R$ is defined by

$$
{ }^{t} \alpha=\sum_{n=0}^{\infty} \alpha_{n} t^{n},
$$

where $\left\{\alpha_{n}\right\}_{n \in \mathbb{N}_{2} 0}$ forms a sequence with elements $\alpha_{n} \in R$. The set of formal power series in variable $t$ with coefficients in $R$ is denoted by $R[[t]]$.

Remark 3. Let $R$ be a ring with addition and multiplication given by maps $a: A \times A \rightarrow A$ and $m: A \times A \rightarrow A$, respectively. The set $R[[t]]$ of formal power series in variable $t$ with coefficients in the ring $R$ forms a ring over $R$ with addition $\bar{a}$ and multiplication $\bar{m}$ induced by $R$ as follows: Let ${ }^{t} \alpha$ and ${ }^{t} \beta$ be formal power series in formal parameter $t$ with coefficients sequences $\left\{\alpha_{n}\right\}_{n \in \mathbb{N}_{2} 0}$ and $\left\{\beta_{n}\right\}_{n \in \mathbb{N}_{2} 0}$, respectively, both with elements $\alpha_{n}, \beta_{n} \in R$. The addition $\tilde{A}: R[[t]] \times R[[t]] \rightarrow R[[t]]$, extended from $a$, is given by

$$
\bar{a}:\left({ }^{t} \alpha,{ }^{t} \beta\right) \mapsto \sum_{i=0}^{\infty} t^{i} a\left(\alpha_{i}, \beta_{i}\right)
$$

and the multiplication $m_{t}: R[[t]] \times R[[t]] \rightarrow R[[t]]$, extended from $m$, is given by

$$
\bar{m}:\left({ }^{t} \alpha,{ }^{t} \beta\right) \mapsto \sum_{j=0}^{\infty} \sum_{\substack{k, \geq 0 \\ k+l=j}} t^{j} m\left(\alpha_{k}, \beta_{l}\right),
$$

where terms are collected using the previously defined addition $\bar{a}$. Note that $R$ forms a subring of $R[[t]]$, by construction.

Definition 4 ([5]). The ring $R[[t]]$, defined in Remark 3 is called the power series ring in formal parameter $t$ over $R$.
Definition 5 ([15]). Let $k$ be a field, $k[[t]]$ its power series ring, and $k((t))$ its field of fractions. Let $A$ be an associative $k$-algebra ${ }^{1}$ and let maps $a: A \times A \rightarrow A$ and

[^0]$m_{0}: A \times A \rightarrow A$ denote its addition and multiplication operation, respectively. Then a formal deformation of $A$ is, by definition, an associative $k((t))$-algebra, denoted by $\tilde{A}$ with the following properties:
(1) $\tilde{A}=A \otimes_{k} k((t))$ as sets.
(2) Addition in $\tilde{A}$ is given by $\bar{a}: \tilde{A} \times \tilde{A} \rightarrow \tilde{A}$, the extension of $a$ in $A$, like $a$ in $R$ was extended to $\bar{a}$ in Remark 3.
(3) The associative multiplication in $\tilde{A}$ is given by a $k((t))$-bilinear map $\tilde{m}: \tilde{A} \times \tilde{A} \rightarrow$ $\tilde{A}$ of the form
$$
\tilde{m}=\sum_{i=0}^{\infty} t^{i} \bar{m}_{i}=\bar{m}_{0}+t \bar{m}_{1}+t^{2} \bar{m}_{2}+\cdots,
$$
where each $\bar{m}_{i}$ is the $k((t))$-bilinear map $\tilde{A} \times \tilde{A} \rightarrow \tilde{A}$ which is obtained by extending some $k$-bilinear map $m_{i}: A \times A \rightarrow A$ like $m$ was extended to $\bar{m}$ in Remark 3. These $k$-bilinear and $k((t))$-bilinear maps form two respective sequences $\left\{m_{n}\right\}_{n \in \mathbb{N}, 0}$ and $\left\{\bar{m}_{n}\right\}_{n \in \mathbb{N}, 0}$. Here $\bar{m}_{0}$ is the extension of $m_{0}$, the multiplication of $A$. More explicitly,
$$
\tilde{m}:\left({ }^{t} \alpha,{ }^{t} \beta\right) \mapsto \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{\substack{k, \geq 0 \\ k+l=j}} t^{i+j} m_{i}\left(\alpha_{k}, \beta_{l}\right),
$$
for ${ }^{t} \alpha,{ }^{t} \beta \in A[[t]]$ with coefficients sequences $\left\{\alpha_{n}\right\}_{n \in \mathbb{N}_{2} 0}$ and $\left\{\beta_{n}\right\}_{n \in \mathbb{N}_{2} 0}$, respectively.

Note that $\tilde{A}$ is indeed closed under multiplication $\tilde{m}$ (in the sense that the product of two formal power series in $\tilde{A}$ is again a formal power series in $\tilde{A}$ ).

Remark 4. Given an associative algebra $A$, the sequence $\left\{m_{n}\right\}_{n \in \mathbb{N}_{\geq 0}}$ of $k$-bilinear maps $m_{i}: A \times A \rightarrow A$ appearing in Definitions 2 and 5 defines the respective objects, family $\mathcal{A}$ and algebra $\tilde{A}$, uniquely (since each $m_{n}$ induces a unique extension to $\bar{m}$, like $m$ is extended to $\bar{m}$ in Remark 3).

Now follows an immediate consequence of Remarks 3 and 4.
Lemma 1. Let $A$ be an associative algebra and let $\left\{m_{n}\right\}_{n \in \mathbb{N} \geq 0}$ be a sequence of $k$-bilinear maps $A \times A \rightarrow A$ with $m_{0}$ the multiplication of $A$. If $\left\{m_{n}\right\}_{n \in \mathbb{N}, 0}$ defines a deformation family $\mathcal{A}$ of $A$, then it defines a formal deformation $\tilde{A}$ of $A$ as well.

The converse does not always hold. If $\left\{m_{n}\right\}_{n \in \mathbb{N} \geq 0}$ defines a formal deformation of $A$, the convergence requirement of Definition 2 pointed out in Remark 2 is not necessarily satisfied.

Definition 6 ([15]). Let $A$ be an associative algebra with addition and multiplication operation as given in Definition 5. Then an infinitesimal deformation of $A$ is a $k$ bilinear map $M_{1}: A \times A \rightarrow A$ satisfying

$$
m_{0}\left(M_{1}(\alpha, \beta), \gamma\right)+M_{1}\left(m_{0}(\alpha, \beta), \gamma\right)=m_{0}\left(\alpha, M_{1}(\beta, \gamma)\right)+M_{1}\left(\alpha, m_{0}(\beta, \gamma)\right)
$$

for all $\alpha, \beta, \gamma \in A$.
Remark 5. In case map $M_{1}$ succeeds $m_{0}$ as an element of a sequence of maps that (see Remark 4) defines $\tilde{A}$, a formal deformation of associative algebra $A$ - as in Definition 5 and Remark 4 - we call $M_{1}$ the infinitesimal deformation (or the differential) of the formal deformation $\tilde{A}$ ([15]). In that case we call multiplication given by $m_{0}+M_{1}$ integrable: the pair of maps $m_{0}$ and $M_{1}$ can be completed with a sequence of $k$-bilinear maps, such that the new multiplication that they define, like the sequence $\left\{m_{n}\right\}_{n \in \mathbb{N} \geq 0}$ does in Definitions 2 and 5, is again associative.

In the literature, the word "deformation" can refer to a new mathematical object, or it can refer to a new structure on a mathematical object (like a multiplication in the case of an algebra). Indeed, a formal deformation of an associative algebra is an associative algebra, but an infinitesimal deformation of an associative algebra is a map. The reason why the terminology is used in this way follows from Remark 4: Formal deformations are uniquely defined by a sequence of maps that induce the multiplicative operation of the new algebra. Hypothetically, one could have introduced the notion of a "deformation of the multiplication operation of an associative algebra" instead of a deformation of the associative algebra itself.

Definition 7 (A similar concept is defined in [15]). Let $A$ be an associative algebra. Then its multiplication operation $m_{0}$ satisfies

$$
\operatorname{Assoc}_{m_{0}}(\alpha, \beta, \gamma):=m_{0}\left(m_{0}(\alpha, \beta), \gamma\right)-m_{0}\left(\alpha, m_{0}(\beta, \gamma)=0,\right.
$$

for all $\alpha, \beta, \gamma \in A$. The left hand side of the equation is the associator of $m_{0}$ at $\alpha, \beta, \gamma \in$ $A$, denoted by $\mathrm{Assoc}_{m_{0}}(\alpha, \beta, \gamma)$.

Note that the associator of a map vanishes on $A \times A \times A$ if and only if the map is associative.

Definition 8 (A similar concept is defined in [15]). Consider a sequence $\left\{M_{n}\right\}_{n \in \mathbb{N}_{20}}$ of $k$-bilinear maps $A \times A \rightarrow A$, with first element $M_{0}=m_{0}$, the associative multiplication operation of the associative algebra $A$. Let us define the following multiplication operation on the set $A[[t]]$,

$$
\tilde{M}:=\sum_{i=0}^{\infty} t^{i} \bar{M}_{i}=\bar{m}_{0}+t \bar{M}_{1}+t^{2} \bar{M}_{2}+\cdots
$$

where each $\bar{M}_{i}$ is the $k[[t]]$-bilinear map $A[[t]] \times A[[t]] \rightarrow A[[t]]$ which is obtained by extending the $k$-bilinear map $M_{i}: A \times A \rightarrow A$ (like $m$ was extended to $\bar{m}$ in Remark 3). Assume $\tilde{M}$ is associative. Then for all ${ }^{t} \alpha,{ }^{t} \beta,{ }^{t} \gamma \in A[[t]]$ it satisfies the equation

$$
\left.\operatorname{Assoc}_{\tilde{M}}{ }^{t} \alpha,{ }^{t} \beta,{ }^{t} \gamma\right)=\tilde{M}\left(\tilde{M}\left({ }^{t} \alpha,{ }^{t} \beta\right),{ }^{t} \gamma\right)-\tilde{M}\left({ }^{t} \alpha, \tilde{M}\left({ }^{t} \beta,{ }^{t} \gamma\right)\right)=0,
$$

a formal power series equated to zero. A formal power series is equal to zero if and only if the coefficients of all powers of $t$ vanish. At $t^{0}$ the coefficient is exactly the
associator of $m_{0}$, which vanishes since $A$ is associative. At $t^{1}$ the coefficient is equal to the following expression

$$
m_{0}\left(M_{1}\left(\alpha_{0}, \beta_{0}\right), \gamma_{0}\right)+M_{1}\left(m_{0}\left(\alpha_{0}, \beta_{0}\right), \gamma_{0}\right)-m_{0}\left(\alpha_{0}, M_{1}\left(\beta_{0}, \gamma_{0}\right)\right)-M_{1}\left(\alpha_{0}, m_{0}\left(\beta_{0}, \gamma_{0}\right)\right)
$$

If this expression is equal to zero for all $\alpha_{0}, \beta_{0}, \gamma_{0} \in A \subset A[[t]]$, we say that the infinitesimal condition for $\tilde{M}$ is satisfied. If this is the case, than the associator of $\tilde{M}$ vanishes up to $\bar{o}(t)$, i. e. $\operatorname{Assoc}_{\bar{m}_{0}+t \bar{M}_{1}+\bar{o}(t)}\left(\alpha_{0}, \beta_{0}, \gamma_{0}\right)=\bar{o}(t)$ for all ${ }^{t} \alpha,{ }^{t} \beta,{ }^{t} \gamma \in A[[t]]$.

The infinitesimal condition is a necessary condition for $\tilde{M}$ to be associative (for the given associative multiplication operation $m_{0}$ of the given associative algebra $A$ ). Note that there are infinitely many such conditions for $\tilde{M}$ to be associative, since all coefficients of powers of $t$ in $\operatorname{Assoc}_{\tilde{M}}$ are required to vanish in that case. ${ }^{2}$ Seen in the perspective of deformation families, the derivative with respect to real parameter $t$ of a deformation family corresponds to the infinitesimal deformation or differential of an algebra.

## 3. Deformations of Poisson algebras

It will turn out that a Poisson algebra is a specific kind of Lie algebra. Lie algebras might be a bit more familiar to the reader, that is why we introduce them first and state this remark.

Definition 9 ([6]). Let $k$ be a field. A Lie algebra is a $k$-algebra $L$ with a bracket operation $[\cdot, \cdot]: L \times L \rightarrow L$ defined on it, called the Lie bracket, which satisfies the following properties:
(1) $[f, g]$ is $k$-bilinear with respect to both arguments.
(2) $[f, g]=-[g, f]$ (skew-symmetry)
(3) $[f,[g, h]]+[g,[h, f]]+[h,[f, g]]=0$ (Jacobi identity)
for $f, g, h \in L$.
Example 2. Let $k$ be a field and let $n \in \mathbb{N}$. Then the $k$-algebra $\mathfrak{g l}_{n}(k)$ of $n \times n$ matrices with elements in $k$ forms a Lie algebra with the commutator bracket $[A, B]:=A B-B A$ for $A, B \in \mathfrak{g l}_{n}(k)$.

Definition 10 ([23]). A smooth Poisson manifold is a smooth manifold $M$ equipped with a bracket operation $\{\cdot, \cdot\}: C^{\infty}(M) \times C^{\infty}(M) \rightarrow C^{\infty}(M)$ defined on its function space, which satisfies the following properties:
(1) $\{f, g\}$ is $\mathbb{R}$-bilinear with respect to both arguments.
(2) $\{f, g\}=-\{g, f\}$ (skew-symmetry)
(3) $\{f g, h\}=f\{g, h\}+\{f, h\} g$ (Leibniz rule)
(4) $\{f,\{g, h\}\}+\{g,\{h, f\}\}+\{h,\{f, g\}\}=0$ (Jacobi identity)

[^1]for any $f, g, h \in C^{\infty}(M)$. The bracket operation is called a Poisson bracket of manifold $M$ (also called Poisson structure). The commutative $\mathbb{R}$-algebra $C^{\infty}(M)$, endowed with a Poisson bracket $\{\cdot, \cdot\}: C^{\infty}(M) \times C^{\infty}(M) \rightarrow C^{\infty}(M)$, is called the Poisson algebra of smooth manifold $M$.

Remark 6. Let $M$ be a smooth manifold. Then $C^{\infty}(M)$ is a commutative $\mathbb{R}$-algebra.
Remark 7. If $C^{\infty}(M)$ is additionally endowed with a structure of a Poisson algebra, with Poisson bracket $\{\cdot, \cdot\}: C^{\infty}(M) \times C^{\infty}(M) \rightarrow C^{\infty}(M)$, then $C^{\infty}(M)$ is a Lie algebra, endowed with bracket $\{\cdot, \cdot\}$, since this Poisson bracket is a Lie bracket as well.

Poisson structures play an important role in physics. They are used to to describe classical and quantum mechanical systems (see [22] and [2]). Deformations of Poisson structures could play a role in describing the bridge between the classical mechanical systems and the quantum mechanical systems (see [12]). That is why it is interesting to deform them.

A family of examples of Poisson structures can be found in Example 4, in [4]. ${ }^{3}$ More examples can be found in [2] and [22].

Remark 8. The structure $\{\cdot, \cdot\}$ is a derivation in each argument. Hence, to calculate the bracket $\{f, g\}$ for $f, g \in C^{\infty}(M)$, it suffices to know the values of the bracket at any local coordinate functions $x^{i}$ in a chart containing that point. Indeed, $\{f, g\}(\mathbf{x})=$ $\frac{\partial f}{\partial x^{i}}(\mathbf{x})\left\{x^{i}, x^{j}\right\}\left|\left.\right|_{\mathbf{x}} \frac{\partial g}{\partial x^{j}}(\mathbf{x})\right.$, see [2] and [22]. We denote by $\mathcal{P}=\left(P^{i j}\right)$ the skew symmetric matrix with entries $P^{i j}:=\left\{x^{i}, x^{j}\right\}(\mathbf{x})$ of coefficients of a given Poisson bracket $\{\cdot, \cdot\}$, expressed using some system of local coordinates.

Remark 9. In this text, a Poisson algebra is always the function space of some affine Poisson manifold, which we define below. We restict ourselves to affine Poisson manifolds because we search for deformations of Poisson algebras via methods (explained on page 8) described in the last part of [18]. These methods only apply to affine Poisson manifolds. ${ }^{4}$

Definition 11. An affine transformation for vector spaces $V$ and $W$ over $\mathbb{R}$ is a map $\phi: V \rightarrow W$ such that, for every weighted sum $\sum_{i \in I} \lambda_{i} v_{i}$ of vectors $v_{i}$ in $V$ and scalars $\lambda_{i}$ in $\mathbb{R}$ with $\sum_{i \in I} \lambda_{i}=1$ we have

$$
\phi\left(\sum_{i \in I} \lambda_{i} v_{i}\right)=\sum_{i \in I} \lambda_{i} \phi\left(v_{i}\right)
$$

An affine manifold is a real smooth manifold equipped with an atlas such that all transition functions between charts are affine transformations.

[^2]In exactly the same way how it has been done for associative algebras in Definitions 2, 5 one introduces the notions of deformation families and formal deformations of Poisson algebras (of affine manifolds). Specifically, for a Poisson algebra $C^{\infty}(M)$ of a given affine manifold $M$, a deformation family of $C^{\infty}(M)$ is a family $\left\{C^{\infty}(M)_{\epsilon}: \epsilon \in \mathcal{I}\right\}$ (where $\mathcal{I}$ is an interval around zero in $\mathcal{R}$ ) of Poisson algebras with identically the same addition, + , and multiplication, $\cdot$, as defined on $C^{\infty}(M)$, but where the bracket operation $\{\cdot, \cdot\}=\mathcal{P}_{0}$ is changed into a new bracket operation, namely a sum $\mathcal{P}=$ $\sum_{m=0}^{\infty} \epsilon \mathcal{P}_{m}$ of bracket operations depending on $\epsilon$, such that the new bracket operation $\mathcal{P}$ (the sum) satisfies the property to be a Poisson bracket. In the same spirit formal deformations of Poisson algebras are introduced (see [15] and [16]). Analogous to the formal deformation of an associative algebra, it is a Poisson algebra that is as well uniquely defined by its new bracket operation $\mathcal{P}$. We give the definition of an infinitesimal deformation of a Poisson algebra explicitely.

Definition 12 ([15]). Let the function space $C^{\infty}(M)$ be a Poisson algebra of an affine manifold $M$ with Poisson bracket $\mathcal{P}$. An infinitesimal deformation of the Poisson algebra $C^{\infty}(M)$ is a map $Q: C^{\infty}(M) \times C^{\infty}(M) \rightarrow C^{\infty}(M)$ satisfying
(1) $Q$ is $\mathbb{R}$-bilinear with respect to both arguments;
(2) $Q(f, g)=-Q(g, f)$ (skew-symmetry);
(3) $Q(f g, h)=f Q(g, h)+Q(f, h) g$ (Leibniz rule);
(4) $\llbracket P, Q \rrbracket(f, g, h)=0$ (compatibility w.r.t. the Schouten bracket, see [4]), that is

$$
\begin{aligned}
& \mathcal{P}(f, \mathcal{Q}(g, h))+\mathcal{P}(g, Q(h, f))+\mathcal{P}(h, Q(f, g)) \\
&+ Q(f, \mathcal{P}(g, h))+\mathcal{Q}(f, \mathcal{P}(g, h))+\mathcal{Q}(f, \mathcal{P}(g, h))=0
\end{aligned}
$$

for all $f, g, h \in C^{\infty}(M)$.
Remark 10. The bracket operation defined by $\mathcal{P}+\epsilon Q+\bar{o}(\epsilon)$ satisfies the properties $1-$ 3 from Definition 10. Like the associativivity of the multiplication in the infinitesimal deformation of an associative algebra, this bracket operation satisfies condition 4 (which is the Jacobi identity in the form $\llbracket \mathcal{P}, \mathcal{P} \rrbracket=0$ ) only infinitesimally, i. e. $\llbracket \mathcal{P}+\epsilon Q+\bar{o}(\epsilon), \mathcal{P}+$ $\epsilon Q+\bar{o}(\epsilon) \rrbracket=\bar{o}(\epsilon)$.

## 4. Kontsevich' graph complexes

Definition 13 ([5]). Let $k$ be a field and $V$ be a $k$-vector space endowed with a grading Grad : $V \rightarrow \mathbb{Z}$ such that $V=\oplus_{n \in \mathbb{Z}} V_{n}$. Let d: $V \rightarrow V$ be a linear map. We say that $d$ is a differential on $V$ if for every $n \in \mathbb{Z}$ we have that $\mathrm{d}\left(V_{n}\right) \subset V_{n+1}$ and if it satisfies the following property

$$
\mathrm{d} \circ \mathrm{~d}=0 .
$$

The vector space endowed with differential d is a differential complex and an element in the kernel of $d$ is a cocycle in $V$.

In 1993 and 1994, Kontsevich defined in [19] and [20] several differential complexes, in particular the unoriented graph complex and the oriented graph complex. Their
underlying vector spaces are each vector spaces of formal sums of specific type of graphs. That is why we also refer to them as graph complexes. In [18], in 1996 he discovered a relation between the cocycles in those two graph complexes and infinitesimal deformations of Poisson algebras of affine manifolds. He claimed the existence of the orientation mapping Or from the unoriented graph complex to the oriented graph complex, that would allow one to obtain a cocycle in the oriented graph complex from a cocycle in the unoriented graph complex. Moreover, he claimed that (under certain conditions on the graphs) one obtains universal infinitesimal deformations of Poisson algebras of affine manifolds via the map that, in this text, we call the translation mapping. Kontsevich gave an example of a cocycle in the unoriented graph complex: namely, the tetrahedron $\gamma_{3}$ (see [18] and [4]). By using that example, Kontsevich illustrated how one can pass to the corresponding deformation via the orientation and language mapping. The tetrahedron cocycle $\operatorname{Or}\left(\gamma_{3}\right)$ in the oriented graph complex and the deformation are given explicitely in [4]. Kontsevich and T. Wilwacher found another cocycle in the unoriented graph complex: the pentagon-wheel cocycle $\gamma_{5}$. T. Wilwacher has shown in [27] that there exist infinitely many nontrivial cocycles in the unoriented graph complex. Namely, he found an infinite sequence of cocycles all of a specific form. For each $l \in \mathbb{N}$ there exists such cocycle that contains a $(2 l+1)$-wheel graph. We call the cocycles in this sequence $(2 l+1)$-wheel cocycles.

## 5. Content of the thesis

This thesis is organized in four chapters. Chapter 1 consists of the paper [9] about the unoriented graph complex and the cocycles therein. In Chapter 2 we give rigorous proofs of several statements that have already been used in [9] (as well as in the literature it is based on). So far those claims were taken for granted. Chapter 3 consists of the paper [10] where we present many algorithms that are used in the search for cocycles in the oriented graph complex. Chapter 4 consists of the paper [8] where the cocycle $\operatorname{Or}\left(\gamma_{5}\right)$ in the oriented graph complex is obtained and the respective deformation is given explicitly.

The definitions of the unoriented and oriented graph complexes with their differentials are recalled in [10] and [9]. In Chapter 2, written by me, we first show that the differential $d$ of the unoriented graph complex satisfies the defining property of a differential: $\mathrm{d} \circ \mathrm{d}=0$. Secondly we show that the differential applied to a zero graph is zero. This result is a necessary condition for the differential $d$ to be a well defined map on the quotient space of formal sums of graphs with an ordered set of edges modulo the equivalence relation induced by the wedge product. Also the definition of the Lie bracket on the unoriented graph complex is recalled in [9]. We prove that the Lie bracket applied to a zero graph is zero. This result is a necessary condition for the Lie bracket to be a well defined map on the quotient space of of formal sums of graphs with an ordered set of edges modulo the equivalence relation induced by the wedge product. (The proofs of the latter two statements will be combined.)

The three publications, [10], [9] and [8], co-authored with R. Buring and A. V. Kiselev, are part of my thesis as well. Here I give an overview of the content of those papers and what my contribution to them was. In [8] the corresponding cocycle $\operatorname{Or}\left(\gamma_{5}\right)$ in the oriented graph complex and the deformation are given in explicit form. The heptagon-wheel cocycle $\gamma_{7}$ (succeding the tetrahedron cocycle $\gamma_{3}$ and the pentagon-wheel cocycle $\gamma_{5}$ in the sequence) is given explicitely in [9]. We performed an extensive search for cocycles in the oriented graph complex, i.e. independently from the unoriented graph complex and the orientation mapping. Here computer assisted techniques from [7] and [10] have been used. So far, it is confirmed that - for oriented graphs on $n \leq 4$ vertices, possibly including eyes, but excluding tadpoles - there do not exist cocycles other then the ones that were known. ${ }^{5}$

Jointly with A. V. Kiselev I designed efficient algorithms to generate the set of all bi-vector graphs in the oriented graph complex on $n$ vertices using subsets of the set of graphs on $n-2$ vertices in the oriented graph complex. I used that algorithm and the software written by R. Buring to establish that in the oriented graph complex there are no cocycles other than those that were already known for graphs on $n \leq 4$ internal vertices.

Jointly with R. Buring and A. V. Kiselev I designed an iterative algorithm that generates the Leibniz graphs that are used to factorize via the Jacobi identity. (This was a modification of the non-iterative algorithm that was used in [4].) These algorithms became part of [10]. The iterative algorithm was used to obtain the pentagon wheel cocycle in explicit form. This cocycle is presented in [8].

Jointly with R. Buring and A. V. Kiselev I designed an efficient algorithm that obtains all the oriented bi-vector versions of an unoriented graph which are admissible in the oriented graph complex. This is as well presented in [10].

Jointly with A. V. Kiselev I formulated and proved the "Handshake lemma" and we jointly verified the cocycle condition explicitly for the tetrahedron $\gamma_{3}$ and the pentagon wheel cocycle $\gamma_{5}$ in the unoriented graph complex and we jointly wrote the chapters concerning this in [9].

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# THE HEPTAGON-WHEEL COCYCLE IN THE KONTSEVICH GRAPH COMPLEX 

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#### Abstract

The real vector space of non-oriented graphs is known to carry a differential graded Lie algebra structure. Cocycles in the Kontsevich graph complex, expressed using formal sums of graphs on $n$ vertices and $2 n-2$ edges, induce - under the orientation mapping - infinitesimal symmetries of classical Poisson structures on arbitrary finite-dimensional affine real manifolds. Willwacher has stated the existence of a nontrivial cocycle that contains the $(2 \ell+1)$-wheel graph with a nonzero coefficient at every $\ell \in \mathbb{N}$. We present detailed calculations of the differential of graphs; for the tetrahedron and pentagon-wheel cocycles, consisting at $\ell=1$ and $\ell=2$ of one and two graphs respectively, the cocycle condition $d(\gamma)=0$ is verified by hand. For the next, heptagon-wheel cocycle (known to exist at $\ell=3$ ), we provide an explicit representative: it consists of 46 graphs on 8 vertices and 14 edges.


Introduction. The structure of differential graded Lie algebra on the space of nonoriented graphs, as well as the cohomology groups of the graph complex, were introduced by Kontsevich in the context of mirror symmetry [10, 11]. It can be shown that by orienting a graph cocycle on $n$ vertices and $2 n-2$ edges (and by adding to every graph in that cocycle two new edges going to two sink vertices) in all such ways that each of the $n$ old vertices is a tail of exactly two arrows, and by placing a copy of a given Poisson bracket $\mathcal{P}$ in every such vertex, one obtains an infinitesimal symmetry of the space of Poisson structures. This construction is universal with respect to all finite-dimensional affine real manifolds (see [12] and [2]). ${ }^{1}$ Until recently two such differential-polynomial symmetry flows were known (of nonlinearity degrees 4 and 6 respectively). Namely, the tetrahedral graph flow $\dot{\mathcal{P}}=\mathcal{Q}_{1: \frac{6}{2}}(\mathcal{P})$ was proposed in the seminal paper [12] (see also $[2,3]$ ). Consisting of 91 oriented bi-vector graphs on $5+1=6$ vertices, the Kontsevich-Willwacher pentagon-wheel flow will presently be described in [7].

[^4]The cohomology of the graph complex in degree 0 is known to be isomorphic to the Grothendieck-Teichmüller Lie algebra $\mathfrak{g r t}$ (see [9] and [16]); under the isomorphism, the $\mathfrak{g r t}$ generators correspond to nontrivial cocycles. Using this correspondence, Willwacher gave in [16, Proposition 9.1] the existence proof for an infinite sequence of the Deligne-Drinfel'd nontrivial cocycles on $n$ vertices and $2 n-2$ edges. (Formulas which describe these cocycles in terms of the $\mathfrak{g r t}$ Lie algebra generators are given in the preprint [15].) To be specific, at each $\ell \in \mathbb{N}$ every cocycle from that sequence contains the $(2 \ell+1)$-wheel with nonzero coefficient (e.g., the tetrahedron alone making the cocycle $\gamma_{3}$ at $\ell=1$ ), and possibly other graphs on $2 \ell+2$ vertices and $4 \ell+2$ edges. For instance, at $\ell=2$ the pentagon-wheel cocycle $\gamma_{5}$ consists of two graphs, see Fig. 1 on p. 6 below.

In this paper we describe the next one, the heptagon-wheel cocycle $\gamma_{7}$ from that sequence of solutions to the equation

$$
\mathrm{d}\left(\sum_{\{\text {graphs }\}}(\text { coefficient } \in \mathbb{R}) \cdot(\text { graph with an ordering of its edge set })\right)=0 .
$$

Our representative of the cocycle $\gamma_{7}$ consists of 46 connected graphs on 8 vertices and 14 edges. (This number of nonzero coefficients can be increased by adding a coboundary.) This solution has been obtained straightforwardly, that is, by solving the graph equation $\mathrm{d}\left(\boldsymbol{\gamma}_{7}\right)=0$ directly. One could try reconstructing the cocycle $\gamma_{7}$ from a set of the $\mathfrak{g r t}$ Lie algebra generators, which are known in low degrees. Still an explicit verification that $\gamma_{7} \in \operatorname{ker} d$ would be appropriate for that way of reasoning.

In this paper we also confirm that the three cocycles known so far - namely the tetrahedron and pentagon- and heptagon-wheel solutions - span the space of nontrivial cohomology classes which are built of connected graphs on $n \leqslant 8$ vertices and $2 n-2$ edges. At $n=9$, there is a unique nontrivial cohomology class with graphs on nine vertices and sixteen edges: namely, the Lie bracket $\left[\gamma_{3}, \gamma_{5}\right]$ of the previously found cocycles. (Brown showed in [4] that the elements $\sigma_{2 \ell+1}$ in the Lie algebra $\mathfrak{g r t}$ which - under the Willwacher isomorphism - correspond to the wheel cocycles $\gamma_{2 \ell+1}$ generate a free Lie algebra; hence it was expected that the cocycle $\left[\gamma_{3}, \boldsymbol{\gamma}_{5}\right]$ is non-trivial.) To verify that the list of currently known d-cocycles is exhaustive - under all the assumptions which were made about the graphs at our disposal - at every $n \leqslant 9$ we count the dimension of the space of cocycles minus the dimension of the space of respective coboundaries. ${ }^{2}$ Our findings fully match the dimensions from [14, Table 1].

This text is structured as follows. Necessary definitions and some notation from the graph complex theory are recalled in $\S 1$. These notions are illustrated in $\S 2$ where a step-by-step calculation of the (vanishing) differentials $\mathrm{d}\left(\boldsymbol{\gamma}_{3}\right)$ and $\mathrm{d}\left(\boldsymbol{\gamma}_{5}\right)$ is explained. Our main result is Theorem 7 with the heptagon-wheel solution of the equation $\mathrm{d}\left(\gamma_{7}\right)=0$. Also in $\S 3$, in Proposition 8 we verify the count of number of cocycles modulo coboundaries which are formed by all connected graphs on $n$ vertices and $2 n-2$ edges (here $4 \leqslant n \leqslant 9$ ). The graphs which constitute $\gamma_{7}$ are drawn on pp. 13-19 in Appendix A. The code in SAGE programming language, allowing one to calculate the differential for

[^5]a given graph $\gamma$ and ordering $E(\gamma)$ on the set of its edges, is contained in Appendix B; the same code can be run to calculate the dimension of graph cohomology groups.

The main purpose of this paper is to provide a pedagogical introduction into the subject. ${ }^{3}$ Besides, the formulas of the three cocycle representatives will be helpful in the future search of an easy recipe to calculate all the wheel cocycles $\gamma_{2 \ell+1}$. (No general recipe is known yet, except for a longer reconstruction of those cohomology group elements from the generators of Lie algebra $\mathfrak{g r t}$.) Thirdly, our present knowledge of both the cocycles $\gamma_{i}$ and the respective flows $\mathcal{P}=\mathcal{Q}_{i}(\mathcal{P})$ on the spaces of Poisson structures will be important for testing and verifying explicit formulas of the orientation mapping $\operatorname{Or}$ such that $\mathcal{Q}_{i}=\mathrm{O} \overrightarrow{\mathrm{r}}\left(\gamma_{i}\right)$.

## 1. THE NON-ORIENTED GRAPH COMPLEX

We work with the real vector space generated by finite non-oriented graphs ${ }^{4}$ without multiple edges nor tadpoles and endowed with a wedge ordering of edges: by definition, an edge swap $e_{i} \wedge e_{j}=-e_{j} \wedge e_{i}$ implies the change of sign in front of the graph at hand. Topologically equal graphs are equal as vector space elements if their edge orderings E differ by an even permutation; otherwise, the graphs are opposite to each other (i.e. they differ by the factor -1 ).
Definition 1. A graph which equals minus itself - under a symmetry that induces a parity-odd permutation of edges - is called a zero graph. In particular (view $\bullet \bullet \bullet$ ), every graph possessing a symmetry which swaps an odd number of edge pairs is a zero graph.

Notation. For a given labelling of vertices in a graph, we denote by $i j$ (equivalently, by $j i$ ) the edge connecting the vertices $i$ and $j$. For instance, both 12 and 21 is the notation for the edge between the vertices 1 and 2. (No multiple edges are allowed, hence 12 is the edge. Indeed, by Definition 1 all graphs with multiple edges would be zero graphs.) We also denote by $N(v)$ the valency of a vertex $v$.

Example 1. The 4 -wheel $12 \wedge 13 \wedge 14 \wedge 15 \wedge 23 \wedge 25 \wedge 34 \wedge 45=I \wedge \cdots \wedge$ VIII or likewise, the $2 \ell$-wheel at any $\ell>1$ is a zero graph; here, the reflection symmetry is $I \rightleftarrows I I I, V \rightleftarrows V I I$, and $V I \rightleftarrows V I I I$.

Note that every term in a sum of non-oriented graphs $\gamma$ with real coefficients is fully encoded by an ordering $\mathbf{E}$ on the set of adjacency relations for its vertices $v$ (if $N(v)>0$ ). From now on, we assume $N(v) \geqslant 3$ unless stated otherwise explicitly.

Example 2. The tetrahedron (or 3 -wheel) is the full graph on four vertices and six edges (enumerated in the ascending order: $12=I, \ldots, 34=V I$ ),

$$
\gamma_{3}=12 \wedge 13 \wedge 14 \wedge 23 \wedge 24 \wedge 34=I \wedge \cdots \wedge V I=
$$



This graph is nonzero. (The axis vertex is labelled 4 in this figure.)

[^6]Example 3. The linear combination $\boldsymbol{\gamma}_{5}$ of two 6 -vertex 10 -edge graphs, namely, of the pentagon wheel and triangular prism with one extra diagonal (here, $12=I$ and so on),

$$
\begin{aligned}
\gamma_{5}=12 \wedge 23 \wedge 34 \wedge 45 \wedge 51 \wedge 16 & \wedge 26 \\
\wedge & 36 \wedge 46 \wedge 56 \\
& +\frac{5}{2} \cdot 12 \wedge 23 \wedge 34 \wedge 41 \wedge 45 \wedge 15 \wedge 56 \wedge 36 \wedge 26 \wedge 13
\end{aligned}
$$

is drawn in Fig. 1 on p. 6 below (cf. [1]).
Let $\gamma_{1}$ and $\gamma_{2}$ be connected non-oriented graphs. The definition of insertion $\gamma_{1} \circ_{i} \gamma_{2}$ of the entire graph $\gamma_{1}$ into vertices of $\gamma_{2}$ and the construction of Lie bracket $[, \cdot$,$] of graphs$ and differential d in the non-oriented graph complex, referring to a sign convention, are as follows (cf. [12] and $[8,14,16])$; these definitions apply to sums of graphs by linearity.
Definition 2. The insertion $\gamma_{1} \circ_{i} \gamma_{2}$ of an $n_{1}$-vertex graph $\gamma_{1}$ with ordered set of edges $\mathbf{E}\left(\gamma_{1}\right)$ into a graph $\gamma_{2}$ with $\# \mathbf{E}\left(\gamma_{2}\right)$ edges on $n_{2}$ vertices is a sum of graphs on $n_{1}+n_{2}-1$ vertices and \#E $\left(\gamma_{1}\right)+\# \boldsymbol{E}\left(\gamma_{2}\right)$ edges. Topologically, the sum $\gamma_{1} \circ_{i} \gamma_{2}=\sum\left(\gamma_{1} \rightarrow v\right.$ in $\left.\gamma_{2}\right)$ consists of all the graphs in which a vertex $v$ from $\gamma_{2}$ is replaced by the entire graph $\gamma_{1}$ and the edges touching $v$ in $\gamma_{2}$ are re-attached to the vertices of $\gamma_{1}$ in all possible ways. ${ }^{5}$ By convention, in every new term the edge ordering is $\boldsymbol{E}\left(\gamma_{1}\right) \wedge \boldsymbol{E}\left(\gamma_{2}\right)$.

To simplify sums of graphs, first eliminate the zero graphs. Now suppose that in a sum, two non-oriented graphs, say $\alpha$ and $\beta$, are isomorphic (topologically, i.e. regardless of the respective vertex labellings and edge orderings $\mathrm{E}(\alpha)$ and $\mathrm{E}(\beta)$ ). By using that isomorphism, which establishes a 1-1 correspondence between the edges, extract the sign from the equation $\mathrm{E}(\alpha)= \pm \mathrm{E}(\beta)$. If " + ", then $\alpha=\beta$; else $\alpha=-\beta$. Collecting similar terms is now elementary.

Lemma 1. The bi-linear graded skew-symmetric operation,

$$
\left[\gamma_{1}, \gamma_{2}\right]=\gamma_{1} \circ_{i} \gamma_{2}-(-)^{\# \mathrm{E}\left(\gamma_{1}\right) \cdot \# \mathbf{E}\left(\gamma_{2}\right)} \gamma_{2} \circ_{i} \gamma_{1},
$$

is a Lie bracket on the vector space $\mathfrak{G}$ of non-oriented graphs. ${ }^{6}$
Lemma 2. The operator $\mathrm{d}($ graph $)=[\bullet \bullet$, graph $]$ is a differential: $\mathrm{d}^{2}=0$.
In effect, the mapping d blows up every vertex $v$ in its argument in such a way that whenever the number of adjacent vertices $N(v) \geqslant 2$ is sufficient, each end of the inserted edge $\bullet$ is connected with the rest of the graph by at least one edge.

Theorem 3 ([12]). The real vector space $\mathfrak{G}$ of non-oriented graphs is a differential graded Lie algebra (dgLa) with Lie bracket $[\cdot, \cdot]$ and differential $\mathrm{d}=[\bullet \bullet, \cdot]$. The differential d is a graded derivation of the bracket $[\cdot, \cdot]$ (due to the Jacobi identity for this Lie algebra structure).

[^7]The graphs $\gamma_{3}$ and $\gamma_{5}$ from Examples 2 and 3 are d-cocycles (this will be shown in $\S 2$ ). Therefore, their commutator $\left[\gamma_{3}, \gamma_{5}\right]$ is also in ker d. Neither $\gamma_{3}$ nor $\gamma_{5}$ is exact, hence marking a nontrivial cohomology class in the non-oriented graph complex.
Theorem 4 ([8, Th. 5.5]). At every $\ell \in \mathbb{N}$ in the connected graph complex there is a nontrivial d-cocycle on $2 \ell+1$ vertices and $4 \ell+2$ edges. Such cocycle contains the $(2 \ell+1)$-wheel in which, by definition, the axis vertex is connected with every other vertex by a spoke so that each of those $2 \ell$ vertices is adjacent to the axis and two neighbours; the cocycle marked by the $(2 \ell+1)$-wheel graph can contain other $(2 \ell+1,4 \ell+2)$-graphs.
Example 4. For $\ell=3$ the heptagon wheel cocycle $\gamma_{7}$, which we present in this paper, consists of the heptagon-wheel graph on $(2 \cdot 3+1)+1=8$ vertices and $2(2 \cdot 3+1)=14$ edges and forty-five other graphs with equally many vertices and edges (hence of the same number of generators of their homotopy groups, or basic loops: $7=14-(8-1)$ ), and with real coefficients. All these weighted graphs are drawn in Appendix A (see pp. 13-19). The chosen - lexicographic - ordering of edges in each term is read from the encoding of every such graph (see also Table 1 on p. 10; each entry of that table is a listing $I \prec \cdots \prec X I V$ of the ordered edge set, followed by the coefficient of that graph). A verification of the cocycle condition $\mathrm{d}\left(\gamma_{7}\right)=0$ for this solution is computer-assisted; it has been performed by using the code (in SAGE programming language) which is contained in Appendix B.

## 2. Calculating the differential of graphs

Example $5\left(\mathrm{~d} \boldsymbol{\gamma}_{3}=0\right)$. The tetrahedron $\boldsymbol{\gamma}_{3}$ is the full graph on $n=4$ vertices; we are free to choose any ordering of the six edges in it, so let it be lexicographic:

$$
\mathrm{E}\left(\gamma_{3}\right)=12 \wedge 13 \wedge 14 \wedge 23 \wedge 24 \wedge 34=I \wedge I I \wedge I I I \wedge I V \wedge V \wedge V I
$$

The differential of this graph is equal to

$$
\mathrm{d}\left(\boldsymbol{\gamma}_{3}\right)=\left[\bullet \bullet, \gamma_{3}\right]=\bullet \bullet \circ_{i} \gamma_{3}-(-)^{\# E(\bullet) \cdot \# E\left(\gamma_{3}\right)} \gamma_{3} \circ_{i} \bullet \bullet=\bullet \bullet \circ_{i} \gamma_{3}-\gamma_{3} \circ_{i} \bullet \bullet
$$

since $\# \mathrm{E}\left(\boldsymbol{\gamma}_{3}\right)=6$. Note that every vertex of valency one appears twice in $\mathrm{d}\left(\boldsymbol{\gamma}_{3}\right)$ : namely in the minuend (where the edge ordering is $E \wedge I \wedge \cdots \wedge V I$ by definition of $\circ_{i}$ ) and subtrahend (where the edge ordering is $I \wedge \cdots \wedge V I \wedge E$ ). Because these edge orderings differ by a parity-even permutation, such graphs in $\bullet \circ_{i} \gamma_{3}$ and $\gamma_{3} \circ_{i} \bullet$ carry the same sign. Hence they cancel in the difference $\bullet \bullet \circ_{i} \gamma_{3}-\gamma_{3} \circ_{i} \bullet \bullet$, and no longer shall we pay any attention to the leaves, absent in the differential of any graph. It is readily seen that the twenty-four graphs $\left(24=4\right.$ vertices $\cdot\binom{3}{1} \cdot 2$ ends of $\left.\bullet \bullet\right)$ we are left with in $\mathrm{d}\left(\boldsymbol{\gamma}_{3}\right)$ are of the shape drawn here. A vertex is blown up to the new edge $E=$

whose ends are both attached to the rest of the graph along the old edges. This shape can be obtained in two ways: by blowing up $v_{i}$, so that edge' is the newly inserted edge, or by blowing up $v_{j}$, so that edge" is the newly inserted edge. By Lemma 5 below we conclude that $\mathrm{d}\left(\gamma_{3}\right)=0$.

Remark 1. Incidentally, every graph which was obtained in $\mathrm{d}\left(\gamma_{3}\right)$ itself is a zero graph. Indeed, it is symmetric with respect to a flip over the vertical line and this symmetry swaps three edge pairs (see Definition 1).
Lemma 5 (handshake). In the differential of any graph $\gamma$ such that the valency of all vertices in $\gamma$ is strictly greater than two, the graphs in which one end of the newly inserted edge $\bullet$ has valency two, all cancel.
Proof. Let $v$ be such a vertex in $\mathrm{d}(\gamma)$, i.e. the vertex $v$ is an end of the inserted edge
 respective graphs in $\mathrm{d}(\gamma)$ the rest, consisting only of old edges and vertices of valency $\geqslant 3$ from $\boldsymbol{\gamma}$, is the same. Yet the two graphs are topologically equal; furthermore, they have the same ordering of edges except for $E^{\prime}=\operatorname{Old}^{\prime \prime}$ and $\mathrm{Old}^{\prime}=E^{\prime \prime}$. Recall that by construction, the edge ordering of the first graph is $E^{\prime} \wedge \cdots \wedge \operatorname{Old}^{\prime} \wedge \cdots$, whereas for the second graph it is $E^{\prime \prime} \wedge \cdots \wedge \operatorname{Old}^{\prime \prime} \wedge \cdots$; the new edge always goes first. So effectively, two edges are swapped. Therefore,

$$
E^{\prime \prime} \wedge \cdots \wedge \operatorname{Old}^{\prime \prime} \wedge \cdots=\operatorname{Old}^{\prime} \wedge \cdots \wedge E^{\prime} \wedge \cdots=-E^{\prime} \wedge \cdots \wedge \operatorname{Old}^{\prime} \wedge \cdots
$$

Hence in every such pair in $\mathrm{d}(\gamma)$, the graphs occur with opposite signs. Moreover, the initial hypothesis $N(a) \geqslant 3$ about the valency of all vertices $a$ in the graph $\gamma$ guarantees that the cancelling pairs of graphs in $\mathrm{d}(\gamma)$ do not intersect, ${ }^{7}$ and thus all cancel.
Corollary 6 (to Lemma 5). In the differential of any graph with vertices of valency $>2$, the blow up of a vertex of valency 3 produces only the handshakes, that is the graphs which cancel out by Lemma 5 (cf. footnote 9 on p. 11 below).
Example $6\left(\mathrm{~d} \boldsymbol{\gamma}_{5}=0\right)$. The pentagon-wheel cocycle is the sum of two graphs with real coefficients which is drawn in Fig. 1. The edges in every term are ordered by


Figure 1. The Kontsevich-Willwacher pentagon-wheel cocycle $\boldsymbol{\gamma}_{5}$.
$I \wedge \cdots \wedge X$. The differential of a sum of graphs is the sum of their differentials; this is why we calculate them separately and then collect similar terms. By the above, neither contains any leaves; likewise by the handshake Lemma 5, all the graphs -in which a new vertex (of valency 2) appears as midpoint of the already existing edgecancel. By Corollary 6 it remains for us to consider the blow-ups of only the vertices of valency $\geqslant 4$ (cf. [12]). Such are the axis vertex of the pentagon wheel and vertices

[^8]labelled 1 and 3 in the other graph (the prism). By blowing up the pentagon wheel axis we shall obtain the (nonzero) 'human' and the (zero) 'monkey' graphs, presented in what follows. Likewise from the prism graph in $\gamma_{5}$ one obtains the 'human', the 'monkey', and the (zero) 'stone'. Let us now discuss this in full detail.

From the pentagon wheel we obtain $2 \cdot 5 \mathrm{Da}$ Vinci's 'human' graphs, two of which are portrayed in Fig. 2. (The factor 2 occurs from the two distinct ways to attach three versus two old edges in the wheel to the loose ends of the inserted edge $\bullet \bullet$.) We claim


Figure 2. Two of the fourteen Da Vinci's 'human' graphs occurring with weights in $\mathrm{d} \boldsymbol{\gamma}_{5}$.
that all the five 'human' graphs (i.e. standing with their feet on the edges $I, \ldots, V$ in the pentagon wheel) carry the same sign, providing the overall coefficient $+10=2 \cdot(+5)$ of such graph in the differential of the wheel. The graph (b) is topologically equal to the graph (a); indeed, the matching of their edges is $I^{(b)}=V^{(a)}, I I^{(b)}=I^{(a)}, I I I^{(b)}=$ $I I^{(a)}, I V^{(b)}=I I I^{(a)}, V^{(b)}=I V^{(a)}, V I^{(b)}=X^{(a)}, V I I^{(b)}=V I^{(a)}, V I I I^{(b)}=V I I^{(a)}$, $I X^{(b)}=V I I I^{(a)}$, and $X^{(b)}=I X^{(a)}$; also $E^{(b)}=E^{(a)}$. Hence the postulated ordering of edges in (b) is

$$
\begin{align*}
E^{(b)} \wedge I^{(b)} & \wedge \cdots \wedge X^{(b)}=E^{(a)} \wedge V^{(a)} \wedge I^{(a)} \wedge I I^{(a)} \wedge I I I^{(a)} \wedge I V^{(a)} \wedge \\
& \wedge X^{(a)} \wedge V I^{(a)} \wedge V I I^{(a)} \wedge V I I I^{(a)} \wedge I X^{(a)}=+E^{(a)} \wedge I^{(a)} \wedge \cdots \wedge X^{(a)} \tag{1}
\end{align*}
$$

which equals the edge ordering of the graph (a). For the other three graphs of this shape the equalities of wedge products are similar: a parity-even permutation of edges works out the mapping of graphs, e.g., to the graph (a) which we take as the reference.

From the pentagon wheel we also obtain $2 \cdot 5$ 'monkey' graphs, a specimen of which is shown in Fig. 3 below. Note that the 'monkey' graph is mirror-symmetric, see the


Figure 3. The 'monkey' graph: animal touches earth with its palm; this is an example of zero graph.
redrawing. This symmetry induces a permutation of edges which swaps 5 pairs, so (since 5 is odd) the 'monkey' graph is equal to zero.

Now consider the graphs obtained by blowing up vertices 1 and 3 in the prism graph. How are the four old neighbors distributed over the ends of the inserted edge? Whenever those four old neighbours are distributed in proportion $4=3+1$ (i.e. with valencies 4 and 2 for the two ends of the inserted edge), there is no contribution from the resulting graphs to d (prism) by the handshake Lemma 5 . So the graphs which could contibute are only those with the $4=2+2$ distribution (i.e. with valency 3 for either of the ends of the inserted edge). For one fixed neighbour of one of the new edge's ends there are three ways to choose the second neighbour of that vertex. This is how the 'human', 'monkey', and 'stone' graphs are presently obtained.

Let us blow up vertex 1 in the prism in these three different ways. First we make the end (now marked 1) of the inserted edge adjacent to 2 and 3 , and the other end (marked $1^{\prime}$ ) to vertices 4 and 5; the resulting graph is the 'human' graph shown in Fig. 4. From the prism graph we obtain $2 \cdot 2=4$ such 'human' graphs. One of the


Figure 4. One of the 'human' graphs obtained by blowing up according to a scenario discussed in the text - a vertex of valency four in the prism graph from $\gamma_{5}$.
factors 2 is obtained like before, namely by attaching a given set of old edges to one or the other end of the inserted edge $\bullet \bullet$, see p. 7 ; the other factor 2 comes by the rotational symmetry of the prism graph. Indeed, the prism with one diagonal is symmetric under the rotation by angle $\pi$ that transposes the vertices $1 \rightleftarrows 3,2 \rightleftarrows 4$, and $5 \rightleftarrows 6$. This is why the same 'human' graph is obtained when the vertex 3 is blown up according to a similar scenario. We claim that the permutation of edges that relates the two graphs is parity-even (similar to (1)), so they do not cancel but add up. Summarizing, the overal coefficient of the 'human' graph - produced in $\mathrm{d}($ prism ) for the edge ordering $E \wedge I \wedge \cdots \wedge X$ shown in Fig. $4-$ equals $2 \cdot 2=+4$.

The count of an overall contribution $10+\frac{5}{2} \cdot(+4) \cdot(-1$ from edge ordering $)=0$ to the differential $\mathrm{d}\left(\boldsymbol{\gamma}_{5}\right)$ of the cocycle $\gamma_{5}$ will be performed using Eq. (2); right now let us inspect the vanishing of contributions from the other two types of graphs wich are obtained by the two possible edge distribution scenarios (with respect to the ends of the new edge $\bullet$ that replaces the blown-up vertex 1 or 3 in the prism).

The 'monkey' graph is obtained by blowing up the vertex 1 (or 3) in the prism and then attaching the new edge's end, still marked 1 , to the vertices 2 and 4 . The other end, now marked $1^{\prime}$, of the new edge becomes adjacent to the vertices 3 and 5 . We keep in mind that every 'monkey' graph itself is equal to zero, hence no contribution to d(prism) occurs.


So far, the new vertex 1 has always been a fixed neighbour of vertex 2, and it was made adjacent to 3 in the 'human' and to 4 in the 'monkey' graphs, respectively. The overall set of neigbours of the new edge $1-1^{\prime}$, apart from the fixed vertex 2 , consists of vertices 3, 4 and 5 . So the third scenario to consider is the 'stone' graph in which the new vertex 1 is adjacent to $1^{\prime}, 2$, and 5 , whereas the new vertex $1^{\prime}$ neighbours 1 , 3 , and 4. This graph is mirror-symmetric under the transposition of vertices $1^{\prime} \rightleftarrows 2$

and $4 \rightleftarrows 6$, which induces the swaps in five edge pairs, namely, $I I \rightleftarrows I I I, E \rightleftarrows X$, $V I \rightleftarrows V I I I, V \rightleftarrows I X$, and $I \rightleftarrows I V$. Arguing as before, we deduce that every such 'stone' graph (obtained by a blow up of either 1 or 3 in the prism) is zero.

Our final task in the calculation of $\mathrm{d}\left(\boldsymbol{\gamma}_{5}\right)$ is collecting the coefficients of the 'human' graphs from $\mathrm{d}(5$-wheel $)$ and d (prism), coming not only with coefficients 10 and 4 respectively, but also with the respective edge orderings. To discriminate edges between the two pictures, that is originating from the pentagon wheel and the prism, let us use the superscripts $(a)$ and $(z)$, see Fig. 4. The edge matching is $E^{(z)}=I I I^{(a)}$, $I^{(z)}=I I^{(a)}, I I^{(z)}=V I I^{(a)}, I I I^{(z)}=E^{(a)}, I V^{(z)}=I X^{(a)}, V^{(z)}=X^{(a)}, V I^{(z)}=I V^{(a)}$, $V I I^{(z)}=V^{(a)}, V I I I^{(z)}=V I^{(a)}, I X^{(z)}=I^{(a)}$, and $X^{(z)}=V I I I^{(a)}$. Consequently, for the edge orderings we have

$$
\begin{align*}
& E^{(z)} \wedge I^{(z)} \wedge \cdots \wedge X^{(z)}= \\
& I I I^{(a)} \wedge I I^{(a)} \wedge V I I^{(a)} \wedge E^{(a)} \wedge I X^{(a)} \wedge X^{(a)} \wedge I V^{(a)} \wedge V^{(a)} \wedge V I^{(a)} \wedge I^{(a)} \wedge V I I I^{(a)} \\
& \quad=(-)^{23} E^{(a)} \wedge I^{(a)} \wedge \cdots \wedge X^{(a)} . \tag{2}
\end{align*}
$$

This argument shows that the graph differential of the linear combination $(+1)$ •pentagonwheel $+\frac{5}{2} \cdot$ prism, with either graph's edge ordering specified as in Example 3, vanishes. In other words, $\boldsymbol{\gamma}_{5}$ is a d-cocycle.

## 3. A representative of the heptagon-wheel cocycle $\boldsymbol{\gamma}_{7}$

It is already known that the heptagon-wheel cocycle $\gamma_{7}$, the existence of which was stated in Theorem 4, is unique modulo d-trivial terms in the respective cohomology group of connected graphs on 8 vertices and 14 edges (hence with 7 basic loops), cf. [14].

Theorem 7. The encoding of every term in a representative of the cocycle $\gamma_{7}$ is given in Table 1, the format of lines in which is the lexicographic-ordered list of fourteen edges $I \wedge \cdots \wedge X I V$ followed by the nonzero real coefficient. The forty-six graphs that form this representative of the d-cohomology class $\boldsymbol{\gamma}_{7}$ are shown on pages 13-19.

Proof scheme. This reasoning is computer-assisted. First, all connected graphs on 8 vertices and 14 edges, and without multiple edges were generated. (There are 1579 such graphs; note that arbitrary valency $N(v) \geqslant 1$ of vertices was allowed.) The coefficient

Table 1. The heptagon-wheel graph cocycle $\gamma_{7}$.

| Graph encoding | Coeff. | Graph encoding |  |  |  |  |  |  |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| 1617 | 18 | 23 | 25 | 28 | 34 | 38 | 46 | 48 |

of the heptagon wheel was set equal to +1 , all other coefficients still to be determined. After calculating the differential of the sum of all these weighted graphs (we used a program in Sage, see Appendix B), zero graphs were eliminated and the remaining terms were collected (in the same way as is explained in $\S 2$ ). In the resulting sum of weighted graphs on 9 vertices and 15 edges, we equated each coefficient to zero. We solved this linear algebraic system w.r.t. the coefficients of graphs in $\gamma_{7}$. There are $N_{\mathrm{im}}(7)=35$ free parameters in the general solution; such parameters count the coboundaries which cannot modify the cohomology class marked by any particular representative (see Table 2 on p. 11 below). Therefore the solution $\boldsymbol{\gamma}_{7}$ is unique modulo d-exact terms. All those free parameters are now set to zero and the resulting nonzero values of the graph coefficients are listed in Table 1.

Proposition 8 (see [14, Table 1]). The space of nontrivial d-cocycles which are built of connected graphs on $n$ vertices and $2 n-2$ edges at $1 \leqslant n \leqslant 9$ is spanned by the terahedron $\boldsymbol{\gamma}_{3}$, pentagon-wheel cocycle $\boldsymbol{\gamma}_{5}$ that consists of two graphs (see Example 3), heptagon-wheel cocycle $\gamma_{7}$ from Theorem 7, and the Lie bracket $\left[\gamma_{3}, \gamma_{5}\right]$. At the same time, for either $n=5$ or $n=7$, the respective graph cohomology groups are trivial. ${ }^{8}$

Verification. The dimension $N_{\text {ker }}$ of the space of cocycles built of connected graphs $\gamma$ on $n$ vertices and $2 n-2$ edges is equal to the number of free parameters in the general solution to the linear system $\mathrm{d}($ sum of such graphs $\gamma$ with undetermined coefficients) $=$ 0 . At the same time, to determine the dimension $N_{\mathrm{im}}$ of the subspace of coboundaries $\gamma=\mathrm{d}(\delta)$, i.e. of those cocycles which are the differentials of connected graphs on $n-1$ vertices and $2 n-3$ edges, we first count the number of $N_{\delta}$ of nonzero

[^9]connected graphs $\delta$ in that vertex-edge bi-grading. Then we subtract from $N_{\delta}$ the number $N_{0}$ of free parameters in the general solution to the linear algebraic system d (sums of such graphs $\delta$ with undetermined coefficients) $=0$. This subtrahend counts the number of relations between exact terms $\gamma=\mathrm{d}(\delta)$; for $n<9$ it is zero. The dimension of cohomology group $H^{*}(n)$ in bi-grading $(n, 2 n-2)$ is then $N_{\text {ker }}-N_{\text {im }}=$ $N_{\text {ker }}-\left(N_{\delta}-N_{0}\right)$.

Our present count of the overall number of connected graphs (and of the zero graphs among them) and the dimensions $N_{\text {ker }}, N_{\delta}, N_{0}$ and $N_{\text {im }}$ of the respective vector spaces are summarized in Tables 2 and 3.

TABLE 2. Dimensions of connected graph spaces and cohomology groups.

| $n$ |  | $\# E$ | $\#$ (graphs) | $\#(=0)$ | $\#(\neq 0), N_{\delta}$ | $N_{\text {ker }}, N_{0}$ | $N_{\mathrm{im}}$ | $\operatorname{dim} H^{*}(n)$ |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 4 |  | 6 | 1 | 0 | 1 |  | 1 |  |  | 1 |
|  | 3 | 5 | 0 | - | - | - |  | - | - |  |
| 5 | 8 | 2 | 2 | 0 |  | - |  |  | 0 |  |
|  | 4 | 7 | 0 | - | - | - |  | - | - |  |
| 6 | 10 | 14 | 8 | 6 |  | 1 |  |  | 1 |  |
|  | 5 | 9 | 1 | 1 | - | 0 |  | - | - |  |
| 7 | 12 | 126 | 78 | 48 |  | 1 |  |  | 0 |  |
|  | 6 | 11 | 9 | 8 | - | 1 |  | 0 | 1 |  |
| 8 | 14 | 1579 | 605 | 974 |  | 36 |  |  | 1 |  |
|  | 7 | 13 | 95 | 60 | - | 35 |  | 0 | 35 |  |
| 9 | 16 | 26631 | 7557 | 19074 |  | 883 |  |  | 1 |  |
|  | 8 | 15 | 1515 | 602 | - | 913 |  | 31 | 882 |  |

Remark 2. This reasoning covers all the connected graphs with specified number of vertices and edges, meaning that the valency $N(v)$ of every graph vertex $v$ can be any positive number (if $n>1$ ). By Lemma 5 on p. 6 it is seen that for the subspaces $V_{>2}$ of connected graphs restricted by $N(v)>2$ for all $v$, the inclusion $\mathrm{d}\left(V_{>2}\right) \subseteq V_{>2}$ holds. Therefore, the dimensions of cohomology groups for graphs with such restriction on valency cannot exceed the dimension of respective cohomology groups for all the graphs under study (i.e. $N(v)>0$ ). ${ }^{9}$ This means that trivial cohomology groups remain trivial under the extra assumption $N(v)>2$ on valency; yet we already know the generators $\boldsymbol{\gamma}_{3}, \boldsymbol{\gamma}_{5}, \boldsymbol{\gamma}_{7}$, and $\left[\boldsymbol{\gamma}_{3}, \boldsymbol{\gamma}_{5}\right]$ of all the nontrivial cohomology groups at $n \leqslant 9$. This is confirmed in Table 3.

We finally note that the numbers of nonzero graphs with a specified number of vertices and edges (and $N(v)>2$ ), which we list in Table 3, all coincide with the respective entries in Table II in the paper [17].

Remark 3. We expect that there are many d-cocycles on $n$ vertices and $2 n-2$ edges other than the ones containing the $(2 \ell+1)$-wheel graphs (which Theorem 4 provides) or their iterated commutators. Namely, some terms in a weighted sum $\gamma \in \operatorname{ker} d$ can

[^10]Table 3. Dimensions of connected graph spaces with $N(v)>2$ and dimensions of cohomology groups in bi-degree ( $n, 2 n-2$ ).

| $n$ |  | $\# E$ | $\#$ (graphs) | $\#(=0)$ | $\#(\neq 0), N_{\delta}$ | $N_{\text {ker }}, N_{0}$ | $N_{\mathrm{im}}$ | $\operatorname{dim} H^{*}(n)$ |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 4 | 6 | 1 | 0 | 1 |  | 1 |  |  | 1 |  |
|  | 3 | 5 | 0 | - | - | - |  | - | - |  |
| 5 | 8 | 1 | 1 | 0 |  | - |  |  | 0 |  |
|  | 4 | 7 | 0 | - | - | - |  | - | - |  |
| 6 | 10 | 4 | 2 | 2 |  | 1 |  |  | 1 |  |
|  | 5 | 9 | 1 | 1 | - | 0 |  | - | - |  |
| 7 | 12 | 18 | 12 | 6 |  | 1 |  |  | 0 |  |
|  | 6 | 11 | 5 | 4 | - | 1 |  | 0 | 1 |  |
| 8 | 14 | 136 | 61 | 75 |  | 11 |  |  | 1 |  |
|  | 7 | 13 | 30 | 20 | - | 10 |  | 0 | 10 |  |
| 9 | 16 | 1377 | 498 | 879 |  | 164 |  |  | 1 |  |
|  | 8 | 15 | 309 | 130 | - | 179 |  | 16 | 163 |  |

be disjoint graphs; moreover, the vertex-edge bi-grading of a connected component of a given term can be other than $(m, 2 m-2)$ for $m \in \mathbb{N}$. Indeed, for any tuple of dcocycles $\gamma_{i}$ on $n_{i}$ vertices and $E_{i}$ edges satisfying $\sum_{i} n_{i}=n$ and $\sum_{i} E_{i}=2 n-2$, one has that $\gamma:=\bigsqcup_{i} \gamma_{i} \in$ kerd. The graphs $\gamma_{i}$ can be restricted by a requirement that each of them belongs to the domain of the orientation mapping $\mathrm{O} \vec{r}$, so that $\mathrm{O} \vec{r}(\gamma)$ is a Kontsevich bi-vector graph (see [12] and [2, 7]). In this way new classes of generators of infinitesimal symmetries $\dot{\mathcal{P}}=\operatorname{Or}(\gamma)(\mathcal{P})$ are obtained for Poisson structures $\mathcal{P}$.

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## Appendix A. The heptagon-wheel cocycle $\gamma_{7}$

In each term, the ordering of edges is lexicographic (cf. Table 1).












The sum of graphs $\gamma_{7}$ is a d-cocycle because when the differential $\mathrm{d}\left(\gamma_{7}\right)$ is constructed, the images of many terms from $\gamma_{7}$ overlap in $\mathrm{d}\left(\boldsymbol{\gamma}_{7}\right)$ (by graphs on 9 vertices and 15 edges). Finding out what the resulting adjacency table is for the forty-six graphs in $\gamma_{7}$ and -more generally - exploring whether such 'meta-graphs', the vertices of which themselves are graphs that constitute d-cocycles modulo coboundaries, are in any sense special, is an intriguing open problem. (We claim that for $\gamma_{7}$, its meta-graph is connected.)

## Appendix B. Sage code for the graph differential

The following script, written in SAGE version 7.2, can calculate the differential of an arbitrary sum of non-oriented graphs with a specified ordering on the set of edges for every term, and reduce sums of graphs modulo vertex and edge labelling. ${ }^{10}$ As an illustration, it is shown how this can be used to find cocycles in the graph complex.

```
import itertools
def insert(user, victim, position):
    result = []
    victim = victim.relabel({k : k + position - 1 for k in victim.vertices()},
                    inplace=False)
    victim = victim.copy(immutable=False)
    for edge in victim.edges():
        victim.set_edge_label(edge[0], edge[1], edge[2] + len(user.edges()))
    user = user.relabel({k : k if k <= position else k + len(victim) - 1 for k in user.vertices()},
                inplace=False)
    for attachment in itertools.product(victim, repeat=len(user.edges_incident(position))):
        new_graph = user.union(victim)
        edges_in = user.edges_incident(position)
        new_graph.delete_edges(edges_in)
        new_edges = [(k if a == position else a, k if b == position else b, c)
                                    for ((a,b,c), k) in zip(edges_in, attachment)]
        new_graph.add_edges(new_edges)
        result.append((1, new_graph))
    return result
def graph_bracket(graph1, graph2):
    result = []
    for v in graph2:
        result.extend(insert(graph2, graph1, v))
    sign_factor = 1 if len(graph1.edges()) % 2 == 1 and len(graph2.edges()) % 2 == 1 else -1
    for v in graph1:
        result.extend([(sign_factor*c, g) for (c,g) in insert(graph1, graph2, v)])
    return result
def graph_differential(graph):
    edge = Graph([(1,2,1)])
    return graph_bracket(edge, graph)
def differential(graph_sum):
    result = []
    for (c,g) in graph_sum:
        result.extend([(c*d,h) for (d,h) in graph_differential(g)])
    return result
def is_zero(graph):
    for sigma in graph.automorphism_group():
        edge_permutation = Permutation([graph.edge_label(sigma(i), sigma(j))
                                    for (i,j,l) in sorted(graph.edges(), key=lambda (a,b,c): c)])
        if edge_permutation.sign() == -1:
            return True
    return False
def reduce(graph_sum):
    graph_table = {}
    for (c,g) in graph_sum:
        if is_zero(g): continue
```

[^11]```
    # canonically label vertices:
    g_canon, relabeling = g.canonical_label(certify=True)
    # shift labeling up by one:
    g_canon.relabel({k : k + 1 for k in g_canon.vertices()})
    # canonically label edges (keeping track of the edge permutation):
    count = 1
    edges_seen = set([])
    edge_relabeling = {}
    for v in g_canon:
        edges_in = sorted(g_canon.edges_incident(v), key = lambda (a,b,c): a if b == v else b)
        for e in edges_in:
            if frozenset([e[0], e[1]]) in edges_seen: continue
            edge_relabeling[count] = e[2]
            g_canon.set_edge_label(e[0], e[1], count)
            edges_seen.add(frozenset([e[0], e[1]]))
            count += 1
            permutation = Permutation([edge_relabeling[i] for i in range(1, len(g.edges())+1)])
            g_canon = g_canon.copy(immutable=True)
            if g_canon in graph_table:
            graph_table[g_canon] += permutation.sign()*c
        else:
            graph_table[g_canon] = permutation.sign()*c
    return [(graph_table[g], g) for g in graph_table if not graph_table[g] == 0]
# Examples of graphs:
def wheel(n):
    return Graph([(k, 1, k-1) for k in range(2, n+2)] + [(k, k+1 if k <= n else 2, n+k-1)
                for k in range(2, n+2)])
tetrahedron = wheel(3)
fivewheel = wheel(5)
print "The differential of the tetrahedron is", reduce(graph_differential(tetrahedron))
# Finding all cocycles on 6 vertices and 10 edges:
n = 6
graph_list = list(filter(lambda G: G.is_connected() and len(G.edges()) == 2*n - 2, graphs(n)))
# shift labeling up by one
for g in graph_list:
    g.relabel({k : k+1 for k in g.vertices()})
    for (k, (i,j,_)) in enumerate(g.edges()):
        g.set_edge_label(i, j, k+1)
# build an ansatz for a cocycle, with undetermined coefficients
nonzeros = filter(lambda g: not is_zero(g), graph_list)
coeffs = [var('c%d' % k) for k in range(0, len(nonzeros))]
cocycle = zip(coeffs, nonzeros)
# calculate its differential and reduce it
d_cocycle = []
for cocycle_term in cocycle:
    d_cocycle.extend(reduce(differential([cocycle_term])))
d_cocycle = reduce(d_cocycle)
# set the coefficients of the graphs in the reduced sum to zero, and solve
linsys = []
for (c,g) in d_cocycle:
    linsys.append(c==0)
print solve(linsys, coeffs)
```

We finally recall that, to the best of our knowledge, the routines by McKay [1] for graph automorphism computation are now used in SAGE (hence by the above program).

# THE KONTSEVICH UNORIENTED GRAPH COMPLEX: PROOFS 

NINA J. RUTTEN

We introduce some notation, so that we can give a more detailed version of Definition 2 from [3]. This is convenient for the proofs that will follow. The definition of unoriented graph complex Gra is recalled in [3].

Notation. A sum of graphs $\gamma^{x}$ in the space Gra of formal sums of graphs consisting of one term is called a single graph. By convention, it has $n_{x}$ vertices and $k_{x}$ edges. Denote by $\mathrm{V}\left(\gamma^{x}\right)$ the set of vertices, by $\mathrm{E}\left(\gamma^{x}\right)$ the set of edges, and by $X_{\gamma^{x}}:=\mathrm{V}\left(\gamma^{x}\right) \sqcup \mathrm{E}\left(\gamma^{x}\right)$ the set of vertices and edges of graph $\gamma^{x}$. Unless stated otherwise, the wedge ordered set of edges is given by $\mathrm{E}\left(\gamma^{x}\right)=I^{(x)} \wedge I I^{(x)} \wedge \cdots \wedge K^{(x)}$ and any arbitrarily chosen labelling of vertices is denoted by $1^{(x)}, 2^{(x)}, \ldots, n_{x}^{(x)}$. Let $v$ be a vertex of graph $\gamma^{x}$. We denote by $N(v)$ the set of neighbouring vertices of $v$ and by $\bar{N}(v)$ the set of edges attached to $v$. For a given number $n \in \mathbb{N}$ we denote by $\Pi_{n}(v)$ the set of all possible ordered partitions of $\bar{N}(v)$ into $n$ disjoint sets (possibly, empty). Any element of $\Pi_{n}(v)$ is of the form $\pi=\left(S_{1}, S_{2}, \cdots, S_{n}\right)$ such that $S_{1} \sqcup S_{2} \sqcup \cdots \sqcup S_{n}=\bar{N}(v)$.

Now let us recall the definition of a symmetry of a graph in terms of sets of edges attached to a fixed vertex. (Note that it is equivalent to the definition of a graph automorphism.)

Definition 1. Let $\gamma$ be a single graph in Gra. A symmetry of the graph $\gamma$ is a permutation $\sigma: X_{\gamma} \rightarrow X_{\gamma}$ such that $\sigma(\mathrm{V}(\gamma))=\mathrm{V}(\gamma)$ and $\sigma(\mathrm{E}(\gamma))=\mathrm{E}(\gamma)$ as sets and such that $\sigma(\bar{N}(v))=\bar{N}(\sigma(v))$ for all vertices $v$ of $\gamma$.

Notation. For a symmetry of a single graph $\gamma$, we denote by $\sigma_{V}: \mathrm{V}(\gamma) \rightarrow \mathrm{V}(\gamma)$ the permutation of vertices induced by $\sigma$ and by $\sigma_{E}: \mathrm{E}(\gamma) \rightarrow \mathrm{E}(\gamma)$ the permutation of edges induced by $\sigma$.

We recall from [3] that, by definition, the insertion of a sum of graphs into another sum of graphs is linear with respect to both arguments. Therefore it suffices to define insertion of a single graph in Gra into another single graph in Gra. From linearity it follows how insertion is defined for sums of graphs in Gra.
Definition 2. Consider single graphs $\gamma^{1}$ and $\gamma^{2}$ in Gra. The $\gamma^{1}$-blow-up $B_{\gamma^{1}}(v)$ of a vertex $v$ in $\gamma^{2}$ is the following sum of graphs:

$$
B_{\gamma^{1}}(v):=\sum_{\pi \in \Pi_{n_{1}}(v)} \gamma_{v, \pi}^{2}
$$

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where $\gamma_{v, \pi}^{2}$ denotes the graph obtained from graph $\gamma^{2}$ by replacing the vertex $v$ in it by the entire graph $\gamma^{1}$ and where the edges $\bar{N}(v)$ are reattached to the vertices in the inserted graph $\gamma^{1}$ via the correspondence given by a vertex labelling from $\gamma^{1}$ and the ordered partition $\pi=\left(S_{1}, S_{2} \ldots, S_{n_{1}}\right)$ of edges $\bar{N}(v)$ in $\gamma^{2}$. That is, for each vertex $w$ in $\gamma^{1}$, the edges in $S_{w}$ in $\pi$ is are reattached to the vertex $w$ in the new graph $\gamma_{v, \pi}^{2}$. We define the insertion $\gamma^{1} \circ_{i} \gamma^{2}$ of graph $\gamma^{1}$ into graph $\gamma^{2}$ to be the sum

$$
\gamma^{1} \circ_{i} \gamma^{2}:=\sum_{v \in \gamma^{2}} B_{\gamma^{1}}(v)=\sum_{v \in \gamma^{2}} \sum_{\pi \in \Pi_{n_{1}}(v)} \gamma_{v, \pi}^{2} .
$$

By convention, in every term of the sum $\gamma^{1} o_{i} \gamma^{2}$, the edge ordering is $\mathrm{E}\left(\gamma^{1}\right) \wedge \mathrm{E}\left(\gamma^{2}\right)$.
Proposition 1 (Lemma 2 from [3]). For any sum of graphs $\gamma \in \mathrm{Gra}$, the (linear) operator d which acts by the formula

$$
\begin{equation*}
\mathrm{d}(\gamma):=[\bullet \bullet, \gamma]=\bullet \circ_{i} \gamma-(-)^{\# \mathrm{E}(\bullet \bullet) \# \mathrm{E}(\gamma)} \gamma \circ_{i} \bullet \bullet \tag{1}
\end{equation*}
$$

is a differential: $\mathrm{d} \circ \mathrm{d}=0$.
Since the operator $d$ is the sum of two linear operators it is linear itself. Therefore it suffices to prove the statement for single graphs instead of sums of graphs.

Definition 3. Let $\Gamma$ be a graph. The edges with a valency one vertex attached to them are called leaves of the graph $\Gamma$. Let $\gamma$ be a sum of graphs in the space Gra. The graphs in the sum $\mathrm{d}(\gamma)$ where the new edge $\bullet$ appears as a leaf are leaved graphs.

Lemma 2. Let $\gamma$ be a sum of graphs in the space Gra. All leaved graphs in $\mathrm{d}(\gamma)$ cancel.
Proof. Again, since d is linear it suffices to show the statement for single graphs. We consider the edge $\bullet$ as the first graph. Denote by $E^{(1)}$ the edge in the graph $\bullet \bullet$, then the wedge ordered singleton set $\mathrm{E}(\bullet \bullet)=E^{(1)}$. We may introduce any labelling of vertices for the graph $\bullet$, let it be $1^{(1)}, 2^{(1)}$. Then graph $\bullet$ is represented by $1_{1^{(1)}}^{E^{(1)}} 2^{(1)}$. Now let $\gamma^{2}$ be a single graph in the space Gra. We can represent each blow-up $B_{\gamma^{2}}\left(i^{(1)}\right)$ of a vertex $i$ in $\bullet$ by $_{i^{(1)}} \gamma^{2} \longrightarrow{ }_{j^{(1)}}$, where $j=3-i$ (the vertex other than $i$ ). Now the subtrahend, (-) $\# \mathrm{E}(\bullet) \neq \mathrm{E}\left(\gamma^{2}\right) \gamma^{2} \circ_{i} \bullet$, in equation (1) applied to the graphs we consider, can be expressed as follows

$$
\begin{aligned}
(-)^{\# \mathrm{E}(\bullet \bullet) \# \mathrm{E}\left(\gamma^{2}\right)} \gamma^{2} \circ_{i} \bullet \bullet & =(-)^{k_{2}}\left(B_{\gamma^{2}}\left(1^{(1)}+B_{\gamma^{2}}\left(2^{(1)}\right)\right)\right. \\
& =(-)^{k_{2}}{ }_{i^{(1)}} \gamma^{2}-\bullet_{2^{(1)}}+(-)^{k_{2}} i_{i^{(1)}} \bullet \gamma_{2^{(1)}},
\end{aligned}
$$

from which it is clear that it is a sum consisting of leaved graphs only. The edge ordering of all graphs in this sum is given by $I^{(2)} \wedge I I^{(2)} \wedge \cdots \wedge K^{(2)} \wedge E^{(1)}$. Note that graphs with exactly the same topology appear in the minuend $\bullet \circ_{i} \gamma^{2}$ of equation (1) as well. Namely precisely the graphs of the form $\gamma_{v,(\bar{N}(v), \varphi)}^{2}$ and $\gamma_{v,(\theta, \bar{N}(v))}^{2}$, all with edge ordering
$E^{(1)} \wedge I^{(2)} \wedge I I^{(2)} \wedge \cdots \wedge K^{(2)}$. We remark that in the minuend there do not appear leaved graphs other than these and that

$$
\begin{aligned}
(-)^{k_{2}} I^{(2)} \wedge I I^{(2)} \wedge \cdots \wedge K^{(2)} \wedge E^{(1)} & =(-)^{k_{2}}(-)^{k_{2}} E^{(1)} \wedge I^{(2)} \wedge I I^{(2)} \wedge \cdots \wedge K^{(2)} \\
& =E^{(1)} \wedge I^{(2)} \wedge I I^{(2)} \wedge \cdots \wedge K^{(2)} .
\end{aligned}
$$

For all leaved graphs appearing in the differential $\mathrm{d}\left(\gamma^{2}\right)$ - see equation (1) - we obtain

$$
\sum_{v \in \gamma^{2}}\left(\gamma_{v,(\bar{N}(v), \varphi)}^{2}+\gamma_{v,(\varphi, \bar{N}(v))}^{2}\right)-(-)^{k_{2}}(-)^{k_{2}} \sum_{v \in \gamma^{2}}\left(\gamma_{v,(\bar{N}(v), \varphi)}^{2}+\gamma_{v,(\varphi, \bar{N}(v))}^{2}\right)=0 .
$$

Hence all leaved graphs cancel in $\mathrm{d}\left(\gamma^{2}\right)$.
We give an equivalent definition for the differential d in the lemma below. It is a direct consequence of Lemma 2.
Lemma 3. The differential d applied to a single graph $\gamma \in$ Gra is equal to

$$
\mathrm{d}(\gamma)=\sum_{v \in \mathrm{~V}(\gamma)} \sum_{\pi \in \tilde{\mathrm{I}}_{2}(v)} \gamma_{v, \pi},
$$

where $\tilde{\Pi}_{2}(v)$ is the set of ordered partitions of $\bar{N}(v)$ into two non-empty sets.
For the proof of Proposition 1 it is convenient to make a distinction between the two kinds of graphs that can appear in the sum $\mathrm{d} \circ \mathrm{d}(\gamma)$ which is obtained by applying the operator $\mathrm{d} \circ \mathrm{d}$ to a single graph $\gamma$ in Gra.

Definition 4. Let $\gamma^{2}$ be a single graph in Gra and let $v$ be one of its vertices. Let $\gamma_{v, \pi}^{2}$ be a graph appearing in the edge-blow-up $B \ldots$. $(v)$ of $v$, that is, in a sum of graphs that contributes to $\mathrm{d}(\gamma)$. We introduce new labellings $1^{(0)} \stackrel{E}{(0)}_{2^{(0)}}$ for the vertices and edges of the edge $\bullet$ that we insert when we apply the differential $d$ for the second time, so that we can distinguish it from the edge that was inserted firstly. (We choose zero labellings because it induces a natural order of labellings in the edge ordering of any graph $\gamma$ in $\mathrm{d} \circ \mathrm{d}\left(\gamma^{2}\right)$, namely $\mathrm{E}(\gamma)=E^{(0)} \wedge E^{(1)} \wedge I^{(2)} \wedge I I^{(2)} \cdots \wedge K^{(2)}$.) We say that graphs in the sum $\mathrm{d} \circ \mathrm{d}\left(\gamma^{(2)}\right)$ where the second edge-blow-up is applied to an old vertex $w$ in $\gamma_{v, \pi}^{2}$ (i.e. not vertices $1^{(1)}$ or $\left.2^{(1)}\right)$ are distant blown-up graphs. These graphs are of the form $\left(\gamma_{v, \pi}^{2}\right)_{w, \tau}$, where $\pi=\left(S_{1}, S_{2}\right) \in \tilde{\Pi}_{2}(v)$ and $\tau=\left(T_{1}, T_{2}\right) \in \tilde{\Pi}_{2}(w)$. In turn, we say that graphs in the sum $\mathrm{d} \circ \mathrm{d}\left(\gamma^{(2)}\right)$, where the second edge-blow-up is applied to one of the two new vertices, $1^{(1)}$ or $2^{(1)}$ in $\gamma_{v, \pi}^{2}$, are nested blown-up graphs. These graphs are of the form $\left(\gamma_{v, \pi}^{2}\right)_{i^{(1)}, \tau}$, for $i=1$ or $i=2$, where $\pi=\left(S_{1}, S_{2}\right) \in \tilde{\Pi}_{2}(v)$ and $\tau=\left(T_{1}, T_{2}\right) \in \tilde{\Pi}_{2}\left(i^{(1)}\right)$.

The definition of a zero graph is recalled in [3]. Now we have the tools to prove Proposition 1.

Proof. As we remarked, it suffices to prove the proposition for single graphs in Gra and for d given as in Lemma 3. In order to prove that $\mathrm{d} \circ \mathrm{d}(\gamma)$ vanishes for any single graph $\gamma \in$ Gra, we prove first that all distant blown-up graphs cancel in $\mathrm{d} \circ \mathrm{d}(\gamma)$, and second, that all nested blown-up graphs cancel in $\mathrm{d} \circ \mathrm{d}(\gamma)$. Let $\gamma^{2}$ be a single graph in the space

Gra and consider the sum $\operatorname{dod}\left(\gamma^{2}\right)$. Let $\gamma^{a}:=\left(\gamma_{v, \pi}^{2}\right)_{w, \tau}$ be a distant blown-up graph in this sum. Here vertex $v$ is blown up first and vertex $w$ is blown up second, so $\gamma^{a}$ is obtained in the sum $\mathrm{d}\left(B_{\ldots} .(v)\right)$. Then locally, near the newly inserted edges, the graph $\gamma^{a}$ looks as follows:

where the sets $A, B, C, D$ next to the vertices are the sets of edges attached to the respective vertices. By the definition of a blow-up we have that $A \cup B=\bar{N}(v)$ and $C \cup D=\bar{N}(w)$. (Note that $\bar{N}(v)$ and $\bar{N}(w)$ intersect if and only if the vertices $v$ and $w$ are neighbours in the graph $\gamma^{2}$.) We pair the graph $\gamma^{a}=\left(\gamma_{v, \pi}^{2}\right)_{w, \tau}$ with the graph $\left(\gamma_{w, \tau}^{2}\right)_{v, \pi}=: \gamma^{b}$ in $\mathrm{d}(B \ldots(w))$. Locally, near the inserted edges, the graph $\gamma^{b}$ looks as follows:


Note that a distant blown-up graph is paired to precisely one other distant blown-up graph via this pairing, ${ }^{1}$ and the set of all blow-ups is a disjoint union of such pairs. Since the labelling of vertices can be chosen arbitrarily, the graphs $\gamma^{a}$ and $\gamma^{b}$ are topologically equal. Namely, they are equal under the following matching of edges: $E^{(0)(a)}=$ $E^{(1)(b)}, E^{(1)(a)}=E^{(0)(b)}$ and $\#^{(2)(a)}=\#^{(2)(b)}$ for all other edges $\#^{(2)(a)}$ of $\gamma^{a}$ and $\#^{(2)(b)}$ of $\gamma^{b}$ (since the sets $A, B, C, D$ are fixed). Then the wedge ordered set of edges $\mathrm{E}\left(\gamma^{a}\right)$ of $\gamma^{a}$ satisfies

$$
\begin{aligned}
\mathrm{E}\left(\gamma^{a}\right) & =E^{(0)(a)} \wedge E^{(1)(a)} \wedge I^{(2)(a)} \wedge I I^{(2)(a)} \wedge \cdots \wedge K^{(2)(a)} \\
& =E^{(1)(b)} \wedge E^{(0)(b)} \wedge I^{(2)(b)} \wedge I I^{(2)(b)} \wedge \cdots \wedge K^{(2)(b)} \\
& =-E^{(0)(b)} \wedge E^{(1)(b)} \wedge I^{(2)(b)} \wedge I I^{(2)(b)} \wedge \cdots \wedge K^{(2)(b)} \\
& =-\mathrm{E}\left(\gamma^{b}\right) .
\end{aligned}
$$

Hence the graph $\gamma^{a}$ is cancelled by the graph $\gamma^{b}$. Thus all distant blown-up graphs cancel in $\mathrm{d} \circ \mathrm{d}\left(\gamma^{2}\right)$.

Let $\gamma^{c}:=\left(\gamma_{v, \pi}^{2}\right)_{i^{(1)}, \tau}$ be a nested blown-up graph in the sum $\mathrm{d} \circ \mathrm{d}\left(\gamma^{2}\right)$, for $i=1$ or $i=2$. Here vertex $v$ is blown up first and the vertex $i^{(1)}$, produced by the first blow up, is blown up second. Without loss of generality, say $i=1$. Then $\gamma^{c}$ is obtained in the sum $\mathrm{d}(B \ldots(v))$. Locally, near the newly inserted edges, the graph $\gamma^{c}$ looks as follows:


[^12]where the sets $A, B, C$ next to the vertices are the sets of edges attached to the respective vertices. By the definition of a blow-up we have that
\[

$$
\begin{aligned}
\bar{N}(v) & =A \sqcup B \sqcup C, \\
\bar{N}\left(1^{(0)}\right) & =A \sqcup\left\{2^{(0)}\right\}, \\
\bar{N}\left(2^{(0)}\right) & =B \sqcup\left\{1^{(0)}, 2^{(1)}\right\}, \\
\bar{N}\left(2^{(1)}\right) & =C \sqcup\left\{2^{(0)}\right\} .
\end{aligned}
$$
\]

We can rewrite the ordered partitions $\pi$ and $\tau$ as

$$
\pi=(A \sqcup B, C) \quad \text { and } \quad \tau=\left(A, B \sqcup\left\{E^{(1)}\right\}\right) .
$$

Now we pair the graph $\gamma^{c}=\left(\gamma_{v, \pi}^{2}\right)_{i^{(1)}, \tau}$ with the graph $\gamma^{d}:=\left(\gamma_{v, \pi^{\prime}}^{2}\right)_{2^{(1)}, \tau^{\prime}}$, where the ordered partitions are given by

$$
\pi^{\prime}=(A, B \sqcup C) \quad \text { and } \quad \tau^{\prime}=\left(\left\{E^{(1)}\right\} \sqcup B, C\right) .
$$

Locally, near the inserted edges, the graph $\gamma^{b}$ looks as follows


Since the labelling of vertices can be chosen arbitrarily, the graphs $\gamma^{c}$ and $\gamma^{d}$ are topologically equal under the following matching of edges: $E^{(0)(c)}=E^{(1)(d)}, E^{(1)(c)}=E^{(0)(d)}$ and $\#^{(2)(c)}=\#^{(2)(d)}$ for all other edges $\#^{(2)(c)}$ of $\gamma^{c}$ and $\#^{(2)(d)}$ of $\gamma^{d}$ (since the sets $A, B, C$ are fixed). Then the wedge ordered set of edges $\mathrm{E}\left(\gamma^{c}\right)$ of $\gamma^{c}$ satisfies

$$
\begin{aligned}
\mathrm{E}\left(\gamma^{c}\right) & =E^{(0)(c)} \wedge E^{(1)(c)} \wedge I^{(2)(c)} \wedge I I^{(2)(c)} \wedge \cdots \wedge K^{(2)(c)} \\
& =E^{(1)(d)} \wedge E^{(0)(d)} \wedge I^{(2)(d)} \wedge I I^{(2)(d)} \wedge \cdots \wedge K^{(2)(d)} \\
& =-E^{(0)(d)} \wedge E^{(1)(d)} \wedge I^{(2)(d)} \wedge I I^{(2)(d)} \wedge \cdots \wedge K^{(2)(d)} \\
& =-\mathrm{E}\left(\gamma^{d}\right)
\end{aligned}
$$

Hence the graph $\gamma^{c}$ is cancelled by the graph $\gamma^{d}$. Thus all nested blown-up graphs cancel in $\mathrm{d} \circ \mathrm{d}\left(\gamma^{2}\right)$.

The definition of the Lie bracket is recalled in [3]. A necessary condition for it to be well defined is the statement given in Theorem 4. We recall some properties of group actions because they are used in the proof of this theorem.

Remark 1. Let $\gamma \in$ Gra be a zero graph, i.e. it has a symmetry given by a permutation $\sigma$ of vertices and edges that induces a parity odd permutation of edges. Since
the permutation $\sigma \operatorname{acts}^{2}$ on the set $X_{\gamma}$ of vertices and edges of graph $\gamma$, it acts ${ }^{3}$ naturally on the power set $P\left(X_{\gamma}\right)$ of $X_{\gamma}$, namely as follows: $\sigma(S)=\{\sigma(s): s \in S\}$ for $S \in P\left(X_{\gamma}\right)$. Hence $\sigma$ acts ${ }^{3}$ naturally on any finite product space of the powerset $P\left(X_{\gamma}\right)$. Namely as follows: $\sigma\left(\left(S_{1}, S_{2}, \cdots, S_{n}\right)\right)=\left(\sigma\left(S_{1}\right), \sigma\left(S_{2}\right), \cdots, \sigma\left(S_{n}\right)\right)$ for $\left.\left(S_{1}, S_{2}, \cdots, S_{n}\right)\right) \in P\left(X_{\gamma}\right) \times \cdots \times P\left(X_{\gamma}\right)$. (This enables us to speak of the orbit $O_{\left(S_{1}, S_{2}, \cdots, S_{n}\right)}$ of an ordered set $\left(S_{1}, S_{2}, \cdots, S_{n}\right) \in P\left(X_{\gamma}\right)$.)

Theorem 4. Let $\gamma^{z}$ and $\gamma^{2}$ be formal sums of graphs in Gra and let $\gamma^{z}$ be a sum of zero graphs. Then the Lie bracket $\left[\gamma^{Z}, \gamma^{2}\right]$ vanishes in Gra.

Corollary 5. For any zero graph $\gamma^{z}$, its differential $\mathrm{d}\left(\gamma^{z}\right)=\left[\bullet \bullet, \gamma^{z}\right]$ vanishes in Gra.
Proof of Theorem 4. By linearity of the Lie bracket, it suffices to show the statement for sums of graphs consisting of one term. Let $\gamma^{z}$ and $\gamma^{2}$ be (single) graphs in the space Gra. Let $\gamma^{Z}$ be a zero graph. Then the graph has a symmetry, a permutation $\sigma$ of edges and vertices, that induces a parity-odd permutation $\sigma_{E}$ of edges. The Lie bracket of $\gamma^{z}$ and $\gamma^{2}$ can be expressed in terms of blow-ups as follows

$$
\begin{equation*}
\left[\gamma^{Z}, \gamma^{2}\right]=\gamma^{Z} o_{i} \gamma^{2}-(-)^{\# \mathrm{E}\left(\gamma^{Z}\right) \# \mathrm{E}\left(\gamma^{2}\right)} \gamma^{2} o_{i} \gamma^{Z}=\sum_{v \in \mathrm{~V}\left(\gamma^{2}\right)} B_{\gamma^{Z}}(v)-(-)^{k_{z} k_{2}} \sum_{v \in \mathrm{~V}\left(\gamma^{Z}\right)} B_{\gamma^{2}}(v) \tag{2}
\end{equation*}
$$

First we prove that the subtrahend $(-)^{\# \mathrm{E}\left(\gamma^{z}\right) \# \mathrm{E}\left(\gamma^{2}\right)} \sum_{v \in \mathrm{~V}\left(\gamma^{z}\right)} B_{\gamma^{2}}(v)$, is equal to zero in Gra.

Let $v_{1}, v_{2}$ be vertices in graph $\gamma^{\neq}$and denote by $O_{v_{1}}$ and $O_{v_{2}}$ their respective orbits under $\sigma_{V}$. Recall that if the orbits $O_{v_{1}}$ and $O_{v_{2}}$ intersect, then they coincide (see [4]). Denote by $\mathbf{O}^{z}$ the set of all orbits of vertices in $\gamma^{z}$ under $\sigma_{V}$. Note that the set $\mathrm{V}\left(\gamma^{z}\right)$ of vertices in $\gamma^{z}$ is equal to the disjoint union of orbits in $\mathbf{O}^{Z}$. Then we can rewrite the subtrahend of equation (2) as follows

$$
(-)^{k_{z} k_{2}} \sum_{v \in \mathrm{~V}\left(\gamma^{z}\right)} B_{\gamma^{2}}(v)=(-)^{k_{z} k_{2}} \sum_{O \in \mathbf{O}^{z}} \sum_{v \in O} B_{\gamma^{2}}(v) .
$$

Hence, in order to show that the subtrahend vanishes, it suffices to show that for each orbit $O \in \mathbf{O}^{Z}$ the sum of $\gamma^{2}$-blow-ups $\sum_{v \in O} B_{\gamma^{2}}(v)$ is equal to zero in space Gra. Let $O \in \mathbf{O}^{z}$ be an orbit under $\sigma_{V}$ and denote the length of this orbit by $\ell$. Let $v_{\circ}$ be a vertex in the orbit $O$. Then $O$ can be seen as the orbit $O_{v_{0}}$ of $v_{0}$. We can rewrite and expand the

[^13]sum of $\gamma^{2}$-blow-ups as follows,
\[

$$
\begin{equation*}
\sum_{v \in O} B_{\gamma^{2}}(v)=\sum_{i=0}^{\ell-1} B_{\gamma^{2}}\left(\sigma^{i}\left(v_{o}\right)\right)=\sum_{i=0}^{\ell-1} \sum_{\pi \in \Pi_{n_{2}}\left(\sigma^{i}\left(v_{o}\right)\right)} \gamma_{\sigma^{i}\left(v_{o}\right), \pi}^{\nexists} \tag{3}
\end{equation*}
$$

\]

By the definition of a symmetry and by the properties of orbits we have that $\Pi_{n_{2}}\left(\sigma^{i}\left(v_{\mathrm{o}}\right)\right)=$ $\left\{\sigma^{i}(\pi): \pi \in \Pi_{n_{2}}\left(v_{0}\right)\right\}$, where $\sigma$ acts on ordered set $\pi$ like it does on ordered sets in Remark 1. Using this, we can rewrite equation (3) once again:

$$
\begin{equation*}
\sum_{v \in O} B_{\gamma^{2}}(v)=\sum_{i=0}^{\ell-1} \sum_{\pi \in \Pi_{n_{2}}\left(\sigma^{i}\left(v_{o}\right)\right)} \gamma_{\sigma^{i}\left(v_{0}\right), \pi}^{Z}=\sum_{\pi \in \Pi_{n_{2}}\left(v_{0}\right)} \sum_{i=0}^{\ell-1} \gamma_{\sigma^{i}\left(v_{0}\right), \sigma^{i}(\pi)}^{\not} . \tag{4}
\end{equation*}
$$

In order to show that the right hand side of the above equation vanishes, it suffices to show that

$$
\begin{equation*}
\sum_{i=0}^{\ell-1} \gamma_{\sigma^{i}\left(v_{0}\right), \sigma^{i}(\pi)}^{Z}=0 \tag{5}
\end{equation*}
$$

for all $\pi \in \Pi_{n_{2}}\left(v_{\mathrm{o}}\right)$. Let $\pi_{\circ}:=\left(S_{1}, S_{2}, \cdots, S_{n_{2}}\right)$ be an ordered partition in $\Pi_{n_{2}}\left(v_{\mathrm{o}}\right)$. Consider the terms $\gamma_{v_{0}, \pi_{o}}^{z}=: \gamma^{a}$ and $\gamma_{\sigma(v), \sigma\left(\pi_{o}\right)}^{z}=: \gamma^{b}$ in the above sum. By definition of a symmetry we have that $\gamma^{a}$ is topologically equal to $\gamma^{b}$ under the following matching of edges: $I^{(z)(a)}=\sigma_{E}\left(I^{(Z)}\right)^{(b)}, I I^{(z)(a)}=\sigma_{E}\left(I I^{(Z)}\right)^{(b)}, \ldots, K^{(z)(a)}=\sigma_{E}\left(K^{(Z)}\right)^{(b)}$ and $I^{(2)(a)}=$ $I^{(2)(b)}, I I^{(2)(a)}=I I^{(2)(b)}, \ldots, K^{(2)(a)}=K^{(2)(b)}$. Since $\sigma_{E}$ is a parity-odd permutation, the wedge ordered set of edges of $\gamma^{a}$ satisfies

$$
\begin{aligned}
\mathrm{E}\left(\gamma^{a}\right) & =I^{(z)(a)} \wedge I I^{(z)(a)} \wedge \cdots \wedge K^{(z)(a)} \wedge I^{(2)(a)} \wedge I I^{(2)(a)} \wedge \cdots \wedge K^{(2)(a)} \\
& =\sigma_{E}\left(I^{(z)}\right)^{(b)} \wedge \sigma_{E}\left(I I^{(Z)}\right)^{(b)} \wedge \cdots \wedge \sigma_{E}\left(K^{(z)}\right)^{(b)} \wedge I^{(2)(b)} \wedge I I^{(2)(b)} \wedge \cdots \wedge K^{(2)(b)} \\
& =-I^{(z)(b)} \wedge I I^{(z)(b)} \wedge \cdots \wedge K^{(z)(b)} \wedge I^{(2)(b)} \wedge I I^{(2)(b)} \wedge \cdots \wedge K^{(2)(b)} \\
& =-\mathrm{E}\left(\gamma^{b}\right)
\end{aligned}
$$

Therefore $\gamma_{v_{o}, \pi_{\circ}}^{Z}=\gamma^{a}=-\gamma^{b}=-\gamma_{\sigma\left(v_{0}\right), \sigma\left(\pi_{\circ}\right)}^{Z}$. Since $\nu_{o} \in O$ is chosen arbitrarily and since the elements in $O$ can be written as $\sigma^{i}\left(\nu_{\mathrm{o}}\right)$ for $0 \leq i<\ell$, it follows that

$$
\begin{equation*}
\gamma_{\sigma^{i}\left(v_{o}\right), \sigma^{i}\left(\pi_{0}\right)}^{Z}=-\gamma_{\sigma^{i+1}\left(v_{0}\right), \sigma^{i+1}\left(\pi_{0}\right)}^{Z} \tag{6}
\end{equation*}
$$

Assume $\ell$ is even. We make a (disjoint) pairing of all graphs in equation (5) and show that all pairs cancel. Specifically, we pair all graphs in equation (5) as follows: $\gamma_{\sigma^{i}\left(v_{0}\right), \sigma^{i}\left(\pi_{0}\right)}^{Z}$ is paired with $\gamma_{\sigma^{i+1}\left(v_{0}\right), \sigma^{i+1}\left(\pi_{0}\right)}^{Z}$ for $i=0,2,4, \ldots, \ell-2$. All these pairs cancel by equation (6), so $\sum_{v \in O} B_{\gamma^{2}}(v)=0$.

Assume $\ell$ is odd. Now we cannot organize all graphs in cancelling pairs, since the sum (5) consists of an odd amount of terms. We shall rewrite equation (4) in such a way that we sum over the orbits of ordered sets. If the orbits have even length we can organize graphs over cancelling pairs again, as in the case above. If the orbits have odd length the graphs turn out to be zero graphs.

Denote by $\lambda$ the length of the orbit $O_{\left(v_{\circ}, \pi_{0}\right)}$ of the ordered set $\left(v_{0}, \pi_{\circ}\right):=\left(\{v\}, S_{1}, S_{2}, \cdots, S_{n_{2}}\right)$, see Remark 1.

First assume $\lambda$ is odd. Then we claim that $\gamma_{v_{0}, \pi_{o}}^{z}$ is a zero graph. Namely,

$$
\gamma_{v_{o}, \pi_{o}}^{7}=\gamma_{\sigma^{\lambda}\left(v_{\mathrm{o}}\right), \sigma^{2}\left(\pi_{o}\right)}^{7}=-\gamma_{v_{\mathrm{o}}, \pi_{o}}^{7},
$$

where the second equality follows from equation (6).
Note that $\lambda$ is a multiple of $\ell$, since we must have $\sigma^{\lambda}\left(v_{0}\right)=\nu_{0}$ in order to have $\sigma^{\lambda}\left(\left(v_{\mathrm{o}}, \pi_{\mathrm{o}}\right)\right)=\left(v_{\mathrm{o}}, \pi_{\mathrm{o}}\right)$.

Now assume $\lambda$ is even. Then $\frac{\lambda}{\ell}$ is even. Denote by $\tilde{\mathbf{O}}^{z}\left(v^{\prime}\right)$ the set of orbits - under $\sigma$ - of ordered sets of the form $\left(\left\{v^{\prime}\right\}, S_{1}^{\prime}, S_{2}^{\prime}, \cdots, S_{n_{2}}^{\prime}\right)$, where $v^{\prime} \in \mathrm{V}\left(\gamma^{z}\right)$ and $\pi^{\prime}:=$ $\left(S_{1}, S_{2}, \cdots, S_{n_{2}}\right) \in \Pi_{n_{2}}\left(v^{\prime}\right)$. Below we rewrite $\sum_{v \in O} B_{\gamma^{2}}(v)$ again, this time by summing over the disjoint orbits of the pairs ( $v_{0}, \pi$ ) under $\sigma$, that is summing over the elements in $\tilde{\mathbf{O}}^{z}\left(v_{o}\right)$. (These orbits can be longer than the orbit $O_{v_{c} i r c}$ of $v_{o}$ since the orbit of ordered partition $\pi$ can be longer.) Again by the definition of a symmetry and by the properties of orbits of ordered sets (see again Remark 1) we have that

$$
\begin{aligned}
& \left\{\left(\sigma^{i}\left(v_{\mathrm{o}}\right), \sigma^{i}(\pi)\right): \pi \in \Pi_{n_{2}}\left(v_{\mathrm{o}}\right), i=0,1, \ldots, \ell-1\right\} \\
= & \left\{\left(\sigma^{i}\left(v_{\mathrm{o}}\right), \sigma^{i}(\pi)\right): O_{\left(v_{\mathrm{o}}, \pi\right)} \in \tilde{\mathbf{O}}^{z}\left(v_{\mathrm{o}}\right), i=1,2, \ldots, \lambda\left(v_{\mathrm{o}}, \pi\right)-1\right\},
\end{aligned}
$$

where $\lambda\left(v_{0}, \pi\right)$ is the (long) length of the orbit $O_{\left(v_{0}, \pi\right)}$ of the ordered set $\left(v_{0}, \pi\right)$. Now we rewrite equation (4) as

$$
\sum_{v \in O} B_{\gamma^{2}}(v)=\sum_{\pi \in \Pi_{n_{2}}\left(v_{0}\right)} \sum_{i=0}^{\ell-1} \gamma_{\sigma^{i}\left(v_{0}\right), \sigma^{i}(\pi)}^{\neq}=\sum_{O_{\left(v_{0}, \pi\right)} \in \widetilde{\mathbf{O}}^{\neq}} \sum_{\left(v_{0}\right)} \sum_{i=0}^{\lambda\left(v_{0}, \pi\right)-1} \gamma_{\sigma^{i}\left(v_{0}\right), \sigma^{i}(\pi)}^{\not} .
$$

In order to show that the above sum vanishes, it suffices to show that for every orbit $O_{\left(v_{o}, \pi\right)} \in \tilde{\mathbf{O}}^{\nexists}\left(v_{o}\right)$ the sum

$$
\sum_{i=0}^{\lambda\left(v_{v}, \pi\right)-1} \gamma_{\sigma^{i}\left(v_{0}\right), \sigma^{i}(\pi)}^{z}
$$

is equal to zero. Let $O_{\left(v_{o}, \pi_{0}\right)} \in \tilde{\mathbf{O}}^{z}\left(v_{\mathrm{o}}\right)$. Again by the definition of a symmetry and by the properties of orbits of ordered sets we have

$$
\begin{aligned}
O_{\left(v_{o}, \pi_{\mathrm{o}}\right)} & =\left\{\left(\sigma^{i}\left(v_{\mathrm{o}}\right), \sigma^{i}(\pi)\right): i=0,1, \ldots, \lambda\left(v_{\circ}, \pi\right)-1\right\} \\
& =\bigsqcup_{0 \leq j<\ell}\left\{\left(\sigma^{j+i \ell}\left(v_{\mathrm{o}}\right), \sigma^{j+i \ell}\left(\pi_{\circ}\right)\right): i=0,1, \ldots, \frac{\lambda\left(v_{\circ}, \pi_{\circ}\right)}{\ell}-1\right\} .
\end{aligned}
$$

Now it suffices to show that, for all $0 \leq j<\ell$, the sum

$$
\sum_{i=0}^{\frac{\lambda\left(v_{0}, \pi_{0}\right)}{t}-1} \gamma_{\sigma^{j+i i_{( }}\left(v_{0}\right), \sigma^{j+i l}\left(\pi_{0}\right)}^{z}
$$

vanishes. Since $\ell$ is odd, by equation (6) we have that $\gamma_{v_{0}, \pi_{o}}^{z}=-\gamma_{\sigma^{\ell}\left(v_{0}\right), \sigma^{\ell}\left(\pi_{o}\right)}^{7}$. Since $\frac{\lambda}{\ell}$ is even we can pair $\gamma_{v_{o}, \pi_{\circ}}^{7}$ with $\gamma_{\sigma^{\ell}\left(v_{0}\right), \sigma^{\ell}\left(\pi_{\circ}\right)}^{7}$ for $i=0,2,4, \ldots, \frac{\lambda}{\ell}-2$. All pairs cancel, so $\sum_{v \in O} B_{\gamma^{2}}(v)=0$ and the subtrahend vanishes.

There is left to show that the minuend,

$$
\sum_{v \in \mathrm{~V}\left(\gamma^{2}\right)} B_{\gamma^{z}}(v)=\sum_{v \in \mathrm{~V}\left(\gamma^{2}\right)} \sum_{\pi \in \Pi_{n Z}(v)} \gamma_{v, \pi}^{2}
$$

of equation (2) is equal to zero in Gra as well. It suffices to show that $\sum_{\pi \in \Pi_{n Z}(v)} \gamma_{v_{\circ}, \pi}^{2}$ is zero in Gra for any fixed vertex $v_{\circ} \in \mathrm{V}(\gamma)$. Denote by $\mathbf{O}^{\Pi_{n \mathcal{Z}}\left(v_{o}\right)}$ the set of all orbits of ordered partitions in $\Pi_{n \mathcal{Z}}\left(v_{\circ}\right)$. Then

$$
\sum_{\pi \in \Pi_{n Z}(v)} \gamma_{v_{o}, \pi}^{2}=\sum_{O \in \mathbf{O}^{\Pi_{n Z}\left(v_{0}\right)}} \sum_{\pi \in O} \gamma_{v_{o}, \pi}^{2}
$$

Hence, it suffices to show that $\sum_{\pi \in O} \gamma_{v_{0}, \pi}^{2}$ is equal to zero in Gra for any fixed orbit $O$ in $\mathbf{O}^{\Pi_{n \mathcal{E}}\left(v_{0}\right)}$.

Let $O_{\circ}$ in $\mathbf{O}^{\Pi_{n} \mathcal{E}^{\left(v_{\circ}\right)}}$ and let $\pi_{\circ} \in O_{\circ}$. The orbit $O_{\circ}$ can be seen as the orbit $O_{\pi_{\circ}}$ of $\pi_{\circ}$ and

$$
\begin{equation*}
\sum_{\pi \in O} \gamma_{v_{\circ}, \pi}^{2}=\sum_{i=0}^{\ell_{\pi_{0}}-1} \gamma_{v_{o}, \sigma^{i}\left(\pi_{\circ}\right)}^{2} \tag{7}
\end{equation*}
$$

where $\ell_{\pi_{\circ}}$ is the length of the orbit $O_{\pi_{\circ}}$.
Assume $\ell_{\pi_{\circ}}$ is even. Consider $\gamma_{v_{o}, \pi_{\circ}}^{2}=: \gamma^{a}$ and $\gamma_{v_{o}, \sigma\left(\pi_{\circ}\right)}^{2}=: \gamma^{b}$. The graphs $\gamma^{a}$ and $\gamma^{b}$ are topologically equal under the following matching of edges: $I^{(Z)(b)}=\sigma\left(I^{(Z)(a)}\right), I I^{(Z)(b)}=$ $\sigma\left(I I^{(Z)(a)}\right), \ldots, K^{(Z)(b)}=\sigma\left(K^{(Z)(a)}\right)$ and $\#^{(2)(b)}=\#^{(2)(a)}$ for all other respective edges in $\gamma^{b}$ and $\gamma^{a}$. The edge ordering of $\gamma^{b}$ satisfies

$$
\begin{aligned}
\mathrm{E}\left(\gamma^{b}\right) & =I^{(Z)(b)} \wedge I I^{(Z)(b)} \wedge \cdots \wedge K^{(Z)(b)} \wedge I^{(2)(b)} \wedge I I^{(2)(b)} \wedge \cdots \wedge K^{(2)(b)} \\
& =\sigma\left(I^{(Z)}\right)^{(a)} \wedge \sigma\left(I I^{(Z)}\right)^{(a)} \wedge \cdots \wedge \sigma\left(K^{(Z)}\right)^{(a)} \wedge I^{(2)(a)} \wedge I I^{(2)(a)} \wedge \cdots \wedge K^{(2)(a)} \\
& =-I^{(Z)(a)} \wedge I I^{(Z)(a)} \wedge \cdots \wedge K^{(Z)(a)} \wedge I^{(2)(a)} \wedge I I^{(2)(a)} \wedge \cdots \wedge K^{(2)(a)} \\
& =-\mathrm{E}\left(\gamma^{a}\right) .
\end{aligned}
$$

Hence $\gamma_{v_{\circ}, \pi_{\circ}}^{2}=\gamma^{a}=-\gamma^{b}=-\gamma_{v_{\circ}, \sigma\left(\pi_{\circ}\right)}^{2}$. Since the elements in $O$ can be written as $\sigma^{i}\left(\pi_{\circ}\right)$ for $0 \leq i<\ell_{\pi_{0}}$ we have that

$$
\begin{equation*}
\gamma_{v_{\mathrm{o}}, \sigma^{i}\left(\pi_{\circ}\right)}^{2}=-\gamma_{v_{\mathrm{o}}, \sigma^{i+1}\left(\pi_{\circ}\right)}^{2} \tag{8}
\end{equation*}
$$

for all $i=0, \ldots, \ell_{\pi_{\circ}}-2$. From the assumption that $\ell_{\pi_{\circ}}$ is even it follows that we can make a disjoint paring of all graphs in the right hand side of equation (7) such that all pairs cancel by equation (8).

Assume that $\ell_{\pi_{0}}$ is odd. Then the graph $\gamma_{v_{0}, \pi_{\circ}}^{2}=: \gamma^{a}$ is topologically equal to itself via the following matching of edges: $I^{(Z)(a)}=\sigma^{\ell_{\pi_{0}}}\left(I^{(Z)(b)}\right), I I^{(\mathcal{Z})(a)}=\sigma^{\ell_{\pi_{0}}}\left(I I^{(Z)(b)}\right), \ldots, K^{(Z)(a)}=$ $\sigma^{\ell_{\pi_{0}}}\left(K^{(Z)(b)}\right)$ and $\#^{(2)(a)}=\#^{(2)(b)}$ for all other respective edges in $\gamma^{a}$ and $\gamma^{b}:=\gamma_{v_{0}, \sigma^{\ell \pi_{\circ}}}^{2}\left(\pi_{\circ}\right)$. The permutation $\sigma^{\ell_{0}}$ indeed satisfies $\sigma^{\ell_{\pi_{0}}}(\bar{N}(v))=\bar{N}\left(\sigma^{\ell_{\pi_{0}}}(v)\right)$ since the newly attached vertices are left invariant, therefore it defines a symmetry. The edge ordering of $\gamma^{b}$
satisfies:

$$
\begin{aligned}
& \mathrm{E}\left(\gamma^{a}\right)=I^{(Z)(a)} \wedge I I^{(Z)(a)} \wedge \cdots \wedge K^{(Z)(a)} \wedge I^{(2)(a)} \wedge I I^{(2)(a)} \wedge \cdots \wedge K^{(2)(a)} \\
& =\sigma^{\ell_{\pi_{0}}}\left(I^{(Z)}\right)^{(b)} \wedge \sigma^{\ell_{\pi_{0}}}\left(I I^{(Z)}\right)^{(b)} \wedge \cdots \wedge \sigma^{\ell_{\pi_{0}}}\left(K^{(Z)}\right)^{(b)} \wedge I^{(2)(b)} \wedge I I^{(2)(b)} \wedge \cdots \wedge K^{(2)(b)} \\
& =-I^{(Z)(b)} \wedge I I^{(\mathcal{Z})(b)} \wedge \cdots \wedge K^{(Z)(b)} \wedge I^{(2)(b)} \wedge I I^{(2)(b)} \wedge \cdots \wedge K^{(2)(b)} \\
& =-\mathrm{E}\left(\gamma^{b}\right) \text {, }
\end{aligned}
$$

where the minus sign follows from the assumption that $\ell_{\pi_{0}}$ is odd. Then the graph $\gamma_{v_{o}, \pi_{o}}^{2}=\gamma^{a}=-\gamma^{b}=-\gamma_{v_{o}, \sigma^{\ell \pi_{0}}\left(\pi_{0}\right)}^{2}$ and hence $\gamma_{v_{0}, \pi}^{2}$ is a zero graph. Since $v_{\circ}$ is chosen arbitrarily all graphs in the sum $\sum_{\pi \in O} \gamma_{v_{0}, \pi}^{2}$ are zero graphs so this sum is equal to zero.

It follows that the minuend vanishes since it is equal to a sum of zero graphs and/or graphs with a vanishing coefficient and the statement follows.

The proof of Corollary 5 is completely analogous to a special case of the the proof of Theorem 4 , where $\gamma^{2}=-\bullet$ and the set of ordered partitions $\Pi$ is restricted to the subset $\bar{\Pi}$ of ordered partitions without empty components. (Here we use that the leaved graphs cancel.)

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# Infinitesimal deformations of Poisson bi-vectors using the Kontsevich graph calculus 

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#### Abstract

Let $\mathcal{P}$ be a Poisson structure on a finite-dimensional affine real manifold. Can $\mathcal{P}$ be deformed in such a way that it stays Poisson? The language of Kontsevich graphs provides a universal approach - with respect to all affine Poisson manifolds - to finding a class of solutions to this deformation problem. For that reasoning, several types of graphs are needed. In this paper we outline the algorithms to generate those graphs. The graphs that encode deformations are classified by the number of internal vertices $k$; for $k \leqslant 4$ we present all solutions of the deformation problem. For $k \geqslant 5$, first reproducing the pentagon-wheel picture suggested at $k=6$ by Kontsevich and Willwacher, we construct the heptagon-wheel cocycle that yields a new unique solution without 2-loops and tadpoles at $k=8$.


Introduction. This paper contains a set of algorithms to generate the Kontsevich graphs that encode polydifferential operators - in particular multi-vectors - on Poisson manifolds. We report a result of implementing such algorithms in the problem of finding symmetries of Poisson structures. Namely, continuing the line of reasoning from [1, 2, we find all the solutions of this deformation problem that are expressed by the Kontsevich graphs with at most four internal vertices. Next, we present one six-vertex solution (based on the previous work by Kontsevich 10] and Willwacher [13]). Finally, we find a heptagon-wheel eight-vertex graph which, after the orientation of its edges, gives a new universal Kontsevich flow. We refer to [8, 9 for motivations, to [2, 4] for an exposition of basic theory, and to [6] and [5] for more details about the pentagonwheel ( $5+1$ )-vertex and heptagon-wheel $(7+1)$-vertex solutions respectively. Let us remark that all the algorithms outlined here can be used without modification in the course of constructing all $k$-vertex Kontsevich graph solutions with higher $k \geqslant 5$ in the deformation problem under study.

Basic concept. We work with real vector spaces generated by finite graphs of the following two types: (1) $k$-vertex non-oriented graphs, without multiple edges nor tadpoles, endowed with a wedge ordering of edges, e.g., $E=e_{1} \wedge \cdots \wedge e_{2 k-2}$; (2) oriented graphs on $k$ internal vertices and $n$ sinks such that every internal vertex is a tail of two edges with a given ordering Left $\prec$ Right. Every connected component of a non-oriented graph $\gamma$ is fully encoded by an ordering $E$
on the set of adjacency relations for its vertices 1 Every such oriented graph is given by the list of ordered pairs of directed edges. An edge swap $e_{i} \wedge e_{j}=-e_{j} \wedge e_{i}$ and the reversal Left $\leftrightarrows$ Right of those edges' order in the tail vertex implies the change of sign in front of the graph at hand 2

Example 1. The sum $\gamma_{5}$ of two 6 -vertex 10 -edge graphs,

$$
\begin{aligned}
& \gamma_{5}=12^{(I)} \wedge 23^{(I I)} \wedge 34^{(I I I)} \wedge 45^{(I V)} \wedge 51^{(V)} \wedge 16^{(V I)} \wedge 26^{(V I I)} \wedge 36^{(V I I I)} \wedge 46^{(I X)} \wedge 56^{(X)} \\
& +\frac{5}{2} \cdot 12^{(I)} \wedge 23^{(I I)} \wedge 34^{(I I I)} \wedge 41^{(I V)} \wedge 45^{(V)} \wedge 15^{(V I)} \wedge 56^{(V I I)} \wedge 36^{(V I I I)} \wedge 26^{(I X)} \wedge 13^{(X)}
\end{aligned}
$$

is drawn in Theorem 7 on p .10 below.
Example 2. The sum $\mathcal{Q}_{1: \frac{6}{2}}$ of three oriented 8-edge graphs on $k=4$ internal vertices and $n=2$ sinks (enumerated using 0 and 1 , see the notation on p. 4),

$$
\mathcal{Q}_{1: \frac{6}{2}}=241 \quad 012442523-3(241 \quad 03142523+241
$$

is obtained from the non-oriented tetrahedron graph $\gamma_{3}=12^{(I)} \wedge 13^{(I I)} \wedge 14^{(I I I)} \wedge 23^{(I V)} \wedge$ $24^{(V)} \wedge 34^{(V I)}$ on four vertices and six edges by taking all the admissible edge orientations (see Theorem 4 and Remark (1).
I.1. Let $\gamma_{1}$ and $\gamma_{2}$ be connected non-oriented graphs. The definition of insertion $\gamma_{1} \circ_{i} \gamma_{2}$ of the entire graph $\gamma_{1}$ into vertices of $\gamma_{2}$ and the construction of Lie bracket $[\cdot, \cdot]$ of graphs and differential d in the non-oriented graph complex, referring to a sign convention, are as follows (cf. [8] and [7, 11, 12]); these definitions apply to sums of graphs by linearity.

Definition 1. The insertion $\gamma_{1} \circ_{i} \gamma_{2}$ of a $k_{1}$-vertex graph $\gamma_{1}$ with ordered set of edges $E\left(\gamma_{1}\right)$ into a graph $\gamma_{2}$ with $\# E\left(\gamma_{2}\right)$ edges on $k_{2}$ vertices is a sum of graphs on $k_{1}+k_{2}-1$ vertices and $\# E\left(\gamma_{1}\right)+\# E\left(\gamma_{2}\right)$ edges. Topologically, the sum $\gamma_{1} \circ_{i} \gamma_{2}=\sum\left(\gamma_{1} \rightarrow v\right.$ in $\left.\gamma_{2}\right)$ consists of all the graphs in which a vertex $v$ from $\gamma_{2}$ is replaced by the entire graph $\gamma_{1}$ and the edges touching $v$ in $\gamma_{2}$ are re-attached to the vertices of $\gamma_{1}$ in all possible ways ${ }_{3}^{3}$ By convention, in every new term the edge ordering is $E\left(\gamma_{1}\right) \wedge E\left(\gamma_{2}\right)$.

To simplify sums of graphs, first eliminate the zero graphs. Now suppose that in a sum, two non-oriented graphs, say $\alpha$ and $\beta$, are isomorphic (topologically, i.e. regardless of the respective vertex labellings and edge orderings $E(\alpha)$ and $E(\beta))$. By using that isomorphism, which establishes a $1-1$ correspondence between the edges, extract the sign from the equation $E(\alpha)= \pm E(\beta)$. If " + ", then $\alpha=\beta$; else $\alpha=-\beta$. Collecting similar terms is now elementary.
Lemma 1. The bi-linear graded skew-symmetric operation,

$$
\left[\gamma_{1}, \gamma_{2}\right]=\gamma_{1} \circ_{i} \gamma_{2}-(-)^{\# E\left(\gamma_{1}\right) \cdot \# E\left(\gamma_{2}\right)} \gamma_{2} \circ_{i} \gamma_{1}
$$

[^14]is a Lie bracket on the vector space $\mathfrak{G}$ of non-oriented graphs $4_{4}^{4}$
Lemma 2. The operator $\mathrm{d}($ graph $)=[\bullet \bullet$, graph $]$ is a differential: $\mathrm{d}^{2}=0$.
In effect, the mapping $d$ blows up every vertex $v$ in its argument in such a way that whenever the number of adjacent vertices $N(v) \geqslant 2$ is sufficient, each end of the inserted edge $\bullet \bullet$ is connected with the rest of the graph by at least one edge.

Summarising, the real vector space $\mathfrak{G}$ of non-oriented graphs is a differential graded Lie algebra (dgLa) with Lie bracket $[\cdot, \cdot]$ and differential $\mathrm{d}=[\bullet \bullet, \cdot]$. The graphs $\gamma_{5}$ and $\gamma_{3}$ from Examples 1 and 2 are d-cocycles. Neither is exact, hence marking a nontrivial cohomology class in the non-oriented graph complex.

Theorem 3 ([7, Th. 5.5]). At every $\ell \in \mathbb{N}$ in the connected graph complex there is a d-cocycle on $2 \ell+1$ vertices and $4 \ell+2$ edges. Such cocycle contains the $(2 \ell+1)$-wheel in which, by definition, the axis vertex is connected with every other vertex by a spoke so that each of those $2 \ell$ vertices is adjacent to the axis and two neighbours; the cocycle marked by the $(2 \ell+1)$-wheel graph can contain other $(2 \ell+1,4 \ell+2)$-graphs (see Example 1 and [5]).
I.2. The oriented graphs under study are built over $n$ sinks from $k$ wedges $\stackrel{i_{\alpha}}{\longleftrightarrow} \stackrel{j_{\alpha}}{\longrightarrow}$ (here $\stackrel{i_{\alpha}}{\longleftrightarrow} \prec \xrightarrow{j_{\alpha}}$ ) so that every edge is decorated with its own summation index which runs from 1 to the dimension of a given affine Poisson manifold $(\mathcal{N}, \mathcal{P})$. Each edge $\xrightarrow{i}$ encodes the derivation $\partial / \partial x^{i}$ of the arrowhead object with respect to a local coordinate $x^{i}$ on $\mathcal{N}$. By placing an $\alpha$ th copy $P^{i_{\alpha} j_{\alpha}}(\boldsymbol{x})$ of the Poisson bi-vector $\mathcal{P}$ in the wedge top $(1 \leqslant \alpha \leqslant k)$, by taking the product of contents of the $n+k$ vertices (and evaluating all objects at a point $\boldsymbol{x} \in \mathcal{N}$ ), and summing over all indices, we realise a polydifferential operator in $n$ arguments; the operator coefficients are differential-polynomial in $\mathcal{P}$. Totally skew-symmetric operators of differential order one in each argument are well-defined $n$-vectors on the affine manifolds $\mathcal{N}$.

The space of multi-vectors $G$ encoded by oriented graphs is equipped with a graded Lie algebra structure, namely the Schouten bracket $\llbracket \cdot, \cdot \rrbracket$. Its realisation in terms of oriented graphs is shown in [2, Remark 4]. Recall that by definition the bi-vectors $\mathcal{P}$ at hand are Poisson by satisfying the Jacobi identity $\llbracket \mathcal{P}, \mathcal{P} \rrbracket=0$. The Poisson differential $\partial_{\mathcal{P}}=\llbracket \mathcal{P}, \cdot \rrbracket$ now endows the space of multi-vectors on $\mathcal{N}$ with the differential graded Lie algebra (dgLa) structure. The cohomology groups produced by the two dgLa structures introduced so far are correlated by the edge orientation mapping $\mathrm{O} \overrightarrow{\mathrm{r}}$.

Theorem 4 ([8] and [12, App. K]). Let $\gamma \in \operatorname{ker} \mathrm{d}$ be a cocycle on $k$ vertices and $2 k-2$ edges in the non-oriented graph complex. Denote by $\{\Gamma\} \subset G$ the subspace spanned by all those bi-vector graphs $\Gamma$ which are obtained from (each connected component in) $\gamma$ by adding to it two edges to the new sink vertices and then by taking the sum of graphs with all the admissible orientations of the old $2 k-2$ edges (so that a set of Kontsevich graphs built of $k$ wedges is produced). Then in that subspace $\{\Gamma\}$ there is a sum of graphs that encodes a nonzero Poisson cocycle $Q(\mathcal{P}) \in \operatorname{ker} \partial_{\mathcal{P}}$.

Consequently, to find some cocycle $Q(\mathcal{P})$ in the Poisson complex on any affine Poisson manifold it suffices to find a cocycle in the non-oriented graph complex and then consider the sum of graphs which are produced by the orientation mapping Or. On the other hand, to list all the $\partial_{\mathcal{P}}$-cocycles $Q(\mathcal{P})$ encoded by the bi-vector graphs made of $k$ wedges $\leftarrow \bullet \rightarrow$, one must generate all the relevant oriented graphs and solve the equation $\partial_{\mathcal{P}}(Q) \doteq 0$ via $\llbracket \mathcal{P}, \mathcal{P} \rrbracket=0$, that is, solve

[^15]graphically the factorisation problem $\llbracket \mathcal{P}, Q(\mathcal{P}) \rrbracket=\diamond(\mathcal{P}, \llbracket \mathcal{P}, \mathcal{P} \rrbracket)$ in which the cocycle condition in the left-hand side holds by virtue of the Jacobi identity in the right. Such construction of some and classification (at a fixed $k>0$ ) of all universal infinitesimal symmetries of Poisson brackets are the problems which we explore in this paper.
Remark 1. To the best of our knowledge [10], in a bi-vector graph $Q(\mathcal{P})=\operatorname{Or}(\gamma)$, at every internal vertex which is the tail of two oriented edges towards other internal vertices, the edge ordering Left $\prec$ Right is inherited from a chosen wedge product $E(\gamma)$ of edges in the nonoriented graph $\gamma$. How are the new edges towards the sinks ordered, either between themselves at a vertex or with respect to two other oriented edges, coming from $\gamma$ and issued from different vertices in $Q(\mathcal{P})$ ? Our findings in [6] will help us to verify the order preservation claim and assess answers to this question.

## 1. The Kontsevich graph calculus

Definition 2. Let us consider a class of oriented graphs on $n+k$ vertices labelled $0, \ldots, n+k-1$ such that the consecutively ordered vertices $0, \ldots, n-1$ are sinks, and each of the internal vertices $n, \ldots, n+k-1$ is a source for two edges. For every internal vertex, the two outgoing edges are ordered using $L \prec R$ : the preceding edge is labelled $L$ (Left) and the other is $R$ (Right). An oriented graph on $n$ sinks and $k$ internal vertices is a Kontsevich graph of type ( $n, k$ ).

For the purpose of defining a graph normal form, we now consider a Kontsevich graph $\Gamma$ together with a sign $s \in\{0, \pm 1\}$, denoted by concatenation of the symbols: $s \Gamma$.
Notation (Encoding of the Kontsevich graphs). The format to store a signed graph $s \Gamma$ for a Kontsevich graph $\Gamma$ is the integer number $n>0$, the integer $k \geqslant 0$, the sign $s$, followed by the (possibly empty, when $k=0$ ) list of $k$ ordered pairs of targets for edges issued from the internal vertices $n, \ldots, n+k-1$, respectively. The full format is then ( $n, k, s$; list of ordered pairs).
Definition 3 (Normal form of a Kontsevich graph). The list of targets in the encoding of a graph $\Gamma$ can be considered as a $2 k$-digit integer written in base- $(n+k)$ notation. By running over the entire group $S_{k} \times\left(\mathbb{Z}_{2}\right)^{k}$, and by this over all the different re-labellings of $\Gamma$, we obtain many different integers written in base- $(n+k)$. The absolute value $|\Gamma|$ of $\Gamma$ is the re-labelling of $\Gamma$ such that its list of targets is minimal as a nonnegative base- $(n+k)$ integer. For a signed graph $s \Gamma$, the normal form is the signed graph $t|\Gamma|$ which represents the same polydifferential operator as $s \Gamma$. Here we let $t=0$ if the graph is zero (see Example 3 below).
Example 3 (Zero Kontsevich graph). Consider the graph with the encoding $\begin{array}{llllllll}2 & 3 & 0 & 1 & 1 & 3\end{array}$. The swap of vertices $2 \rightleftarrows 3$ is a symmetry of this graph, yet it also swaps the ordered edges $(4 \rightarrow 2) \prec(4 \rightarrow 3)$, producing a minus sign. Equal to minus itself, this Kontsevich graph is zero.


Notation. Every Kontsevich graph $\Gamma$ on $n$ sinks (or every sum $\Gamma$ of such graphs) yields the sum Alt $\Gamma$ of Kontsevich graphs which is totally skew-symmetric with respect to the $n$ sinks content $s_{1}, \ldots, s_{n}$. Indeed, let

$$
\begin{equation*}
(\operatorname{Alt} \Gamma)\left(s_{1}, \ldots, s_{n}\right)=\sum_{\sigma \in \mathcal{S}_{n}}(-)^{\sigma} \Gamma\left(s_{\sigma(1)}, \ldots, s_{\sigma(n)}\right) \tag{1}
\end{equation*}
$$

Due to skew-symmetrisation, the sum of graphs Alt $\Gamma$ can contain zero graphs or repetitions.
Example 4 (The Jacobiator). The left-hand side of the Jacobi identity is a skew sum of Kontsevich graphs (e.g. it is obtained by skew-symmetrizing the first term)


The default ordering of edges is the one which we see.

Definition 4 (Leibniz graph). A Leibniz graph is a graph whose vertices are either sinks, or the sources for two arrows, or the Jacobiator (which is a source for three arrows). There must be at least one Jacobiator vertex. The three arrows originating from a Jacobiator vertex must land on three distinct vertices. Each edge falling on a Jacobiator works by the Leibniz rule on the two internal vertices in it.

Example 5. The Jacobiator itself is a Leibniz graph (on one tri-valent internal vertex).
Definition 5 (Normal form of a Leibniz graph with one Jacobiator). Let $\Gamma$ be a Leibniz graph with one Jacobiator vertex Jac. From (2) we see that expansion of Jac into a sum of three Kontsevich graphs means adding one new edge $w \rightarrow v$ (namely joining the internal vertices $w$ and $v$ within the Jacobiator). Now, from $\Gamma$ construct three Kontsevich graphs by expanding Jac using (2) and letting the edges which fall on Jac in $\Gamma$ be directed only to $v$ in every new graph. Next, for each Kontsevich graph find the relabelling $\tau$ which brings it to its normal form and re-express the edge $w \rightarrow v$ using $\tau$. Finally, out of the three normal forms of three graphs pick the minimal one. By definition, the normal form of the Leibniz graph $\Gamma$ is the pair: normal form of Kontsevich graph, that edge $\tau(w) \rightarrow \tau(v)$.

We say that a sum of Leibniz graphs is a skew Leibniz graph Alt $\Gamma$ if it is produced from a given Leibniz graph $\Gamma$ by alternation using formula (1).
Definition 6 (Normal form of a skew Leibniz graph with one Jacobiator). Likewise, the normal form of a skew Leibniz graph Alt $\Gamma$ is the minimum of the normal forms of Leibniz graphs (specifically, of the graph but not edge encodings) which are obtained from $\Gamma$ by running over the group of permutations of the sinks content.
Lemma 5 ([3]). In order to show that a sum $S$ of weighted skew-symmetric Kontsevich graphs vanishes for all Poisson structures $\mathcal{P}$, it suffices to express $S$ as a sum of skew Leibniz graphs: $S=\diamond(\mathcal{P}, \operatorname{Jac}(\mathcal{P}))$.

### 1.1. Formulation of the problem

Let $\mathcal{P} \mapsto \mathcal{P}+\varepsilon \mathcal{Q}(\mathcal{P})+\bar{o}(\varepsilon)$ be a deformation of bi-vectors that preserves their property to be Poisson at least infinitesimally on all affine manifolds: $\llbracket \mathcal{P}+\varepsilon \mathcal{Q}+\bar{o}(\varepsilon), \mathcal{P}+\varepsilon \mathcal{Q}+\bar{o}(\varepsilon) \rrbracket=\bar{o}(\varepsilon)$. Expanding and equating the first order terms, we obtain the equation $\llbracket \mathcal{P}, \mathcal{Q}(\mathcal{P}) \rrbracket \doteq 0$ via $\llbracket \mathcal{P}, \mathcal{P} \rrbracket=0$. The language of Kontsevich graphs allows one to convert this infinite analytic problem within a given set-up $\left(\mathcal{N}^{n}, \mathcal{P}\right)$ in dimension $n$ into a set of finite combinatorial problems whose solutions are universal for all Poisson geometries in all dimensions $n<\infty$.

Our first task in this paper is to find the space of flows $\dot{\mathcal{P}}=\mathcal{Q}(\mathcal{P})$ which are encoded by the Kontsevich graphs on a fixed number of internal vertices $k$, for $1 \leqslant k \leqslant 4$. Specifically, we solve the graph equation

$$
\begin{equation*}
\llbracket \mathcal{P}, \mathcal{Q}(\mathcal{P}) \rrbracket=\diamond(\mathcal{P}, \operatorname{Jac}(\mathcal{P})) \tag{3}
\end{equation*}
$$

for the Kontsevich bi-vector graphs $\mathcal{Q}(\mathcal{P})$ and Leibniz graphs $\diamond$. We then factor out the Poissontrivial and improper solutions, that is, we quotient out all bi-vector graphs that can be written in the form $\mathcal{Q}(\mathcal{P})=\llbracket \mathcal{P}, X \rrbracket+\nabla(\mathcal{P}, \operatorname{Jac}(\mathcal{P}))$, where $X$ is a Kontsevich one-vector graph and $\nabla$ is a Leibniz bi-vector graph. (The bi-vectors $\llbracket \mathcal{P}, X \rrbracket$ make $\llbracket \mathcal{P}, \mathcal{Q}(\mathcal{P}) \rrbracket$ vanish since $\llbracket \mathcal{P}, \cdot \rrbracket$ is a differential. The improper graphs $\nabla(\mathcal{P}, \operatorname{Jac}(\mathcal{P}))$ vanish identically at all Poisson bi-vectors $\mathcal{P}$ on every affine manifold.

Before solving factorisation problem (3) with respect to the operator $\diamond$, we must generate - e.g., iteratively as described below - an ansatz for expansion of the right-hand side using skew Leibniz graphs with undetermined coefficients.

### 1.2. How to generate Leibniz graphs iteratively

The first step is to construct a layer of skew Leibniz graphs, that is, all skew Leibniz graphs which produce at least one graph in the input (in the course of expansion of skew Leibniz graphs using formula (1) and then in the course of expansion of every Leibniz graph at hand to a sum of Kontsevich graphs). For a given Kontsevich graph in the input $S$, one such Leibniz graph can be constructed by contracting an edge between two internal vertices so that the new vertex with three outgoing edges becomes the Jacobiator vertex. Note that these Leibniz graphs, which are designed to reproduce $S$, may also produce extra Kontsevich graphs that are not given in the input. Clearly, if the set of Kontsevich graphs in $S$ coincides with the set of such graphs obtained by expansion of all the Leibniz graphs in the ansatz at hand, then we are done: the extra graphs, not present in $S$, are known to all cancel. Yet it could very well be that it is not possible to express $S$ using only the Leibniz graphs from the set accumulated so far. Then we proceed by constructing the next layer of skew Leibniz graphs that reproduce at least one of the extra Kontsevich graphs (which were not present in $S$ but which are produced by the graphs in the previously constructed layer(s) of Leibniz graphs). In this way we proceed iteratively until no new Leibniz graphs are found; of course, the overall number of skew Leibniz graphs on a fixed number of internal vertices and sinks is bounded from above so that the algorithm always terminates. Note that the Leibniz graphs obtained in this way are the only ones that can in principle be involved in the vanishing mechanism for $S$.
Notation. Let $v$ be a graph vertex. Denote by $N(v)$ the set of neighbours of $v$, by $H(v)$ the (possibly empty) set of arrowheads of oriented edges issued from the vertex $v$, and by $T(v)$ the (possibly empty) set of tails for oriented edges pointing at $v$. For example, $\# N(\bullet)=2$, $\# H(\bullet)=2$, and $T(\bullet)=\varnothing$ for the top $\bullet$ of the wedge graph $\leftarrow \bullet \rightarrow$.

Algorithm Consider a skew-symmetric sum $S_{0}$ of oriented Kontsevich graphs with real coefficients. Let $S_{\text {total }}:=S_{0}$ and create an empty table $L$. We now describe the $i$ th iteration of the algorithm $(i \geqslant 1)$.

Loop $\circlearrowright$ Run over all Kontsevich graphs $\Gamma$ in $S_{i-1}$ : for each internal vertex $v$ in a graph $\Gamma$, run over all vertices $w \in T(v)$ in the set of tails of oriented edges pointing at $v$ such that $v \notin T(w)$ and $H(v) \cap H(w)=\varnothing$ for the sets of targets of oriented edges issued from $v$ and $w$. Replace the edge $w \rightarrow v$ connecting $w$ to $v$ by Jacobiator (2), that is, by a single vertex Jac with three outgoing edges and such that $T(\mathrm{Jac})=(T(v) \backslash w) \cup T(w)$ and $H(\mathrm{Jac})=H(v) \cup(H(w) \backslash v)=:\{a, b, c\}$.

Because we shall always expand the skew Leibniz graphs in what follows, we do not actually contract the edge $w \rightarrow v$ (to obtain a Leibniz graph explicitly) in this algorithm but instead we continue working with the original Kontsevich graphs containing the distinct vertices $v$ and $w$.

For every edge that points at $w$, redirect it to $v$. Sum over the three cyclic permutations that provide three possible ways to attach the three outgoing edges for $v$ and $w$ (excluding $w \rightarrow v$ ) - now seen as the outgoing edges of the Jacobiator - to the target vertices $a, b$, and $c$ depending on $w$ and $v$. Skew-symmetrise ${ }^{5}$ each of these three graphs with respect to the sinks by applying formula (11).

For every marked edge $w \rightarrow v$ indicating the internal edge in the Jacobiator vertex in a graph, replace each sum of the Kontsevich graphs which is skew with respect to the sink content by using the normal form of the respective skew Leibniz graph, see Definition 6. If this skew Leibniz graph is not contained in $L$, apply the Leibniz rule(s) for all the derivations acting on the Jacobiator vertex Jac. Otherwise speaking, sum over all possible ways to attach the incoming edges of the target $v$ in the marked edge $w \rightarrow v$ to its source $w$ and target. To each Kontsevich

[^16]graph resulting from a skew Leibniz graph at hand assign the same undetermined coefficient, and add all these weighted Kontsevich graphs to the sum $S_{i}$. Further, add a row to the table $L$, that new row containing the normal form of this skew Leibniz graph (with its coefficient that has been made common to the Kontsevich graphs).

By now, the new sum of Kontsevich graphs $S_{i}$ is fully composed. Having thus finished the current iteration over all graphs $\Gamma$ in the set $S_{i-1}$, redefine the algebraic sum of weighted graphs $S_{\text {total }}$ by subtracting from it the newly formed sum $S_{i}$. Collect similar terms in $S_{\text {total }}$ and reduce this sum of Kontsevich graphs modulo their skew-symmetry under swaps $L \rightleftarrows R$ of the edge ordering in every internal vertex, so that all zero graphs (see Example (3) also get eliminated.
$\circlearrowright$ end loop
Increment $i$ by 1 and repeat the iteration until the set of weighted (and skew) Leibniz graphs $L$ stabilizes. Finally, solve - with respect to the coefficients of skew Leibniz graphs - the linear algebraic system obtained from the graph equation $S_{\text {total }}=0$ for the sum of Kontsevich graphs which has been produced from its initial value $S_{0}$ by running the iterations of the above algorithm.

Example 6. For the skew sum of Kontsevich graphs in the right-hand side of (21), the algorithm would produce just one skew Leibniz graph: namely, the Jacobiator itself.
Example 7 (The 3 -wheel). For the Kontsevich tetrahedral flow $\dot{\mathcal{P}}=\mathcal{Q}_{1: 6 / 2}(\mathcal{P})$ on the spaces of Poisson bi-vectors $\mathcal{P}$, see [8, 9] and [1, 2], building a sufficient set of skew Leibniz graphs in the r.-h.s. of factorisation problem (3) requires two iterations of the above algorithm: 11 Leibniz graphs are produced at the first step and 50 more are added by the second step, making 61 in total. One of the two known solutions of this factorisation problem [2] then consists of 8 skew Leibniz graphs (expanding to 27 Leibniz graphs). In turn, as soon as all the Leibniz rules acting on the Jacobiators are processed and every Jacobiator vertex is expanded via (2), the right-hand side $\diamond(\mathcal{P}, \operatorname{Jac}(\mathcal{P}))$ equals the sum of 39 Kontsevich graphs which are assembled into the 9 totally skew-symmetric terms in the left-hand side $\llbracket \mathcal{P}, \mathcal{Q}_{1: 6 / 2} \rrbracket$.
Example 8 (The 5 -wheel). Consider the factorisation problem $\llbracket \mathcal{P}, \mathrm{O} \overrightarrow{\mathrm{r}}\left(\gamma_{5}\right) \rrbracket=\diamond(\mathcal{P}, \operatorname{Jac}(\mathcal{P}))$ for the pentagon-wheel deformation $\dot{\mathcal{P}}=\operatorname{Or}\left(\gamma_{5}\right)(\mathcal{P})$ of Poisson bi-vectors $\mathcal{P}$, see [10, 13] and 6]. The ninety skew Kontsevich graphs encoding the bi-vector $\operatorname{Or}\left(\gamma_{5}\right)$ are obtained by taking all the admissible orientations of two $(5+1)$-vertex graphs $\gamma_{5}$, one of which is the pentagon wheel with five spokes, the other graph complementing the former to a cocycle in the non-oriented graph complex. By running the iterations of the above algorithm for self-expanding construction of the Leibniz tri-vector graphs in this factorisation problem, we achieve stabilisation of the number of such graphs after the seventh iteration, see Table 1 below.

Table 1. The number of skew Leibniz graphs produced iteratively for $\llbracket \mathcal{P}, \mathrm{O} \overrightarrow{\mathrm{r}}\left(\gamma_{5}\right) \rrbracket$.

| No. iteration $i$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| \# of graphs | 1518 | 14846 | 41031 | 54188 | 56318 | 56503 | 56509 | 56509 |
| of them new | all | +13328 | +26185 | +13157 | +2130 | +185 | +6 | none |

## 2. Generating the Kontsevich multi-vector graphs

Let us return to problem (31): it is the ansatz for bi-vector Kontsevich graphs $\mathcal{Q}(\mathcal{P})$ with $k$ internal vertices, as well as the Kontsevich 1 -vectors $X$ with $k-1$ internal vertices (to detect trivial terms $\mathcal{Q}(\mathcal{P})=\llbracket \mathcal{P}, \mathcal{X} \rrbracket)$ which must be generated at a given $k$. (At $1 \leqslant k \leqslant 4$, one can still
expand with respect to all the Leibniz graphs in the r.-h.s. of (3), not employing the iterative algorithm from $\$ 1.2$, So, a generator of the Kontsevich (and Leibniz) tri-vectors will also be described presently.)

The Kontsevich graphs corresponding to $n$-vectors are those graphs with $n$ sinks (each containing the respective argument of $n$-vector) in which exactly one arrow comes into each sink, so that the order of the differential operator encoded by an $n$-vector graph equals one w.r.t. each argument, and which are totally skew-symmetric in their $n$ arguments. Let us explain how one can economically obtain the set of one-vectors and skew-symmetric bi- and tri-vectors with $k$ internal vertices in three steps (including graphs with eyes $\bullet \rightleftarrows \bullet$ but excluding graphs with tadpoles). This approach can easily be extended to the construction of $n$-vectors with any $n \geqslant 1$.

### 2.1. One-vectors

Each one-vector under study is encoded by a Kontsevich graph with one sink. Since the sink has one incoming arrow, there is an internal vertex as the tail of this incoming arrow. The target of another edge issued from this internal vertex can be any internal vertex other then itself.
Step 1. Generate all Kontsevich graphs on $k-1$ internal vertices and one sink (i.e. graphs including those with eyes yet excluding those with tadpoles, and not necessarily of differential order one with respect to the sink content).
Step 2. For every such graph with $k-1$ internal vertices, add the new sink and make it a target of the old sink, which itself becomes the $k$ th internal vertex. Now run over the $k-1$ internal vertices excluding the old sink and - via the Leibniz rule - make every such internal vertex the second target of the old sink.

### 2.2. Bi-vectors

There are two cases in the construction of bi-vectors encoded by the Kontsevich graphs. At all $k \geqslant 1$ the first variant is referred to those graphs with an internal vertex that has both sinks as targets.
Variant 1: Step 1. Generate all $k$-vertex graphs on $k-1$ internal vertices and one sink.
Variant 1: Step 2. For every such graph, add two new sinks and proclaim them as targets of the old sink.

Note that the obtained graphs are skew-symmetric.
The second variant produces those graphs which contain two internal vertices such that one has the first sink as target and the other has the second sink as target. The second target of either such internal vertex can be any internal vertex other then itself. Note that for $k=1$ only the first variant applies.
Variant 2: Step 1. Generate all $k$-vertex Kontsevich graphs on $k-2$ internal vertices and two sinks. These sinks now become the $(k-1)$ th and $k$ th internal vertices.
Variant 2: Step 2. For every such graph, add two new sinks, make the first new sink a target of the first old sink and make the second new sink a target of the second old sink. Now run over the $k-1$ internal vertices excluding the first old sink, each time proclaiming an internal vertex the second target of the first old sink. Simultaneously, run over the $k-1$ internal vertices excluding the second old sink and likewise, declare an internal vertex to be the second target of the second old sink.
Variant 2: Step 3. Skew-symmetrise each graph with respect to the content of two sinks using (11).

### 2.3. Tri-vectors

For $k \geqslant 3$, there exist two variants of tri-vectors. The first variant at all $k \geqslant 2$ yields those Kontsevich graphs with two internal vertices ${\underset{8}{8}}^{8}$ that one has two of the three sinks as its
targets while another internal vertex has the third sink as one of its targets. The second target of this last vertex can be any internal vertex other then itself. The second variant contains those graphs with three internal vertices such that the first one has the first sink as a target, the second one has the second sink as a target, and the third one has the third sink as a target. For each of these three internal vertices with a sink as target, the second target can be any internal vertex other then itself.
Variant 1: Step 1. Generate all $k$-vertex Kontsevich graphs on $k-2$ internal vertices and two sinks.
Variant 1: Step 2. For every such graph, add three new sinks, make the first two new sinks the targets of the first old sink and make the third new sink a target of the second old sink. Now run over the $k-1$ internal vertices excluding the second old sink and every time declare an internal vertex the second target of the second old sink.
Variant 1: Step 3. Skew-symmetrise all graphs at hand by applying formula (1) to each of them.
Note that for $k=1$ there are no tri-vectors encoded by Kontsevich graphs and also note that for $k=2$ only the first variant applies.
Variant 2: Step 1. Generate all Kontsevich graphs on $k-3$ internal vertices and three sinks.
Variant 2: Step 2. For every such graph, add three new sinks, make the first new sink a target of the first old sink, make the second new sink a target of the second old sink and make the third new sink a target of the third old sink. Now run over the $k-1$ internal vertices excluding the first old sink and declare every such internal vertex the second target of the first old sink. Independently, run over the $k-1$ internal vertices excluding the second old sink and declare each internal vertex to be the second target of the second old sink. Likewise, run over the $k-1$ internal vertices excluding the third old sink and declare each internal vertex to be the second target of the third old sink.
Variant 2: Step 3 Skew-symmetrise all the graphs at hand using (11).

### 2.4. Non-iterative generator of the Leibniz n-vector graphs

The following algorithm generates all Leibniz graphs with a prescribed number of internal vertices and sinks. Note that not only multi-vectors, but also all graphs of arbitrary differential order with respect to the sinks can be generated this way.
Step 1: Generate all Kontsevich graphs of prescribed type on $k-1$ internal vertices and $n$ sinks, e.g., all $n$-vectors.

Step 2: Run through the set of these Kontsevich graphs and in each of them, run through the set of its internal vertices $v$. For every vertex $v$ do the following: re-enumerate the internal vertices so that this vertex is enumerated by $k-1$. This vertex already targets two vertices, $i$ and $j$, where $i<j<k-1$. Proclaim the last, $(k-1)$ th vertex to be the placeholder of the Jacobiator (see (21), so we must still add the third arrow. Let a new index $\ell$ run up to $i-1$ starting at $n$ if only the $n$-vectors are produced ${ }^{6}$ For every admissible value of $\ell$, generate a new graph where the $\ell$ th vertex is proclaimed the third target of the Jacobiator vertex $k-1$. (Restricting $\ell$ by $<i$, we reduce the number of possible repetitions in the set of Leibniz graphs. Indeed, for every triple $\ell<i<j$, the same Leibniz graph in which the Jacobiator stands on those three vertices would be produced from the three Kontsevich graphs: namely, those in which the $(k-1)$ th vertex targets at the $\ell$ th and $i$ th, at the $\ell$ th and $j$ th, and at the $i$ th and $j$ th vertices. In these three cases it is the $j$ th, $i$ th, and $\ell$ th vertex, respectively, which would be appointed by the algorithm as the Jacobiator's third target.)

We use this algorithm to generate the Leibniz tri- and bi-vector graphs: to establish Theorem[6, we list all possible terms in the right-hand side of factorisation problem (3) at $k \leqslant 4$

[^17]and then we filter out the improper bi-vectors in the found solutions $\mathcal{Q}(\mathcal{P})$.
Remark 2. There are at least 265,495 Leibniz graphs on 3 sinks and 6 internal vertices of which one is the Jacobiator vertex. When compared with Table 1 on p . 7 , this estimate suggests why at large $k \gtrsim 5$, the breadth-first-search iterative algorithm from $\$ 1.2$ generates a smaller number of the Leibniz tri-vector graphs, namely, only the ones which can in principle be involved in the factorisation under study.

## 3. Main result

Theorem $6(k \leqslant 4)$. The few-vertex solutions of problem (3) are these (note that disconnected Kontsevich graphs in $\mathcal{Q}(\mathcal{P})$ are allowed!):

- $k=1$ : The dilation $\dot{\mathcal{P}}=\mathcal{P}$ is a unique, nontrivial and proper solution.
- $k=2$ : No solutions exist (in particular, neither trivial nor improper).
- $k=3$ : There are no solutions: neither Poisson-trivial nor Leibniz bi-vectors.
- $k=4:$ A unique nontrivial and proper solution is the Kontsevich tetrahedral flow $\mathcal{Q}_{1: \frac{6}{2}}(\mathcal{P})$ from Example 2 (see [8, 9] and [1, 2]). There is a one-dimensional space of Poisson trivial (still proper) solutions $\llbracket \mathcal{P}, X \rrbracket$; the Kontsevich 1-vector $X$ on three internal vertices is drawn in [2, App. F]. Intersecting with the former by $\{0\}$, there is a three-dimensional space of improper (still Poisson-nontrivial) solutions of the form $\nabla(\mathcal{P}, \operatorname{Jac}(\mathcal{P}))$.

None of the solutions $\mathcal{Q}(\mathcal{P})$ known so far contains any 2 -cycles (or "eyes" $\bullet \bullet) \cdot 7$
We now report a classification of Poisson bi-vector symmetries $\dot{\mathcal{P}}=\mathcal{Q}(\mathcal{P})$ which are given by those Kontsevich graphs $\mathcal{Q}=\mathrm{O} \overrightarrow{\mathbf{r}}(\gamma)$ on $k$ internal vertices that can be obtained at $5 \leqslant k \leqslant 9$ by orienting $k$-vertex connected graphs $\gamma$ without double edges. By construction, this extra assumption keeps only those Kontsevich graphs which may not contain eyes.

We first find such graphs $\gamma$ that satisfy $\mathrm{d}(\gamma)=0$, then we exclude the coboundaries $\gamma=\mathrm{d}\left(\gamma^{\prime}\right)$ for some graphs $\gamma^{\prime}$ on $k-1$ vertices and $2 k-3$ edges.

Theorem $7(5 \leqslant k \leqslant 8)$. Consider the vector space of non-oriented connected graphs on $k$ vertices and $2 k-2$ edges, without tadpoles and without multiple edges. All nontrivial dcocycles for $5 \leqslant k \leqslant 8$ are exhausted by the following ones:

- $k=5,7$ : No solutions.
- $k=6$ : A unique solution 8 is given by the Kontse-vich-Willwacher pentagon-wheel cocycle (see Example 11). The established factorisation
 $\llbracket \mathcal{P}, \mathrm{O}\left(\gamma_{5}\right) \rrbracket=\diamond(\mathcal{P}, \operatorname{Jac}(\mathcal{P}))$ will be addressed in a separate paper $($ see [6]).
$\bullet k=8$ : The only solution $\gamma_{7}$ consists of the heptagon-wheel and 45 other graphs (see Table 2, in which the coefficient of heptagon graph is $\mathbf{1}$ in bold, and [5]).
Remark 3. The wheel graphs are built of triangles. The differential d cannot produce any triangle since multiple edges are not allowed. Therefore, all wheel cocycles are nontrivial. Note also that every wheel graph with $2 \ell$ spokes is invariant under a mirror reflection with respect to a diagonal consisting of two edges attached to the centre. Hence there exists an edge permutation that swaps $2 \ell-1$ pairs of edges. By footnote 1 such graph equals zero.


## Appendix A. How the orientation mapping $O \vec{r}$ is calculated

The algorithm lists all ways in which a given non-oriented graph can be oriented in such a way that it becomes a Kontsevich graph on two sinks. It consists of two steps:

[^18]Table 2. The heptagon-wheel graph cocycle $\gamma_{7}$.

| Graph encoding | Coeff. | Graph encoding | Coeff. |
| :---: | :---: | :---: | :---: |
| 1617182325283438464857586878 | 1 | 1213182526373845464756576878 | -7 |
| 1214182327353746485758676878 | -21/8 | 1214162325363745485758676878 | 77/8 |
| 1314182325283746485657676878 | -77/4 | 1316172425263537454858676878 | $-7$ |
| 1213152427353646485758676878 | -35/8 | 1415172326283738464856576878 | 49/4 |
| 1213182426373846475657586878 | 49/8 | 1216182728343638464756575878 | $-147 / 8$ |
| 1417182325263537464856586778 | 77/8 | 1215162728353638454647576878 | -21/8 |
| 1213182627353845464756576878 | $-105 / 8$ | 1214182327353645465758676878 | $-35 / 8$ |
| 1214182327363846485657586778 | 7/8 | 1415162326283738464857586778 | -49/4 |
| 1214152327353646485758676878 | 35/8 | 1215182328343746485657676878 | 105/8 |
| 1213142728363846475657586878 | $-49 / 8$ | 1214172326373846485657586878 | -49/8 |
| 1213182527343647485658676878 | 35/4 | 1216182527353637454648576878 | 49/16 |
| 1213142526363845475758676878 | -119/16 | 1213182527353646474856576878 | 7 |
| 1213152428363847485657676878 | 49/8 | 1214182528343638475758676878 | $-7$ |
| 1213142328374648565758676878 | 77/4 | 1216182527353637454648586778 | $-77 / 16$ |
| 1215172526353638454748676878 | -49/8 | 1214182327353846475758676878 | 77/4 |
| 1315182426283738464756576878 | -49/4 | 1214152327363846485758676878 | $35 / 2$ |
| 1314182526283638474856576778 | -49/4 | 1213182527343646485758676878 | $-105 / 8$ |
| 1214182328353746485657676878 | -7 | 1215162527353638464748576878 | -7 |
| 1214182328363846475657586778 | -7 | 1213162528343747485758676878 | $-147 / 16$ |
| 1215162527353638464748586778 | 49/8 | 1213172526353745464858676878 | $-77 / 4$ |
| 1214182328363746475657586878 | 49/8 | 1214172327353846485758676878 | $-49 / 8$ |
| 1213152627353645474858676878 | $-7$ | 1213152628353745464758676878 | $-7 / 4$ |
| 1213182428353846475758676878 | 7 | 1214182326363847485657586778 | $-7$ |

(i) choosing the source(s) of the two arrows pointing at the first and second sink, respectively;
(ii) orienting the edges between the internal vertices in all admissible ways, so that only Kontsevich graphs are obtained.

Step 1. Enumerate the $k$ vertices of a given non-oriented, connected graph using $2, \ldots, k+1$. They become the internal vertices of the oriented graph. Now add the two sinks to the nonoriented graph, the sinks enumerated using 0 and 1 . Let $a$ and $b$ be a non-strictly ordered $(a \leqslant b)$ pair of internal vertices in the graph. Extend the graph by oriented edges $a \rightarrow 0$ and $b \rightarrow 1$ from vertices $a$ and $b$ to the sinks 0 and 1 , respectively.
Remark 4. The choice of such a base pair, that is, the vertex or vertices from which two arrows are issued to the sinks, is an external input in the orientation procedure. Let us agree that if, at any step of the algorithm, a contradiction is achieved so that a graph at hand cannot be of Kontsevich type, the oriented graph draft is discarded; one proceeds with the next options in that loop, or if the former loop is finished, with the next level-up loops, or - having returned to the choice of base vertices - with the next base. In other words, we do not exclude in principle a possibility to have no admissible orientations for a particular choice of the base for a given non-oriented graph.

Notation. Let $v$ be an internal vertex. Recalling from p . 6 the notation for the set $N(v)$ of neighbours of $v$, the (initially empty) set $H(v)$ of arrowheads of oriented edges issued from the vertex $v$, and the (initially empty) set $T(v)$ of tails for oriented edges pointing at $v$, we now put by definition $F(v):=N(v) \backslash(H(v) \cup T(v))$. In other words, $F(v)$ is the subset of neighbours connected with $v$ by a non-oriented edge.
Step 2.1. Inambiguous orientation of (some) edges. Here we use that every internal vertex of a Kontsevich graph should be the tail of exactly two outgoing arrows. We run over the
set of all internal vertices $v$. For every vertex such that the number of elements $\# H(v)=2$, proclaim $T(v):=N(v) \backslash H(v)$, whence $F(v)=\varnothing$. If for a vertex $v$ we have that $\# H(v)=1$ and $\# F(v)=1$, then include $F(v) \hookrightarrow H(v)$, that is, convert a unique non-oriented edge touching $v$ into an outgoing edge issued from this vertex. If $\# H(v)=0$ and $\# F(v)=2$, also include $F(v) \hookrightarrow H(v)$, effectively making both non-oriented edges outgoing from $v$.

Repeat the three parts of Step 2.1 while any of the sets $F(v), T(v)$, or $S(v)$ is modified for at least one internal vertex $v$ unless a contradiction is revealed. Summarising, Step 2.1 amounts to finding the edge orientations which are implied by the choice of the base pair $a, b$ and by all the orientations of edges fixed earlier.
Step 2.2. Fixing the orientation of (some) remaining edges. Choose an internal vertex $v$ such that $H(v)<2$ and such that $H(v) \neq \varnothing$ or $T(v) \neq \varnothing$, that is, choose a vertex that is not yet equipped with two outgoing edges and that is attached to an oriented edge. If $\# H(v)=1$, then run over the non-empty set $F(v)$ : for every vertex $w$ in $F(v)$, include $\{w\} \hookrightarrow H(v)$ and start over at Step 2.1. Otherwise, i.e. if $H(v)=\varnothing$, run over all ordered pairs $(u, v)$ of vertices in the set $F(v)$ : for every such pair, make $H(v):=\{u, w\}$ and start over at Step 2.1.

By realising Steps 1 and 2 we accumulate the sum of fully oriented Kontsevich graphs.

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# Poisson brackets symmetry from the pentagon-wheel cocycle in the graph complex 

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#### Abstract

Kontsevich designed a scheme to generate infinitesimal symmetries $\dot{\mathcal{P}}=\mathcal{Q}(\mathcal{P})$ of Poisson brackets $\mathcal{P}$ on all affine manifolds $M^{r}$; every such deformation is encoded by oriented graphs on $n+2$ vertices and $2 n$ edges. In particular, these symmetries can be obtained by orienting sums of non-oriented graphs $\gamma$ on $n$ vertices and $2 n-2$ edges. The bi-vector flow $\dot{\mathcal{P}}=\mathrm{O}(\gamma)(\mathcal{P})$ preserves the space of Poisson structures if $\gamma$ is a cocycle with respect to the vertex-expanding differential in the graph complex.

A class of such cocycles $\gamma_{2 \ell+1}$ is known to exist: marked by $\ell \in \mathbb{N}$, each of them contains a $(2 \ell+1)$-gon wheel with a nonzero coefficient. At $\ell=1$ the tetrahedron $\gamma_{3}$ itself is a cocycle; at $\ell=2$ the Kontsevich-Willwacher pentagon-wheel cocycle $\gamma_{5}$ consists of two graphs. We reconstruct the symmetry $\mathcal{Q}_{5}(\mathcal{P})=\mathrm{O} \overrightarrow{\mathrm{r}}\left(\gamma_{5}\right)(\mathcal{P})$ and verify that $\mathcal{Q}_{5}$ is a Poisson cocycle indeed: $\llbracket \mathcal{P}, \mathcal{Q}_{5}(\mathcal{P}) \rrbracket \doteq 0$ via $\llbracket \mathcal{P}, \mathcal{P} \rrbracket=0$.


Generic classical Poisson brackets $\mathcal{P}$ can be deformed along no less than countably many directions (in the spaces of bi-vectors) such that they stay Poisson at least infinitesimally and the change of brackets is not necessarily induced by a diffeomorphim along integral curves of a vector field on the Poisson manifold at hand The use of graphs converts this infinite analytic problem into a set of finite combinatorial problems of finding cocycles $\gamma \in \operatorname{ker} \mathrm{d}$ in the graph complex and orienting them: $\mathcal{Q}(\mathcal{P})=\operatorname{Or}(\gamma)(\mathcal{P})$, see the diagram.

| cocycles $\in$ kerd: sums of $n$-vertex ( $2 n-2$ )-edge nonoriented graphs with $\mathrm{E}(\gamma)=\bigwedge_{i} e_{i}$ and coeff $\in \mathbb{R}$ | $\xrightarrow[\substack{\text { make } \\ \text { skew }}]{\mathrm{O} \overrightarrow{\mathrm{r}}}$ | \|sums of Kontsevich graphs $\mathcal{Q}$ on 2 sinks, $n$ internal vertices, and $2 n$ edges in $n \times(\underset{L}{\overleftarrow{ }} \bullet \underset{R}{\longrightarrow})$ with Left $\prec$ Right | $\xrightarrow[\text { into }]{\stackrel{\text { put }}{\mathcal{P}}}$ | $\left\lvert\, \begin{aligned} & \text { bi-vector fields } \\ & \mathcal{Q}(\mathcal{P})=\mathrm{Or}(\gamma)(\mathcal{P}) \text { : } \\ & \text { Poisson 2-cocycles } \\ & \in \operatorname{ker} \partial_{\mathcal{P}}=\llbracket \mathcal{P}, \cdot \rrbracket \end{aligned}\right.$ |
| :---: | :---: | :---: | :---: | :---: |

1. Graph complex theory. There are several ways to introduce a differential on the space of non-oriented graphs (see [7, [8]). We consider the real vector space of finite non-oriented graphs such that each of them is equipped with a wedge product of edges, i.e. we suppose that for every graph its edges $e_{i}$ are enumerated $I, I I, \ldots$ and proclaimed parity-odd, so that $\mathrm{E}(\gamma):=\bigwedge_{i} e_{i}$ and $(\gamma, I \wedge I I \wedge I I I \wedge \ldots)=-(\gamma, I I \wedge I \wedge I I I \wedge \ldots)$, etc.
[^19]Suppose also that all vertices are at least tri-valent (cf. [4, 9]). On this subspace (which we study here), the differential amounts to a blow-up - via the Leibniz rule - of vertices in a graph $\gamma$; every vertex $v$ at hand is replaced by the new edge $E$ such that every edge which was incident to $v$ in $\gamma$ is now re-directed to one of the two ends of $E$. The choice where to direct a given edge does not depend on a similar choice for other such edges, but overall, the valency of either end of $E$ must be at least two 2 By construction, the new edge $E$ is placed firstmost in the wedge product of edges in every graph $g$ in $\mathrm{d}(\gamma)$ : whenever $\mathrm{E}(\gamma)=I \wedge I I \wedge \ldots$, let $\mathrm{E}(g)=E \wedge I \wedge I I \wedge \ldots$. Now one has $\mathrm{d}^{2}=0$.
Example 1. Let $w_{4}:=\square$, and let the edge ordering in these graphs be lexicographic:


A flip over a diagonal in $w_{4}$ swaps three pairs of edges; 3 is odd, so by this symmetry, $\mathrm{E}\left(w_{4}\right)=-\mathrm{E}\left(w_{4}\right)$, i.e. $w_{4}$ is a zero graph $\sqrt[3]{ }$ By this, $\mathrm{d}\left(w_{4}\right)=0$. Because $\mathrm{d}^{2}=0$, one has $\mathrm{d}\left(\delta_{6}\right)=0$ for the coboundary $\delta_{6} \in$ imd. Put $\gamma_{3}:=\Delta$; another example of nontrivial cocycle, $\boldsymbol{\gamma}_{5} \notin \mathrm{imd}$, also on $n$ vertices and $2 n-2$ edges, is given on p. 3.

The notion of oriented Kontsevich graphs from [7] was recalled in [1, 2, 5]. Every such graph is built over $m$ ordered sinks from $n$ wedges $\stackrel{L_{L}}{\leftarrow} \bullet \xrightarrow{R}$ : each top $\bullet$ of the wedge is the source of exactly two arrows (which are ordered by Left $\prec \operatorname{Right}$ ). Let ( $M^{r}, \mathcal{P}$ ) be a real affine Poisson manifold of dimension $r$; let $x^{1}, \ldots, x^{r}$ be local coordinates. By decorating each edge with its own summation index that runs from 1 to $r$, by identifying every such edge $\xrightarrow{i}$ with $\partial / \partial x^{i}$ acting on the content of arrowhead vertex, and by placing a copy of the Poisson bivector $\mathcal{P}=\left(\mathcal{P}^{i j}\right)$ at the top $\bullet$ of each wedge $\stackrel{i}{\leftarrow} \bullet \stackrel{j}{\rightarrow}$, we associate a polydifferential operator (e.g., an $m$-vector) with every such graph. The arguments of the operator are contained in the $m$ respective sinks. The resulting polydifferential operators are differential-polynomial in the coefficients $\mathcal{P}^{i j}$ of a given Poisson structure $\mathcal{P}$. It is known that for $\mathcal{P}$ Poisson (hence $\llbracket \mathcal{P}, \mathcal{P} \rrbracket=0$ under the Schouten bracket), its adjoint action $\partial_{\mathcal{P}}:=\llbracket \mathcal{P}, \rrbracket \rrbracket$ is a differential on the space of multi-vectors. One can try finding Poisson cohomology cocycles $\mathcal{Q} \in \operatorname{ker} \partial_{\mathcal{P}}$ by assuming they are realized using the Kontsevich oriented graphs.

Now let us note that certain sums $\mathcal{Q}$ of oriented graphs built on two sinks from $n$ wedges can be obtained by taking all admissible ways to orient graphs $\gamma$ on $n$ vertices and $2 n-2$ edges (clearly, two sinks and two edges into them are added). Moreover, suppose that $\gamma \in \operatorname{ker} \mathrm{d}$ in vertex-edge bi-grading $(n, 2 n-2)$ is such that this sum of graphs can be oriented to yield a sum of Kontsevich graphs on two sinks, $n$ internal vertices and $2 n$ edges. Then, in fact, these oriented graphs, taken with suitable coefficients $\in \mathbb{R}$, do assemble to a Poisson cocycle $\mathcal{Q}(\mathcal{P}) \in \operatorname{ker} \partial_{\mathcal{P}}$. Let this orientation mapping be denoted by $\mathrm{Or}\left(\mathrm{cf}\right.$. 7 ] and [1, 5]), $\mathbf{4}^{4}$
2. The pentagon-wheel cocycle. The mechanism of factorization $\llbracket \mathcal{P}, \mathcal{Q}(\mathcal{P}) \rrbracket \doteq 0$ via $\llbracket \mathcal{P}, \mathcal{P} \rrbracket=0$ for the cocycle condition $\mathcal{Q}(\mathcal{P}) \in \operatorname{ker} \partial_{\mathcal{P}}$ is known from [2], where it is used in a similar problem of the $\star$-product associativity (cf. [3]). In [1] this mechanism is applied to the Kontsevich tetrahedral flow $\mathcal{Q}_{3}(\mathcal{P})=\operatorname{Or}\left(\gamma_{3}\right)(\mathcal{P})$. Would the mapping Or be known, the verification $\operatorname{Or}(\gamma) \in \operatorname{ker} \partial_{\mathcal{P}}$ is still compulsory (e.g., by using a factorization via the Jacobi identity for $\mathcal{P})$. But for us now, the factorization $\llbracket \mathcal{P}, \mathcal{Q}_{5}(\mathcal{P}) \rrbracket=\diamond(\mathcal{P}, \llbracket \mathcal{P}, \mathcal{P} \rrbracket)$ is the way to

[^20]find the right formula of the flow $\dot{\mathcal{P}}=\mathcal{Q}_{5}(\mathcal{P})$ that should correspond to the KontsevichWillwacher pentagon-wheel cocycle $\boldsymbol{\gamma}_{5}$ under the orientation mapping, $\mathcal{Q}_{5}=\mathrm{O} \overrightarrow{\mathrm{r}}\left(\boldsymbol{\gamma}_{5}\right)$, giving one solution $\mathcal{Q}_{5}$ yet not necessarily unique operator $\diamond$.
Example 2. There are only two essentially different admissible ways to orient (and skewsymmetrize with respect to sinks) the tetrahedron $\gamma_{3} \in \operatorname{ker} \mathrm{~d}$. Each of the three oriented graphs in the flow $\mathcal{Q}_{3}$ is encoded by the list of targets for the ordered pair of edges issued from the $i$ th vertex $(m=2 \leqslant i \leqslant 5=m+n-1)$, and a coefficient $\in \mathbb{Z}$. Specifically, we have that $\mathcal{Q}_{3}=1 \cdot(0,1 ; 2,4 ; 2,5 ; 2,3)-3 \cdot(0,3 ; 1,4 ; 2,5 ; 2,3+0,3 ; 4,5 ; 1,2 ; 2,4)$; the analytic formula of the respective bi-differential operators acting on the sinks content $f$, $g$ is $\mathcal{Q}_{3}(f, g)=\partial_{k m p} \mathcal{P}^{i j} \partial_{q} \mathcal{P}^{k \ell} \partial_{\ell} \mathcal{P}^{m n} \partial_{n} \mathcal{P}^{p q} \cdot \partial_{i} f \partial_{j} g-3 \partial_{m p} \mathcal{P}^{i j} \partial_{j q} \mathcal{P}^{k \ell} \partial_{\ell} \mathcal{P}^{m n} \partial_{n} \mathcal{P}^{p q} \cdot \partial_{i} f \partial_{k} g-$ $3 \partial_{n p} \mathcal{P}^{i j} \partial_{j} \mathcal{P}^{k \ell} \partial_{k q} \mathcal{P}^{m n} \partial_{\ell} \mathcal{P}^{p q} . \partial_{i} f \partial_{m} g$. A factorization of $\llbracket \mathcal{P}, \mathcal{Q}_{3}(\mathcal{P}) \rrbracket$ via 8 tri-vector graphs containing $\llbracket \mathcal{P}, \mathcal{P} \rrbracket$ is explained in [1], based on [2].

Now consider the pentagon-wheel cocycle $\gamma_{5} \in \operatorname{ker} \mathrm{~d}$, see [4]. By orienting both graphs in $\gamma_{5}$ (i.e. by shifting the vertex labelling by $+1=m-1$, adding two edges
 to the sinks 0 , 1 , and keeping only those oriented graphs out of $1024=2^{\# \text { edges }}$ which are built from $\leftarrow \bullet \rightarrow$ ) and skew-symmetrizing with respect to $0 \rightleftarrows 1$, we obtain 91 parameters for Kontsevich graphs on 2 sinks, 6 internal vertices, and 12 ( $=6$ pairs) of edges. We take the sum $\mathcal{Q}$ of these 91 bi-vector graphs (or skew differences of Kontsevich graphs) with their undetermined coefficients, and for the set of tri-vector graphs occurring in $\llbracket \mathcal{P}, \mathcal{Q} \rrbracket$, we generate all the possibly needed tri-vector "Leibniz" graphs with $\llbracket \mathcal{P}, \mathcal{P} \rrbracket$ inside. 5 This yields 41031 such Leibniz graphs, which, with undetermined coefficients, provide the ansatz for the r.-h.s. of the factorization problem $\llbracket \mathcal{P}, \mathcal{Q}(\mathcal{P}) \rrbracket=\diamond(\mathcal{P}, \llbracket \mathcal{P}, \mathcal{P} \rrbracket)$. This gives us an inhomogeneous system of 463,344 linear algebraic equations for both the coefficients in $\mathcal{Q}$ and $\diamond$. In its l.-h.s., we fix the coefficient of one bi-vector graph ${ }^{6}$ by setting it to $+\mathbf{2}$.

Claim. For $\gamma_{5}$, the factorization problem $\llbracket \mathcal{P}, \mathcal{Q}(\mathcal{P}) \rrbracket=\diamond(\mathcal{P}, \llbracket \mathcal{P}, \mathcal{P} \rrbracket)$ has a solution $\left(\mathcal{Q}_{5}, \widehat{\nabla}_{5}\right)$; the sum $\mathcal{Q}_{5}$ of 167 Kontsevich graphs (on $m=2$ sinks 0,1 and $n=6$ internal vertices 2 , $\ldots, 7)$ with integer coefficients is given in the table below.7

| 012425364724 | 10 | 034512262724 | -10 | 031456232745 | 5 | 031425263725 | 5 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 012425264734 | -10 | 031456233723 | $-10$ | 034526471236 | 5 | 034512464725 | -5 |
| 031425672434 | 10 | 034526271226 | 10 | 031456235724 | -5 | 031425262735 | 5 |
| 034512672334 | -10 | 012425262723 | 2 | 034512672446 | -5 | 031456263723 | -5 |
| 031425264734 | 10 | 012425263734 | -5 | 031425672326 | -5 | 034526472712 | -5 |
| 034512463723 | -10 | 012425363724 | 5 | 031456235723 | -5 | 031456232734 | -5 |
| 031425364724 | -10 | 012425263745 | -5 | 034526471226 | 5 | 034526671223 | -5 |
| 034512263734 | -10 | 012425264735 | -5 | 031425672334 | 5 | 031456233724 | -5 |
| 031456235725 | -10 | 031456275723 | 5 | 034512672324 | -5 | 034526271236 | 5 |
| 034526471246 | 10 | 034556672712 | 5 | 031425364723 | -5 | 031425362735 | -5 |
| 034516245725 | 10 | 031425672436 | 5 | 034512263724 | -5 | 034512264725 | 5 |
| 034526461724 | -10 | 034512672734 | -5 | 031425672336 | -5 | 031425363725 | -5 |
| 034526472714 | -10 | 031425263745 | 5 | 034512672426 | -5 | 034512262745 | 5 |
| 034516243723 | 10 | 034512462735 | -5 | 034512672434 | -5 | 034556671226 | -5 |
| 034526671323 | -10 | 031425264735 | 5 | 031425672324 | 5 | 031456262723 | 5 |
| 034526271336 | 10 | 034512463725 | -5 | 034512463724 | -5 | 012425262734 | -5 |
| 034516472323 | -10 | 034512672346 | 5 | 031425264723 | -5 | 012425263725 | -5 |
| 034515262745 | 10 | 031425672734 | 5 | 012425672734 | -5 | 012425262735 | -5 |
| 034516272334 | 10 | 034512264735 | 5 | 012425362745 | 5 | 034526671246 | 5 |
| 034515264725 | 10 | 031425362745 | -5 | 012425364725 | 5 | 031456232725 | -5 |
| 034512464724 | -10 | 034512263745 | 5 | 012425362735 | 5 | 034512464723 | -5 |
| 031425262723 | -10 | 031425364725 | -5 | 012425363725 | 5 | 031425262734 | 5 |
| 031425363723 | -10 | 034526671234 | 5 | 034512462745 | -5 | (see next page) |  |

[^21]| 034512264734 | -5 | 034515672426 | 5 | 034516272346 | -5 | 034516242725 | 5 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 031425362724 | -5 | 034526271536 | 5 | 034515264723 | 5 | 034516462724 | 5 |
| 031456233725 | -5 | 034516263724 | 5 | 034515262734 | -5 | 034516242723 | 5 |
| 034526271246 | 5 | 034526261734 | -5 | 034516472326 | -5 | 034526475712 | 5 |
| 031456273723 | -5 | 034526471526 | -5 | 034516242745 | -5 | 031456263725 | 5 |
| 034526672712 | -5 | 034516272534 | 5 | 034516272724 | -5 | 034525671246 | -5 |
| 031425363724 | -5 | 034516472526 | 5 | 034516245724 | 5 | 031456273526 | 5 |
| 034512262734 | -5 | 034516472723 | -5 | 034526261724 | -5 | 034525671236 | -5 |
| 031425263734 | 5 | 034516462725 | 5 | 034515264724 | 5 | 031456273524 | 5 |
| 034512462723 | -5 | 034516273524 | -5 | 034516272324 | 5 | 034526673712 | 5 |
| 034516275724 | -5 | 034525671426 | -5 | 034516242734 | 5 | 031456273724 | 5 |
| 034526461725 | -5 | 034526472713 | -5 | 034516262724 | -5 | 034556671223 | 5 |
| 034516272546 | 5 | 034525671326 | -5 | 034516243724 | 5 | 031456262734 | 5 |
| 034516472523 | -5 | 034526671724 | 5 | 034526271526 | 5 | 034512264724 | -5 |
| 034516262745 | 5 | 034516245723 | 5 | 034526671326 | -5 | 031425362723 | -5 |
| 034516272734 | 5 | 034526672714 | -5 | 034526271326 | 5 | 034526671226 | 5 |
| 034526671723 | -5 | 034516243725 | 5 | 034516472324 | -5 | 031456232723 | -5 |
| 034515672324 | 5 | 034526271346 | 5 | 034515262724 | -5 | 034512462724 | -5 |
| 034526461723 | -5 | 034526671324 | -5 | 034516472724 | 5 | 031425263723 | -5 |

Remark. To establish the formula for the morphism Or that would be universal with respect to all cocycles $\gamma \in$ ker d, we are accumulating a sufficient number of pairs (d-cocycle $\gamma$, $\partial_{\mathcal{P}}$-cocycle $\mathcal{Q}$ ), in which $\mathcal{Q}$ is built exactly from graphs that one obtains from orienting the graphs in $\gamma$. Let us remember that not only nontrivial cocycles (e.g., $\gamma_{3}, \gamma_{5}$, or $\gamma_{7}$ from 4], cf. [6, 9]) but also d-trivial, like $\delta_{6}$ on p. 2, or even the 'zero' non-oriented graphs are suited for this purpose: e.g., a unique $\operatorname{Or}\left(w_{4}\right)(\mathcal{P}) \equiv 0$ constrains $\mathrm{O} \overrightarrow{\mathrm{r}}$. In every such case, the respective $\partial_{\mathcal{P}}$-cocycle is obtained ${ }^{d}$ by solving the factorization problem $\llbracket \mathcal{P}, \mathcal{Q}(\mathcal{P}) \rrbracket \doteq 0$ via $\llbracket \mathcal{P}, \mathcal{P} \rrbracket=0$. The formula of the orientation morphism Or will be the object of another paper.
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[^22] $\mathrm{O} \overrightarrow{\mathrm{r}}\left(\gamma_{5}\right)(\mathcal{P})$, uniqueness is not claimed for the operator $\diamond$ in the r.-h.s. of the factorization.)

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## A The pentagon-wheel flow: analytic formula

Here is the value $\mathcal{Q}_{5}(\mathcal{P})(f, g)$ of bi-vector $\mathcal{Q}_{5}$ at two functions $f, g$ :

$$
\begin{aligned}
& 10 \partial_{t} \partial_{m} \partial_{k} \mathcal{P}^{i j} \partial_{p} \mathcal{P}^{k \ell} \partial_{v} \partial_{r} \partial_{\ell} \mathcal{P}^{m n} \partial_{n} \mathcal{P}^{p q} \partial_{q} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{j} g \\
& -10 \partial_{p} \partial_{m} \partial_{k} \mathcal{P}^{i j} \partial_{t} \mathcal{P}^{k \ell} \partial_{v} \partial_{r} \partial_{\ell} \mathcal{P}^{m n} \partial_{n} \mathcal{P}^{p q} \partial_{q} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{j} g \\
& +10 \partial_{r} \partial_{m} \mathcal{P}^{i j} \partial_{t} \partial_{j} \mathcal{P}^{k \ell} \partial_{v} \partial_{s} \partial_{\ell} \mathcal{P}^{m n} \partial_{n} \mathcal{P}^{p q} \partial_{p} \mathcal{P}^{r s} \partial_{q} \mathcal{P}^{t v} \partial_{i} f \partial_{k} g \\
& -10 \partial_{r} \partial_{n} \mathcal{P}^{i j} \partial_{t} \partial_{s} \partial_{j} \mathcal{P}^{k \ell} \partial_{v} \partial_{k} \mathcal{P}^{m n} \partial_{\ell} \mathcal{P}^{p q} \partial_{p} \mathcal{P}^{r s} \partial_{q} \mathcal{P}^{t v} \partial_{i} f \partial_{m} g \\
& +10 \partial_{p} \partial_{m} \mathcal{P}^{i j} \partial_{t} \partial_{j} \mathcal{P}^{k \ell} \partial_{v} \partial_{r} \partial_{\ell} \mathcal{P}^{m n} \partial_{n} \mathcal{P}^{p q} \partial_{q} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{k} g \\
& -10 \partial_{t} \partial_{n} \mathcal{P}^{i j} \partial_{v} \partial_{r} \partial_{j} \mathcal{P}^{k \ell} \partial_{p} \partial_{k} \mathcal{P}^{m n} \partial_{\ell} \mathcal{P}^{p q} \partial_{q} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{m} g \\
& -10 \partial_{t} \partial_{m} \mathcal{P}^{i j} \partial_{p} \partial_{j} \mathcal{P}^{k \ell} \partial_{v} \partial_{r} \partial_{\ell} \mathcal{P}^{m n} \partial_{n} \mathcal{P}^{p q} \partial_{q} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{k} g \\
& -10 \partial_{p} \partial_{n} \mathcal{P}^{i j} \partial_{t} \partial_{r} \partial_{j} \mathcal{P}^{k \ell} \partial_{v} \partial_{k} \mathcal{P}^{m n} \partial_{\ell} \mathcal{P}^{p q} \partial_{q} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{m} g \\
& -10 \partial_{t} \partial_{p} \mathcal{P}^{i j} \partial_{q} \partial_{j} \mathcal{P}^{k \ell} \partial_{\ell} \mathcal{P}^{m n} \partial_{v} \partial_{r} \partial_{m} \mathcal{P}^{p q} \partial_{n} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{k} g \\
& +10 \partial_{s} \partial_{m} \mathcal{P}^{i j} \partial_{j} \mathcal{P}^{k \ell} \partial_{t} \partial_{p} \partial_{k} \mathcal{P}^{m n} \partial_{\ell} \mathcal{P}^{p q} \partial_{v} \partial_{n} \mathcal{P}^{r s} \partial_{q} \mathcal{P}^{t v} \partial_{i} f \partial_{r} g \\
& +10 \partial_{t} \partial_{p} \mathcal{P}^{i j} \partial_{j} \mathcal{P}^{k \ell} \partial_{q} \partial_{k} \mathcal{P}^{m n} \partial_{v} \partial_{r} \partial_{\ell} \mathcal{P}^{p q} \partial_{n} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{m} g \\
& -10 \partial_{t} \partial_{m} \mathcal{P}^{i j} \partial_{j} \mathcal{P}^{k \ell} \partial_{v} \partial_{p} \partial_{k} \mathcal{P}^{m n} \partial_{\ell} \mathcal{P}^{p q} \partial_{q} \partial_{n} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{r} g \\
& -10 \partial_{r} \partial_{m} \mathcal{P}^{i j} \partial_{j} \mathcal{P}^{k \ell} \partial_{v} \partial_{p} \partial_{k} \mathcal{P}^{m n} \partial_{\ell} \mathcal{P}^{p q} \partial_{n} \mathcal{P}^{r s} \partial_{s} \partial_{q} \mathcal{P}^{t v} \partial_{i} f \partial_{t} g \\
& +10 \partial_{t} \partial_{p} \mathcal{P}^{i j} \partial_{v} \partial_{r} \partial_{j} \mathcal{P}^{k \ell} \partial_{q} \partial_{k} \mathcal{P}^{m n} \partial_{\ell} \mathcal{P}^{p q} \partial_{n} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{m} g \\
& -10 \partial_{t} \partial_{m} \mathcal{P}^{i j} \partial_{v} \partial_{s} \partial_{j} \mathcal{P}^{k \ell} \partial_{k} \mathcal{P}^{m n} \partial_{\ell} \mathcal{P}^{p q} \partial_{p} \partial_{n} \mathcal{P}^{r s} \partial_{q} \mathcal{P}^{t v} \partial_{i} f \partial_{r} g \\
& +10 \partial_{p} \partial_{m} \mathcal{P}^{i j} \partial_{t} \partial_{s} \partial_{j} \mathcal{P}^{k \ell} \partial_{k} \mathcal{P}^{m n} \partial_{\ell} \mathcal{P}^{p q} \partial_{v} \partial_{n} \mathcal{P}^{r s} \partial_{q} \mathcal{P}^{t v} \partial_{i} f \partial_{r} g \\
& -10 \partial_{t} \partial_{r} \mathcal{P}^{i j} \partial_{v} \partial_{s} \partial_{j} \mathcal{P}^{k \ell} \partial_{p} \partial_{k} \mathcal{P}^{m n} \partial_{\ell} \mathcal{P}^{p q} \partial_{n} \mathcal{P}^{r s} \partial_{q} \mathcal{P}^{t v} \partial_{i} f \partial_{m} g \\
& +10 \partial_{r} \partial_{p} \mathcal{P}^{i j} \partial_{j} \mathcal{P}^{k \ell} \partial_{t} \partial_{k} \mathcal{P}^{m n} \partial_{v} \partial_{n} \partial_{\ell} \mathcal{P}^{p q} \partial_{q} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{m} g \\
& +10 \partial_{r} \partial_{p} \mathcal{P}^{i j} \partial_{t} \partial_{s} \partial_{j} \mathcal{P}^{k \ell} \partial_{v} \partial_{k} \mathcal{P}^{m n} \partial_{\ell} \mathcal{P}^{p q} \partial_{n} \mathcal{P}^{r s} \partial_{q} \mathcal{P}^{t v} \partial_{i} f \partial_{m} g \\
& +10 \partial_{t} \partial_{p} \mathcal{P}^{i j} \partial_{j} \mathcal{P}^{k \ell} \partial_{r} \partial_{k} \mathcal{P}^{m n} \partial_{v} \partial_{n} \partial_{\ell} \mathcal{P}^{p q} \partial_{q} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{m} g \\
& -10 \partial_{t} \partial_{n} \mathcal{P}^{i j} \partial_{j} \mathcal{P}^{k \ell} \partial_{v} \partial_{r} \partial_{p} \partial_{k} \mathcal{P}^{m n} \partial_{\ell} \mathcal{P}^{p q} \partial_{q} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{m} g \\
& -10 \partial_{t} \partial_{r} \partial_{p} \partial_{m} \mathcal{P}^{i j} \partial_{v} \partial_{j} \mathcal{P}^{k \ell} \partial_{\ell} \mathcal{P}^{m n} \partial_{n} \mathcal{P}^{p q} \partial_{q} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{k} g \\
& -10 \partial_{t} \partial_{m} \mathcal{P}^{i j} \partial_{v} \partial_{r} \partial_{p} \partial_{j} \mathcal{P}^{k \ell} \partial_{\ell} \mathcal{P}^{m n} \partial_{n} \mathcal{P}^{p q} \partial_{q} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{k} g \\
& -10 \partial_{t} \partial_{r} \partial_{p} \partial_{n} \mathcal{P}^{i j} \partial_{j} \mathcal{P}^{k \ell} \partial_{v} \partial_{k} \mathcal{P}^{m n} \partial_{\ell} \mathcal{P}^{p q} \partial_{q} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{m} g \\
& -10 \partial_{t} \partial_{p} \mathcal{P}^{i j} \partial_{v} \partial_{r} \partial_{q} \partial_{j} \mathcal{P}^{k \ell} \partial_{\ell} \mathcal{P}^{m n} \partial_{m} \mathcal{P}^{p q} \partial_{n} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{k} g \\
& +10 \partial_{t} \partial_{s} \partial_{p} \partial_{m} \mathcal{P}^{i j} \partial_{j} \mathcal{P}^{k \ell} \partial_{k} \mathcal{P}^{m n} \partial_{\ell} \mathcal{P}^{p q} \partial_{v} \partial_{n} \mathcal{P}^{r s} \partial_{q} \mathcal{P}^{t v} \partial_{i} f \partial_{r} g \\
& +2 \partial_{t} \partial_{r} \partial_{p} \partial_{m} \partial_{k} \mathcal{P}^{i j} \partial_{v} \mathcal{P}^{k \ell} \partial_{\ell} \mathcal{P}^{m n} \partial_{n} \mathcal{P}^{p q} \partial_{q} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{j} g \\
& -5 \partial_{p} \partial_{m} \partial_{k} \mathcal{P}^{i j} \partial_{t} \partial_{r} \mathcal{P}^{k \ell} \partial_{v} \partial_{\ell} \mathcal{P}^{m n} \partial_{n} \mathcal{P}^{p q} \partial_{q} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{j} g \\
& +5 \partial_{t} \partial_{m} \partial_{k} \mathcal{P}^{i j} \partial_{r} \partial_{p} \mathcal{P}^{k \ell} \partial_{v} \partial_{\ell} \mathcal{P}^{m n} \partial_{n} \mathcal{P}^{p q} \partial_{q} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{j} g \\
& -5 \partial_{p} \partial_{m} \partial_{k} \mathcal{P}^{i j} \partial_{r} \mathcal{P}^{k \ell} \partial_{t} \partial_{\ell} \mathcal{P}^{m n} \partial_{v} \partial_{n} \mathcal{P}^{p q} \partial_{q} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{j} g \\
& -5 \partial_{p} \partial_{m} \partial_{k} \mathcal{P}^{i j} \partial_{t} \mathcal{P}^{k \ell} \partial_{r} \partial_{\ell} \mathcal{P}^{m n} \partial_{v} \partial_{n} \mathcal{P}^{p q} \partial_{q} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{j} g \\
& +5 \partial_{t} \partial_{p} \mathcal{P}^{i j} \partial_{v} \partial_{j} \mathcal{P}^{k \ell} \partial_{\ell} \mathcal{P}^{m n} \partial_{r} \partial_{m} \mathcal{P}^{p q} \partial_{n} \mathcal{P}^{r s} \partial_{s} \partial_{q} \mathcal{P}^{t v} \partial_{i} f \partial_{k} g \\
& +5 \partial_{v} \partial_{r} \mathcal{P}^{i j} \partial_{j} \mathcal{P}^{k \ell} \partial_{k} \mathcal{P}^{m n} \partial_{m} \partial_{\ell} \mathcal{P}^{p q} \partial_{p} \partial_{n} \mathcal{P}^{r s} \partial_{s} \partial_{q} \mathcal{P}^{t v} \partial_{i} f \partial_{t} g
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& +5 \partial_{r} \partial_{m} \mathcal{P}^{i j} \partial_{t} \partial_{j} \mathcal{P}^{k \ell} \partial_{s} \partial_{\ell} \mathcal{P}^{m n} \partial_{n} \mathcal{P}^{p q} \partial_{v} \partial_{p} \mathcal{P}^{r s} \partial_{q} \mathcal{P}^{t v} \partial_{i} f \partial_{k} g \\
& -5 \partial_{r} \partial_{n} \mathcal{P}^{i j} \partial_{t} \partial_{j} \mathcal{P}^{k \ell} \partial_{v} \partial_{k} \mathcal{P}^{m n} \partial_{\ell} \mathcal{P}^{p q} \partial_{p} \mathcal{P}^{r s} \partial_{s} \partial_{q} \mathcal{P}^{t v} \partial_{i} f \partial_{m} g \\
& +5 \partial_{p} \partial_{m} \mathcal{P}^{i j} \partial_{r} \partial_{j} \mathcal{P}^{k \ell} \partial_{t} \partial_{\ell} \mathcal{P}^{m n} \partial_{v} \partial_{n} \mathcal{P}^{p q} \partial_{q} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{k} g \\
& -5 \partial_{r} \partial_{n} \mathcal{P}^{i j} \partial_{t} \partial_{j} \mathcal{P}^{k \ell} \partial_{p} \partial_{k} \mathcal{P}^{m n} \partial_{v} \partial_{\ell} \mathcal{P}^{p q} \partial_{q} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{m} g \\
& +5 \partial_{p} \partial_{m} \mathcal{P}^{i j} \partial_{t} \partial_{j} \mathcal{P}^{k \ell} \partial_{r} \partial_{\ell} \mathcal{P}^{m n} \partial_{v} \partial_{n} \mathcal{P}^{p q} \partial_{q} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{k} g \\
& -5 \partial_{t} \partial_{n} \mathcal{P}^{i j} \partial_{r} \partial_{j} \mathcal{P}^{k \ell} \partial_{p} \partial_{k} \mathcal{P}^{m n} \partial_{v} \partial_{\ell} \mathcal{P}^{p q} \partial_{q} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{m} g \\
& +5 \partial_{r} \partial_{n} \mathcal{P}^{i j} \partial_{s} \partial_{j} \mathcal{P}^{k \ell} \partial_{t} \partial_{k} \mathcal{P}^{m n} \partial_{\ell} \mathcal{P}^{p q} \partial_{v} \partial_{p} \mathcal{P}^{r s} \partial_{q} \mathcal{P}^{t v} \partial_{i} f \partial_{m} g \\
& +5 \partial_{r} \partial_{m} \mathcal{P}^{i j} \partial_{t} \partial_{j} \mathcal{P}^{k \ell} \partial_{v} \partial_{\ell} \mathcal{P}^{m n} \partial_{n} \mathcal{P}^{p q} \partial_{p} \mathcal{P}^{r s} \partial_{s} \partial_{q} \mathcal{P}^{t v} \partial_{i} f \partial_{k} g \\
& +5 \partial_{p} \partial_{n} \mathcal{P}^{i j} \partial_{t} \partial_{j} \mathcal{P}^{k \ell} \partial_{r} \partial_{k} \mathcal{P}^{m n} \partial_{v} \partial_{\ell} \mathcal{P}^{p q} \partial_{q} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{m} g \\
& -5 \partial_{r} \partial_{m} \mathcal{P}^{i j} \partial_{p} \partial_{j} \mathcal{P}^{k \ell} \partial_{t} \partial_{\ell} \mathcal{P}^{m n} \partial_{v} \partial_{n} \mathcal{P}^{p q} \partial_{q} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{k} g \\
& +5 \partial_{p} \partial_{n} \mathcal{P}^{i j} \partial_{r} \partial_{j} \mathcal{P}^{k \ell} \partial_{t} \partial_{k} \mathcal{P}^{m n} \partial_{v} \partial_{\ell} \mathcal{P}^{p q} \partial_{q} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{m} g \\
& -5 \partial_{t} \partial_{m} \mathcal{P}^{i j} \partial_{p} \partial_{j} \mathcal{P}^{k \ell} \partial_{r} \partial_{\ell} \mathcal{P}^{m n} \partial_{v} \partial_{n} \mathcal{P}^{p q} \partial_{q} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{k} g \\
& +5 \partial_{s} \partial_{m} \mathcal{P}^{i j} \partial_{t} \partial_{j} \mathcal{P}^{k \ell} \partial_{v} \partial_{k} \mathcal{P}^{m n} \partial_{\ell} \mathcal{P}^{p q} \partial_{p} \partial_{n} \mathcal{P}^{r s} \partial_{q} \mathcal{P}^{t v} \partial_{i} f \partial_{r} g \\
& +5 \partial_{r} \partial_{p} \mathcal{P}^{i j} \partial_{q} \partial_{j} \mathcal{P}^{k \ell} \partial_{t} \partial_{\ell} \mathcal{P}^{m n} \partial_{v} \partial_{m} \mathcal{P}^{p q} \partial_{n} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{k} g \\
& +5 \partial_{s} \partial_{m} \mathcal{P}^{i j} \partial_{t} \partial_{j} \mathcal{P}^{k \ell} \partial_{p} \partial_{k} \mathcal{P}^{m n} \partial_{\ell} \mathcal{P}^{p q} \partial_{v} \partial_{n} \mathcal{P}^{r s} \partial_{q} \mathcal{P}^{t v} \partial_{i} f \partial_{r} g \\
& -5 \partial_{t} \partial_{p} \mathcal{P}^{i j} \partial_{q} \partial_{j} \mathcal{P}^{k \ell} \partial_{v} \partial_{\ell} \mathcal{P}^{m n} \partial_{r} \partial_{m} \mathcal{P}^{p q} \partial_{n} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{k} g \\
& -5 \partial_{r} \partial_{n} \mathcal{P}^{i j} \partial_{j} \mathcal{P}^{k \ell} \partial_{t} \partial_{s} \partial_{k} \mathcal{P}^{m n} \partial_{\ell} \mathcal{P}^{p q} \partial_{v} \partial_{p} \mathcal{P}^{r s} \partial_{q} \mathcal{P}^{t v} \partial_{i} f \partial_{m} g \\
& -5 \partial_{t} \partial_{r} \partial_{m} \mathcal{P}^{i j} \partial_{s} \partial_{j} \mathcal{P}^{k \ell} \partial_{\ell} \mathcal{P}^{m n} \partial_{n} \mathcal{P}^{p q} \partial_{v} \partial_{p} \mathcal{P}^{r s} \partial_{q} \mathcal{P}^{t v} \partial_{i} f \partial_{k} g \\
& -5 \partial_{t} \partial_{p} \mathcal{P}^{i j} \partial_{v} \partial_{q} \partial_{j} \mathcal{P}^{k \ell} \partial_{\ell} \mathcal{P}^{m n} \partial_{r} \partial_{m} \mathcal{P}^{p q} \partial_{n} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{k} g \\
& +5 \partial_{t} \partial_{s} \partial_{m} \mathcal{P}^{i j} \partial_{j} \mathcal{P}^{k \ell} \partial_{p} \partial_{k} \mathcal{P}^{m n} \partial_{\ell} \mathcal{P}^{p q} \partial_{v} \partial_{n} \mathcal{P}^{r s} \partial_{q} \mathcal{P}^{t v} \partial_{i} f \partial_{r} g \\
& +5 \partial_{r} \partial_{m} \mathcal{P}^{i j} \partial_{t} \partial_{s} \partial_{j} \mathcal{P}^{k \ell} \partial_{v} \partial_{\ell} \mathcal{P}^{m n} \partial_{n} \mathcal{P}^{p q} \partial_{p} \mathcal{P}^{r s} \partial_{q} \mathcal{P}^{t v} \partial_{i} f \partial_{k} g \\
& -5 \partial_{t} \partial_{r} \partial_{n} \mathcal{P}^{i j} \partial_{s} \partial_{j} \mathcal{P}^{k \ell} \partial_{v} \partial_{k} \mathcal{P}^{m n} \partial_{\ell} \mathcal{P}^{p q} \partial_{p} \mathcal{P}^{r s} \partial_{q} \mathcal{P}^{t v} \partial_{i} f \partial_{m} g \\
& -5 \partial_{t} \partial_{m} \mathcal{P}^{i j} \partial_{v} \partial_{p} \partial_{j} \mathcal{P}^{k \ell} \partial_{r} \partial_{\ell} \mathcal{P}^{m n} \partial_{n} \mathcal{P}^{p q} \partial_{q} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{k} g \\
& -5 \partial_{t} \partial_{p} \partial_{n} \mathcal{P}^{i j} \partial_{r} \partial_{j} \mathcal{P}^{k \ell} \partial_{v} \partial_{k} \mathcal{P}^{m n} \partial_{\ell} \mathcal{P}^{p q} \partial_{q} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{m} g \\
& -5 \partial_{r} \partial_{m} \mathcal{P}^{i j} \partial_{t} \partial_{s} \partial_{j} \mathcal{P}^{k \ell} \partial_{\ell} \mathcal{P}^{m n} \partial_{n} \mathcal{P}^{p q} \partial_{v} \partial_{p} \mathcal{P}^{r s} \partial_{q} \mathcal{P}^{t v} \partial_{i} f \partial_{k} g \\
& -5 \partial_{t} \partial_{r} \partial_{n} \mathcal{P}^{i j} \partial_{j} \mathcal{P}^{k \ell} \partial_{s} \partial_{k} \mathcal{P}^{m n} \partial_{\ell} \mathcal{P}^{p q} \partial_{v} \partial_{p} \mathcal{P}^{r s} \partial_{q} \mathcal{P}^{t v} \partial_{i} f \partial_{m} g \\
& -5 \partial_{r} \partial_{n} \mathcal{P}^{i j} \partial_{t} \partial_{j} \mathcal{P}^{k \ell} \partial_{v} \partial_{s} \partial_{k} \mathcal{P}^{m n} \partial_{\ell} \mathcal{P}^{p q} \partial_{p} \mathcal{P}^{r s} \partial_{q} \mathcal{P}^{t v} \partial_{i} f \partial_{m} g \\
& +5 \partial_{t} \partial_{r} \partial_{m} \mathcal{P}^{i j} \partial_{s} \partial_{j} \mathcal{P}^{k \ell} \partial_{v} \partial_{\ell} \mathcal{P}^{m n} \partial_{n} \mathcal{P}^{p q} \partial_{p} \mathcal{P}^{r s} \partial_{q} \mathcal{P}^{t v} \partial_{i} f \partial_{k} g \\
& -5 \partial_{t} \partial_{n} \mathcal{P}^{i j} \partial_{r} \partial_{j} \mathcal{P}^{k \ell} \partial_{v} \partial_{p} \partial_{k} \mathcal{P}^{m n} \partial_{\ell} \mathcal{P}^{p q} \partial_{q} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{m} g \\
& -5 \partial_{t} \partial_{p} \partial_{m} \mathcal{P}^{i j} \partial_{v} \partial_{j} \mathcal{P}^{k \ell} \partial_{r} \partial_{\ell} \mathcal{P}^{m n} \partial_{n} \mathcal{P}^{p q} \partial_{q} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{k} g \\
& -5 \partial_{r} \partial_{m} \partial_{k} \mathcal{P}^{i j} \partial_{t} \mathcal{P}^{k \ell} \partial_{v} \partial_{\ell} \mathcal{P}^{m n} \partial_{n} \mathcal{P}^{p q} \partial_{p} \mathcal{P}^{r s} \partial_{s} \partial_{q} \mathcal{P}^{t v} \partial_{i} f \partial_{j} g \\
& +5 \partial_{r} \partial_{m} \partial_{k} \mathcal{P}^{i j} \partial_{p} \mathcal{P}^{k \ell} \partial_{t} \partial_{\ell} \mathcal{P}^{m n} \partial_{v} \partial_{n} \mathcal{P}^{p q} \partial_{q} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{j} g \\
& +5 \partial_{t} \partial_{m} \partial_{k} \mathcal{P}^{i j} \partial_{p} \mathcal{P}^{k \ell} \partial_{r} \partial_{\ell} \mathcal{P}^{m n} \partial_{v} \partial_{n} \mathcal{P}^{p q} \partial_{q} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{j} g \\
& +5 \partial_{r} \partial_{m} \partial_{k} \mathcal{P}^{i j} \partial_{t} \partial_{p} \mathcal{P}^{k \ell} \partial_{\ell} \mathcal{P}^{m n} \partial_{v} \partial_{n} \mathcal{P}^{p q} \partial_{q} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{j} g \\
& +5 \partial_{t} \partial_{m} \partial_{k} \mathcal{P}^{i j} \partial_{r} \partial_{p} \mathcal{P}^{k \ell} \partial_{\ell} \mathcal{P}^{m n} \partial_{v} \partial_{n} \mathcal{P}^{p q} \partial_{q} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{j} g \\
& -5 \partial_{r} \partial_{n} \mathcal{P}^{i j} \partial_{j} \mathcal{P}^{k \ell} \partial_{t} \partial_{p} \partial_{k} \mathcal{P}^{m n} \partial_{v} \partial_{\ell} \mathcal{P}^{p q} \partial_{q} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{m} g
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& +5 \partial_{t} \partial_{p} \partial_{m} \mathcal{P}^{i j} \partial_{r} \partial_{j} \mathcal{P}^{k \ell} \partial_{\ell} \mathcal{P}^{m n} \partial_{v} \partial_{n} \mathcal{P}^{p q} \partial_{q} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{k} g \\
& -5 \partial_{t} \partial_{n} \mathcal{P}^{i j} \partial_{j} \mathcal{P}^{k \ell} \partial_{r} \partial_{p} \partial_{k} \mathcal{P}^{m n} \partial_{v} \partial_{\ell} \mathcal{P}^{p q} \partial_{q} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{m} g \\
& +5 \partial_{r} \partial_{p} \partial_{m} \mathcal{P}^{i j} \partial_{t} \partial_{j} \mathcal{P}^{k \ell} \partial_{\ell} \mathcal{P}^{m n} \partial_{v} \partial_{n} \mathcal{P}^{p q} \partial_{q} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{k} g \\
& -5 \partial_{t} \partial_{p} \mathcal{P}^{i j} \partial_{v} \partial_{r} \partial_{j} \mathcal{P}^{k \ell} \partial_{\ell} \mathcal{P}^{m n} \partial_{m} \mathcal{P}^{p q} \partial_{q} \partial_{n} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{k} g \\
& -5 \partial_{v} \partial_{r} \partial_{m} \mathcal{P}^{i j} \partial_{j} \mathcal{P}^{k \ell} \partial_{p} \partial_{k} \mathcal{P}^{m n} \partial_{\ell} \mathcal{P}^{p q} \partial_{n} \mathcal{P}^{r s} \partial_{s} \partial_{q} \mathcal{P}^{t v} \partial_{i} f \partial_{t} g \\
& -5 \partial_{r} \partial_{p} \mathcal{P}^{i j} \partial_{t} \partial_{q} \partial_{j} \mathcal{P}^{k \ell} \partial_{v} \partial_{\ell} \mathcal{P}^{m n} \partial_{m} \mathcal{P}^{p q} \partial_{n} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{k} g \\
& -5 \partial_{t} \partial_{s} \partial_{m} \mathcal{P}^{i j} \partial_{v} \partial_{j} \mathcal{P}^{k \ell} \partial_{k} \mathcal{P}^{m n} \partial_{\ell} \mathcal{P}^{p q} \partial_{p} \partial_{n} \mathcal{P}^{r s} \partial_{q} \mathcal{P}^{t v} \partial_{i} f \partial_{r} g \\
& -5 \partial_{t} \partial_{p} \mathcal{P}^{i j} \partial_{r} \partial_{q} \partial_{j} \mathcal{P}^{k \ell} \partial_{v} \partial_{\ell} \mathcal{P}^{m n} \partial_{m} \mathcal{P}^{p q} \partial_{n} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{k} g \\
& +5 \partial_{s} \partial_{p} \partial_{m} \mathcal{P}^{i j} \partial_{t} \partial_{j} \mathcal{P}^{k \ell} \partial_{k} \mathcal{P}^{m n} \partial_{\ell} \mathcal{P}^{p q} \partial_{v} \partial_{n} \mathcal{P}^{r s} \partial_{q} \mathcal{P}^{t v} \partial_{i} f \partial_{r} g \\
& -5 \partial_{r} \partial_{m} \mathcal{P}^{i j} \partial_{t} \partial_{p} \partial_{j} \mathcal{P}^{k \ell} \partial_{\ell} \mathcal{P}^{m n} \partial_{v} \partial_{n} \mathcal{P}^{p q} \partial_{q} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{k} g \\
& +5 \partial_{t} \partial_{p} \partial_{n} \mathcal{P}^{i j} \partial_{j} \mathcal{P}^{k \ell} \partial_{r} \partial_{k} \mathcal{P}^{m n} \partial_{v} \partial_{\ell} \mathcal{P}^{p q} \partial_{q} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{m} g \\
& -5 \partial_{t} \partial_{m} \mathcal{P}^{i j} \partial_{r} \partial_{p} \partial_{j} \mathcal{P}^{k \ell} \partial_{\ell} \mathcal{P}^{m n} \partial_{v} \partial_{n} \mathcal{P}^{p q} \partial_{q} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{k} g \\
& +5 \partial_{r} \partial_{p} \partial_{n} \mathcal{P}^{i j} \partial_{j} \mathcal{P}^{k \ell} \partial_{t} \partial_{k} \mathcal{P}^{m n} \partial_{v} \partial_{\ell} \mathcal{P}^{p q} \partial_{q} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{m} g \\
& -5 \partial_{t} \partial_{s} \mathcal{P}^{i j} \partial_{j} \mathcal{P}^{k \ell} \partial_{k} \mathcal{P}^{m n} \partial_{m} \partial_{\ell} \mathcal{P}^{p q} \partial_{v} \partial_{p} \partial_{n} \mathcal{P}^{r s} \partial_{q} \mathcal{P}^{t v} \partial_{i} f \partial_{r} g \\
& +5 \partial_{t} \partial_{r} \partial_{p} \mathcal{P}^{i j} \partial_{v} \partial_{j} \mathcal{P}^{k \ell} \partial_{\ell} \mathcal{P}^{m n} \partial_{m} \mathcal{P}^{p q} \partial_{q} \partial_{n} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{k} g \\
& -5 \partial_{r} \partial_{p} \partial_{m} \partial_{k} \mathcal{P}^{i j} \partial_{t} \mathcal{P}^{k \ell} \partial_{v} \partial_{\ell} \mathcal{P}^{m n} \partial_{n} \mathcal{P}^{p q} \partial_{q} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{j} g \\
& -5 \partial_{t} \partial_{p} \partial_{m} \partial_{k} \mathcal{P}^{i j} \partial_{r} \mathcal{P}^{k \ell} \partial_{\ell} \mathcal{P}^{m n} \partial_{v} \partial_{n} \mathcal{P}^{p q} \partial_{q} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{j} g \\
& -5 \partial_{r} \partial_{p} \partial_{m} \partial_{k} \mathcal{P}^{i j} \partial_{t} \mathcal{P}^{k \ell} \partial_{\ell} \mathcal{P}^{m n} \partial_{v} \partial_{n} \mathcal{P}^{p q} \partial_{q} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{j} g \\
& +5 \partial_{s} \partial_{m} \mathcal{P}^{i j} \partial_{j} \mathcal{P}^{k \ell} \partial_{t} \partial_{k} \mathcal{P}^{m n} \partial_{\ell} \mathcal{P}^{p q} \partial_{v} \partial_{p} \partial_{n} \mathcal{P}^{r s} \partial_{q} \mathcal{P}^{t v} \partial_{i} f \partial_{r} g \\
& -5 \partial_{t} \partial_{r} \partial_{p} \mathcal{P}^{i j} \partial_{q} \partial_{j} \mathcal{P}^{k \ell} \partial_{\ell} \mathcal{P}^{m n} \partial_{v} \partial_{m} \mathcal{P}^{p q} \partial_{n} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{k} g \\
& -5 \partial_{t} \partial_{n} \mathcal{P}^{i j} \partial_{v} \partial_{j} \mathcal{P}^{k \ell} \partial_{r} \partial_{p} \partial_{k} \mathcal{P}^{m n} \partial_{\ell} \mathcal{P}^{p q} \partial_{q} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{m} g \\
& +5 \partial_{r} \partial_{p} \partial_{m} \mathcal{P}^{i j} \partial_{t} \partial_{j} \mathcal{P}^{k \ell} \partial_{v} \partial_{\ell} \mathcal{P}^{m n} \partial_{n} \mathcal{P}^{p q} \partial_{q} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{k} g \\
& -5 \partial_{p} \partial_{n} \mathcal{P}^{i j} \partial_{t} \partial_{j} \mathcal{P}^{k \ell} \partial_{v} \partial_{r} \partial_{k} \mathcal{P}^{m n} \partial_{\ell} \mathcal{P}^{p q} \partial_{q} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{m} g \\
& -5 \partial_{t} \partial_{r} \partial_{m} \mathcal{P}^{i j} \partial_{p} \partial_{j} \mathcal{P}^{k \ell} \partial_{v} \partial_{\ell} \mathcal{P}^{m n} \partial_{n} \mathcal{P}^{p q} \partial_{q} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{k} g \\
& -5 \partial_{t} \partial_{p} \mathcal{P}^{i j} \partial_{r} \partial_{q} \partial_{j} \mathcal{P}^{k \ell} \partial_{\ell} \mathcal{P}^{m n} \partial_{v} \partial_{m} \mathcal{P}^{p q} \partial_{n} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{k} g \\
& +5 \partial_{s} \partial_{p} \partial_{m} \mathcal{P}^{i j} \partial_{j} \mathcal{P}^{k \ell} \partial_{t} \partial_{k} \mathcal{P}^{m n} \partial_{\ell} \mathcal{P}^{p q} \partial_{v} \partial_{n} \mathcal{P}^{r s} \partial_{q} \mathcal{P}^{t v} \partial_{i} f \partial_{r} g \\
& -5 \partial_{t} \partial_{p} \mathcal{P}^{i j} \partial_{v} \partial_{r} \partial_{j} \mathcal{P}^{k \ell} \partial_{\ell} \mathcal{P}^{m n} \partial_{m} \mathcal{P}^{p q} \partial_{n} \mathcal{P}^{r s} \partial_{s} \partial_{q} \mathcal{P}^{t v} \partial_{i} f \partial_{k} g \\
& -5 \partial_{v} \partial_{r} \partial_{m} \mathcal{P}^{i j} \partial_{j} \mathcal{P}^{k \ell} \partial_{k} \mathcal{P}^{m n} \partial_{\ell} \mathcal{P}^{p q} \partial_{p} \partial_{n} \mathcal{P}^{r s} \partial_{s} \partial_{q} \mathcal{P}^{t v} \partial_{i} f \partial_{t} g \\
& -5 \partial_{t} \partial_{m} \mathcal{P}^{i j} \partial_{r} \partial_{p} \partial_{j} \mathcal{P}^{k \ell} \partial_{v} \partial_{\ell} \mathcal{P}^{m n} \partial_{n} \mathcal{P}^{p q} \partial_{q} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{k} g \\
& -5 \partial_{r} \partial_{p} \partial_{n} \mathcal{P}^{i j} \partial_{t} \partial_{j} \mathcal{P}^{k \ell} \partial_{v} \partial_{k} \mathcal{P}^{m n} \partial_{\ell} \mathcal{P}^{p q} \partial_{q} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{m} g \\
& +5 \partial_{p} \partial_{m} \mathcal{P}^{i j} \partial_{t} \partial_{r} \partial_{j} \mathcal{P}^{k \ell} \partial_{v} \partial_{\ell} \mathcal{P}^{m n} \partial_{n} \mathcal{P}^{p q} \partial_{q} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{k} g \\
& -5 \partial_{t} \partial_{r} \partial_{n} \mathcal{P}^{i j} \partial_{v} \partial_{j} \mathcal{P}^{k \ell} \partial_{p} \partial_{k} \mathcal{P}^{m n} \partial_{\ell} \mathcal{P}^{p q} \partial_{q} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{m} g \\
& -5 \partial_{t} \partial_{p} \mathcal{P}^{i j} \partial_{j} \mathcal{P}^{k \ell} \partial_{v} \partial_{k} \mathcal{P}^{m n} \partial_{r} \partial_{\ell} \mathcal{P}^{p q} \partial_{n} \mathcal{P}^{r s} \partial_{s} \partial_{q} \mathcal{P}^{t v} \partial_{i} f \partial_{m} g \\
& -5 \partial_{t} \partial_{m} \mathcal{P}^{i j} \partial_{j} \mathcal{P}^{k \ell} \partial_{p} \partial_{k} \mathcal{P}^{m n} \partial_{v} \partial_{\ell} \mathcal{P}^{p q} \partial_{q} \partial_{n} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{r} g \\
& +5 \partial_{r} \partial_{p} \mathcal{P}^{i j} \partial_{j} \mathcal{P}^{k \ell} \partial_{t} \partial_{k} \mathcal{P}^{m n} \partial_{s} \partial_{\ell} \mathcal{P}^{p q} \partial_{v} \partial_{n} \mathcal{P}^{r s} \partial_{q} \mathcal{P}^{t v} \partial_{i} f \partial_{m} g \\
& -5 \partial_{t} \partial_{r} \mathcal{P}^{i j} \partial_{v} \partial_{j} \mathcal{P}^{k \ell} \partial_{p} \partial_{k} \mathcal{P}^{m n} \partial_{s} \partial_{\ell} \mathcal{P}^{p q} \partial_{n} \mathcal{P}^{r s} \partial_{q} \mathcal{P}^{t v} \partial_{i} f \partial_{m} g
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& +5 \partial_{r} \partial_{p} \mathcal{P}^{i j} \partial_{j} \mathcal{P}^{k \ell} \partial_{t} \partial_{k} \mathcal{P}^{m n} \partial_{v} \partial_{\ell} \mathcal{P}^{p q} \partial_{q} \partial_{n} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{m} g \\
& +5 \partial_{r} \partial_{p} \mathcal{P}^{i j} \partial_{t} \partial_{j} \mathcal{P}^{k \ell} \partial_{v} \partial_{k} \mathcal{P}^{m n} \partial_{\ell} \mathcal{P}^{p q} \partial_{n} \mathcal{P}^{r s} \partial_{s} \partial_{q} \mathcal{P}^{t v} \partial_{i} f \partial_{m} g \\
& -5 \partial_{t} \partial_{m} \mathcal{P}^{i j} \partial_{v} \partial_{j} \mathcal{P}^{k \ell} \partial_{k} \mathcal{P}^{m n} \partial_{\ell} \mathcal{P}^{p q} \partial_{p} \partial_{n} \mathcal{P}^{r s} \partial_{s} \partial_{q} \mathcal{P}^{t v} \partial_{i} f \partial_{r} g \\
& +5 \partial_{t} \partial_{r} \mathcal{P}^{i j} \partial_{s} \partial_{j} \mathcal{P}^{k \ell} \partial_{v} \partial_{k} \mathcal{P}^{m n} \partial_{n} \partial_{\ell} \mathcal{P}^{p q} \partial_{p} \mathcal{P}^{r s} \partial_{q} \mathcal{P}^{t v} \partial_{i} f \partial_{m} g \\
& -5 \partial_{t} \partial_{m} \mathcal{P}^{i j} \partial_{v} \partial_{j} \mathcal{P}^{k \ell} \partial_{p} \partial_{k} \mathcal{P}^{m n} \partial_{\ell} \mathcal{P}^{p q} \partial_{q} \partial_{n} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{r} g \\
& +5 \partial_{t} \partial_{r} \mathcal{P}^{i j} \partial_{j} \mathcal{P}^{k \ell} \partial_{s} \partial_{k} \mathcal{P}^{m n} \partial_{n} \partial_{\ell} \mathcal{P}^{p q} \partial_{v} \partial_{p} \mathcal{P}^{r s} \partial_{q} \mathcal{P}^{t v} \partial_{i} f \partial_{m} g \\
& +5 \partial_{p} \partial_{m} \mathcal{P}^{i j} \partial_{t} \partial_{j} \mathcal{P}^{k \ell} \partial_{k} \mathcal{P}^{m n} \partial_{s} \partial_{\ell} \mathcal{P}^{p q} \partial_{v} \partial_{n} \mathcal{P}^{r s} \partial_{q} \mathcal{P}^{t v} \partial_{i} f \partial_{r} g \\
& +5 \partial_{t} \partial_{p} \mathcal{P}^{i j} \partial_{r} \partial_{j} \mathcal{P}^{k \ell} \partial_{v} \partial_{k} \mathcal{P}^{m n} \partial_{\ell} \mathcal{P}^{p q} \partial_{q} \partial_{n} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{m} g \\
& -5 \partial_{p} \partial_{m} \mathcal{P}^{i j} \partial_{t} \partial_{j} \mathcal{P}^{k \ell} \partial_{v} \partial_{k} \mathcal{P}^{m n} \partial_{\ell} \mathcal{P}^{p q} \partial_{q} \partial_{n} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{r} g \\
& -5 \partial_{t} \partial_{m} \mathcal{P}^{i j} \partial_{j} \mathcal{P}^{k \ell} \partial_{p} \partial_{k} \mathcal{P}^{m n} \partial_{s} \partial_{\ell} \mathcal{P}^{p q} \partial_{v} \partial_{n} \mathcal{P}^{r s} \partial_{q} \mathcal{P}^{t v} \partial_{i} f \partial_{r} g \\
& +5 \partial_{r} \partial_{p} \mathcal{P}^{i j} \partial_{t} \partial_{j} \mathcal{P}^{k \ell} \partial_{v} \partial_{k} \mathcal{P}^{m n} \partial_{s} \partial_{\ell} \mathcal{P}^{p q} \partial_{n} \mathcal{P}^{r s} \partial_{q} \mathcal{P}^{t v} \partial_{i} f \partial_{m} g \\
& +5 \partial_{t} \partial_{r} \mathcal{P}^{i j} \partial_{j} \mathcal{P}^{k \ell} \partial_{p} \partial_{k} \mathcal{P}^{m n} \partial_{s} \partial_{\ell} \mathcal{P}^{p q} \partial_{v} \partial_{n} \mathcal{P}^{r s} \partial_{q} \mathcal{P}^{t v} \partial_{i} f \partial_{m} g \\
& -5 \partial_{t} \partial_{r} \mathcal{P}^{i j} \partial_{v} \partial_{j} \mathcal{P}^{k \ell} \partial_{p} \partial_{k} \mathcal{P}^{m n} \partial_{\ell} \mathcal{P}^{p q} \partial_{n} \mathcal{P}^{r s} \partial_{s} \partial_{q} \mathcal{P}^{t v} \partial_{i} f \partial_{m} g \\
& +5 \partial_{t} \partial_{r} \mathcal{P}^{i j} \partial_{j} \mathcal{P}^{k \ell} \partial_{p} \partial_{k} \mathcal{P}^{m n} \partial_{v} \partial_{\ell} \mathcal{P}^{p q} \partial_{q} \partial_{n} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{m} g \\
& -5 \partial_{t} \partial_{p} \mathcal{P}^{i j} \partial_{r} \partial_{j} \mathcal{P}^{k \ell} \partial_{v} \partial_{k} \mathcal{P}^{m n} \partial_{s} \partial_{\ell} \mathcal{P}^{p q} \partial_{n} \mathcal{P}^{r s} \partial_{q} \mathcal{P}^{t v} \partial_{i} f \partial_{m} g \\
& -5 \partial_{t} \partial_{m} \mathcal{P}^{i j} \partial_{j} \mathcal{P}^{k \ell} \partial_{s} \partial_{k} \mathcal{P}^{m n} \partial_{n} \partial_{\ell} \mathcal{P}^{p q} \partial_{v} \partial_{p} \mathcal{P}^{r s} \partial_{q} \mathcal{P}^{t v} \partial_{i} f \partial_{r} g \\
& -5 \partial_{r} \partial_{m} \mathcal{P}^{i j} \partial_{v} \partial_{j} \mathcal{P}^{k \ell} \partial_{p} \partial_{k} \mathcal{P}^{m n} \partial_{\ell} \mathcal{P}^{p q} \partial_{n} \mathcal{P}^{r s} \partial_{s} \partial_{q} \mathcal{P}^{t v} \partial_{i} f \partial_{t} g \\
& -5 \partial_{t} \partial_{m} \mathcal{P}^{i j} \partial_{s} \partial_{j} \mathcal{P}^{k \ell} \partial_{k} \mathcal{P}^{m n} \partial_{n} \partial_{\ell} \mathcal{P}^{p q} \partial_{v} \partial_{p} \mathcal{P}^{r s} \partial_{q} \mathcal{P}^{t v} \partial_{i} f \partial_{r} g \\
& +5 \partial_{t} \partial_{m} \mathcal{P}^{i j} \partial_{j} \mathcal{P}^{k \ell} \partial_{v} \partial_{k} \mathcal{P}^{m n} \partial_{\ell} \mathcal{P}^{p q} \partial_{p} \partial_{n} \mathcal{P}^{r s} \partial_{s} \partial_{q} \mathcal{P}^{t v} \partial_{i} f \partial_{r} g \\
& +5 \partial_{t} \partial_{p} \mathcal{P}^{i j} \partial_{v} \partial_{j} \mathcal{P}^{k \ell} \partial_{q} \partial_{k} \mathcal{P}^{m n} \partial_{r} \partial_{\ell} \mathcal{P}^{p q} \partial_{n} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{m} g \\
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& -5 \partial_{t} \partial_{m} \mathcal{P}^{i j} \partial_{s} \partial_{j} \mathcal{P}^{k \ell} \partial_{v} \partial_{k} \mathcal{P}^{m n} \partial_{\ell} \mathcal{P}^{p q} \partial_{p} \partial_{n} \mathcal{P}^{r s} \partial_{q} \mathcal{P}^{t v} \partial_{i} f \partial_{r} g \\
& -5 \partial_{r} \partial_{p} \mathcal{P}^{i j} \partial_{s} \partial_{j} \mathcal{P}^{k \ell} \partial_{t} \partial_{k} \mathcal{P}^{m n} \partial_{\ell} \mathcal{P}^{p q} \partial_{v} \partial_{n} \mathcal{P}^{r s} \partial_{q} \mathcal{P}^{t v} \partial_{i} f \partial_{m} g \\
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& -5 \partial_{t} \partial_{r} \mathcal{P}^{i j} \partial_{s} \partial_{j} \mathcal{P}^{k \ell} \partial_{p} \partial_{k} \mathcal{P}^{m n} \partial_{\ell} \mathcal{P}^{p q} \partial_{v} \partial_{n} \mathcal{P}^{r s} \partial_{q} \mathcal{P}^{t v} \partial_{i} f \partial_{m} g \\
& -5 \partial_{r} \partial_{p} \mathcal{P}^{i j} \partial_{j} \mathcal{P}^{k \ell} \partial_{t} \partial_{q} \partial_{k} \mathcal{P}^{m n} \partial_{v} \partial_{\ell} \mathcal{P}^{p q} \partial_{n} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{m} g \\
& -5 \partial_{t} \partial_{r} \partial_{p} \mathcal{P}^{i j} \partial_{j} \mathcal{P}^{k \ell} \partial_{v} \partial_{k} \mathcal{P}^{m n} \partial_{\ell} \mathcal{P}^{p q} \partial_{n} \mathcal{P}^{r s} \partial_{s} \partial_{q} \mathcal{P}^{t v} \partial_{i} f \partial_{m} g \\
& +5 \partial_{t} \partial_{p} \mathcal{P}^{i j} \partial_{j} \mathcal{P}^{k \ell} \partial_{v} \partial_{q} \partial_{k} \mathcal{P}^{m n} \partial_{r} \partial_{\ell} \mathcal{P}^{p q} \partial_{n} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{m} g \\
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& +5 \partial_{t} \partial_{p} \mathcal{P}^{i j} \partial_{j} \mathcal{P}^{k \ell} \partial_{v} \partial_{r} \partial_{k} \mathcal{P}^{m n} \partial_{n} \partial_{\ell} \mathcal{P}^{p q} \partial_{q} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{m} g \\
& +5 \partial_{t} \partial_{r} \partial_{p} \mathcal{P}^{i j} \partial_{s} \partial_{j} \mathcal{P}^{k \ell} \partial_{v} \partial_{k} \mathcal{P}^{m n} \partial_{\ell} \mathcal{P}^{p q} \partial_{n} \mathcal{P}^{r s} \partial_{q} \mathcal{P}^{t v} \partial_{i} f \partial_{m} g \\
& +5 \partial_{r} \partial_{p} \mathcal{P}^{i j} \partial_{t} \partial_{j} \mathcal{P}^{k \ell} \partial_{v} \partial_{q} \partial_{k} \mathcal{P}^{m n} \partial_{\ell} \mathcal{P}^{p q} \partial_{n} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{m} g
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& +5 \partial_{t} \partial_{p} \mathcal{P}^{i j} \partial_{r} \partial_{j} \mathcal{P}^{k \ell} \partial_{v} \partial_{q} \partial_{k} \mathcal{P}^{m n} \partial_{\ell} \mathcal{P}^{p q} \partial_{n} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{m} g \\
& +5 \partial_{t} \partial_{p} \partial_{m} \mathcal{P}^{i j} \partial_{j} \mathcal{P}^{k \ell} \partial_{k} \mathcal{P}^{m n} \partial_{s} \partial_{\ell} \mathcal{P}^{p q} \partial_{v} \partial_{n} \mathcal{P}^{r s} \partial_{q} \mathcal{P}^{t v} \partial_{i} f \partial_{r} g \\
& -5 \partial_{t} \partial_{m} \mathcal{P}^{i j} \partial_{s} \partial_{j} \mathcal{P}^{k \ell} \partial_{k} \mathcal{P}^{m n} \partial_{\ell} \mathcal{P}^{p q} \partial_{v} \partial_{p} \partial_{n} \mathcal{P}^{r s} \partial_{q} \mathcal{P}^{t v} \partial_{i} f \partial_{r} g \\
& +5 \partial_{t} \partial_{p} \partial_{m} \mathcal{P}^{i j} \partial_{s} \partial_{j} \mathcal{P}^{k \ell} \partial_{k} \mathcal{P}^{m n} \partial_{\ell} \mathcal{P}^{p q} \partial_{v} \partial_{n} \mathcal{P}^{r s} \partial_{q} \mathcal{P}^{t v} \partial_{i} f \partial_{r} g \\
& -5 \partial_{t} \partial_{r} \mathcal{P}^{i j} \partial_{s} \partial_{j} \mathcal{P}^{k \ell} \partial_{v} \partial_{p} \partial_{k} \mathcal{P}^{m n} \partial_{\ell} \mathcal{P}^{p q} \partial_{n} \mathcal{P}^{r s} \partial_{q} \mathcal{P}^{t v} \partial_{i} f \partial_{m} g \\
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& +5 \partial_{v} \partial_{m} \mathcal{P}^{i j} \partial_{j} \mathcal{P}^{k \ell} \partial_{p} \partial_{k} \mathcal{P}^{m n} \partial_{r} \partial_{\ell} \mathcal{P}^{p q} \partial_{n} \mathcal{P}^{r s} \partial_{s} \partial_{q} \mathcal{P}^{t v} \partial_{i} f \partial_{t} g \\
& +5 \partial_{t} \partial_{p} \mathcal{P}^{i j} \partial_{r} \partial_{j} \mathcal{P}^{k \ell} \partial_{\ell} \mathcal{P}^{m n} \partial_{v} \partial_{m} \mathcal{P}^{p q} \partial_{q} \partial_{n} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{k} g \\
& -5 \partial_{s} \partial_{m} \mathcal{P}^{i j} \partial_{j} \mathcal{P}^{k \ell} \partial_{t} \partial_{k} \mathcal{P}^{m n} \partial_{n} \partial_{\ell} \mathcal{P}^{p q} \partial_{v} \partial_{p} \mathcal{P}^{r s} \partial_{q} \mathcal{P}^{t v} \partial_{i} f \partial_{r} g \\
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& +5 \partial_{r} \partial_{p} \mathcal{P}^{i j} \partial_{t} \partial_{j} \mathcal{P}^{k \ell} \partial_{v} \partial_{\ell} \mathcal{P}^{m n} \partial_{m} \mathcal{P}^{p q} \partial_{q} \partial_{n} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{k} g \\
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& +5 \partial_{t} \partial_{s} \partial_{m} \mathcal{P}^{i j} \partial_{j} \mathcal{P}^{k \ell} \partial_{k} \mathcal{P}^{m n} \partial_{\ell} \mathcal{P}^{p q} \partial_{v} \partial_{p} \partial_{n} \mathcal{P}^{r s} \partial_{q} \mathcal{P}^{t v} \partial_{i} f \partial_{r} g \\
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& -5 \partial_{t} \partial_{p} \partial_{m} \mathcal{P}^{i j} \partial_{v} \partial_{r} \partial_{j} \mathcal{P}^{k \ell} \partial_{\ell} \mathcal{P}^{m n} \partial_{n} \mathcal{P}^{p q} \partial_{q} \mathcal{P}^{r s} \partial_{s} \mathcal{P}^{t v} \partial_{i} f \partial_{k} g .
\end{aligned}
$$

In every term, the Einstein summation convention works for each repeated index (i.e. once upper and another time lower), the indices running from 1 to the dimension $r<\infty$ of the affine Poisson manifold $M^{r}$ at hand.


[^0]:    ${ }^{1}$ In fact, for any full subcategory of the category of rings such that the inclusion functor has a left adjoint, deformations can be defined, see [16]. In this thesis we do not elaborate on category theoretical topics.

[^1]:    ${ }^{2}$ If the coefficients of powers of $t$ vanish in $\operatorname{Assoc}_{\tilde{M}}$ up to power $p$ of $t$, we say that $\tilde{M}$ is integrable up to order $p$. We only discuss infinitesimal deformations in this thesis, so we do not elaborate on this. More information can be found in [16] and [15].

[^2]:    ${ }^{3}$ In fact, every symplectic structure is a particular example of a Poisson structure, see [23]. We do not elaborate on symplectic structures in this thesis.
    ${ }^{4}$ Methods to find deformations of Poisson algebras of arbitrary smooth manifolds are discussed in the first part of [18].

[^3]:    ${ }^{5}$ Both papers provide a set of algorithms that can be used to continue this search for graphs with a greater amount of vertices.

[^4]:    Date: 24 November 2017.
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    Key words and phrases. Non-oriented graph complex, differential, cocycle, symmetry, Poisson geometry.
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    ${ }^{1}$ The dilation $\dot{\mathcal{P}}=\mathcal{P}$, also universal with respect to all Poisson manifolds, is obtained by orienting the graph $\bullet$ on one vertex and no edges, yet that graph is not a cocycle, $d(\bullet)=-\bullet \bullet 0$. The singleedge graph $\bullet \in$ ker d on two vertices is a cocycle but its bi-grading differs from $(n, 2 n-2)$. However, by satisfying the zero-curvature equation $d(\bullet \bullet)+\frac{1}{2}[\bullet \bullet, \bullet \bullet]=0$ the graph $\bullet$ is a Maurer-Cartan element in the graph complex.

[^5]:    ${ }^{2}$ The proof scheme is computer-assisted (cf. $[2,6]$ ); it can be applied to the study of other cocycles: either on higher number of vertices or built at arbitrary $n \geqslant 2$ from not necessarily connected graphs.

[^6]:    ${ }^{3}$ The first example of practical calculations of the graph cohomology - with respect to the edge contracting differential - is found in [1]; a wide range of vertex-edge bi-degrees is considered there.
    ${ }^{4}$ The vector space of graphs under study is infinite dimensional; however, it is endowed with the bi-grading (\#vertices, \#edges) so that all the homogeneous components are finite dimensional.

[^7]:    ${ }^{5}$ Let the enumeration of vertices in every such term in the sum start running over the enumerated vertices in $\gamma_{2}$ until $v$ is reached. Now the enumeration counts the vertices in the graph $\gamma_{1}$ and then it resumes with the remaining vertices (if any) that go after $v$ in $\gamma_{2}$.
    ${ }^{6}$ The postulated precedence or antecedence of the wedge product of edges from $\gamma_{1}$ with respect to the edges from $\gamma_{2}$ in every graph within $\gamma_{1} \circ_{i} \gamma_{2}$ produce the operations $\circ_{i}$ which coincide with or, respectively, differ from Definition 2 by the sign factor $(-) \# \mathrm{E}\left(\gamma_{1}\right) \cdot \# \mathrm{E}\left(\gamma_{2}\right)$. The same applies to the Lie bracket of graphs [ $\gamma_{1}, \gamma_{2}$ ] if the operation $\gamma_{1} \circ_{i} \gamma_{2}$ is the insertion of $\gamma_{2}$ into $\gamma_{1}$ (as in [14]). Anyway, the notion of d-cocycles which we presently recall is well defined and insensitive to such sign ambiguity.

[^8]:    ${ }^{7}$ This is why the assumption $N(v) \geqslant 3$ is important. Indeed, the disjoint-pair cancellation mechanism does work only for chains with even numbers of valency-two vertices $v$ in $\gamma$. Here is an example (of one such vertex $v$ between $a$ and $b$ ) when it actually does not: in the differential of a graph that
     term can be cancelled against either the first or the last one but not with both of them simultaneously.

[^9]:    ${ }^{8}$ None of the results in Theorem 7 and Proposition 8 involves floating point operations in the way how it is obtained; hence even if computer-assisted, both the claims are exact.

[^10]:    ${ }^{9}$ Indeed, we recall that these cohomology dimensions - in the count with versus without restriction $N(v)>2$ of the valency - are the same (e.g., see [16, Proposition 3.4] with a sketch of the proof).

[^11]:    ${ }^{10}$ Another software package for numeric computation of the graph complex cohomology groups in various degrees and loop orders is available from https://github.com/wilthoma/GHoL.

[^12]:    ${ }^{1}$ Graphs $\gamma^{a}$ and $\gamma^{b}$ appear in sets of four in $\mathrm{d} \circ \mathrm{d}\left(\gamma^{2}\right)$. Namely, the edges can both be inserted in two ways - still producing exactly the same graph - by exchanging the roles of vertex $1^{(i)}$ and $2^{(i)}$ for $i=0,1$, that is, by swapping the set $A$ with the set $B$ (for $i=1$ ) and swapping the set $C$ with the set $D$ (for $i=0$ ).

[^13]:    ${ }^{2}$ Any permutation $\sigma$ that defines a symmetry of graph $\gamma$ acts on the space $X_{\gamma}$ of vertices and edges in a particular way: vertices are sent to vertices and edges are sent to edges. Hence we could also consider $\sigma$ acting on the product space $\mathrm{V}(\gamma) \times \mathrm{E}(\gamma)$ of the set of vertices with the set of edges. However, it turns out that this would be inconvenient for the notation in the proof of Theorem 4.
    ${ }^{3}$ Permutation $\sigma$ acts here on the power set $P\left(X_{\gamma}\right)$ in the sense that the power set $P\left(X_{\gamma}\right)$ itself is left invariant under $\sigma$, not the elements of the power set. In turn, permutation $\sigma$ acts on the product set $P\left(X_{\gamma}\right) \times \cdots \times P\left(X_{\gamma}\right)$ in the sense that this product set is left invariant under $\sigma$, not the elements of the product set.

[^14]:    ${ }^{1}$ The edges are antipermutable so that a graph which equals minus itself - under a symmetry that induces a parity-odd permutation of edges - is proclaimed to be equal to zero. In particular (view $\bullet \bullet \bullet$ ), every graph possessing a symmetry which swaps an odd number of edge pairs is a zero graph. For example, the 4 -wheel $12 \wedge 13 \wedge 14 \wedge 15 \wedge 23 \wedge 25 \wedge 34 \wedge 45=I \wedge \cdots \wedge V I I I$ or the $2 \ell$-wheel at any $\ell>1$ is such; here, the reflection symmetry is $I \rightleftarrows I I I, V \rightleftarrows V I I$, and $V I \rightleftarrows V I I I$.
    ${ }_{2}$ An oriented graph equals minus itself, hence it is a zero graph if there is a permutation of labels for its internal vertices such that the adjacency tables for the two vertex labellings coincide but the two realisations of the same graph differ by the ordering of outgoing edges at an odd number of internal vertices (see Example 3 below).
    ${ }^{3}$ Let the enumeration of vertices in every such term in the sum start running over the enumerated vertices in $\gamma_{2}$ until $v$ is reached. Now the enumeration counts the vertices in the graph $\gamma_{1}$ and then it resumes with the remaining vertices (if any) that go after $v$ in $\gamma_{2}$.

[^15]:    ${ }^{4}$ The postulated precedence or antecedence of the wedge product of edges from $\gamma_{1}$ with respect to the edges from $\gamma_{2}$ in every graph within $\gamma_{1} \circ_{i} \gamma_{2}$ produce the operations $\circ_{i}$ which coincide with or, respectively, differ from Definition by the sign factor $(-)^{\# E\left(\gamma_{1}\right) \cdot \# E\left(\gamma_{2}\right)}$. The same applies to the Lie bracket of graphs $\left[\gamma_{1}, \gamma_{2}\right]$ if the operation $\gamma_{1} \circ_{i} \gamma_{2}$ is the insertion of $\gamma_{2}$ into $\gamma_{1}$ (as in [11). Anyway, the notion of d-cocycles which we presently recall is well defined and insensitive to such sign ambiguity.

[^16]:    ${ }^{5}$ This algorithm can be modified so that it works for an input which is not skew, namely, by replacing skew Leibniz graphs by ordinary Leibniz graphs (that is, by skipping the skew-symmetrisation). For example, this strategy has been used in 3, 4] to show that the Kontsevich star product $\star \bmod \bar{o}\left(\hbar^{4}\right)$ is associative: although the associator $(f \star g) \star h-f \star(g \star h)=\diamond(\mathcal{P}, \operatorname{Jac}(\mathcal{P}))$ is not skew, it does vanish for every Poisson structure $\mathcal{P}$.

[^17]:    ${ }^{6}$ If we want to generate not only $n$-vectors but all graphs of arbitrary differential orders, then we let $\ell$ start at 0 (so that the sinks are included).

[^18]:    ${ }^{7}$ Finding solutions $\mathcal{Q}(\mathcal{P})$ with tadpoles or extra sinks - with fixed arguments - is a separate problem.
    8 There are only 12 admissible graphs to build cocycles from; of these 12 , as many as 6 are zero graphs. This count shows to what extent the number of graphs decreases if one restricts to only the flows $\mathcal{Q}=\mathrm{O} \overrightarrow{\mathrm{r}}(\gamma)$ obtained from cocycles $\gamma \in \operatorname{ker}(\mathrm{d})$ in the non-oriented graph complex.

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    ${ }^{1}$ The dilation $\dot{\mathcal{P}}=\mathcal{P}$ is an example of symmetry for Jacobi identity; we study nonlinear flows $\dot{\mathcal{P}}=\mathcal{Q}(\mathcal{P})$ which are universal w.r.t. all affine manifolds and should persist under the quantization $\frac{\hbar}{i}\{\cdot, \cdot\}_{\mathcal{P}} \mapsto[\cdot, \cdot]$.

[^20]:    ${ }^{2}$ In earnest, graphs with valency 1 of an end of $E$ cancel out in the action of this differential d, cf. 4, 8,
    ${ }^{3}$ One proves that $\mathrm{d}($ zero graph $)=$ sum of zero graphs and graphs with zero coefficients.
    ${ }^{4}$ The present paper is aimed to help us reveal the general formula of the morphism O $\vec{r}$ which connects the two graph complexes.

[^21]:    ${ }^{5}$ The algorithm from [5], §1.2] produces 41031 Leibniz graphs in $\nu=3$ iterations and 56509 at $\nu \geqslant 7$.
    ${ }^{6}$ This is done because it is anticipated that, counting the number of ways to obtain a given bi-vector while orienting the nonzero cocycle $\gamma_{5}$, none of the coefficients in a solution $\mathcal{Q}_{5}$ vanishes.
    ${ }^{7}$ The analytic formula of degree-six nonlinear differential polynomial $\mathcal{Q}_{5}(\mathcal{P})$ is given in App. A. The encoding of 8691 Leibniz tri-vector graphs containing the Jacobiator $\llbracket \mathcal{P}, \mathcal{P} \rrbracket$ for the Poisson structure $\mathcal{P}$ that occur in the r.-h.s. $\diamond(\mathcal{P}, \llbracket \mathcal{P}, \mathcal{P} \rrbracket)$ is available at https://rburing.nl/Q5d5.txt. The machine format to encode such graphs (with one tri-valent vertex for the Jacobiator) is explained in [5] (see also [1, 3]).

[^22]:    ${ }^{a}$ The actually found $\partial_{\mathcal{P}}$-cocycle $\mathcal{Q}$ might differ from the value $\mathrm{O} \overrightarrow{\mathrm{r}}(\gamma)$ by $\partial_{\mathcal{P}}$-trivial or improper terms, i.e. $\mathcal{Q}=\mathrm{O} \overrightarrow{\mathrm{r}}(\gamma)+\partial_{\mathcal{P}}(X)+\nabla(\mathcal{P}, \llbracket \mathcal{P}, \mathcal{P} \rrbracket)$ for some vector field $X$ realized by Kontsevich graphs and for some "Leibniz" bi-vector graphs $\nabla$ vanishing identically at every Poisson structure $\mathcal{P}$.
    ${ }^{b}$ As soon as the expression of 167 Kontsevich graph coefficients in $\mathcal{Q}_{5}$ via the 91 integer parameters was obtained, the linear system in factorization $\llbracket \mathcal{P}, \mathcal{Q}_{5}(\mathcal{P}) \rrbracket=\diamond(\mathcal{P}, \llbracket \mathcal{P}, \mathcal{P} \rrbracket)$ for the pentagon-wheel flow $\mathcal{P}=\mathcal{Q}_{5}(\mathcal{P})$ was solved independently by A. Steel (Sydney) using the Markowitz pivoting run in Magma. The flow components $\mathcal{Q}_{5}$ of all the known solutions $\left(\mathcal{Q}_{5}, \widehat{\wedge}_{5}\right)$ match identically. (For the flow $\dot{\mathcal{P}}=\mathcal{Q}_{5}(\mathcal{P})=$

