MULTIVECTORS WITH TRIVIAL VALUES AND THE INVERSE SCATTERING TRANSFORM

FLORIS WISSE

Abstract. This is a research concerning 2 related but separate subjects in mathematics and theoretical physics. The mathematical aspects concern geometry, algebra and analysis. The physical aspects concern Hamiltonian and Lagrangian systems. The first subject is variational multivectors on a jet space. We prove that variational multivectors with trivial values are themselves trivial. We accomplish this by finding a homotopy operator for variational multivectors. We find this homotopy operator by identifying variational multivectors with horizontal differential forms on a jet space. To get more insight into homotopy operators, we first consider examples on manifolds, jet spaces and super jet spaces. The second subject is the Inverse Scattering Transform as a method to solve a class of non-linear partial differential equations. We will use this method to find solutions of the Korteweg-de Vries equation.

1. Introduction

In the mathematical part of this thesis we will study variational multivectors on a jet space. They are used in Hamiltonian mechanics, where Poisson brackets are variational bivectors. We consider a variational 2-vector P, and let $[\![\cdot,\cdot]\!]$ denote the Schouten bracket. By the classical master equation P is Hamiltonian if and only if $[\![P,P]\!]=0$. We define a homotopy operator as an operator on differential forms. Let Λ be a set of differential forms. The differential k-forms are denoted by Λ^k. Let d be a differential. The operator $h:\Lambda^k\to\Lambda^{k-1}$ is an homotopy operator if for all $\omega\in\Lambda^k$, $d\omega=0$ implies that $dh(\omega)=\omega$. Whenever $d\omega=0$, ω is called closed. Whenever there exists an $\eta\in\Lambda^{k-1}$ such that $d\eta=\omega$, ω is called exact. The existence of a homotopy operator shows that every closed differential form is exact. We will find a homotopy operator for variational multivectors.

In the physics part of this thesis we will inspect the inverse scattering transform by applying it to the KdV equation. We will use it to find and study solutions of the KdV equation. We will also use it to find conserved values and symmetries of the KdV equation.
Sections 2-3 concern themselves with the mathematical problem discussed in this thesis. Sections 4-5 concern themselves with the physical subject. The conclusion concerns itself with both.

Date: January 2017.

We will introduce structures on jet spaces in section 2. In particular we will introduce the jet space in section 2.1, differentiation in section 2.2 and differential forms in section 2.3. In section 2.4 we will introduce the jet superbundle, and in section 2.5 we will introduce multivectors as Hamiltonians on a specific jet superbundle, and state the main theorem of this thesis. In 2.6 we will consider multivectors as skew symmetric horizontal differential operators. In the subsequent section 3 we will study homotopy operators corresponding to several differentials. In section 3.1 we will study a homotopy operator corresponding to the differential on manifolds. In section 3.2 we will study the Cartan homotopy operator corresponding to the Cartan differential on jet spaces. In section 3.3 we will study the horizontal homotopy operator corresponding to the horizontal differential on jet spaces. In section 3.4 we will proof the main theorem.

In section 4 we will introduce the KdV equation. In section 4.1 we will introduce the KdV equation. In sections 4.2 we will consider a derivation of the KdV equation, and in section 4.3 we will study the Hamiltonian and Lagrangian structures of the KdV equation. In section 5 we will study the mathematics behind the solitons. First we will focus on the time independent scattering of the Schrödinger eigenvalue equation in section 5.1. After this, we will study the inverse scattering problem in section 5.2. In section 5.3 we will study the time dependence of the solutions of the Schrödinger eigenvalue equations. In section 5.4 to find and study properties of specisolutions of the KdV equation using the inverse scattering transform. We will specifically look at the long term behaviour of the solutions, the conserved quantities and the symmetries of the solutions. We will look at several physical In section 8 we will consider multid

2. The Geometry of Jet Spaces

In this section we will introduce the geometry of the problem. We will do this along the lines of [1] and [2].

2.1. The definition of a jet space.

First, we will recall the definition of a jet space and define useful objects that are to be used.

Definition 2.1. Let M be a smooth manifold of dimension n and $\pi : E \to M$ a locally trivial smooth vector bundle over M with dimension $m+n$. The set of sections s of the bundle π will be denoted as $\Gamma(\pi)$. We will use the multi-indices to denote derivatives w.r.t. manifold coordinates $\boldsymbol{x} = (x_1, ..., x_n)$ in the following way. Let $s \in \Gamma(\pi)$ and $\alpha := (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n_{\geq 0}$. Then we define

$$\left(\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{x}}\right)^{\alpha} s = \frac{\mathrm{d}^{\alpha_1}}{\mathrm{d}x_1^{\alpha_1}} \cdots \frac{\mathrm{d}^{\alpha_n}}{\mathrm{d}x_n^{\alpha_n}} s.$$

We define $|\alpha| = \sum_{i=1}^{n} \alpha_i$. Two sections $s_1, s_2 \in \Gamma(\pi)$ are tangent at $\boldsymbol{x}$ with order k if for all $\alpha \in \mathbb{N}^n$ such that $|\alpha| \leq k$

$$\left(\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{x}}\right)^{\alpha} s_1(\boldsymbol{x}) = \left(\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{x}}\right)^{\alpha} s_2(\boldsymbol{x}).$$

Tangency is an equivalence relation. The equivalence class $[s]_{\boldsymbol{x}}^k$ of sections that are tangent to s at $\boldsymbol{x}$ with order k is called the k-jet of s at $\boldsymbol{x}$. Each k-jet $[s]_{\boldsymbol{x}}^k$ can be labelled by the values of its derivatives, which we denote by $\boldsymbol{u}_\alpha := \left(\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{x}}\right)^{\alpha} s(\boldsymbol{x}), \quad |\alpha| \leq n$. The set of all k-jets of π is called the manifold of k-jets of π:

$$J^k(\pi) := \{[s]_{\boldsymbol{x}}^k \quad | \quad s \in \Gamma(\pi), \boldsymbol{x} \in M\}.$$

The derivatives u_α^j can be used as coordinates for $J^k(\pi)$. It is endowed with a natural structure of a vector bundle $\pi_k : J^k(\pi) \to M$ and the vector bundle $\pi_{k,l} : J^k(\pi) \to J^l(\pi)$ for $l < k$. The jet space of π is defined as the projective limit

$$J^\infty(\pi) := \varprojlim_{k\to\infty} J^k(\pi)$$

It has an infinite chain of epimorphisms $\pi_{\infty,k} : J^\infty(\pi) \to J^k(\pi)$ and a vector bundle structure $\pi_\infty : J^\infty(\pi) \to M$ [1, p. 10].

Remark 2.2. In a coordinate neighbourhood $U \subset M$ such that the bundle π is trivial, coordinates naturally arise [2, p. 5]. Each point $[s]_{\boldsymbol{x}}^k \in J^\infty(\pi|_U)$ can be labelled by coordinates

$$(\boldsymbol{x}, [\boldsymbol{u}]) := (\boldsymbol{u}, \boldsymbol{u}_{\boldsymbol{x}}, ..., \boldsymbol{u}_\alpha, ...),$$

where u_α^j denotes the derivative of any section $s \in [s]_{\boldsymbol{x}}^k$ w.r.t α at point $\boldsymbol{x}$.

We will now introduce the set of functions on the jet space.

Definition 2.3. The ring of smooth functions $C^\infty(J^k(\pi))$ is denoted by

$$\mathcal{F}_k(\pi) := C^\infty(J^k(\pi)).$$

The ring of smooth functions on the jet space $J^\infty(\pi)$ is defined as the direct limit

$$\mathcal{F}(\pi) := \varinjlim_{k\to\infty} \mathcal{F}_k(\pi).$$

Definition 2.4. For any section $s \in \Gamma(\pi)$ we define the map $j_\infty : \Gamma(\pi) \to \Gamma(\pi_\infty)$ by

$$j_\infty(s)(\boldsymbol{x}) := [s]_{\boldsymbol{x}}^\infty.$$

$j_\infty(s)$ is called the infinite jet of s. Each function $f \in \mathcal{F}(\pi)$ defines a nonlinear differential operator $\Delta_f : \Gamma(\pi) \to C^\infty(M)$,

$$\Delta_f(s) := j_\infty^*(f)(s).$$

We denote the set of nonlinear differential operators $\Gamma(\xi_1) \to \Gamma(\xi_2)$ as $\mathrm{Diff}(\xi_1, \xi_2)$.

Lemma 2.5. *$j_\infty^* : \mathcal{F}(\pi) \to \mathit{Diff}(\pi, \pi)$ is an isomorphism.*

This is shown in [3].

2.2. **Differentiation on the jet space.** We will now define certain vector fields on the jet space.

Definition 2.6. We define horizontal vector fields as $\mathcal{C}(\pi) = \mathcal{F}(\pi) \otimes_{C^\infty(M)} \Gamma(TM)$. A horizontal vector field $\sum_{i \in I} f_i v^i \in \mathcal{F}(\pi) \otimes_{C^\infty(M)} \Gamma(TM)$ acts on $\mathcal{F}(\pi)$ via the rule

$$\sum_{i \in I} f_i j_\infty^* \circ v^i = \sum_{i \in I} f_i v^i \circ j_\infty^*$$

Remark 2.7. Let $1_i := (0, ..., 0, 1, 0, ..., 0) \in \mathbb{N}^n$, where the one is placed in the ith entry. In jet space coordinates we write $\frac{\mathrm{d}}{\mathrm{d}x^i}$ as

$$\frac{\mathrm{d}}{\mathrm{d}x^i} = \frac{\partial}{\partial x^i} + \sum_\alpha u_{\alpha+1_i} \frac{\partial}{\partial u_\alpha}$$

Remark 2.8. A horizontal vector field can be uniquely determined by its action on $\mathcal{C}^\infty(M)$. The set of horizontal vector fields $C(\pi)$ is also called the Cartan distribution.

Definition 2.9. We denote the set of horizontal differentiations of $\mathcal{F}(\pi)$ as $\mathcal{C}\mathrm{Diff}(\mathcal{F}(\pi), \mathcal{F}(\pi))$. In coordinates we write for $\Delta \in \mathcal{C}\mathrm{Diff}(\mathcal{F}(\pi), \mathcal{F}(\pi))$ the finite sum $\Delta(\boldsymbol{x}, \boldsymbol{u}) = \sum_\alpha f_\alpha(\boldsymbol{x}, \boldsymbol{u}) \left(\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{x}}\right)^\alpha$.

We will now define the vertical vector fields.

Definition 2.10. Let $X \in \Gamma(TJ^k(\pi))$. We call X compatible if it commutes with any horizontal vector field. We call a compatible vector field vertical if for all $f \in C^\infty(M)$, $X(f) = 0$. The set of all vertical vector fields over $J^k(\pi)$ is denoted as $V_k(\pi)$. We define the vertical vector fields as the projective limit

$$V(\pi) = \varprojlim_{x \to \infty} V_k(\pi)$$

We will now introduce the related concept of horizontal modules.

Definition 2.11. Let ξ be a vector bundle $\xi : X \to M$, we denote the pullback of ξ along π_∞ as $\pi_\infty^*(\xi)$. The set of sections of $\pi_\infty^*(\xi)$ is defined as the direct limit

$$\Gamma(\pi_\infty^*(\xi)) := \varinjlim_{k \to \infty} \Gamma(\pi_k^*(\xi)).$$

It is called the horizontal module of ξ.

Remark 2.12. By definition, $\Gamma(\pi_k^*(\xi)) = \mathcal{F}_k(\pi) \otimes_{C^\infty(M)} \Gamma(\xi)$. Therefore, the horizontal module of ξ is given by $\Gamma(\pi_\infty^*(\xi)) = \mathcal{F}(\pi) \otimes_{C^\infty(M)} \Gamma(\xi)$.

Remark 2.13. An interesting horizontal module is the module of generating sections $\varkappa(\pi) := \Gamma(\pi_\infty^*(\pi))$. Each vertical vector field is uniquely defined by a generating section $\varphi \in \varkappa(\pi)$ in the following way:

$$\partial_\varphi^{(u)} := \sum_{j=1}^{m} \sum_{\alpha} \frac{\mathrm{d}}{\mathrm{d}x^i}(\varphi^j) \frac{\partial}{\partial u_\alpha^j},$$

as is shown in [3, p.147]. $\partial_\varphi^{(u)}$ is the called evolutionary derivative of φ.

We will now extend the precomposition j_∞^* to horizontal modules.

Definition 2.14. Let $\sum_{i\in I} f_i v^i \in \mathcal{F}(\pi) \otimes_{C^\infty(M)} \Gamma(\xi)$. Then $j_\infty^* : \mathcal{F}(\pi) \otimes_{C^\infty(M)} \Gamma(\xi) \to \mathrm{Diff}(\pi, \xi)$ is defined by

$$j_\infty^*(\sum_{i\in I} f^i v^i) = \sum_{i\in I} j_\infty^*(f^i) v^i. \tag{1}$$

We will now extend the notion of horizontal differential operators to horizontal modules.

Definition 2.15. Let $P_1 = \Gamma(\pi^*(\xi_1))$ and $P_2 = \Gamma(\pi^*(\xi_2))$ be horizontal modules. The set of linear maps between $\Gamma(\xi_1)$ and $\Gamma(\xi_2)$ is denoted as $\mathrm{Hom}(\xi_1, \xi_2)$. The horizontal module of linear maps between P_1 and P_2 is defined by

$$\mathrm{Hom}(P_1, P_2) := \mathcal{F}(\pi) \otimes_{C^\infty(M)} \mathrm{Hom}(\xi_1, \xi_2).$$

Similarly, the linear differential operators between $\Gamma(\xi_1)$ and $\Gamma(\xi_2)$ is denoted as $\mathrm{Diff}^{lin}(\xi_1, \xi_2)$. The module of horizontal differential operators between P_1 and P_2 is defined by

$$\mathcal{C}\mathrm{Diff}(P_1, P_2) = \mathcal{F}(\pi) \otimes_{C^\infty(M)} \mathrm{Diff}^{lin}(\xi_1, \xi_2).$$

The set of multilinear horizontal differential operators $\Delta : P_1 \times ... \times P_k \to Q$ can be constructed as

$$\mathcal{C}\mathrm{Diff}(P_1; ...; P_k, Q) = \mathcal{C}\mathrm{Diff}(P_1, \mathcal{C}\mathrm{Diff}(P_2, ..., \mathcal{C}\mathrm{Diff}(P_k, Q)...).$$

Specifically

$$\mathcal{C}\mathrm{Diff}_k(P, Q) := \mathcal{C}\mathrm{Diff}_k(P; ...; P, Q).$$

Remark 2.16. By equation (1), the precomposition j_∞^* maps elements of $\mathcal{C}\mathrm{Diff}(P_1, P_2)$ to the set $\mathrm{Diff}(\pi, \mathrm{Diff}^{lin}(\xi_1, \xi_2)) \subset \mathrm{Diff}(\pi \oplus \xi_1, \xi_2)$. Elements of $\mathcal{C}\mathrm{Diff}(P_1; ...; P_k, P_{k+1})$ are mapped to the set

$$\mathrm{Diff}(\pi, \mathrm{Diff}^{lin}(\xi_1, \mathrm{Diff}^{lin}(\xi_2, ..., \mathrm{Diff}^{lin}(\xi_k, \xi_{k+1})...) \subset \mathrm{Diff}(\pi \oplus \bigoplus_{i=1}^{k} \xi_i, \xi_{k+1}).$$

We will now define the linearization of sections of horizontal modules.

Definition 2.17. The linearization $\ell^{(u)}_{f^i v^i} \in \mathcal{C}\mathrm{Diff}(\varkappa(\pi), P)$ of a section $\sum_{i\in I} f^i v^i \in P$ applied to $\varphi \in \varkappa(\pi)$ is defined by

$$\ell^{(u)}_{\sum_{i\in I} f^i v^i}(\varphi) := \sum_{i\in I} \partial^{(u)}_{\varphi}(f^i) v^i$$

2.3. **Differential forms.** The horizontal differential is defined as

$$\bar{\mathrm{d}} := \mathrm{d}x^i \frac{\mathrm{d}}{\mathrm{d}x^i}$$

The Cartan differential d_C is defined as the difference between the de Rham differential d_{dR} and the horizontal differential.

$$\mathrm{d}_C := \mathrm{d}_{dR} - \bar{\mathrm{d}} = \sum_{j,\alpha} \big[\, \mathrm{d}u^j_\alpha + \sum_{i=1}^{n} \mathrm{d}x^i \; u^j_{\alpha+1_i} \big] \frac{\partial}{\partial u^j_\alpha}$$

The module of horizontal differential p-forms is

$$\bar{\Lambda}^p(\pi) := \mathcal{F}(\pi) \otimes_{C^\infty(M)} \Lambda^p(M).$$

The module of vertical differential forms is generated by

$$w^j_\alpha := \mathrm{d}_C(u^j_\alpha).$$

The module of vertical differential q-forms is denoted as $C\Lambda^q(\pi)$. Let $\boldsymbol{\alpha} := (\alpha_1, ..., \alpha_n) \in (\mathbb{N}^n)^q$ and $\boldsymbol{j} := (j_1, ..., j_q)$ with $1 \le j_i \le m$. We introduce the notation $w^{\boldsymbol{j}}_{\boldsymbol{\alpha}} := w^{j_1}_{\alpha_1} \wedge ... \wedge w^{j_q}_{\alpha_q}$. The space of differential k-forms on $J^\infty(\pi)$ can be defined as the sum of products of horizontal differential p-forms and vertical differential q-form such that $p+q = k$:

$$\Lambda^k(\pi) := \bigoplus_{p+q=k} \bar{\Lambda}^p(\pi) \otimes_{\mathcal{F}(\pi)} C\Lambda^q(\pi)$$

This gives us the following bi-complex:

$$\begin{array}{ccccc}
\bar{\Lambda}^n(\pi) & \xrightarrow{\mathrm{d}_C} & \bar{\Lambda}^n(\pi) \otimes_{\mathcal{F}(\pi)} C^1\Lambda(\pi) & \xrightarrow{\mathrm{d}_C} & \dots \\
\bar{\mathrm{d}}\Big\uparrow & & \bar{\mathrm{d}}\Big\uparrow & & \\
\vdots & & \vdots & & \\
\bar{\mathrm{d}}\Big\uparrow & & \bar{\mathrm{d}}\Big\uparrow & & \\
\bar{\Lambda}^1(\pi) & \xrightarrow{\mathrm{d}_C} & \bar{\Lambda}^1(\pi) \otimes_{\mathcal{F}(\pi)} C^1\Lambda(\pi) & \xrightarrow{\mathrm{d}_C} & \dots \\
\bar{\mathrm{d}}\Big\uparrow & & \bar{\mathrm{d}}\Big\uparrow & & \\
\mathcal{F}(\pi) & \xrightarrow{\mathrm{d}_C} & C^1\Lambda(\pi) & \xrightarrow{\mathrm{d}_C} & \dots
\end{array} \tag{2}$$

We also define the horizontal cohomology

$$\bar{E}^{p.q}(\pi) := \frac{\ker \bar{d} : \bar{\Lambda}^p(\pi) \otimes_{\mathcal{F}(\pi)} C\Lambda^q(\pi) \to \bar{\Lambda}^{p+1}(\pi) \otimes_{\mathcal{F}(\pi)} C\Lambda^q(\pi)}{\mathrm{im}\, \bar{d} : \bar{\Lambda}^{p-1}(\pi) \otimes_{\mathcal{F}(\pi)} C\Lambda^q(\pi) \to \bar{\Lambda}^(\pi) \otimes_{\mathcal{F}(\pi)} C\Lambda^q(\pi)}.$$

For the cohomology of the horizontal differential forms we will use the notation $\bar{H}^i(\pi) := E^{i,0}(\pi)$. $\bar{H}^n(\pi)$ is called the set of Lagrangians or Hamiltonians. The equivalence class corresponding to $\omega \in \bar{\Lambda}^p(\pi) \otimes_{\mathcal{F}(\pi)} C\Lambda^q(\pi)$ is denoted as $[\omega]$.

Remark 2.18. Since $\bar{\Lambda}^n(\pi)$ is a horizontal module, the map $j_\infty^* : \bar{\Lambda}^n(\pi) \to \mathrm{Diff}(\Gamma(\pi), \Lambda^n(M))$ exists. $[\omega]$ can be evaluated as a functional using the map $s \to \int_M j_\infty^*(\omega)(s)$.

Remark 2.19. Let P be a horizontal module. The adjoint module of P is denoted by $\widehat{P} := \mathrm{Hom}(P, \bar{\Lambda}^n(\pi))$. $<,>: \widehat{P} \times P \to \bar{\Lambda}^n(\pi)$ denotes the natural coupling between $\widehat{P}$ and P. Specifically, $\widehat{\mathcal{F}(\pi)} = \bar{\Lambda}^n(\pi)$.

2.4. **The Jet Superbundle.** We will first introduce the jet superbundle. The definition will be very similar to our introduction of the infinite jet bundle, and a lot of definitions will carry directly over from the Jet bundle. In this section we follow a setup similar to [4].

Definition 2.20. Let M^n be a manifold with dimension n and E a supermanifold of superdimension $(n + m_0)|m_1$ with the structure of a vector bundle $\pi : E \to M^n$. If π can be split into two separate bundles $\pi = \pi^0 \oplus \pi^1$ such that the fibres of π^0 are even and the fibres of π^1 are odd, then we π is called a superbundle over M^n. The superbundle can be extended to an infinite jet superbundle by setting

$$\pi^0_\infty := (\pi^0)_\infty$$
$$\pi^1_\infty := (((\pi^1)^\Pi)_\infty)^\Pi$$

π^0_∞ is the usual infinite jet bundle over π^0. Π denotes the parity operator. It declares odd vector bundles to be even, and even vector bundles to be odd. The set $\pi_\infty := \pi^0_\infty \oplus_M \pi^1_\infty$ is called a jet superbundle.

We define the set of sections of the superbundle as

$$\Gamma(\pi_\infty) := \Gamma(\pi_\infty)^0 \oplus \Gamma(\pi_\infty)^1 := \Gamma(\pi^0_\infty) \oplus (\Gamma((\pi^1_\infty)^\Pi))^\Pi$$

Any section $\Gamma(\pi) \ni \boldsymbol{s} = \boldsymbol{s}^0 + \boldsymbol{s}^1$ can be identified with a section $\Gamma(\pi_\infty) \ni j_\infty(\boldsymbol{s}) := j_\infty(\boldsymbol{s}^0) + (j_\infty((\boldsymbol{s}^1)^\Pi))^\Pi$. With $\mathcal{F}(\pi)$ we denote a superalgebra of differentiable functions on $J^\infty(\pi)$. We define it as

$$\mathcal{F}(\pi) \cong \sum_{k=0}^{\infty} \mathcal{F}(\pi^0) \otimes_{C^\infty(M)} \mathfrak{G}(\mathcal{F}_{lin}((\pi^1)^\Pi))$$

Where $\mathcal{F}_{lin}((\pi^1)^\Pi)$ is the subset of $\mathcal{F}((\pi^1)^\Pi)$ containing functions that are linear in the fibre of $(\pi^1_\infty)^\Pi$. $\mathfrak{G}(\mathcal{F}_{lin}((\pi^1)^\Pi))$ is the Grassmann algebra generated by these functions. A function $f \in \mathcal{F}$ is called homogeneous whenever it is homogeneous w.r.t. the Grassmann algebra $\mathfrak{G}(\mathcal{F}_{lin}((\pi^1)^\Pi))$. The degree of a homogeneous function f is denoted by D_f.

The definitions of horizontal and vertical vector fields on jet superbundles are identical to the definition of these fields on jet bundles.

Let $\xi = \xi_0 \oplus \xi_1$ be $\xi : X \to M$ be a vector superbundle over M, splitting in the even bundle ξ_0 and the odd bundle ξ_1. We define the pullback $\pi_\infty^*(\xi) := \pi_\infty^*(\xi_0) \oplus (\pi_\infty^*(\xi_1^\Pi))^\Pi$. The $\mathcal{F}(\pi)$-supermodules are defined in exactly the same way as $\mathcal{F}(\pi)$-modules. Left sided vertical vector fields can be written as

$$\overrightarrow{\partial}_\varphi^{(\boldsymbol{u},\boldsymbol{b})} = \sum_{j,\alpha}^{m} \frac{\mathrm{d}^{|\alpha|}}{\mathrm{d}x^\alpha}(\varphi^j)\frac{\overrightarrow{\partial}}{\partial u_\alpha^j}$$

with $\varphi \in \varkappa(\pi)$. For the evolutionary derivative we also have a right-sided variant, denoted with a arrow to the left. We define the linearization as $\ell_\omega^{(\boldsymbol{u},\boldsymbol{b})}(\varphi) = Sign(\varphi\omega)\overrightarrow{\partial}_\varphi^{(\boldsymbol{u},\boldsymbol{b})}(\omega)$.

2.5. **Hamiltonian Structures.** In this section we will define the Hamiltonian structures on a jet space. We will define and review the properties of the Schouten-Nijenhuis bracket, the Poisson bracket and discuss when a variational bivector defines a Hamiltonian equation. We use the same construction as is used in [4]. We start out with the definition of the horizontal jet bundle and consider an example: an infinite jet version of Kupershmidt's cotangent bundle to a vector bundle.
The following definition comes directly from [5].

Definition 2.21. Let ξ be a vector bundle over $J^\infty(\pi)$. Two sections $\boldsymbol{s}_1, \boldsymbol{s}_2 \in \Gamma(\xi)$ are horizontally equivalent at $\theta \in J^\infty(\pi)$if $\forall \alpha \in \mathbb{N}_{\geq 0}^n$ if $D_\alpha(\boldsymbol{s}_1^\beta) = D_\alpha(\boldsymbol{s}_2^\beta)$ at θ for all multi-indices α and fibre-indices β. Denote the equivalence class by $[\boldsymbol{s}]_\theta$. The set

$$\overline{J_\pi^\infty}(\xi) := \{[\boldsymbol{s}]_\theta | s \in \Gamma(\xi), \theta \in J^\infty(\pi)\}$$

is called the horizontal jet space of ξ.

We will now define Kupershmidt's cotangent bundle to a vector bundle and the infinite jet version.

Definition 2.22. Let $\pi : E \to M$ be a locally trivial smooth vector bundle over M. Let $\pi^* : E^* \to M$ be the dual bundle to π. Let $\hat{\pi}$ be the vector bundle $\hat{\pi} : E^* \otimes_{M^n} \Lambda^n(T^*M^n) \to M^n$. The superbundle $\mathcal{K} = \mathcal{K}^{0*}(\mathcal{K}^1)$, where $\mathcal{K}^0 = \pi$ and $\mathcal{K}^1 = \hat{\pi}$ is called Kupershmidt's cotangent bundle to π. The horizontal jet superbundle $\mathcal{K}_\infty = J_{\mathcal{K}_0}^\infty(\mathcal{K}_0^*(\mathcal{K}_1))$ is called the cotangent bundle of π_∞.

We will use $\boldsymbol{u}$ to refer to even coordinates and $\boldsymbol{b}$ to refer to odd coordinates of Kupershmidt's jet bundle. We will now define variational multivectors as the Hamiltonians of Kupershmidt's jet bundle.

Definition 2.23. Elements $P \in \bar{H}^n(\mathcal{K})$ is are variational multivectors. If P is homogeneous with degree $D_f(P) = k$, then P is a variational k-vector. The set of variational k-vectors is denoted by $\bar{H}_k^n(\mathcal{K})$.

We will now define the Schouten bracket, which will give us a way to evaluate multivectors.

Definition 2.24. Let $F, H \in \bar{H}^n_p(\mathcal{K})$. The variational Schouten bracket $[\![\cdot,\cdot]\!] : \bar{H}^n_p(\mathcal{K}) \times \bar{H}^n_q(\mathcal{K}) \to \bar{H}^n_{p+q-1}(\mathcal{K})$ is defined as

$$[\![F,H]\!] = \sum_j \frac{\delta F}{\delta u^j}\frac{\delta H}{\delta b^j} - (-1)^{(D_F-1)(D_H-1)}\frac{\delta H}{\delta b^j}\frac{\delta F}{\delta u^j}$$

Remark 2.25. The bracket is graded commutative:

$$[\![F,H]\!] = -(-1)^{(D_F-1)(D_H-1)}[\![H,F]\!]$$

It also satisfies a graded version of the Jacobi identity:

$$(-1)^{(D_F-1)(D_H-1)}[\![[\![F,G]\!],H]\!] + (-1)^{(D_G-1)(D_F-1)}[\![[\![G,H]\!],F]\!]$$
$$+(-1)^{(D_H-1)(D_G-1)}[\![[\![H,F]\!],G]\!] = 0$$

This implies that it forms a shift graded Lie algebra.

Evaluation of a variational multivector $P \in \bar{H}^n_k(\mathcal{K})$ at the densities $H^1, ..., H^k \in \bar{H}^n(\pi)$ is defined in the following manner:

$$P(H^1, ..., H^k) := (-)^k[\![[\![[\![[\![P, H^1]\!], H^2]\!], ...]\!], H^k]\!]$$

Remark 2.26. Note that, since $H^1, ..., H^k$ are Hamiltonians of $J^\infty(\pi) \simeq \bar{H}^n_0(\mathcal{K})$, the variational Schouten bracket simplifies to

$$[\![P, H^i]\!] = \sum_j (-1)^k \frac{\delta P}{\delta b^j}\frac{\delta H^i}{\delta u^j}$$

We will now state the central theorem of the thesis:

Theorem 2.27. *Let $\mathcal{P} \in \bar{H}^n_k(\pi)$. If $\mathcal{P}(H^1, ..., H^k) \equiv 0$ for all $H^i \in \bar{H}^n_0$, then $\mathcal{P} \equiv 0$.*

2.6. **Variational multivectors as differential operators.** We will now introduce an equivalent definition of variational multivectors. We will first define the relevant differential operators.

Definition 2.28. Let P and Q be $\mathcal{F}(\pi)$-modules. We denote $P^k := \overbrace{P \oplus ... \oplus P}^{\text{k times}}$. We denote the set of k-linear horizontal differential operators $\Delta : P^k \to Q$ as $\mathcal{C}\mathrm{Diff}_k(P, Q)$. We define $\mathcal{C}\mathrm{Diff}_0(P, Q) := Q$. A horizontal differential operator $\Delta \in \mathcal{C}\mathrm{Diff}_k(P, Q)$ is skew symmetric if for all $\sigma \in S_k$,

$$\Delta(p_1, ..., p_k) = (-1)^\sigma \Delta(p_{\sigma(1)}, ..., p_{\sigma(k)}).$$

We denote the set of skew symmetric multi-linear differential operators by $\mathcal{C}\mathrm{Diff}^{\mathrm{skew}}(P, Q)$. In the special case $P = \hat{Q}$, we call $\Delta \in \mathcal{C}\mathrm{Diff}(\hat{Q}, Q)$ self adjoint if it is self adjoint in each argument, i.e.

$$< \Delta(p_1, ..., p_j, ..., p_k), p_{k+1} >=< \Delta(p_1, ..., p_{j-1}, p_{k+1}, p_{j+1}, ..., p_k), p_j > .$$

We denote the set of all skew adjoint symmetric differential operators as $\mathcal{C}\mathrm{Diff}_k^{\text{sk-ad}}(P, Q)$.

Lemma 2.29. *The set of variational multivectors $\bar{H}_k^n(\mathcal{K})$ is isomorphic to the set $\mathcal{C}Diff_{k-1}^{sk-ad}(\hat{\varkappa}, \varkappa)$. There is an isomorphism $f : \bar{H}_k^n(\mathcal{K}) \leftrightarrow \mathcal{C}Diff_{k-1}^{sk-ad}(\hat{\varkappa}, \varkappa)$ satisfying the following property:*

$$P(H^1, ..., H^k) = [< \frac{\delta H^k}{\delta u}, f(P)(\frac{\delta H^1}{\delta u}, ..., \frac{\delta H^{k-1}}{\delta u}) >]$$

Furthermore, for each operator $\Delta_P \in \mathcal{C}Diff_k^{skew}(\widehat{\varkappa(\pi)}, \bar{\Lambda}^n(\pi))$ there exists a unique multivector $P \in \mathcal{C}Diff_{k-1}^{sk-ad}(\hat{\varkappa}, \varkappa)$ such that

$$< p_1, P(p_2, ..., p_k) >= [\Delta_P(p_1, ..., p_k)].$$

This is shown in [6]. The main theorem of this thesis can be restated by interpreting multivectors as differential operators.

Theorem 2.30. *Let $\mathcal{P} \in \mathcal{C}Diff_k^{sk-ad}(\hat{\varkappa}, \varkappa)$. If $\mathcal{P}(p^1, ..., p^k) = 0$ for all $p^i \in \widehat{\varkappa(\pi)}$, then $\mathcal{P} = 0$.*

3. Homotopy operators on star-shaped domains

In this section we will now show a proof of the Poincaré lemma on star-shaped manifolds an jet spaces.

3.1. The de Rham differential.

We will first show a proof of the Poincaré lemma for differential k-forms on manifolds using a homotopy operator. We follow the proof as outlined in [7, p. 63]. We call a set $V \subset \mathbb{R}^n$ star-shaped if $\forall x, \forall \lambda \in [0, 1], \lambda x \in V$. In other words, every point $x \in V$ is connected to the origin via a straight line. We will prove the lemma on a star shaped domain $V \subset R^n$. This then extends to manifolds diffeomorphic to V. We will prove the following statement

Theorem 3.1. *The Poincaré lemma on a Manifold*
Let ω be a differential k-form over the star shaped domain V of dimension n, $0 \leq k \leq n$. ω is exact whenever ω is closed.

Proof. We will show this by constructing a homotopy operator $h : \Lambda^{k+1}(V) \to \Lambda^k$. We will begin by recalling a few concepts and results from Lie theory. A vector field $v : M^n \to TM^n$ has a flow which we can denote by $e^{\epsilon v} : M^n \to M^n, p \to e^{\epsilon v}p$. The pullback $e^{\epsilon v *}$ of $e^{\epsilon v}$ of a tangent vector u at point $e^{\epsilon v}p$ is given by $e^{\epsilon v *} : T_{e^{\epsilon v}p}M \to T_pM, u \to (\mathrm{d}e^{-\epsilon v})^{-1}(u_1)$. The pullback $e^{\epsilon v *} : \Lambda^k(T^*_{e^{\epsilon v}p}M) \to \Lambda^k(T^*_pM)$ of $e^{\epsilon v}$ on differential k-forms is defined by $e^{\epsilon v *}(\omega|_{e^{\epsilon v}p})(u_1, ..., u_k) = (\omega|_p)(e^{\epsilon v *}(u_1), ..., e^{\epsilon v *}(u_k))$. We will make use of the Lie derivative L_v. We define it as

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}(e^{\epsilon v})^*(\omega|_{exp(\epsilon v)x}) = (e^{\epsilon v})^*(L_v(\omega)|_{exp(\epsilon v)(x)}). \quad (3)$$

We define the inner product $\iota : \Lambda^{k+1}(M) \to \Lambda^k(M)$ by setting $\iota_v\omega(v_1, ..., v_{k-1}) = \omega(v, v_1, ..., v_{k-1})$ Cartan's identity tells us that we can express the Lie derivative working on a k-form $\omega \in \Lambda^k(M^n)$ as follows:

$$L_v(\omega) = \mathrm{d}(\iota_v\omega) + \iota_v(\mathrm{d}\omega)$$

We can now construct a homotopy operator by integrating the Lie derivative.

We integrate equation 3 from 0 to $\epsilon < 0$:

$$\begin{aligned}(e^{\epsilon v})^*(\omega|_{exp(\epsilon v)(x)}) - \omega_x &= \int_0^\epsilon \frac{\mathrm{d}}{\mathrm{d}\epsilon}(e^{\mu v})^*(\omega|_{exp(\mu v)(x)})\,\mathrm{d}\mu \\ &= \int_0^\epsilon (e^{\mu v})^*(L_v(\omega|_{exp(\mu v)(x)})\,\mathrm{d}\mu \\ &= \int_0^\epsilon (e^{\mu v})^*(\mathrm{d}((\iota_v\omega)|_{exp(\mu v)(x)}) + (e^{\mu v})^*\iota_v(\mathrm{d}\omega|_{exp(\mu v)(x)})\,\mathrm{d}\mu \\ &= \mathrm{d}\int_0^\epsilon (e^{\mu v})^*((\iota_v(\omega|_{exp(\mu v)(x)}))\,\mathrm{d}\mu + \int_0^\epsilon (e^{\mu v})^*\iota_v(\mathrm{d}\omega|_{exp(\mu v)(x)})\,\mathrm{d}\mu\end{aligned} \tag{4}$$

This formula looks a lot like our required homotopy operator. To see this more clearly we define the operator h_v^ϵ:

$$h_v^\epsilon(\omega) = -\int_0^\epsilon (e^{\mu v})^*(\iota_v(\omega_{exp(\mu v)(x)}))\,\mathrm{d}\mu \tag{5}$$

Written in terms of this operator, we have

$$\omega|_x - (e^{\epsilon v})^*(\omega|_{exp(\epsilon v)(x)}) = \mathrm{d}h_v^\epsilon(\omega)|_x + h_v^\epsilon(\mathrm{d}\omega)|_x \tag{6}$$

To prove the theorem, we now choose the tangent vector field $v = \sum_{i=1}^n (x_i \frac{\partial}{\partial x_i}) \in \Gamma(TV)$. The flow of this vector field is given by $e^{\epsilon v}x = e^\epsilon x$.

We assume $x \in V$. Therefore, for $\epsilon < 0$ we know that the flow $e^\epsilon x \in V$ since we are on a star shaped domain. This implies that the exponential map is well defined for $\epsilon < 0$.

The pull-back of the flow is given by

$$(e^{\epsilon v})^*(\omega|_x(v_1, ..., v_k)) = \omega|_{e^\epsilon x}(e^\epsilon v_1, ..., e^\epsilon v_k)) = e^{k\epsilon}\omega|_{e^\epsilon x}(v_1, ..., v_k))$$

If we now take the limit $\epsilon \to -\infty$ we obtain

$$\begin{aligned}\lim_{\epsilon\to-\infty} \omega|_x - (e^{\epsilon v})^*(\omega|_{exp(\epsilon v)(x)}) &= \omega|_x - \lim_{\epsilon\to-\infty} e^{k\epsilon}\omega|_{e^\epsilon x}(v_1, ..., v_k)) \\ &= \omega|_x - \lim_{\epsilon\to-\infty} \sum_\sigma f_\sigma(e^\epsilon x)e^{k\epsilon}\,\mathrm{d}x_{\sigma(1)} \wedge ... \wedge \mathrm{d}x_{\sigma(k)} \\ &= \omega|_x - \lim_{\epsilon\to-\infty} \sum_\sigma f_\sigma(0)e^{k\epsilon}\,\mathrm{d}x_{\sigma(1)} \wedge ... \wedge \mathrm{d}x_{\sigma(k)} \\ &= \omega|_x\end{aligned}$$

Thus, if the right limit exists whenever $\mathrm{d}\omega = 0$, we obtain the equation

$\omega = \lim_{\epsilon \to -\infty} \mathrm{d}h_v^\epsilon(\omega) + h_v^\epsilon(\mathrm{d}\omega)$, proving the Poincaré lemma. To investigate the limit of the right terms, we look at the limit of our operator h_v^ϵ:

$$\begin{aligned} h_v(\omega) = \lim_{\epsilon \to -\infty} h_v^\epsilon(\omega) &= \lim_{\epsilon \to -\infty} -\int_0^\epsilon (\iota_v \omega)[e^\mu x])\,\mathrm{d}\mu \\ &= -\int_0^{-\infty} (\iota_v \omega)[e^\mu x])\,\mathrm{d}\mu \\ &= \int_0^1 (\iota_v \omega)[\lambda x])\frac{\mathrm{d}\lambda}{\lambda} \end{aligned}$$

In the last sentence we used the substitution $\mu = \ln \lambda$.

This allows us to state our final result: the construction of a homotopy operator $h : \Lambda^k(M) \to \Lambda^{k-1}(M)$ such that $h(\eta) = \omega$ whenever $\mathrm{d}\omega = 0$:

$$h(\omega) := \int_0^1 (\iota_v \omega)[\lambda x])\frac{\mathrm{d}\lambda}{\lambda} \tag{7}$$

This shows that any closed form is exact. □

Remark 3.2. In this proof, we have used two properties of the vector field $v = \sum_{i=1}^n (x_i \frac{\partial}{\partial x_i})$. Firstly, the flow $e^{\epsilon v}\boldsymbol{x}$ of any point $\boldsymbol{x}$ is defined for all $\epsilon \leq 0$. Secondly, for all $\boldsymbol{x} \in M$, $\lim_{\epsilon \to -\infty} e^{\epsilon v}\boldsymbol{x} = \boldsymbol{x}_0$ for some constant $\boldsymbol{x}_0 \in M$. We call vector fields satisfying these properties dilations. Any dilation X can be used to construct a global chart such that $X = \lambda_i \frac{\partial}{\partial \lambda_i}$. Therefore it can be used to construct a homotopy operator. This implies that any manifold with a dilation has an exact differential complex.

We will now apply the homotopy operator in an elementary example.

Example 3.3 (Homotopy operator on manifolds)**.** We will consider the 2-form $\omega = \mathrm{d}x \wedge \mathrm{d}y$.

$$\begin{aligned} \eta &= h(\mathrm{d}x \wedge \mathrm{d}y) \\ &= \int_0^1 \iota_v(\mathrm{d}x \wedge \mathrm{d}y)[\lambda \boldsymbol{x}]\frac{\mathrm{d}\lambda}{\lambda} \\ &= \int_0^1 (x\,\mathrm{d}y - y\,\mathrm{d}x)[\lambda \boldsymbol{x}]\frac{\mathrm{d}\lambda}{\lambda} \\ &= (x\,\mathrm{d}y - y\,\mathrm{d}x)\int_0^1 \lambda\,\mathrm{d}\lambda \\ &= \frac{1}{2}x\,\mathrm{d}y - \frac{1}{2}y\,\mathrm{d}x \end{aligned}$$

Now we can see that, indeed, $\mathrm{d}\eta = \omega$

3.2. **The Poincaré Lemma on Jet Spaces: the Cartan differential.** We will now prove the Poincaré Lemma for the Cartan differential. In coordinates, the Cartan differential has the following representation:

$$\mathrm{d}_C = \sum_j \left(\mathrm{d}u^j_\alpha \frac{\partial}{\partial u^j_\alpha} - \sum_i \mathrm{d}x^i u^i_{\alpha+1_i} \frac{\partial}{\partial u^j_\alpha} \right)$$

We will restrict in our proof the domain of d_C to the set $\bar{\Lambda}^n(\pi) \otimes C\Lambda(\pi)$, the top row of bi-complex (2). For any $\omega \in \bar{\Lambda}^n(\pi) \otimes C\Lambda(\pi)$ we have $dx^i \wedge \omega = 0$. Therefore

$$d_C(\omega) = \sum_j (\mathrm{d}u^j_\alpha \frac{\partial}{\partial u^j_\alpha})\omega. \tag{8}$$

Theorem 3.4. *The Poincaré Lemma: the Cartan differential*
Let $\pi : E \to M$ be a trivial vector bundle. Let $\omega \in \bar{\Lambda}^n(\pi) \otimes C\Lambda^k(\pi)$, $k > 0$. Then ω is d_C-exact whenever it is d_C-closed.

Proof. We consider the vector space $V = \pi^{-1}_\infty(\boldsymbol{x})$. V is an infinite-dimensional vector space with coordinates $[\boldsymbol{u}]$. Let $\omega|_{\boldsymbol{x}} \in C\Lambda^k(\pi) \otimes \Lambda^n(T^*_{\boldsymbol{x}}M)$ be the restriction of ω to $\pi^{-1}(\boldsymbol{x})$. We define the Cartan differential on V as $d_C\omega([\boldsymbol{u}]) = \sum_j (\mathrm{d}u^j_\alpha \frac{\partial}{\partial u^j_\alpha})\omega([u])$. Since V is a vector space, V is star shaped. Any $\omega|_{\boldsymbol{x}} \in C\Lambda^k(\pi) \otimes \Lambda^n(T^*_{\boldsymbol{x}}M)$ can be written as

$$\omega|_{\boldsymbol{x}} = \sum_{j_1,\dots,j_k} \sum_{\alpha_1,\dots,\alpha_k} f([\boldsymbol{u}])\, \mathrm{d}u^{j_1}_{\alpha_1} \wedge \dots \wedge \mathrm{d}u^{j_k}_{\alpha_k} \wedge dvol \tag{9}$$

Any term in this sum is of finite differential order, and can therefore be further restricted to $\pi^{-1}_p(\boldsymbol{x})$, the manifold of p-jets over $\boldsymbol{x}$. The de Rham differential on this manifold is d_C. Therefore we can apply the Poincaré lemma for manifolds. Let

$$\omega^{\boldsymbol{\alpha}}_{\boldsymbol{j}}|_{\boldsymbol{x}} = f([\boldsymbol{u}])\, \mathrm{d}u^{j_1}_{\alpha_1} \wedge \dots \wedge \mathrm{d}u^{j_k}_{\alpha_k} dvol \in \Lambda^k(\pi^{-1}_p(\boldsymbol{x})) \otimes \Lambda^n(T^*_{\boldsymbol{x}}M).$$

The dilation of $\pi^{-1}_k(\boldsymbol{x})$ is given by $\partial^{\boldsymbol{u}}_{\boldsymbol{u}}$. The homotopy operator is therefore

$$\begin{aligned} h : &\Lambda^k(\pi^{-1}_k(\boldsymbol{x})) \otimes \Lambda^n(T^*_{\boldsymbol{x}}M) \to \Lambda^{k-1}(\pi^{-1}_k(\boldsymbol{x})) \otimes \Lambda^n(T^*_{\boldsymbol{x}}M), \\ &\omega^{\boldsymbol{\alpha}}_{\boldsymbol{j}}|_{\boldsymbol{x}} \to \int_0^1 (\iota_{\partial^{\boldsymbol{u}}_{\boldsymbol{u}}} \omega^{\boldsymbol{\alpha}}_{\boldsymbol{j}}|_{\boldsymbol{x}})[\lambda \boldsymbol{x}]) \frac{\mathrm{d}\lambda}{\lambda}. \end{aligned}$$

We define the homotopy operator on $\pi^{-1}_\infty(\boldsymbol{x})$ as

$$\begin{aligned} h_C : &C\Lambda^k(\pi) \otimes \Lambda^n(T^*_{\boldsymbol{x}}M) \to C\Lambda^{k-1}(\pi) \otimes \Lambda^n(T^*_{\boldsymbol{x}}M), \\ &\sum_{j_1,\dots,j_k} \sum_{\boldsymbol{\alpha} \in (\mathbb{N}^n)^k} \omega^{\boldsymbol{\alpha}}_{\boldsymbol{j}}|_{\boldsymbol{x}} \to \sum_{j_1,\dots,j_k} \sum_{\boldsymbol{\alpha} \in (\mathbb{N}^n)^k} h(\omega^{\boldsymbol{\alpha}}_{\boldsymbol{j}}|_{\boldsymbol{x}}) \end{aligned}$$

Clearly, h_C is a d_C-homotopy operator for $C\Lambda^k(\pi) \otimes \Lambda^n(T^*_{\boldsymbol{x}}M)$. We define the Cartan homotopy operator of $C\Lambda^k(\pi) \otimes \bar{\Lambda}^n(\pi)$ by its evaluation

$$h_C(\omega)(\boldsymbol{x}, [\boldsymbol{u}]) = h_C(\omega|_{\boldsymbol{x}})([\boldsymbol{u}]).$$

Since d_C has a corresponding homotopy operator, ω is exact if and only if it is closed. □

Remark 3.5. We require that π is trivial to ensure the global existence of the vector field $\partial^{\boldsymbol{u}}_{\boldsymbol{u}}$. This proof does not work if there is no vertical vector field X such that $X|_{\boldsymbol{x}}$ is a dilation of $\pi_k^{-1}(\boldsymbol{x})$ at every point $\boldsymbol{x} \in M$.

Remark 3.6. Note that the Cartan homotopy operator commutes with the horizontal differential. Therefore it is also a homotopy operator for the spaces $E^{n,q}$. On this space we can consider the operator

$$\begin{aligned}\delta &= \sum_j \mathrm{d}u^j \left(-\frac{\mathrm{d}}{\mathrm{d}x^i}\right)^{\alpha} \frac{\partial}{\partial u^j_\alpha} \\ &= \sum_j \mathrm{d}u^j \frac{\delta}{\delta u^j}\end{aligned}$$

The principle of least action can be stated as $\delta\mathcal{L} = 0$. Since $\delta L \equiv d_C L$ we have $d_C(\delta L) \equiv 0$. Since $\delta\omega \equiv \mathrm{d}_C(\omega)$, we have $h_C(\delta\omega) \equiv h_C(\mathrm{d}_C(\omega))$. This means that inverting the Cartan homotopy operator allows us to construct Lagrangians from Euler-Lagrange equations. Working on $\mathbf{F}\delta\boldsymbol{u} \in \widehat{\varkappa(\pi)}$, h_C simplifies to

$$\begin{aligned}h_C(\mathbf{F}\,\mathrm{d}\boldsymbol{u}) &= \int_0^1 (\iota_{\partial^{\boldsymbol{u}}_{\boldsymbol{u}}}(\mathbf{F}\,\mathrm{d}\boldsymbol{u}))(\boldsymbol{x}, [\lambda\boldsymbol{u}])\frac{\mathrm{d}\lambda}{\lambda} \\ &= \int_0^1 (\mathbf{F}\cdot\boldsymbol{u})|_{(\boldsymbol{x},[\lambda\boldsymbol{u}])}\frac{\mathrm{d}\lambda}{\lambda} \\ &= \int_0^1 \mathbf{F}(\boldsymbol{x}, [\lambda u])\cdot\boldsymbol{u}\,\mathrm{d}\lambda \qquad (10)\end{aligned}$$

As an example, we can apply this to the hyperbolic Liouville equation, which is given by $u_{xy} - e^{2u} = 0$

Example 3.7. Applying h_C to $(u_{xy} - e^{2u})\,\mathrm{d}u$, we obtain

$$\begin{aligned}\mathcal{L} &= h_C((u_{xy} - e^{2u})\,\mathrm{d}u) \\ &= \int_0^1 u(\lambda u_{xy} - e^{2\lambda u})\frac{\mathrm{d}\lambda}{\lambda} \\ &= \frac{1}{2}uu_{xy} - \frac{1}{2}e^{2u} + \frac{1}{2}\end{aligned}$$

If we calculate $\frac{\delta}{\delta u}(h(\omega))$, we find that this indeed gives us the original equation.

3.3. **The Poincaré Lemma on Jet Spaces: The Horizontal Differential.** In this section we will show the construction of a homotopy operator to the horizontal differential $\bar{d}$ on the set $\bar{\Lambda}^n(\pi)$ of volume forms over a jet space $J^\infty(\pi)$. In other words, we will define an operator $\bar{h} : \bar{\Lambda}^n(\pi) \to \bar{\Lambda}^{n-1}(\pi)$ such that whenever $\omega \in \bar{\Lambda}^n(\pi)$ is $\bar{\mathrm{d}}$ exact, $\bar{d}(\bar{h}(\omega)) = \omega$. By finding the homotopy operator we will show which volume forms are exact. Any $\omega \in \bar{\Lambda}^n(\pi)$ is $\bar{\mathrm{d}}$-closed. However, not every horizontal volume is exact.

Lemma 3.8. *Let $\eta \in \bar{\Lambda}^{n-1}(\pi)$. Then $\delta(\mathrm{d}\eta) = 0$.*

Proof. We will show that $\frac{\delta}{\delta u^j} \circ \frac{\mathrm{d}}{\mathrm{d}x_i} = 0$.

$$\frac{\delta}{\delta u^j}\left(\frac{\mathrm{d}}{\mathrm{d}x_i}(f)\right) = \sum_\alpha \left(-\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{x}}\right)^\alpha \frac{\partial}{\partial u^j_\alpha}\left(\frac{\mathrm{d}}{\mathrm{d}x_i}(f)\right)$$
$$= \sum_\alpha - \left(-\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{x}}\right)^{\alpha+1_i} \frac{\partial}{\partial u^j_\alpha}(f) + \left(-\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{x}}\right)^\alpha \frac{\partial}{\partial u^j_\alpha}\left(\frac{\mathrm{d}}{\mathrm{d}x_i}\right)(f) \tag{11}$$

Applying $\frac{\partial}{\partial u^j_\alpha}$ directly to $\frac{d}{dx^i}$, we find

$$\frac{\partial}{\partial u^j_\alpha}\left(\frac{\mathrm{d}}{\mathrm{d}x_i}\right) = \frac{\partial}{\partial u^j_\alpha}\left(\frac{\mathrm{d}}{\partial x_i} + \sum_{j',\beta} u^{j'}_{\beta+1_i}\frac{\partial}{\partial u^{j'}_\beta}\right)$$
$$= \sum_\beta \delta^\alpha_{\beta+1_i}\frac{\partial}{\partial u^j_\beta}$$
$$= \begin{cases} \frac{\partial}{\partial u^j_{\alpha-1_i}} & \text{if } \alpha_i > 0 \\ 0 & \text{if } \alpha_i = 0 \end{cases}$$

Applying this to equation (11) we find

$$\frac{\delta}{\delta u^j}\left(\frac{\mathrm{d}}{\mathrm{d}x_i}(f)\right) = -\sum_\alpha \left(-\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{x}}\right)^{\alpha+1_i} \frac{\partial}{\partial u^j_\alpha}(f) + \sum_{\alpha'\geq 1_i}\left(-\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{x}}\right)^{\alpha'} \frac{\partial}{\partial u^j_{\alpha'-1_i}}(f)$$
$$= -\sum_\alpha \left(-\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{x}}\right)^{\alpha+1_i} \frac{\partial}{\partial u^j_\alpha}(f) + \sum_\alpha \left(-\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{x}}\right)^{\alpha+1_i} \frac{\partial}{\partial u^j_\alpha}(f) = 0$$

□

This shows that a volume form can only be $\bar{d}$-exact whenever it is δ-closed.

Theorem 3.9. *The Poincaré Lemma: the horizontal differential*
Let $\pi : E \to V$ be the trivial vector bundle over a star-shaped domain V. Let ω be a horizontal differential form $\omega \in \bar{\Lambda}^n(\pi)$. Then ω is $\bar{d}$-exact whenever ω is δ-closed.

Proof. We will prove this by inverting the differential using a homotopy operator, in a way similar to the proof of the Poincaré lemma on manifolds. To construct our homotopy operator, we first find $\partial_{\varphi}^{\boldsymbol{u}}(\eta)$ and then integrate it to obtain η. To do this, we first notice that $\bar{\mathrm{d}}(\partial_{\varphi}^{\boldsymbol{u}}(\eta)) = \partial_{\varphi}^{\boldsymbol{u}}(\bar{\mathrm{d}}\eta)$. Should we be able to integrate along the flow of $\partial_{\varphi}^{\boldsymbol{u}}$, we can obtain η from $\partial_{\varphi}^{\boldsymbol{u}}(\eta)$. Thus, if we can invert the differential on $\bar{\mathrm{d}}(\partial_{\varphi}^{\boldsymbol{u}}(\eta))$, we are practically finished with our proof.

3.3.1. *Reversing the differential on horizontal differential operators.* In this proof we will use the space of horizontal differential form valued operators, $\mathcal{C}\mathrm{Diff}(F(\pi), \bar{\Lambda}^p(\pi))$. The fibre $\mathfrak{A}_\theta$ of $\mathcal{C}\mathrm{Diff}(F(\pi), \bar{\Lambda}(\pi))$ at any point in of $J^\infty(\pi)$ consists of vectors

$$\sum_{\tau \in S^n} \sum_{\beta} \sum_{p=0}^{n} a_{\beta,p}^{\tau} \, \mathrm{d}x^{\tau(1)} \wedge ... \wedge \mathrm{d}x^{\tau(p)} \cdot \frac{\mathrm{d}^{|\beta|}}{\mathrm{d}x^{\beta}},$$

where $\tau \in S^n$. It forms an algebra with multiplication

$$\mathrm{d}x^{\tau_1(1)} \wedge ... \wedge \mathrm{d}x^{\tau_1(p_1)} \cdot \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{\beta_1} \wedge \mathrm{d}x^{\tau_2(1)} \wedge ... \wedge \mathrm{d}x^{\tau_2(p_2)} \cdot \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{\beta_2} =$$
$$= (\mathrm{d}x^{\tau_1(1)} \wedge ... \wedge \mathrm{d}x^{\tau_1(p_1)} \wedge \mathrm{d}x^{\tau_2(1)} \wedge ... \wedge \mathrm{d}x^{\tau_2(p_2)}) \cdot \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{\beta_1+\beta_2}$$

The horizontal differential on $\mathfrak{A}_\theta$ is defined by

$$\bar{d}a_{\beta,p}^{\tau} \, \mathrm{d}x^{\tau(1)} \wedge ... \wedge \mathrm{d}x^{\tau(p)} \cdot \frac{\mathrm{d}^{|\beta|}}{\mathrm{d}x^{\beta}} = a_{\beta,p}^{\tau} \sum_{i=1}^{n} \mathrm{d}x^i \wedge \mathrm{d}x^{\tau(1)} \wedge ... \wedge \mathrm{d}x^{\tau(p)} \cdot \frac{\mathrm{d}^{|\beta+1_i|}}{\mathrm{d}x^{\beta+1_i}}$$

We will first invert the horizontal differential on this algebra, as described in [1, p. 50]. We will later see that we can use this inversion to construct a homotopy operator for differential forms.

Inverting the differential is easier to do if we rewrite our operators using a vector space automorphism. Specifically, we will use the algebra generated by the even symbols $D_1, ..., D_n$ and the odd symbols $\xi_1, ..., \xi_n$. We define an automorphism by

$$Aut(\mathrm{d}x_{\tau(1)} \wedge ... \wedge \mathrm{d}x_{\tau(p)} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{\beta}) = (-1)^{\tau} \xi_{\tau(p+1)} \cdot ... \cdot \xi_{\tau(n)} \cdot D_{\beta} \cdot (-1)^{s_p} \quad (12)$$

Where $\tau \in S_n$ and s_p is a sequence that determines the sign. On this algebra, we define an operator $\bar{\mathrm{d}}'$ that works by the graded Leibniz rule as follows:

$$\bar{\mathrm{d}}'(\xi_i) = D_i, \;\; \bar{\mathrm{d}}'(D_i) = 0 \quad (13)$$

Recall that the graded Leibniz rule is given by $\bar{\mathrm{d}}'(a \cdot b) = \bar{\mathrm{d}}'(a) \cdot b + (-1)^{deg(a)} a \cdot \bar{\mathrm{d}}'(b)$. We claim that if we choose the sequence s_p correctly, $\bar{\mathrm{d}}' \circ Aut = Aut \circ \bar{\mathrm{d}}$. In other words, it will just be the horizontal

differential on our algebra. To see why, we will calculate $Aut \circ \bar{\mathrm{d}}(\mathrm{d}x_{\tau(1)} \wedge \cdots \wedge \mathrm{d}x_{\tau(p)})$ and $\bar{\mathrm{d}}' \circ Aut(\mathrm{d}x_{\tau(1)} \wedge \cdots \wedge \mathrm{d}x_{\tau(p)})$. To make the expressions slightly less cumbersome, we use the notation $\xi_{\tau(p+1)} \cdot ... \cdot \xi_{\tau(l-1)} \cdot \xi_{\tau(l+1)} \cdot ... \cdot \xi_{\tau(n)} = \xi_{\tau(p+1)} \cdot ... \cdot \widehat{\xi_{\tau(l)}} \cdot ... \cdot \xi_{\tau(n)}$.
First we calculate $Aut \circ \bar{\mathrm{d}}(\mathrm{d}x_{\tau(1)} \wedge ... \wedge \mathrm{d}x_{\tau(p)})$:

$$\begin{aligned} Aut \circ \bar{\mathrm{d}}(\mathrm{d}x_{\tau(1)} \wedge ... \wedge \mathrm{d}x_{\tau(p)}) &= Aut(\sum_{l=p+1}^{n} \mathrm{d}x_{\tau(l)} \wedge \mathrm{d}x_{\tau(1)} \wedge ... \wedge \mathrm{d}x_{\tau(p)} \frac{\mathrm{d}}{\mathrm{d}x_{\tau(l)}}) \\ &= Aut(\sum_{l=p+1}^{n} (-1)^p \, \mathrm{d}x_{\tau(1)} \wedge ... \wedge \mathrm{d}x_{\tau(p)} \wedge \mathrm{d}x_{\tau(l)} \frac{\mathrm{d}}{\mathrm{d}x_{\tau(l)}}) \end{aligned}$$

To apply Aut, we have to rewrite this term in a way that will allow us to apply definition 12. To do this, we define

$$\tau_l = \tau(l \;\; l-1)(l-1 \;\; l-2)...(p+2 \;\; p+1)$$

Where we assume that $n \geq l > p$. Note that its sign is $(-1)^{N(\tau_l)} = (-1)^{N(\tau)+l-(p+1)}$, where we define $N(\tau)$ as the number of inversions of τ. This permutation looks as follows:

$$\tau_l = \left(\begin{smallmatrix} 1 & 2 & \cdots & p & p+1 & p+2 & \cdots & l & l+1 & \cdots & n \\ \tau(1) & \tau(2) & \cdots & \tau(p) & \tau(l) & \tau(p+1) & \cdots & \tau(l-1) & \tau(l+1) & \cdots & \tau(n) \end{smallmatrix}\right)$$

so that

$$\begin{aligned} & Aut(\sum_{l=p+1}^{n} (-1)^p \, \mathrm{d}x_{\tau(1)} \wedge ... \wedge \mathrm{d}x_{\tau(p)} \wedge \mathrm{d}x_{\tau(l)} \frac{\mathrm{d}}{\mathrm{d}x_{\tau(l)}}) \\ &= Aut(\sum_{l=p+1}^{n} (-1)^p \, \mathrm{d}x_{\tau_l(1)} \wedge ... \wedge \mathrm{d}x_{\tau_l(p)} \wedge \mathrm{d}x_{\tau_l(p+1)} \frac{\mathrm{d}}{\mathrm{d}x_{\tau_l(p+1)}}) \\ &= \sum_{l=p+1}^{n} (-1)^{p+N(\tau_l)+s_{p+1}} \xi_{\tau_l(p+2)} \cdot ... \cdot \xi_{\tau_l(n)} D_{\tau_l(p+1)} \\ &= \sum_{l=p+1}^{n} (-1)^{p+N(\tau)+l-(p+1)+s_{p+1}} \xi_{\tau(p+1)} \cdot ... \cdot \widehat{\xi_{\tau(l)}} \cdot ... \cdot \xi_{\tau(n)} D_{\tau(l)} \\ &= \sum_{l=p+1}^{n} (-1)^{N(\tau)+l-1+s_{p+1}} \xi_{\tau(p+1)} \cdot ... \cdot \widehat{\xi_{\tau(l)}} \cdot ... \cdot \xi_{\tau(n)} D_{\tau(l)} \end{aligned}$$

Now we calculate $\bar{\mathrm{d}}' \circ Aut(\mathrm{d}x_{\tau(1)} \wedge ... \wedge \mathrm{d}x_{\tau(p)})$:

$$\begin{aligned}
&\bar{\mathrm{d}}' \circ Aut(\mathrm{d}x_{\tau(1)} \wedge ... \wedge \mathrm{d}x_{\tau(p)}) \\
&= \bar{\mathrm{d}}'(-1)^{N(\tau)+s_p} \cdot \xi_{\tau(p+1)} \cdot ... \cdot \xi_{\tau(n)} \\
&= \sum_{l=p+1}^{n} (-1)^{N(\tau)+l-(p+1)+s_p} \cdot \xi_{\tau(p+1)} \cdot ... \cdot \widehat{\xi_{\tau(l)}} \cdot ... \cdot \xi_{\tau(n)} \cdot D_{\tau(l)}
\end{aligned}$$

We will now choose the sign given by s_p. By equating $\bar{\mathrm{d}}' \circ Aut$ and $Aut \circ \bar{\mathrm{d}}$ we obtain the recurrence relation

$$s_{p+1} = s_p - p$$

Solving this recurrence relation for $s_0 = 0$, we obtain

$$s_p = -\frac{p(p-1)}{2} \tag{14}$$

This gives the following sequence : $((-1)^{s_p}) = (1, 1, -1, -1, 1, 1, -1, -1, ...)$ For this choice of signs, we obtain $\bar{\mathrm{d}}' \circ Aut = Aut \circ \bar{\mathrm{d}}$. From now on, we will no longer differentiate between $\bar{\mathrm{d}}'$ and $\bar{\mathrm{d}}$.

We will now define the Koszul differential on this algebra

Definition 3.10. Let s be the derivation satisfying the graded Leibniz rule and

$$s(\xi_i) = 0, \;\; s(D_i) = \xi_i$$

This differential has a interesting property. If we apply $\bar{\mathrm{d}} \circ s$ to a term, we obtain:

$$\begin{aligned}
\bar{\mathrm{d}} \circ s(\xi_{\tau(1)} \cdot ... \cdot \xi_{\tau(p)} \cdot D_\beta) &= \bar{\mathrm{d}} \sum_{k=1}^{n} \beta_k \cdot \xi_k \cdot \xi_{\tau(1)} \cdot ... \cdot \xi_{\tau(p)} D_{\beta - 1_k} \\
&= \sum_{l=1}^{p} \sum_{k=1}^{n} (-1)^l \beta_k \cdot \xi_k \cdot \xi_{\tau(1)} \cdot ... \cdot \widehat{\xi_{\tau(l)}} \cdot ... \cdot \xi_{\tau(p)} D_{\beta - 1_k + 1_{\tau(l)}} \\
&\quad + \sum_{k=1}^{n} \beta_k \cdot \xi_{\tau(1)} \cdot ... \cdot \xi_{\tau(p)} \cdot D_\beta
\end{aligned} \tag{15}$$

The second sum in the final expression arises when $\bar{\mathrm{d}}$ is applied to ξ_k. If we apply $s \circ \bar{\mathrm{d}}$ to a term, we obtain:

$$
\begin{aligned}
s\circ \bar{\mathrm{d}}(\xi_{\tau(1)}\cdot \ldots \cdot \xi_{\tau(p)}\cdot D_\beta) &= s\sum_{l=1}^{p}(-1)^{l-1}\xi_{\tau(1)}\cdot \ldots \cdot \widehat{\xi_{\tau(l)}}\cdot \ldots \cdot \xi_{\tau(p)} D_{\beta+1_{\tau(l)}} \\
&= \sum_{k=1}^{p}\sum_{l=1}^{|\beta|}(-1)^{l-1}\beta_k\cdot \xi_k\cdot \xi_{\tau(1)}\cdot \ldots \cdot \widehat{\xi_{\tau(l)}}\cdot \ldots \cdot \xi_{\tau(p)}\cdot D_{\beta+1_{\tau(l)}-1_k} \\
&\quad + \sum_{l=1}^{p}\xi_{\tau(1)}\cdot \ldots \cdot \xi_{\tau(p)}\cdot D_\beta \qquad (16)
\end{aligned}
$$

The second sum in the final expression arises when s is applied to $D_{\tau(l)}$. We then add equations (15) and (16) to obtain

$$
\bar{\mathrm{d}}\circ s(\xi_{\tau(1)}\cdot\ldots\cdot\xi_{\tau(p)}\cdot D_\beta)+s\circ\bar{\mathrm{d}}(\xi_{\tau(1)}\cdot\ldots\cdot\xi_{\tau(p)}\cdot D_\beta) = (p+|\beta|)\xi_{\tau(1)}\cdot\ldots\cdot\xi_{\tau(p)}\cdot D_\beta \qquad (17)
$$

This means that the anticommutator of s and $\bar{\mathrm{d}}$, $[s,\bar{\mathrm{d}}]_+$, is a weight counting operator. We define $\bar{s}=\frac{1}{p+|\beta|}s$ whenever $p+|\beta|\neq 0$, $\bar{s}=0$ whenever $p+|\beta|\neq 0$. If $\bar{\mathrm{d}}(\xi_{\tau(1)}\cdot\ldots\cdot\xi_{\tau(p)}\cdot D_\beta)=0$, we see that

$$
\bar{\mathrm{d}}(\bar{s}(\xi_{\tau(1)}\cdot \ldots \cdot \xi_{\tau(p)}\cdot D_\beta)) = \xi_{\tau(1)}\cdot \ldots \cdot \xi_{\tau(p)}\cdot D_\beta. \qquad (18)
$$

We call $\bar{s}$ the Koszul differential. We will use its properties to construct the homotopy operator.
We note that for $p=0$, $\bar{d}\bar{s}(\sum_\alpha f_\beta D_\beta)=\sum_\alpha f_\beta D_\beta - f_\emptyset$

Let $f^i\frac{\mathrm{d}}{\mathrm{d}x^\alpha}\in \mathcal{C}\mathrm{Diff}(\mathcal{F}(\pi),\bar{\Lambda}(\pi))$. We define

$$
G(f^i\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^\alpha \mathrm{d}x^{i_1}\wedge \mathrm{d}x^{i_p}) := -\bar{s}(\left(-\frac{\mathrm{d}}{\mathrm{d}x}\right)^\alpha \mathrm{d}x^{i_1}\wedge\ldots\wedge\mathrm{d}x^{i_p})(f^i)
$$

To show that $\partial^{\boldsymbol{u}}_\varphi(\omega)$ is invertible whenever $\ell^{u\dagger}_\omega=0$, we consider $\partial^{\boldsymbol{u}}_\varphi(\omega)=\ell^{\boldsymbol{u}}_\omega(\varphi)$, we note that

$$
\bar{d}G(\ell^{\boldsymbol{u}}_\omega\circ\varphi) = -\ell^{u\dagger}_\omega(\varphi)+\ell^{\boldsymbol{u}}_\omega(\varphi) \qquad (19)
$$

We recall that we assumed

$$
\ell^{u\dagger}_\omega(1) = (-\frac{d}{dx})^\alpha(\frac{\partial\omega}{\partial \boldsymbol{u}_\alpha}) = \frac{\delta\omega}{\delta\boldsymbol{u}} = 0 \qquad (20)
$$

Therefore we have $\bar{d}G(\ell^{\boldsymbol{u}}_\omega\circ\varphi)=\ell^{\boldsymbol{u}}_\omega(\varphi)$. We can find an explicit formula for $G(\ell^{\boldsymbol{u}}_\omega\circ\varphi)$ by investigating the Koszul differential.

For $p=n$ we will explicitly write down $\bar{s}$.
We obtain

$$
\begin{aligned}
&\bar{s}(\frac{\mathrm{d}^{|\beta|}}{\mathrm{d}\boldsymbol{x}^{\beta}}\,\mathrm{d}x_1\wedge\ldots\wedge\mathrm{d}x_n)) \qquad (21)\\
&=(-1)^{s_n}Aut^{-1}\circ\bar{s}\circ Aut(\frac{\mathrm{d}^{|\beta|}}{\mathrm{d}\boldsymbol{x}^{\beta}}\,\mathrm{d}x_1\wedge\ldots\wedge\mathrm{d}x_n)\\
&=(-1)^{s_n}Aut^{-1}\circ\bar{s}(D_\beta)\\
&=(-1)^{s_n}Aut^{-1}\sum_{k=1}^{n}\frac{\beta_k}{|\beta|}\xi_k\cdot D_{\beta-1_k}\\
&=(-1)^{s_n}Aut^{-1}\sum_{k=1}^{n}\frac{\beta_k}{|\beta|}\xi_{(k\ \ k+1)\ldots(n-1\ \ n)(n)}\cdot D_{\beta-1_k}\\
&=(-1)^{s_n-s_{n-1}}\sum_{k=1}^{n}\frac{\beta_k}{|\beta|}(-1)^{(n-k)}\,\mathrm{d}x_1\wedge\ldots\wedge\mathrm{d}x_{k-1}\wedge\mathrm{d}x_{k+1}\wedge\ldots\wedge\mathrm{d}x_n\frac{\mathrm{d}^{|\beta|-1}}{\mathrm{d}x^{\beta-1_k}}\\
&=\sum_{k=1}^{n}\frac{\beta_k}{|\beta|}(-1)^{k-1}\,\mathrm{d}x_1\wedge\ldots\wedge\widehat{\mathrm{d}x_k}\wedge\ldots\wedge\mathrm{d}x_n\left(\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{x}}\right)^{\beta-1_k} \qquad (22)
\end{aligned}
$$

For $\omega=f\,\mathrm{d}x^1\wedge\ldots\wedge\mathrm{d}x^n\in\bar{\Lambda}^n$ we define $\omega[\frac{\mathrm{d}}{\mathrm{d}x^i}]:=(-1)^{k-1}f\,\mathrm{d}x^1\wedge\ldots\wedge\widehat{\mathrm{d}x^i}\wedge\ldots\wedge\mathrm{d}x^n\in\bar{\Lambda}^n$. Using this notation we can write

$$
G(\omega\left(\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{x}}\right)^{\beta})=\sum_{k=1}^{n}\frac{\beta_k}{|\beta|}\left(-\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{x}}\right)^{\beta-1_k}(\omega[\frac{\mathrm{d}}{\mathrm{d}x^i}]) \qquad (23)
$$

3.3.2. *The construction of a horizontal homotopy operator.* We will now move on to the final part of the proof. We choose $\varphi=\boldsymbol{u}$, so that the flow A_ϵ is given by $A_\epsilon(\boldsymbol{x},\boldsymbol{u}_\alpha)=(\boldsymbol{x},e^\epsilon\boldsymbol{u}_\alpha)$. The derivative of A_ϵ works as follows: $\frac{\mathrm{d}}{\mathrm{d}\epsilon}A_\epsilon(\omega)=A_\epsilon(\partial_{\boldsymbol{u}}^{\boldsymbol{u}}\omega)=A_\epsilon(\ell_\omega^{\boldsymbol{u}}(\boldsymbol{u}))$. We can now integrate the derivative of the flow of ω:

$$
\begin{aligned}
\omega-\omega(\boldsymbol{x},[0])&=A_0(\omega)-A_{-\infty}(\omega)\\
&=\int_{-\infty}^{0}\frac{\mathrm{d}}{\mathrm{d}\epsilon}A_\epsilon(\omega)d\epsilon\\
&=\int_{-\infty}^{0}A_\epsilon(\ell_\omega^{\boldsymbol{u}}(\boldsymbol{u}))d\epsilon\\
&=\int_{-\infty}^{0}A_\epsilon(\bar{\mathrm{d}}G(\ell_\omega^{\boldsymbol{u}}\circ\boldsymbol{u}))d\epsilon\\
&=\int_{-\infty}^{0}\bar{\mathrm{d}}A_\epsilon(G(\ell_\omega^{\boldsymbol{u}}\circ\boldsymbol{u}))d\epsilon\\
&=\bar{\mathrm{d}}\int_{-\infty}^{0}(G(\ell_\omega^{\boldsymbol{u}}\circ\boldsymbol{u}))[e^\epsilon\boldsymbol{u}]d\epsilon\\
&=\bar{\mathrm{d}}\int_{0}^{1}(G(\ell_\omega^{\boldsymbol{u}}\circ\boldsymbol{u}))[\lambda\boldsymbol{u}]\frac{1}{\lambda}d\lambda. \qquad (24)
\end{aligned}
$$

$\omega(\boldsymbol{x},[0])$ is exact by the original Poincaré lemma on manifolds. We can use the homotopy operator h to reconstruct the corresponding $(n-1)$-form. Therefore, our homotopy operator is:

$$\bar{h}(\omega) = \int_0^1 (G(\ell_\omega^{\boldsymbol{u}} \circ \varphi))[\lambda \boldsymbol{u}]\frac{1}{\lambda}d\lambda + h(\omega|_{[\boldsymbol{u}]=0}) \tag{25}$$

□

Remark 3.11. Let $\eta \in \bar{\Lambda}^{k-1}(\pi)$, and $\mathbf{F} \in \widehat{\varkappa(\pi)}$. For a conserved current η of an equation $\mathbf{F}=0$ we have $\bar{d}\eta \equiv < \psi_\eta, F >= 0$. Noether's theorem tells us that if $\mathbf{F}=0$ is an Euler-Lagrange equation, $\partial_{\psi_\eta}^{\boldsymbol{u}}$ is a differentiable symmetry of the equation, as is explained in [1]. Therefore, inverting the horizontal differential will allow us to derive conserved currents from differentiable symmetries of Euler-Lagrange equations.

Example 3.12. We will consider the same equation as in example 3.7, the hyperbolic Liouville equation: $u_{xy} - e^{2u} = 0$. A conserved current η of this equation with corresponding symmetry ψ_η has the property: $\bar{\mathrm{d}}\eta = \psi_\eta(u_{xy} - e^{2u})\,\mathrm{d}x \wedge \mathrm{d}y$. To construct the conserved current corresponding to the symmetry u_x of x-translation, we will apply the homotopy operator $\bar{h}$ to $\omega = u_x(u_{xy} - e^{2u})\,\mathrm{d}x \wedge \mathrm{d}y$. First we consider $\frac{\delta\omega}{\delta u}$.

$$\begin{aligned}\frac{\delta\omega}{\delta u} &= \sum_{\alpha\in\mathbb{N}^2}(-\frac{d}{dx})^\alpha \frac{\partial}{\partial u_\alpha}(u_x(u_{xy} - e^{2u}))dx \wedge dy \\ &= \big[-2u_x e^{2u} - u_{xyx} + 2u_x e^{2u} + u_{xyx}\big]dx \wedge dy = 0\end{aligned}$$

Therefore we can conclude that η indeed exists. We also note that $\omega(\boldsymbol{x},[0]) = 0$. This implies that we need not concern ourselves with the contribution of this term. To apply the homotopy operator, we first calculate the operator ℓ_ω^u:

$$\ell_\omega^u = \mathrm{d}x \wedge \mathrm{d}y\big[-2u_x e^{2u} + (u_{xy} - e^{2u})\frac{\mathrm{d}}{\mathrm{d}x} + u_x\frac{\mathrm{d}^2}{\mathrm{d}x\,\mathrm{d}y}\big]$$

We can integrate by parts to find $\bar{d}\nu$, where we apply ℓ_ω^u to a test function φ:

$$\ell_\omega^u(\varphi) = \big[-\frac{\mathrm{d}}{\mathrm{d}x}(e^{2u}\varphi) + \frac{d}{dy}(u_x\varphi_x)\big]\,\mathrm{d}x \wedge \mathrm{d}y$$

Terms outside of total derivatives cancel since $\frac{\delta\omega}{\delta u} = 0$. This allows us to easily find

$$G(\ell_\omega^u \circ \varphi))$$

:

$$G(\ell_\omega^u \circ \varphi)) = -e^{2u}\varphi dy + u_x\varphi_x dx$$

Note we also could have used the formula for operator $G(\ell^u_\omega \circ \varphi))$, written down in equation (23). We now take $\varphi = u$ and integrate it in the manner of equation (24) to find

$$\begin{aligned}\eta &= \int_0^1 (G(\ell^u_\omega \circ u))[\lambda u] \frac{\mathrm{d}\lambda}{\lambda} \\ &= \int_0^1 (-e^{2u} u dy + u_x^2 dx)[\lambda u] \frac{\mathrm{d}\lambda}{\lambda} \\ &= \int_0^1 (-e^{2u\lambda} u dy + \lambda u_x^2 dx)\, \mathrm{d}\lambda \\ &= -\frac{1}{2} e^{2u} + \frac{1}{2} dy + \frac{1}{2} u_x^2 dx\end{aligned}$$

We can now see that indeed, $\bar{d}\eta = \omega$.

3.4. **The Poincaré lemma on variational multivectors.**

Theorem 3.13. *Let P be a variational multivector such that $\forall p_1, ..., p_k \in \widehat{\varkappa(\pi)}$ $P(p^1, ..., p^k) \equiv 0$. Then $P \equiv 0$.*

Proof. Since the variational multivectors are isomorphic to $[\Delta_P]$, it suffices to show that for all $\Delta_P \in \mathcal{C}\mathrm{Diff}_k(\varkappa(\pi), \bar{\Lambda}^n(\pi))$, $\Delta_P \equiv 0$ whenever all its values $\Delta_P(p_1, ..., p_k)$ are trivial.

Two different jet bundles will be considered in this proof. Therefore, to avoid confusion, the map $j_\infty : \Gamma(\xi) \to J^\infty(\xi)$ will be denoted as $j_\infty[\xi]$. By remark 2.16 we state that the map

$$\begin{aligned}j_\infty[\pi]^* : \mathcal{C}\mathrm{Diff}_k(\widehat{\varkappa(\pi)}, \bar{\Lambda}^p(\pi)) &\to \mathrm{Diff}(\pi, \mathrm{Diff}^{lin}(\hat{\pi}, ..., \mathrm{Diff}^{lin}(\hat{\pi}, \bar{\Lambda}^p(M))...) \\ &\subset \mathrm{Diff}(\pi \oplus \hat{\pi}^k, \bar{\Lambda}^p(M))\end{aligned}$$

exists. By Lemma 2.5 it is an isomorphism. Therefore $j_\infty[\pi]^*$ restricted to $\mathcal{C}\mathrm{Diff}_k(\widehat{\varkappa(\pi)}, \bar{\Lambda}^p(\pi))$ defines an injective homomorphism to $\mathrm{Diff}(\pi \oplus \hat{\pi}^k, \bar{\Lambda}^p(M))$, with a left inverse that we will denote by $j_\infty[\pi]^{*-1}$. The map

$$j_\infty[\pi \oplus \hat{\pi}^k]^{*-1} : \mathrm{Diff}(\pi \oplus \hat{\pi}^k, \bar{\Lambda}^p(M)) \to \bar{\Lambda}^p(\pi \oplus \hat{\pi}^k))$$

establishes an isomorphism. We denote by

$$\omega_P := j_\infty[\pi \oplus \hat{\pi}^k]^{*-1}(j_\infty[\pi]^*(P)) \in \bar{\Lambda}^p(\pi \oplus \hat{\pi}^k))$$

the volume form corresponding to P. We have for any $\hat{s}_1, ..., \hat{s}_k \in \Gamma(\hat{\pi})$

$$\frac{\delta}{\delta u} \omega_P(\hat{s}_1, ..., \hat{s}_k) = \frac{\delta}{\delta u} \Delta_P(\hat{s}_1, ..., \hat{s}_k) = 0.$$

Furthermore, locally we have

$$\frac{\delta}{\delta u}(\Delta_P(\hat{s}_1, ..., \hat{s}_{j-1}, u, \hat{s}_{j+1}, ..., \hat{s}_k))(x, s)$$
$$= \frac{\delta}{\delta u}(\omega_P)(x, s(x), \hat{s}_1(x), ..., s(x), ..., \hat{s}_k(x)) + \frac{\delta}{\delta p_j}(\omega_P)(x, s(x), \hat{s}_1, ..., \hat{s}_j, ..., \hat{s}_k)$$
$$= \frac{\delta}{\delta p_j}(\omega_P)(x, s(x), \hat{s}_1(x), ..., \hat{s}_k(x)) = 0$$

Therefore, by theorem (3.9), we have

$$\overline{dh}(\omega_P) = \omega_P.$$

Note that by the linearity of $\bar{h}$, $\bar{h}(\omega)$ is multilinear and skew symmetric in $\hat{\pi}^k$. Therefore

$$j_\infty[\pi \oplus \hat{\pi}^k]^*(\bar{h}(\omega)) \in \mathrm{Diff}(\pi, \mathrm{Diff}^{lin}(\hat{\pi}, ..., \mathrm{Diff}^{lin}(\hat{\pi}, \bar{\Lambda}^{n-1}(M))...).$$

Therefore the map

$$\bar{h}(\Delta_P) \coloneqq j_\infty[\pi]^{*-1}(j_\infty[\pi \oplus \hat{\pi}^k]^*(\bar{h}(j_\infty[\pi \oplus \hat{\pi}^k]^{*-1}(j_\infty[\pi]^*(\Delta_P)))))$$

has the property

$$\overline{dh}(\Delta_P) = \Delta_P$$

Whenever all values of P are trivial. This completes the proof □

We will now show an example in analogue to our proof.

Example 3.14. Let $\Delta(p(x,u)) = (u_x p(x,u) + u p_x)(x,u)\,\mathrm{d}x \in \mathcal{C}\mathrm{Diff}_1(\widehat{\varkappa(\pi)}, \bar{\Lambda}^p(\pi))$. This corresponds to the differential form ω with the same formula

$$\omega = (u_x p(x) + u p_x(x))\,\mathrm{d}x$$

Clearly $\frac{\delta}{\delta u}\omega = \frac{\delta}{\delta u}\Delta(p(x)) = 0$. Furthermore,

$$\frac{\delta}{\delta p}\omega = \frac{\delta}{\delta u}\Delta(u) = \frac{\delta}{\delta u}(u_x u + u u_x) = 0$$

Since ω is a closed differential form we can apply the horizontal homotopy operator to ω, $\bar{h}(\omega) = \int_0^1 G(\ell_\omega^{(u,p)} \circ (u,p))\frac{\mathrm{d}\lambda}{\lambda}$.

$$G(\ell_u^{(u,p)} \circ \omega) = G(2u_x p(x) + 2u p_x(x) + 2up\frac{\mathrm{d}}{\mathrm{d}x}) = 2up$$

We calculate $\bar{h}(\omega) = \int_0^1 2up\lambda\,\mathrm{d}\lambda = up$. This gives our final result

$$\bar{d}(up) = u_x p + u p_x = \omega = \Delta.$$

The theorems (2.30) and (2.27) follow as a corollary of this proof. We assume $\Delta_P \equiv 0$. Then the corresponding multivector $A \in \mathcal{C}\mathrm{Diff}_{k-1}(\widehat{\varkappa(\pi)}, \varkappa(\pi))$ is equal to 0. Theorem (2.27) follows directly from the one-to-one relation stated in lemma (2.29). This completes our proof of the main theorem.

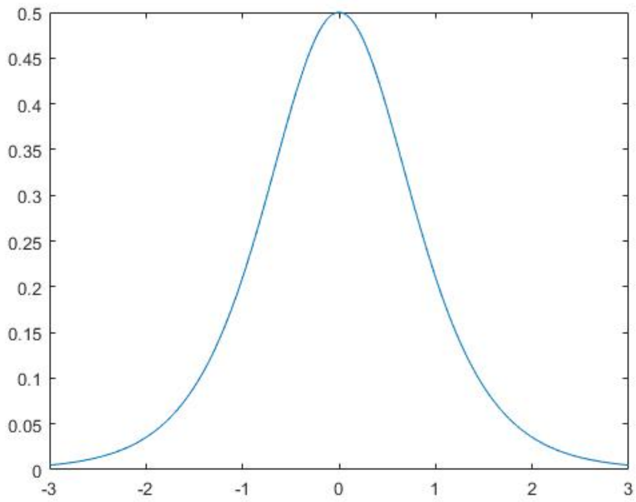

FIGURE 1. A single soliton

4. SOLITONS, IST AND THE KDV EQUATION

In this section we will consider non-linear equations. Specifically we will analyse the Korteweg de Vries equation (KdV)using the inverse scattering transform (IST), as is described in [8].

4.1. **The KdV equation.** The KdV equation is given by

$$u_t + u_{xxx} + 6uu_x = 0. \tag{26}$$

The KdV equation was derived by Korteweg and de Vries (1895). It governs the evolution of long unidirectional water waves with small elevation, collisionless plasma magnetohydrodynamic waves, and long waves in anharmonic crystals. It has wave solutions of the form

$$u(x, t) = 2\kappa^2 \operatorname{sech}^2(\kappa x + \omega t).$$

The graph of $u(x, 0)$ is shown in figure 1. In 1955, Fermi, Pasta and Ulam studied the motion of a one-dimensional anharmonic lattice [9]. They found that as time passes, the energy of the motion is stored in low vibrational modes for extended periods of time, i.e. the energy does not disperse to higher vibrational modes. In 1965, Kruskal and Zabusky related this behaviour to the KdV equation, which they found to have elastically interacting wave pulses [10]. That is, the profile and velocity of the wave pulses before the interaction are identical before and after the interaction. The only difference is a phase difference. An example of this interaction is shown in figure 2. A solution u(x,t) with 2 wave pulses is graphed at four different times t. It can be clearly seen that the wave pulses do not just pass through each other. They termed these wave pulses "solitary-wave pulses" or solitons. In

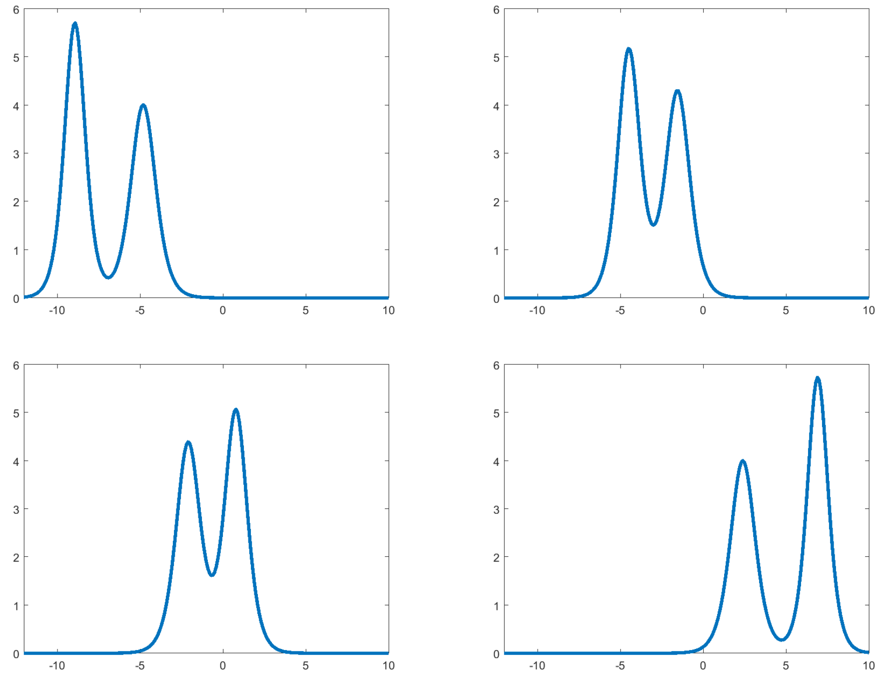

FIGURE 2. Two interacting solitons

1967, Miura, Kruskal, Gardner and Greene developed a method for solving the KdV equation [11], the inverse scattering transform. In 1968, Miura, Kruskal and Gardner proved that the KdV equation has an infinite amount of conserved quantities [12]. The inverse scattering transform can be used to solve a large number of non-linear partial differential equations. Its function is analogous to the Fourier transform for linear partial differential equations. Many equations that can be solved by the inverse scattering transform have soliton solutions and an infinite amount of conserved values.

To get an intuitive grasp of the differential equation involved we will first relate the KdV equation to the anharmonic lattice used by Fermi, Pasta and Ulam.

4.2. **A derivation of the KdV equation.** We will consider a one dimensional anharmonic lattice consisting of identical particles, connected by non-linear springs. The force law will be given by $F(q) = -K(q + \alpha q^2)$. We will use a nearest neighbour approximation. For each particle labelled by $i \in \mathbb{Z}$ with displacement $q_i(t)$ from a state with equidistant particles, we have the equation

$$\begin{aligned}\frac{m}{K} q_{i;tt} &= q_{i-1} - 2q_i + q_{i+1} + \alpha((q_{i+1} - q_i)^2 - (q_{i-1} - q_i)^2) \\ &= q_{i-1} - 2q_i + q_{i+1} + \alpha(q_{i+1} - q_{i-1})(q_{i+1} + q_{i-1} - 2q_i). \qquad (27)\end{aligned}$$

Calling the typical distance between particles h, we derive the KdV equation by using a continuum model $q_i \to q(ih)$. Specifically, we use the Taylor expansion

$$q(x+h) = q(x) + hq_x(x) + \frac{1}{2}h^2 q_{xx}(x) + \frac{h^3}{6}q_{xxx}(x) + \frac{h^4}{24}q_{xxxx}(x) + O(h^5)$$

and replace $q_{i-1}(t)$ in equation (27) by $q((i-1)h, t)$ and q_{i+1} by $q((i+1)h, t)$ to obtain the partial differential equation

$$\begin{aligned}\frac{m}{K}q_{tt} &= h^2 q_{xx} + \frac{h^4}{12}q_{xxxx} + O(h^6) + \alpha(2hq_x + O(h^3))(h^2 q_{xx} + O(h^4)) \\ &= h^2 q_{xx} + \frac{h^4}{12}q_{xxxx} + 2\alpha h^3 q_x q_{xx} + O(\alpha h^5, h^6).\end{aligned}$$

We now use the coordinate transformation $t \to ht\sqrt{\frac{m}{K}}$ to obtain

$$q_{tt} = q_{xx} + \frac{h^2}{12}q_{xxxx} + 2\alpha h q_x q_{xx} + O(\alpha h^3, h^4).$$

We then use another change of coordinates, $T = \alpha h t$, $X = x - t$, to obtain

$$0 = q_{XT} + \frac{h}{24\alpha}q_{XXXX} + q_X q_{XX} + O(h^2, \frac{h^3}{\alpha}, \alpha h). \tag{28}$$

We can then substitute $u = q_X$ into the equation to obtain

$$u_T + \frac{h}{24\alpha}u_{XXX} + uu_X + O(h^2, \frac{h^3}{\alpha}, \alpha h) = 0.$$

For $\alpha << 1$, and $h \simeq \alpha$ or $h < \alpha$, the PDE reduces to

$$u_T + \frac{h}{24\alpha}u_{XXX} + uu_X = 0,$$

which differs from the KdV equation only in the constants before the terms. By appropriately rescaling u, X and T we obtain the KdV equation, given by

$$u_T + u_{XXX} + 6uu_X = 0.$$

This shows that solutions of the KdV equation can be used to study the motion of this one-dimensional anharmonic lattice.

4.3. **Lagrangian and Hamiltonian structures.** From this point on we will only use the coordinates previously denoted by X and T, and denote them by x and t. As an application of the Poincaré lemma we can find a Lagrangian of the KdV equation by using equation (10). The KdV-equation itself is not an Euler-Lagrange equation (ELE). However, if q_x satisfies the KdV equation, the equation for q is an ELE, given by $q_{xt} + q_{xxxx} + 6q_x q_{xx} = 0$. Note that q describes the transmission of waves through the anharmonic lattice, see equation (28). For the Lagrangian we find

$$\begin{aligned}\mathcal{L} &= h_C(F\,\mathrm{d}q)\\ &= \int_0^1 q(\lambda q_{tx} + \lambda q_{xxxx} + \lambda^2 6 q_x q_{xx})\,\mathrm{d}\lambda\\ &= \frac{1}{2} q q_{tx} + \frac{1}{2}\lambda q q_{xxxx} + 2 q q_x q_{xx}.\end{aligned}$$

The KdV equation is also a Hamiltonian equation, with Poisson bracket

$$\{P, Q\}_{\frac{\mathrm{d}}{\mathrm{d}x}} := \frac{\delta P}{\delta u}\frac{\mathrm{d}}{\mathrm{d}x}\frac{\delta Q}{\delta u}$$

and Hamiltonian $H_1 = \int(-u^3 + \frac{1}{2}u_x^2)\,\mathrm{d}x$. The KdV equation is given by

$$u_t = \{u, H_1\}_{\frac{\mathrm{d}}{\mathrm{d}x}}$$

5. Scattering transform of the Schrödinger eigenvalue Equation

The inverse scattering transform is a non-linear analogue of the Fourier transform. It allows us to solve Initial Value Problems (IVP's) of certain non-linear equations. It does this by transforming the initial data to *scattering data* which evolve following uncomplicated differential equations. This transformation is called direct scattering. Given the scattering data at any time t, we can then use the inverse scattering transform to obtain the solution at time t. The inverse scattering transform was first developed for the KdV-equation by C. S. Gardner, J. M. Greene, M. D. Kruskal and R. M. Miu [11].

5.1. **The Schrodinger scattering problem.** We will first consider the scattering problem for the KdV-equation. The full derivation with proofs was originally done by Faddeev [13] in 1964 and corrected by Deift and Trubowitz [14] in 1979. Any proof that is left out in this text can be found in these articles. To solve the KdV equation, it is useful to consider a related equation: the one-dimensional time-independent Schrödinger eigenvalue equation with a potential $q(x,t)$. We will require a function $v(x,t)$ to satisfy this equation, i.e.

$$v_{xx} + (\lambda + q)v = 0. \tag{29}$$

$\lambda \in \mathbb{R}$ is an eigenvalue. We also require v to satisfy the time dependence equation

$$v_t = Av + Bv_x. \tag{30}$$

These two equations are related to the KdV equation by the compatibility condition $v_{txx} = v_{xxt}$. This gives us the compatibility conditions

$$\begin{aligned} q_t &= 2B_x(\lambda + q) + Bq_x - A_{xx}\\ 2A_x + B_{xx} &= 0.\end{aligned}$$

We can choose $B = 4\lambda - 2q$, $A = q_x$ to obtain the KdV equation:

$$q_t = -q_{xxx} - 6qq_x$$

Different choices for A and B give different equations. We, however, are only interested in the KdV equation. To summarise: for the system given by equations (29) and (30) to be compatible for our choice of A and B, we require q to satisfy the KdV equation.

We will assume that q has a finite L^2-norm, and $\lim_{x\to\infty} q_x = 0$. We define $\phi, \psi.\bar{\psi}$ as specific solutions of equation (29). We do this by setting their boundary conditions to

$$\begin{aligned}
\lim_{x\to-\infty} \phi(x, \sqrt{\lambda}, t)e^{-i\sqrt{\lambda}x} &= 1 \\
\lim_{x\to+\infty} \psi(x, \sqrt{\lambda}, t)e^{+i\sqrt{\lambda}x} &= 1 \\
\lim_{x\to+\infty} \bar{\psi}(x, \sqrt{\lambda}, t)e^{-i\sqrt{\lambda}x} &= 1.
\end{aligned}$$

The existence of these functions is shown by Deift and Trubowitz in [14]. We will, for now, ignore the time dependence of these functions and consider them at a fixed time t. We can separate three different cases: $\lambda = k^2 > 0$, $\lambda = 0$ and $\lambda = -\kappa^2 < 0$. For now we will ignore the case $\lambda = 0$. For $\lambda = k^2$ we find the solutions satisfying

$$\begin{aligned}
\lim_{x\to-\infty} \phi(x, k)e^{-ikx} &= 1 \\
\lim_{x\to+\infty} \psi(x, k)e^{+ikx} &= 1 \\
\lim_{x\to+\infty} \bar{\psi}(x, k)e^{-ikx} &= 1
\end{aligned} \tag{31}$$

For $\lambda = -\kappa^2$ we find the solutions satisfying

$$\begin{aligned}
\lim_{x\to-\infty} \phi(x, i\kappa)e^{+\kappa x} &= 1 \\
\lim_{x\to+\infty} \psi(x, i\kappa)e^{-\kappa x} &= 1 \\
\lim_{x\to+\infty} \bar{\psi}(x, i\kappa)e^{+\kappa x} &= 1.
\end{aligned}$$

The space of solutions of equation (29) is two-dimensional, and $\psi, \bar{\psi}$ are linearly independent. Therefore we can write

$$\phi(x, k) = a(k)\bar{\psi}(x, k) + b(k)\psi(x, k). \tag{32}$$

The functions a and b give us information about the potential q, which is called the scattering data. To clarify this term, we can rewrite the equation by dividing by a:

$$\frac{1}{a}\phi = \bar{\psi} + \frac{b}{a}\psi.$$

In this equation $\bar{\psi}$ can be interpreted as an incoming polarised wave of amplitude 1 from $+\infty$. This wave is then scattered by the potential q

into $\frac{1}{a}\phi$, i.e. the transmitted wave going towards $-\infty$, and $\frac{b}{a}\psi$, i.e. the reflected wave going towards $+\infty$. For this reason we will call $\tau := \frac{1}{a}$ the transmission coefficient and $\rho := \frac{b}{a}$ the reflection coefficient. Using these coefficients, the equation reads as

$$\tau\phi = \bar{\psi} + \rho\psi. \tag{33}$$

We will define bound states of the time-independent Schrödinger as functions with a finite L^2-norm. For $\lambda = k^2$ we have $\lim_{x\to\infty} |\phi(x,k)| = 1$. All solutions of this type therefore have an infinite L^2 norm. For $\lambda = -\kappa^2$ the solution ϕ can have a finite L^2-norm, since $\int_{-\infty}^{0} \phi(x,k)^2 \,\mathrm{d}x$ converges. This however also requires that $\int_0^{+\infty} \phi(x,k)^2 \,\mathrm{d}x$ converges. For this to be the case we must have $\phi(x,i\kappa) = b(i\kappa)\psi(x,i\kappa)$, since $\bar{\psi}(x,i\kappa)$ diverges whenever $x \to \infty$. This implies that the zeroes of $a(i\kappa)$ give the bound states of the Schrödinger eigenvalue equation. For the Schrödinger eigenvalue equation, all bound states have eigenvalues $i\kappa$ located on the imaginary axis, and there is a finite amount of eigenvalues. We label the bound states by $i\kappa_n$. For reasons concerning residues in complex integration we will define the constant $C_n = b(i\kappa_n)/a'(i\kappa_n)$. $S(q) = \{\{(\kappa_n, C_n)\}, \rho(k)\}$ is called the scattering data of q. It turns out that, using the scattering data of q, q can be determined. This process is called inverse scattering. Furthermore, the scattering data evolves in an uncomplicated manner.

5.2. **Inverse scattering for the Schrödinger scattering problem.** In this section we will derive the equations of inverse scattering from equation (33). This was first done by Gel'fand and Levitan in [15]. We can rewrite $\bar{\psi}$ and ψ as

$$\begin{aligned} \bar{\psi}(x) &= e^{+ikx} + \int_x^\infty \bar{K}(x,z)e^{+ikz}dz \\ \psi(x) &= e^{-ikx} + \int_x^\infty K(x,z)e^{-ikz}dz. \end{aligned} \tag{34}$$

$K(x,z)$ determines the difference between $\psi(x)$ and e^{-ikx}. As we will see in the next section, $K(x,z)$ is independent of the eigenvalue. If we substitute equation 34 for ψ into the Schrödinger equation, we obtain

$$\begin{aligned} &\int_x^\infty [e^{-iks}(\partial_x^2 - \partial_s^2 + q(x))K(x,s)]ds + \lim_{s\to\infty} e^{-iks}\big[\partial_x K(x,s) + \partial_s K(x,s)\big] \\ &- \partial_x \lim_{s\to\infty} K(x,s)e^{-iks} - e^{-ikx}\big[2\partial_x K(x,s) + 2\partial_s K(x,s) - q(x)\big]|_x = 0. \end{aligned} \tag{35}$$

It is sufficient for $K(x,s)$ to satisfy

$$\begin{aligned}(\partial_x^2 - \partial_s^2 + q(x))K(x,s) &= 0 \\ \lim_{s\to\infty} \partial_x K(x,s) + \partial_s K(x,s) &= 0 \\ \lim_{s\to\infty} K(x,s)e^{-iks} &= 0 \\ 2\partial_x K(x,x) - q(x) &= 0. \end{aligned} \tag{36}$$

Since $K(x,s)$ is unique, it necessarily satisfies these equations. This gives us an easy way to calculate q, given K. We will next show that $K(x,s)$ can be determined from the scattering data.

Since the boundary condition of $\bar{\psi}$ is conjugate to ψ and equation (29) is real, we have the relation $\bar{\psi}^* = \psi$. This can be seen by noticing that $\bar{\psi} + \psi$ is a real solution. This directly implies that $K = \bar{K}^*$. We can substitute this into equation (33) to obtain

$$\tau\phi = e^{+ikx} + \int_x^\infty K^*(x,z)e^{+ikz}dz + \rho e^{-ikx} + \int_x^\infty \rho K(x,z)e^{-ikz}dz.$$

We can further manipulate this equation by applying the integral transform $\int_c \mathrm{d}k e^{-iky}$, where the contour c lies in the upper half of complex plane, starts at $-\infty$, ends at $+\infty$ and passes over all zeroes of a. Furthermore, we choose $y > x$:

$$\begin{aligned}\int_c e^{-iky}\tau\phi\,\mathrm{d}k &= \int_c e^{ik(x-y)}\,\mathrm{d}k + \int_c\int_x^\infty K^*(x,z)e^{ik(z-y)}dz\,\mathrm{d}k \\ &\quad + \int_c \rho e^{-ik(x+y)}\,\mathrm{d}k + \int_c\int_x^\infty \rho K(x,z)e^{-ik(z+y)}dz\,\mathrm{d}k \\ &= 0 + K^*(x,y) + \int_c \rho e^{-ik(x+y)}\,\mathrm{d}k + \int_c\int_x^\infty \rho K(x,z)e^{-ik(z+y)}dz\,\mathrm{d}k. \end{aligned} \tag{37}$$

By using the residue theorem we find that

$$\int_c e^{-iky}\tau\phi\,\mathrm{d}k = i\sum_{n=1}^N C_n e^{-\kappa_n x}.$$

If we define the function

$$F(x) \coloneqq \frac{1}{2\pi}\int_{-\infty}^\infty \rho e^{-ikx}\,\mathrm{d}k - i\sum_{n=1}^N C_n e^{-\kappa_n x},$$

equation (37) reduces to

$$K(x,y) + F(x+y) + \int_x^\infty K(x,z)F(y+z)\,\mathrm{d}z = 0. \tag{38}$$

Thus, since we can calculate F from the scattering data, we can determine K from this integral equation. Using equation (36) we can calculate $q(x)$, solving the inverse scattering problem.

5.3. **Time Dependence.** To investigate the time dependent scattering data $S(q(t))$, we investigate the time dependence of $\phi(x,k,t)$, $\psi(x,k,t)$ and $\bar{\psi}(x,k,t)$ at the boundaries corresponding to their definition.

We then define dependent functions $\phi^{(t)}(x,k)$, $\psi^{(t)}(x,k)$, $\bar{\psi}^{(t)}(x,k)$ as the functions that satisfy the Schrödinger eigenvalue equation (29) and the time dependence equation (30), as well as the initial value conditions

$$\begin{aligned}\phi^{(0)}(x,k) &= \phi(x,k)\\ \psi^{(0)}(x,k) &= \psi(x,k)\\ \bar{\psi}^{(0)}(x,k) &= \bar{\psi}(x,k).\end{aligned}$$

We define the functions ϕ, ψ and $\bar{\psi}$ by setting

$$\begin{aligned}\phi(x,k,t) &:= \phi^{(t)}(x,k)e^{-4ik^3t}\\ \psi(x,k,t) &:= \psi^{(t)}(x,k)e^{4ik^3t}\\ \bar{\psi}(x,k,t) &:= \bar{\psi}^{(t)}(x,k)e^{-4ik^3t}.\end{aligned}$$

The time evolution of $\phi(x,k,t)$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi(x,k,t) = \frac{\mathrm{d}}{\mathrm{d}t}\phi^{(t)}e^{-i4k^3t} = (q_x - i4k^3)\phi + (4k^2 - q)\phi_x. \tag{39}$$

We have

$$\lim_{x\to-\infty}\phi_x - ike^{ikx} = 0.$$

This implies that

$$\lim_{x\to-\infty}\frac{\mathrm{d}}{\mathrm{d}t}\phi^{(t)}e^{-i4k^3t} = \lim_{x\to\infty}(4ik^3 - 4ik^3)e^{-ikx} = 0.$$

Therefore $\phi(x,k,t)$ satisfies the boundary condition (31) at all times. Similarly, $\bar{\psi}(x,k,t) := \phi^{(t)}e^{-i4k^3t}$ and $\psi(x,k,t) := \psi^{(t)}e^{i4k^3t}$ satisfy their corresponding boundary conditions at all times.

Since the Schrödinger eigenvalue equation is linear, we have

$$\phi^{(t)}(x,k) = a(k,0)\bar{\phi}^{(t)}(x,k) + b(k,0)\psi^{(t)}(x,k)$$

Therefore

$$\phi(x,k,t) = a(k,0)e^{-i4k^3t}\bar{\phi}(x,k,t) + b(k,0)e^{-i8k^3t}\psi(x,k,t)$$

We obtain

$$a(k,t) = a(k,0) \qquad\qquad b(k,t) = b(k,0)e^{-i8k^3} \tag{40}$$

For the reflection coefficient we find

$$\rho(k,t) = \rho(k,0)e^{-8ik^3t}.$$

We obtain the results for negative λ by substituting $i\kappa$ for k. Doing this we obtain

$$C_n(t) = \frac{b(i\kappa_n,t)}{a(i\kappa_n,t)} = C_ne^{-8\kappa^3t}.$$

In conclusion, the scattering data is given by

$$S(q(t)) = \{\{(\kappa_n, C_n e^{-8\kappa^3 t})\}, \rho(k)e^{-8ik^3 t}\}.$$

5.4. **Soliton solutions.** We will now consider solutions of the KdV equation for a given initial scattering data. Specifically, we will solve the case in which we have N eigenvalues. However, we will first solve the equation for $N = 1$. We will assume a reflectionless potential by setting $\rho(k) = 0$ and we have 1 eigenvalue $i\kappa$, with a corresponding constant $-iC = c^2$. We will assume $-iC$ to be positive. We will later see that this is required for obtaining a real solution. We then have

$$F(x) = c^2 e^{-\kappa x}.$$

Substituting this into equation (38) we obtain

$$K(x, y) + c^2 e^{-\kappa y}(e^{-\kappa x} + \int_x^\infty K(x, z)e^{-\kappa z}\, \mathrm{d}z) = 0. \tag{41}$$

We make the ansatz $K(x, y) = ce^{-\kappa y}\psi(x)$. This yields an equation for $\psi(x)$:

$$\psi(x) + ce^{-\kappa x} + c^2\psi(x) \int_x^\infty e^{-2\kappa z}\, \mathrm{d}z = 0.$$

This equation can be easily solved

$$\psi(x) = -\frac{ce^{-\kappa x}}{1 + \frac{c^2}{2\kappa}e^{-2\kappa x}}.$$

Therefore, we can calculate $K(x, x)$ to be

$$K(x, x) = -c^2 \frac{e^{-\kappa x}}{1 + \frac{c^2}{2\kappa}e^{-2\kappa x}}.$$

The time independent solution of the KdV equation is

$$\begin{aligned} q(x) &= 2\frac{\mathrm{d}}{\mathrm{d}x}K(x, x) \\ &= 2\kappa^2 \operatorname{sech}^2(\kappa x + \frac{1}{2}\log\frac{c^2}{2k}). \end{aligned}$$

We have $c^2(t) = c_0^2 e^{-8\kappa^3 t}$. If we define $x_0 = -\frac{1}{2\kappa}\log\frac{c_0^2}{2k}$, we obtain the time dependent solution

$$q(x, t) = 2\kappa^2 \operatorname{sech}^2(\kappa(x - 4\kappa^2 t - x_0)). \tag{42}$$

Thus a solution with one eigenvalue κ corresponds to a soliton of height $2\kappa^2$, a velocity of $4\kappa^2$ and a characteristic width of $\frac{1}{\kappa}$. This shows that the solitons of the KdV equation have a very specific wave profile. The graph of a single soliton with eigenvalue $\kappa = 1$ is shown in figure 1.

We can also explicitly calculate the solution of the KdV equation for N eigenvalues κ_n. To do this, we need to solve equation (38) with $F(x) = \sum_{n=1}^{N} \kappa_n e^{-\kappa_n x}$. We obtain the equation

$$K(x,y) + \sum_{n=1}^{N} c_n^2 e^{-\kappa_n(x+y)} + \sum_{n=1}^{N} c_n^2 e^{-\kappa_n y} \int_x^\infty K(x,z) e^{-\kappa_n z} \, \mathrm{d}z = 0.$$

We make the ansatz

$$K(x,y) = -\sum_{n=1}^{N} c_n e^{-\kappa_n y} \psi_n(x).$$

Substituting this into the equation for K yields

$$-\sum_{n=1}^{N} c_n e^{-\kappa_n y} \left(\psi_n(x) - c_n e^{-\kappa_n x} + \sum_{m=1}^{N} c_m c_n \psi_m(x) \int_x^\infty e^{-(\kappa_m+\kappa_n)z} \, \mathrm{d}z \right) = 0.$$

Calculating the integral and take the coefficients of $c_n e^{-\kappa_n y}$ to obtain

$$\psi_n(x) + \sum_{m=1}^{N} c_m c_n \psi_m(x) \frac{e^{-(\kappa_m+\kappa_n)x}}{(\kappa_m + \kappa_n)} = c_n e^{-\kappa_n x}.$$

We define the matrix C by setting its coefficients to

$$C_{mn} = c_m c_n \frac{e^{-(\kappa_m+\kappa_n)x}}{(\kappa_m + \kappa_n)},$$

so that we can write

$$(I + C)\psi = \phi,$$

where $\phi(x) = (c_1 e^{-\kappa_1 x}, ..., c_N e^{-\kappa_N x})$ and $\psi = (\psi_1, ..., \psi_N)$. C can be shown to be positive definite. Let $x \in \mathbb{R}^N$ such that $x \neq 0$, then

$$\begin{aligned}
x^T C x &= \sum_{n=1}^{N} \sum_{m=1}^{N} c_m c_n \frac{e^{-(\kappa_m+\kappa_n)x}}{(\kappa_m + \kappa_n)} x_m x_n \\
&= \sum_{n=1}^{N} \sum_{m=1}^{N} c_m c_n \int_x^\infty e^{-(\kappa_m+\kappa_n)x} \, \mathrm{d}x \; x_m x_n \\
&= \int_x^\infty \sum_{m=1}^{N} c_m x_m e^{-\kappa_m} \sum_{n=1}^{N} c_n x_n e^{-\kappa_n x} \, \mathrm{d}x \\
&= \int_x^\infty (\sum_{n=1}^{N} c_n x_n e^{-\kappa_n x})^2 \, \mathrm{d}x > 0.
\end{aligned}$$

Since a sum of positive definite matrices is positive definite, $I + C$ is invertible and we can write

$$\psi(x) = (I + C)^{-1} \phi(x).$$

Thus we obtain

$$K(x,x) = -\sum_{n=1}^{N}\sum_{m=1}^{N} c_n e^{-\kappa_n x}(I+C)^{-1}_{mn} c_m e^{-\kappa_m x}.$$

With this equation we can calculate $q(x) = 2\frac{d}{\mathrm{d}x}K(x,x)$, which solves the inverse scattering problem.

We can rewrite this by noticing that we have $\frac{d(I+C)_{nm}}{dx} = \frac{dC_{mn}}{dx} = -c_m c_n e^{-(\kappa_m+\kappa_n)x}$. These are the same coefficients that we find in our formula for $K(x,x)$. Writing $I+C=A$, we have

$$\begin{aligned} K(x,x) &= \sum_{n=1}^{N}\sum_{m=1}^{N}(A)^{-1}_{mn}\frac{d}{\mathrm{d}x}A_{mn} \\ &= tr((\frac{d}{\mathrm{d}x}A)A^{-1}) \\ &= \frac{d}{\mathrm{d}x}tr(\log(A)) \\ &= \frac{d}{\mathrm{d}x}\log(\det(A)). \end{aligned}$$

Where we note that the logarithm of a positive definite matrix exists and is real. Thus we can write down

$$q(x) = 2\frac{d^2}{\mathrm{d}x^2}\log(\det(A)) \tag{43}$$

We will now consider a 2 soliton solution. To do this we take two eigenvalues $0 > \kappa_1 > \kappa_2$, and a reflectionless potential. We can use this to directly calculate $\det(A)$:

$$\begin{aligned} \det(A) &= \begin{vmatrix} 1 + c_1^2\frac{e^{-2\kappa_1 x}}{2\kappa_1} & c_1c_2\frac{e^{-(\kappa_1+\kappa_2)x}}{\kappa_1+\kappa_2} \\ c_1c_2\frac{e^{-(\kappa_1+\kappa_2)x}}{\kappa_1+\kappa_2} & 1 + c_2^2\frac{e^{-2\kappa_2 x}}{2\kappa_2} \end{vmatrix} \\ &= 1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_1+\eta_2+A_{12}}. \end{aligned}$$

Where we define $\eta_n = -2\kappa_n(x - 4\kappa_n^2 t - x_{n0})$, with $x_{n0} = \frac{1}{2\kappa_n}\log(\frac{c_{n,0}^2}{2\kappa_n})$ and $e^{A_{12}} = \left(\frac{\kappa_1-\kappa_2}{\kappa_1+\kappa_2}\right)^2$.

To see how the 2 solitons affect each other, we study the solution as $t \to \pm\infty$. We will move along a trajectory such that η_1 is constant. For $t \to \infty$ we see that $\eta_2 \to -\infty$. Thus we have

$$\det(A) \simeq 1 + e^{\eta_1}.$$

Substituting this into equation (43) gives us

$$q_\infty(x,t) = 2\kappa_1^2\operatorname{sech}^2(\kappa_1(x - 4\kappa_1^2 t - x_{10})),$$

which we recognise as a single soliton solution. For $t \to -\infty$ we see that $\eta_2 \to \infty$ and so

$$\det(A) \simeq e^{\eta_2}(1 + e^{\eta_1+A_{12}}).$$

This gives the solution

$$q_{-\infty}(x,t) = 2\kappa_1^2 \operatorname{sech}^2(\kappa_1(x - 4\kappa_1^2 t - x_{10} - \frac{A_{12}}{2}),$$

which is just the q_∞ solution with x shifted by $\frac{A_{12}}{\kappa_1}$. We see that the only result of the interaction between the two solitons is a phase shift in the solution. In figure 2 we can see how two solitons interact. It displays $q(x)$ at different times during the interaction.

In general, as is shown in [16], for reflectionless multi-soliton solutions we obtain a similar result, where the solitons obtain a phase shift ¨between" $t = -\infty$ and $t = +\infty$ equal to the sum of each phase shift obtained when interacting individually. The interaction of solitons with continuous spectra ($\rho \neq 0$) also produces a phase shift. Furthermore, the continuous spectra dissipates as $t \to \infty$.

5.5. **Conservation laws.** From the fact that $a(k)$ is time-independent, we can find an infinite set of conserved quantities [8]. The following values are conserved for the KdV equation.

$$\begin{aligned} \mu_0 &= q, \mu_1 = 0 \\ \mu_n &= \sum_{i=0}^{n-2} \mu_i \mu_{n-2-i} + q \left(\frac{\mu_{n-1}}{q} \right)_x \qquad \forall n \geq 2. \end{aligned} \tag{44}$$

They are locally conserved, in the sense that $\frac{d}{dt} \int_{x_0}^{x_1} \mu_n(x,t)\, \mathrm{d}x = j(x_0,t) - j(x_1,t)$ for some current j. Only the even values of this sequence give nontrivial conservation laws The first values are given by

$$\begin{aligned} \mu_0 &= q \\ \mu_1 &= 0 \\ \mu_2 &= q^2 \\ \mu_3 &= qq_x \\ \mu_4 &= qq_{xx} + 2q^3 \\ \mu_5 &= qq_{xxx} + 6q_x q^2 \\ \mu_6 &= q\partial_x^4 q + 6qq_x^2 + 8q^2 q_{xx} + 5q^4. \end{aligned} \tag{45}$$

To each conserved quantity corresponds the conservation law given by

$$\partial_t \mu_n = -\partial_x(\mu_{n+2} - q_x \frac{\mu_{n+1}}{q} + \frac{\mu_n}{q}(2q^2 + q_{xx}).$$

Equivalently, we have the following conserved currents

$$\omega_n = \mu_n dx - (\mu_{n+2} - q_x \frac{\mu_{n+1}}{q} + \frac{\mu_n}{q}(2q^2 + q_{xx}))dt. \tag{46}$$

The first nontrivial conserved currents are given by

$$\begin{aligned}\omega_0 &= (q)dx - (3q^2 + q_{xx})dt \\ \omega_2 &= (q^2)dx - (2qq_{xx} - q_x^2 + 4q^3)dt \\ \omega_4 &= (qq_{xx} + 2q^3)dx - (q\partial_x^4 q + 12q^2 q_{xx} + 9q^4 - q_x q_{xxx} + q_{xx}^2)dt. \quad (47)\end{aligned}$$

In general, we have the relation $\mathrm{d}\omega_i = (\nabla F)dt \wedge dx$ for some total differential operator $\nabla = \sum_i f_i([q])\frac{\mathrm{d}^{n_i}}{\mathrm{d}x^{n_i}}$. The generating section is determined by $\phi_i = \nabla^\dagger(1)$. More specifically, due to the structure of the conserved currents, we have

$$\frac{d\mu_i}{dt} = \nabla q_t.$$

Using this, we obtain

$$\begin{aligned}\phi_0 &= 1 \\ \phi_2 &= 2q \\ \phi_4 &= 2q_{xx} + 6q^2 \\ \phi_6 &= 2\partial_x^4 q + 20qq_{xx} + 10q_x^2 + 5q^4.\end{aligned}$$

When we physically interpret the conservation laws, we can do this in two different ways. Firstly we can do this for the KdV equation itself, which can describe water waves. Secondly we can interpret the conservation laws for u, with $u_x = q$, which is related to the thermal conductivity of an anharmonic lattice. Since u satisfies a Lagrangian equation we can use Noether's theorem to find symmetries of this equation, given by $\partial^u_{\phi_i}$.

The corresponding symmetries for the KdV equation are given by $\partial^q_{\phi_{i;x}}$ The first few symmetries are given by

$$\begin{aligned}S_0 &= 0 \\ S_2 &= \partial^q_{q_x} = q_x\frac{\partial}{\partial q} \simeq \frac{\partial}{\partial x} \\ S_4 &= \partial_{2q_{xxx}+12qq_x} = 2\partial^q_{-q_t} \simeq -2\frac{\partial}{\partial t} \\ S_6 &= 2\partial_x^5 q + 20qq_{xxx} + 40q_x q_{xx} + 20q_x q^3.\end{aligned}$$

The first locally conserved quantity is the function q itself, which corresponds to conservation of the area or volume of the wave. Interpreting the solution as a water wave, this, together with conservation of mass, reflects the incompressibility of the fluid. We see here that the symmetry corresponding to the first conserved quantity is the trivial symmetry. The volume of a single soliton is given by

$$\begin{aligned} V &= \int_{-\infty}^{\infty} q \, \mathrm{d}x \\ &= \int_{-\infty}^{\infty} 2\kappa_1^2 \operatorname{sech}^2(\kappa x) \, \mathrm{d}x \quad = \lim_{x\to\infty} 2\kappa \tanh(\kappa x) - \lim_{x\to-\infty} 2\kappa \tanh(\kappa x) \\ &= 4\kappa. \end{aligned}$$

The second conserved quantity can be interpreted as the gravitational energy of the solution, $U = \int \rho g q \, \mathrm{d}x \, \mathrm{d}q = \int \frac{1}{2}\rho g q^2 \, \mathrm{d}x$. The corresponding symmetry is the spatial translation $x \to x + \delta x$. We note here that this not imply that the conserved quantity corresponds to momentum. This is a property of Lagrangian equations, which the KdV equation is not. However, the conserved quantity $\frac{1}{3}\omega_0 + \omega_2$ corresponds to the notion of momentum [17]. The gravitational energy for a single soliton is given by

$$\begin{aligned} V &= \int_{-\infty}^{\infty} q^2 \, \mathrm{d}x \\ &= \frac{16}{3}\kappa^3. \end{aligned}$$

It's momentum is given by

$$\begin{aligned} V &= \int_{-\infty}^{\infty} q^2 + q \, \mathrm{d}x \\ &= \frac{16}{3}\kappa^3 + \frac{4}{3}\kappa. \end{aligned}$$

The third conserved quantity is harder to interpret. The third symmetry corresponds to the transformation $t \to t + \delta t$. The third conserved quantity of a single soliton is

$$\begin{aligned} T &= \frac{1}{2}\int_{-\infty}^{\infty} q q_{xx} + 2q^3 \, \mathrm{d}x \\ &= \int_{-\infty}^{\infty} 4\kappa_1^6 \cosh(2x) \operatorname{sech}^6(\kappa x) \, \mathrm{d}x \\ &= \frac{32}{3}\kappa^5. \end{aligned}$$

Higher order symmetries and conserved quantity are more difficult to interpret, since they contain terms with high powers of q and high order derivatives of q.

6. Conclusion

In the first part of this thesis we have looked at the geometry of the jet space and multivectors. Furthermore, we have stated the main theorem of the thesis. After that, we reproduced existing proofs of the

Poincaré lemma for the total differential of a manifold, and the Cartan and horizontal differential of the jet space. After this, we considered a proof of our main theorem. It would be interesting to investigate in under which conditions the horizontal homotopy operator of the jet space of the bundle $\pi \oplus \hat{\pi}^k$. Is the condition that the horizontal homotopy operator exists for $J^\infty(\pi)$ sufficient? Furthermore, it would be interesting to research whether this proof van be extended to the setting of non-commutative, see [18] for more information on this subject. In the second part of this thesis we have looked at the Schrödinger inverse scattering problemand showed how to solve it using the inverse scattering transform. We used this method to find specific solutions to the Korteweg de Vries equation. Furthermore, we looked at properties of more general solutions of the Korteweg de Vries equation. We also discussed the conserved quantities of the Korteweg de Vries equation.It would be interesting to further investigate the properties of the Korteweg de Vries equation in more detail. We could for example consider how a soliton interacts with a continuous spectrum.

It would also be interesting to view the inverse scattering transform in the frame of a change in coordinates of a Hamiltonian system. Specifically, a change to action angle coordinates.

Lastly, it would also be interesting to consider multiple space dimensions for the Korteweg de Vries equation. Specifically the Kadomtsev–Petviashvili equation would be interesting to study, since it is also solvable by the IST.

References

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