Growth and Decay in Networks

Huseynov Ibrahim

University of Groningen

Faculty of Science and Engineering

IE&M Bachelor Thesis

June 2018

Supervisors: Nima Monshizadeh, Gunn Larsen

Noorderstationsstraat 70 B

9717KR Groningen

(06) 30857215

i.huseynov@student.rug.nl

student number : s2709538

university of groningen

Abstract

The purpose of this research project is to find the relation between the stability of direct current electric circuits and the magnitudes of their resistances. Electric circuits which are analyzed are depicted in form of indirect graphs, containing nodes and edges. The circuits which are the goal of investigation in the project are 2-node and 3-node circuits. These circuits are described in specific cases, and each case includes a circuit with different topology and different signs for ground resistors. Each type of circuit is analyzed separately, its stability conditions are found by performing analytical calculations and they are represented in terms of effective resistance. Matlab plots are included in order to verify the the results from analytical calculations. After all of the cases are investigated and the stability conditions are found, the results from both Matlab plots and analytical calculations are used in order to deduce how the the magnitude of negative and positive resistors influences the stability of direct current electric circuits.

Table of contents

List of tables

List of figures

Introduction

Research topic

Electric circuits are crucial as they carry energy throughout electronic appliances. Many branches of electrical engineering, such as power, electric machines, control, electronics, communications, and instrumentation, are based on electric circuit theory [1]. The design procedure of electronic appliances includes electric circuit design. Electric circuit should be designed in a way that it fulfills its specified function: carrying the current to various parts of an electronic appliance. In order for an electric circuit to achieve this function, it has to be stable. Definition of stability concerning electric circuits will be discussed further in the report, however, it needs to be mentioned that stability will depend on the magnitude of resistors, which are one of the major components of an electric circuit. Therefore, it is crucial to comprehend the correlation between resistance values and circuit stability prior to design process of electric circuits.

Problem context

In order to comprehend the essence of the "problem", it is essential to represent the circuit in a mathematical way. One of the ways to represent the circuit is to use a state-space model. The concept of the state of a dynamic system refers to a minimum set of variables, known as state variables that fully describe the system and its response to any given set of inputs. State-space model represents the change of a variable in the system by the current state of that variable and the input of the system [2]. The general model for state-space can be represented as:

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

If state-space model is applied to an electric circuit, it can be represented by Laplacian matrix and written in a form which is depicted below:

$$\dot{x} = -L\,x - \Delta\,x$$

This formula represents an electric circuit as a control system, where the change of the state "$\dot{x}$" is depicted by the current state of the system – "x". In an electric circuit, the state "x" indicates the voltage, the entries of the matrix "$-(L+\Delta)$" represents inverse resistance (conductance) values throughout the circuit and "$\dot{x}$" is the current. This representation is justified by fundamental laws of electricity that will be discussed later in the research.

In the expression "$-(L+\Delta)$", the Laplacian matrix "L" will only have positive resistance values whereas Delta matrix "Δ" can have both positive and negative resistance values. If the Delta matrix has only positive resistance values, then the circuit will always be stable and if it has only negative resistance values then the circuit will always be unstable (proof of this statement is given in the Appendix B). If the Delta matrix has both positive and negative resistances, then the circuit can be stable or unstable, depending on the magnitude of positive and negative resistors throughout the circuit. In this mixed case, the total positive resistances will always outnumber the negative ones. Therefore, the notion of effective resistance will be beneficial, where a single resistor can replace these positive resistances, with some other resistance. In this case, it can be stated that the stability of the circuit will depend on the magnitude of negative resistance(s) and the effective resistance. However, it is unknown how each one of these components affects stability of the circuit.

Potential applications

The circuits which will be the goal of focus in this research will be called direct current resistor circuits. Direct current circuits are utilized in DC (direct current) power grids [3], which have recently gained popularity because of their efficiency in energy usage [4]. Therefore, understanding the DC circuit design process by gaining more knowledge about the circuit's stability conditions will be beneficial for the engineers who design DC power grids. Today, with few exceptions, the electric grid is predominately AC. However, it appears that DC power grids may be on the verge of a comeback. Digital equipment, solar PV, storage batteries, electric vehicles and other end-use devices all require DC power. Data centres are chockfull of such devices and for several years, there has been a movement toward DC data centres. It is likely that in the future DC power grid applications will be enabled more frequently; therefore, understanding the DC circuit design process by gaining more knowledge about the circuit's stability conditions will be beneficial for the engineers who design DC power grids.

Theoretical Background

Electric circuit

Prior to analyzing the stability of the circuit, it is essential to study main variables which define the circuit. Electric circuit allows negatively charged electrons to flow through the wires. The amount of charges that passes through surface per unit time is called current, which is depicted by “I”. In order to control the current throughout the circuit, resistors are used, and they are denoted by “R”. Electrical circuit’s power source pushes charged electrons (current) through the circuit by applying pressure on them, which is called voltage, indicated by “V”. Ohm’s law states that the voltage across a resistor is proportional to the current that passes through the same resistor [5]. Ohm’s law is given in the formula below.

$$V = IR$$

If any node at the circuit is analysed, the algebraic sum of current(s) entering the node and current(s) leaving the node is zero. Therefore:

$$\sum I_{in} = \sum I_{out}$$

It is crucial to mention the case where there are several resistors among two nodes. In this situation, the notion of effective resistance will be beneficial, as mentioned earlier in the report. The calculation of effective resistance differs between the circuits in series and in parallel, and is described below.

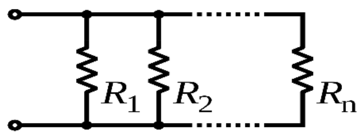


Figure 1: Resistors in parallel

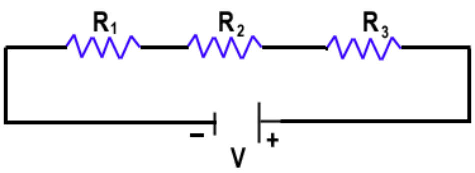


Figure 2: Resistors in series

$$\frac{1}{R(eff)} = \frac{1}{R1} + \frac{1}{R2} + \ldots + \frac{1}{Rn}$$

$$R_{eff} = R_1 + R_2 + R_3$$

Circuits as graphs

It is possible to represent an electric circuit as an undirected graph, which contains nodes and edges. If the Figure 3 is considered, 4 distinct nodes of the circuit can be distinguished. In an undirected graph each edge will have a weight rank. In a circuit, these weight ranks will be represented by conductance-inverse resistance values. Displaying the circuit in the form of a graph is beneficial because in that case Laplacian matrix of the circuit can be formed, which is the part of the "*–(L +Δ)*" matrix which describes the circuit in a mathematical way. Formation of Laplacian and Delta matrices are given in the subsection below.

Matrix formation

In order to get familiar with the stability of electric circuits, it is important to comprehend how the Laplacian matrix is achieved, as it describes the circuit. Following circuit can be taken as an example.

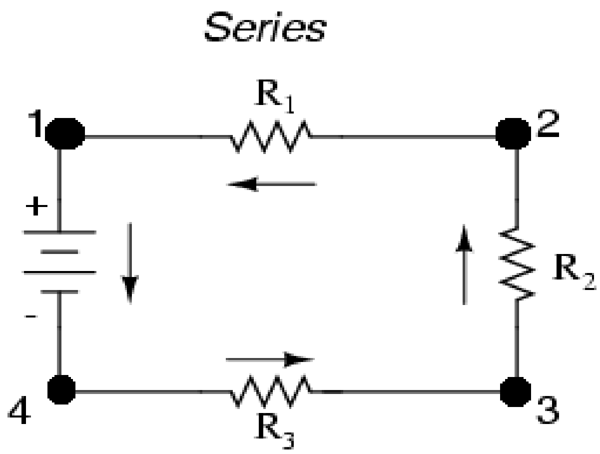


Figure 3: sample RC circuit

Following rules should be applied in order to construct a Laplacian matrix [6]:

1. Diagonal matrix "D" should be gained.
2. Adjacency matrix "A" is calculated.
3. Laplacian matrix "L = D - A" is derived.

Diagonal matrix

The elements of the diagonal matrix "D" represent the sum of the conductance values associated with a specific node. For instance, in the figure X, the associated conductance values with node 1 will be $\frac{1}{R1}$, with node 2 it will be $\frac{1}{R1} + \frac{1}{R2}$. This principle will be applied to all the nodes in the procedure of formation of diagonal matrices. All of the non-diagonal elements of the diagonal matrix will equal to zero.

Adjacency matrix

Formation of adjacency matrices includes considering the conductance values among each pair of nodes. For example, the conductance value among node 2 and node 3 will equal to $\frac{1}{R2}$. This indicates that in the 4x4 Adjacency matrix, the value of $\frac{1}{R2}$ will be inserted in the third column of the second row and second column of the third row. The conductance value between nodes 3 and 4 equals to $\frac{1}{R3}$. This means that conductance value of $\frac{1}{R3}$ will be entered to the fourth column of the third row and third column of the fourth row. This rule will apply to all the adjacency matrix elements in its formation process.

Laplacian matrix

As mentioned, Laplacian matrix is derived from the subtraction of Adjacency matrix from the Diagonal matrix. Since the formation procedure of both of these matrices are discussed, Laplacian matrix of a sample circuit given in Figure 3 can be given.

$$D = \begin{vmatrix} 1/R1 & 0 & 0 & 0 \\ 0 & 1/R1+1/R2 & 0 & 0 \\ 0 & 0 & 1/R2+1/R3 & 0 \\ 0 & 0 & 0 & 1/R3 \end{vmatrix}$$

$$A = \begin{vmatrix} 0 & 1/R1 & 0 & 0 \\ 1/R1 & 0 & 1/R2 & 0 \\ 0 & 1/R2 & 0 & 1/R3 \\ 0 & 0 & 1/R3 & 0 \end{vmatrix}$$

$$\mathbf{L= D\text{-}A =} \begin{vmatrix} 1/R1 & -1/R1 & 0 & 0 \\ -1/R1 & 1/R1+1/R2 & -1/R2 & 0 \\ 0 & -1/R2 & 1/R2+1/R3 & -1/R3 \\ 0 & 0 & -1/R3 & 1/R3 \end{vmatrix}$$

Delta matrix

Elements of the Delta matrix ”*Δ*” represent the conductance-inverse resistance values among the nodes and the electrical ground. These type of resistors are called ground resistors and they are shown in the Figure X.

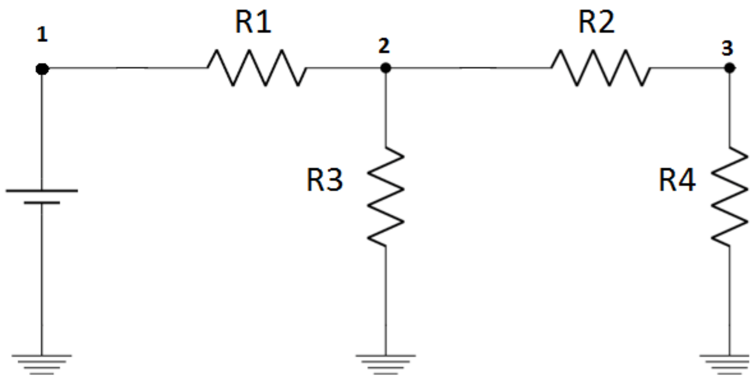


Figure 4 : Sample 3 node RC circuit

This matrix will also have only diagonal elements, as the Diagonal matrix mentioned previously. In the circuit above, ground resistors exist between the electrical ground and the nodes 2 and 3. In this case, Delta matrix will look like the following :

$$\Delta = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 1/R3 & 1/R4 \\ 0 & 0 & 1/R4 \end{vmatrix}$$

Theorems from Linear Algebra

The procedure of finding the stability condition for electric circuit requires calculations with the matrix. The matrices that represent the circuits in all cases are Hermitian, and it is already known that for the circuit to be stable all of the eigenvalues should be negative [9]. As the matrix is Hermitian and it needs to be negative definite, theories about negative definite Hermitian matrices will be applied in these calculations. The ones that can be used in the calculations are listed below:

1. If the matrix is represented as A, then the roots of the determinant “λI-A”, need to be negative.

$$\lambda_n < 0 \quad n = \{1,2,3 \ldots\}$$

2. If A has diagonal elements a_{ii}, $\mathbf{a_{ii}} < 0$ for all $\boldsymbol{i}$.

3. If A_1, A_2, and A_3 are leading principal sub-minors of A, then their determinants should have following signs :

   Det $(A_1) < 0$ Det $(A_2) > 0$ Det$(A_3) < 0$

Control theory and Circuit differential equations

As already mentioned in the problem statement, an electric circuit will be represented in the state-space form:

$$\dot{x} = -L\,x - \Delta\,x$$

The circuit is referred as a dynamical system. Dynamical systems describe processes in motion, try to predict the future of these systems and understand the limitations of these predictions [7]. In the expression above, future state of the circuit will depend on the matrix "$-(L+\Delta)$". When the matrix has negative eigenvalues, the circuit will be stable and the voltage in the circuit will get a constant value, over time. However, in an unstable circuit, voltage will grow and go to infinity, therefore causing excessive heat and damaging the circuit, which is not desirable. In order to understand this phenomenon better, the differential equations of Resistor Capacitor circuits can be analysed.

In order to understand the phenomenon of stability and instability better in an electric circuit, step response of first order RC circuits can be analysed. If the circuit closes, the voltages and currents in the circuit elements adjust to the new conditions. If the change is an abrupt step, the response of the voltages and currents is called the step response. It tells us quite a lot about the properties of the circuit. Following equation describes the step response of the RC circuit to the applied voltage:

$$v_{\mathbf{R}}(t) = 1 - v_{\mathrm{C}}(t) = \mathrm{e}^{-t/\tau}\ \mathrm{V}\ \text{for}\ t \geq 0^{+}$$

$$v_{\mathrm{C}}(t) = \int_{0^{+}}^{t} \frac{1}{RC}\mathrm{e}^{-t/\tau}\mathrm{d}t = (1 - \mathrm{e}^{-t/\tau})\ \mathrm{V}\ \text{for}\ t \geq 0^{+}$$

The solution to the homogeneous system $\dot{x} = Ax$ can be given in the following form:

$$\mathbf{X} = \boldsymbol{\eta} e^{\lambda t}$$

As the electric circuit is also given in the same form, we will see that when the eigenvalue is negative and time "t" goes to infinity, V_c will equal to V, which means that voltage at the capacitor will equal to the voltage applied to the circuit. At the same time time, voltage at the resistor will get close to zero. However, notice from both expressions that if the power of "e", hence, the eigenvalue is positive, voltage at the resistor will go to infinity, therefore creating excessive heat and damaging the circuit.

Now from the formulas we see that with the negative eigenvalue V_R will get closer to zero while with a positive one V_R will get significantly larger. The terms growth and decay are associated with this phenomenon.

Research Problem Analysis

Goal

As the problem statement is formulated and all of the scientific theory concerning electric circuits is revised, the research goal can be set. It is known that the stability of the matrix will be dictated by the values of the matrix "$-(L+\Delta)$" which contains positive and negative resistance values. Therefore, it can be stated that there is a correlation between stability/instability of the circuit and positive/negative resistances in the circuit. The main goal of this project will be stated as:

Determine how the magnitude of effective resistance and negative resistances influence the stability of the circuit.

It can be claimed that reaching the goal will solve the research problem, as understanding the correlation between stability of the circuit and magnitudes of effective resistance and negative resistance(s) will give new insights about the stability of the circuits to the engineers and researchers. The deliverable is pure scientific knowledge, and can be used by engineers and researchers after the successful completion of this project.

Scope

Scope of the research should be narrowed down to the scientific fields which are actually related to the research. For instance, graph theory includes the investigation of directed and undirected graphs. As the electric circuit is depicted as an undirected graph, any theory concerning directed graphs will be ignored in the research. The subfields of the chosen fields can be considered as the scope of the project, and they are given in the table below:

Electrical Engineering	**Graph theory**	**Stability theory**
Kirchhoff law	Directed graphs	Stability of linear systems
Ohm law	-	-
Effective resistance	-	-

Table 1: Relevant scientific fields concerning the research

Research (sub) questions

As the main goal is stated, research questions can be formulated in order to lead the research to the goal.

How does the magnitude of effective resistance and negative resistance(s) affect the stability of the electric circuit?

Sub questions and their relation to the main research question are listed below:

1. ***What are the sufficient conditions that assure stability of the circuit?***

   It is extremely crucial to answer this sub question in order to reach the answer of the main research question. Stability conditions can be given in terms of inequalities, which are derived from linear algebra theorems. The influence of effective resistance and negative resistances to stability of the circuit can be found only after deriving the stability conditions/inequalities.

2. ***How does the magnitude of effective resistance of the circuit affect its stability?***

   This question will explicitly focus on the effective resistance and stability correlation. It can only be found after calculating the effective resistance of the circuit and depicting the stability condition in a way that effective resistance can be differentiated in the condition.

3. ***How does the magnitude of negative resistance of the circuit affect its stability?***

   This question will primarily focus on the stability/negative resistance correlation.

Cycle choice / Research methods

Cycle choice

As the research questions are formulated, some theoretical information about different types of research projects can be analysed. Hevner differentiates among 3 cycles that can be used in Research Project and gives specific guidelines for each one of them [10]. These include Rigor cycle, Relevance cycle, and Design cycle. Relevance cycle initiates design science research with an application context that not only provides the requirements for the research (e.g., the opportunity/problem to be addressed) as inputs but also defines acceptance criteria for the ultimate evaluation of the research results. The rigor cycle provides past knowledge to the research project to ensure its innovation and the design cycle focuses on construction of an artefact, its evaluation, and subsequent feedback to refine the design further.

A researcher can choose among these three cycles for application to a specific project. It is crucial to analyse the specifications of design cycle and rigor cycle in order to decide which one suits the project more.

	Design Cycle	**Rigor Cycle**
Output	Construct	Knowledge
Expected results	Feasible process/product	Rigor (paper)
Connection to context	Stakeholders	Isolated

Table 2: Cycles of Hevner

It can be noticed that the expected results of this research project is a paper (rigor) and the output is knowledge, therefore, the main focus of the researcher should be aimed at Rigor Cycle. From the figure, it is noticed that stakeholders are more in distance in this project, so there will be less connection to stakeholders compare to design cycle.

Research methods

After the research goal and the research questions are defined, appropriate research methods can be selected which will be beneficial in finding the necessary information to reach the research goal. The appropriate research methods that can be used in this research project are literature search and experiments.

Literature search is beneficial from the perspective that it contains all the information about electric circuits, stability theory, graph theory and linear algebra. The process of determining the conditions for stability of an electric circuit requires revision of the related articles to the stability of systems and working principles of an electric circuit. Therefore, literature search is an essential method to utilize.

Experiments will be helpful in order to observe how different magnitudes for resistors affect stability of the circuit. By entering different values for resistor on Matlab and finding the signs of the eigenvalues of the matrix depicting the circuit, some common patterns about the stability and effective/negative resistance correlation can be observed.

Research execution

In order to find the answer to the stated research question, the research process should be executed properly. Several steps which are implemented in research execution are given below and their benefit to the research is explained.

1. Several circuits need to be analyzed.

Investigation of several circuits in order to find a common phenomenon regarding circuit stability is necessary. This is to assure that the effect of effective/negative resistance on stability is the same for all type of circuits. In this research project, circuits with 2 nodes and 3 nodes are discussed. These circuits also have either a different topology or different sign for resistances.

2. Stability conditions need to be checked with Matlab plots

After the analytical calculations for the specific circuit are done and the stability conditions are found, it is necessary to check the results on Matlab. This is possible by plotting Matlab graphs and observing whether the condition which is found is right. The stability conditions can also be checked by inserting different values which satisfy these conditions to the matrix "$-(L+\Delta)$" and checking whether all the eigenvalues remain negative. However, this procedure is time consuming and therefore is avoided.

3. Found conditions should be represented in a proper form

This means that the conditions should be shown in a way where the influence of effective and negative resistances on stability could be visible. For instance, if inequality depicting the stability condition is found and the impact of the effective resistances on stability needs to be analyzed, the effective resistance should be on one side of the inequality, otherwise it can't be differentiated in the condition and its correlation with stability cannot be found.

Results

Several type of circuits are taken into consideration and stability condition(s) are derived for each type. Results of the analytical calculations are followed by the results of Matlab plots as the plots are essential in order to check whether the found conditions are met or not. Initially, 2*2 matrices are investigated, and they are followed by 3*3 matrices.

Analytical calculations

Case 1

Initially, 2*2 matrices can be investigated in order to find out the stability conditions of the electric circuit. If the matrix is 2*2, this means that there are two nodes in the circuit. This type of circuit is depicted below in Figure 5. The resistance R1 is positive and the resistance R2 is negative, in this case scenario.

Given conditions:

R1 > 0

R2 < 0

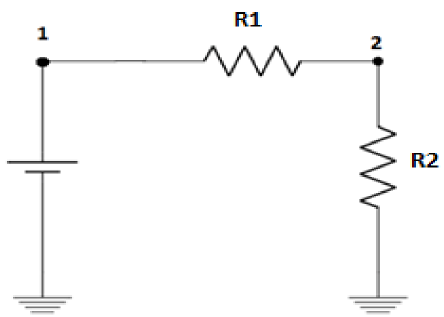


Figure 5: 2 node RC circuit with 2 total resistors

Stability conditions found from the calculations:

The system is always unstable for positive R1 and negative R2.

Case 2

This case is different from the first case in a way that now there are 3 resistors in the circuit, instead of 2 in the previous case. Since only one positive resistor could not compensate for the negative one in the last case, now we set two resistors to be positive and see in which condition the circuit is stable.

Given conditions:

R1 > 0;   R2 < 0;   R3 > 0

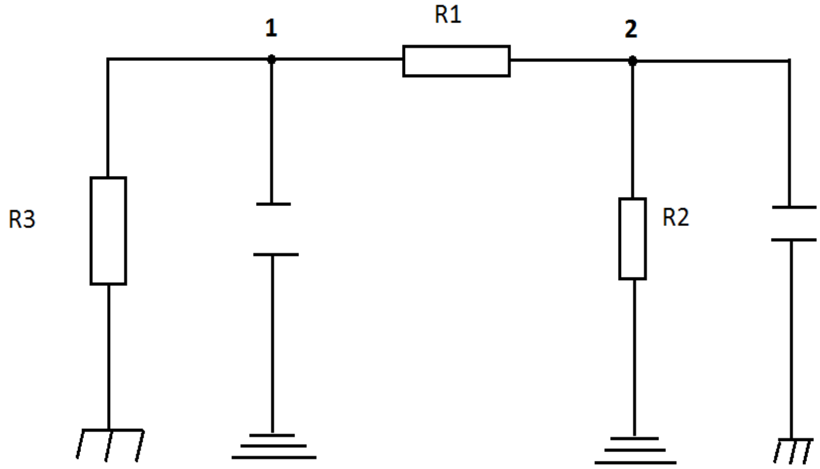


Figure 6: 2 node RC circuit with 3 total resistors

Stability conditions found from the calculations:

$|R2| > R1+R3$ (1)

Effective resistance of positive resistances:

$R_{eff} = R1+R3$

Stability condition in terms of effective resistance:

$|R2| > R_{eff}$ (2)

Interpretation of the condition:

In order to achieve the stability of the circuit, the magnitude of the effective resistance should always be less than the absolute value of the negative resistor R2.

Case 3

For this circuit, 3*3 matrix needs to be formulated since it has 3 nodes, which adds to the complexity of the case. Moreover, the calculation for determining the sufficient conditions for stability is much more complicated as well and they are given in the appendix. As in the previous case, only the second resistance value is negative, while other resistances R1, R3, R4 and R5 are positive.

Given conditions:

R1 > 0 ;   R2 < 0 ;   R3 > 0 ;   R4 > 0 ;   R5 > 0

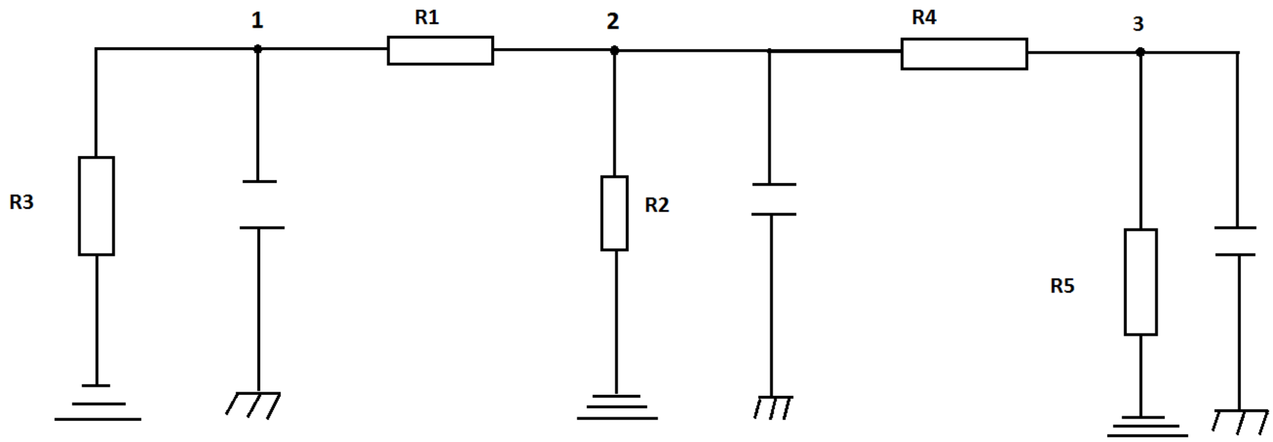


Figure 7: 3 node RC circuit with negative resistor R2

Stability conditions found from the calculations:

$$|R2| > \frac{R3R5+R3R4+R1R5+R1R4}{R5+R3+R1+R4} \quad (3)$$

Effective resistance of positive resistances:

$$R_{eff} = \frac{R3R5+R3R4+R1R5+R1R4}{R5+R3+R1+R4}$$

Stability condition in terms of effective resistance:

$$|R2| > R_{eff} \quad (4)$$

Interpretation of the condition:

In order to achieve the stability of the circuit, the magnitude of the effective resistance should always be less than the absolute value of the negative resistor R2.

Case 4

This case is quite similar to the previous one, the only difference is that this time R3 is the resistor with negative resistance value, instead of R2 in the previous case.

Given conditions:

R1 > 0 ;   R2 > 0 ;   R3 < 0 ;   R4 > 0 ;   R5 > 0 ;

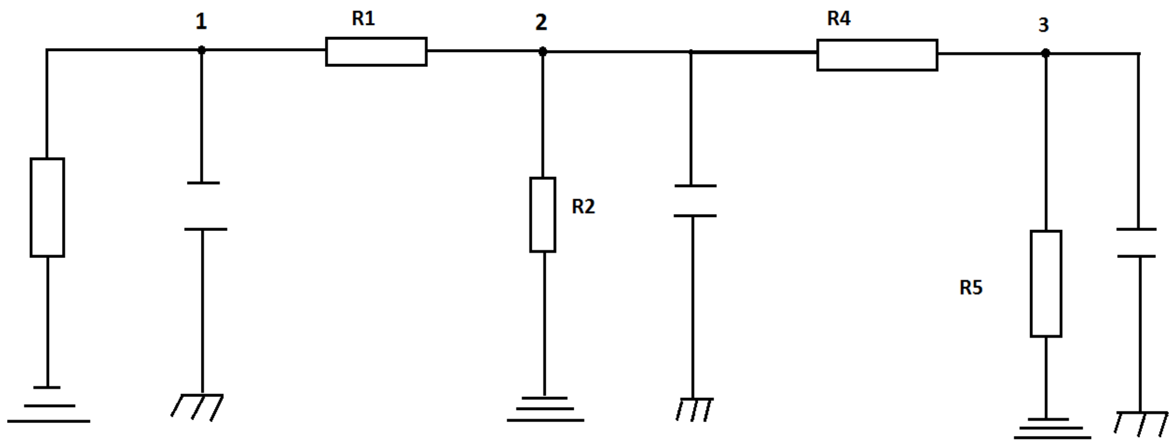


Figure 8: 3 node RC circuit with negative resistor R3

Stability condition found from the calculations:

$$\mathbf{R3} > \frac{R2R5+R2R4+R1R2+R1R5+R1R4}{R2+R4+R5} \quad (5)$$

Effective resistance of positive resistors:

$$R_{eff} = \frac{R2R5+R2R4+R1R2+R1R5+R1R4}{R2+R4+R5}$$

Stability condition in terms of effective resistance:

$$|R3| > R_{eff} \quad (6)$$

Interpretation of the condition:

In order to achieve the stability of the circuit, the magnitude of the effective resistance should always be less than the absolute value of the negative resistor R3.

Case 5

In all of the previous cases only one resistor were set to have a negative value. This changes in Case 5, as two resistors, R2 and R3, are set to have negative values.

Given conditions:

R1 > 0 ; R2 < 0 ; R3 < 0 ; R4 > 0 ; R5 > 0

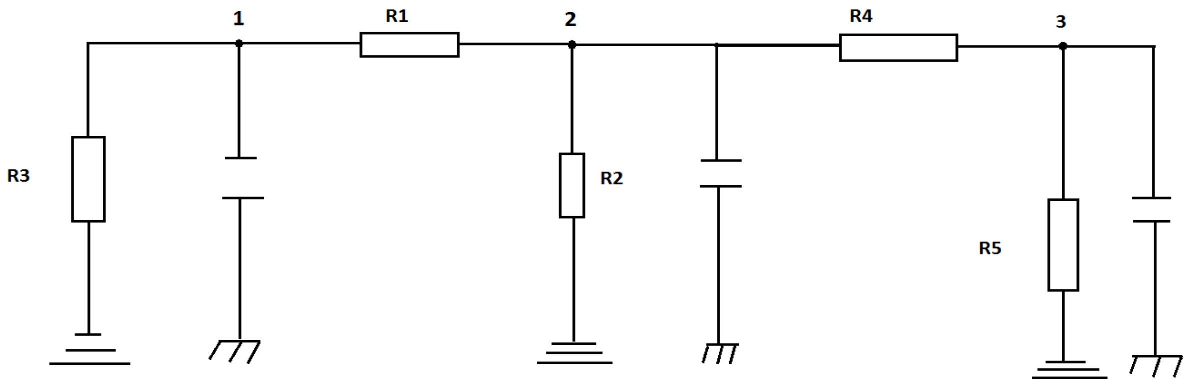


Figure 9: 3 node RC circuit with negative resistances R2 and R3

Stability conditions found from the calculations:

$|R3| > R1+R4+R5$; $|R2| > \frac{R3R5+R3R4+R1R5+R1R4}{R5+R3+R1+R4}$

Effective resistance of positive resistors:

$R_{eff} = R1+R4+R5$

Stability condition in terms of effective resistance:

$|R3| > R_{eff}$

Note that this condition is necessary, but not sufficient for stability. Necessary condition in this case means that if the system is stable, it indicates that the condition is met, however, if the condition is met, this does not necessarily mean that the system is stable. Sufficient condition means that if the condition is met, then the system is stable, and vice versa. The two conditions that are found are necessary if taken separately, but sufficient if taken together [11].

Interpretation of the conditions:

R3 needs to have a greater magnitude than the effective resistance of positive resistors. However, this condition is not enough for stability, as the magnitude of R2 should also be greater than the effective resistance of the rest of the circuit (including the negative resistance R3 as well this time). The circuit is stable when both of these conditions hold.

Matlab plots

Notice that an arbitrary value will be chosen for positive resistances in order to see in which values of the negative resistance the circuit will become unstable. In each case, the values chosen for positive resistances will be displayed. It is also worth to mention that the eigenvalue given on the “y” axis is the eigenvalue which can get positive or negative values depending on the magnitude of the resistors. In 2*2 matrices this eigenvalue is the second eigenvalue of the circuit, while in 3*3 matrices it is the third eigenvalue.

Case 1

Set values for positive resistance(s):

R1 = 1

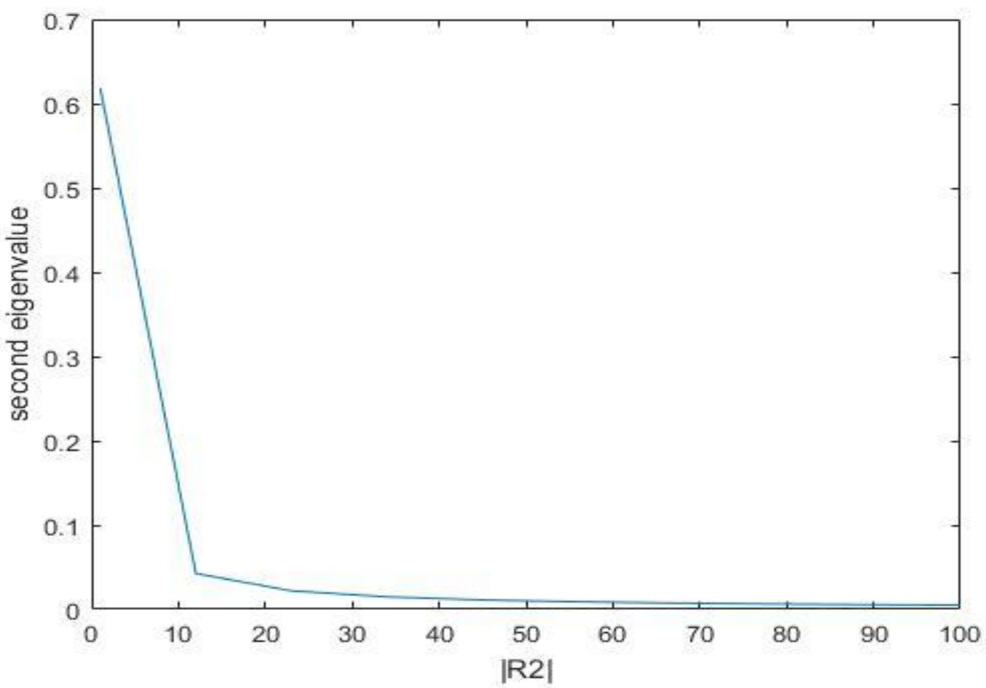


Figure 10: Relationship between second eigenvalue and the magnitude of resistor R2 (Case 1)

Result: $\lambda_2 > 0$ for $|R2| \in (0; \infty)$

Interpretation of the result:

The sign of the second eigenvalue remains positive as the absolute value of |R2| goes to infinity.

Case 2

Set values for positive resistances:

R1 = 1

R3 = 1

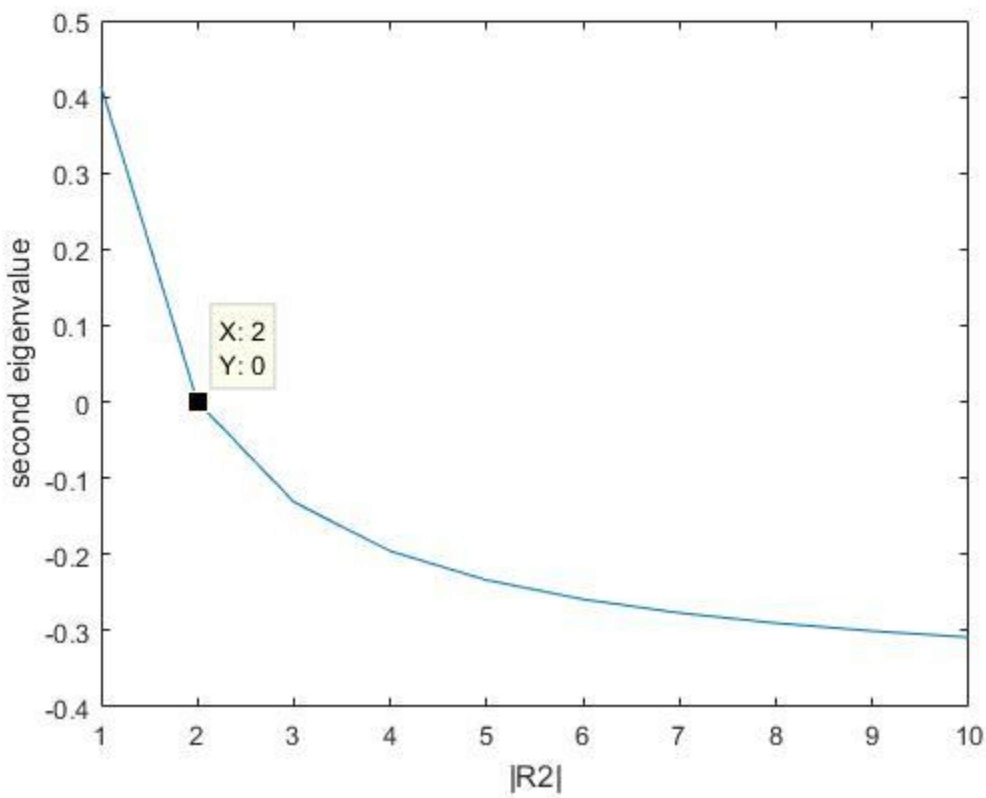


Figure 11: Relationship between second eigenvalue and the magnitude of resistor R2 (Case 2)

Result: $\lambda_2 > 0$ for $|R2| \in (0;2)$

$\lambda_2 < 0$ for $|R2| \in (2;\infty)$

Interpretation of the results:

The sign of the second eigenvalue is positive when the absolute value of |R2| is between 0 and 2. When absolute value of |R2| is greater than 2, then the second eigenvalue is negative.

Case 3

Set values for positive resistance(s):

R1=1 R4=1

R3=1 R5=1

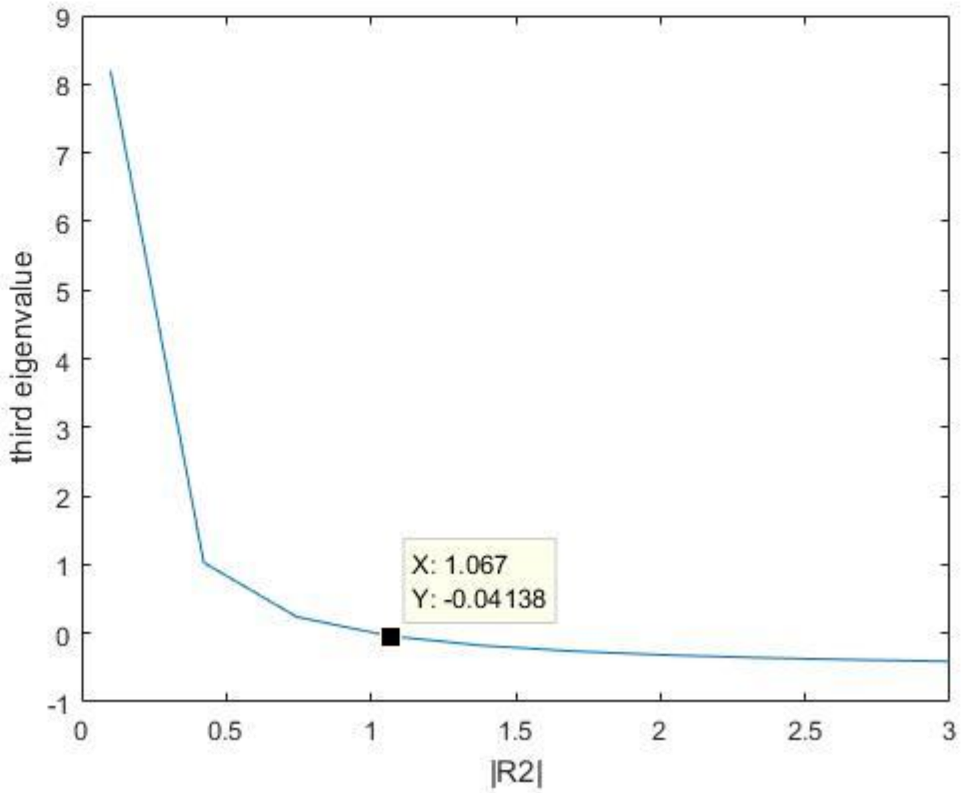


Figure 12: Relationship between third eigenvalue and the magnitude of resistor R2 (Case 3)

Results: $\lambda_3 > 0$ for $|R2| \in (0; 1)$

$\lambda_3 < 0$ for $|R2| \in (1; \infty)$

Interpretation of the results:

The third eigenvalue is positive when absolute value of |R2| is between 0 and 1. If |R2| is greater than 1, then the third eigenvalue is negative.

Case 4

Set values for positive resistance(s):

R1=1 R4=1

R2=1 R5=1

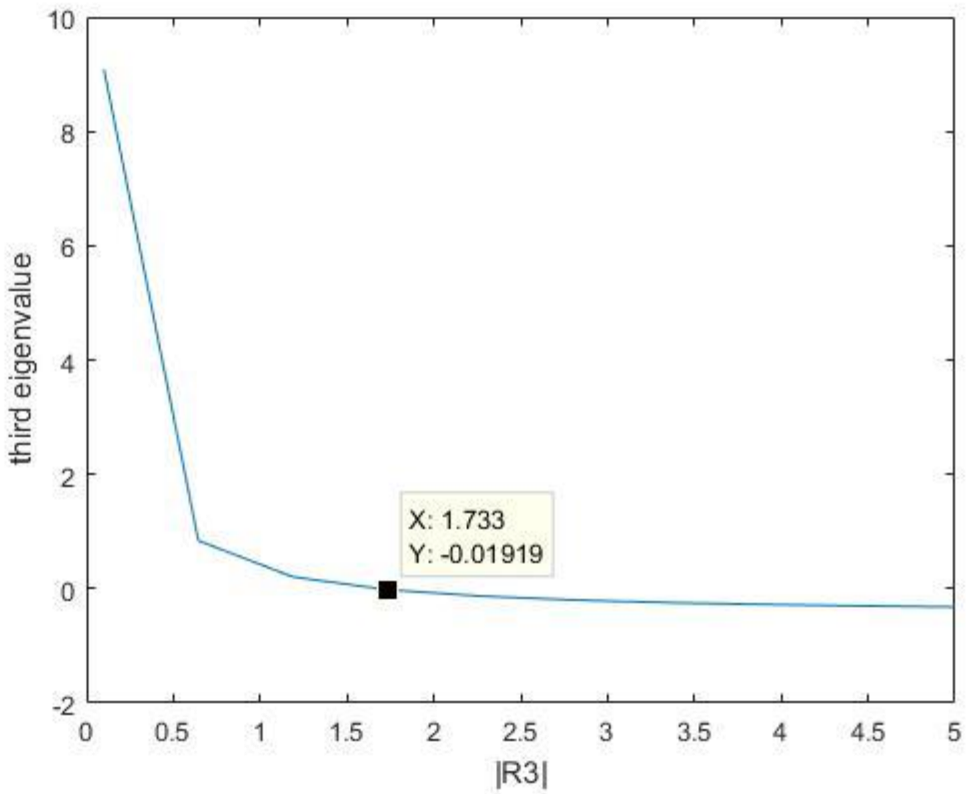


Figure 13: Relationship between third eigenvalue and the magnitude of resistor R3 (Case 4)

Results: $\lambda_3 > 0$ for $|R3| \in (0,1.67)$

$\lambda_3 < 0$ for $|R3| \in (1.67; \infty)$

Interpretation of the results:

The third eigenvalue is positive when |R3| is between 0 and 1.67. if |R3| is greater than 1.67, the the third eigenvalue is negative

Case 5 (a)

Set values for positive resistances and the negative resistance R3:

R1=1 R2=-1

R4=1 R5=1

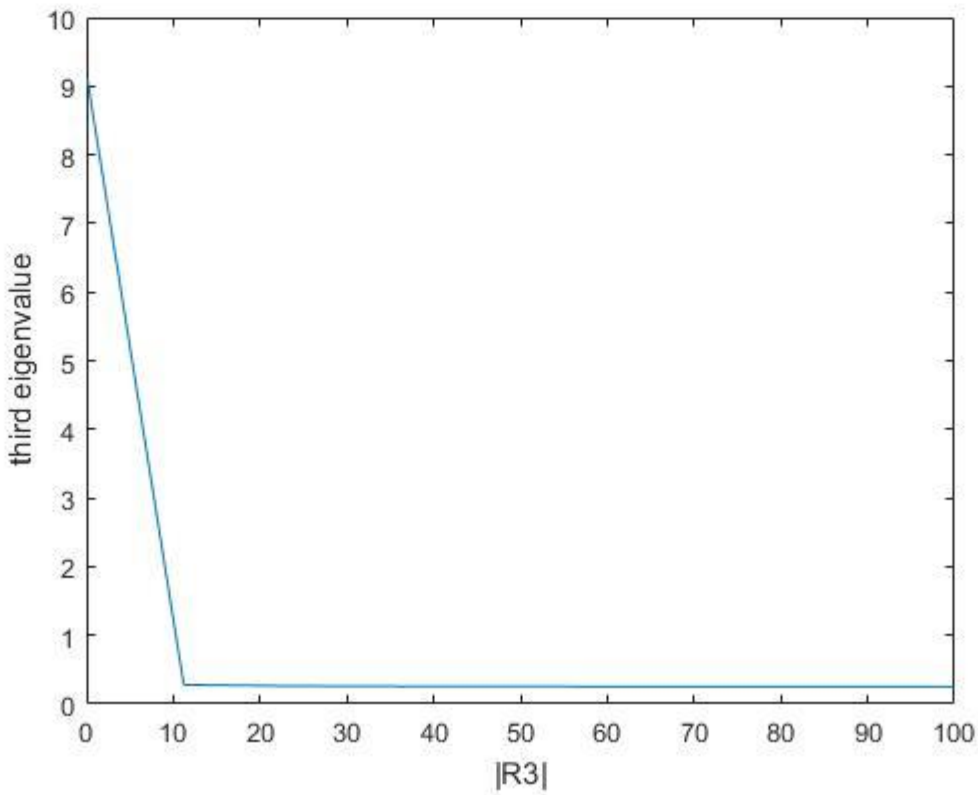


Figure 14: Relationship between third eigenvalue and the magnitude of resistor R3 (Case 5a)

Results: $\lambda_3 > 0$ for $|R3| \in (0, \infty)$

Interpretation of the results:

The third eigenvalue is always positive when |R3| goes to infinity.

Case 5 (b)

Set values for positive resistances and the negative resistance R2:

R1=1 R3= -1

R4=1 R5=1

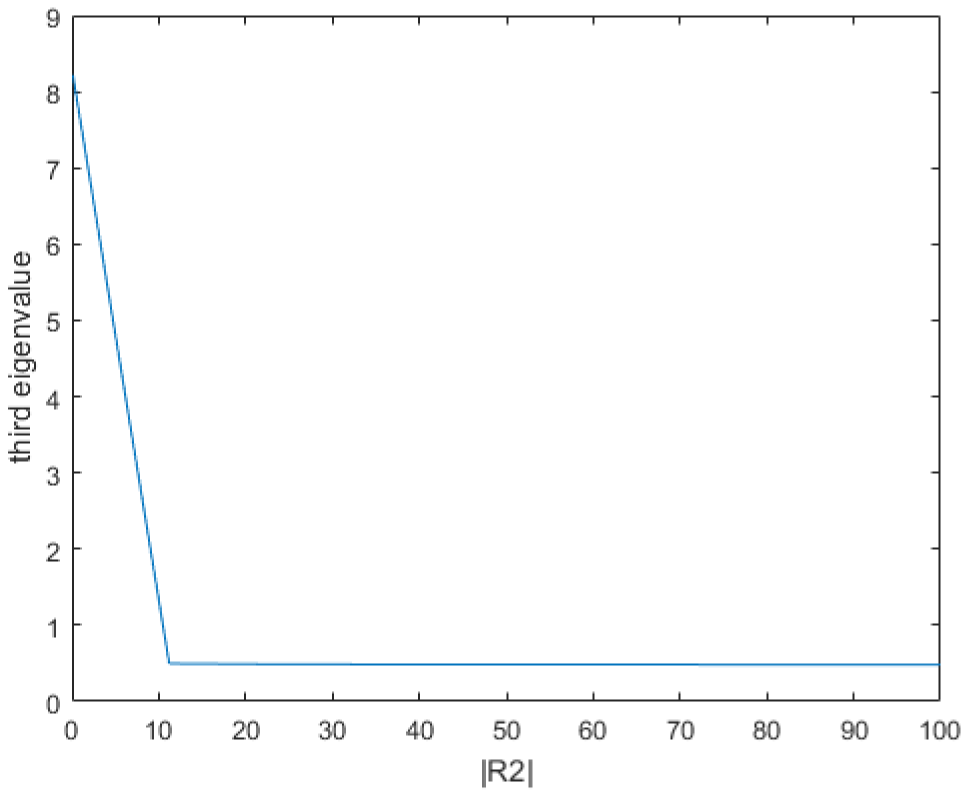


Figure 15: Relationship between third eigenvalue and the magnitude of resistor R2 (Case 5b)

Results: $\lambda_3 > 0$ for $|R2| \in (0, \infty)$

Interpretation of the results:

The third eigenvalue is always positive when |R2| goes to infinity.

Case 5 (c)

Set values for positive resistances and negative resistance R3:

R1=1 R3= - 10

R4=1 R5=1

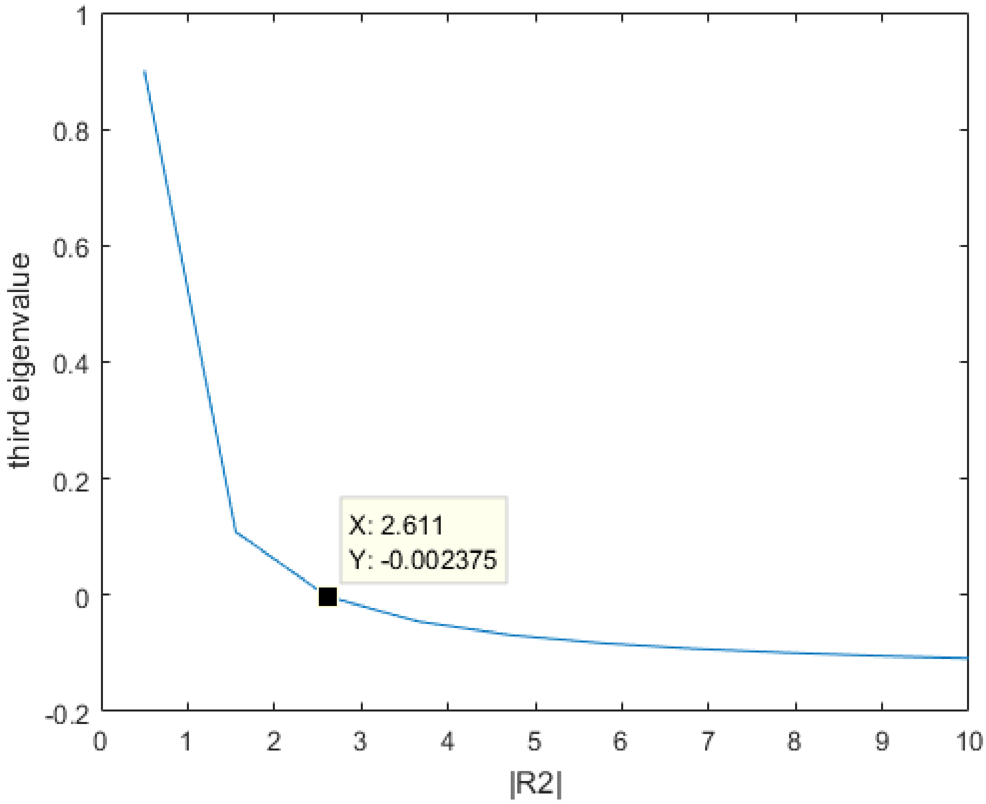


Figure 16: Relationship between third eigenvalue and the magnitude of resistor R2 (Case 5c)

Results: $\lambda_3 > 0$ for $|R2| \in (0, 2.58)$

$\lambda_3 < 0$ for $|R2| \in (2.58, \infty)$

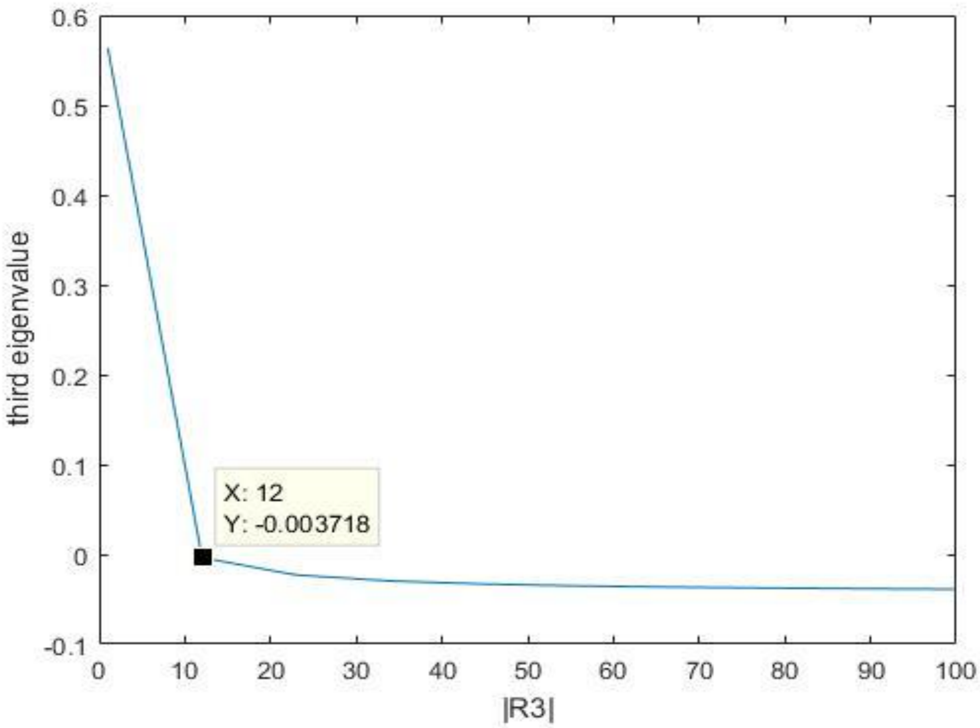
Interpretation of the results:

The third eigenvalue is positive when |R2| is between 0 and 2.58. It is negative when |R2| is greater than 2.58.

Case 5 (d)

Set values for positive resistances and negative resistance R2:

R1=1 R2=-2.5 R4=1 R5=1



5

Figure 17: Relationship between third eigenvalue and the magnitude of resistor R3 (Case 5d)

Results: $\lambda_3 > 0$ for $|R3| \in (0, 11)$

$\lambda_3 < 0$ for $|R3| \in (11, \infty)$

Interpretation of the results:

The third eigenvalue is positive when |R3| is between 0 and 11. It is negative when |R3| is greater than 11.

Note 1:

The plot shows that a value higher than 11 leads the matrix to have a negative eigenvalue. When |R3| is greater than 11 by just a small margin (11.1 for instance, the third eigenvalue) the third eigenvalue is negative.

Note 2:

The values of -2.5 for the R2 and -10 for R3 are chosen arbitrarily. While entering these arbitrary chosen these values for negative resistances in case 5c and case 5d, it was discovered that the circuit does become stable at a specific value, therefore they were chosen in order to demonstrate this phenomenon.

Discussion

In the results section, 5 different circuits were analysed and their stability conditions were found. These stability conditions were displayed in terms of inequalities that had the absolute value(s) of negative resistors(s) on the left side and effective resistance of the positive resistors on the right side. Representation of stability conditions in such a way was helpful to see the how the absolute values of negative resistance(s) and the magnitude of effective resistance affect stability. Matlab plots displayed in which ranges for the absolute value of the negative resistance the circuit becomes stable/unstable. The results obtained for each circuit are discussed separately.

Case 1

The circuit will always be unstable because the Delta matrix only has a negative element and a zero. There is no positive resistance between any node and electrical ground to compensate for the negative resistance, therefore, this circuit is unstable for any positive R1 and negative R2.

Matlab plot for this case also displays that the second eigenvalue remains positive when |R2| goes to infinity, which means that the circuit is always unstable.

Case 2

From the obtained equation for stability, $|R2| > R_{eff}$, it can be deducted that to assure the stability of the circuit, the ground resistor should have a relatively large resistance, so it could overpower the effective resistance. It can be stated that if the effective resistance and the ground resistance R2 has close values, increasing the effective resistance makes the circuit unstable, while increasing the negative resistance leads to a stable circuit.

Matlab plot for the case also demonstrates that when when the |R2| gets larger than the effective resistance, the second eigenvalue becomes negative, which ensures stability.

Cases 3 and 4

The same idea, which is discussed for the explanation of case 2, applies to cases 3 and 4 as well. The larger the magnitude of the negative resistor, the greater the chance of the circuit to be stable. Both of the circuits have the same topology, however, in case 3 the ground resistor which is negative is R2, while in case 4 it is R3. Even though the negative resistance has a different location in those two cases, the same phenomenon remains, as the negative resistor should have a greater value than the effective resistance of positive resistors. If somewhat moderate values are given for the resistors R1,R3,R4 and R5, meanwhile R2 is chosen to be relatively large, then the matrix “$-(L+\Delta)$” will have negative eigenvalues, therefore assuring stability.

Matlab plots for both cases verify the results obtained from analytical calculations. The circuit is stable when the absolutes value of negative resistors are greater than the effective resistance of the rest of the circuit.

Case 5

As this case contains two negative resistors, two conditions need to be satisfied in order for the system to be stable. The condition $|R2| > \frac{R3R5+R3R4+R1R5+R1R4}{R5+R3+R1+R4}$ is the same as found in case 3, which implies that the effective resistance of the rest of the circuit should be smaller than the absolute value of the negative resistor R2. Moreover, the resistance of R3 should be greater than the effective resistance of the circuit even excluding R2, which means that if the effective resistance of the resistors in series, R1, R4 and R5 exceed R3, this will lead to an unstable circuit. This indicates that no matter how large the R2 is chosen in this 3-node circuit design, if R3 is relatively small, then the circuit will always be unstable.

Four different Matlab plots were formed for this case. In the first 2 cases, it can be seen that the third eigenvalue never becomes negative. It can be explained by stating that given values for resistances do not meet the stability criteria found from analytical calculations. In the former 2 cases, the other negative resistor (R3 in case 5c and R2 in case 5d) is given a larger magnitude, so the circuit does become stable after a specific point.

Conclusion

Calculations and results from Matlab simulations led to finding the necessary conditions for 2 node and 3 node electric circuits. It was discovered that for the stability of both circuits, the absolute value of negative resistance should be greater than the effective resistance of the rest of the circuit. In order to assure the stability of the circuit in the circuit design process, negative resistors with large magnitudes should be chosen, so they would meet the satisfactory conditions for stability.

Case number	Given Conditions	Derived stability conditions
1	R1 > 0 R2 < 0	Always unstable
2	R1 > 0 R2 < 0 R3 > 0	$\lvert R2 \rvert > R1+R3$
3	R1 > 0 R2 < 0 R3 > 0 R4 > 0 R5 > 0	$\lvert R2 \rvert > \frac{R3R5+R3R4+R1R5+R1R4}{R5+R3+R1+R4}$
4	R1 > 0 R2 > 0 R3 < 0 R4 > 0 R5 > 0	$\lvert R3 \rvert > \frac{R2R5+R2R4+R1R2+R1R5+R1R4}{R2+R4+R5}$
5	R1 > 0 R2 < 0 R3 < 0 R4 < 0 R5 < 0	$\lvert R3 \rvert > R1+R4+R5$ $\lvert R2 \rvert > \frac{R3R5+R3R4+R1R5+R1R4}{R5+R3+R1+R4}$

Table 3: Stability conditions for 2-node and 3-node circuits

This analysis can be done for larger circuits as well, in order to find how effective resistance and magnitude of negative resistance(s) affect stability in those cases. Intuitively, the same phenomenon found from 2 and 3-node cases should apply to the larger circuits as well, however, it would be useful to check this statement mathematically.

References

1. *Salam, M. A. and Rahman, Q. M. (2018) Fundamentals of electrical circuit analysis. Singapore: Springer. doi: 10.1007/978-981-10-8624-3.*
2. *Mughal, A. M. (2016) Real time modeling, simulation and control of dynamical systems. Switzerland: Springer. doi: 10.1007/978-3-319-33906-1*
3. *Zlatanov, Nikola. (2017). DC Power Supplies, Applications and Measurements. 10.13140/RG.2.2.27236.17288.*
4. *John, Jeff St. "Time to Rethink the Use of DC power for the Energy-Smart Home? " Gtm, Greenthech media, 3 Dec, 2013*
5. *Lurch, E. N. (1979) Electric circuit fundamentals. Englewood Cliffs, N.J.: Prentice-Hall.*
6. *Mirzaev I and Gunawardena J (2013) "Laplacian Dynamics on General Graphs," Bulletin of mathematical biology, 75(11), pp. 2118–49. doi: 10.1007/s11538-013-9884-8.*
7. *Knill, Oliver. Dynamical Systems. Harvard University, 2005.*
8. *Kumar,K.S.Suresh. Electric Circuits and Networks, 2009*
9. *Prasolov, V. V. and Ivanov, S. (1994) Problems and theorems in linear algebra. Providence, R.I.: American Mathematical Society (Translations of mathematical monographs, v. 134)*
10. *Alan R.Hevner, A Three Cycle View of Design Science Research*
11. *Jeffreys, H. and Jeffreys, B. S. "Necessary: Sufficient." §1.036 in Methods of Mathematical Physics Cambridge, England: Cambridge University Press, pp. 10-11, 1988.*

Appendix A

Analytical calculations

Case 1

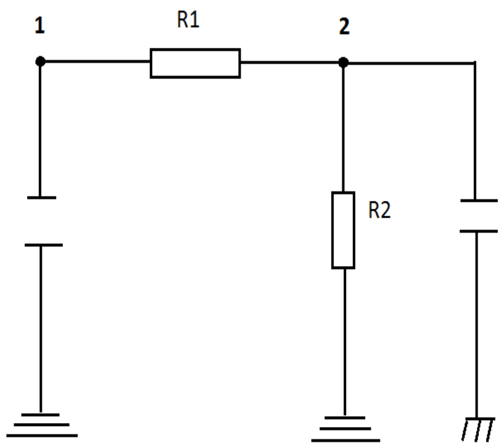


Figure 5

Conditions:

R1 > 0

R2 < 0

Matrices:

D = $\begin{pmatrix} 1/R1 & 0 \\ 0 & 1/R1 \end{pmatrix}$

$A = \begin{pmatrix} 0 & 1/R1 \\ 1/R1 & 0 \end{pmatrix}$

L = $\begin{pmatrix} 1/R1 & -1/R1 \\ -1/R1 & 1/R1 \end{pmatrix}$

$\Delta = \begin{pmatrix} 0 & 0 \\ 0 & 1/R2 \end{pmatrix}$

L+ Δ = $\begin{pmatrix} 1/R1 & -1/R1 \\ -1/R1 & 1/R1 + 1/R2 \end{pmatrix}$

- (L+ Δ) = $\begin{pmatrix} -1/R1 & 1/R1 \\ 1/R1 & -1/R1 - 1/R2 \end{pmatrix}$

Calculations:

1. Finding the characteristic polynomial of the circuit

λI- (-(L+ Δ)) = $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} + \begin{pmatrix} 1/\mathrm{R1} & -1/\mathrm{R1} \\ -1/\mathrm{R1} & 1/\mathrm{R1}+1/\mathrm{R2} \end{pmatrix} = \begin{pmatrix} \lambda+1/\mathrm{R1} & -1/\mathrm{R1} \\ -1/\mathrm{R1} & \lambda+1/\mathrm{R1}+1/\mathrm{R2} \end{pmatrix}$

For simplicity, 1/R1 = a,   1/R2 = b.

In that case*:* $\begin{pmatrix} \lambda+1/\mathrm{R1} & -1/R1 \\ -1/R1 & \lambda+1/\mathrm{R1}+1/\mathrm{R2} \end{pmatrix} = \begin{pmatrix} \lambda+\mathrm{a} & -a \\ -a & \lambda+\mathrm{a}+\mathrm{b} \end{pmatrix}$

Det(λI+(L+ *Δ))* = $\lambda^2+\lambda a+\lambda b+\lambda a+a^2+ab-a^2 = \lambda^2+\lambda(2a+b)+ab$

2. Finding the roots of the polynomial

$D = (2a+b)^2-4ab = 4a^2+4ab+b^2-4ab = 4a^2+b^2$

$\lambda_1 = \frac{-(2a+b)+\sqrt{D}}{2}$

$\lambda_2 = \frac{-(2a+b)-\sqrt{D}}{2}$

3. Analysing conditions for stability

First root can be analysed first. For stability of the circuit, the roots of the characteristic polynomial should be negative. In the first case:

$-(2a+b)+\sqrt{D} < 0$

$\sqrt{4a^2+b^2} < 2a+b$

$4a^2+b^2 > 0$

$2a+b > 0$

$b > -2a$

$4a^2+b^2 < 4a^2+4ab+b^2$

$0 < 4ab$

Now, it should be noted that we inserted variable "a" instead of 1/R1. It is given that R1 is positive therefore "a" will be positive as well. However, R2 is negative so "b" will have a negative value. It can be concluded that "4ab" will always be less than 0 since the product of positive and negative values is negative. One of the eigenvalues will have a positive value, hence the circuit will always be unstable as long as R2<0.

Case 2

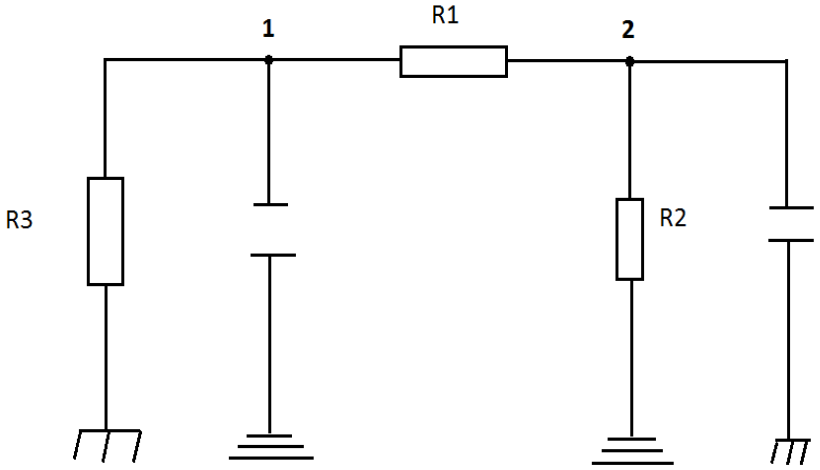


Figure 6

Conditions:

R1 > 0

R2 < 0

R3 > 0

Matrices:

D = $\begin{pmatrix} 1/\mathrm{R1} & 0 \\ 0 & 1/\mathrm{R1} \end{pmatrix}$

A = $\begin{pmatrix} 0 & 1/\mathrm{R1} \\ 1/\mathrm{R1} & 0 \end{pmatrix}$

L = $\begin{pmatrix} 1/\mathrm{R1} & -1/\mathrm{R1} \\ -1/\mathrm{R1} & 1/\mathrm{R1} \end{pmatrix}$

Δ = $\begin{pmatrix} 1/\mathrm{R3} & 0 \\ 0 & 1/\mathrm{R2} \end{pmatrix}$

L+ Δ = $\begin{pmatrix} 1/R1 + 1/R3 & -1/R1 \\ -1/R1 & 1/R1 + 1/R2 \end{pmatrix}$

- (L+ Δ) = $\begin{pmatrix} -1/\mathrm{R1} - 1/\mathrm{R3} & 1/\mathrm{R1} \\ 1/\mathrm{R1} & -1/\mathrm{R1} - 1/\mathrm{R2} \end{pmatrix}$

Analytical calculations:

1. **Finding the characteristic polynomial**

λI- (-(L+ *Δ*)) = $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$+$\begin{pmatrix} 1/R1 + 1/R3 & -1/R1 \\ -1/R1 & 1/R1 + 1/R2 \end{pmatrix}$= $\begin{pmatrix} \lambda + 1/\mathrm{R1} + 1/\mathrm{R3} & -1/R1 \\ -1/R1 & \lambda + 1/\mathrm{R1} + 1/\mathrm{R2} \end{pmatrix}$

Again, for simplicity, 1/R1=a, 1/R2=b, 1/R3=c.

In that case:

$\lambda I - (-(L+\Delta)) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} + \begin{pmatrix} a+c & -a \\ -a & a+b \end{pmatrix} = \begin{pmatrix} \lambda + a + c & -a \\ -a & \lambda + a + b \end{pmatrix}$

$\det(\lambda I - (-(L+\Delta))) = \lambda^2 + \lambda a + \lambda b + \lambda a + a^2 + ab + \lambda c + ac + bc - a^2 = \lambda^2 + \lambda(2a+b+c) + ab + ac + bc$

2. Finding the roots of the polynomial

$\lambda^2 + \lambda(2a+b+c) + ab + ac + bc = 0$

$D = (2a+b+c)^2 - 4(ab+ac+bc) = 4a^2+b^2+c^2-2bc$

$\lambda_1 = -(2a+b+c) + \sqrt{D}$

$\lambda_2 = -(2a+b+c) - \sqrt{D}$

3. Finding the values which make the roots negative

a) $\lambda_1 = -(2a+b+c) + \sqrt{D} < 0$

$\sqrt{D} < 2a+b+c$

$\sqrt{4a^2 + b^2 + c^2 - 2bc} < 2a+b+c$

$4a^2+b^2+c^2-2bc > 0$ ($a>0, b<0, c>0$)

$2a+b+c > 0$

$b > -2a - c$

$4a^2+b^2+c^2-2bc < 4a^2+b^2+c^2+4ab+4ac+2bc$

$4ab+4ac+4bc > 0$

$ab+ac + bc > 0$

$b(a + c) > -ac$

$b > \frac{-ac}{a+c}$

b) $\lambda_2 = -(2a+b+c) - \sqrt{D} < 0$

$-(2a+b+c) < \sqrt{4a^2 + b^2 + c^2 - 2bc}$

$-(2a+b+c) < 0$ (inequality **a**)

-(2a+b+c) < $\sqrt{4a^2 + b^2 + c^2 - 2bc}$
If inequality **a** holds, inequality above will holds as well.

Substituting the variables from previous inequalities, following results were obtained:

- b > $\frac{-ac}{a+c}$

- b > -2a - c

Following the substitution:

- $\frac{1}{R2} > \frac{-\frac{1}{R1}\frac{1}{R3}}{\frac{1}{R1}+\frac{1}{R3}}$

$\frac{1}{R2} > \frac{-\frac{1}{R1}\frac{1}{R3}}{\frac{R1+R3}{R1R3}}$

$\frac{1}{R2} > \frac{-1}{R1+R3}$

$R_2 < -(R_1+R_3)$

- $\frac{1}{R2} > -\frac{2}{R1} - \frac{1}{R3}$

$R_2 < -\frac{R1R3}{2R3+R1}$

The answer of the first inequality is the range (- ∞ ; - (R1+R3)), the second inequality has the range (-∞; $-\frac{R1R3}{2R3+R1}$).

It can be assumed that $\frac{R1R3}{2R3+R1}$ is smaller than (R1+R3), since it is a ratio, when the values of these resistors are positive, as given. The assumption was proven mathematically correct:

$\frac{R1R3}{2R3+R1} < R_1+R_3$

$R_1R_3 < 3R_1R_3+2R_3^2+R_1^2$

$0 < 2R_3^2+ 2R_1R_3+R_1^2$

$0 < R_3^2 + (R_3+R_1)^2$

The inequality above will always hold, since the square values are always positive, it can be claimed that for any positive values of R_1 and R_3, sum of these resistance values will be greater than ratio $\frac{R1R3}{2R3+R1}$.

Since both of these values have negative signs in front of them, smaller value will be chosen, which is the negative sum of the resistors R_1 and R_3. Any value greater than that will not be applicable since it will not hold for both of the equations.

Therefore, it can be concluded that the circuit is stable when:

$$R_2 < -(R_1+R_3)$$

$$|R2| > R1+R3$$

$$R_1 > 0,$$

$$R_2 < 0,$$

$$R_3 > 0.$$

Effective resistance and stability

The effective resistance of the circuit, excluding the negative resistor R2, will equal to the sum of two resistors, R1 and R3, which are connected in series. Therefore:

$$R_{eff} = R1+R3$$

We observe that this expression is the same as the right hand side of the stability inequality. Therefore, it can be concluded that:

$$|R2| > R_{eff}$$

Case 3

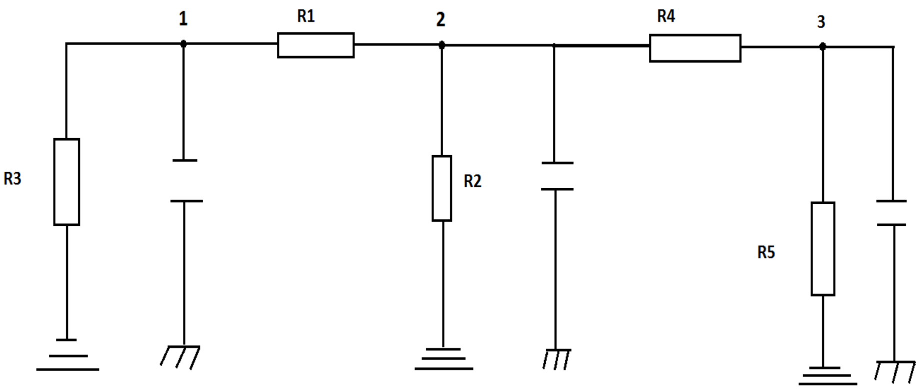


Figure 7

Conditions:

R2 < 0

R1 > 0

R3 > 0

R4 > 0

R5 > 0

Matrices:

$$D = \left\langle \begin{array}{c|c|c} 1/R1 & 0 & 0 \\ 0 & 1/R1 + 1/R4 & 0 \\ 0 & 0 & 1/R4 \end{array} \right\rangle \qquad L = \left\langle \begin{array}{c|c|c} 1/R1 & -1/R1 & a3 \\ -1/R1 & 1/R1 + 1/R4 & -1/R4 \\ 0 & -1/R4 & 1/R4 \end{array} \right\rangle$$

$$A = \left\langle \begin{array}{c|c|c} 0 & 1/R1 & 0 \\ 1/R1 & 0 & 1/R4 \\ 0 & 1/R4 & 0 \end{array} \right\rangle \qquad \Delta = \left\langle \begin{array}{c|c|c} 1/R3 & 0 & 0 \\ 0 & 1/R2 & 0 \\ 0 & 0 & 1/R5 \end{array} \right\rangle$$

$$L+\Delta = \left\langle \begin{array}{c|c|c} 1/R1 + 1/R3 & -1/R1 & 0 \\ -1/R1 & 1/R1 + 1/R4 + 1/R2 & -1/R4 \\ 0 & -1/R4 & 1/R4 + 1/R5 \end{array} \right\rangle$$

-(L+Δ) = $\begin{pmatrix} -1/R1-1/R3 & 1/R1 & 0 \\ 1/R1 & -1/R1-1/R4-1/R2 & 1/R4 \\ 0 & 1/R4 & -1/R4-1/R5 \end{pmatrix}$

(λI- (-(L+ Δ)= (λI+(L+ Δ) = $\begin{pmatrix} \lambda+1/R1+1/R3 & -1/R1 & 0 \\ -1/R1 & \lambda+1/R1+1/R4+1/R2 & -1/R4 \\ 0 & -1/R4 & \lambda+1/R4+1/R5 \end{pmatrix}$

This time, 1/R1 = **a**, 1/R2 = **b**, 1/R3 = **c** , 1/R4 = **d** , 1/R5 = **e**.
In that case :

(λI- (-(L+ Δ)= (λI+(L+ Δ) = $\begin{pmatrix} \lambda+a+c & -a & 0 \\ -a & \lambda+a+d+b & -d \\ 0 & -d & \lambda+d+e \end{pmatrix}$

Det(λI+(L+ Δ) = (λ $+a+c)((\lambda+a+d+b)(\lambda+d+e)$-$d^2$))-(-a)(-a(λ $+d+e$))

3[rd] degree polynomial. Too complex to deduct a result!

-(L+Δ) = $\begin{pmatrix} -1/R1-1/R3 & 1/R1 & 0 \\ 1/R1 & -1/R1-1/R4-1/R2 & 1/R4 \\ 0 & 1/R4 & -1/R4-1/R5 \end{pmatrix}$ = $\begin{pmatrix} -a-c & a & 0 \\ a & -a-d-b & d \\ 0 & d & -d-e \end{pmatrix}$

Matrix is symmetric! $-(L+\Delta)^T$ = -(L+Δ)

A negative definite matrix is a symmetric matrix A for which all eigenvalues are negative.

Which values of a,b,c,d,e make the matrix negative definite?

Det(A1) < 0 , Det(A2) > 0 Det(A3) < 0

Finding determinants of submatrices

3 matrices are formed :

A1 = (-a-c)

A2 = $\begin{pmatrix} -a-c & a \\ a & -a-d-b \end{pmatrix}$

A3= $\begin{pmatrix} -a-c & a & 0 \\ a & -a-d-b & d \\ 0 & d & -d-e \end{pmatrix}$

Det(A1) = (-a-c) < 0 a > 0, c > 0, condition always met!

Det(A2) = (-a-c)(-a-d-b)-a^2 = ad+ab+ac+cd+bc > 0

Det(A3) = (-a-c)((-a-d-b)(-d-e)-d^2))-a(a(-d-e)) = -ade-abd-abe-acd-ace-cde-cbd-cbe < 0

Derivation of the stability condition(s)

$b > -\frac{ad+ac+cd}{a+c}$

$b > -\frac{ade+acd+ace+cde}{ad+ae+cd+ce}$

$\frac{ad+ac+cd}{a+c} > \frac{ade+acd+ace+cde}{ad+ae+cd+ce}$ *(maximum of these 2 expressions)*

$b > -\frac{ade+acd+ace+cde}{ad+ae+cd+ce}$

$\frac{1}{R2} > -\frac{\frac{R5+R3+R4+R1}{R1R3R4R5}}{\frac{R3R5+R3R4+R1R5+R1R4}{R1R3R4R5}}$

$\frac{1}{R2} > -\frac{R5+R3+R4+R1}{R3R5+R3R4+R1R5+R1R4}$

$R2 < -\frac{R3R5+R3R4+R1R5+R1R4}{R5+R3+R1+R4}$

$|R2| > \frac{R3R5+R3R4+R1R5+R1R4}{R5+R3+R1+R4}$ *Stability condition*

Effective resistance and stability

In this circuit, 4 resistors which have a positive resistance value can be distinguished, which are R1,R3,R4,R5. Calculations to find their effective resistance are given below :

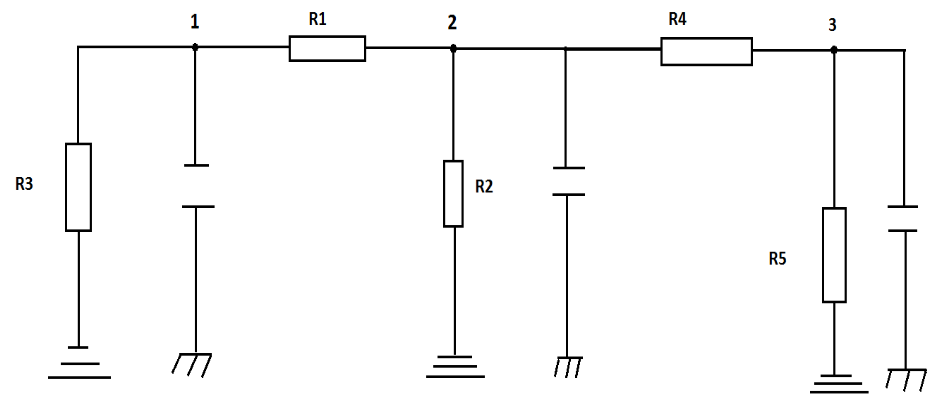


Figure 8

R_{eff1}=R4+R5

$$\frac{1}{Reff2} = \frac{1}{R2} + \frac{1}{R4+R5}$$

$$\frac{1}{Reff2} = \frac{R2+R4+R5}{R2(R4+R5}$$

$$R_{eff2} = \frac{R2(R4+R5)}{R2+R4+R5}$$

$$R_{eff} = R1 + \frac{R2(R4+R5)}{R2+R4+R5}$$

$$R_{eff} = \frac{R3R5+R3R4+R1R5+R1R4}{R5+R3+R1+R4}$$

It can be seen that the expression for effective resistance equals to the right hand side of the stability condition. Therefore, stability condition can be given in terms of effective resistance :

$$|R2| > R_{eff}$$

Case 4

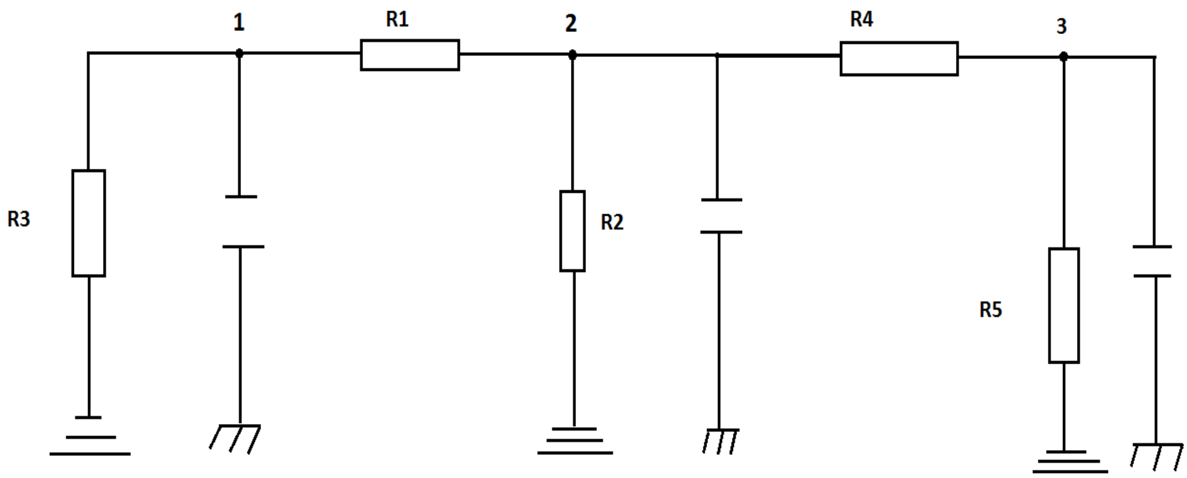


Figure 9

This case is similar to case 3, in terms of structure of the electric circuit. However, in this case, resistance R3 will be negative, and other resistances will be positive.

Conditions:

R1 > 0

R2 > 0

R3 < 0

R4 > 0

R5 > 0

The same equation found which set the determinant of the 3rd submatrix to be less than zero in case 3 will be used. That equation is:

-ade-abd-abe-acd-ace-cde-cbd-cbe < 0

ade+abd+abe+acd+ace+cde+cbd+cbe > 0

c(ad+ae+de+bd+be)+ade+abd+abe) > 0

$$c > -\frac{ade+abd+abe}{ad+ae+de+bd+be}$$

$$\frac{1}{R3} < -\frac{\frac{R2+R5+R4}{R1R2R4R5}}{\frac{R2R5+R2R4+R1R2+R1R5+R1R4}{R1R2R4R5}}$$

$$\frac{1}{R3} < -\frac{R2+R4+R5}{R2R5+R2R4+R1R2+R1R5+R1R4}$$

$$R3 < -\frac{R2R5+R2R4+R1R2+R1R5+R1R4}{R2+R4+R5}$$

R3 > $\left|\frac{R2R5+R2R4+R1R2+R1R5+R1R4}{R2+R4+R5}\right|$ **Stability condition**

Effective resistance and stability

The effect of effective resistance on the stability condition can be investigated. Prior to this procedure, effective resistance of part of the circuit which lays on the right side of the node 1 can be calculated.

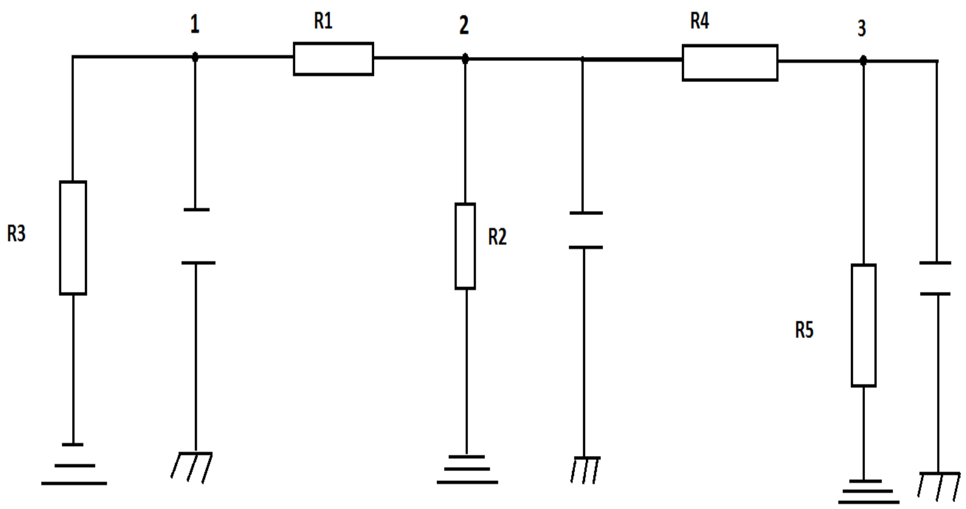


Figure 10

Calculations :

R_{eff1} = R4+R5

$$\frac{1}{Reff2} = \frac{1}{R2} + \frac{1}{R4+R5}$$

$$\frac{1}{Reff2} = \frac{R2+R4+R5}{R2(R4+R5)}$$

$$Reff2 = \frac{R2(R4+R5)}{R2+R4+R5}$$

$$Reff3 = R1+\frac{R2(R4+R5)}{R2+R4+R5} = \frac{R2(R4+R5)+R1R2+R1R4+R1R5}{R2+R4+R5} = \frac{R1R2+R1R4+R1R5+R2R4+R2R5}{R2+R4+R5}$$

As effective resistance is found, now it can be analysed how the effective resistance affects stability.

R3 > $\left|\frac{R2R5+R2R4+R1R2+R1R5+R1R4}{R2+R4+R5}\right|$ Stability condition

Reff = $\frac{R1R2+R1R4+R1R5+R2R4+R2R5}{R2+R4+R5}$ Effective resistance

By substitution:

R3 > $|R_{eff}|$ **Stability condition in terms of effective resistance**

Case 5

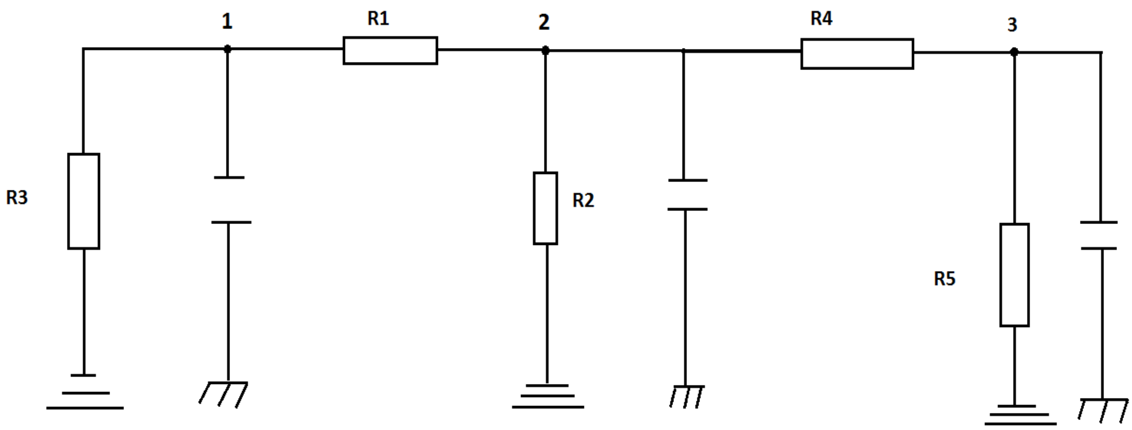


Figure 11

Conditions:

R2 < 0

R3 < 0

R1 > 0

R4 > 0

R5 > 0

Submatrices:

A1 = (-a-c) Det(A1) < 0

A2 = $\begin{pmatrix} -a-c & a \\ a & -a-d-b \end{pmatrix}$ Det(A2) > 0

A3 = $\begin{pmatrix} -a-c & a & 0 \\ a & -a-d-b & d \\ 0 & d & -d-e \end{pmatrix}$ Det(A3) < 0

1. (-a-c) < 0

   -a < c

2. ad+ab+ac+cd+bc > 0

$$b > - \frac{ad+ac+cd}{a+c}$$

3. ade+abd+abe+acd+ace+cde+cbd+cbe > 0

$$b > - \frac{ade+acd+ace+cde}{ad+ae+cd+ce}$$

Replacement of all the variables, a, b, c, d, e to with R1,R2,R3,R4 and R5 will give us all the conditions for stability, when R2 and R3 are negative and other resistance values are positive.

Conditions for submatrices:

1. -a-c < 0

$$- \frac{1}{R1} - \frac{1}{R3} < 0$$

R1<-R3

|R3|> R1

2. ad+ab+ac+cd+bc > 0

$$b > - \frac{ad+ac+cd}{a+c}$$

$$\frac{1}{R2} > -\frac{R1+R3+R4}{R4(R1+R3)}$$

R1+R3+R4 < 0

|R3| > R1+R4

$$R2 < - \frac{R4(R1+R3)}{R1+R3+R4}$$

$$|R2| > \frac{R4(R1+R3)}{R1+R3+R4}$$

3. ade+abd+abe+acd+ace+cde+cbd+cbe > 0

$$b > - \frac{ade+acd+ace+cde}{ad+ae+cd+ce}$$

$$\frac{1}{R2} > - \frac{R5+R3+R4+R1}{(R1+R3)(R4+R5)}$$

R1+R3+R4+R5 < 0

|R3| > R1+R4+R5

$$R2 < - \frac{R3R5+R3R4+R1R5+R1R4}{R5+R3+R1+R4}$$

$|R2| > \frac{R3R5+R3R4+R1R5+R1R4}{R5+R3+R1+R4}$

Note as already mentioned, in this case, two resistors will have a negative value. The inequalities can be shown either with R2 on the right side or R3 on the right side. An arbitrary choice can be made, as the crucial step is to ensure that the determinant of the submatrix is more (less) than zero, and we can can show that inequality in terms of any of the variables R2 or R3. It should be mentioned that if one of these resistances are given on the right side of the inequality, the other one needs to satisfy a specific condition so that inequality holds. For instance, we can analyse condition 2, given above. The inequality $\frac{1}{\mathrm{R2}} > -\frac{\mathrm{R1+R3+R4}}{\mathrm{R4(R1+R3)}}$ can only be transformed into *R2* < - $\frac{R4(R1+R3)}{R1+R3+R4}$ when the expression $\frac{R4(R1+R3)}{R1+R3+R4}$ has a positive value. In order for that to happen, the denominator should be less than zero since it is already known that the nominator will be negative, according to condition 1. Therefore, it can be stated that in order for |R2| > $\frac{R4(R1+R3)}{R1+R3+R4}$ to hold, the inequality |R3| > R1+R4 should hold too. The same pattern is followed in condition 3 as well.

Derivation of sufficient conditions

Several conditions were found in the calculations above and they were given in terms of R2 and R3.

The goal is to find the right hand expression for each of the resistors which will result in the maximum value. So if the condition with the maximum value on the right hand side is satisfied, other conditions are satisfied as well.

Conditions for R3

1. |R3|> R1

2. |R3| > R1+R4

3. ***|R3| > R1+R4+R5***

This case will be relatively easy. It is know that R1, R4 and R5 are all positive. Therefore, the one with the maximum value will be the third condition.

Conditions for R2

1. |R2| > $\frac{R4(R1+R3)}{R1+R3+R4}$

2. ***|R2|*** **>** $\boldsymbol{\frac{R3R5+R3R4+R1R5+R1R4}{R5+R3+R1+R4}}$

It will arbitrarily be assumed that the second condition will be the maximum of these two and this assumption can be mathematically checked. If the second expression has the maximum value, the subtraction of the first one from the second one will be greater than zero.

$$\frac{R3R5+R3R4+R1R5+R1R4}{R5+R3+R1+R4} - \frac{R4(R1+R3)}{R1+R3+R4} > 0$$

$$\frac{(R1+R3)(R4+R5)(R1+R3+R4)-(R1+R3+R4+R5)(R1R4+R3R4)}{(R1+R3+R4)(R1+R3+R4+R5)} > 0$$

$$\frac{(R1)^2R5+2R1R3R5+(R3)^2R5}{(R1+R3+R4)(R1+R3+R4+R5)} > 0$$

$$\frac{R5(R1+R3)^2}{(R1+R3+R4)(R1+R3+R4+R5)} > 0$$

Condition above is satisfied since both nominator and denominator are positive. Nominator is positive because it includes a positive variable (R5) and a quadratic equation. Denominator is positive because the expression in both brackets are negative, as it was found in Conditions for R3.

Final satisfactory conditions for stability:

$|R3| > R1+R4+R5$

$|R2| > \frac{R3R5+R3R4+R1R5+R1R4}{R5+R3+R1+R4}$

Stability conditions given in terms of effective resistance

The positive resistors of the circuit are R1,R4 and R5, which are connected in series. Therefore, the effective resistance of positive resistors will equal to:

$$R_{eff} = R1+R4+R5$$

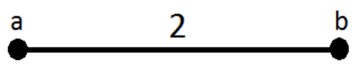
It can be seen that the expression equals to the right hand side of the first inequality. Therefore, one of the necessary conditions for stability can be represented as:

$$|R3| > R_{eff}$$

Appendix B

Cases with always stable and always unstable circuits

It was stated in the problem statement that if the Delta matrix has only positive or only negative resistors, then the matrix "–(*L* +*Δ*)" is stable and unstable, accordingly. In order to explain these statements, sample 2 node network will be demonstrated. This network is given below .



If the Adjacency and Diagonal matrices are formed, following results will be obtained.

D = $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ L = $\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$

A = $\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$

The characteristic polynomial of the Laplacian matrix can be found, by calculating det(λI-L).

Det (λI-L) = $\begin{pmatrix} \lambda - 2 & -2 \\ -2 & \lambda - 2 \end{pmatrix}$ = $(\lambda-2)^2 - 4 = \lambda(\lambda-4)$

$\lambda_1 = 0$ $\lambda_2 = 4$

If the matrix "-(*L* +*Δ*)" needs to have negative eigenvalues for stability, it means that the matrix "(*L* +*Δ*)" needs to have positive eigenvalues for the circuit to be stable. In the eigenvalues obtained above, we see that none of the eigenvalues is negative, so the circuit is stable. However, one of the eigenvalues is 0 so it is on the verge of becoming negative. As it is already mentioned, Delta matrix has only diagonal elements; therefore, if the Delta matrix is added to "L", it will affect only the diagonal elements of "L". Let us consider a Delta matrix, which contains only a positive resistor with resistance "a". In the case, the characteristic polynomial of the matrix "(*L* +*Δ*)" will look like the following:

$\Delta = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ $L = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$

$L + \Delta = \begin{pmatrix} 2 + a & -2 \\ -2 & 2 \end{pmatrix}$

λ I - (L +Δ) = $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ - $\begin{pmatrix} 2+a & -2 \\ -2 & 2 \end{pmatrix}$ = $\begin{pmatrix} \lambda - 2 - \mathrm{a} & 2 \\ 2 & \lambda - 2 \end{pmatrix}$

Det(λ I - (L +Δ)) = (λ-2-a)(λ-2)-4

Det(λ I - (L +Δ)) = λ^2- $\lambda(4+a)+2a$

$\lambda^2- \lambda(4+a)+2a = 0$

D = $(4+a)^2$-8a = a^2+16

λ_1 = 4+a+$\sqrt{a^2+16}$

λ_2 = 4+a-$\sqrt{a^2+16}$

We can see that the first root will always be positive when a>0. The second root can be given in form of an inequality and checked whether it will be stable in every value of ”a”.

4+a-$\sqrt{a^2+16}$ > 0

4+a > $\sqrt{a^2+16}$

16+8a+a^2 > a^2+16

8a > 0

It can be observed that indeed, when a > 0, both of the eigenvalues will be positive, which means that the eigenvalues of *“–(L +Δ)”* will be negative and therefore lead to stability. It has been proven that when Delta matrix has only positive elements, then the circuit will always be stable. Similar calculations can be made for the case where Delta matrix will only have negative elements, and the result can be achieved in a similar way.