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Predictability of extreme events in one-dimensional maps

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Abstract

The predictability of extremes in dynamical models can be compared to that of non-extremes. Past research has evaluated this predictability using methods based either on statistics or on dynamical systems theory, leading to different conclusions. This thesis assesses predictability in the context of dynamical systems, by studying distributions of finitetime Lyapunov exponents (FTLEs). These FTLEs measure error growth rates. The predictability of the extreme is quantified by comparing the distribution of FTLEs of all other initial conditions. We focus the research on one-dimensional maps. Three maps are studied in detail, through the combination of an analytical approach and a numerical approximation. We come to the conclusion that the predictability of extremes depends on both the map and the definition of the extreme. Finally, we discuss the validity and the implications of the method we used.

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1 Introduction

1.1 Literature review

Extreme value theory Extreme value theory classically is concerned with the statistics of extreme deviations. Predominantly statistical processes have been studied thoroughly in this field. Developments like the generalized extreme value distribution (GEVD) have allowed for studying extremes through the block maximum method. Alternatively extreme events can be qualified as values which exceed a certain threshold. The method of studying these extreme events is often referred to as peak over threshold (POT). This method compares distributions of 'standard data' (not exceeding the threshold) to 'peaks' (exceeding the threshold) [3].

The block maximum method has the disadvantage of potentially ignoring information on other values in a block, since it does not distinguish blocks with many extreme values from blocks containing only a few. The POT method has the disadvantage of having to pre-specify a suitable threshold. Which method is used is therefore determined according to which best suits the situation.

The POT research has expanded towards stochastic-deterministic and deterministic processes [6]. The main argument is the fact that many practical applications of extreme values occur in geophysics. Dynamical models (often chaotic, deterministic systems) are frequently central in describing these geophysical processes [11]. The approach of analyzing these processes therefore shifts from pure statistics towards a combination of statistics with dynamical and stochasticdynamical systems theory. Extreme value statistics thereby becomes a method of evaluating the predictability of extreme events in these dynamical systems. This article will also evaluate the predictability of extreme events in dynamical systems.

Predictability There are different methods for evaluating the predictability of extreme events in systems. One of these methods is through receiver operating characteristic (ROC) curves. This method treats states of the system as binary events and compares their true positive rates to their false positive rates. Classically the ROC curves of different diagnostic tests/predictors are compared to determine which is most suitable in a given situation. In extreme value theory, the ROC curve of the entire system is compared to the ROC curve of the entire system is compared to the ROC curve of the entire system is compared to the ROC curve of the extreme event.

The advantage of this method is that it is both applicable to measurementbases datasets and distributions generated from dynamical models. Thereby it can compare theory to practice. A disadvantage is that though it can evaluate dynamical systems, it still mostly treats them stochastically. By treating the orbit of a system as a sequence of binary events, one ignores that in essence these events are not independent from each other. Researchers who used the ROC curve approach include Franzke who, after reducing the order of his model by stochastic mode reduction, found that extreme events are predictable [4], basing his approach on Hallerberg et al. [10]. His studies find better predictability for larger event sizes than smaller ones [5]. Hallerberg herself wrote multiple papers on this topic, also finding that large threshold crossings are better predictable [7]. Together with Kantz, she also investigated extreme increments in time-series and found that extreme events are better predictable [9, 8].

A somewhat similar method to the ROC curves is the equitable threat score (ETS), where not only the false positive but also the false negative rate is incorporated, whilst also compensating for random hits. Stephenson et al. presented a variant of this method more specified on extreme events, called the extreme dependence score (EDS). Since the ETS tends to zero for vanishingly rare events, they propose that their EDS is better for assessing the skill in deterministic forecasts of rare events [13]. Unfortunately they did not apply the research to extremes in dynamical systems.

An entirely different approach to the predictability of extremes analyses the growth rate of errors. This is fundamentally different from the previous methods since it has its origins in dynamical systems theory, rather than stochastics. Typically the growth rate is expressed in Lyapunov exponents or singular values and compared for different initial conditions and prediction times.

The upside of this approach is the fact that it analyses characteristics of the underlying dynamical system, rather than treating the successive (semi-)dependent points on the orbit as random events. The downside of this method however is that it is only applicable to the dynamical system itself and not to a dataset of observations. It therefore can never account for model errors [1].

Sterk et al. investigated growth rates of errors via finite-time Lyapunov exponents and evaluated them using invariant probability measures. Their research into multiple systems showed general statements on the predictability of extreme events cannot be made, since it is dependent on the geometry of the attractor, the prediction time and the observable in question [15, 14].

From the above it becomes apparent that the difference in assumptions between the two approaches results in different conclusions. Ghil et al. briefly show this by predicting the cumulative probability distribution of extremes of the cusp map with both a statistical and a deterministic method [6].

1.2 Research question

In this paper we will approach predictability in the deterministic setting, in the spirit of Sterk et al. This approach is more suitable for a Master Project in the field of Dynamical Systems. Sterk et al. investigated dynamical systems like the Lorenz-96 model [15] and the barotropic voritcity equation [14]. This paper will attempt to add to the information gathered in these articles, by using similar methods to investigate other chaotic dynamical systems. This leads to the following research question:

What is the predictability of extremes compared to non-extremes for one-dimensional maps?

The reason for choosing one-dimensional systems is twofold. On the one hand, it allows for an easier application of the methods described by Sterk et al. On the other hand, it allows us to not only approach the research question in a numerical way, but also analytically.

2 Methodology

2.1 Finite-time Lyapunov exponents

As explained in section 1.2, we aim to study the relative predictability of extreme events inside a one-dimensional system. In order to approach this question we will inspect multiple one-dimensional systems and choose a suitable definition of an extreme event for each. Subsequently, we will compare its relative predictability to that of non-extremes for different prediction times. Firstly however, we need a quantification of what relative predictability is.

The definition of predictability used in this paper is based on comparing the growth of small perturbations in initial conditions [15, 14]. We apply this by finding the extreme event, going back to its pre-image for that prediction time and checking how the growth rate of small perturbations for these initial conditions compare to the growth rate of small perturbations for all other initial conditions.

The focus of this paper will be on one-dimensional maps, which means any system we study can be written as

$$f^t(x) = \left(\overbrace{f \circ f \circ \cdots \circ f \circ f}^{t \text{ times}}\right)(x),$$

where $f: I \to I$ for some $I \subseteq \mathbb{R}$ and $t \in \mathbb{N} \cup \{0\}$.

Definition 2.1. If f is a one-dimensional mapping, the expression λ as defined by

$$\lambda_{t}(x) = \frac{1}{t} \sum_{k=0}^{t-1} \ln\left(\left|f'\left(f^{k}\left(x\right)\right)\right|\right)$$

is called a finite-time Lyapunov exponent (for here on denoted by FTLE). Assuming perturbations are infinitesimally small, it describes the growth rate of errors at initial condition x.

The growth rate of a perturbation ϵ at some point x after time-step t is given by

$$\frac{|f^t(x+\epsilon) - f^t(x)|}{\epsilon}.$$
 (1)

The errors being infinitesimally small allows us to take a limit of expression (1), from ϵ tending to 0. This is equal to the derivative of f^t at x:

$$\lim_{\epsilon \to 0} \frac{|f^t(x+\epsilon) - f^t(x)|}{\epsilon} = \left| \frac{d}{dx} f^t(x) \right|.$$

The implications of the assumption that errors are infinitesimally small will be discussed in section 4.3. We can expand the derivative using the chain rule and find that

$$\left. \frac{d}{dx} f^t(x) \right| = \left| \prod_{k=0}^{t-1} f'\left(f^k\left(x\right) \right) \right|,$$

where f' denotes the derivative of f. We equate this derivative to an exponentially increasing function $e^{t\lambda}$,

$$e^{t\lambda} = \left|\prod_{k=0}^{t-1} f'\left(f^k\left(x\right)\right)\right|,$$

which means λ will denote the magnitude of the growth of perturbations. Rewriting the equation allows us to express λ as follows

$$\lambda_{t}(x) = \frac{1}{t} \ln \left(\prod_{k=0}^{t-1} \left| f'(f^{k}(x)) \right| \right)$$

= $\frac{1}{t} \sum_{k=0}^{t-1} \ln \left(\left| f'(f^{k}(x)) \right| \right),$ (2)

using the logarithm chain rule.

For a given time step t, there is a distribution of FTLEs for all different values of x on the attractor of the map. It is this distribution that we want to compare to a distribution resulting from FTLEs of extremes. In order make this comparison we need an invariant probability measure μ .

2.2 Invariant probability measures

This section is largely based on the exposition in [2]. If a set does not have countably many elements, it may be hard to make statements about its size. In this case measures can be used to say something mathematically meaningful. **Definition 2.2.** \mathcal{A} is a collection of subsets of set A, which is called a σ -algebra if the following properties are satisfied

- 1. $\emptyset, A \in \mathcal{A},$
- 2. if $X \in \mathcal{A}$, then also $A \setminus X \in \mathcal{A}$,
- 3. if $X_1, X_2, X_3, \ldots \in \mathcal{A}$, then also $\bigcup_{n=1}^{\infty} X_n \in \mathcal{A}$.

Definition 2.3. Let \mathcal{A} be a σ -algebra of A. Then $\mu : \mathcal{A} \to \mathbb{R}$ is called a probability measure for the set A if for all $X \in \mathcal{A}$, the following holds

- 1. $\mu(X) \in [0,1],$
- 2. $\mu(\emptyset) = 0$,
- 3. $\mu(A) = 1$ and
- 4. if X_1, X_2, X_3, \ldots are all disjoint, then

$$\mu\left(\bigcup_{n=1}^{\infty} X_n\right) = \sum_{n=1}^{\infty} \mu\left(X_n\right).$$

Moreover, if μ is a probability measure for A, the integral

$$\int_A \varphi \,\mathrm{d}\mu$$

is defined and represents the space average of map φ .

Definition 2.4. Assume that set A has probability measure μ . A map $f : A \to A$ is called a measure preserving transformation for A and μ if it follows that for any measurable $E \subseteq A$,

$$\mu\left(E\right) = \mu\left(f^{-1}E\right).$$

In this case μ is called an invariant probability measure for A and f.

Via the σ -algebra, definition 2.2 gives us a method to construct a set of sets containing a parent set's different subsets and their unions. We thereafter define a measure which maps each of these subsets towards a value between 0 and 1 in definition 2.3. In definition 2.4 we signify a relation between a measure and map such that the measure of a subset does not change if the map is applied.

In order to determine an integral with respect to a measure, we will often apply the Radon-Nikodym theorem.

Definition 2.5. Say measures μ and ν are defined for the same σ -algebra \mathcal{A} . μ is said to be absolutely continuous with respect to ν if $\nu(A) = 0$ implies $\mu(A) = 0$, for all $A \in \mathcal{A}$, . **Theorem 2.1** (Radon-Nikodym theorem). Say μ and ν are both invariant probability measures with respect to the same σ -algebra and set A. φ is a μ -integrable function. If μ is absolutely continuous with respect to ν , then there exists a function g such that

$$\int_A \varphi \,\mathrm{d}\mu = \int_A \varphi g \,\mathrm{d}\nu \,.$$

In this case we often write that

$$g = \frac{\mathrm{d}\mu}{\mathrm{d}\nu}.$$

In our case ν will often be the standard Lebesgue measure. If the invariant probability measures of our systems are absolutely continuous with respect to the Lebesgue measure, we can use that

$$\int_{A} \varphi \, \mathrm{d}\mu = \int_{A} \varphi \frac{\mathrm{d}\mu}{\mathrm{d}x} \, \mathrm{d}x \, .$$

We will end this section by pointing out Birkhoff's ergodic theorem:

Definition 2.6. Let $f : A \to A$ be a measure preserving transformation for set A and invariant probability measure μ . Let \mathcal{A} be the σ -algebra of A. We say f is ergodic iff for every $X \in \mathcal{A}$, it follows that $f^{-1}(X) = X$ implies that either $\mu(X) = 0$ or $\mu(X) = 1$.

Theorem 2.2 (Birkhoff's ergodic theorem). Let $f : A \to A$ be a measure preserving transformation for A and invariant probability measure μ . Let $\varphi : A \to \mathbb{R}$ be a measurable function for measure μ . If f is ergodic, then

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n} \varphi \circ f^k(a) = \int_A \varphi \,\mathrm{d}\mu\,,\tag{3}$$

for almost all $a \in A$ (the collection of values for which this does not hold has measure 0).

Definition 2.6 defines ergodicity, a formal method of denoting a map which visits every part of system from any part of that system. Theorem 2.2 uses this ergodicity and finds an equality between the limit on the left side of equation (3) (which is the time average) and the constant on the right side (which is the space average). Intuitively this theorem tells us that after a sufficient number of iterations an ergodic system will have 'forgotten' its initial state.

2.3 Distribution functions of FTLEs

The different definitions and theorems from sections 2.1 and 2.2 are used in two distribution functions which we will be using to determine the relative predictability of extremes and non-extremes of a system. Firstly, the distribution function of the FTLEs λ of the attractor A is given by

$$P_t \left(\lambda \le s\right) = \mu \left(\left\{ x \in A \mid \lambda_t(x) \le s \right\} \right). \tag{4}$$

This distribution can be compared to the conditional distribution function for the FTLEs leading to the extreme region E, which is given by

$$P_t (\lambda \le s \mid E) = \frac{\mu \left(\{ x \in A \mid \lambda_t(x) \le s \} \cap f^{-t}(E) \right)}{\mu(E)}.$$
 (5)

In the case of this paper both the set A and the map f are given. In order to make f a measure preserving transformation, we will therefore need the corresponding invariant probability measure μ .

The observant reader will have noticed that the formula for the space average in Birkhoff's ergodic theorem is very similar to the formula in definition 2.1. If in equation (2) $\ln (|f'|)$ is integrable and f is ergodic, we can apply theorem 2.2 and conclude that the distribution (4) and (5) approach a constant value independent of x, the space average.

2.4 Numerical method

This analytic method has two practical disadvantages. Firstly, a system has to have a known invariant probability measure in order for us to be able to construct the distribution functions. In practice, invariant probability measures are only known for some systems. Secondly, with each iteration the maps become more complex and therefore the distribution functions become harder to determine by hand.

To counter these impracticalities, we present a method to approximate distribution (4) and (5) numerically. The boxplots found in section 3 are all generated using this method. For the following steps let A be the region where f is defined.

- 1. Choose some $x \in A$.
- 2. Run $x = f^{1}(x)$ for a sufficiently large number of times, we do not use this transient.
- 3. Run $x = f^{1}(x)$ again, now for *n* times and put the iterations x_{1}, \ldots, x_{n} in a list *L*.
- 4. For some time-step t, replace the n entries of list L by their image $f^t(x_i)$ for i = 1, ..., n.
- 5. For some extreme threshold u, find the values $x_i \in L \mid x_i \geq u$ and delete the other values from L.
- 6. Replace the remaining m entries of list L by their pre-image $f^{-t}(x_j)$, where $x_j \in L$ for j = 1, ..., m.
- 7. Replace the new *m* entries of list *L* by their FTLE $\lambda_t(x_j)$, where $x_j \in L$ for $j = 1, \ldots, m$.

- 8. Plot the m FLTEs from list L in a boxplot.
- 9. Repeat all of the above for other values of u and t.

This method only makes sense if Birkhoff's Ergodic Theorem can be applied. The first three steps are needed to create a list of initial conditions in the distribution of the system. In order to do this they depend on the 'forgetting' of initial state, since this method would otherwise be biased towards the initial random value chosen in step 1. When using this method, we will therefore always first check that theorem 2.2 can be applied.

3 Results

3.1 The logistic map

The first map we apply the general methodology of section 2 on, is the logistic map. The logistic map is one of the simplest maps which shows chaotic behavior and is therefore suited for studying the effects of extreme events in chaotic dynamical systems. The map is the discrete-time variant of the logistic function and is given as

$$f(x) = rx\left(1 - x\right).$$

It was popularized amongst others by biologist Robert May as a simple model to look at the growth and decline of a population [12]. Here $x \in [0, 1]$ and represents the fraction of the maximal population, whilst $r \in [0, 4]$ is a determinable scalar dependent on the type of system. When concerning biological models, this scalar is often called the 'fertility parameter'.

Depending on the value of r, this system is periodic or chaotic. This paper will focus on the chaotic regions. Which values of r give rise to chaos can be best visualized via a bifurcation diagram, as given by figure 1. This figure should be seen as a collection of vertical attractor plots of the logistic map for each value of r.

Looking at figure 1, we see that chaos occurs mostly for values of r which are larger than 3.5. Additionally we see a succession of pitchfork bifurcations, fist at $r \approx 3.0$, the second at $r \approx 3.45$, the third at $r \approx 3.54$, the fourth at $r \approx 3.56$, etc. In the chaotic region of the diagram, these bifurcations often come accompanied with periodic windows which start at $r \approx 3.62$, $r \approx 3.73$ and $r \approx 3.83$.

A second observation from figure 1 is that the density doesn't seem to be equally distributed across the attractor. For r = 4, we are able to determine this density analytically and approximate it numerically, resulting in figure 2. The observation that the outer regions of the attractor have the highest density is in accordance with figure 1.



Figure 1: A bifurcation diagram of the logistic map for $2.5 \leq r \leq 4.0.$



Figure 2: A numerically generated probability histogram of the logistic map for r = 4, plotted alongside its analytic density function $\frac{1}{\pi\sqrt{x(1-x)}}$.

3.1.1 Analytical results

In this section we will approach the predictability of the logistic map analytically. Unfortunately, the invariant probability measure for the logistic map is known only for r = 4, which is why we will focus on the r = 4 case. Despite providing just a partial answer to the research question, the analytic answers will give us insight in the characteristics of the distribution equations. More importantly, it will serve as a benchmark to check the validity and accuracy of the numerical method when we approach other values of r and larger values of t.

Lemma 3.1. The corresponding invariant probability measure for the logistic map with r = 4 is given by

$$\mu\left([a,b]\right) = \int_{a}^{b} \frac{1}{\pi\sqrt{x(1-x)}} \,\mathrm{d}x$$

Proof. We have to show that $\mu([0,1]) = 1$ and $\mu(f^{-1}[a,b]) = \mu([a,b])$. Indeed we find that

$$\mu([0,1]) = \int_0^1 \frac{1}{\pi\sqrt{x(1-x)}} dx = \left[\frac{1}{\pi}\arcsin\left(2x-1\right)\right]_0^1 = 1.$$
 (6)

We can identify the pre-image of f and see that

$$\mu\left(f^{-1}\left[a,b\right]\right) = \mu\left(\left[\frac{1}{2} - \frac{1}{2}\sqrt{1-a}, \frac{1}{2} - \frac{1}{2}\sqrt{1-b}\right] \cup \left[\frac{1}{2} + \frac{1}{2}\sqrt{1-b}, \frac{1}{2} + \frac{1}{2}\sqrt{1-a}\right]\right)$$
$$= \frac{2}{\pi}\left(\arcsin\left(\sqrt{1-a}\right) - \arcsin\left(\sqrt{1-b}\right)\right).$$

Using the trigonometric relation that $\sin(\arccos(x)) = \sqrt{1 - x^2}$, this expression is equal to

$$\frac{2}{\pi} \left(\arccos\left(\sqrt{a}\right) - \arccos\left(\sqrt{b}\right) \right)$$

Since $0 \le x \le 1$ for the logistic map, we can use that $\arccos(x) = \frac{1}{2}\arccos(2x^2) - 1$ to rewrite the expression above into

$$\frac{1}{\pi}\left(\arccos\left(2a-1\right)-\arccos\left(2b-1\right)\right).$$

Now since $\arccos(x) = \frac{\pi}{2} - \arcsin(x)$, this expression can finally be rewritten into

$$\frac{1}{\pi} \left(\arcsin\left(2b - 1\right) - \arcsin\left(2a - 1\right) \right) = \mu\left([a, b]\right).$$

Now that we have verified the invariant probability measure, we continue the analytic investigation. As discussed in section 2.4, the distributions rapidly become more complex as t increases, which is why we will mostly focus on t = 1. Findings for t = 2 will be presented as well, albeit briefly.



Figure 3: The FTLE formula for the logistic map with t = 1 and r = 4.



Figure 4: The distribution function of FTLEs for logistic map with t = 1 and r = 4.

Lemma 3.2. The distribution function of the FTLEs for the logistic map with r = 4 and t = 1 is given by

$$P_1 (\lambda \le s) = \frac{2}{\pi} \arcsin\left(e^{s-\ln(4)}\right)$$

Proof. In equation (6) we already used that the state space of the logistic map at r = 4 is A = [0, 1], which is also in accordance with figure 1. In order to determine the distribution function of the FTLEs, we combine this A with the invariant probability measure from lemma 3.1 and the FTLE formula from equation (2), applied to t = 1. The latter reduces to one term, namely:

$$\lambda_1(x) = \ln(|4 - 8x|).$$
(7)

Figure 3 shows what equation (7) looks like when plotted on [0, 1], it is defined everywhere except at $x = \frac{1}{2}$. This is due to the fact that at $x = \frac{1}{2}$, it follows that $\lambda(\frac{1}{2}) = \ln(0)$, which is not defined.

Equation (4) now reduces to a function dependent only on s, since

$$P_{1} (\lambda \leq s) = \mu \left(\left\{ x \in [0,1] \mid \lambda_{1}(x) \leq s \right\} \right)$$

$$= \mu \left(\left\{ x \in [0,1] \mid \frac{-e^{s}+4}{8} \leq x \leq \frac{e^{s}+4}{8} \right\} \right)$$

$$= \mu \left(\left[\frac{-e^{s}+4}{8}, \frac{1}{2} \right] \cup \left(\frac{1}{2}, \frac{e^{s}+4}{8} \right] \right)$$

$$= \mu \left(\left[\frac{-e^{s}+4}{8}, \frac{e^{s}+4}{8} \right] \right)$$

$$= \frac{1}{\pi} \left(\arcsin \left(\frac{e^{s}}{4} \right) - \arcsin \left(\frac{-e^{s}}{4} \right) \right)$$

$$= \frac{2}{\pi} \arcsin \left(\frac{e^{s}}{4} \right)$$

$$= \frac{2}{\pi} \arcsin \left(e^{s-\ln(4)} \right),$$
(8)

where $s \in (-\infty, \ln(4)]$. In equation (8) we are allowed to the intervals since a single point has measure zero $(\mu(\frac{1}{2}) = 0)$. Plotting equation (8), we find figure 4, which is in accordance with figure 3.

In order to look at the conditional distribution function, we need to define the extreme event space. We choose to let the extreme event E depend on a parameter u, in the following way:

$$E_u = \left[u, 1\right],$$

with $u \in [0, 1]$.



Figure 5: The t = 1 iteration of the logistic map for r = 4.



Figure 6: The conditional distribution function of FTLEs for the logistic map with t = 1 and r = 4, for four values of u.

Lemma 3.3. The conditional distribution function of the FTLEs for the logistic map with r = 4 and t = 1 is given by

$$P_1\left(\lambda \le s \mid E_u\right) = \begin{cases} \frac{2 \operatorname{arcsin}\left(\frac{1}{4}e^s\right)}{\frac{\pi}{2} - \operatorname{arcsin}(2u-1)} & \text{if } s \le \ln\left(4\sqrt{1-u}\right)\\ 1 & \text{if } s > \ln\left(4\sqrt{1-u}\right) \end{cases}.$$

Proof. In figure 5 we observe that an extreme interval $E_u = [u, 1]$ has an inverse image $f^{-1}(E_u)$ which is somewhere central in [0, 1]. Indeed we find that

$$4x(1-x) \ge u$$
$$x^{2} - x + \frac{u}{4} \le 0$$
$$\frac{1}{2} - \frac{1}{2}\sqrt{1-u} \le x \le \frac{1}{2} + \frac{1}{2}\sqrt{1-u}.$$

Using μ we determine that

$$\mu([u, 1]) = \frac{1}{\pi} (\arcsin(2 - 1) - \arcsin(2u - 1))$$
$$= \frac{1}{2} - \frac{1}{\pi} \arcsin(2u - 1).$$

We can now determine equation (5) and find that

$$P_{1} (\lambda \leq s \mid E_{u}) = \frac{\mu \left(\{x \in [0,1] \mid \lambda(x) \leq s\} \cap f^{-1} (E_{u}) \right)}{\mu (E_{u})}$$

$$= \frac{\mu \left(\left[\frac{-e^{s}+4}{8}, \frac{e^{s}+4}{8} \right] \cap \left[\frac{1}{2} - \frac{1}{2}\sqrt{1-u}, \frac{1}{2} + \frac{1}{2}\sqrt{1-u} \right] \right)}{\frac{1}{2} - \frac{1}{\pi} \arcsin(2u-1)}$$

$$= \begin{cases} \frac{\frac{2}{\pi} \arcsin(\frac{1}{4}e^{s})}{\frac{1}{2} - \frac{1}{\pi} \arcsin(2u-1)} & \text{if } \frac{e^{s}+4}{8} \leq \frac{1}{2} + \frac{1}{2}\sqrt{1-u} \\ \frac{\frac{2}{\pi} \arcsin(\sqrt{1-u})}{\frac{1}{2} - \frac{1}{\pi} \arcsin(2u-1)} & \text{if } \frac{e^{s}+4}{8} > \frac{1}{2} + \frac{1}{2}\sqrt{1-u} \end{cases}$$

$$= \begin{cases} \frac{2 \arcsin(\sqrt{1-u})}{\frac{1}{2} - \frac{1}{\pi} \arcsin(2u-1)} & \text{if } \frac{e^{s}+4}{8} > \frac{1}{2} + \frac{1}{2}\sqrt{1-u} \\ \frac{1}{2} - \frac{1}{\pi} \arcsin(2u-1)} & \text{if } s \leq \ln \left(4\sqrt{1-u} \right) \\ 1 & \text{if } s > \ln \left(4\sqrt{1-u} \right). \end{cases}$$

We see equation (9) reduces to a function dependent on the relation of u and s, which we plot in figure 6. The Mathematica code for this figure can be found in appendix A.

Proposition 3.1. For r = 4 and t = 1, extremes of the logistic map are better predictable than non-extremes.

Proof. From figure 6 we get the sense that extremes seem to be better predictable than non-extremes. We however want to check that this is true in general and not only for the three values of u which are plotted. We therefore



Figure 7: The first, second and third quartiles of the conditional distribution function of FTLEs for the logistic map with t = 1 and r = 4.



Figure 8: The formula for FTLEs for the logistic map with t = 2 and r = 4.

determine the first, second and third quartiles. We find the following functions dependent on u (and plotted in figure 7):

$$P_1 \left(\lambda \le s \mid E_u \right) = \frac{1}{4}$$

$$s = \ln \left(4 \sin \left(\frac{\pi}{16} - \frac{1}{8} \arcsin \left(2u - 1 \right) \right) \right),$$

$$P_1 \left(\lambda \le s \mid E_u \right) = \frac{1}{2}$$

$$s = \ln \left(4 \sin \left(\frac{\pi}{8} - \frac{1}{4} \arcsin \left(2u - 1 \right) \right) \right)$$

and

$$P_1 \left(\lambda \le s \mid E_u \right) = \frac{3}{4}$$
$$s = \ln \left(4 \sin \left(\frac{3\pi}{16} - \frac{3}{8} \arcsin \left(2u - 1 \right) \right) \right).$$

We find that a higher u will lead to a higher $P_1(\lambda \leq s \mid E_u)$ for any s. We can therefore conclude that for t = 1 and r = 4, extremes are better predictable than non-extremes.

Proposition 3.2. For r = 4 and t = 2, extremes of the logistic map are better predictable than non-extremes.

Proof. In case of t = 2, the equations become more complex. In an effort to spare the reader of any arduously long equations, we will shortly describe the process of determining equations (4) and (5). At t = 2, the formula for the FTLEs (equation (2)) becomes

$$\lambda_2(x) = \frac{1}{2} \ln \left(|256x^3 - 384x^2 + 160x - 16| \right), \tag{10}$$

which is plotted in figure 8. We see that $\lambda_2 \leq s$ will result in either one large or three smaller intervals. The point where this changes can be found by calculating the derivative of equation (10) and equating in to 0. This results in five roots corresponding to the downward peaks (at $x = \frac{1}{2}, \frac{1}{2} - \frac{1}{2\sqrt{2}}$ and $\frac{1}{2} + \frac{1}{2\sqrt{2}}$) and the local maxima (at $x = \frac{1}{2} - \frac{1}{2\sqrt{6}}$ and $\frac{1}{2} + \frac{1}{2\sqrt{6}}$). We use the roots of

$$\frac{1}{2}\ln\left(|256x^3 - 384x^2 + 160x - 16|\right) \le s$$
$$-e^{2s} \le 256x^3 - 384x^2 + 160x - 16 \le e^{2s}$$

together with the position of the local maxima to construct equation (4), which results in figure 9. In figure 10, we can see that the reserve image of any extreme interval will typically consist of two intervals. Each interval lies around one of the local maxima of f^2 (although this maximum is not necessarily in the



Figure 9: The distribution function of FTLEs for logistic map with t = 2 and r = 4.



Figure 10: The t = 2 iteration of the logistic map for r = 4.

center of the interval). The location of the maxima can again be calculated by equating the derivative of f^2 to 0, from which we find the maxima are found at $x = \frac{1}{2} - \frac{1}{2\sqrt{2}}$ and $\frac{1}{2} + \frac{1}{2\sqrt{2}}$.

Using the information we gathered, we can now construct a plot for equation (5). Fortunately, we know that the upper and lower boundaries of the outer intervals of both $f^{-2}(E_u)$ and $\lambda_2(x) \leq s$ will be greater or smaller than $\frac{1}{2} - \frac{1}{2\sqrt{2}}$ or $\frac{1}{2} + \frac{1}{2\sqrt{2}}$ respectively. This diminishes the number of possible intersections of intervals. The result can be seen in figure 11 and a Mathematica document for the piecewise function underlying this plot can be found in Appendix A. Again we observe that extremes seem to be better predictable than non-extremes for t = 2 and r = 4. This function is also used to calculate first, second and third quantiles for t = 2, which are used in section 3.1.2.

3.1.2 Numerical results

In this section we will attempt to give an answer to the research question for larger values of t and other values of r. Before being able to use the numerical methodology from section 2.4, we have to show that

- 1. f(x) = rx(1-x) is ergodic and
- 2. $\ln(|r-2rx|)$ is integrable with respect to μ .

Since we need an invariant measure to prove ergodicity, we will only be able to provide this proof for the r = 4 case.

Lemma 3.4. f(x) = 4x(1-x) is ergodic.

f is ergodic iff $f^{-1}(E) = E$ for some $E \in \mathcal{A}$ implies that $\mu(E) = 0$ or $\mu(E) = 1$. We therefore assume we have an $E \in [0, 1]$ such that $f^{-1}(E) = E$. We can expand this equality to the closures of both sets,

$$\overline{f^{-1}(E)} = \overline{E}.$$

The closure of any $E \in [0, 1]$ will add its limit points and can be written as

 $\overline{E} = [e_1, e_2] \cup [e_3, e_4] \cup \ldots \cup [e_{n-1}, e_n],$

where we can order the e_i 's such that $e_1 < e_2 < \ldots < e_n$, for some $n \in 2\mathbb{N}$. For this E, it follows that

$$\overline{f^{-1}(E)} = \left[\frac{1}{2} - \frac{1}{2}\sqrt{1 - e_1}, \frac{1}{2} - \frac{1}{2}\sqrt{1 - e_2}\right] \cup \dots \cup \left[\frac{1}{2} - \frac{1}{2}\sqrt{1 - e_{n-1}}, \frac{1}{2} - \frac{1}{2}\sqrt{1 - e_n}\right]$$
$$\cup \left[\frac{1}{2} + \frac{1}{2}\sqrt{1 - e_n}, \frac{1}{2} + \frac{1}{2}\sqrt{1 - e_{n-1}}\right] \cup \dots \cup \left[\frac{1}{2} + \frac{1}{2}\sqrt{1 - e_2}, \frac{1}{2} + \frac{1}{2}\sqrt{1 - e_1}\right]$$

which is also ordered from small to large due to the ordering of the e_i 's. We know that $e_i \neq e_{i+1}$ for any *i*, which implies there are only three cases for which \overline{E} and $\overline{f^{-1}(E)}$ have an equal number of intervals:

- 1. $\overline{E} = \emptyset$,
- 2. $\overline{E} = \{1\}$ and
- 3. $\overline{E} = [e, 1]$ for some $e \in [0, 1)$.

In the first two cases, $\mu(E) = \mu(\overline{E}) = 0$. In the third case

$$\begin{split} \overline{E} &= \overline{f^{-1}(E)} \\ [e,1] &= [\frac{1}{2} - \frac{1}{2}\sqrt{1-e}, \frac{1}{2}] \cup [\frac{1}{2}, \frac{1}{2} + \frac{1}{2}\sqrt{1-e}] \\ &= [\frac{1}{2} - \frac{1}{2}\sqrt{1-e}, \frac{1}{2} + \frac{1}{2}\sqrt{1-e}] \\ 1 &= \frac{1}{2} + \frac{1}{2}\sqrt{1-e} \\ e &= 0, \end{split}$$

which means $\mu(E) = \mu(\overline{E}) = \mu([0,1]) = 1$. We conclude that f(x) = 4x(1-x) is ergodic.

Lemma 3.5. $\varphi = \ln (|4 - 8x|)$ is integrable with respect to μ and

$$\int_0^1 \varphi \,\mathrm{d}\mu = \ln\left(2\right).$$

Proof. μ is absolutely continuous with respect to the Lebesgue measure, which means that we can use theorem 2.1 to state that

$$\int_0^1 \ln \left(|4 - 8x| \right) \mathrm{d}\mu = \int_0^1 \ln \left(|4 - 8x| \right) \frac{\mathrm{d}\mu}{\mathrm{d}x} \, \mathrm{d}x \, .$$

 $\frac{\mathrm{d}\mu}{\mathrm{d}x}$ is the density of the r=4 logistic map, which means the expression above is equal to

$$\int_0^1 \frac{\ln\left(|4-8x|\right)}{\pi\sqrt{x(1-x)}} \,\mathrm{d}x = \int_0^{\frac{1}{2}} \frac{\ln\left(4-8x\right)}{\pi\sqrt{x(1-x)}} \,\mathrm{d}x + \int_{\frac{1}{2}}^1 \frac{\ln\left(8x-4\right)}{\pi\sqrt{x(1-x)}} \,\mathrm{d}x \,.$$

Using Mathematica we find that this expression is equal to

$$\frac{\ln(2)}{2} + \frac{\ln(2)}{2} = \ln(2).$$

Corollary 3.1. Any $x \in [0,1]$ of the logistic map with r = 4 has infinite time Lyapunov exponent

$$\lambda_{\infty}(x) = \lim_{t \to \infty} \frac{1}{t} \sum_{k=0}^{t-1} \ln\left(\left|f'\left(f^{t}\left(x\right)\right)\right|\right) = \int_{0}^{1} \varphi \,\mathrm{d}\mu = \ln\left(2\right).$$



Figure 11: The conditional distribution function of FTLEs for the logistic map with t = 2 and r = 4, for four values of u.



Figure 12: The numerical approximation of the conditional distribution of the FTLEs of the logistic map with r = 4, for $u \in \{0, 0.9, 0.95, 0.975\}$ and $t \in \{1, 2, 4, 8\}$.

Proof. Combine lemma 3.5 with theorem 2.2.

We have shown Birkhoff's ergodic theorem to hold for r = 4 case. Due to the lack of invariant probability measure for other values of r, we are unable to show that this also holds for the other cases. Those values however do not change the characteristics of the system significantly, which is why we will also apply the numerical method from section 2.4 for these cases.

The first boxplot we want to make, is for r = 4. Comparing this with the findings from section 3.1.1 will give us a way to check if the distributions resulting from this numerical method in fact represent the actual distributions accurately. The result can be found in figure 12. This figure is created using n = 40000, for which the resulting first, second and third quartiles of t = 1 and t = 2 are all within 0.05 of their analytic value calculated in section 3.1.1. We will use this value of n in subsequent boxplots. Additionally we find that if we increase t, the distributions approach the value described in lemma 3.5, as expected.

Conjecture 3.1. For the chaotic regions of the logistic map, extremes are better predictable than non-extremes.

Having verified the numerical method, we now have the tools to inspect the predictability of extremes of the logistic map in general. After reviewing the distributions resulting from other values of r inside the chaotic region and larger values of t, we find similar results as before. A Mathematica code for generating these boxplots can be found in appendix A. To give a sense of the results that may be found, we present the r = 3.9 and r = 3.8 cases in figures 13 and 14. We can conclude that in all chaotic cases extremes seem to have lower FLTE's than the non-extremes. We thereby make the general conclusion that extremes are better predictable than non-extremes for the logistic map.

3.2 The cusp map

The second map we will be studying, is the cusp map. Like the logistic map, the cusp is a chaotic one-dimensional map. It is given by

$$f(x) = 1 - 2\sqrt{|x|}.$$

If $x \in [-1, 1]$, then $f(x) \in [-1, 1]$, which is why we will be focusing on this region. The cusp can be found trough the Lorenz-63 system. If you plot the local maxima of the z variable for each n against the maximum at n - 1, the cusp map will be the function running through these points.

The fact that this map is extremely dependent on initial conditions, is visualized in figure 15. After a few iterations three very close initial conditions already have significantly different orbits. From figure 15 we get the sense that values close to -1 tend to remain in the same neighborhood. We confirm this by looking the density distribution of the cusp map, visualized in figure 16.



Figure 13: The numerical approximation of the conditional distribution of the FTLEs of the logistic map with r = 3.9, for four values of u and $t \in \{1, 2, 4, 8\}$.



Figure 14: The numerical approximation of the conditional distribution of the FTLEs of the logistic map with r = 3.8, for four values of u and $t \in \{1, 2, 4, 8\}$.



Figure 15: The orbit of the cusp map for initial conditions x = 0.199, x = 0.2 and x = 0.201.



Figure 16: A numerically generated probability histogram of the cusp map, plotted alongside its analytic density function $\frac{1-x}{2}$.

3.2.1 Analytical results

This section will employ the same method as section 3.1.1. It will therefore consist mostly of results, for the underlying calculations, visit section 3.1.1. t = 1 will again be the time step for our analytic approach, t = 2 will also be presented in brief.

Lemma 3.6. The corresponding invariant probability measure for the cusp map is given by

$$\mu([a,b]) = \int_{a}^{b} \frac{1-x}{2} \, \mathrm{d}x = \left[\frac{1}{2}x - \frac{1}{4}x^{2}\right]_{a}^{b}.$$

Proof. Firstly,

$$\mu\left([-1,1]\right) = \left[\frac{1}{2}x - \frac{1}{4}x^2\right]_{-1}^1 = 1.$$

Additionally

$$\begin{split} \mu\left(f^{-1}\left[a,b\right]\right) &= \mu\left(\left[-\frac{1}{4}(a-1)^2, -\frac{1}{4}(b-1)^2\right] \cup \left[\frac{1}{4}(b-1)^2, \frac{1}{4}(a-1)^2\right]\right) \\ &= -\frac{1}{8}(b-1)^2 - \frac{1}{64}(b-1)^4 + \frac{1}{8}(a-1)^2 + \frac{1}{64}(a-1)^4 \\ &\quad + \frac{1}{8}(a-1)^2 - \frac{1}{64}(a-1)^4 - \frac{1}{8}(b-1)^2 + \frac{1}{64}(b-1)^4 \\ &= \frac{1}{4}(a-1)^2 - \frac{1}{4}(b-1)^2 \\ &= \frac{1}{2}b - \frac{1}{4}b^2 - \frac{1}{2}a + \frac{1}{4}a^2 \\ &= \mu\left([a,b]\right) \end{split}$$

Lemma 3.7. The distribution function of the FTLEs for the cusp map with t = 1 is given by

$$P_1\left(\lambda \le s\right) = 1 - \frac{1}{e^{2s}} \tag{11}$$

Proof. For t = 1, the sum formula of the FTLEs reduces to

$$\lambda_1(x) = \ln\left(\left|\frac{1}{\sqrt{x}}\right|\right).$$

Figure 17 shows what equation (11) looks like when plotted on [-1, 1]. Equation (4) can now be rewritten to

$$P_{1} (\lambda \leq s) = \mu \left(\{ x \in [0,1] \mid \lambda(x) \leq s \} \right)$$

= $\mu \left(\{ x \in [0,1] \mid -\frac{1}{e^{2s}} \geq x \geq \frac{1}{e^{2s}} \} \right)$
= $\mu \left(\left[-1, -\frac{1}{e^{2s}}, \right] \cup \left[\frac{1}{e^{2s}}, 1 \right] \right)$
= $1 - \frac{1}{e^{2s}},$ (12)



Figure 17: The formula for the FTLEs for the cusp map with t = 1.



Figure 18: The distribution function of FTLEs for cusp map with t = 1.

where $s \in [0, \infty)$. Plotting equation (12), we find figure 18, which is in accordance with figure 17.

In order to approach equation (5), we again need to define an extreme region. Consulting figure 19, we define the extreme region as $E_u = [u, 1]$, where $u \in [-1, 1]$.

Lemma 3.8. The conditional distribution function of the FTLEs for the cusp map with t = 1 is given by

$$P_1 \left(\lambda \le s \mid E_u \right) = \begin{cases} 0 & s < \ln\left(\frac{4}{(u-1)^2}\right) \\ 1 - \frac{4}{e^{2s}(u-1)^2} & s \ge \ln\left(\frac{4}{(u-1)^2}\right) \end{cases}.$$
 (13)

Proof. The pre-image of E_u will be somewhere in the middle of [-1, 1], specifically we see that

$$1 - 2\sqrt{|x|} \ge u$$

$$-\frac{1}{4}(u-1)^2 \le x \le \frac{1}{4}(u-1)^2$$

Additionally

$$\mu\left([u,1]\right) = \frac{1}{2} - \frac{1}{4} - \frac{1}{2}u + \frac{1}{4}u^2 = \frac{1}{4}\left(u-1\right)^2.$$

Equation (5) now can be defined as

$$P_{1} (\lambda \leq s \mid E_{u}) = \frac{\mu \left(\left\{ x \in [0,1] \mid \lambda_{1}(x) \leq s \right\} \cap f^{-1} (E_{u}) \right)}{\mu (E_{u})}$$
$$= \frac{\mu \left(\left\{ \left[-1, -\frac{1}{e^{2s}} \right] \cup \left[\frac{1}{e^{2s}}, 1 \right] \right\} \cap \left[-\frac{1}{4} (u-1)^{2}, \frac{1}{4} (u-1)^{2} \right] \right)}{\frac{1}{2} - \frac{1}{\pi} \arcsin (2u-1)}$$
$$= \begin{cases} 0 \qquad s < \ln \left(\frac{4}{(u-1)^{2}} \right) \\ 1 - \frac{4}{e^{2s}(u-1)^{2}} \qquad s \geq \ln \left(\frac{4}{(u-1)^{2}} \right) \end{cases}.$$

Equation (13) is plotted in figure 20. In appendix A the reader can find the Mathematica document to reproduce this figure. \Box

Proposition 3.3. For t = 1, extremes of the cusp map are less predictable than non-extremes.

 $Proof.\,$ The first, second and third quartiles are presented in figure 21) and given by the formula's

$$P_1 \left(\lambda \le s \mid E_u \right) = \frac{1}{4}$$
$$s = \frac{1}{2} \ln \left(\frac{16}{3 \left(u - 1 \right)^2} \right),$$



Figure 19: The t = 1 iteration of the cusp map.



Figure 20: The conditional distribution function of FTLEs for the cusp map with t = 1., for four values of u



Figure 21: The first, second and third quartiles of the conditional distribution function of FTLEs for the cusp map with t = 1.



Figure 22: The formula for the FTLEs for the cusp map with t = 2.

$$P_1 \left(\lambda \le s \mid E_u \right) = \frac{1}{2}$$
$$s = \frac{1}{2} \ln \left(\frac{16}{2 \left(u - 1 \right)^2} \right)$$

and

$$P_1 \left(\lambda \le s \mid E_u \right) = \frac{3}{4}$$
$$s = \frac{1}{2} \ln \left(\frac{16}{\left(u - 1 \right)^2} \right).$$

A larger u leads to a smaller $P_1(\lambda \leq s \mid E_u)$ for any s. We can therefore conclude that for t = 1, extremes are less predictable than non-extremes.

Proposition 3.4. For t = 2, extremes of the cusp map are less predictable than non-extremes.

Proof. At t = 2, The formula for the FTLEs, equation (2), becomes

$$\lambda_2(x) = \frac{1}{2} \ln \left(\left| \frac{1}{\sqrt{x - 2x\sqrt{|x|}}} \right| \right), \tag{14}$$

which is plotted in figure 22. We observe that $\lambda_2 \leq s$ will result in either one large or three smaller intervals. The local minima are found at $-\frac{1}{9}$ and $\frac{1}{9}$. The peaks where the function is not defined are found at $-\frac{1}{4}$, 0 and $\frac{1}{4}$. In order to find the boundaries for the interval(s), we look at

$$\frac{1}{2} \ln \left(\left| \frac{1}{\sqrt{x - 2x\sqrt{|x|}}} \right| \right) \le s$$
$$-\frac{1}{e^{4s}} \ge x - 2x\sqrt{|x|} \ge \frac{1}{e^{4s}}$$

whose roots are used to construct the distribution function, plotted in figure 23. The inverse image of any extreme interval will typically consist of two intervals, as can be seen in figure 24. Each interval lies around one of the points where this function is not defined, $x = -\frac{1}{4}$ and $x = \frac{1}{4}$.

The information above is used to construct the conditional distribution, of which the result is displayed in figure 25. As before, a Mathematica document in added in appendix A to reproduce this result. As for t = 1, the extremes seem to be less predictable for t = 2 as well.

3.2.2 Numerical results

As in section 3.1.2, we use the numerical method to determine the predictability of extremes of the cusp map for larger values of t. First, we check the conditions for Birkhoff's ergodic theorem.



Figure 23: The distribution function of FTLEs, plotted to s for cusp map with t = 2.



Figure 24: The t = 2 iteration of the cusp map.

Lemma 3.9. $f(x) = 1 - 2\sqrt{|x|}$ is ergodic.

Proof. As in lemma 3.4, we use

 $\overline{f^{-1}(E)} = \overline{E}$

and argue that both sides of the equation can only exist of an equal amount of disjoint intervals if

1. $\overline{E} = \emptyset$ 2. $\overline{E} = \{1\}$ 3. $\overline{E} = [e, 1]$, for some $e \in [0, 1)$.

The first two cases have measure 0. For the third case

$$[e,1] = \left[-\frac{1}{4}(e-1)^2, \frac{1}{4}(e-1)^2\right].$$

This means that

$$\frac{1}{4} (e-1)^2 = 1$$

 $e = -1,$

which implies that the third case has measure 1.

Lemma 3.10. $\varphi = \ln \left(\left| \frac{1}{\sqrt{x}} \right| \right)$ is integrable with respect to μ and

$$\int_{-1}^{1} \varphi \,\mathrm{d}\mu = \frac{1}{2}.$$

Proof. μ is again absolutely continuous with respect to the Lebesgue measure, which means by theorem 2.1:

$$\int_{-1}^{1} \ln\left(\left|\frac{1}{\sqrt{x}}\right|\right) d\mu = \int_{-1}^{1} \frac{1}{2} (1-x) \ln\left(\left|\frac{1}{\sqrt{x}}\right|\right) dx = \frac{1}{2},$$

using Mathematica to determine the integral.

Corollary 3.2. Any $x \in [-1,1]$ of the cusp map has infinite time Lyapunov exponent

$$\lambda_{\infty}(x) = \lim_{t \to \infty} \frac{1}{t} \sum_{k=0}^{t-1} \ln(|f'(f_t(x))|) = \int_{-1}^{1} \varphi \, \mathrm{d}\mu = \frac{1}{2}.$$

Proof. Combine lemma 3.10 with theorem 2.2.

Conjecture 3.2. Extremes of the cusp map are less predictable than nonextremes.



Figure 25: The conditional distribution function of FTLEs for the cusp map with t = 2, for four values of u.



Figure 26: The numerical approximation of the conditional distribution of the FTLEs for the cusp map for $u \in \{0, 0.8, 0.9, 0.95\}$ and $t \in \{1, 2, 4, 8\}$.

Constructing a boxplot of the distributions of FTLEs for different values of t, we immediately check if for t = 1 and t = 2 the first, second and third quartiles match with the analytic values from section 3.2.1. We find that for n = 70000, they are within 0.05 of their analytic value. This n is used for figure 26. As we increase t, the distributions approach $\frac{1}{2}$, as we expect from corollary 3.2.

We can conclude that in all cases, extremes seem to have higher FLTE's than the non-extremes. We can hereby conclude extremes of the cusp map are in general less predictable than non-extremes.

3.3 The divergent Newton map

The third one-dimensional map we will be studying is given by

$$f(x) = \frac{1}{2}\left(x - \frac{1}{x}\right),$$

where $x \in \mathbb{R}$. It is found when Newton's method is applied to

$$f(x) = x^2 + 1,$$

which does not tend towards one value since f has no root. Unfortunately this map does not have a name. Since this map originates from a divergent case of Netwon's method, we will call it the 'divergent Netwon map' in this paper.

The fact that this map is chaotic can be seen in figure 27. We can also show the density distribution of the divergent Newton map, visualized in figure 28. We see that the system is densest in the regions around the origin.

3.3.1 Analytical results

As in the previous sections, we approach t = 1 analytically.

Lemma 3.11. The invariant probability measure for the divergent Newton map is given by

$$\mu([a,b]) = \frac{1}{\pi} \int_{a}^{b} \frac{1}{1+x^{2}} \, \mathrm{d}x = \frac{1}{\pi} \Big[\arctan(x) \Big]_{a}^{b}$$

Proof. This map is defined on the entire real line $(A = \mathbb{R})$, which means that

$$\mu((-\infty,\infty)) = \frac{1}{\pi} \left(\frac{\pi}{2} + \frac{\pi}{2}\right) = 1.$$

Additionally

$$\mu \left(f^{-1} [a, b] \right) = \mu \left(\left[a - \sqrt{a^2 + 1}, b - \sqrt{b^2 + 1} \right] \cup \left[a + \sqrt{a^2 + 1}, b + \sqrt{b^2 + 1} \right] \right)$$
$$= \frac{1}{\pi} \left(\arctan \left(b - \sqrt{b^2 + 1} \right) - \arctan \left(a - \sqrt{a^2 + 1} \right) \right)$$
$$+ \frac{1}{\pi} \left(\arctan \left(b + \sqrt{b^2 + 1} \right) - \arctan \left(a + \sqrt{a^2 + 1} \right) \right).$$



Figure 27: The orbit of the divergent Newton map for initial conditions x = 0.399, x = 0.4 and x = 0.401.



Figure 28: A numerically generated probability histogram of the divergent Newton map, plotted alongside its analytic density function $\frac{1}{1+x^2}$.

We use the arctan sum

$$\arctan(\alpha) + \arctan(\beta) = \arctan\left(\frac{\alpha+\beta}{1-\alpha\beta}\right)$$

to rewrite the expression above into

$$\frac{1}{\pi} \left(\arctan\left(\frac{b - \sqrt{b^2 + 1} + b + \sqrt{b^2 + 1}}{1 - (b - \sqrt{b^2 + 1})(b + \sqrt{b^2 + 1})}\right) - \arctan\left(\frac{a - \sqrt{a^2 + 1} + a + \sqrt{a^2 + 1}}{1 - (a - \sqrt{a^2 + 1})(a + \sqrt{a^2 + 1})}\right) \right)$$
$$= \frac{1}{\pi} \left(\arctan\left(b\right) - \arctan\left(a\right)\right) = \mu\left([a, b]\right),$$

which together prove that μ is an invariant probability measure.

Lemma 3.12. The distribution function of the FTLEs for the divergent Newton map with t = 1 is given by

$$P_1\left(\lambda \le s\right) = 1 - \frac{2}{\pi} \arctan\left(\frac{1}{\sqrt{2e^s - 1}}\right),\tag{15}$$

where $s \in \left[\ln\left(\frac{1}{2}\right), \infty\right)$.

Proof. For t = 1, the sum formula of the FTLEs reduces to

$$\lambda_1(x) = \ln\left(\left|\frac{1}{2} + \frac{1}{2x^2}\right|\right). \tag{16}$$

Figure 29 shows what equation (16) looks like when plotted on [-10, 10]. Equation (4) can now be rewritten to

$$\begin{split} P_1 \left(\lambda \le s \right) &= \mu \left(\{ x \in [0,1] \mid \lambda(x) \le s \} \right) \\ &= \mu \left(\{ x \in [0,1] \mid -\frac{1}{\sqrt{2e^s - 1}} \ge x \ge \frac{1}{\sqrt{2e^s - 1}} \} \right) \\ &= \mu \left(\left(-\infty, -\frac{1}{\sqrt{2e^s - 1}} \right] \cup \left[\frac{1}{\sqrt{2e^s - 1}}, \infty \right) \right) \\ &= 1 - \frac{2}{\pi} \arctan \left(\frac{1}{\sqrt{2e^s - 1}} \right), \end{split}$$

where $s \in \left[\ln\left(\frac{1}{2}\right), \infty\right)$. Plotting equation (15), we find figure 30, which is in accordance with figure 29.

Defining an extreme region for this map is somewhat less trivial than for the previous maps. f is anti-symmetric around x = 0, which is why we will define the extreme region as $E_u = (-\infty, -u] \cup [u, \infty)$, with $u \in [0, \infty)$.



Figure 29: The formula for the FTLEs for the divergent Newton map with t = 1.

Figure 30: The distribution function of FTLEs for the divergent Newton map with t = 1.

Figure 31: $f_1(x) = \frac{1}{2} \left(x - \frac{1}{x} \right)$, the first iteration of the divergent Newton map.

Figure 32: The conditional distribution function of FTLEs for the divergent Newton map with t = 1, for four values of u.

Lemma 3.13. The conditional distribution function of the FTLEs of the divergent Newton map for t = 1 is given by

$$P_{1} \left(\lambda \leq s \mid E_{u} \right) = \begin{cases} \frac{1}{2} & I_{1} < s \leq I_{2} \\ \frac{1 - \frac{2}{\pi} \arctan\left(\frac{1}{\sqrt{2e^{s} - 1}}\right)}{1 - \frac{2}{\pi} \arctan(u)} & s \leq I_{1} \\ \frac{1 - \frac{2}{\pi} \left(\arctan\left(u + \sqrt{u^{2} + 1}\right) + \arctan\left(-u + \sqrt{u^{2} + 1}\right) - \arctan\left(\frac{1}{\sqrt{2e^{s} - 1}}\right)\right)}{1 - \frac{2}{\pi} \arctan(u)} & s > I_{2}, \end{cases}$$

$$(17)$$

where

$$I_1 = \ln\left(\frac{1}{2} + \frac{1}{4u^2 + 2 - 4x\sqrt{u^2 + 1}}\right)$$

and

$$I_2 = \ln\left(\frac{1}{2} + \frac{1}{4u^2 + 2 + 4x\sqrt{u^2 + 1}}\right).$$

Proof. We find that

$$\begin{split} \left| \frac{1}{2} \left(x - \frac{1}{x} \right) \right| \geq u \\ -u + \sqrt{u^2 + 1} \geq |x| \geq u + \sqrt{u^2 + 1}, \end{split}$$

implying that f^{-1} will consist of one central interval and an interval on each outer side, which agrees with our expectations from figure 31. Additionally

$$\mu\left((-\infty, -u] \cup [u, \infty)\right) = 1 - \frac{2}{\pi} \arctan\left(u\right).$$

Determining equation (5) we find that it will result in a piecewise function consisting of three parts, given by equation (17). This distribution is presented in figure 32, the Mathematica code for it can again be found in appendix A. \Box

We note that the second quartile will be difficult to determine analytically, since for $u \neq 0$, the conditional distribution $P_1(\lambda \leq s \mid E_u) = \frac{1}{2}$ gives a horizontal part of the graph. We can however determine the functions for the first and third quartiles which are presented in figure 33 and given by the formula's:

$$P_1 \left(\lambda \le s \mid E_u \right) = \frac{1}{4}$$
$$s = \ln \left(\frac{1}{2} + \frac{1}{2 \tan^2 \left(\frac{3\pi}{8} + \frac{1}{4} \arctan\left(u \right) \right)} \right)$$

and

$$P_1 \left(\lambda \le s \mid E_u \right) = \frac{3}{4}$$

$$s = \ln \left(\frac{1}{2} + \frac{1}{2 \tan^2 \left(\arctan \left(-u + \sqrt{u^2 + 1} \right) + \frac{1}{4} \arctan \left(u \right) - \frac{\pi}{8} \right)} \right).$$

Interestingly, a quick conclusion on the predictability of the t = 1 case cannot be made analytically. We can only conclude that a higher u will lead to a lower first quartile, but a higher third quartile. It is up to the numerical method presented in section 3.3.2 to provide us with the answer to whether extremes are better predictable or not. Since the distributions functions for t = 1 were already quite arduous to calculate and since our numerical method has already been verified in sections 3.1.2 and 3.2.2, we will not work out the analytical approach for the t = 2 case of the divergent Newton map.

3.3.2 Numerical results

In this section we will again approximate the distribution functions for larger values of t. First we check the ergodicity of f and find the integral of φ with respect to μ .

Lemma 3.14. $f(x) = \frac{1}{2} \left(x - \frac{1}{x} \right)$ is ergodic.

Proof. In the case of the divergent Newton map, the right and left side of equation

$$\overline{f^{-1}(E)} = \overline{E}$$

can only have an equal amount of intervals if

1.
$$E = \emptyset$$

2. $E = (-\infty, \infty)$.

The second case is true since both intervals resulting from f^{-1} have to approach x = 0, which is only true if $E = (-\infty, \infty)$.

Lemma 3.15. $\varphi = \ln \left(\left| \frac{1}{2} + \frac{1}{2x^2} \right| \right)$ is integrable with respect to μ and

$$\int_{-\infty}^{\infty} \varphi \,\mathrm{d}\mu = \ln\left(2\right).$$

Proof. The fact that μ is absolutely continuous with respect to the Lebesgue measure allows us to use theorem 2.1 to determine that

$$\int_{-\infty}^{\infty} \ln\left(\left|\frac{1}{2} + \frac{1}{2x^2}\right|\right) d\mu = \int_{-\infty}^{\infty} \frac{\ln\left(\left|\frac{1}{2} + \frac{1}{2x^2}\right|\right)}{\pi \left(1 + x^2\right)} dx = \ln\left(2\right),$$

again using Mathematica to determine the integral.

Corollary 3.3. Any $x \in \mathbb{R}$ of the divergent Newton map has infinite time Lyapunov exponent

$$\lambda_{\infty}(x) = \lim_{t \to \infty} \frac{1}{t} \sum_{k=0}^{t-1} \ln\left(\left|f'\left(f^{t}(x)\right)\right|\right) = \int_{-\infty}^{\infty} \varphi \,\mathrm{d}\mu = \ln(2).$$

Proof. Combine lemma 3.15 with theorem 2.2.

Figure 33: The first and third quartiles of the conditional distribution function of FTLEs, plotted to u for the divergent Newton map with t = 1.

Figure 34: The numerical approximation of the conditional distribution of the FTLEs for the divergent Newton map for $u \in \{0, 0.8, 0.9, 0.95\}$ and $t \in \{1, 2, 4, 8\}$.

Conjecture 3.3. *Extremes of the divergent Newton map are less predictable than non-extremes.*

We construct the boxplot of the distributions of FTLEs as before and check whether it complies with the first and third quantiles for t = 1 we found in section 3.2.1. We find that for n = 20000, the first and third quantile are within 0.05 of their analytic value. This n is used for figure 34. We also find that the distributions approach $\ln(2)$, which we expect from corollary 3.3. The reader finds a Mathematica document in appendix A to reproduce the results.

What is particularly interesting about this figure is that for t = 1 the numerically approximated second quantile typically seems to coincide with either the left or the right end of the horizontal part at $\frac{1}{2}$ of figure 32. We observe that for t > 1, extremes seem in general to have higher FLTE's than the non-extremes. We can hereby conclude that in general extremes probably are less predictable than non-extremes for the divergent Newton map.

4 Conclusion and discussion

4.1 Generalization

In this section we will attempt to generalize the results from section 3, starting with the t = 1 case. We note that we restrict ourselves to ergodic, chaotic, one-dimensional maps. The goal is to see if we can make statements on the predictability of a region, just from inspecting f^1 and its graph.

The first thing to note is that it does not matter whether E is an extreme region or not. We can say something about the predictability for any $E \subseteq A$, with A being the state space of f.

Secondly, there are two situations in which we can immediately make conclusions. The first situation occurs when $f^{-1}(E)$ is the part of the graph whose derivative is closest to zero. Since λ is essentially a rescaling of the absolute value of the derivative of f, the FTLEs of this region will be the smallest. The region will therefore be better predictable than the map in general. The extreme region we used for the logistic map in section 3.1 is an example of this case.

The second situation is the opposite of the first case. If the pre-image of E is the steepest part of the graph, the FTLEs will be the largest. E will then be less predictable than the map in general. The extreme region we used for the cusp map in section 3.2 fits this example.

The reach of the conclusions above is fairly limited, since it only allows conclusions on the predictability of very specific regions for each map. We can however take these findings and turn them around. Most maps we look at have some form of local extrema. Say these local extrema have the form of mountains/valleys (with a flat slope). If $\overline{f^{-1}(E)}$ does not contain the x values of these extrema, the conditional distribution of the FTLEs of E will not include the relative low λ 's associated with these flat slopes. E will therefore be less predictable than the entire set.

Alternatively, extrema can occur in the form of upward/downward peaks (with a steep slope). Now if $\overline{f^{-1}(E)}$ does not contain the x values of these peaks, it does not include some relatively high FTLEs in its distribution. It will therefore be better predictable than the entire set.

Unfortunately, we cannot generalize this finding for larger values of t and make a-priori conclusions based on the t = 1 condition. The t = 1 graph may look completely different from the t = 2 graph and its extrema can be found in entirely different places. In order to make any statements on the predictability of t = i, we will therefore have to examine the graph of f^i . We however want to note that the type of extrema a map has, is likely to stay the same. If f^i has similar extrema at the same height, the predictability of E is bound to stay the same for t = i as it is for t = 1.

4.2 Conclusion

In an effort to answer the research question, we can conclude that the predictability of extremes of one-dimensional chaotic maps depends on the map itself and the definition of the extremes. Although determining the precise distributions of the FTLE's requires substantial calculations and some numerical methodology, observing the plot of the map can often give a general idea of the predictability in most cases.

4.3 Discussion

This section will be devoted to providing nuance to the findings and putting them into context. We will also provide some ideas for further research.

The first item we want to discuss, is the relevance of studying one-dimensional maps as we did in this paper. We note that one-dimensional maps are only scarcely used in prediction models for practical purposes (such as weather forecasting). In the theoretical sense of the word relevance, we note that the interest of mathematicians and physicists often goes to far more complex systems of equations. It is however relevant to note that the process of determining the predictability of more complex systems requires the same methodology as used in this paper. The relevance of this paper therefore is not so much found in the direct applicability of the results, but as a more intuitive and easier-to-grasp demonstration of a methodology to determine the predictability of extremes of systems of equations. Additionally the predictability of one-dimensional maps is determinable analytically, which is extremely complex and often impossible for higher-dimensional systems. The one-dimensional case therefore gives us the possibility to compare numerical methods to analytic calculations.

The second item of this discussion is the use of FTLE's. In section 2.1 we used the assumption of infinitesimally small errors as a crucial step to construct the expression of an FTLE. It is important to realize that in practical measurements, this assumption never holds. Additionally, the FTLE is the result of a comparison of the growth rate of errors to an exponentially increasing function. In practice the error of a prediction does not increase to infinity due to the system being bounded. We therefore have to conclude that observing FTLE's is not a perfect tool to approach predictability, which one has to keep in mind when using them.

Thirdly, we want to discuss the fact that the numerical approximation requires the map to be ergodic. This means that the conclusions we make are only applicable to ergodic systems. In practice most systems are ergodic. Ergodicity is therefore often assumed without explicitly mentioning it. Nonetheless, for completeness, we want to point it out here.

The combination of ergodicity and the formula of the FTLE's makes sure Birkhoff's ergodic theorem can be applied, from which we can conclude that the FTLE's distributions approach a single point. The method is therefore inherently less suited for assessing the long term predictability of systems. When applying the methods from this paper, one should focus on the short term predictability of systems. The speed at which the distributions converge, can however vary greatly. In the case of the cusp map, we saw that the distributions converge slowly compared to the other maps. This is due to the fact that iterates of initial conditions close to -1 spend long times near -1.

Finally, we would like to suggest a direction that one could take to further research into the predictability of extremes of one-dimensional maps. In section 1.1 we explained that different approaches on the predictability of extremes resulted in fundamentally different conclusions. We therefore suggest to apply these other approaches (such as ROC curves) to one-dimensional systems as well. Comparing the results of such a study to the conclusions of this paper might give some insight into why the approaches result in different conclusions.

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Appendices

A Mathematica codes

Mathematica code for plotting the analytical conditional distribution function of the logistic map for t = 1

```
Clear ["Global '*"]
f[x_{-}, y_{-}] :=
 Piecewise [\{ \{ ((2/Pi)) \}
          \operatorname{ArcSin}[(1/4) * \operatorname{Exp}[x]]) / (0.5 - (1/\operatorname{Pi}) *
           \operatorname{ArcSin}[2*y - 1]), \quad (\operatorname{Exp}[x] + 4)/8 <=
       0.5 + 0.5 * Sqrt[1 - y], {((2/Pi)*
          \operatorname{ArcSin}[\operatorname{Sqrt}[1 - y]]) / (0.5 - (1/\operatorname{Pi}) * \operatorname{ArcSin}[2 * y - 1]), (\operatorname{Exp}[x] +
           4)/8 >= 0.5 + 0.5 * Sqrt[1 - y] \}
Plot [\{f[x, 0], f[x, 0.9], f[x, 0.95], f[x, 0.975]\}, \{x, -4, 2\},
 PlotStyle \rightarrow Thick, Exclusions \rightarrow None,
 AxesLabel \rightarrow {Style ["s", 20],
    Style ["P(\[Lambda] \[LessEqual] s)", 20]},
 PlotLegends \rightarrow
  Placed [{ Style [0, 20], Style [0.9, 20], Style [0.95, 20],
     Style [0.975, 20], {Scaled [\{0.9, 0.25\}], \{0, 0.5\}\}],
 AxesStyle \rightarrow Directive [20], ImageSize \rightarrow Full]
```

Mathematica code for plotting the analytical conditional distribution function of the logistic map for t = 2

```
Clear ["Global '*"]
eq1 = 256 x^3 - 384 x^2 + 160 x - 16 = Exp[2*s];
eq2 = 256 x^3 - 384 x^2 + 160 x - 16 = -Exp[2*s];
eq3 = -64 x^{4} + 128 x^{3} - 80 x^{2} + 16 x = u;
sol1 = Solve[eq1, x, Reals];
sol2 = Solve[eq2, x, Reals];
sol3 = Solve[eq3, x, Reals];
Rootlist1 = Flatten [x /. {sol1, sol2}];
Rootlist2 = Flatten [x /. {sol3}];
a = Part[Rootlist1, 1];
b = Part[Rootlist1, 5];
c = Part[Rootlist1, 6];
d = Part[Rootlist1, 3];
e = Part[Rootlist2, 3];
f = Part[Rootlist2, 4];
t = (1/2) * Log[((16 * Sqrt[2/3])/3)];
```

equ1 = ((2/Pi)*(ArcSin[2*f - 1] - ArcSin[2*e - 1]))/(0.5 - (1/Pi)*) $\operatorname{ArcSin}[2 * u - 1]);$ equ2 = ((2/Pi)*(ArcSin[2*a - 1] - ArcSin[2*e - 1]))/(0.5 - (1/Pi)* $\operatorname{ArcSin}[2 * u - 1]);$ equ3 = ((2/Pi)*(ArcSin[2*f - 1] - ArcSin[2*e - 1]))/(0.5 - (1/Pi)* $\operatorname{ArcSin}[2 * u - 1]);$ equ4 = ((2/Pi)*(ArcSin[2*f - 1] - ArcSin[2*c - 1]))/(0.5 - (1/Pi)* $\operatorname{ArcSin}[2 * u - 1]);$ equ5 = ((2/Pi)*(ArcSin[2*d - 1] - ArcSin[2*e - 1]))/(0.5 - (1/Pi)* $\operatorname{ArcSin}[2 * u - 1]);$ equ6 = ((2/Pi)*(ArcSin[2*d - 1] - ArcSin[2*c - 1]))/(0.5 - (1/Pi)* $\operatorname{ArcSin}[2 * u - 1]);$ equ7 = ((2/Pi)*(ArcSin[2*b - 1] - ArcSin[2*e - 1]))/(0.5 - (1/Pi)* $\operatorname{ArcSin}[2 * u - 1]);$ $l[s_-] := Piecewise[{$ $\{equ1, s >= t \&\& f < a\},\$ $\{equ2, s >= t \&\& f >= a\},\$ $\{equ3, s < t \&\& b < e \&\& e >= c \&\& f < d\},\$ $\{equ4, s < t \&\& b < e \&\& e < c \&\& f < d\},\$ $\{equ5, s < t \&\& b < e \&\& e >= c \&\& f >= d\},\$ $\{equ6, s < t \&\& b < e \&\& e < c \&\& f >= d\},\$ $\{equ3 + equ7, s < t \&\& b >= e \&\& e >= c \&\& f < d\},\$ $\{equ4 + equ7, s < t \&\& b >= e \&\& e < c \&\& f < d\},\$ $\{equ5 + equ7, s < t \&\& b >= e \&\& e >= c \&\& f >= d\},\$ $\{equ6 + equ7, s < t \&\& b >= e \&\& e < c \&\& f >= d\}$ }] u = 0.0000001;g = 1 [s];u = 0.9;h = l[s];u = 0.95;i = 1[s];u = 0.975;j = 1 [s]; $Plot[\{g, h, i, j\}, \{s, -4, 2\}, Exclusions \rightarrow None, PlotStyle \rightarrow Thick,$ AxesLabel \rightarrow {Style["s", 20], Style ["P($[Lambda] \setminus [LessEqual] s$)", 20]}, $PlotLegends \rightarrow$ Placed [{ Style [0, 20], Style [0.9, 20], Style [0.95, 20], Style [0.975, 20], {Scaled $[\{0.9, 0.25\}], \{0, 0.5\}\}$], AxesStyle \rightarrow Directive [20], ImageSize \rightarrow Full]

Mathematica code for numerically approximating the conditional distribution function of the logistic map

```
Clear ["Global '*"]
r = 4;
n = 40000;
t1 = 1;
t2 = 2;
t3 = 4;
t4 = 8;
u1 = 0.9 * (r/4);
u2 = 0.95 * (r/4);
u3 = 0.975 * (r/4);
tlist = \{t1, t2, t3, t4\};
totallist = \{\};
For[k = 1, k <= 4, k++, t = Part[tlist, k];
 list = Range[n];
 beginlist = Range[n + t];
 x = 0.123456789;
 For [i = 1, i \le 100, i++, x = r * x * (1 - x)];
 For [i = 1, i \le n + t, i++, x = r*x*(1 - x); Part [beginlist, i] = x];
 For [i = t + 1, i \le n + t, i++, Lambda = 0;
  For [j = 1, j <= t, j++,
   Lambda = Lambda + Log[Abs[r - 2*r*Part[beginlist, i - j]]];
  Part[list, i - t] = \{Part[beginlist, i], (Lambda/t)\}];
 list = Sort[list, #1[[1]] > #2[[1]] \&];
 u1list = \{\};
 u2list = \{\};
 u3list = \{\};
 For [i = 1, i \le n, i++,
  If [Part [Part [list, i], 1] >= u1,
   AppendTo[u1list, Part[Part[list, i], 2]], Break[] ];
 For [i = 1, i \le n, i++,
  If [Part [Part [list, i], 1] \ge u2,
   AppendTo[u2list, Part[Part[list, i], 2]], Break[]];
 For [i = 1, i <= n, i++,
  If [Part [Part [list, i], 1] \ge u3,
   AppendTo[u3list, Part[Part[list, i], 2]], Break[]];
 For [i = 1, i \leq n, i++, Part[list, i] = Part[Part[list, i], 2]];
 AppendTo[totallist, {list, u1list, u2list, u3list}]]
BoxWhiskerChart[totallist, {{"MedianMarker", "\[FilledSmallCircle]",
   Black}, {"Whiskers", Black}, {"Fences", Black}},
 ChartLegends \rightarrow
  Placed [{ Style [0, 20], Style [u1, 20], Style [u2, 20],
    Style [u3, 20], {Right, Center}],
```

```
ChartLabels \ -> \ \left\{ \left\{ Style \left[ \, t1 \, , \ 20 \right] \, , \ Style \left[ \, t2 \, , \ 20 \right] \, , \ Style \left[ \, t3 \, , \ 20 \right] \, , \right. \right. \\
```

```
Style[t4, 20]}, None}, BarSpacing -> {0.1, 2},
ChartBaseStyle -> EdgeForm[Black],
FrameLabel -> {{Style["\[Lambda]", 20],
Style["u", 20]}, {Style["t", 20], Style[r, 20]}},
RotateLabel -> False, FrameStyle -> Directive[20], ImageSize -> Full]
```

Mathematica code for plotting the analytical conditional distribution function of the cusp map for t = 1

```
 \begin{array}{l} \text{Clear} \left[ \text{"Global `*"} \right] \\ \text{f} \left[ \text{s}_{-} , \text{ u}_{-} \right] := \\ \text{Piecewise} \left[ \left\{ \left\{ 0, \\ \text{s} < 0.5 \ \text{Log} \left[ \left( 4 / \left( \left( \text{u} - 1 \right)^2 2 \right) \right) \right] \right\} \right] \\ \text{s} = 0.5 \ \text{Log} \left[ \left( 4 / \left( \left( \text{u} - 1 \right)^2 2 \right) \right) \right] \right\} \right] \\ \text{Plot} \left[ \left\{ \text{f} \left[ \text{s}, -1 \right], \ \text{f} \left[ \text{s}, 0.8 \right], \ \text{f} \left[ \text{s}, 0.9 \right], \ \text{f} \left[ \text{s}, 0.95 \right] \right\}, \ \left\{ \text{s}, 0, 8 \right\}, \\ \text{PlotRange} \to \text{Full}, \ \text{PlotStyle} \to \text{Thick}, \ \text{Exclusions} \to \text{None}, \\ \text{AxesLabel} \to \left\{ \text{Style} \left[ \text{"s"}, 20 \right], \\ \text{Style} \left[ \text{"P}(\backslash [\text{Lambda}] \ \lfloor \text{LessEqual} \right] \ \text{s} \ \mid E) \text{", 20} \right] \right\}, \\ \text{PlotLegends} \to \\ \text{Placed} \left[ \left\{ \text{Style} \left[ -1, 20 \right], \ \text{Style} \left[ 0.8, 20 \right], \ \text{Style} \left[ 0.9, 20 \right], \\ \text{Style} \left[ 0.95, 20 \right] \right\}, \ \left\{ \text{Scaled} \left[ \left\{ 0.9, 0.25 \right\} \right], \ \left\{ 0, 0.5 \right\} \right\} \right], \\ \text{AxesStyle} \to \text{Directive} \left[ 20 \right], \ \text{ImageSize} \to \text{Full} \end{array} \right]
```

Mathematica code for plotting the analytical conditional distribution function of the cusp map for t = 2

```
Clear ["Global '*"]
eq1 = x - 2 x * Sqrt[Abs[x]] = 1/Exp[4 * s];
eq2 = x - 2 x * Sqrt[Abs[x]] = -1/Exp[4 * s];
eq3 = 1 - 2 Sqrt [Abs [1 - 2 Sqrt [Abs [x]]] = u;
sol1 = Solve[eq1, x, Reals];
sol2 = Solve[eq2, x, Reals];
sol3 = Solve[eq3, x, Reals];
Rootlist1 = Flatten [x /. {sol1, sol2}];
Rootlist2 = Flatten [x /. {sol3}];
a = Part[Rootlist1, 6];
b = Part[Rootlist1, 3];
c = Part[Rootlist1, 2];
d = Part[Rootlist2, 4];
e = Part[Rootlist2, 3];
t = 3 * Log[3] / 4;
equ1 = 4 (d - a)/((u - 1)^2);
```

```
equ2 = 4 (d - e)/((u - 1)^2);
equ3 = 4 (b - e)/((u - 1)^2);
equ4 = 4 (d - c)/((u - 1)^2);
equ5 = 4 (b - c)/((u - 1)^2);
l[s_-] := Piecewise[{
   \{equ1, s < t \&\& a < d\},\
    \{0, s < t \&\& a >= d\},\
    \{equ2, s >= t \&\& c < e \&\& b >= d \&\& a >= d\},\
    \{equ3, s >= t \&\& c < e \&\& b < d \&\& a >= d\},\
    \{equ4, s >= t \&\& c >= e \&\& b >= d \&\& a >= d\},\
    \{equ5, s >= t \&\& c >= e \&\& b < d \&\& a >= d\},\
    \{equ2 + equ1, s >= t \&\& c < e \&\& b >= d \&\& a < d\},\
    \{equ3 + equ1, s >= t \&\& c < e \&\& b < d \&\& a < d\},\
    \{equ4 + equ1, s >= t \&\& c >= e \&\& b >= d \&\& a < d\},\
    \{equ5 + equ1, s >= t \&\& c >= e \&\& b < d \&\& a < d\}
   }]
u = -0.999999;
g = 1 [s];
u = 0.8;
h = 1 [s];
u = 0.9;
i = 1 [s];
u = 0.95;
j = l[s];
Plot[\{g, h, i, j\}, \{s, 0, 8\}, PlotRange \rightarrow \{0, 1\}, Exclusions \rightarrow None,
  PlotStyle \rightarrow Thick,
 AxesLabel \rightarrow {Style ["s", 20],
   Style ["P(\[Lambda] \[LessEqual] s | E)", 20] },
 PlotLegends \rightarrow
  Placed[{Style[-1, 20], Style[0.8, 20], Style[0.9, 20]},
     Style [0.95, 20], {Scaled [\{0.9, 0.25\}], \{0, 0.5\}\}],
 AxesStyle \rightarrow Directive [20], ImageSize \rightarrow Full]
```

Mathematica code for numerically approximating the conditional distribution function of the cusp map

Clear ["Global '*"] n = 70000; t1 = 1; t2 = 2; t3 = 4; t4 = 8; u1 = 0.8; u2 = 0.9; u3 = 0.95; $tlist = \{t1, t2, t3, t4\};$ $totallist = \{\};$ For $[k = 1, k \le 4, k++, t = Part [tlist, k];$ list = Range[n];beginlist = Range[n + t];x = 0.123456789;For $[i = 1, i \le 100, i++, x = 1 - 2*Sqrt[Abs[x]]];$ For[i = 1, i <= n + t, i++, x = 1 - 2*Sqrt[Abs[x]];Part[beginlist, i] = x];For $[i = t + 1, i \le n + t, i++, Lambda = 0;$ For $[j = 1, j \le t, j++,$ Lambda = Lambda + Log[Abs[(1/Sqrt[Part[beginlist, i - j]])]]; $Part[list, i - t] = \{Part[beginlist, i], (Lambda/t)\}];$ list = Sort[list, #1[[1]] > #2[[1]] &]; $u1list = \{\};$ $u2list = \{\};$ $u3list = \{\};$ For $[i = 1, i \le n, i++,$ If [Part [Part [list, i], 1] >= u1, AppendTo[u1list, Part[Part[list, i], 2]], Break[]]; For $[i = 1, i \le n, i++,$ If [Part [Part [list, i], 1] >= u2, AppendTo[u2list, Part[Part[list, i], 2]], Break[]]; For $[i = 1, i \le n, i++,$ If [Part [Part [list, i], 1] >= u3, AppendTo[u3list, Part[Part[list, i], 2]], Break[]]; For $[i = 1, i \leq n, i++, Part[list, i] = Part[Part[list, i], 2]];$ AppendTo[totallist, {list, u1list, u2list, u3list}]] BoxWhiskerChart[totallist, {{"MedianMarker", "\[FilledSmallCircle]", Black }, {"Whiskers", Black }, {"Fences", Black } }, ChartLegends \rightarrow $Placed [\{Style[-1, 20], Style[u1, 20], Style[u2, 20], \}$ Style [u3, 20], {Right, Center}], $ChartLabels \rightarrow \{ \{ Style [t1, 20], Style [t2, 20], Style [t3, 20], \} \}$ Style [t4, 20], None}, BarSpacing -> $\{0.1, 2\}$, $ChartBaseStyle \rightarrow EdgeForm[Black],$ $FrameLabel \rightarrow \{Style["t", 20], Style[" | Lambda]", 20] \},\$ RotateLabel \rightarrow False, FrameStyle \rightarrow Directive [20], ImageSize \rightarrow Full]

Mathematica code for plotting the analytical conditional distribution function of the lonely map for t = 1

Clear ["Global '*"]

```
a = Log[(1/2) + (1/(4*u^2 + 2 + 4*u*Sqrt[1 + u^2]))];
\mathbf{b} = \mathrm{Log}[(1/2) + (1/(4 * u^2 + 2 - 4 * u * \mathrm{Sqrt}[1 + u^2]))];
equ1 = ((1/2) - (1/Pi)*ArcTan[u + Sqrt[1 + u^2]]);
equ2 = ((1/2) - (1/Pi) * ArcTan[1/(Sqrt[-1 + 2*Exp[s]])]);
equ3 = ((1/Pi)*(ArcTan[-u + Sqrt[1 + u^2]) -
       \operatorname{ArcTan}[1/(\operatorname{Sqrt}[-1 + 2*\operatorname{Exp}[s]])]);
equ4 = 2*((1/2) - (1/Pi)*ArcTan[u]);
f[s_-] := Piecewise[{
     \{2*equ1/equ4, s < b \&\& s >= a\},\
     \{2*equ2/equ4, s < a\},\
     \{2*(equ1 + equ3)/equ4, s >= b\}
     }];
u = 0;
g = f[s];
u = 0.5;
h = f[s];
u = 1;
i = f[s];
u = 2;
j = f[s];
Plot[\{g, h, i, j\}, \{s, Log[0.5], 10\}, PlotRange \rightarrow \{0, 1\},
 Exclusions -> None, PlotStyle -> Thick,
 AxesLabel \rightarrow {Style ["s", 20],
   Style ["P(\[Lambda] \[LessEqual] s | E)", 20] },
 PlotLegends \rightarrow
  Placed [{ Style [0, 20], Style [0.5, 20], Style [1, 20],
     Style [2, 20], {Scaled [\{0.9, 0.25\}], \{0, 0.5\}\}],
 AxesStyle -> Directive [20], ImageSize -> Full]
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Mathematica code for numerically approximating the conditional distribution function of the lonely map

```
Clear ["Global '*"]

n = 20000;

t1 = 1;

t2 = 2;

t3 = 4;

t4 = 8;

u1 = 0.5;

u2 = 1;

u3 = 2;

tlist = {t1, t2, t3, t4};

totallist = {};

For [k = 1, k <= 4, k++, t = Part[tlist, k];
```

list = Range[n];beginlist = Range[n + t];x = 0.123456789;For $[i = 1, i \le 500, i++, x = (1/2)*(x - (1/x))];$ For $[i = 1, i \le n + t, i++, x = (1/2)*(x - (1/x));$ Part[beginlist, i] = x];For $[i = t + 1, i \le n + t, i++, Lambda = 0;$ For $[j = 1, j \le t, j++,$ Lambda = Lambda + Log[Abs](1/2) + $(1/(2*Part[beginlist, i - j]^2))$]]; $Part[list, i - t] = \{Part[beginlist, i], (Lambda/t)\}];$ list = Sort[list, #1[[1]] > #2[[1]] &]; $u1list = \{\};$ $u2list = \{\};$ $u3list = \{\};$ For $[i = 1, i \le n, i++,$ If [Part[Part[list, i], 1] >= u1, AppendTo[u1list, Part[Part[list, i], 2]], Break[]]; For $[i = 1, i \le n, i++,$ If [Part [Part [list, i], 1] >= u2, AppendTo[u2list, Part[Part[list, i], 2]], Break[]]; For $[i = 1, i \le n, i++,$ $\label{eq:interm} {\rm If} \, [\, {\rm Part} \, [\, {\rm Part} \, [\, {\rm list} \ , \ i \,] \ , \ 1 \,] \ >= \ u 3 \ ,$ AppendTo[u3list, Part[Part[list, i], 2]], Break[]]; list = Sort[list, #1[[1]] < #2[[1]] &];For $[i = 1, i \le n, i++,$ If $[Part [Part [list, i], 1] \leq -u1$, AppendTo[u1list, Part[Part[list, i], 2]], Break[]]; For $[i = 1, i \le n, i++,$ If $[Part [Part [list, i], 1] \leq -u2$, AppendTo[u2list, Part[Part[list, i], 2]], Break[]]; For $[i = 1, i \le n, i++,$ If $[Part [Part [list, i], 1] \leq -u3$, AppendTo[u3list, Part[Part[list, i], 2]], Break[]]; For $[i = 1, i \leq n, i++, Part[list, i] = Part[Part[list, i], 2]];$ AppendTo[totallist, {list, u1list, u2list, u3list}]] BoxWhiskerChart[totallist, {{"MedianMarker", "\[FilledSmallCircle]", Black }, {"Whiskers", Black }, {"Fences", Black } }, ChartLegends \rightarrow $Placed[{Style[0, 20], Style[u1, 20], Style[u2, 20],}$ Style[u3, 20], {Right, Center}], ChartLabels \rightarrow {{Style [t1, 20], Style [t2, 20], Style [t3, 20], Style [t4, 20], None}, BarSpacing $\rightarrow \{0.1, 2\}$, $ChartBaseStyle \rightarrow EdgeForm[Black],$ FrameLabel \rightarrow {Style ["t", 20], Style ["\[Lambda]", 20]}, RotateLabel \rightarrow False, FrameStyle \rightarrow Directive [20], ImageSize \rightarrow Full]