Strong Structural Controllability of Systems on Colored Graphs

Master Project Applied Mathematics

April 2018

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Abstract

In this thesis, strong structural controllability of networks on graphs is studied. In the study of strong structural controllability it is often assumed that the parameters appearing in the pattern matrix of the qualitative class are independent. In this thesis, we allow appearances of the same parameter in different locations, which leads to a constrained qualitative class. Networks with this underlying structure are represented by so-called colored graphs. Sufficient conditions for strong structural controllability of the systems associated with these colored graphs are given, by means of perfect matchings and balancing sets.

Keywords: Strong structural controllability, qualitative class, zero forcing, colored graph, perfect matching

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Chapter 1

Introduction

Recently, the study of networks of dynamical systems became a very popular topic in the systems and control community. A network of dynamical systems can be seen as a dynamical system itself ([1], [2]). Often, the network structure is represented using graphs. To study structural properties of the network, the topology of the associated graph is then used. Examples of networks of dynamical systems include chemical and engineering networks, power grids and robotic networks ([3], [4]).

In the study of controllability of systems defined on graphs, often linear input/state systems of the following form are considered:

$$\dot{x} = Ax + Bu, \tag{1.1}$$

where $x \in \mathbb{R}^n$ is the state and $u \in \mathbb{R}^m$ is the input. Here, the system matrix A is associated to the graph and represents the graph structure, while the input matrix B encodes the control nodes, by which we mean that we can apply external inputs through these nodes ([5]). Sometimes, these input nodes are called the leaders of the network, and (1.1) is refered to as a leader/follower system ([6]).

One line of research focuses on specific dynamics of the network. In this case, the system matrix A is a constant matrix with all entries being fixed values. The nonzero off-diagonal entries represent the weights of the edges in the graph. Examples of this are adjacency matrices and (the negative of) the Laplacian matrix ([7], [8], [9], [10]). There is, however, one downside to this approach and that is that the study of controllability is not very robust. For example, results of controllability do not take into account small deviations that might occur on the edges of the graph.

Therefore, another line of research has erected where the weights of the edges can be arbitrary. As such, the system matrix A is not a fixed matrix anymore. Sometimes this matrix A is referred to as a pattern matrix ([11]). In this case, the family of system matrices assuming a certain structure is studied in the controllability analysis. This thread is known as structural controllability ([12]). Structural controllability studies controllability of a family

of pairs (A, B), as opposed to a particular instance. Two types of structural controllability can be distinguished. Weak structural controllability asks whether there exists a controllable pair in the family, while strong structural controllability deals with the question whether all members of the family are controllable.

Weak structural controllability was first introduced by Lin, [12], for single input systems. In that paper, also a graph theoretic test is given to study weak structural controllability, by means of cacti. The results of Lin were extended to multi-input systems by Shields and Pearson, [13]. Other references on weak structural controllability can be found in [14], [15] and [16].

Strong structural controllability was first introduced by Mayeda and Yamada, [17]. Recently, in [18] a necessary and sufficient condition for strong structural controllability of the network is given by investigating its underlying graph topology. This is done by representing the network as a bipartite graph, and then by looking at the perfect matchings in this graph. Another tool in the study of strong structural controllability is by means of zero forcing, see Monshizadeh et al., [19]. In that paper again a family of matrices, called the qualitative class, is assigned to the given network. A necessary and sufficient condition on the graph topology is then given for strong structural controllability of this family. Other references for this line of research are [20], [21], [22], [23] and [24].

A basic assumption in the study of structural controllability is that the nonzero parameters appearing in the pattern system matrix A are independent. In practice this is however not always satisfied. For example, if the underlying graph of the network is an undirected graph, the system matrix A is an adjacency matrix that is always symmetric. Another example is a power grid, in which often some outflows or power supplies are equal to each other ([25]).

In a recent paper by Morse and Liu ([26]), weak structural controllability is studied with the distinguishing feature that the same parameter can appear on multiple locations in the system matrix. In this paper a necessary and sufficient condition for weak structural controllability is given based on the graph topology, by looking at so-called colored subgraphs.

The purpose of the present thesis is now as follows. Inspired by [26], we also consider families of matrices where the same parameter can appear in multiple occasions. This induces a constrained qualitative class. For this constrained qualitative class, we aim to find graph theoretic conditions for strong structural controllability.

The remainder of this thesis is now organized as follows. First, in Chapter 2 the mathematical preliminaries are explained. In Chapter 3 the problem is formulated. In Chapter 4 we will investigate and extend the results of [19]. The ideas obtained in that chapter can then be used to solve the main problem, which is done in Chapters 5 and 6. Finally, the conclusion is presented in Chapter 7.

Chapter 2

Mathematical Preliminaries

In this chapter we will introduce some preliminaries from graph theory and systems theory.

2.1 Graph theory

A *directed graph* $G = (V, E)$ is a pair of sets, the *node set* $V = \{1, 2, \ldots, n\}$ and the *edge set* E, which consists of ordered pairs $(i, j) \in V \times V$. The cardinality of the node set is denoted by $|V| = n$. An *undirected graph* is a graph whose edge set consists of unordered pairs $\{i, j\} \in E$, that is, both (i, j) and (j, i) are in E. A graph is called a *simple* graph if we do not allow self loops, i.e., $(i, i) \notin E$ for all $1 \leq i \leq n$.

Let $G = (V, E)$ be a simple directed graph. For a node $i \in V$, we say that the node j is an *out-neighbor of i* if $(i, j) \in E$. Furthermore we call i a *mother node* of j. The *neighbor set* of a node i, denoted as N_i, is defined as the set of nodes $N_i = \{j \in V \mid (i, j) \in E\}$. If we consider an undirected graph, we simply say that i and j are *neighbors* if $\{i, j\} \in E$.

Let $G = (V, E)$ be a simple directed graph. To this graph we associate the following set of matrices

$$\mathcal{W}(G) = \{W \in \mathbb{R}^{n \times n} \mid W_{i,j} \neq 0 \iff (j, i) \in E\}.$$

Any matrix $W \in \mathcal{W}(G)$ is called a *weighted adjacency matrix* of the graph G. For a given weighted adjacency matrix, the nonzero entry $W_{i,j}$ is called the *weight* of the edge (j, i). Note that the diagonal entries of W are always zero, since the graph is simple. For a given $W \in \mathcal{W}(G)$, we call $G = (V, E, W)$ a *weighted graph*.

A graph $G' = (V', E')$ is called a *subgraph* of a graph $G = (V, E)$, denoted as $G' \subseteq G$, if $V' \subseteq V$ and $E' \subseteq E$.

Consider now an undirected graph $G = (V, E)$. This graph is called a *bipartite graph* if there exists two nonempty disjoint subsets of nodes $S, T \subset V$ such that $S \cup T = V$ and for any edge $\{i, j\} \in E$ we have that $i \in S$ and $j \in T$. Sometimes we also write $E = E_{S,T}$ to emphasize that all edges are between the node sets S and T. We denote the bipartite graph by either $G = (S, T, E)$ or $G = (S, T, E_{S,T})$.

Let $G = (S, T, E)$ be a bipartite graph, and denote the node sets by $S = \{s_1, s_2, \ldots, s_p\}$ and $T = \{t_1, t_2, \ldots, t_q\}$. To this bipartite graph we can assign a set of matrices, called a *pattern matrix*, as follows

$$\mathcal{P}(G) = \{A \in \mathbb{C}^{q \times p} \mid A_{j,i} \neq 0 \iff \{s_i, t_j\} \in E\}.$$

Note that, contrary to the set of adjacency matrices, we now allow complex matrices. Let $X \subseteq S$ and $Y \subseteq T$ be two subsets of nodes. The submatrix $A_{Y,X}$ is the matrix obtained by keeping the columns with indices corresponding to nodes in X, while the row indices correspond to nodes in Y.

Let $G = (V, E)$ be an undirected graph and let $X, Y \subset V$ denote two nonempty disjoint subsets of nodes. Let $E_{X,Y}$ denote the subset of edges between X and Y, so $E_{X,Y} = \{\{i, j\} \in E \mid i \in X, j \in Y\}$. A *matching* between X and Y is a subset of edges of $E_{X,Y}$, with no common nodes. We say that a node is *matched* if it appears in one of the edges. A *d-matching* is a matching that consists of d edges. A *maximum* matching is a matching with maximum cardinality. For $|X| = |Y| = k$, any k-matching is also called a *perfect matching*. We say that X and Y are *perfect neighbors* if there exists a perfect matching between X and Y.

Let $G = (V, E)$ now be a simple directed graph. Consider any two nonempty disjoint subsets of nodes, $X, Y \subset V$. Let $E_{X,Y}$ denote the set of pairs $\{i, j\}$ such that (i, j) is an edge going from X to Y, so

$$E_{X,Y} = \{\{i, j\} \mid (i, j) \in E, i \in X, j \in Y\}.$$

Then $(X, Y, E_{X,Y})$ is a bipartite graph. For any two nonempty disjoint subsets of nodes $X, Y \subset V$, we say that $(X, Y, E_{X,Y})$ is the *associated bipartite graph*.

In the remainder of this thesis, unless stated otherwise, if we talk about a graph $G = (V, E)$, we mean that the graph is simple and directed.

2.2 Systems theory

Consider the following linear time-invariant (LTI) system:

$$\dot{x} = Ax + Bu, \tag{2.1}$$

where $x \in \mathbb{R}^n$ represents the state, $u \in \mathbb{R}^m$ is the input, and $A \in \mathbb{R}^{n \times n}$ is the system matrix and $B \in \mathbb{R}^{n \times m}$ is the input matrix. We say that the pair (A, B) is controllable if the system (2.1) is controllable. To check if the system is controllable, we can apply one of the following tests.

Theorem 2.1 (Kalman, [2], [27]). (A, B) *is controllable if and only if* $\mathcal{C} = [B, AB, ..., A^{n-1}B]$ *has full row rank.*

Theorem 2.2 (Hautus, [28]). (A, B) *is controllable if and only if for all* $\lambda \in \mathbb{C}$, $[A - \lambda I, B]$ *has full row rank.*

Theorem 2.3 (Popov-Belevich-Hautus, [29, 30, 28]). (A, B) *is controllable if and only if for all left eigenvectors* v *of* A, *i.e.,* $v \neq 0$ *and* $v^T A = \lambda v^T$ *for some* $\lambda \in \mathbb{C}$, *we have* $v^T B \neq 0$.

2.3 Systems on graphs

Consider a network of agents. The structure of a network can be captured mathematically in a simple directed graph $G = (V, E)$. Here, each agent is represented by a node, and whenever there is a link between two agents this is represented by an edge between the nodes representing these agents. To study the behavior of the network we consider again the following LTI system:

$$\dot{x} = Ax + Bu, \tag{2.1}$$

where the input matrix $B \in \mathbb{R}^{n \times m}$ corresponds to the input nodes. To this end, we write $V_L \subseteq V$ for the leader (input) nodes. The matrix $B = B(V; V_L)$ is then given by

$$B_{i,j} = \begin{cases} 1 & \text{if } i = v_j, \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

For the given graph $G = (V, E)$, we consider a class of matrices, called the *qualitative class* of G, as follows

$$\mathcal{Q}(G) = \{A \in \mathbb{R}^{n \times n} \mid \text{ for } i \neq j, \text{we have } A_{i,j} \neq 0 \iff (j, i) \in E\}.$$

Note that the off-diagonal entries of $A \in \mathcal{Q}(G)$ correspond to the edges in G: an entry is nonzero if and only if there is a corresponding edge. There are no restrictions on the diagonal entries of $A \in \mathcal{Q}(G)$, they can take any real value. The system matrix A appearing in (2.1) is now allowed to be any $A \in \mathcal{Q}(G)$. Any particular choice of such $A \in \mathcal{Q}(G)$ is called a *realization*.

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Let the graph $G = (V, E)$ and the set of input nodes V_L be given. This then determines $\mathcal{Q}(G)$ and $B = B(V; V_L)$. For a realization $A \in \mathcal{Q}(G)$, we can test if the pair (A, B) is controllable by either the Kalman test, the Hautus Lemma or the Popov-Belevitch-Hautus test. With slight abuse of notation, we say that (A, V_L) is controllable if (A, B) is controllable. We say that (G, V_L) is *weakly structurally controllable* if there exists $A \in \mathcal{Q}(G)$ such that (A, V_L) is controllable. It has been shown that if there exists such A, then (G, V_L) is controllable for almost all $A \in \mathcal{Q}(G)$ ([14]). We say that (G, V_L) is *strongly structurally controllable* if (A, V_L) is controllable for all $A \in \mathcal{Q}(G)$. In the latter case, we also say that (G, V_L) is controllable.

We will now state the following fact, taken from [19]:

Theorem 2.4. *Let $G = (V, E)$ be a graph and $V_L \subseteq V$ be the leader set. Then, (G, V_L) is controllable if and only if the matrix $[A, B]$ has full row rank for all $A \in \mathcal{Q}(G)$, where $B = B(V; V_L)$.*

2.4 Zero forcing

In minimal rank problems, the concept of zero forcing is often encountered [19]. In this section, we will explain what zero forcing is and how it can be applied to study controllability of the graph $G = (V, E)$ with leader set $V_L \subseteq V$.

To that end, consider the graph $G = (V, E)$ with leader set $V_L \subseteq V$. Every node $i \in V$ is either colored black or white. Initially, all nodes $i \in V_L$ are colored black and all nodes $i \in V \setminus V_L$ are set to be white. We consider now the following coloring rule:

Color change rule: If i is a black node and j is the only white out-neighbor of i, we color j black.

A single application of the color change rule is called a *forcing*. We also say that i forces j, and we write $i \rightarrow j$. Suppose we have a subset of black nodes C of the node set V and there exists an $i \in C$ such that i forces j for some $j \in V \setminus C$. The updated set of black nodes is now $C \cup \{j\}$. If we apply the color change rule repeatedly until no more changes can be made, we arrive at a subset of V denoted by $D^{zf}(C)$. This set is called the *derived set*. If $D^{zf}(C) = V$ we say that C is a *zero forcing set* (ZFS) of the graph G.

Zero forcing is a very powerful tool for the study of strong structural controllability. In [19] a necessary and sufficient condition for strong structural controllability is given.

Theorem 2.5. *Let $G = (V, E)$ be a graph and let $V_L \subseteq V$. Then (G, V_L) is controllable if and only if V_L is a zero forcing set.*

Consider the following example as depicted in Figure 2.1. It is a graph consisting of 5 nodes and the leader set is $V_L = \{1, 2\}$.
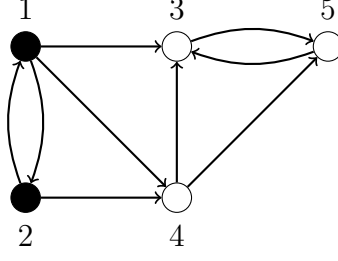
Figure 2.1: Example of zero forcing.

Any matrix $A \in \mathcal{Q}(G)$ of the qualitative class is of the following form:

$$
A = \begin{bmatrix}
? & * & 0 & 0 & 0 \\
* & ? & 0 & 0 & 0 \\
* & 0 & ? & * & * \\
* & * & 0 & ? & 0 \\
0 & 0 & * & * & ?
\end{bmatrix},
$$

where the stars $*$ denote nonzero free parameters, and the diagonal entries can take any real value. We also write:

$$
A = \begin{bmatrix}
\xi_1 & x_{12} & 0 & 0 & 0 \\
x_{21} & \xi_2 & 0 & 0 & 0 \\
x_{31} & 0 & \xi_3 & x_{34} & x_{35} \\
x_{41} & x_{42} & 0 & \xi_4 & 0 \\
0 & 0 & x_{53} & x_{54} & \xi_5
\end{bmatrix},
$$

where the x_{ij}'s are nonzero free parameters, and the ξ_i's are allowed to take any real value (including zero). Note that the variable x_{ij} correspond to edge (j, i).

We claim that the leader set, $V_L = \{1, 2\}$, is a zero forcing set. To see this, note that node 2 has only one white out-neighbor, namely 4. Hence, $2 \to 4$. After that, node 1 has only one white out-neighbor, namely node 3, so we color 3 black. Then, node 3 forces node 5, and we see that $D^{zf}(V_L) = V$. The sequence of forcings is depicted in Figure 2.2.
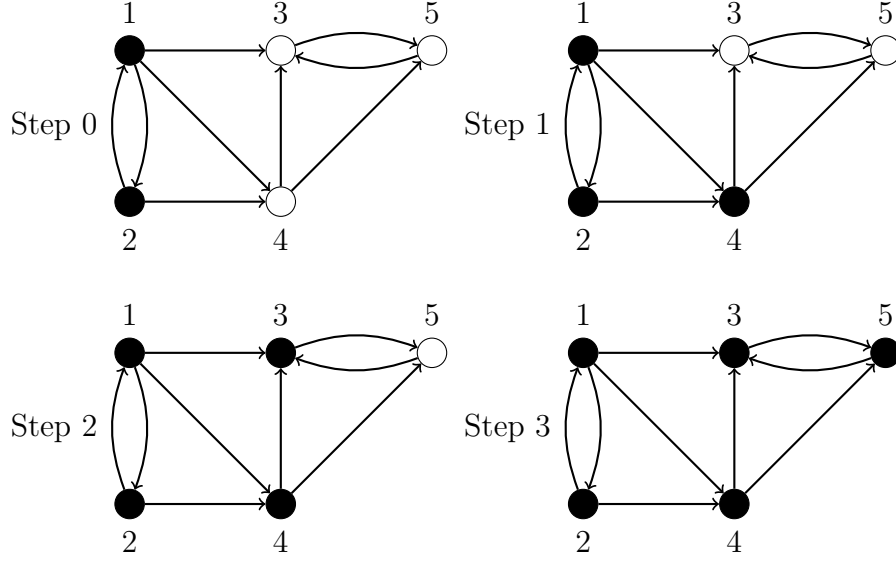
Figure 2.2: The sequence of forcings of the network depicted in Figure 2.1, with leader set $V_L = \{1, 2\}$.

Since $D(V_L) = V$, by Theorem 2.5 we have that (G, V_L) is controllable. Finally we would like to remark that this could also be concluded from Theorem 2.4, since the matrix

$$[A, \ B] = \begin{bmatrix} \xi_1 & x_{12} & 0 & 0 & 0 & 1 & 0 \\ x_{21} & \xi_2 & 0 & 0 & 0 & 0 & 1 \\ x_{31} & 0 & \xi_3 & x_{34} & x_{35} & 0 & 0 \\ x_{41} & x_{42} & 0 & \xi_4 & 0 & 0 & 0 \\ 0 & 0 & x_{53} & x_{54} & \xi_5 & 0 & 0 \end{bmatrix},$$

has full row rank for any $A \in \mathcal{Q}(G)$. To see this, note that the top two rows are always linearly independent. In the bottom three rows we can find a nonsingular submatrix:

$$\begin{bmatrix} x_{31} & 0 & \xi_3 \\ x_{41} & x_{42} & 0 \\ 0 & 0 & x_{53} \end{bmatrix},$$

since this matrix has determinant equal to $x_{31}x_{42}x_{53}$, which is always nonzero since the x_{ij}'s are nonzero free parameters.

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2.5   Zero extension

Let us now consider a directed weighted graph $G(W) = (V, E, W)$. We assign to it the following set of matrices:

$$\mathcal{Q}_W(G) = \{A \in \mathbb{R}^{n \times n} \mid A_{i,j} = W_{i,j} \text{ for all } i \neq j\}.$$

That is, each off-diagonal entry is either zero or equal to the given nonzero fixed weight, and the diagonal entries can take any value and are free. Suppose we are also given a set of leaders V_L. To study controllability of $(G(W), V_L)$, i.e., whether the pair (A, B) is controllable for all $A \in \mathcal{Q}_W(G)$ and $B = B(V; V_L)$, we make use of the concept known as *zero extension*. Before introducing that, we have the following intermediate result.

Theorem 2.6. *Let $G(W) = (V, E, W)$ be a weighted graph and let $V_L \subseteq V$ be the leader set. Then, $(G(W), V_L)$ is controllable if and only if $[A, \ B]$ has full row rank for all $A \in \mathcal{Q}_W(G)$, where $B = B(V; V_L)$.*

Proof. The proof is heavily inspired by the proof of Theorem 2.4, taken from [19]. Note that the only difference between the theorems is that in Theorem 2.4 the qualitative class $\mathcal{Q}(G)$ is considered, while we consider $\mathcal{Q}_W(G)$ now in Theorem 2.6. The proof is as follows.

The 'only if' direction is straightforward. Suppose $(G(W), V_L)$ is controllable. This means that (A, V_L) is controllable for all $A \in \mathcal{Q}_W(G)$. Hence, by the Hautus Test (Theorem 2.2), the matrix $[A - \lambda I, \ B]$ has full row rank for all $\lambda \in \mathbb{C}$. In particular, take $\lambda = 0$. Then the result follows.

Next, we will prove the 'if' direction. Suppose $[A, \ B]$ has full row rank for all $A \in \mathcal{Q}_W(G)$. We will now prove that (A, V_L) is controllable for any $A \in \mathcal{Q}_W(G)$. To do so, we need to prove that $[A - \lambda I, \ B]$ has full row rank for all $\lambda \in \mathbb{C}$. Let $\lambda \in \mathbb{C}$ and $z \in \mathbb{C}^n$ be such that $z^H[A - \lambda I, \ B] = 0$ for some $A \in \mathcal{Q}_W(G)$. Here, z^H denotes the Hermitian transpose of z. If we can now prove that this implies $z = 0$, we are done.

To do so, we write $z = p + jq$, where j denotes the imaginary unit and $p, q \in \mathbb{R}^n$. We construct now the vector $x = p + \alpha q$, where $\alpha \in \mathbb{R}$ is such that

$$\alpha \notin \left\{ -\frac{p_i}{q_i} \mid q_i \neq 0, \ 1 \leq i \leq n \right\}.$$

Here, p_i and q_i denote the ith element of the vectors p and q, respectively. We now have that $x_i = 0$ if and only if $z_i = 0$. To see this, note that if $z_i = 0$, then $p_i = q_i = 0$ and hence $x_i = 0$. On the other hand, if $x_i = 0$, then we have that $p_i + \alpha q_i = 0$, and by construction of α, we have that this implies that $q_i = 0$ and $p_i = 0$. Consequently, $z_i = 0$. Furthermore, the following implication is true:

$$x_i = 0 \implies (x^T A)_i = 0.$$

To see this, suppose $x_i = 0$. Then also $z_i = 0$ by the previous reasoning. We assumed that $z^H[A - \lambda I, \ B] = 0$ so in particular we have $z^H A = \lambda z^H$. Since $z_i = 0$, we also have $(z^H A)_i = 0$. Recall that $z = p + jq$, so furthermore we have that $(p^T A)_i = 0$ and $(q^T A)_i = 0$. Since $x = p + \alpha q$, we find that $(x^T A)_i = 0$ as well.

We now define the following matrix: $D = \text{diag}(d_1, \ldots, d_n)$, where each d_i is given by

$$d_i = \begin{cases} 0 & \text{if } x_i = 0, \\ \frac{(x^T A)_i}{x_i} & \text{otherwise .} \end{cases}$$

Note that D is a real matrix. Now suppose that $x_i = 0$. Then we have that $(x^T A)_i = 0$ by the previous implication. Also, if $x_i = 0$, then $(x^T D)_i = x_i d_i = 0$ as well. On the other hand, suppose that $x_i \neq 0$. Then by definition of d_i, we have that $(x^T D)_i = x_i d_i = (x^T A)_i$. In either case, we have that $(x^T A)_i = (x^T D)_i$, so we derive $x^T A = x^T D$.

Let now $\bar{A}$ be given by $\bar{A} = A - D$. Clearly $\bar{A} \in \mathcal{Q}_W(G)$. Note that $x^T \bar{A} = x^T(A - D) = 0$.

Furthermore, from $z^H[A - \lambda I, \ B] = 0$ we have in particular that $z^H B = 0$. This in turn implies that both $p^T B = 0$ and $q^T B = 0$, and consequently, $x^T B = 0$.

Combining these two facts, we see that $x^T[\bar{A}, \ B] = 0$. Since we assumed that $[A, \ B]$ has full row rank for all $A \in \mathcal{Q}_W(G)$, this implies that $x = 0$. This in turn implies that $z = 0$, which completes the proof. $\qquad \square$

The concept of zero extension is now as follows. Let us assign a variable x_i to each node i. Suppose we set $x_i = 0$ for all nodes $i \in C$, where C is some subset of V called the set of *zero nodes*. For each node $i \in C$, we now consider the *balance equation*

$$\sum_{j \in N_i \setminus C} x_j W_{j,i} = 0.$$

If this system of $|C|$ balance equations implies that $x_k = 0$ for all $k \in Y$, we say that C forces Y and we add the nodes in Y to the set of zero nodes C. The new set of zero nodes is now $C \cup Y$. Note that zero extension has a very elegant linear algebraic representation. Let $A_{V \setminus C, C}$ denote the submatrix of $A \in \mathcal{Q}_W(G)$ whose columns are the columns of A indexed by nodes in C, and whose rows are indexed by nodes in $V \setminus C$. Then the set of balance equations can be written as the matrix equation $x_{V \setminus C}^T A_{V \setminus C, C} = 0$ and if this equation implies $x_Y^T = 0$ for some $Y \subseteq V \setminus C$, we add the nodes of Y to our set of zero nodes. We repeat the application of zero extension and this process is known as the *zero extension process*. If we can not add any more zero nodes, we arrive at a set of nodes called the *derived* set $D^{ze}(C)$. We say that C is a *balancing set* if $D^{ze}(C) = V$. In [22] the following theorem was proven for undirected graphs:

Theorem 2.7. *For a given weighted undirected graph $G(W) = (V, E, W)$ and leader set $V_L \subseteq V$, $(G(W), V_L)$ is controllable if and only if V_L is a balancing set.*

Since only undirected graphs are considered in [22] while we study directed graphs in this thesis, we now wish to extend the result of Theorem 2.7 to directed graphs. Before stating the main theorem, we first state and prove the following intermediate result:

Lemma 2.8. *Let $G(W) = (V, E, W)$ be a weighted simple and directed graph, and let $C \subseteq V$ be the set of zero nodes. Suppose the balance equations imply that $Y \subseteq V \setminus C$ becomes zero as well. Then, $(G(W), C)$ is controllable if and only if $(G(W), C \cup Y)$ is controllable.*

What this lemma shows is that by applying zero extension, controllability is preserved. We will now give the proof.

Proof. First, we prove the 'only if' direction. Suppose $(G(W), C)$ is controllable. By Theorem 2.6, we know that the matrix $[A, \ B]$ has full row rank for all $A \in \mathcal{Q}_W(G)$, where B is given by $B = B(V; C)$. Let now $B' = B(V; C \cup Y)$, then clearly $[A, \ B']$ has full row rank for any $A \in \mathcal{Q}_W(G)$ as well, since we simply added a couple of columns and that does not lower the row rank. Then, by Theorem 2.6, also $(G(W), C \cup Y)$ is controllable.

We will now prove the 'if' direction. Suppose that $(G(W), C \cup Y)$ is controllable. Then we want to prove that $(G(W), C)$ is controllable. To do so, by Theorem 2.6, we need to show that the matrix $[A, \ B]$ has full row rank for all $A \in \mathcal{Q}_W(G)$ and where $B = B(V; C)$. So we want to prove that if $x^T[A, \ B] = 0$ for some vector $x \in \mathbb{R}^n$, this implies that $x = 0$. We make the following partition of $x^T[A, \ B] = 0$:

$$(x_C^T, \ x_Y^T, \ x_{V \setminus (C \cup Y)}^T) \begin{bmatrix} A_{11} & A_{12} & A_{13} & I \\ A_{21} & A_{22} & A_{23} & 0 \\ A_{31} & A_{32} & A_{33} & 0 \end{bmatrix} = 0. \tag{2.3}$$

Where the first block row of A corresponds to nodes in C, the second block row corresponds to nodes in Y and the third block row corresponds to nodes in $V \setminus (C \cup Y)$. Since A is a square matrix, the block columns are indexed in the same way as the block rows. Now we want to prove that (2.3) implies that $x^T = (x_C^T, \ x_Y^T, \ x_{V \setminus (C \cup Y)}^T) = 0$. To do so, we make use of the fact that $(G(W), C \cup Y)$ is controllable and that the balance equations of C imply that the nodes in Y are zero as well. Controllability of $(G(W), C \cup Y)$ gives by Theorem 2.6 that $x^T[A, \ B'] = 0$ implies $x = 0$, where $B = B(V; C \cup Y)$ now. Using the same partition as in (2.3), we get:

$$(x_C^T, \ x_Y^T, \ x_{V \setminus (C \cup Y)}^T) \begin{bmatrix} A_{11} & A_{12} & A_{13} & I & 0 \\ A_{21} & A_{22} & A_{23} & 0 & I \\ A_{31} & A_{32} & A_{33} & 0 & 0 \end{bmatrix} = 0, \tag{2.4}$$

and these equations imply $(x_C^T, \ x_Y^T, \ x_{V \setminus (C \cup Y)}^T) = 0$. Taking a closer look at the equations of (2.4), we see immediately that $x_C = 0$ and $x_Y = 0$ by $x^T B' = 0$. Then, we also have the following implication:

$$x_{V \setminus (C \cup Y)}^T[A_{31}, \ A_{32}, \ A_{33}] = 0 \implies x_{V \setminus (C \cup Y)}^T = 0. \tag{2.5}$$

We also have that the balance equations of C imply that the nodes of Y are zero. Writing down the balance equations as a homogeneous system of equations, we get that $x_{V\setminus C}^T A_{V\setminus C, C} = 0$ implies $x_Y^T = 0$. Taking a closer look at this implication, we can see that the matrix $A_{V\setminus C, C}$ can be partitioned using the same labels as in (2.3), and we obtain

$$x_{V\setminus C}^T A_{V\setminus C, C} = (x_Y^T,\ x_{V\setminus(C\cup Y)}^T) \begin{bmatrix} A_{Y,C} \\ A_{V\setminus(C\cup Y),C} \end{bmatrix} = (x_Y^T,\ x_{V\setminus(C\cup Y)}^T) \begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix} = 0.$$

Thus we also have the following implication:

$$(x_Y^T,\ x_{V\setminus(C\cup Y)}^T) \begin{bmatrix} A_{21} \\ A_{31} \end{bmatrix} = 0 \implies x_Y^T = 0. \tag{2.6}$$

Let us now consider the equations $x^T[A, B] = 0$ as in (2.3) again. We see immediately that $x_C^T = 0$ since $x^T B = 0$. Looking at the first block column of equations of (2.3), which is $x_C^T A_{11} + x_Y^T A_{21} + x_{V\setminus(C\cup Y)}^T A_{31} = 0$, we see that this reduces to $x_Y^T A_{21} + x_{V\setminus(C\cup Y)}^T A_{31} = 0$. By implication (2.6), we obtain $x_Y^T = 0$ now. Finally, now that $x_C^T = 0$ and $x_Y^T = 0$, by implication (2.5) we now get that $x_{V\setminus(C\cup Y)} = 0$ as well. In other words, $x = 0$, so $[A, B]$ has full row rank. Thus $(G(W), C)$ is controllable, which completes the proof. $\qquad\square$

Of course this lemma can be applied repeatedly, so that we derive:

Corollary 2.9. *Let $G(W) = (V, E, W)$ be a weighted directed graph, and let C denote the set of zero nodes. Then, $(G(W), C)$ is controllable if and only if $(G(W), D^{ze}(C)$ is controllable.*

We will now establish that also for weighted directed graphs there is a one-to-one relation between balancing sets and controllability of $(G(W), V_L)$. This is formulated in the following theorem.

Theorem 2.10. *Let $G(W) = (V, E, W)$ be a weighted directed graph and let $V_L \subseteq V$ denote the leader set. Then, $(G(W), V_L)$ is controllable if and only if V_L is a balancing set.*

Proof. First, we will prove the 'if' direction. Assume V_L is a balancing set, so $D^{ze}(V_L) = V$. By Corollary 2.9 we have that $(G(W), V_L)$ is controllable if and only if $(G(W), D^{ze}(V_L)) = (G(W), V)$ is controllable. Note that $(G(W), V)$ is trivially controllable by Theorem 2.6, since the matrix $[A, I]$ has always full row rank for any $A \in \mathcal{Q}_W(G)$.

We will now prove the 'only if' direction. We do that by using reductio ad absurdum. Suppose that $(G(W), V_L)$ is controllable and assume, for a contradiction, that $D^{ze}(V_L) \neq V$. Without loss of generality, we label $V_L = \{1, 2, \ldots, m\}$ and $D^{ze}(V_L) = \{1, 2 \ldots, m, m+1, \ldots, m+r\}$ with $m+r < n$. Then $V \setminus D^{ze}(V_L) = \{m+r+1, m+r+2, \ldots, n\}$. Note that by Corollary 2.9 we have that $(G(W), D^{ze}(V_L))$ is controllable, so by Theorem 2.6 we know that the matrix $[A, B]$ has full row rank for any $A \in \mathcal{Q}_W(G)$ with $B = (V; D^{ze}(V_L))$. We partition $[A, B]$ as follows now:

$$[A,\ B] = \begin{bmatrix} A_{11} & A_{12} & I \\ A_{21} & A_{22} & 0 \end{bmatrix},$$

where the first block row corresponds to nodes in $D^{ze}(V_L)$ and the second block row corresponds to nodes in $V \setminus D^{ze}(V_L)$. The block columns are indexed in the same way. Note that $A_{11} \in \mathbb{R}^{(m+r) \times (m+r)}$ and $A_{22} \in \mathbb{R}^{(n-m-r) \times (n-m-r)}$. Let now $x \in \mathbb{R}^n$ be such that $x^T[A, B] = 0$. We partition $x^T = (x^T_{D^{ze}(V_L)}, x^T_{V \setminus D^{ze}(V_L)})$. Then we have

$$(x^T_{D^{ze}(V_L)}, \ x^T_{V \setminus D^{ze}(V_L)}) \begin{bmatrix} A_{11} & A_{12} & I \\ A_{21} & A_{22} & 0 \end{bmatrix} = 0.$$

Note that this directly implies that $x^T_{D^{ze}(V_L)} = 0$. Considering the equations obtained from the first block column, we see that this now reads $x^T_{V \setminus D^{ze}(V_L)} A_{21} = 0$. Note that $A_{21} = A_{V \setminus D^{ze}(V_L), D^{ze}(V_L)}$. Hence, $x^T_{V \setminus D^{ze}(V_L)} A_{V \setminus D^{ze}(V_L), D^{ze}(V_L)} = 0$ are precisely the balance equations for every zero node in $D^{ze}(V_L)$. Note that these equations imply that every entry of $x^T_{V \setminus D^{ze}(V_L)}$ is nonzero, because if it were zero, it would be in the derived set $D^{ze}(V_L)$. Since $(x^T_{V \setminus D^{ze}(V_L)})_i \neq 0$ for all $m + r + 1 \leq i \leq n$, we can always choose the diagonal entries of A_{22} such that $x^T_{V \setminus D^{ze}(V_L)} A_{22} = 0$. So there exists a matrix $A \in \mathcal{Q}_W(G)$ such that

$$(0, \ x^T_{V \setminus D^{ze}(V_L)}) \begin{bmatrix} A_{11} & A_{12} & I \\ A_{21} & A_{22} & 0 \end{bmatrix} = 0,$$

where $x^T_{V \setminus D^{ze}(V_L)}$ is nonzero. This violates the fact that $[A, B]$ has full row rank for all $A \in \mathcal{Q}_W(G)$ with $B = B(V; D^{ze}(V_L))$. We reached a contradiction and this completes the proof. $\square$

Hence, V_L being a balancing set for $(G(W), V_L)$ is a necessary and sufficient condition for strong structural controllability of $(G(W), V_L)$. Note that, obviously,

$$\mathcal{Q}(G) = \bigcup_{W \in \mathcal{W}(\mathcal{G})} \mathcal{Q}_W(G).$$

Using this equality and the notion of balancing set, we have the following fact:

Fact 2.11. Let $G = (V, E)$ be a directed graph and let $V_L \subseteq V$ be the leader set. Then, (G, V_L) is controllable if and only if V_L is a balancing set for $(G(W), V_L)$ for all $W \in \mathcal{W}(G)$.

Note that it is infeasible to check whether V_L is a balancing set for all weighted graphs $(G(W), V_L)$, so a prefered method to study strong structural controllability is by means of zero forcing sets.

Chapter 3

Problem Formulation

Suppose that for a given simple directed graph $G = (V, E)$ and leader set V_L the system (G, V_L) is not strongly structurally controllable. It might happen that if we add certain restrictions on the qualitative class $\mathcal{Q}(G)$, so effectively making the set of matrices we consider smaller, any system matrix satisfying these additional constraints together with the input nodes is controllable. For example, suppose a node has two neighbors. Then the corresponding weights of the two edges are arbitrary, and we have two degrees of freedom. But now suppose that we impose that the two weights of the edges are equal: in a sense the two edges are then identical. Of course the weight of the first edge can still be anything, as long as it is nonzero, but now the weight of the second edge is equal to the weight of the first edge. We have now effectively reduced the degrees of freedom by one. By imposing such restrictions on the qualitative class of the graph, we obtain a subset of the original qualitative class. It can happen that this smaller subset yields strong structural controllability, and we will now give an example of this.

Consider the graph $G = (V, E)$ and leader set $V_L = \{1, 2, 3\}$, depicted in Figure 3.1.
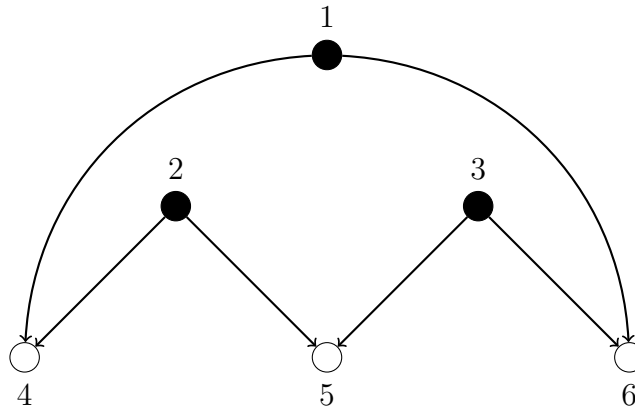

Figure 3.1: A graph consisting of six nodes with three leader nodes.

The qualitative class associated with this graph is equal to:

$$\mathcal{Q}(G) = \{A \in \mathbb{R}^{6\times 6} \mid \text{ for } i \neq j, A_{i,j} \neq 0 \iff (j,i) \in E\}.$$

This means that elements A of $\mathcal{Q}(G)$ are of the form:

$$A = \begin{bmatrix} ? & 0 & 0 & 0 & 0 & 0 \\ 0 & ? & 0 & 0 & 0 & 0 \\ 0 & 0 & ? & 0 & 0 & 0 \\ * & * & 0 & ? & 0 & 0 \\ 0 & * & * & 0 & ? & 0 \\ * & 0 & * & 0 & 0 & ? \end{bmatrix},$$

where the off-diagional entries are either equal to zero or nonzero free parameters, and the diagonal entries can take any value. Since V_L is not a zero forcing set, which is easily seen by the fact that every black node has two white out-neighbors, the system (G, V_L) is not controllable by Theorem 2.5.

Suppose now that we impose that some of the edges are identical, by which we mean that for any realization $A \in \mathcal{Q}(G)$, the entries in A corresponding to those edges are equal. In particular, let us consider the subclass of $\mathcal{Q}(G)$ consisting of all matrices A of the form:

$$A = \begin{bmatrix} \xi_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \xi_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \xi_3 & 0 & 0 & 0 \\ a & b & 0 & \xi_4 & 0 & 0 \\ 0 & b & c & 0 & \xi_5 & 0 \\ a & 0 & c & 0 & 0 & \xi_6 \end{bmatrix},$$

where $a, b, c \neq 0$ are free nonzero parameters, and the ξ_i's denote the diagonal entries that can take arbitrary real values (including zero). Again, the leader set is given by $V_L = \{1, 2, 3\}$. This is depicted in Figure 3.2. Note that we use colors to indicate the edges that are imposed to have equal weight. Thus the study of controllability of systems on such graphs will use the notion of colored graph.
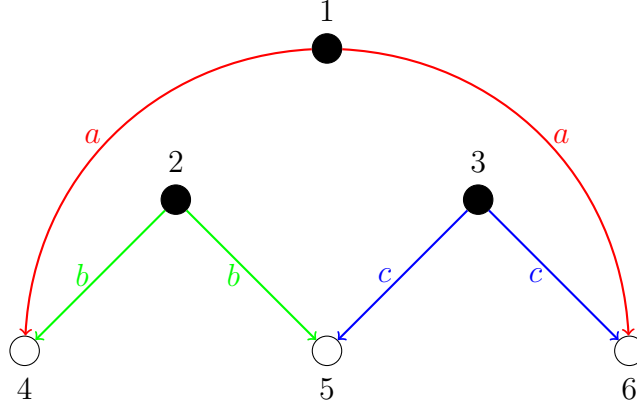
Figure 3.2: The graph of Figure 3.1 but now with added constraints, depicted by the use of colors.

Then we have that for all realizations of this form, (A, V_L) *is* controllable. To see this, we will use Theorem 2.1. We claim that $\mathcal{C} = [B, AB, \ldots, A^5B]$ has full row rank for any realization A. To see this, note that

$$
\mathcal{C} = [B, AB, A^2B, A^3B, A^4B, A^5B] = \begin{bmatrix} 1 & 0 & 0 & \xi_1 & 0 & 0 \\ 0 & 1 & 0 & 0 & \xi_2 & 0 \\ 0 & 0 & 1 & 0 & 0 & \xi_3 \\ 0 & 0 & 0 & a & b & 0 \\ 0 & 0 & 0 & 0 & b & c \\ 0 & 0 & 0 & a & 0 & c \end{bmatrix} \quad A^2B, \quad A^3B, \quad A^4B, \quad A^5B \end{bmatrix}.
$$

And clearly the first three rows are always linearly independent. In addition, in the lower three rows we can find a submatrix of full row rank, since

$$
\det \begin{bmatrix} a & b & 0 \\ 0 & b & c \\ a & 0 & c \end{bmatrix} = abc + bac = 2abc,
$$

and this product is nonzero since a, b, c are nonzero free parameters. Since we had not specified matrix A, we see that $\text{rank}(\mathcal{C}) = 6$ for all A, and hence the constrained qualitative class is strongly structurally controllable.

So, even while the original qualitative class does not yield controllability, if we make restrictions and impose that some edges have equal weights, the obtained constrained subclass can be controllable. In this thesis, we will make this more precise, and try to find a graph theoretical test for strong structural controllability for this new class.

Chapter 4

Generalized zero forcing sets

In this chapter, we will study and extend the results of [19], see also Section 2.4. In that paper, a necessary and sufficient condition for strong structural controllability is given by using the concept of zero forcing. To check if this condition holds, we look at the graph topology and use a so-called color change rule. In this chapter, we will generalize this color change rule and propose another necessary and sufficient condition for strong structural controllability. To do so, we will first take a closer look at bipartite graphs.

4.1 Bipartite graphs

Consider a bipartite graph $G = (S, T, E)$ and suppose $|S| = |T| = k$. We will denote the node sets by $S = \{s_1, \ldots, s_k\}$ and $T = \{t_1, \ldots, t_k\}$. If there exists a k-matching between S and T, we call that matching a *perfect matching*. We denote this matching by $\mathcal{M} = \{\{s_1, t_{\tau(1)}\}, \{s_2, t_{\tau(2)}\}, \ldots, \{s_k, t_{\tau(k)}\}\}$, where τ denotes a permutation of $(1, 2, \ldots, k)$.

To this bipartite graph $G = (S, T, E)$ with $|S| = |T| = k$ we assign the pattern matrix $\mathcal{P}(G)$ as follows:

$$\mathcal{P}(G) = \{A \in \mathbb{C}^{k \times k} \mid A_{j,i} \neq 0 \iff (s_i, t_j) \in E\}.$$

We have the following two well-known facts ([31],[32]):

Theorem 4.1. *Let $G = (S, T, E)$ be a bipartite graph with $|S| = |T| = k$. Denote the pattern matrix by $\mathcal{P}(G)$. We have the following two statements:*

1. *There exists a nonsingular matrix $A \in \mathcal{P}(G)$ if and only if there exists a perfect matching between S and T.*

2. *All matrices $A \in \mathcal{P}(G)$ are nonsingular if and only if there exists exactly one perfect matching between S and T.*

Proof. The main idea of the proof is based on the Leibniz formula for the determinant. We compute the determinant of any of the matrices $A \in \mathcal{P}(G)$ as follows:

$$\det(A) = \sum_{\tau \in S_n} \text{sgn}(\tau) \cdot \prod_{i=1}^{k} A_{\tau(i),i}, \tag{4.1}$$

where S_n denotes the set of all permutations of $(1, 2, \ldots, k)$, and $\text{sgn}(\tau)$ denotes the sign of the permutation τ, i.e., $\text{sgn}(\tau) = (-1)^m$, where m is the number of transpositions one needs to make to transform $(\tau(1), \tau(2), \ldots, \tau(k))$ to $(1, 2 \ldots, k)$. Note that we have not specified the matrix A yet, so the entries $A_{\tau(i),i}$ are indeterminates. It turns out that there is a one-to-one correspondence between a permutation and a perfect matching ([31]). If we denote $\mathbb{P}$ as the set of all perfect matchings in G, we have

$$\prod_{i=1}^{k} A_{\tau(i),i} \neq 0 \iff \{\{s_1, t_{\tau(1)}\}, \{s_2, t_{\tau(2)}\}, \ldots, \{s_k, t_{\tau(k)}\}\} \in \mathbb{P}.$$

We see that (4.1) can be reduced to

$$\det(A) = \sum_{\mathcal{M} \in \mathbb{P}} \text{sgn}(\mathcal{M}) \cdot \prod_{i=1}^{k} A_{\tau(i),i}, \tag{4.2}$$

where $\text{sgn}(\mathcal{M})$ denotes the sign of the perfect matching, which is defined as the sign of the permutation τ. We see that if there does not exists a perfect matching $\mathcal{M}$, i.e. if $\mathbb{P}$ is empty, all products vanish in equation (4.2) and we find that the determinant is equal to zero. On the other hand, there exists at least one perfect matching if and only if the values of the entries in A can be chosen in such a way that (4.2) is nonzero, i.e., there exists a matrix A that is nonsingular. Finally, there is exactly one perfect matching in G if and only if (4.2) reduces to a single product, equivalently, (4.2) is nonzero for all $A \in \mathbb{C}^{k \times k}$. This completes the proof. $\qquad \square$

Recall that if in the bipartite graph $G = (S, T, E)$ with $|S| = |T| = k$ there exists a perfect matching, we call S and T perfect neighbors. Furthermore, we call S and T *strong perfect neighbors* if there is exactly one perfect matching in this bipartite graph. The second statement from Theorem 4.1 together with a result from [32] yield the following theorem:

Theorem 4.2. *Let $G = (S, T, E)$ be a bipartite graph and suppose $|S| = |T| = k$. Let $\mathcal{P}(G)$ be the pattern matrix. Then, the following statements are equivalent:*

    i. *S and T are strong perfect neighbors;*

    ii. *The matrix A is nonsingular for all $A \in \mathcal{P}(G)$;*

    iii. *The nodes in S and T can be relabeled such that every $A \in \mathcal{P}(G)$ is a nonsingular upper triangular matrix.*

This theorem can be used to define the concept of *generalized zero forcing set*, which we will discuss in the next section.

4.2 Generalized color change rule

Let $G = (V, E)$ now be a given directed and simple graph. For any two disjoint nonempty subsets $X, Y \subset V$, denoted by $X = \{x_1, x_2, \ldots, x_p\}$ and $Y = \{y_1, y_2, \ldots, y_q\}$, we consider the associated bipartite graph, $(X, Y, E_{X,Y})$, with $E_{X,Y} = \{\{x_i, y_j\} \mid (x_i, y_j) \in E, x_i \in X, y_j \in Y\}$.

Now suppose $|X| = |Y| = k$. Recall that we have defined X and Y to be *perfect neighbors* if there exists a perfect matching between X and Y, i.e., there exists a k-matching between the two sets. In the sequel, we call X and Y *strong perfect neighbors* if there exists a unique perfect matching.

Suppose we color each node in V either black or white. Initially, let $C \subseteq V$ be the set of black nodes and set $V \setminus C$ to white. Suppose there exists a subset of black nodes $X \subseteq C$ and a subset of white nodes $Y \subseteq V \setminus C$, such that $|X| = |Y|$. Then the notion of strong perfect neighbor leads to the following coloring rule.

Generalized color change rule (1): Let C be the set of black nodes and let $X \subseteq C$. Let $Y \subseteq V \setminus C$ be such that $|Y| = |X|$. If Y is the only white strong perfect neighbor of X, we color Y black. We say that X forces Y, and we write $X \to Y$.

Note that a subset X of black nodes might have many perfect neighbors, and among those also many strong perfect neighbors. We force, however, only if there is exactly one strong perfect neighbor. For example, consider the graph depicted in Figure 4.1.
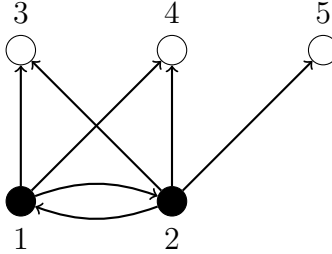

Figure 4.1: Example with perfect neighbors and strong perfect neighbors

We see that $\{1\}$ has two white strong perfect neighbors, namely $\{3\}$ and $\{4\}$. The set $\{2\}$ has three white strong perfect neighbors, $\{3\}$, $\{4\}$ and $\{5\}$. The set $\{1, 2\}$ has as perfect neighbors $\{3, 4\}$, $\{3, 5\}$ and $\{4, 5\}$. Among these, $\{3, 5\}$ and $\{4, 5\}$ are also strong perfect neighbors. Since there is no subset of black nodes with exactly one white strong perfect neighbor, we do not force any of the white nodes to become black.

Note that the above coloring rule is a generalization of the zero forcing rule explained in Section 2.4, because a single black node i has exactly one white out-neighbor j if and only if the set $\{j\}$ is the only white strong perfect neighbor of $\{i\}$.

The above generalized color change rule can also be reformulated in a different way:

(Reformulated) generalized color change rule (2): Let C be the set of black nodes, and let $X \subseteq C$. Denote the white neighbor set of X as $Y = N_{V \setminus C}(X) = \{j \in V \setminus C \mid (i,j) \in E, \text{ for some } i \in X\}$. If Y is a white strong perfect neighbor of X, we color Y black. We say that X forces Y, and we write $X \to Y$.

We claim that this reformulated generalized color change rule (2) is the same as the generalized color change rule (1), in the sense that these two color change rules are equivalent. By that we mean that if a black set of nodes X forces a white set of nodes Y according to rule (1), this same set of black nodes X also forces this same set of white nodes Y, according to rule (2), and vice versa.

As an example, let us consider the graph depicted in Figure 4.1 again, with black node set $\{1, 2\}$. Note that the white neighbor set of $\{1\}$ is given by $\{3, 4\}$. Since this set has cardinality 2, it can never be a strong perfect neighbor of the set $\{1\}$, which has cardinality 1. Likeso the white neighbor set of $\{2\}$ is $\{3, 4, 5\}$ and this can also not be a strong perfect neighbor of $\{2\}$. The white neighbor set of $\{1, 2\}$ is given by $\{3, 4, 5\}$ and since these cardinalities differ too, it can not be a strong perfect neighbor. We see that the reformulated generalized color change rule can not be applied here, i.e., no white set of nodes is forced to become black. We have also seen this when we used the first formulation of the generalized color change rule.

As stated, we claim that these two color change rules are equivalent and the proof will be given in Section 4.3. For the time being, we assume this is true, so that we can use the two formulations of the generalized color change rule interchangeably.

Then, any of the two above generalized color change rules leads to a notion of *derived set*. For a given graph $G = (V, E)$ and colored set $C \subseteq V$, the derived set $D^{gzf}(C)$ is the set of black nodes obtained by repeated application of the generalized color change rule (1) or (2), until no more changes are possible. If $D^{gzf}(C) = V$, we call C a *generalized zero forcing set* (GZFS). We have the following relationship between strong structural controllability and generalized zero forcing sets:

Theorem 4.3. *Let $G = (V, E)$ be a graph and let the leader set be given by $V_L \subseteq V$. Then, (G, V_L) is controllable if and only if V_L is a generalized zero forcing set.*

Before proving this, we need a couple of preliminary results. Recall from Theorem 2.4 in Section 2.3 that (G, V_L) is controllable if and only if the matrix $[A, \ B]$ has full row rank for all $A \in \mathcal{Q}(G)$, with $B = B(V; V_L)$. This theorem will be used to show that the process of coloring nodes according to the generalized color change rule does not affect controllability. This is made concrete in the following lemma.

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Lemma 4.4. *Let $G = (V, E)$ be a graph and $C \subset V$ the set of black nodes. Suppose $X \subseteq C$, $Y \subseteq V \setminus C$ and $X \to Y$. Then (G, C) is controllable if and only if $(G, C \cup Y)$ is controllable.*

Proof. The "only if" direction is trivial. Suppose that $[A, \ B]$ has full row rank for all $A \in \mathcal{Q}(G)$ and $B = B(V; C)$. Then so will $[\ A, \ B'\ ]$ have full row rank for all $A \in \mathcal{Q}(G)$ and $B' = B(V; C \cup Y)$, since we simply added a couple of columns (corresponding to the Y nodes) and this does not lower the row rank.

We now prove the "if" part. Suppose $(G, C \cup Y)$ is controllable, then by a possible re-labeling of the nodes we get that

$$
[\ A, \ B'\ ] = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & I & 0 & 0 \\ A_{21} & A_{22} & A_{23} & A_{24} & 0 & I & 0 \\ A_{31} & A_{32} & A_{33} & A_{34} & 0 & 0 & I \\ A_{41} & A_{42} & A_{43} & A_{44} & 0 & 0 & 0 \end{bmatrix} \tag{4.3}
$$

has full row rank. Here, the first block row corresponds to nodes in Y, the second block row corresponds to nodes in X, the third block row corresponds to nodes in $C \setminus X$ and the fourth block row corresponds to nodes in $V \setminus (C \cup Y)$. Since A is a square matrix, we partition the columns using the same labels.

Because $[A, \ B']$, where $B' = B(V; C \cup Y)$ has full row rank by controllability of $(G, C \cup Y)$, in particular the last block row of (4.3) has full row rank.

We know that X forces Y. By the second formulation of the generalized color change rule, we know that the white neighbor set of X, given by $N_{V \setminus C}(X) = \{j \in V \setminus C \mid \{i, j\} \in E_{X,Y}$, for some $i \in X\}$ is equal to the set Y. Any white neighbor of X is thus contained in Y, and this implies that block A_{42} is zero, since this block corresponds to edges going from X to $V \setminus (C \cup Y)$.

Furthermore, since $X \to Y$, we also know that X and Y are strong perfect neighbors. Then, by Theorem 4.2, block A_{12} is nonsingular for any $A \in \mathcal{Q}(G)$.

Combining these two facts we see that the submatrix

$$
\begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{41} & 0 & A_{43} & A_{44} \end{bmatrix},
$$

has full row rank. Then also

$$
\begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & 0 & 0 & 0 \\ A_{21} & A_{22} & A_{23} & A_{24} & 0 & I & 0 \\ A_{31} & A_{32} & A_{33} & A_{34} & 0 & 0 & I \\ A_{41} & 0 & A_{43} & A_{44} & 0 & 0 & 0 \end{bmatrix} = [\ A, \ B\ ],
$$

has full row rank, so we find that (G, C) is controllable. This completes the proof. $\qquad \square$

Of course we can apply this lemma repeatedly and we find:

Corollary 4.5. *Let $G = (V, E)$ be a graph and let $C \subseteq V$ denote the set of black nodes. Then, (G, C) is controllable if and only if $(G, D^{gzf}(C))$ is controllable.*

We will now state the main result.

Theorem 4.6. *Let $G = (V, E)$ be a graph and let $V_L \subseteq V$ denote the leader set. Then, (G, V_L) is controllable if and only if V_L is a generalized zero forcing set.*

Proof. The "if" direction is straightforward. Indeed, since V_L is a generalized zero forcing set, we have $D^{gzf}(V_L) = V$. Since (G, V) is trivially controllable, because $[A, \ I]$ has full row rank for any $A \in \mathcal{Q}(G)$, we get from the previous corollary that (G, V_L) is controllable.

To prove the "only if" direction, we use reductio ad absurdum (proof by contradiction). Suppose V_L is not a generalized zero forcing set, so $D^{gzf}(V_L) \neq V$. Let us label the nodes as follows: $V_L = \{1, \ldots, m\}$ and $D^{gzf}(V_L) = \{1, \ldots, m, m+1, \ldots, m+r\}$ where $m + r < |V|$. By the previous corollary, we get that $[A, \ B]$ with $B = B(V; D^{gzf}(V_L))$ has full row rank, so

$$[A, \ B] = \begin{bmatrix} A_{11} & A_{12} & I \\ A_{21} & A_{22} & 0 \end{bmatrix},$$

has full row rank, where the first block row corresponds to nodes in $D^{gzf}(V_L)$ and the second block row corresponds to nodes in $V \setminus D^{gzf}(V_L)$. We now show that this is a contradiction, by distinguishing two cases.

First of all, suppose there exists a column of A_{21} with only one nonzero element. Then, there exist nodes $i \in V \setminus D^{gzf}(V_L)$, $j \in D^{gzf}(V_L)$ with $\{i\}$ being the strong perfect neighbor of $\{j\}$. However, this means that j forces i, which is a contradiction to the fact that $D^{gzf}(V_L)$ was the derived set.

Secondly, suppose there does not exist a column in A_{21} with only one nonzero element. Then each column is either the zero column or has at least two nonzero elements. Then we can always find a realization A_{21} such that $\mathbb{1}^T A_{21} = 0$, where $\mathbb{1}^T$ denotes the transpose of the all-ones vector, $\mathbb{1} = (1, \ldots, 1)^T \in \mathbb{R}^m$. Since the diagonal elements of A are arbitrary, we can always choose a realization such that $\mathbb{1}^T A_{22} = 0$. Define $v^T = (0, \mathbb{1}^T)$. Hence, we can always find a matrix $A \in \mathcal{Q}(G)$ such that $v^T[A, \ B] = 0$:

$$\begin{pmatrix} 0, & \mathbb{1}^T \end{pmatrix} \begin{bmatrix} A_{11} & A_{12} & I \\ A_{21} & A_{22} & 0 \end{bmatrix} = 0,$$

this is a contradiction with the full row rank property for all realizations, so the proof is completed. $\square$

4.3   The equivalence of two color change rules

In the previous section, we considered two formulations of the same generalized color change rule. While at first sight the rules might seem different, they are in fact the same. In this section, we will prove that the two rules are indeed equivalent, by which we mean that a black set X forces a white set Y to become black using the first formulation of the generalized color change rule if and only if the black set X forces the white node set Y using the second formulation.

To this end, consider again the qualitative class $\mathcal{Q}(G)$ of a given graph $G = (V, E)$. Every node is either colored black or white, and initially C denotes the set of black nodes and $V \setminus C$ is set to white. In the previous section, we studied the following two color change rules:

Generalized color change rule (1): Let $X \subseteq C$ be a set of black nodes. If Y is the only white strong perfect neighbor of X, we color Y black and we say that X forces Y, and we write $X \to Y$.

The second formulation involved the notion of white neighbor set. For a given set of black nodes $X \subseteq C$, we denote the white neighbor set of X as $N_{V \setminus C}(X) = \{j \in V \setminus C \mid (i, j) \in E \text{ for some } i \in X\}$. We consider now the following color change rule:

Generalized color change rule (2): For a given set of black nodes $X \subseteq C$, denote the white neighbors of X as $Y = N_{V \setminus C}(X)$. If Y is a strong perfect neighbor of X, we color Y black and we say that X forces Y, and we write $X \to Y$.

We will now prove that the above two color change rules are in fact equivalent. This means that X forces Y according to (1) if and only if X forces Y according to (2).

Theorem 4.7. $X \to Y$ *using* (1) *if and only if* $X \to Y$ *using* (2).

Proof. Suppose $X \to Y$ according to the second formulation of the generalized color change rule. Then we know that X and Y are strong perfect neighbors, so in particular $|X| = |Y|$. Furthermore $N_{V \setminus C}(X) = Y$, so this means that Y is the only white neighbor of X. Hence, Y is also the only white strong perfect neighbor of X. This implies that $X \to Y$ according to (1).

On the other hand, suppose that $X \to Y$ using the first formulation of the generalized color change rule. Then we know that Y is the only white strong perfect neighbor of X. We will now prove that $N_{V \setminus C}(X) = Y$. Consider any node j in Y. Because Y is a white neighbor of X, we have that $j \in N_{V \setminus C}(X)$.

Consider now any node $v \in N_{V \setminus C}(X)$. We will now show that $v \in Y$. To do so, suppose, for a contradiction, that $v \notin Y$. Because X and Y are strong perfect neighbors, we have that $|X| = |Y| = k$ and we let $X = \{1, \ldots, k\}$ and $Y = \{k+1, \ldots, 2k\}$. Without loss of

generality, we can relabel the nodes in such a way that $A_{Y,X}$ is a nonsingular upper triangular matrix, by Theorem 4.2. Suppose this relabeling results in the ordered sets $X = \{1, \ldots, k\}$ and $Y = \{k+1, \ldots, 2k\}$. Because $v \in N_{V \setminus C}(X)$, we know that there exists an $i \in X$ that links to v. Furthermore, since $X = \{1, \ldots, k\}$ is an ordered set, let i denote the first node in X that is a mother node of v.

Replace now the ith row of $A_{Y,X}$ by the submatrix $A_{\{v\},X}$. Note that $A_{\{v\},X}$ is a single row, whose row index corresponds to node v and whose columns are indexed by $1, \ldots k$. We see that the first $i - 1$ entries of this row must be zero, since the ith element was the first mother node of v. After making this swap, the newly obtained matrix $A_{Y',X}$, with $Y' = (Y \setminus \{i\}) \cup \{v\}$, is also a nonsingular upper triangular matrix, and hence by Theorem 4.2, X and Y' are also strong perfect neighbors. This is a contradiction. This completes the proof. $\qquad\square$

4.4 Zero forcing and generalized zero forcing

From Theorem 4.6 and Theorem 2.5 from Section 2.4, we obtain immediately the following Corollary:

Corollary 4.8. *Let $G = (V, E)$ be a graph and let $V_L \subseteq V$ denote the leader set. Then, V_L is a zero forcing set if and only if V_L is a generalized zero forcing set.*

We will now investigate the relationship between the color change rule corresponding to zero forcing and the generalized color change rule corresponding to generalized zero forcing. We have already seen that the generalized color change rule induced by generalized zero forcing is a generalization of zero forcing, since if a node j is the only white out-neighbor of a black node i, we also have that $\{j\}$ is the only white strong perfect neighbor of $\{i\}$. We now claim that by applying the generalized color change rule once, this is equivalent to a sequence of single node-disjoint zero forcings. To see this, suppose $X \to Y$ according to the generalized zero forcing rule. Then we know that X and Y are strong perfect neighbors and the white neighbor set of X is given by Y. We label the nodes in X and Y in such a way that every realization $A_{Y,X}$ in the corresponding pattern matrix is in upper triangular form, which is possible by Theorem 4.2. Suppose this relabeling is given by $X = \{1, 2, \ldots, k\}, Y = \{k+1, k+2, \ldots, 2k\}$. Then one sees that instead of using the generalized color change rule, we can also apply the color change rule associated to zero forcing. We see that $1 \to k+1$ since node $k+1$ is the only white out-neighbor of the black node 1. We have now $2 \to k+2$, since $k+2$ is the only white out-neighbor of the black node 2. Repeating this argument, we see that $i \to k+i$ for all $1 \leq i \leq k$. Indeed, if $X \to Y$ using the generalized color change rule, one can regard this as a sequence of node-disjoint zero forcings $1 \to k+1, 2 \to k+2, \ldots, k \to 2k$.

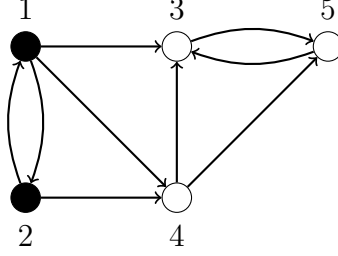As an example, consider the graph depicted in Figure 4.2, with leader set $\{1, 2\}$.

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Figure 4.2: Example of generalized zero forcing.

We claim that $\{1, 2\} \to \{3, 4\}$ by the generalized color change rule. To see this, we consider the corresponding bipartite graph, depicted in Figure 4.3.
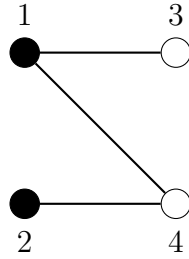

Figure 4.3: The bipartite graph $(X, Y, E_{X,Y})$ with $X = \{1, 2\}$ and $Y = \{3, 4\}$.

Clearly, there is exactly one perfect matching between the two node sets, so $\{1, 2\}$ and $\{3, 4\}$ are strong perfect neighbors. Furthermore, since the white neighbor set of $\{1, 2\}$ is precisely given by $\{3, 4\}$, we have that $\{1, 2\}$ forces $\{3, 4\}$. After we have colored nodes 3 and 4 black, we see that $\{5\}$ is the only white strong perfect neighbor of $\{4\}$, so we color node 5 black as well. We derived that $D^{gzf}(V_L) = D^{gzf}(\{1, 2\}) = V$, so indeed $\{1, 2\}$ is a generalized zero forcing set. In fact, we have already explored this example in Section 2.4, where we showed that $\{1, 2\}$ is a zero forcing set. The sequence of zero forcings is depicted in Figure 2.2. Furthermore, we see that $\{1, 2\} \to \{3, 4\}$ by using the generalized color change rule, is equivalent to applying the color change rule of zero forcing multiple times. The sequence of node-disjoint zero forcings is given by $2 \to 4, 1 \to 3$.

Since generalized zero forcing is equivalent to zero forcing, this raises the question why we consider generalized zero forcing at all. The reason for that is twofold. On the one hand, the generalized color change rule enables us to color multiple nodes black simultaneously in one step. This might be useful for computational considerations. On the other hand, the concept of perfect matchings that we have discussed in great detail in this chapter, will turn out to be very fruitful to the study of strong structural controllability of systems defined on colored graphs, which is the main problem of this thesis, and which will be discussed extensively in the next chapter.

Chapter 5

Networks on colored graphs

In Chapter 2 we have seen how we can assign a qualitative class of matrices, $\mathcal{Q}(G)$, to a given graph $G = (V, E)$. For any $A \in \mathcal{Q}(G)$, the off-diagonal entry $A_{i,j}$ is nonzero if and only if $(j, i) \in E$. Put differently, with a given graph G, we can associate a set of matrices A whose off-diagonal entries are either fixed zeros (if there is no edge) or a free nonzero parameter (if there is an edge), and its diagonal entries can take any value.

We will now define what we mean by a *colored graph* $G = (V, E, \pi)$. For a given graph $G = (V, E)$, we assign to every edge a *color*. Two edges have the same color, if for every realization $A \in \mathcal{Q}(G)$, the two entries in A corresponding to those two edges are equal. Mathematically, we write $G = (V, E, \pi)$ for a colored graph, where $\pi = \{E_1, E_2, \ldots, E_N\}$ is an edge partition. By that we mean that we introduce a partition of the edge set

$$E = \bigcup_{i=1}^{N} E_i, \quad E_i \cap E_j = \emptyset, \forall i \neq j.$$

The partition consist of N disjoint subsets E_i, and to each subset E_i we assign a color, denoted by the symbol α_i. Thus the symbols $\alpha_1, \alpha_2, \ldots, \alpha_N$ denote the colors. We call the set $\{\alpha_1, \ldots, \alpha_N\}$ the *color palette* of the colored graph. In the sequel we will also use $\alpha_1, \ldots, \alpha_N$ as independent nonzero variables. Two edges from the same subset E_i have the same color α_i. For a given partition π, we consider the following subset of the original qualitative class $\mathcal{Q}(G)$:

$$\mathcal{Q}_\pi(G) = \{A \in \mathcal{Q}(G) \mid A_{i,j} = A_{m,n} \text{ if } (j, i), (n, m) \in E_r \text{ for some } 1 \leq r \leq N\}$$

Let $G = (V, E, \pi)$ be a colored graph and $V_L \subseteq V$ a leader set. Take $A \in \mathcal{Q}_\pi(G)$ and $B = B(V; V_L)$ as in (2.2), then we can check whether the pair (A, B) is controllable by Theorem 2.1, 2.2 or 2.3. With slight abuse of notation we also say that (A, V_L) is controllable if (A, B) is controllable. We say that (G, V_L) is *strongly structurally controllable* if (A, V_L) is controllable for all $A \in \mathcal{Q}_\pi(G)$. If (G, V_L) is strongly structurally controllable, we often simply say that (G, V_L) is *controllable*.

The aim of this chapter is now to find conditions for strongly structurally controllable colored graphs. First of all, let $\mathcal{W}_\pi(G)$ denote the set of colored weighted adjacency matrices:

$$\mathcal{W}_\pi(G) = \{W \in \mathcal{W}(G) \mid W_{i,j} = W_{m,n} \text{ if } (j,i),(n,m) \in E_r \text{ for some } 1 \leq r \leq N\}.$$

For any colored weighted adjacency matrix $W \in \mathcal{W}_\pi(G)$, we consider the following associated family of matrices:

$$\mathcal{Q}_W(G) = \{A \in \mathbb{R}^{n \times n} \mid A_{i,j} = W_{i,j} \text{ for all } i \neq j\}.$$

Then, obviously, we have:

$$\mathcal{Q}_\pi(G) = \bigcup_{W \in \mathcal{W}_\pi(G)} \mathcal{Q}_W(G).$$

Using the above equality and Theorem 2.10, we immediately obtain the following fact:

Fact 5.1. Let $G = (V, E, \pi)$ be a colored graph and let $V_L \subseteq V$ be the leader set. Then, (G, V_L) is controllable if and only if V_L is a balancing set for $(G(W), V_L)$ for all $W \in \mathcal{W}_\pi(G)$.

Note that it is infeasible to verify whether V_L is a balancing set for all weighted graphs $G(W) = (V, E, W)$ with $W \in \mathcal{W}_\pi(G)$. That is why we hope to establish a graph theoretic test to conclude strong structural controllability. To this end, let us study colored bipartite graphs in greater detail now.

5.1 Colored bipartite graphs

Consider a colored bipartite graph $G = (S, T, E, \pi)$ with $S = \{s_1, \ldots, s_m\}$ and $T = \{t_1, \ldots, t_n\}$. Here, as in the above, we have a partition of the edge set, $\pi = \{E_1, \ldots, E_N\}$, and we denote the associated colors by $\alpha_1, \ldots, \alpha_N$. Suppose for the time being that $|S| = |T| = k$. To this bipartite graph we assign a pattern matrix $\mathcal{P}_\pi(G)$ as follows

$$\mathcal{P}_\pi(G) = \{A \in \mathcal{P}(G) \mid A_{j,i} = A_{n,m} \iff \{s_i, t_j\}, \{s_m, t_n\} \in E_r \text{ for some } r \in \{1, \ldots, N\}\}.$$

Clearly $\mathcal{P}_\pi(G) \subseteq \mathcal{P}(G)$, with equality if and only if $|\pi| = N = |E|$, that is, $|E_i| = 1$ for all $1 \leq i \leq N$.

Let $E' \subseteq E$ be any subset of edges. The *spectrum* of an edge set, $\sigma(E')$, is the set of colors (counting multiplicity) of the edges in E', and we write $\sigma(E') = \{\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_{|E'|}}\}$. Here, any color $\alpha_{i_l} \in \sigma(E')$ is an element of the color palette $\{\alpha_1, \ldots, \alpha_N\}$. Note that a color can appear multiple times in a spectrum. Let k_i denote the multiplicity of color α_i in $\sigma(E')$, where k_i is zero if it does not appear in $\sigma(E')$. Then to every spectrum of an edge set, $\sigma(E')$, we can assign a *color function*, which we denote by $\mathbb{X}_{E'}(\alpha_1, \ldots, \alpha_N) = \alpha_1^{k_1} \cdot \ldots \cdot \alpha_N^{k_N}$. This color function is a monomial, and the α_i's represent the independent nonzero variables now.

Consider now the colored bipartite graph $G = (S, T, E, \pi)$, with $|S| = |T| = k$. Suppose G permits a perfect matching, i.e., there exists a k-matching. Let us denote this matching by $\mathcal{M} = \{\{s_1, t_{\tau(1)}\}, \{s_2, t_{\tau(2)}\}, \ldots, \{s_k, t_{\tau(k)}\}\}$. Here, τ denotes a permutation of $(1, 2, \ldots, k)$. For every perfect matching, we can write down its spectrum, $\sigma(\mathcal{M})$, and the color function, $\mathbb{X}_{\mathcal{M}}(\alpha_1, \ldots, \alpha_N)$.

We say that two perfect matchings are *equivalent* if they have the same spectrum (equivalently, if they have the same color function). This gives an equivalence relation on the set of perfect matchings of G. Two perfect matchings $\mathcal{M}_1$ and $\mathcal{M}_2$ are in the same equivalence class $\mathbb{P}_i$ if and only if $\sigma(\mathcal{M}_1) = \sigma(\mathcal{M}_2)$. The spectrum of an equivalence class, $\sigma(\mathbb{P}_i)$, is equal to $\sigma(\mathcal{M})$, for any $\mathcal{M} \in \mathbb{P}_i$. For a colored bipartite graph, we can thus partition the set of all perfect matchings into the disjoint union of equivalence classes. In other words, let $\mathbb{P}$ denote the set of all perfect matchings, then we have

$$\mathbb{P} = \bigcup_{i=1}^{m} \mathbb{P}_i, \qquad \mathbb{P}_i \cap \mathbb{P}_j = \emptyset, \ \forall i \neq j.$$

To each perfect matching $\mathcal{M}$, we assign a sign, $\text{sign}(\mathcal{M})$, which is equal to $(-1)^m$, with m the number of swaps one needs to transform $(\tau(1), \tau(2), \ldots, \tau(k))$ into $(1, 2, \ldots, k)$. We call a perfect matching a positive matching if $\text{sign}(\mathcal{M}) = +1$, and we call it a negative matching if $\text{sign}(\mathcal{M}) = -1$. The *signature* of an equivalence class of perfect matchings, $\text{sgn}(\mathbb{P}_i)$, is equal to the sum of the signs of the matchings in the class, i.e., the number of negative matchings subtracted from the number of positive matchings:

$$\text{sgn}(\mathbb{P}_i) := \sum_{\mathcal{M} \in \mathbb{P}_i} \text{sign}(\mathcal{M}).$$

We are now in a position to state and prove the following result.

Theorem 5.2. *Let $G = (S, T, E, \pi)$ be a colored bipartite graph with $|S| = |T|$. Every $A \in \mathcal{P}_\pi(G)$ is nonsingular if and only if there exists at least one perfect matching, and there is exactly one equivalence class of perfect matchings with nonzero signature.*

Proof. By the Leibniz formula for the determinant, for every realization $A \in \mathcal{P}_\pi(G)$ we have the following expression

$$\det(A) = \sum_{\mathcal{M} \in \mathbb{P}} \text{sgn}(\mathcal{M}) \cdot \prod_{i=1}^{k} A_{\tau(i),i}, \tag{5.1}$$

where $\mathbb{P}$ is the set of all perfect matchings in G. If two perfect matchings are equivalent, we know that they have the same spectrum. Like before, $\sigma(\mathbb{P}_i) = \sigma(\mathcal{M})$, for any $\mathcal{M} \in \mathbb{P}_i$. The color function of an equivalence class is simply the color function of any perfect matching in that equivalence class, and we write $\mathbb{X}_{\mathbb{P}_i}(\alpha_1, \ldots, \alpha_N) = \mathbb{X}_{\mathcal{M}}(\alpha_1, \ldots, \alpha_N) = \alpha_1^{k_1} \cdot \ldots \cdot \alpha_N^{k_N}$. Using this, and the notion of signature of an equivalence class, equation (5.1) becomes

$$\det(A) = \sum_{i=1}^{m} \text{sgn}(\mathbb{P}_i) \cdot \mathbb{X}_{\mathbb{P}_i}(\alpha_1, \ldots, \alpha_N) \tag{5.2}$$

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We will now prove the claim. Let us first prove the "if" direction. Suppose there exists at least one perfect matching and there is exactly one equivalence class of perfect matchings with signature nonzero. This means that all except one equivalence class of perfect matchings has signature zero, and hence the ones corresponding to signature zero will vanish in the expression of the determinant (5.2). Thus, the sum reduces to a single monomial in the nonzero variables $\alpha_1, \ldots, \alpha_N$ which is therefore nonzero.

We now prove the "only if" direction. Suppose every matrix $A \in \mathcal{P}_\pi(G)$ is nonsingular. We have to prove that there exists at least one perfect matching, and there is exactly one equivalence class of perfect matchings with nonzero signature.

Assume, for a contradiction, that no perfect matching exists. Then the sum in expression (5.2) is over the empty set, and hence we find that $\det(A) = 0$ for all $A \in \mathcal{P}_\pi(G)$. This is a contradiction.

Since there exists at least one perfect matching, there also exists at least one equivalence class of perfect matchings. We will now prove that there is exactly one equivalence class with nonzero signature. We do that again by *reductio ad absurdum*. Suppose, for a contradiction, there is not exactly one equivalence class with nonzero signature. We then distinguish two cases.

First, suppose that every equivalence class has zero signature. Then equation (5.2) reads $\det(A) = 0$ for any $A \in \mathcal{P}_\pi(G)$, which is a contradiction.

Secondly, suppose there is more than one equivalence class of perfect matchings with nonzero signature. The equivalence classes are denoted by $\mathbb{P}_1, \mathbb{P}_2, \ldots$, and there are at least two equivalence classes. Since at least two of them have nonzero signature, there exists at least one color that does not occur in $\sigma(\mathbb{P}_1)$ and $\sigma(\mathbb{P}_2)$ with the same multiplicity. Without loss of generality we denote this color by α_1 and we let the multiplicities be k_1 and j_1, respectively. The formula of the determinant, equation (5.2), becomes

$$\det(A) = \sum_{i=1}^m \operatorname{sgn}(\mathbb{P}_i) \cdot \mathbb{X}_{\mathbb{P}_i}(\alpha_1, \ldots, \alpha_N)$$

$$= \operatorname{sgn}(\mathbb{P}_1) \mathbb{X}_{\mathbb{P}_1}(\alpha_1, ..., \alpha_N) + \operatorname{sgn}(\mathbb{P}_2) \mathbb{X}_{\mathbb{P}_2}(\alpha_1, ..., \alpha_N) + \sum_{i=3}^m \operatorname{sgn}(\mathbb{P}_i) \mathbb{X}_{\mathbb{P}_i}(\alpha_1, ..., \alpha_N)$$

$$= \operatorname{sgn}(\mathbb{P}_1) \cdot \alpha_1^{k_1} \cdot \alpha_2^{k_2} \cdot \ldots \cdot \alpha_N^{k_N} + \operatorname{sgn}(\mathbb{P}_2) \cdot \alpha_1^{j_1} \cdot \alpha_2^{j_2} \cdot \ldots \cdot \alpha_N^{j_N} + f(\alpha_1)$$

$$= \operatorname{sgn}(\mathbb{P}_1) \cdot \alpha_1^{k_1} \cdot a_1 + \operatorname{sgn}(\mathbb{P}_2) \cdot \alpha_1^{j_1} \cdot a_2 + f(\alpha_1), \tag{5.3}$$

where both $a_1 = \alpha_2^{k_2} \cdot \ldots \cdot \alpha_N^{k_N}$ and $a_2 = \alpha_2^{j_2} \cdot \ldots \cdot \alpha_N^{j_N}$ are monomials that depend on $\alpha_2, \ldots, \alpha_N$ and $f(\alpha_1)$ is a polynomial in α_1. This polynomial $f(\alpha_1)$ corresponds to the remaining equivalence classes. It can happen that there exist spectra of equivalence classes with nonzero signature that also contain color α_1 with multiplicity k_1 or j_1. Then, in expression (5.3), we

move those corresponding monomials to the first two terms, and (5.3) becomes

$$\det(A) = b_1 \cdot \alpha_1^{k_1} + b_2 \cdot \alpha_1^{j_1} + f'(\alpha_1), \tag{5.4}$$

where b_1 and b_2 depend on $\alpha_2, \ldots, \alpha_N$. The first term in (5.4) now corresponds to all equivalence classes that have color α_1 with exactly multiplicity k_1 in their spectrum. The second term in (5.4) corresponds to all equivalence classes that have color α_1 with exactly multiplicity j_1 in their spectrum. Finally, the remaining polynomial $f'(\alpha_1)$ does not contain monomials with $\alpha_1^{k_1}$ or $\alpha_1^{j_1}$ anymore. It is easily see that we can always choose nonzero complex values of $\alpha_2, \ldots, \alpha_N$ such that both b_1 and b_2 are nonzero. For this particular choice of $\alpha_2, \ldots, \alpha_N$, the expression (5.4) can be viewed as a complex polynomial in the variable α_1. By the Fundamental Theorem of Algebra, the equation $b_1 \cdot \alpha_1^{k_1} + b_2 \cdot \alpha_1^{j_1} + f'(\alpha_1) = 0$ has always at least one nonzero root. Hence, there is a choice of $\alpha_1, \alpha_2, \ldots, \alpha_N$ such that (5.2) equals zero. This is a contradiction. $\qquad\square$

Note that, if we only allow real matrices $A \in \mathcal{P}_\pi(G)$, Theorem 5.2 still gives a sufficient condition for nonsingularity. That is, if there exists at least one perfect matching and exactly one equivalence class of perfect matchings has nonzero signature, any real matrix $A \in \mathcal{P}_\pi(G)$ is nonsingular.

5.2   A first look at a colored color change rule

Consider a colored graph $G = (V, E, \pi)$, where the partition of the edge set is given by $\pi = \{E_1, E_2, \ldots, E_N\}$. Recall that to every subset E_i a color α_i is assigned. Consider now any two nonempty disjoint subsets of nodes $X, Y \subset V$. Associated with this, we have a colored bipartite graph $(X, Y, E_{X,Y}, \pi_{X,Y})$, where $E_{X,Y} = \{\{i, j\} \mid (i, j) \in E, i \in X, j \in Y\}$. Here, $\pi_{X,Y}$ denotes the partition of the edges in $E_{X,Y}$. We define

$$E_{X,Y}^r = \{\{i, j\} \in E_{X,Y} \mid (i, j) \in E_i\}, \quad 1 \le r \le N$$

Note that for some r, the edge set $E_{X,Y}^r$ might be the empty set. Removing these empty sets, the partition is now given by $\pi_{X,Y} = \{E_{X,Y}^{h_1}, E_{X,Y}^{h_2}, \ldots, E_{X,Y}^{h_l}\}$, where $l \le N$. The associated colors are now $\alpha_{h_1}, \alpha_{h_2}, \ldots, \alpha_{h_l}$. Without loss of generality, we relabel $\alpha_{h_1}, \alpha_{h_2}, \ldots, \alpha_{h_l}$ as $\alpha_1, \alpha_2, \ldots, \alpha_l$. Edges in $E_{X,Y}^r$ have now color α_r.

Suppose $|X| = |Y| = k$. We say that X and Y are *color perfect neighbors* if there exists a perfect matching between X and Y, and exactly one equivalence class of perfect matchings has nonzero signature.

We now color each node in V either black or white. Let $C \subseteq V$ denote the set of black nodes, initially. The white nodes are $V \setminus C$. A color change rule, once applied, enables us to color some white nodes black. Based on Chapter 4, one would expect the following color change rule. (The first adjective 'colored' refers to the fact that we are dealing with a colored graph.)

Wrong colored color change rule: Let $C \subseteq V$ be the set of black nodes, and let the white nodes be $V \setminus C$. For some $X \subseteq C$ and $Y \subseteq V \setminus C$, we say that X forces Y, denoted as $X \to Y$, if Y is the only white color perfect neighbor of X.

Let $G = (V, E, \pi)$ be a colored graph and let $C \subseteq V$ be the set of nodes initially colored black. Apply the above color change rule to this set of black nodes repeatedly, until no more changes are possible. The obtained set of nodes is the *derived set*, denoted as $D^w(C)$. (The superscript 'w' stands for 'wrong' or 'worthless', since this color change rule will turn out to be of no use to determine controllability.)

For controllability of the colored graph $G = (V, E, \pi)$ with given leader set $V_L \subseteq V$, we would like to find a necessary and sufficient condition. Based on the previous chapters, we expect that this condition is $D^w(V_L) = V$.

It turns out that the fact that the derived set, using the above color change rule, of V_L is equal to the entire node set V, says nothing about strong structural controllability. (This is because the motivation behind this color change rule is completely lost, i.e., the relationship with zero forcing or zero extension is gone. See also Section 6.1)

We will now show that, indeed, if $D^w(V_L) = V$, we can not conclude controllability of the colored network. In other words, $D^w(V_L) = V$ is not a sufficient condition for controllability. For this, consider the following counterexample.

We consider the network depicted in Figure 5.1, a colored network representing 7 nodes with leader set $V_L = \{1, 2, 3\}$.
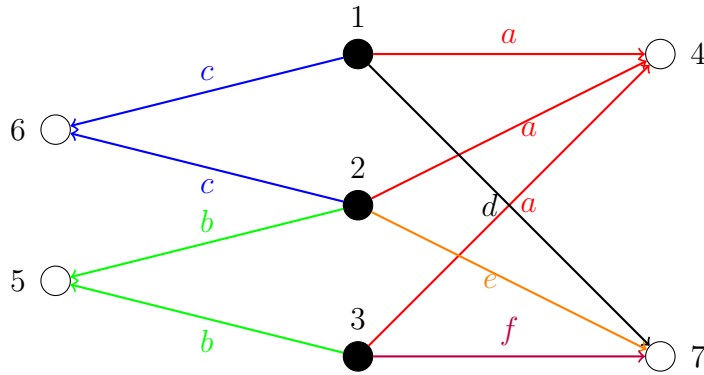

Figure 5.1: An example to show that the wrong colored color change rule does not give a sufficient condition for controllability.

Again the input matrix is $B = B(V; V_L)$. The system matrices $A \in \mathcal{Q}_\pi(G)$ are of the form:

$$A = \begin{bmatrix} \xi_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \xi_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \xi_3 & 0 & 0 & 0 & 0 \\ a & a & a & \xi_4 & 0 & 0 & 0 \\ 0 & b & b & 0 & \xi_5 & 0 & 0 \\ c & c & 0 & 0 & 0 & \xi_6 & 0 \\ d & e & f & 0 & 0 & 0 & \xi_7 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where a, b, c, d, e, f are nonzero free parameters, and the ξ_i's can take any real value. We will now show that $\{4, 5, 6\}$ is a color perfect neighbor of $\{1, 2, 3\}$. To do so, we consider the associated colored bipartite graph $(X, Y, E_{X,Y}, \pi_{X,Y})$ with $X = \{1, 2, 3\}$ and $Y = \{4, 5, 6\}$. There are three perfect matchings in this bipartite graph, which are depicted in Figure 5.2. We also computed the spectrum and signature of every perfect matching, see Table 5.1.

	3-matching	spectrum	signature
$\mathcal{M}_1$	$\{(1, 4), (2, 6), (3, 5)\}$	$\{a, c, b\}$	-1
$\mathcal{M}_2$	$\{(1, 6), (2, 4), (3, 5)\}$	$\{c, a, b\}$	$+1$
$\mathcal{M}_3$	$\{(1, 6), (2, 5), (3, 4)\}$	$\{c, b, a\}$	-1

Table 5.1: Perfect matchings between $X = \{1, 2, 3\}$ and $Y = \{4, 5, 6\}$ in the colored bipartite graph $(X, Y, E_{X,Y}, \pi_{X,Y})$.

We see that all three perfect matchings are equivalent, since they have the same spectrum, and this equivalence class has nonzero signature. By definition then, $\{4, 5, 6\}$ is a color perfect neighbor of $\{1, 2, 3\}$.

Equivalently, we could have considered the following submatrix and its determinant:

$$\det(A_{\{4,5,6\},\{1,2,3\}}) = \det\left(\begin{bmatrix} a & a & a \\ 0 & b & b \\ c & c & 0 \end{bmatrix} \right) = -abc + abc - abc = -abc.$$

What this computation shows, is that the submatrix $A_{\{4,5,6\},\{1,2,3\}}$ is nonsingular for any choice of nonzero variables a, b, c. By Theorem 5.2 we can then also conclude that $\{1, 2, 3\}$ and $\{4, 5, 6\}$ are color perfect neighbors.
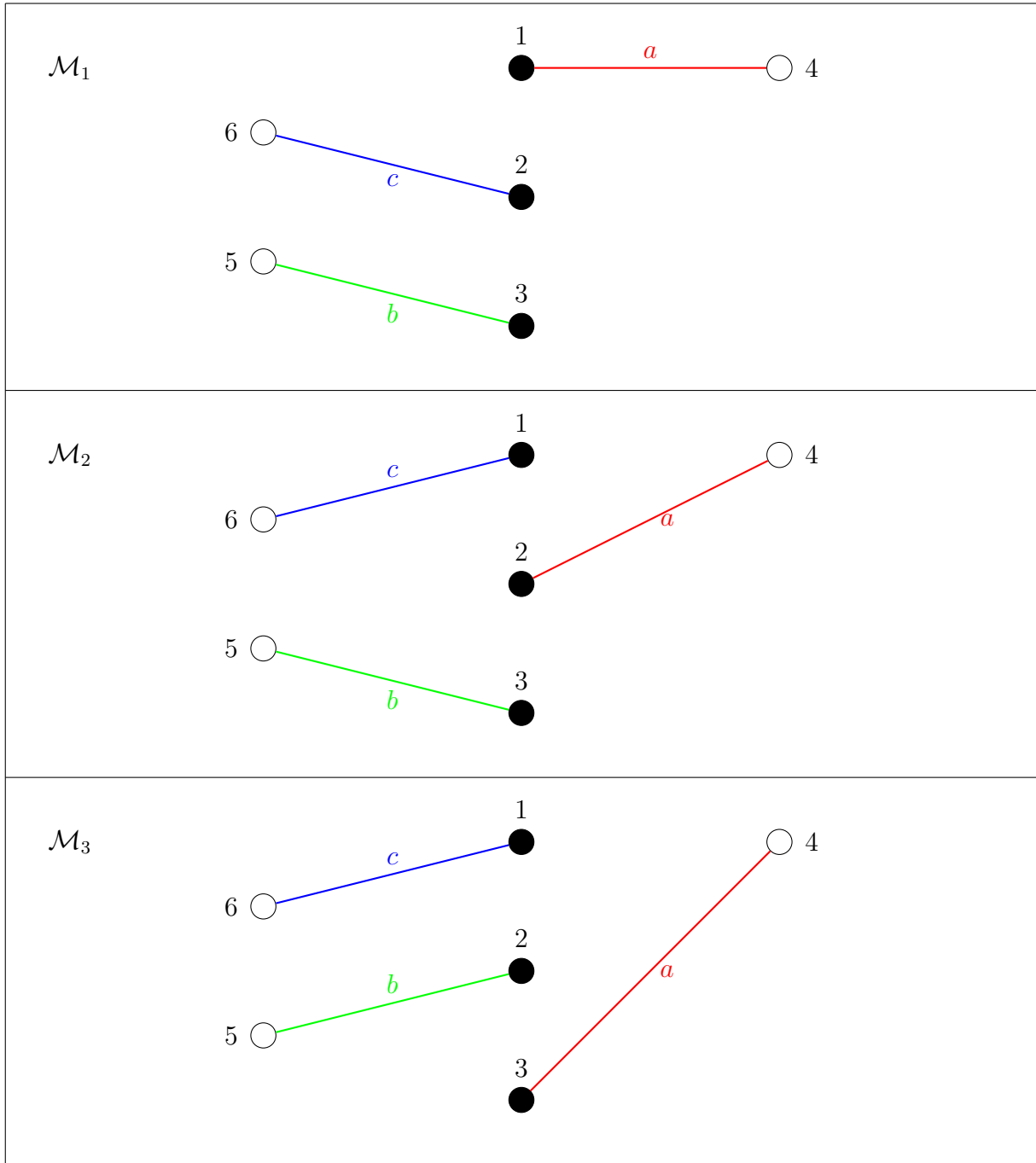
Figure 5.2: All three color perfect matchings in the colored bipartite graph $(X, Y, E_{X,Y}, \pi_{X,Y})$ with $X = \{1, 2, 3\}$ and $Y = \{4, 5, 6\}$, see also Table 5.1.

We will now show that $\{4,5,6\}$ is the only color perfect neighbor of $\{1,2,3\}$, by showing that $\{4,5,7\}$, $\{4,6,7\}$, $\{5,6,7\}$ are not color perfect neighbors of $\{1,2,3\}$. For example, to show that $\{4,5,7\}$ is not a color perfect neighbor of $\{1,2,3\}$, we consider again all perfect matchings between these two node sets. This is done in Table 5.2.

	3-matching	spectrum	signature
$\mathcal{M}_1$	$\{(1,4),(2,5),(3,7)\}$	$\{a,b,f\}$	$+1$
$\mathcal{M}_2$	$\{(1,4),(2,7),(3,5)\}$	$\{a,e,b\}$	-1
$\mathcal{M}_3$	$\{(1,7),(2,4),(3,5)\}$	$\{d,a,b\}$	$+1$
$\mathcal{M}_4$	$\{(1,7),(2,5),(3,4)\}$	$\{d,b,a\}$	-1

Table 5.2: Perfect matchings between $X = \{1,2,3\}$ and $Y = \{4,5,7\}$ in the associated colored bipartite graph $(X, Y, E_{X,Y}, \pi)$.

Equivalently, one could compute the following determinant:

$$\det(A_{\{4,5,7\},\{1,2,3\}}) = \det\left(\begin{bmatrix} a & a & a \\ 0 & b & b \\ d & e & f \end{bmatrix}\right) = abf - abe + abd - abd = abf - abe.$$

In either case, we see that there are three equivalence classes, but two of them have nonzero signature. So $\{4,5,7\}$ and $\{1,2,3\}$ are not color perfect neighbors.

To show that $\{4,6,7\}$ and $\{5,6,7\}$ are not color perfect neighbors, we compute:

$$\det(A_{\{4,6,7\},\{1,2,3\}}) = \det\left(\begin{bmatrix} a & a & a \\ c & c & 0 \\ d & e & f \end{bmatrix}\right) = acf - acf + ace - acd = ace - acd,$$

$$\det(A_{\{5,6,7\},\{1,2,3\}}) = \det\left(\begin{bmatrix} 0 & b & b \\ c & c & 0 \\ d & e & f \end{bmatrix}\right) = -bcf + bce - bcd.$$

Since there are always at least two equivalence classes with nonzero signature, none of these neighbor sets are color perfect neighbors.

Hence, by the above colored color change rule, we color $\{4,5,6\}$ black, since it is the only white color perfect neighbor of $\{1,2,3\}$. After that, we see that $\{7\}$ is the only white color perfect neighbor of $\{3\}$, so we color it black. We see that $D^w(V_L) = D^w(\{1,2,3\}) = V$.

However, the network $G = (V, E, \pi)$ with leader set $V_L = \{1, 2, 3\}$ is not strongly structurally controllable. To see this, we take the following realization $A \in \mathcal{Q}_\pi(G)$:

$$
A = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0
\end{bmatrix}.
$$

Then we compute:

$$
B = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad
AB = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 1 & 1 \\
0 & 1 & 1 \\
1 & 1 & 0 \\
1 & 1 & 1
\end{bmatrix}, \quad
A^2 B = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad A^3 B = A^4 B = A^5 B = A^6 B = 0.
$$

Clearly, $\mathrm{rank}([B, AB, A^2 B, \ldots, A^6 B]) \neq 7$ since there are only six nonzero columns. By Theorem 2.1 then, we conclude that (A, V_L) is not controllable. Since we found a realization $A \in \mathcal{Q}_\pi(G)$ for which the pair (A, V_L) is not controllable, we conclude that (G, V_L) is not controllable.

In fact, one can also show that this color change rule does not give a necessary condition for controllability (so it really is a useless rule!). More specific, if (A, V_L) is controllable for any $A \in \mathcal{Q}_\pi(G)$, we can not conclude that $D^w(V_L) = V$. We will now give an example of this.

Let any $A \in \mathcal{Q}_\pi(G)$ be of the following form:

$$
A = \begin{bmatrix}
\xi_1 & 0 & 0 & 0 & 0 \\
0 & \xi_2 & 0 & 0 & 0 \\
a & a & \xi_3 & 0 & 0 \\
a & a & 0 & \xi_4 & b \\
0 & b & 0 & b & \xi_5
\end{bmatrix}, B = \begin{bmatrix}
1 & 0 \\
0 & 1 \\
0 & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix}.
$$

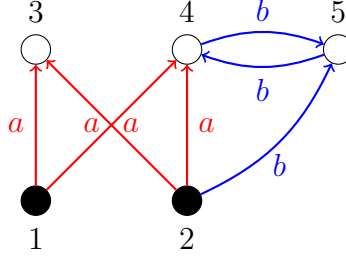The corresponding network is depicted in Figure 5.3.

Figure 5.3: An example to show that the wrong colored color change rule does not give a necessary condition for controllability.

We will now show that (G, V_L) is controllable, by using Fact 5.1. To that end, consider an arbitrary weighted graph $G(W) = (V, E, W)$ for any $W \in \mathcal{W}_\pi(G)$. We will now show that $V_L = \{1, 2\}$ is a balancing set for this graph. To do so, we assign a variable x_i to every node $i \in V$, and we set $x_i = 0$ initially for all $i \in V_L$. The balance equations for the two zero nodes 1 and 2 now read:

$$ax_3 + ax_4 = 0$$
$$ax_3 + ax_4 + bx_5 = 0,$$

and solving this homogeneous system of equations, we see immediately that $x_5 = 0$. So, we add node 5 to our zero nodes. The zero nodes are now $1, 2$ and 5. The balance equation for zero node 5 reads

$$bx_4 = 0,$$

and hence $x_4 = 0$ as well. The zero nodes are now $1, 2, 4$ and 5. Writing down the balance equation for node 1, we get

$$ax_3 = 0,$$

which obviously implies that $x_3 = 0$. We see that the derived set is V, and since we took $W \in \mathcal{W}_\pi(G)$ arbitrary, we see that $V_L = \{1, 2\}$ is a balancing set for any $G(W) = (V, E, W)$ with $W \in \mathcal{W}_\pi(G)$. By Fact 5.1, we can now conclude that (G, V_L) is controllable, as desired.

However, the derived set of $V_L = \{1, 2\}$ using the wrong colored color change rule, is unequal to V. To see this, note that $\{1\}$ has two white color perfect neighbors, namely $\{3\}$ and $\{4\}$. The black node $\{2\}$ has three white color perfect neighbors, namely $\{3\}$, $\{4\}$ and $\{5\}$. The black set of nodes $\{1, 2\}$ has also more than one white color perfect neighbor, since both $\{3, 5\}$ and $\{4, 5\}$ are white color perfect neighbors of $\{1, 2\}$. To see this, we consider the associated colored bipartite graphs depicted in Figure 5.4.
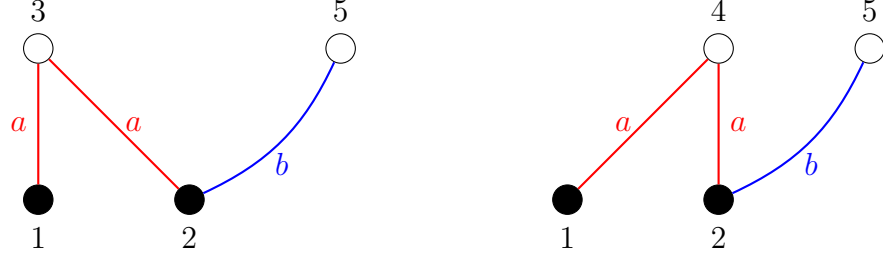
Figure 5.4: Two color perfect neighbors of $V_L = \{1, 2\}$.

We see that in both bipartite graphs there is exactly one perfect matching between $\{1, 2\}$ and the white neighbor set, so indeed $\{3, 5\}$ and $\{4, 5\}$ are both white color perfect neighbors of $\{1, 2\}$. In either case, we have shown that there does not exist a subset of black nodes $X \subseteq \{1, 2\}$ that forces any white set of nodes. We derived $D^w(V_L) = \{1, 2\} \neq V$, and hence the condition $D^w(V_L) = V$ is not a necessary condition for controllability of (G, V_L).

We showed that the condition that the derived set (using the wrong colored color change rule) of V_L is equal to the entire node set V, is neither a sufficient nor a necessary condition for controllability of (G, V_L). In the next section, we will discuss a color change rule that does give a sufficient condition for controllability of (G, V_L).

5.3   Color generalized color change rule

Consider again a colored network $G = (V, E, \pi)$. Again, we color each node either black or white. Initially, let $C \subseteq V$ denote the subset of black nodes, and let $V \setminus C$ be white. The color change rule we consider is now the following:

Color generalized color change rule: Let $X \subseteq C$ be a subset of black nodes. If Y is the only white neighbor set of X, and X and Y are color perfect neighbors, we color Y black. We say that X forces Y and we write $X \to Y$.

Given is a colored graph $G = (V, E, \pi)$. Initially, let $C \subseteq V$ denote the set of black nodes, and let $V \setminus C$ be white. The *derived set* $D^{cgzf}(C)$ is the set of black nodes that we obtain by repeated application of this color generalized color change rule, until no more changes are possible. We call a set $C \subseteq V$ a *color generalized zero forcing set* (CGZFS) if $D^{cgzf}(C) = V$. We now state the main theorem.

Theorem 5.3. *Let $G = (V, E, \pi)$ be a colored graph and $V_L \subseteq V$ be a leader set. Then, (G, V_L) is controllable if V_L is a color generalized zero forcing set.*

Proof. We know that V_L is a color generalized zero forcing set. Hence, there exists a sequence of forcings

$$X_1 \to Y_1, X_2 \to Y_2, \ldots, X_w \to Y_w, \tag{5.5}$$

with the following properties. Each X_i is a subset of black nodes and each Y_i is a subset of white nodes. For the first step we have $X_1 \subseteq V_L$ and $Y_1 \subseteq V \setminus V_L$. Since X_1 forces Y_1 to become black, we have in the second step of (5.5) that $X_2 \subseteq V_L \cup Y_1$, and so on. In general we have $X_i \subseteq V_L \cup \bigcup_{j=1}^{i-1} Y_j$. Finally, since V_L is a color generalized zero forcing set, we have $V_L \cup \bigcup_{j=1}^{w} Y_j = V$.

We will now prove that (G, V_L) is controllable. To do so, consider any $A \in \mathcal{Q}_\pi(G)$. We claim that V_L is also a balancing set, in the sense that the sequence of forces (5.5) also holds if one were to consider the process of zero extension (Section 2.5). By Fact 5.1 we can then conclude controllability.

Consider one step in (5.5), i.e., $X_f \to Y_f$. Since X_f forces Y_f we have that X_f and Y_f are color perfect neighbors, and we have that the set of white neighbors of X_f is precisely the set Y_f. Consider now the process of zero extension. We assign to each node i a variable x_i, and initially we set $x_i = 0$ if i is a black node. For a black node $i \in X_f$ we have the following balance equation:

$$\sum_{j \in Y_f} A_{j,i} x_j = 0,$$

because Y_f is the set of white neighbors (we do not care about black neighbors, since the corresponding values of the variable are already zero). The above equation can be rewritten as

$$x_{Y_f}^T A_{Y_f, \{v_i\}} = 0,$$

where $A_{Y_f, \{v_i\}}$ is a submatrix of A, and it is a single column matrix whose column index corresponds to node i. The vector x_{Y_f} is the vector consisting of all the variables x_j, corresponding to nodes $j \in Y_f$. If we consider the balance equations for all black nodes $i \in X_f$, we obtain the following system of balance equations:

$$x_{Y_f}^T A_{Y_f, X_f} = 0.$$

Recall that X_f and Y_f are color perfect neighbors, so in particular we have that in the associated colored bipartite graph $(X_f, Y_f, E_{X_f, Y_f}, \pi_{X_f, Y_f})$ there exists a perfect matching and there is exactly one equivalence class with nonzero signature. Then, by Theorem 5.2, we conclude that the matrix A_{Y_f, X_f} is invertible. The above homogeneous system of balance equations then implies that $x_{Y_f} = 0$, which means that we have indeed $X_f \to Y_f$ according to zero extension. Since we considered an arbitrary step in the sequence of forcings (5.5), we can conclude by using Theorem 2.7 that (A, V_L) is controllable. Because we took $A \in \mathcal{Q}_\pi(G)$ arbitrary, it follows that (G, V_L) is controllable by Fact 5.1, which completes the proof. $\quad\square$

We will now look at an example and study how the color generalized color change rule can be applied. Consider the colored graph $G = (V, E, \pi)$ depicted in the figure below, which is equal to Figure 3.2. The leader set is $V_L = \{1, 2, 3\}$.
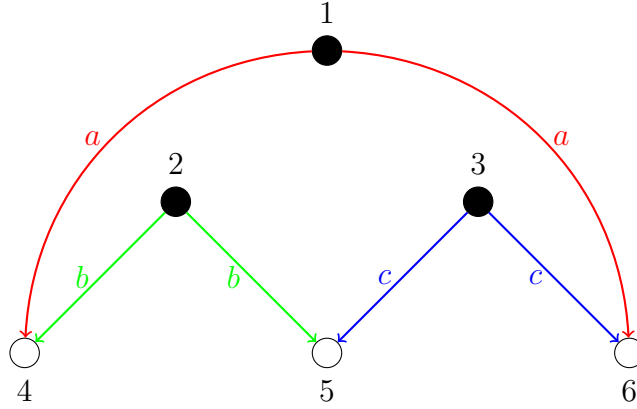

Figure 5.5: The graph of Figure 3.2.

We already showed in Chapter 3 that (G, V_L) is controllable. We will now show that this also follows from Theorem 5.3. To do so, we will apply the color generalized color change rule. Initially, $V_L = \{1, 2, 3\}$ is the set of black nodes and $V \setminus V_L = \{4, 5, 6\}$ is the set of white nodes. To apply this color change rule, we need to take a set of black nodes X and look at the white neighbor set of X, denoted by Y. If X and Y are color perfect neighbors, we color the nodes in Y black. Note that in order to be color perfect neighbors, the cardinalities of X and Y must be the same. In other words, if the cardinalities of X and Y differ we can already conclude that X and Y are not color perfect neighbors. Suppose the subset of black nodes is $\{1\}$. Since the white neighbor set of $\{1\}$ is equal to $\{4, 6\}$, which has a different cardinality, they can not be color perfect neighbors. Likeso, if we let our subset of black nodes equal $\{2\}$, the white neighbor set is given by $\{4, 5\}$ and these sets are also not color perfect neighbors. If we let $X = \{3\}$ we can also not color any white nodes black. Let us now consider subsets of black nodes with cardinality 2. Let X be $\{1, 2\}$, $\{1, 3\}$ or $\{2, 3\}$. In either case we see that the white neighbor set is equal to $\{4, 5, 6\}$. This set has a different cardinality so they can not be color perfect neighbors. Finally, let now $X = \{1, 2, 3\}$. The white neighbor set is then given by $Y = \{4, 5, 6\}$. These sets have the same cardinality so we look at the associated colored bipartite graph $(X, Y, E_{X,Y}, \pi_{X,Y})$ which is depicted in the figure below.
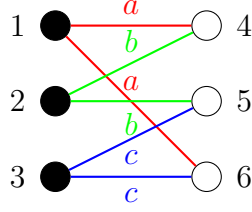
Figure 5.6: The colored bipartite graph $(X, Y, E_{X,Y}, \pi_{X,Y})$ with $X = \{1, 2, 3\}$ and $Y = \{4, 5, 6\}$.

There are two perfect matchings in this colored bipartite graph, depicted in Figure 5.7. The first perfect matching is given by $\{\{1, 4\}, \{2, 5\}, \{3, 6\}\}$. The spectrum is $\{a, b, c\}$ and the sign of this perfect matching is $(-1)^0 = +1$, since there are zero transpositions needed to transform $(4, 5, 6)$ to $(4, 5, 6)$. The second perfect matching is given by $\{\{1, 6\}, \{2, 4\}, \{3, 5\}\}$. The spectrum of this perfect matching is $\{a, b, c\}$. The sign is $(-1)^2 = +1$, since there are 2 transpositions needed to transform $(6, 4, 5)$ to $(4, 5, 6)$.
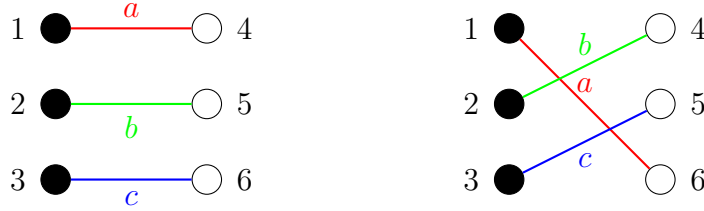

Figure 5.7: The two perfect matchings in the bipartite graph $(X, Y, E_{X,Y}, \pi_{X,Y})$ with $X = \{1, 2, 3\}$ and $Y = \{4, 5, 6\}$, depicted in Figure 5.6.

Since both spectra are $\{a, b, c\}$, the two perfect matchings are equivalent. The signs are both $+1$. We see that there is only one equivalence class of perfect matchings. The signature of this equivalence class is 2. Hence, X and Y are color perfect neighbors by definition. By the color generalized color change rule, we color the nodes in $Y = \{4, 5, 6\}$ black. We see that $D^{cgzf}(V_L) = V$, so V_L is a color generalized zero forcing set. By Theorem 5.3, we can now conclude that (G, V_L) is controllable, as desired.

This color generalized color change rule does however not give a necessary condition for controllability. To see this, we consider the following example. Let any $A \in \mathcal{Q}_\pi(G)$ be of the following form:

$$A = \begin{bmatrix} \xi_1 & 0 & 0 & 0 & 0 \\ 0 & \xi_2 & 0 & 0 & 0 \\ a & a & \xi_3 & 0 & 0 \\ a & a & 0 & \xi_4 & b \\ 0 & b & 0 & b & \xi_5 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

The corresponding network is depicted in Figure 5.3. We have already showed on page 37 that (G, V_L) is controllable.

However, $\{1, 2\}$ is not a color generalized zero forcing set. To see this, note that there does not exist a black node that has exactly one white out-neighbor. Also the set $\{1, 2\}$ has a white neighbor set of cardinality 3, and hence can not be a color perfect neighbor. We see that there does not exists a subset of black nodes $X \subseteq V_L = \{1, 2\}$ that forces any white set of nodes, so $D^{cgzf}(V_L) = \{1, 2\} \neq V$. Hence, $D^{cgzf}(V_L) = V$ is not a necessary condition for controllability.

Chapter 6

Colored zero extension

In the previous chapter, we found a sufficient condition for controllability of systems defined on a colored graph $G = (V, E, \pi)$ with leader set $V_L \subseteq V$. We showed that if V_L is a color generalized zero forcing set, i.e., $D^{cgzf}(C) = V$, we can conclude that (G, V_L) is controllable. The color generalized color change rule we studied to compute $D^{cgzf}(C)$ involved the notion of color perfect neighbor.

Furthermore we showed that a colored color change rule based on color perfect neighbors without taking into account whether a white set is the only neighbor set of a black set, was not of any use to conclude controllability of (G, V_L). We referred to that rule as the wrong colored color change rule. The reason that this rule is worthless, is because the mathematics behind zero extension is lost. To make this more precise, we will study the concept of zero extension in more detail in this chapter, and eventually we will find another sufficient condition for controllability of (G, V_L).

6.1 A colored color change rule using zero extension

We first explain the motivation behind the process that we will later call colored zero extension. Let $G = (V, E, \pi)$ be a colored graph. Again we color each node either black or white, and initially let $C \subseteq V$ be the set of black nodes, and let $V \setminus C$ be the set of white nodes. For any realization $A \in \mathcal{Q}_\pi(G)$ we partition A as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where the first block row corresponds to nodes in C, and the second block row corresponds to nodes in $V \setminus C$. The indices of the columns are in the same order as the indices of the rows. In this way, $A_{11} \in \mathbb{R}^{|C| \times |C|}$ and $A_{22} \in \mathbb{R}^{(n-|C|) \times (n-|C|)}$.

We now assign to each node $i \in V$ a variable x_i, and we set $x_i = 0$ if i is a black node. The values of the white nodes are not known a priori. We now consider the homogeneous system of equations $x^T A = 0$, with $x^T = (x_1, \ldots, x_n)$. We partition $x^T = (x_C^T, x_{V \setminus C}^T)$, such that the first set of variables x_i of x correspond to the black nodes in C, and the second set of variables x_j correspond to the set of white nodes in $V \setminus C$. This linear system of n equations then reads:

$$x^T A = \begin{pmatrix} x_C^T, & x_{V \setminus C}^T \end{pmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = 0.$$

In particular, this yields:

$$x_C^T A_{11} + x_{V \setminus C}^T A_{21} = 0, \tag{6.1}$$

where $x_C \in \mathbb{R}^{|C|}$ corresponds to black nodes and $x_{V \setminus C} \in \mathbb{R}^{n - |C|}$ corresponds to the white nodes. Note that we set $x_C = 0$, so (6.1) becomes:

$$x_{V \setminus C}^T A_{21} = 0. \tag{6.2}$$

Note that (6.2) is a homogeneous system of $|C|$ linear equations in $n - |C|$ unknowns. Furthermore, recall that A_{21} is the submatrix whose entries correspond to edges going from C to $V \setminus C$, since the rows of A_{21} are indexed according to nodes in $V \setminus C$ and the columns of A_{21} are indexed according to nodes in C. If we consider a node $j \in V \setminus C$ and the corresponding row in the matrix A_{21}, we see that this row is nonzero if and only if $j \in N_{V \setminus C}(C)$. Equivalently, this row, corresponding to the node $j \in V \setminus C$, is a zero row if and only if j does not have a mother node i in C. Since zero rows do not add any constraints to the solution set of the homogeneous system of equations (6.2), this system can be simplified to:

$$x_Y^T A_{Y,C} = 0, \tag{6.3}$$

where C is the set of black nodes, and $Y = N_{V \setminus C}(C)$ is the set of white neighbors of C. Note that (6.3) are precisely the balance equations we studied in Section 2.5.

The balance equations (6.3) give rise to a color change rule. For a given realization $A \in \mathcal{Q}_\pi(G)$ and set of black nodes C, let $Y = N_{V \setminus C}(C)$ and consider the equations $x_Y^T A_{Y,C} = 0$. If these equations imply that $x_j = 0$ for all $x_j \in Y' \subseteq Y$, we say that the nodes Y' are forced to become black for this realization A. Recall that this is ordinary zero extension, see also Section 2.5. Instead of looking at the balance equations corresponding to a single realization, we now consider all systems of balance equations for every $A \in \mathcal{Q}_\pi(G)$. This leads to the notion of *color generalized zero extension*, and the colored color change rule we consider is the following. (The first adjective 'colored' refers to the fact that we consider a colored graph $G = (V, E, \pi)$.)

Colored color change rule: Let $G = (V, E, \pi)$ be a colored graph and let $C \subseteq V$ be the set of black nodes. Let $Y = N_{V \setminus C}(X)$. For a given realization $A \in \mathcal{Q}_\pi(G)$, suppose $x_Y^T A_{Y,C} = 0$ implies $x_{Y'} = 0$ for some $Y' \subseteq Y$ (note that Y' depends on the given realization). The set of black nodes C is then updated in the following way

$$C^{(\text{new})} = C^{(\text{old})} \cup \bigcap_{A \in \mathcal{Q}_\pi(G)} Y'.$$

This colored color change rule also leads to the notion of *derived set*. The derived set, $D^{cgze}(C)$, is the set of black nodes obtained after repeated application of the colored color change rule, until no more changes are possible. If $D^{cgze}(C) = V$, we call C a *color generalized balancing set* (CGBS). We state the following theorem:

Theorem 6.1. *Let $G = (V, E, \pi)$ be a colored graph and $V_L \subseteq V$ be a leader set. Then, (G, V_L) is controllable if V_L is a color generalized balancing set.*

Proof. Suppose V_L is a color generalized balancing set, so $D^{cgze}(V_L) = V$. Then there exists a sequence of updates

$$
\begin{aligned}
C^0 &= V_L; \\
C^1 &= V_L \cup Y_1; \\
C^2 &= V_L \cup Y_1 \cup Y_2; \\
&\vdots \\
C^k &= V_L \cup Y_1 \cup \ldots \cup Y_k;
\end{aligned}
$$

such that Y_i denotes the set of white nodes forced to become black in the ith step, using the colored color change rule. Finally, since V_L is a color generalized balancing set, we have that $C^k = V$, for some k.

Consider now any realization $A \in \mathcal{Q}_\pi(G)$. Then, V_L is also a balancing set for this particular realization A, by using the same sequence of forcings. To see this, note that in the first step $C^0 = V_L$ are the black nodes, and we consider the corresponding balance equations. For this particular choice of A, these equations imply that some set, let us say $Y_1' \subseteq V \setminus C^0$, becomes black. We know now that by the colored color change rule explained above, we must have $Y_1 \subseteq Y_1'$. More explicitly, we know that at least the nodes in Y_1 are forced to become black. There might be more nodes that are forced to become black, since the set $Y_1' \setminus Y_1$ can be nonempty, however, for the time being we neglect those nodes. So, for this particular choice of A, the zero nodes are now extended to $V_L \cup Y_1$. We can now write down the balance equations corresponding to the black nodes $V_L \cup Y_1$, and for this particular choice of A, these equations imply that some subset of white nodes, let us say Y_2', are forced to become black. Note that $Y_2 \subseteq Y_2'$. Again, we only color the nodes in Y_2 black now, and we ignore the nodes in $Y_2' \setminus Y_2$. Repeating this argument, we see that the sequence of forcings also yield that $D^{ze}(V_L) = V$, for this particular choice of A, and hence (A, V_L) is controllable by Theorem 2.10. Since we chose $A \in \mathcal{Q}_\pi(G)$ arbitrary, we see that (G, V_L) is controllable by Fact 5.1 which completes the theorem. $\qquad\square$
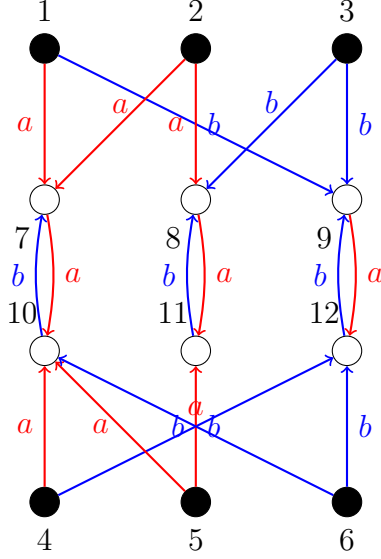
Figure 6.1: A counterexample to the necessity of the colored color change rule

We will now show that V_L being a color generalized balancing set is only a sufficient condition for controllability of (G, V_L). To show that it is not a necessary condition, we consider the graph depicted in Figure 6.1. Let the qualitative class be given by all matrices of the form:

$$A = \left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array} \right) = \left(\begin{array}{cccccc|cccccc} \xi_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \xi_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \xi_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \xi_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \xi_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \xi_6 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline a & a & 0 & 0 & 0 & 0 & \xi_7 & 0 & 0 & b & 0 & 0 \\ 0 & a & b & 0 & 0 & 0 & 0 & \xi_8 & 0 & 0 & b & 0 \\ b & 0 & b & 0 & 0 & 0 & 0 & 0 & \xi_9 & 0 & 0 & b \\ 0 & 0 & 0 & a & a & b & a & 0 & 0 & \xi_{10} & 0 & 0 \\ 0 & 0 & 0 & 0 & a & 0 & 0 & a & 0 & 0 & \xi_{11} & 0 \\ 0 & 0 & 0 & b & 0 & b & 0 & 0 & a & 0 & 0 & \xi_{12} \end{array} \right) \in \mathcal{Q}_\pi(G),$$

where the a and b denote nonzero free parameters, and all ξ_i's denote the diagonal entries which can take arbitrary real values, including zero. The leader set is $V_L = \{1, 2, 3, 4, 5, 6\}$. To every node i we assign a variable x_i. Initially, $x_1 = \ldots = x_6 = 0$ since they correspond to the black nodes. The set of white nodes is $V \setminus V_L$, which we have labeled $\{7, \ldots, 12\}$.

We introduce the notation $X = V_L = \{1, \ldots, 6\}$ and $Y = \{7, \ldots, 12\}$. Note that Y is in this example also equal to the neighbor set of X. Consider the balance equations $x_Y^T A_{Y,X} = 0$:

$$x_Y^T A_{Y,X} = \begin{pmatrix} x_7, & x_8, & x_9, & x_{10}, & x_{11}, & x_{12} \end{pmatrix} \begin{pmatrix} a & a & 0 & 0 & 0 & 0 \\ 0 & a & b & 0 & 0 & 0 \\ b & 0 & b & 0 & 0 & 0 \\ 0 & 0 & 0 & a & a & b \\ 0 & 0 & 0 & 0 & a & 0 \\ 0 & 0 & 0 & b & 0 & b \end{pmatrix} = 0,$$

which yield the following balance equations:

$$ax_7 + bx_9 = 0$$
$$ax_7 + ax_8 = 0$$
$$bx_8 + bx_9 = 0$$
$$ax_{10} + bx_{12} = 0$$
$$ax_{10} + ax_{11} = 0$$
$$bx_{10} + bx_{12} = 0$$

We will now solve this homogeneous system of equations. The second and third equation can be rewritten as $x_7 = -x_8$ and $x_8 = -x_9$, respectively. This implies that $x_7 = x_9$. Combining this with the first equation, we obtain $(a+b)x_7 = 0$. Looking at the fifth and sixth equation, we obtain $x_{10} = -x_{11}$ and $x_{10} = -x_{12}$. These two identities imply $x_{11} = x_{12}$. Furthermore, substituting $x_{10} = -x_{12}$ into the fourth equation, we obtain $(b - a)x_{12} = 0$. We will now consider three different cases:

Case I: $a \neq b$ and $a \neq -b$;

Case II: $a = b$;

Case III: $a = -b$.

The qualitative class $\mathcal{Q}_\pi(G)$ can be partitioned into three subclasses:

$$\mathcal{Q}_\pi(G) = \mathcal{Q}_\pi^{(I)}(G) \cup \mathcal{Q}_\pi^{(II)}(G) \cup \mathcal{Q}_\pi^{(III)}(G), \tag{6.4}$$

where the subclasses are given by:

$$\mathcal{Q}_\pi^{(I)}(G) = \{A \in \mathcal{Q}_\pi(G) \mid a \neq b, a \neq -b\},$$
$$\mathcal{Q}_\pi^{(II)}(G) = \{A \in \mathcal{Q}_\pi(G) \mid a = b\},$$
$$\mathcal{Q}_\pi^{(III)}(G) = \{A \in \mathcal{Q}_\pi(G) \mid a = -b\}.$$

We will now investigate what happens if we apply the colored color change rule once. To do that, we need to look at every possible system of balance equation. First of all, suppose $A \in \mathcal{Q}_\pi^{(I)}(G)$. Then the above equations imply that $x_7 = 0$ and $x_{12} = 0$, and consequently $x_8 = x_9 = 0$ and $x_{10} = x_{11} = 0$. So for all $A \in \mathcal{Q}_\pi^{(I)}(G)$ we see that the balance equations imply that $x_7 = x_8 = x_9 = x_{10} = x_{11} = x_{12} = 0$. We denote $Y^I := \{7, 8, 9, 10, 11, 12\}$.

Secondly, suppose $A \in \mathcal{Q}_\pi^{(II)}(G)$. Then the above equations only imply that $x_7 = x_8 = x_9 = 0$. We denote $Y^{II} := \{7, 8, 9\}$.

Thirdly, suppose $A \in \mathcal{Q}_\pi^{(III)}(G)$. Then the balance equations imply that $x_{10} = x_{11} = x_{12} = 0$. We denote $Y^{III} := \{10, 11, 12\}$.

From this it follows that $\bigcap_{A \in \mathcal{Q}_\pi(G)} Y = Y^{(I)} \cap Y^{(II)} \cap Y^{(III)} = \emptyset$, so the colored color change rule gives $D^{cgze}(X) = D^{cgze}(V_L) = V_L$. Hence, V_L is not a color generalized balancing set.

On the other hand, we can show that (G, V_L) is controllable. To do so, we use Fact 5.1, by showing that V_L is a balancing set for any realization $A \in \mathcal{Q}_\pi(G)$. To see that, we consider three different cases and use the partition (6.4) again.

For any realization $A \in \mathcal{Q}_\pi^{(I)}(G)$, we saw that the balance equations imply $x_7 = x_8 = x_9 = x_{10} = x_{11} = x_{12} = 0$. Hence, $D^{ze}(V_L) = V$ for all $A \in \mathcal{Q}_\pi^{(I)}(G)$.

Now consider a realization $A \in \mathcal{Q}_\pi^{(II)}(G)$. Then the balance equations imply $x_7 = x_8 = x_9 = 0$. The new set of black nodes are $\{1, 2, \ldots, 9\} = X_2$. For these new black nodes, we can write down the balance equations $x_{V \setminus X_2}^T A_{V \setminus X_2, X_2} = 0$, where $A_{V \setminus X_2, X_2}$ is now a 9×3 matrix. By applying zero extension again we see that $\{10, 11, 12\}$ is forced, and hence $D^{ze}(V_L) = V$ for all realizations $A \in \mathcal{Q}_\pi^{(II)}(G)$.

Finally, one can also show that for any $A \in \mathcal{Q}_\pi^{(III)}(G)$, we have that $D^{ze}(V_L) = V$. First of all, we saw that the balance equations imply $x_{10} = x_{11} = x_{12} = 0$. The new set of black nodes is then $\{1, 2, 3, 4, 5, 6, 10, 11, 12\}$. Applying zero extension again, the balance equations imply that $x_7 = x_8 = x_9 = 0$. Hence, $D^{ze}(V_L) = V$ for all $A \in \mathcal{Q}_\pi^{(III)}(G)$.

Because $\mathcal{Q}_\pi(G) = \mathcal{Q}_\pi^{(I)}(G) \cup \mathcal{Q}_\pi^{(II)}(G) \cup \mathcal{Q}_\pi^{(III)}(G)$, we see that $D^{ze}(V_L) = V$ for any realization $A \in \mathcal{Q}_\pi(G)$ using the balance equations. The system (G, V_L) is thus controllable by Fact 5.1, however, as we showed, V_L is not a color generalized balancing set.

Chapter 7

Conclusions

In this thesis, we studied controllability of systems defined on graphs. When we study the controllability of a network of dynamical systems, the topology of the network is represented by a graph. The nodes through which we can apply external inputs are often called the leaders of the network or simply the input nodes. The graph topology gives rise to a family of matrices, called the qualitative class. Each nonzero off-diagonal entry of any matrix from the qualitative class represents the weight of an edge. Every matrix from this qualitative class thus carries the underlying graph structure of the network. The network together with the input nodes is called strongly structurally controllable if for any matrix from the qualitative class the system is controllable.

In previous literature, it has already been shown that there is a one-to-one correspondence between sets of leaders that render a network strongly structurally controllable and zero forcing sets. In the concept of zero forcing, nodes are either colored black or white. A so-called color change rule is given, which can change the color of a single white node to black. A set of input nodes is then called a zero forcing set if all nodes are eventually colored black, by consecutive application of the color change rule. Note that zero forcing only depends on the graph structure itself, and not on a particular choice of the system matrix from the qualitative class. Zero forcing sets give hence a graph theoretic condition for strong structural controllability.

First, we took a closer look at this concept of zero forcing. Previously, the associated color change rule was only able to color a single white node black. We have improved the color change rule and formulated a generalized color change rule, that can color multiple white nodes black simultaneously. In order to do so, our generalized color change rule involved the concept of perfect matchings in bipartite graphs.

We then formulated and solved the main problem considered in this thesis, which is the following. Commonly, an important assumption is made on the qualitative class. This assumption is that every nonzero off-diagonal element of any of the matrices from the qualitative class is independent from the other entries. This is, however, in practice not always

satisfied. Therefore we looked at qualitative classes where the same parameter can appear on multiple locations. This constrained qualitative class gave rise to the concept of a colored graph, by which we mean that every edge has now a color. Two edges have the same color if the two corresponding parameters in the constrained qualitative class are the same. We then aimed at finding conditions for strong structural controllability of systems with underlying network a colored graph.

For two disjoint nonempty subsets of nodes, we were able to write down a corresponding colored bipartite graph. Then, we established a necessary and sufficient condition for the nonsingularity of every complex matrix from the pattern matrix of this colored bipartite graph. This theorem was then used to formulate a colored color change rule, which used the notion of color perfect neighbor. We showed that this gave only a sufficient condition for strong structural controllability of systems defined on colored graphs, and not a necessary condition. For the latter case, we gave an example to illustrate that.

Finally, we looked at another colored color change rule that uses the concept of generalized zero extension. This colored color change rule can also be used to determine strong structural controllability of colored networks. However, again, this condition was only a sufficient condition. We gave an example to illustrate that it is not a necessary condition.

So far, the colored color change rule that uses generalized zero extension is purely algebraic. While generalized zero extension also gave a sufficient condition for strong structural controllability of colored graphs, it is not easy to check if this condition holds. In the future, it might be worthwhile to take a closer look at generalized zero extension and see if it can be interpreted as a graph theoretic tool, since conditions that can be verified by looking at the graph topology are easier to verify.

Acknowledgements

First and foremost I would like to thank prof. dr. H. L. Trentelman for his guidance through the entire process of doing a master research project. Our weekly meetings were not only enlightening and instructive, above all they were fun and I often left the meeting with the feeling that I learned something new. I thank you for investing so much time in me. I could not have asked for a better supervisor!

Secondly I would like to thank Jiajia Jia. It was very nice that we could work simultenously on the problems. Our discussions were sometimes very lengthy and intense, but we never lost our goals out of sight. I think we can both agree that our collaboration was very useful for the both of us!

Finally I would like to thank prof. dr. ir. M. Cao for agreeing on being my second supervisor.

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