BACHELOR THESIS

PHYSICS

Very Special Relativity

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Abstract

Very Special Relativity (VSR) is a proposed reduction of the theory of special relativity (SR), in which effects that break Lorentz invariance would still be VSR invariant. VSR invariant theories could then be implemented to extend the Standard Model in order to incorporate some known phenomena, such as weak CP violation in the weak interaction, or the existence of a neutrino mass. This bachelor thesis starts by studying VSR groups' main characteristics to differentiate them from the SR group. This is first done by determining the isomorphisms that represent the group elements by their actions on a two dimensional plane. Subsequently, further differentiation is achieved by identifying invariant vectors and tensors on which representations of the group elements act. One interesting feature of the groups is that they admit an invariant direction in spacetime. Another peculiar characteristic that is shown is the fact that the complete Lorentz group is obtained when the group elements are conjugated by the parity or time reversal operators. The second part of the thesis is an analysis of the dynamics of spin under VSR, as proposed by two different papers. One of them predicts a VSR Thomas precession which is off by a factor of 10^3 from experimentally established results, while the VSR-extended BMT equation proposed by the other paper is consistent with known results.

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1 Introduction

After Einstein's brilliant insight, following the works of Maxwell's unification of electricity, magnetism and optics [1] , the theory of special relativity started to shape the body of modern physics. In it, two fundamental postulates work as a starting point. Namely, that the speed of light is a universal constant that is independent of the choice of inertial frame of reference and, that the laws of physics are the same in every inertial frame [2].
The Standard Model of particle physics, which is built from the marriage of special relativity and quantum mechanics, is very successful at describing the world at small scales and high energy regimes. It has been able to explain most of the observed particle phenomenology, while being successful at predicting yet unobserved ones (which, later came to be observed). However, it is not a complete theory in the sense that it cannot explain some observations. These include, among others, the problems of CP violations not present in processes involving the strong interaction [3], the existence of dark matter, or the fact that neutrinos exhibit mass [4]. This points out the need of thinking of new physics beyond the standard model that could resolve the problems at hand.
One such line of reasoning, proposed by Cohen and Glashow [5] leads to the consideration of subgroups of the Lorentz group of symmetry transformations. They argue that the symmetries of nature might be described by smaller subsets of Lorentz symmetries. These suggested subgroups, together with spacetime translations, constitute what is called "Very Special Relativity" (VSR). In it, the most important features of special relativity, such as the universally isotropic speed of light, time dilation and length contraction, are preserved, while other features such as space isotropy are violated.

The aim of this bachelor thesis is to investigate how these subgroups differ from the full group of Lorentz transformations, and how it extends the current description of nature. To do so, the first part of the paper will first describe standard special relativity in group theoretical terms, i.e a description of the full Lorentz group. It will then be followed by a description of the specific subgroups proposed by Cohen and Glashow, how they are related to the full Lorentz group, what they preserve and how they differ from it. After this, VSR derivations of spin dynamics, as proposed by different teams, will be investigated. These dynamics are described by modified BMT equations and the resulting Thomas precession. Contradicting results are found, suggesting that one of them might be wrong. Finally, it is argued that the results coupling VSR predicted spin dynamics to established SR models by means of a tunable parameter are more sensible, and that the ones suggesting VSR is incompatible with Thomas Precession may have taken a wrong path leading to their results.

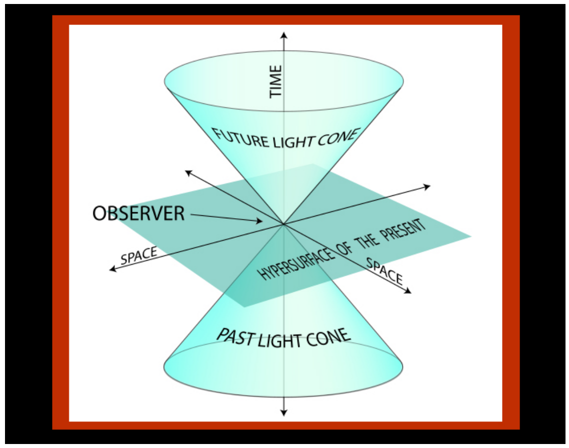

Figure 1: Spacetime diagram, with a two-dimensional spatial surface (reference: Wikipedia image)

2 Special Relativity and the Lorentz Group

2.1 Causal Structure

Special relativity is concerned with relating how different observers in different frames can relate what they observe in their respective frames. To do so, one needs to consider the set of transformations that lead to invariance of observed physical phenomena. In order to do this, one must think about events that take place in a particular position in space, at a particular time. The time ordering of the events is important as causality must be preserved. The most important quantity that will define the ordering of events is what is called the spacetime interval:

Let two events be denoted by the coordinates (ct_1, x_1, y_1, z_1) and (ct_2, x_2, y_2, z_2). The spacetime interval s^2 is defined as:

$$\begin{aligned} s^2 &\equiv c^2\Delta t^2 - |\Delta \mathbf{x}|^2 \\ &= c^2(t_2 - t_1)^2 - \left(|x_2 - x_1|^2 + |y_2 - y_1|^2 + |z_2 - z_1|^2\right). \end{aligned} \tag{1}$$

This definition allows for three kinds of connections between the events. Depending on its sign, the interval s^2 is said to be:

$$\begin{aligned} s^2 > 0 \quad &\rightarrow \quad \text{time-like} \\ s^2 < 0 \quad &\rightarrow \quad \text{space-like} \\ s^2 = 0 \quad &\rightarrow \quad \text{light-like} \end{aligned}$$

Causality is limited by the maximal speed of light, denoted c, in the sense that no events can be causally connected if there has not been enough time for information to propagate between the spatial parts of the coordinates. This is well illustrated by a space-time diagram, in which the light cone separates the causally connected region of spacetime from the unconnected one. A simple example in two spatial dimensions is given in figure 1.

The interval (1) is the quantity that every observer agrees on, regardless of their frame of reference. It is the universal invariant quantity. This leads to considering a vector space endowed with a special kind of inner product, in which Lorentz transformations will act. It is called Minkowski spacetime, denoted in this thesis by $\mathbb{M}$, and is defined to be the four dimensional real space of vectors $\underline{\boldsymbol{x}} = (x_0, x_1, x_2, x_3)$ (where $x_0 \equiv ct$ and from now on, natural units shall be used with $c = 1$), and a Minkowski inner product:

$$\underline{\boldsymbol{x}} \cdot \underline{\boldsymbol{y}} = x_0 y_0 - \mathbf{x} \cdot \mathbf{y} = x_0 y_0 - x_1 y_1 - x_2 y_2 - x_3 y_3. \tag{2}$$

Minkowski four-vectors will be represented by a bold letter with an underbar, while Euclidian space vectors by the regular bold letters. If Einstein's summation convention is used, and the product is viewed from the point of view of matrix multiplication, then it can be written as:

$$x_\mu y^\mu = x_\mu \eta^{\mu\nu} y_\nu,$$

where $\eta = \mathrm{diag}(1, -1, -1, -1)$ is called the metric tensor and determines the geometry of the vector space, x^μ is the component of a contravariant vector and x_μ is its covariant version; they are related by having the metric operate by raising/lowering their indices, $x^\mu = \eta^{\mu\nu} x_\nu$ or $x_\mu = \eta_{\mu\nu} x^\nu$.

2.2 Symmetries of the Lorentz Group

The symmetries of the Minkowski space are given by all transformations that leave the Minkowski inner product unchanged; i.e the set of all Lorentz transformations

$$\mathcal{L} = \{\Lambda | \quad \underline{\boldsymbol{x}}' = \Lambda \underline{\boldsymbol{x}}, \quad \underline{\boldsymbol{x}}'^T \eta' \underline{\boldsymbol{x}}' = \underline{\boldsymbol{x}}^T \eta \underline{\boldsymbol{x}}, \quad \underline{\boldsymbol{x}}, \underline{\boldsymbol{x}}' \in \mathbb{M}\}. \tag{3}$$

Using Einstein's notation, the transformation can be expressed as $x'^\mu = \Lambda^\mu{}_\nu x^\nu$, and what then defines the set of Lorentz transformation are the Λ's satisfying

$$\Lambda^\mu{}_\sigma \eta^{\sigma\tau} \Lambda^\nu{}_\tau = \eta^{\mu\nu}, \tag{4}$$

which allows for the following restriction:

$$\det(\Lambda) = \pm 1. \tag{5}$$

As equation (4) implies, inverses of Lorentz transformations exist and they also satisfy equation (5), meaning they are also part of $\mathcal{L}$. The inverses of the matrices satisfy the condition $\Lambda^{-1} = \eta \Lambda^T \eta$, thus making them part of the set of pseudo-orthogonal matrices, which, along with matrix multiplication form the Lie group $O(3,1)$. When $\det(\Lambda)$=1, they form the subgroup of proper Lorentz transformations $SO(3,1)$ [6] . In fact, $\mathcal{L}$, together with matrix multiplication as a composition law, forms a four dimensional representation of the Lorentz Lie group.
The Lorentz group is a group that has four connected components. They can be distinguished by the sign of the determinant and the sign of the temporal component $\Lambda^0{}_0$. They are [6]:

1. $\mathcal{L}_+^\uparrow$: the set of proper ortochronous Lorentz transformations (also called $SO^+(3,1)$). In this case $\det(\Lambda) = 1$ and $\Lambda^0{}_0 \geq 1$.

2. $\mathcal{L}_-^\uparrow$: The set of non-proper, orthochronous Lorentz transformation; i.e $\det(\Lambda) = -1$ and $\Lambda^0{}_0 \geq 1$.

3. $\mathcal{L}_+^\downarrow$: The set of proper, non-orthochronous Lorentz transformations; i.e $\det(\Lambda) = 1$ and $\Lambda^0{}_0 \leq -1$.

4. $\mathcal{L}_-^\downarrow$: The set of non-proper, non-orthochronous Lorentz transformations; i.e $\det(\Lambda) = -1$ and $\Lambda^0{}_0 \leq -1$.

The first one is of great importance as it is the only subgroup to which every element can be connected to the identity element of the Lorentz group; i.e it is the only connected continuous subgroup of the Lorentz Lie group. The focus from here onward shall therefore be on the group $\mathcal{L}_+^\uparrow$, as the subsequent discussion on the VSR subgroups relies on understanding this one.

2.3 Lorentz Lie Algebra

Lie groups are described by parameters that allow them to change continuously. Taking some parameterization $\boldsymbol{\omega}$, the following expansion in the neighborhood of the identity can be made for the specific four dimensional representation:

$$\Lambda(d\boldsymbol{\omega}) = \mathbb{1} + d\omega_i X^i,$$

where the X^i's are called generators of the group, given by

$$X^i \equiv \left.\frac{\partial \Lambda(\boldsymbol{\omega})}{\partial \omega_i}\right|_{\boldsymbol{\omega}=0}.$$

They can generate all elements of the group. To see how this is the case, the transformation should be taken away from identity [6]. Writing $d\omega_i = \omega_i/k$, group component can be obtained through exponentiation

$$\Lambda(\boldsymbol{\omega}) = \lim_{k\to\infty}\left(\mathbb{1} + \frac{\omega_i X^i}{k}\right)^k = \exp\big(\omega_i X^i\big). \tag{6}$$

If the generators form a commuting set, then all group elements can be written as one exponential. Otherwise, all group elements can be written as a product of matrix exponentials. The generators of the group constitute what is called an algebra. It is these Lie algebras that are of importance as they define the properties of the group they generate. A Lie algebra is given by the set of generators together with what is called a Lie bracket, forming a linear vector space tangent to the identity element of the group. The Lie bracket in the case of the representation used is the usual commutation relations between the matrices. But more generally, the Lie bracket is defined as the non-associative, bilinear map $\mathcal{G} \times \mathcal{G} \to \mathcal{G}$, $(X, Y) \mapsto [X, Y]$ where $\mathcal{G}$ is the Lie algebra, being a vector space over some field F. The Lie bracket satisfies the following conditions

$$\begin{aligned}
[aX + bY, Z] &= a[X, Z] + b[Y, Z] && \text{(Bilinearity)}\\
[X, X] &= 0 && \text{(Alternativity)}\\
[X, [Y, Z]] &= [[X, Y], Z] + [[Z, Y], X] && \text{(Jacobi Identity)}\\
[X, Y] &= -[Y, X] && \text{(Anticommutativity)}
\end{aligned} \tag{7}$$

where $a, b \in F$ are scalars over the field F, and $X, Y, Z \in \mathcal{G}$ are elements of the Lie algebra. In the case of the Lorentz Lie algebra, the infinitesimal transformation

$$\Lambda^\mu{}_\nu = \mathbb{1} + \omega^\mu{}_\nu,$$

together with the condition imposed by equation (4) leads to the infinitesimal matrices ω being antisymmetric in nature, $\omega^{\mu\nu} = -\omega^{\nu\mu}$. This means there are six independent components, representing the three rotations about axes which are orthogonal to each other and three boosts in three orthogonal directions. These can be expanded into a basis of 6 antisymmetric matrices $(\mathcal{M}^{\rho\sigma})^{\mu\nu}$, where the pair of indices $\{\rho, \sigma\}$ are also antisymmetric. For example $(\mathcal{M}^{01})^{\mu\nu} = (-\mathcal{M}^{10})^{\mu\nu}$ and $(\mathcal{M}^{01})^{\mu\nu} = (-\mathcal{M}^{01})^{\nu\mu}$. This basis can be written as [6]

$$(\mathcal{M}^{\rho\sigma})^\mu{}_\nu = i(\eta^{\rho\mu}\delta^\sigma{}_\nu - \eta^{\sigma\mu}\delta^\rho{}_\nu), \tag{8}$$

where $\delta^\mu{}_\nu = \mathrm{diag}(1, 1, 1, 1) = \mathbb{1}$, and the antisymmetry in the pair $\{\mu, \nu\}$ is not imposed anymore due to one index being lowered in order to make use of Einstein's summation convention. These matrices obey the Lorentz Lie algebra commutation relations:

$$[\mathcal{M}^{\rho\sigma}, \mathcal{M}^{\tau\phi}] = i(\eta^{\sigma\tau}\mathcal{M}^{\rho\phi} - \eta^{\rho\tau}\mathcal{M}^{\sigma\phi} + \eta^{\rho\phi}\mathcal{M}^{\sigma\tau} - \eta^{\sigma\phi}\mathcal{M}^{\rho\tau}). \tag{9}$$

Now any of the infinitesimal antisymmetric matrices $\omega^\mu{}_\nu$ can be written as linear combinations of the generators of Lorentz transformations $\mathcal{M}^{\rho\sigma}$ [7]:

$$\omega^\mu{}_\nu = -\frac{i}{2}\zeta_{\rho\sigma}(\mathcal{M}^{\rho\sigma})^\mu{}_\nu, \tag{10}$$

where $\zeta_{\rho\sigma}$ expresses the parametrization that decides which transformation is dealt with, and from which a full transformation is obtained by exponentiation:

$$\Lambda = \exp\left(-\frac{i}{2}\zeta_{\rho\sigma}\mathcal{M}^{\rho\sigma}\right). \tag{11}$$

Due to the nature of the Lorentz group in being non-compact, not all elements of the Lorentz subgroup $\mathcal{L}_+^\uparrow$ can be written in one exponential. A more general expression for a transformation would be given in terms of products of matrix exponentials, as will be shown later in the text (equation (14)). To put these matrices in a more familiar form, the generators of rotation, J^i, and those of boosts, K^i, can be expressed as

$$\begin{aligned} J^i &= \frac{1}{2}\epsilon_{ijk}\mathcal{M}^{jk}, \\ K^i &= M^{0i}, \end{aligned} \tag{12}$$

where the indices $\{i, j, k\}$ run from 1 to 3. Concretely, they take the form

$$J^1 = M^{23} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} \quad K^1 = M^{01} = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$J^2 = M^{31} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} \quad K^2 = M^{02} = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$J^3 = M^{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad K^3 = M^{03} = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}.$$

The algebra of commutation relations between these generators is:

$$[J^i, J^j] = i\epsilon_{ijk}J^k, \qquad [J^i, K^j] = i\epsilon_{ijk}K^k, \qquad [K^i, K^j] = -i\epsilon_{ijk}J^k. \tag{13}$$

It can be shown that any element of $\mathcal{L}_+^\uparrow$ can be uniquely written as a rotation, followed by a boost [8], therefore all group elements can be written as a product of the exponentials

$$\Lambda = \exp(-i\boldsymbol{\chi} \cdot \mathbf{K})\exp(-i\boldsymbol{\theta} \cdot \mathbf{J}), \tag{14}$$

in which the parameters $\boldsymbol{\theta} = (\theta_1,\ \theta_2,\ \theta_3)$ and $\boldsymbol{\chi} = (\chi_1,\ \chi_2,\ \chi_3)$ determine the "size" and nature of the transformations.

In order to obtain the full Lorentz group, extra operators must be included that will connect $\mathcal{L}_+^\uparrow$ to the rest of the group. These are parity transformations as well as time inversions, two elements which are discrete and not smoothly connected to the identity. Their forms in the vector representation is

$$P = \mathrm{diag}(1, -1, -1, -1), \qquad T = \mathrm{diag}(-1, 1, 1, 1).$$

To obtain $\mathcal{L}_-^\uparrow$, $\mathcal{L}_+^\downarrow$ and $\mathcal{L}_-^\downarrow$, one must multiply elements of $\mathcal{L}_+^\uparrow$ with P, T and PT respectively.

2.4 Poincaré Group

The set of transformations that make up special relativity contains all isometries of Minkowski spacetime. Thus on top of the full Lorentz group, the set of all spacetime translation are also part of the symmetries of special relativity. This larger group is called the Poincaré group, and has four extra parameters, making up a total of 10. An element from this group acts on a four-vector $\underline{\boldsymbol{x}}$ as $\mathcal{P}\underline{\boldsymbol{x}} = (\underline{\boldsymbol{a}}|\Lambda)\underline{\boldsymbol{x}} = \Lambda\underline{\boldsymbol{x}} + \underline{\boldsymbol{a}}$, where $\underline{\boldsymbol{a}}$ is a translation four-vector. The group composition law can be expressed as [6]

$$\mathcal{P}_1\mathcal{P}_2 = (\underline{\boldsymbol{a}}_1|\Lambda_1)(\underline{\boldsymbol{a}}_2|\Lambda_2) = (\Lambda_2\underline{\boldsymbol{a}}_1 + \underline{\boldsymbol{a}}_2|\Lambda_1\Lambda_2), \tag{15}$$

The generators for infinitesimal translations along a four-vector $\underline{\boldsymbol{a}}$ are given by four-momentum operators $P_\mu = i\partial_\mu$, such that the full translation is given by the operator $\exp(-ia_\mu P^\mu)$, acting on some spacetime position vector x_μ. The last set of commutation relation of the whole Poincaré group is given by

$$[P^\mu, P^\nu] = 0 \qquad\qquad [P^\mu, \mathcal{M}^{\tau\sigma}] = i(\eta^{\tau\mu}P^\sigma - \eta^{\sigma\mu}P^\tau) \tag{16}$$

which, in terms of boosts and rotation generators give

$$\begin{aligned} &[P^0, K^j] = iP^j \qquad\qquad && [P^k, K^j] = iP^0\delta_{jk} \\ &[P^0, J^k] = 0 && [P^l, J^k] = -i\epsilon_{klm}P^m \end{aligned}$$

The final version of the most general transformation that upholds the symmetry of special relativity is then given by

$$\mathcal{P} = \exp(-ia_\mu P^\mu)\exp(-i\boldsymbol{\chi}\cdot\mathbf{K})\exp(-i\boldsymbol{\theta}\cdot\mathbf{J}) \tag{17}$$

along with the discrete symmetries mentioned earlier.

3 Very Special Relativity

Building on the previous general discussion, Cohen and Glashow propose that the fundamental symmetries of nature do not actually follow the full Lorentz group, but a subset thereof. They identified four of those subgroups, called $T(2)$, $E(2)$, $HOM(2)$ and $SIM(2)$, that share similar properties, such as preserving essential features of special relativity like the constancy of the speed of light, length contraction along the direction of motion and time dilation [5]. On top of that, they share the peculiar property of generating the full Lorentz group when they are adjoined by either of the three discrete transformations P, T or PT. These subgroups will be explored in the following subsections.

3.1 $T(2)$

This group is generated by two generators built from a linear combination of the following Lorentz algebra generators [5]:

$$T_1 \equiv K_x + J_y \qquad\qquad T_2 \equiv K_y - J_x \tag{18}$$

These two form a commuting set:

$$[T_1, T_2] = [K_x, K_y - J_x] + [J_y, K_y - J_x] = -iJ_z + iJ_z = 0 \tag{19}$$

This group is isomorphic to the group of translations on the x_1x_2 plane.The isomorphism is easily identified when exponentiating the algebra generators, and acting on a general four-vector:

$$L = exp(-i\alpha T_1)exp(-i\beta T_2) = \begin{pmatrix} \frac{1}{2}(\alpha^2+\beta^2)+1 & -\alpha & -\beta & -\frac{1}{2}(\alpha^2+\beta^2) \\ -\alpha & 1 & 0 & \alpha \\ -\beta & 0 & 1 & \beta \\ \frac{1}{2}(\alpha^2+\beta^2) & -\alpha & -\beta & -\frac{1}{2}(\alpha^2+\beta^2)+1 \end{pmatrix}$$

$$\begin{aligned} x' = Lx &= \begin{pmatrix} \frac{1}{2}(\alpha^2+\beta^2)+1 & -\alpha & -\beta & -\frac{1}{2}(\alpha^2+\beta^2) \\ -\alpha & 1 & 0 & \alpha \\ -\beta & 0 & 1 & \beta \\ \frac{1}{2}(\alpha^2+\beta^2) & -\alpha & -\beta & -\frac{1}{2}(\alpha^2+\beta^2)+1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \\ &= \begin{pmatrix} x_0 + \frac{1}{2}(\alpha^2+\beta^2)(x_0-x_3) - \alpha x_1 - \beta x_2 \\ x_1 - \alpha(x_0 - x_3) \\ x_2 - \beta(x_0-x_3) \\ x_3 + \frac{1}{2}(\alpha^2+\beta^2)(x_0-x_3) - \alpha x_1 - \beta x_2 \end{pmatrix} \end{aligned}$$

where two independent translation parameters $\epsilon_1 \equiv -\alpha(x_0 - x_3)$ and $\epsilon_2 \equiv -\beta(x_0 - x_3)$ show up and shift the x_1 and x_2 coordinates independently.

The next property that will be shown is the generation of the full Lorentz group when adjoining the $T(2)$ group generators with the discrete symmetries. This property holds for all four VSR groups, as $T(2)$ is a subgroup of the next three groups that will be investigated.
The action of P and T on the $T(2)$ generators is given by the following conjugations

$$\begin{aligned} PT_1P^{-1} &= P(K_x + J_y)P^{-1} = K_x - J_y \\ PT_2P^{-1} &= P(K_y - J_x)P^{-1} = K_y + J_x \\ TT_1T^{-1} &= T(K_x + J_y)T^{-1} = -K_x + J_y \\ TT_2T^{-1} &= T(K_y - J_x)T^{-1} = -K_y - J_x. \end{aligned}$$

Because the algebra forms a linear vector space, any linear combination of the generators will also be a group generator. In particular, the following linear combinations can be formed

$$\frac{1}{2}\left(T_1 + TT_1T^{-1}\right) = J_y \qquad -\frac{1}{2}\left(T_2 + TT_2T^{-1}\right) = J_x$$
$$\frac{1}{2}\left(T_1 - TT_1T^{-1}\right) = K_x \qquad \frac{1}{2}\left(T_2 - TT_2T^{-1}\right) = K_y,$$

and after applying the Lie bracket on these

$$[J_x, J_y] = J_z \qquad [J_x, K_y] = K_z,$$

the proper Lorentz subgroup is obtained. This means that by conjugating the $T(2)$ algebra elements with the discrete operator T (or P), and if P is also included, the full Lorentz Lie algebra, and therefore the full Lorentz group, is obtained.

The next three groups are built from the same two generators T_1 and T_2, adjoined with one or two more Lorentz algebra generators. Each generator added results in additional actions on the plane, on top of translations.

3.2 $E(2)$

This group is generated by $T(2)$, to which the generator of rotation around the z axis is added, making it a three parameter Lie group. The additional commutation relations for the algebra of this group are:

$$[T_1, J_z] = -iT_2 \qquad\qquad [T_2, J_z] = iT_1. \tag{20}$$

It is isomorphic to the group of isometries in a two dimensional plane, i.e the Euclidean group $E(2)$. The group of isometries in 2D involves all transformations that leave the Euclidean distance invariant, meaning all rotations and translations. The isomorphism can be seen, like before, with the action of a group element on a general four-vector. Because $T(2)$ generators commute and the group is isomorphic to that of translations on the plane, the order of operations can be implied from equation (17), i.e $L = \exp(-i\alpha T_1)\exp(-i\beta T_2)\exp(-i\theta J_z)$, and the action on a four-vector takes the form

$$\begin{aligned}
Lx &= \begin{pmatrix} \frac{1}{2}(\alpha^2+\beta^2)+1 & -\alpha\cos\theta-\beta\sin\theta & \alpha\sin\theta-\beta\cos\theta & -\frac{1}{2}(\alpha^2+\beta^2) \\ -\alpha & \cos\theta & -\sin\theta & \alpha \\ -\beta & \sin\theta & \cos\theta & \beta \\ \frac{1}{2}(\alpha^2+\beta^2) & -\alpha\cos\theta-\beta\sin\theta & \alpha\sin\theta-\beta\cos\theta & -\frac{1}{2}(\alpha^2+\beta^2)+1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \\
&= \begin{pmatrix} x_0 + \frac{1}{2}(\alpha^2+\beta^2)(x_0-x_3) - \alpha(x_1\cos\theta - x_2\sin\theta) - \beta(x_1\sin\theta + x_2\cos\theta) \\ x_1\cos\theta - x_2\sin\theta - \alpha(x_0-x_3) \\ x_1\sin\theta + x_2\cos\theta - \beta(x_0-x_3) \\ x_3 + \frac{1}{2}(\alpha^2+\beta^2)(x_0-x_3) - \alpha(x_1\cos\theta - x_2\sin\theta) - \beta(x_1\sin\theta + x_2\cos\theta) \end{pmatrix}.
\end{aligned}$$

The general transformation induces a rotation followed by a translation on the x_1x_2 plane, exposing the isomorphism mentioned earlier.

3.3 $HOM(2)$

This group results from adjoining the generator K_z to T_1 and T_2, also making it a three parameter group. The additional commutation relations are:

$$[T_1, K_z] = iT_1 \qquad\qquad [T_2, K_z] = iT_2. \tag{21}$$

This group is isomorphic to the three-parameter group of orientation preserving similarity transformations, also called homotheties. The general form of such a transformation on the plane is [9]

$$\begin{aligned} x' &= e^{\epsilon_1}x + \epsilon_2, \\ y' &= e^{\epsilon_1}y + \epsilon_3. \end{aligned} \tag{22}$$

The isomorphism in this case is more subtle and not directly obtained by performing a general transformation on a four vector. In order to derive it, a little detour has to be made by expressing the $SO(3,1)$ transformations in terms of the $SL(2,\mathbb{C})$, the special linear group of 2×2 complex matrices with unit determinant.

A general four vector x^μ can be uniquely written as a 2×2 matrix by defining the following:

$$X \equiv x^\mu \sigma_\mu, \tag{23}$$

where $\sigma^\mu = (\mathbb{1}, \boldsymbol{\sigma})$, $\mathbb{1}$ is the unit matrix and $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ is a vector's worth of Pauli matrices. To get the four vector back, a covariant form of Pauli's vector is defined as $\widetilde{\sigma}^\mu = (1, -\boldsymbol{\sigma})$. Using the property $\mathrm{Tr}(\sigma_i\sigma_j) = 2\delta_{ij}$, x^μ can be obtained by taking the following trace:

$$x^\mu = \frac{1}{2}\mathrm{Tr}(\widetilde{\sigma}^\mu X), \tag{24}$$

X then takes the explicit form

$$X = \begin{pmatrix} x_0 - x_3 & -x_1 + ix_2 \\ -x_1 - ix_2 & x_0 + x_3 \end{pmatrix},$$

thus making the determinant a mapping to the invariant dot product

$$\det(X) = (x_0)^2 - (x_1)^2 - (x_2)^2 - (x_3)^2 = x_\mu x^\mu. \tag{25}$$

A general transformation $A \in GL(2, \mathbb{C})$ acts on X in the following way:

$$X' = AXA^\dagger, \tag{26}$$

from which it can be seen that if $\det(X)$ is to be left invariant, $|\det(A)|^2 = 1$. Hence all transformations A of $GL(2,\mathbb{C})$ with $\det(A) = e^{i\phi}$ result in the same determinant invariance. Choosing the determinant to have zero phase makes the transformations be part of the $SL(2,\mathbb{C})$ group. The relation between matrices $A \in SL(2,\mathbb{C})$ and $\Lambda \in SO(3,1)$ is made explicit by the following 2 to 1 mapping:

$$\Lambda : SL(2,\mathbb{C}) \to SO(3,1) \qquad\qquad \Lambda(A)^\mu{}_\nu = \Lambda(-A)^\mu{}_\nu = \frac{1}{2}\mathrm{Tr}(\tilde{\sigma}^\mu A \sigma_\nu A^\dagger). \tag{27}$$

In the case of $HOM(2)$, the matrix A that defines a general transformation is parametrized in the following way

$$A_{HOM} = \begin{pmatrix} e^\chi & 0 \\ \alpha + i\beta & e^{-\chi} \end{pmatrix}. \tag{28}$$

By doing this, and requiring that

$$A(\chi_1, \alpha_1, \beta_1)A(\chi_2, \alpha_2, \beta_2) = A(\chi_3, \alpha_3, \beta_3),$$

the following relations are obtained:

$$\begin{aligned} \chi_3 &= \chi_1 + \chi_2, \\ \alpha_3 &= e^{\chi_2}\alpha_1 + e^{-\chi_1}\alpha_2, \\ \beta_3 &= e^{\chi_2}\beta_1 + e^{-\chi_1}\beta_2. \end{aligned}$$

Upon further reparametrization, the final form of the general homothetic transformation like the one in equation (22) is obtained for some arbitrary, two dimensional spinor plane (α, β):

$$\begin{aligned} \alpha' &= e^{\epsilon_1}\alpha + \epsilon_2, \\ \beta' &= e^{\epsilon_1}\beta + \epsilon_3. \end{aligned} \tag{29}$$

3.4 $SIM(2)$

The last group is obtained when both generators K_z and J_z are adjoined to T_1 and T_2. The last commutation relation to completely specify the algebra (on top of the ones previously given in equations (19),(20) and (21)) is already given in the middle part of equation (13), i.e

$$[J_z, K_z] = 0. \tag{30}$$

This group is isomorphic to the four-parameter group of similitude transformations, or transformations that allow for uniform scalings and rigid motions on the plane. The general form of such a transformation is given by [9]

$$\begin{aligned} x' &= e^{\epsilon_1}(x\cos\epsilon_2 - y\sin\epsilon_2) + \epsilon_3, \\ y' &= e^{\epsilon_1}(x\sin\epsilon_2 + y\cos\epsilon_2) + \epsilon_4. \end{aligned} \tag{31}$$

This is similar to the transformations performed by $HOM(2)$, but rotations are now allowed, whereas in $HOM(2)$, they would violate the orientation preservation. Once again, in order to obtain the isomorphism, the $SL(2,\mathbb{C})$ group comes in handy. The general transformation in this case takes the form

$$A_{SIM} = \begin{pmatrix} e^{\chi + i\phi} & 0 \\ \alpha + i\beta & e^{-\chi - i\phi} \end{pmatrix}, \tag{32}$$

and again, by requiring that

$$A(\chi_1,\phi_1,\alpha_1,\beta_1)A(\chi_2,\phi_2,\alpha_2,\beta_2)=A(\chi_3,\phi_3,\alpha_3,\beta_3),$$

the following relations are obtained:

$$\begin{aligned}
\chi_3 &= \chi_1+\chi_2,\\
\phi_3 &= \phi_1+\phi_2,\\
\alpha_3 &= e^{\chi_2}(\alpha_1\cos\phi_2-\beta_1\sin\phi_2)+e^{-\chi_2}(\alpha_2\cos\phi_2+\beta_2\sin\phi_2),\\
\beta_3 &= e^{\chi_2}(\alpha_1\sin\phi_2+\beta_1\cos\phi_2)+e^{-\chi_2}(\beta_2\cos\phi_2-\alpha_2\sin\phi_2),
\end{aligned}$$

which upon further reparametrization gives the final form of the group of similitude transformation in the (α,β) two dimensional spinor plane, just as in equation (31):

$$\begin{aligned}
\alpha' &= e^{\epsilon_1}(\alpha\cos\epsilon_2-\beta\sin\epsilon_2)+\epsilon_3,\\
\beta' &= e^{\epsilon_1}(\alpha\sin\epsilon_2+\beta\cos\epsilon_2)+\epsilon_4.
\end{aligned}\tag{33}$$

3.5 Geometry

By looking at the defining invariant quantity of special relativity, the inner product $x^2=x_\mu x^\mu=x_0^2-|\mathbf{x}|^2$ for some four vector x, and given more generally as the bilinear form $x^2=x^Tgx$, the form of the rank 2 metric tensor g can be determined from the allowed transformations of the specific subgroups. The process is simplified when it is done in infinitesimal form, where terms of order higher than two are dropped. The following condition is then imposed on the metric

$$x'^Tg'x'=x^T\Lambda^Tg\Lambda x \quad\rightarrow\quad g'=\Lambda^Tg\Lambda.\tag{34}$$

The complete operator is a (product of) matrix exponential, as discussed before, and in infinitesimal form this results in

$$g'=(\mathbb{1}+\alpha L^T)g(\mathbb{1}+\alpha L),$$

with L being one of the algebras' generators. So if the metric is to be invariant, i.e $g'=g$, the following condition must hold (up to and excluding second order terms)

$$gL+L^Tg=0.\tag{35}$$

This results in the following metric freedom for each of the subgroups:

1. $T(2)$:

$$g_{T(2)}=\begin{pmatrix} a & b & c & \frac{1}{2}(d+a)\\ -b & \frac{1}{2}(d-a) & 0 & -b\\ -c & 0 & \frac{1}{2}(d-a) & -c\\ \frac{1}{2}(d+a) & b & c & d\end{pmatrix}.$$

   Upon imposing the condition that the metric tensor must be a symmetric bilinear form, it becomes:

$$g_{T(2)}=\begin{pmatrix} a & 0 & 0 & \frac{1}{2}(d+a)\\ 0 & \frac{1}{2}(d-a) & 0 & 0\\ 0 & 0 & \frac{1}{2}(d-a) & 0\\ \frac{1}{2}(d+a) & 0 & 0 & d\end{pmatrix},$$

   where the matrix entries are real. If one further requires the same metric signature as that of the Minkowski metric, then d and a have more constraints to satisfy. This can be more easily investigated by performing a change of basis that makes the metric diagonal. The diagonalized version of the metric takes the form

$$g^{diag}_{T(2)}=\begin{pmatrix} \frac{1}{2}(d-a) & 0 & 0 & 0\\ 0 & \frac{1}{2}(d-a) & 0 & 0\\ 0 & 0 & \frac{1}{2}(d+a)-\frac{1}{2}\sqrt{2(d^2+a^2)} & 0\\ 0 & 0 & 0 & \frac{1}{2}(d+a)+\frac{1}{2}\sqrt{2(d^2+a^2)}\end{pmatrix}.$$

With its eigenvalues being the column vectors, one just needs to determine what conditions are needed to create the $(3,1)$ signature. An example of a choice that would not be allowed is if $d > a$, with $d, a < 0$ and $(d+a) < -\sqrt{2(d^2+a^2)}$. This would lead to a signature of the form $(++--)$, which is incompatible with the Minkowski metric $(+---)$ (or $(-+++)$, depending on the convention used), and leads to problems with causality and ordering of events as described in section 1.

2. $E(2)$: It is the same as $T(2)$, although it is obtained without having to impose the symmetry requirement:

$$g_{E(2)} = \begin{pmatrix} a & 0 & 0 & \frac{1}{2}(d+a) \\ 0 & \frac{1}{2}(d-a) & 0 & 0 \\ 0 & 0 & \frac{1}{2}(d-a) & 0 \\ \frac{1}{2}(d+a) & 0 & 0 & d \end{pmatrix}.$$

3. $HOM(2)$: apart from a scaling factor, it is the same as the Minkowski metric,

$$g_{HOM(2)} = a \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

4. $SIM(2)$: same as $HOM(2)$,

$$g_{SIM(2)} = a \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

This shows that the two first subgroups $T(2)$ and $E(2)$ could have spacetimes with atypical geometry as their modules, while the last two, $HOM(2)$ and $SIM(2)$, have flat Minkowski spacetimes as their modules.

3.6 Invariants

The next investigation aims to find out which quantities remain invariant under a given subset of Lorentz transformations. Working infinitesimally, a vector $\underline{\boldsymbol{v}} = (a, b, c, d)$ transforms as

$$\underline{\boldsymbol{v}}' = (\mathbb{1} + \alpha L)\underline{\boldsymbol{v}},$$

with L being one of the algebras' generators again. This means that for it to be invariant, the following condition must be imposed

$$L\underline{\boldsymbol{v}} = \underline{\mathbf{0}}. \tag{36}$$

As for rank 2 tensors, (which are represented by four by four matrices in this case), the condition imposed is the same as that in equation (35), without the requirement for it to be symmetric. There may be higher rank tensors which could be invariant as well, but they are not of interest at this point, as they are not needed to further differentiate the VSR subgroups from each other and from the Lorentz group.
The resulting invariant quantities for each of the four subgroups are as follows:

1. $T(2)$:

   (a) any light-like vector of the form

$$v = a(1, 0, 0, 1);$$

(b) any matrix of the form

$$M = \begin{pmatrix} a & b & c & \frac{1}{2}(d+a) \\ -b & \frac{1}{2}(d-a) & 0 & -b \\ -c & 0 & \frac{1}{2}(d-a) & -c \\ \frac{1}{2}(d+a) & b & c & d \end{pmatrix}.$$

2. $E(2)$:

(a) any light-like vector of the same form as in $T(2)$

$$v = a(1,0,0,1);$$

(b) any matrix of the form

$$M = \begin{pmatrix} a & 0 & 0 & \frac{1}{2}(d+a) \\ 0 & \frac{1}{2}(d-a) & 0 & 0 \\ 0 & 0 & \frac{1}{2}(d-a) & 0 \\ \frac{1}{2}(d+a) & 0 & 0 & d \end{pmatrix}.$$

3. $HOM(2)$ and $SIM(2)$: These do not allow for any invariant vector, and the only rank 2 tensor that is invariant is the metric tensor. However, it is argued by Cohen and Glashow that invariant scalar quantities can be built from a fixed, light-like vector that is multiplied with other kinematic variables under the Minkowski dot product [5]. While strictly speaking the light-like vector found for $T(2)$ and $E(2)$ is not invariant under $HOM(2)$ and $SIM(2)$, its *direction* along the preferred spatial axis (z in this case) is conserved. As shown before, $E(2)$ transformations do not affect it, while a boost in the z direction only scales it as $e^{-i\chi K_3}v = e^{\chi}v$. As such, it can be used to construct ratios of the form $\frac{p_1 \cdot n}{p_2 \cdot n}$, where p_1 and p_2 are kinematic variables like momentum. These ratios are invariant under the whole set of transformations from $HOM(2)$ and $SIM(2)$, but not under all Lorentz transformations, as the symmetries of the group do not allow for an invariant direction.

4 Spin Dynamics and Thomas Precession

Back in the days when physicists were constructing quantum mechanics to describe the observed spectrum of the hydrogen atom, a problem persisted in the description they had at hand. There seemed to always be a missing factor of 1/2 in the interaction energy from which the emitted photons acquired their frequency. That is when Llewellyn Thomas' insight proved useful in 1925, when he derived a relativistic correction to the expected frequency of the doublet separation in hydrogen's fine structure, and found that exact missing factor [10].

In this section, the standard derivation of Thomas' precession frequency in a relativistic classical way will be performed. It will then be followed by a derivation of the BMT equation, describing a generalization of the relativistic description of the dynamics of spin pseudo-vectors, from which Thomas precession is a natural consequence. Subsequently, possible derivations of Thomas precession in VSR mimicking the classical SR derivation shall be investigated, together with non-classical derivations of the BMT equation.

Examination and comparison of both leads to two contradictory results. However, the non-classical case is found to be acting as a correction to the established SR predictions, with a parameter setting the scale of VSR contributions, making these results more sensible than the classically derived ones.

4.1 Thomas Precession in SR

A first look at the way the problem was treated before Thomas' insight reveals what it is missing. An electron, with spin $\mathbf{s}$ and mass M, possesses a magnetic moment $\boldsymbol{\mu} = \frac{ge}{2M}\mathbf{s}$. When placed in a magnetic field $\mathbf{B}$, as viewed from the laboratory frame, it will experience a torque, given by

$$\boldsymbol{\tau} = \left(\frac{d\boldsymbol{s}}{dt}\right)_{e^-frame} = \boldsymbol{\mu} \times \mathbf{B}'. \tag{37}$$

As the equation implies, in its rest frame, the electron "sees" a magnetic field $\mathbf{B}'$, which is given by [11]

$$\mathbf{B}' = \gamma(\mathbf{B} - \mathbf{v} \times \mathbf{E}) - \frac{\gamma^2}{1+\gamma}\mathbf{v}(\mathbf{v} \cdot \mathbf{B}) \approx \mathbf{B} - \mathbf{v} \times \mathbf{E},$$

in the non-relativistic approximation, with $|\mathbf{v}| \ll 1$ and $\gamma = (1 - |\mathbf{v}|^2)^{-1/2} \approx 1$. The electric field $\mathbf{E}$ is that of the nucleus, and is approximated as a central field given by $\mathbf{E} \approx -\frac{dV}{dr}\hat{\mathbf{r}} = -\frac{dV}{dr}\frac{\mathbf{r}}{r}$, where V is the electric potential.
An interaction results as $\mathbf{B}'$ is coupled with the magnetic moment of the electron and the energy of interaction is given by

$$\begin{aligned} U = -\boldsymbol{\mu} \cdot \mathbf{B}' &= -\frac{ge}{2M}\mathbf{s} \cdot \mathbf{B} + \frac{ge}{2M}\mathbf{s} \cdot (\mathbf{v} \times \mathbf{E}) \\ &= -\frac{ge}{2M}\mathbf{s} \cdot \mathbf{B} - \frac{ge}{2Mr}\mathbf{s} \cdot (\mathbf{v} \times \mathbf{r})\frac{dV}{dr}. \end{aligned}$$

Using the definition of orbital angular momentum $\mathbf{L} = M\mathbf{r} \times \mathbf{v}$, this can be written as

$$U = -\frac{ge}{2M}\mathbf{s} \cdot \mathbf{B} + \frac{ge}{2M^2r}\mathbf{s} \cdot \mathbf{L}\frac{dV}{dr}. \tag{38}$$

The last term is what makes up the correction due to spin-orbit interaction. The known value of the dimensionless gyromagnetic ratio $g \approx 2$, correctly fits the observation of the Zeeman effect exhibited in the first term. However, it does not fit the observed splitting of the spectral lines, given by the second term, as it is twice as large. Hence the need for a missing $1/2$ factor in the second term of the interaction energy.
That is when Thomas realized that the electron frame cannot be described by an inertial frame, as it is orbiting the nucleus due to the central electric field. The way he treated it is by applying infinitesimal transformations that relate electron inertial frames at different, infinitesimally separated, times, where the velocities also change direction infinitesimally, effectively resulting in rotations [10]. The following text's aim is to derive his results.

Let S be the laboratory frame, coinciding with the nucleus frame. Let S' be the electron rest frame at time t, with velocity $\mathbf{v}$, and S'' be the electron's frame at time $t+\delta t$ and velocity $\mathbf{v}+\delta\mathbf{v}$. S'' and S' are related by the following boost

$$\underline{\boldsymbol{x}}'' = \Lambda_{boost}(\mathbf{v} + \delta\mathbf{v})\underline{\boldsymbol{x}}',$$

while S and S' are related by

$$\underline{\boldsymbol{x}}' = \Lambda_{boost}(\mathbf{v})\underline{\boldsymbol{x}}.$$

Thus in order to relate S'' and S, the following transformation should be performed:

$$\underline{\boldsymbol{x}}'' = \Lambda_{boost}(\mathbf{v} + \delta\mathbf{v})[\Lambda_{boost}(\mathbf{v})]^{-1}\underline{\boldsymbol{x}}. \tag{39}$$

Choosing the orbital plane to be the xy plane, the initial velocity can be taken to be in the x direction, $\mathbf{v} = v\hat{\boldsymbol{x}}$ and the subsequent infinitesimal velocity along both the x and y directions, $\delta\mathbf{v} = \delta v_x\hat{\boldsymbol{x}} + \delta v_y\hat{\boldsymbol{y}}$. The transformation that relates the lab frame S and the the electron frame S' is then given by [12]

$$[\Lambda_{boost}(\mathbf{v})]^{-1} = \Lambda_{boost}(-\mathbf{v}) = \begin{pmatrix} \gamma & v\gamma & 0 & 0 \\ v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

To get the full transformation, the infinitesimal change should be computed. This will be done by using the regular transformation equation [11],

$$\begin{aligned} t' &= \gamma(t - \mathbf{v}\cdot\mathbf{x}), \\ \mathbf{x}' &= \mathbf{x} + \frac{\gamma-1}{v^2}(\mathbf{v}\cdot\mathbf{x})\mathbf{v} - \gamma\mathbf{v}t, \end{aligned} \tag{40}$$

and plugging the velocity $\mathbf{v} + \delta\mathbf{v}$ in it and only keeping linear terms in $\delta\mathbf{v}$. Starting with the gamma factor, it becomes:

$$\begin{aligned} \gamma' &= \frac{1}{\sqrt{1 - |\mathbf{v}+\delta\mathbf{v}|^2}} = \frac{1}{\sqrt{1-v^2}\sqrt{1 - \frac{2\mathbf{v}\cdot\delta\mathbf{v}}{1-v^2}}} \\ \gamma' &\approx \gamma + \gamma^3 \mathbf{v}\cdot\delta\mathbf{v}, \end{aligned} \tag{41}$$

with $\gamma = (1-v^2)^{-1/2}$, and the last equality obtained by expanding $\left(1 - \frac{2\mathbf{v}\cdot\delta\mathbf{v}}{1-v^2}\right)^{-1/2}$. After some calculations, the transformation from S to S'' , in four vector form, is given by

$$\Lambda(\mathbf{v}+\delta\mathbf{v}) = \begin{pmatrix} \gamma + \gamma^3 v\delta v_x & -v\gamma + \gamma^3\delta v_x & -\gamma\delta v_y & 0 \\ -v\gamma + \gamma^3\delta v_x & \gamma + \gamma^3 v\delta v_x & \frac{\gamma-1}{v}\delta v_y & 0 \\ -\gamma\delta v_y & \frac{\gamma-1}{v}\delta v_y & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and the transformation from S' to S'' by

$$\Lambda_{boost}(\mathbf{v}+\delta\mathbf{v})[\Lambda_{boost}(\mathbf{v})]^{-1} = \begin{pmatrix} \gamma + \gamma^3 v\delta v_x & -v\gamma + \gamma^3\delta v_x & -\gamma\delta v_y & 0 \\ -v\gamma + \gamma^3\delta v_x & \gamma + \gamma^3 v\delta v_x & \frac{\gamma-1}{v}\delta v_y & 0 \\ -\gamma\delta v_y & \frac{\gamma-1}{v}\delta v_y & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma & v\gamma & 0 & 0 \\ v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

After simplifying each entry, it results in

$$\Lambda(S \to S'') = \begin{pmatrix} 1 & -\gamma^2\delta v_x & -\gamma\delta v_y & 0 \\ -\gamma^2\delta v_x & 1 & \frac{\gamma-1}{v}\delta v_y & 0 \\ -\gamma\delta v_y & -\frac{\gamma-1}{v}\delta v_y & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{42}$$

This equation can be expressed as an infinitesimal boost, followed by an infinitesimal rotation. This gives, in terms of the boost and rotation generators, and the angular and velocity change:

$$\Lambda(S \to S'') = \mathbb{1} - i\delta\boldsymbol{\theta}\cdot\mathbf{J} - i\delta\mathbf{v}\cdot\mathbf{K}, \tag{43}$$

from which the angular change is identified as

$$\delta\boldsymbol{\theta} = \left(0, 0, -\frac{\gamma-1}{v}\delta v_y\right) = -\frac{\gamma^2}{\gamma+1}\mathbf{v}\times\delta\mathbf{v}.$$

By taking this transformation to happen at each instant δt, the rate at which the angle changes results in the precession frequency and its direction

$$\boldsymbol{\omega}_T = -\frac{1}{\gamma}\frac{\delta\boldsymbol{\theta}}{\delta\tau} = \frac{\gamma}{\gamma+1}\mathbf{v}\times\mathbf{a}, \tag{44}$$

where $\mathbf{a} = \frac{\delta\mathbf{v}}{\delta\tau}$ is the acceleration due to the electric field of the nucleus. The equation of motion for a spinning electron, as observed from the laboratory frame is given by

$$\left(\frac{d\mathbf{s}}{dt}\right)_{Labframe} = \left(\frac{d\boldsymbol{s}}{dt}\right)_{e^- frame} + \boldsymbol{\omega}_T \times \mathbf{s}, \tag{45}$$

so that the incorrect equation (37) becomes

$$\begin{aligned}\left(\frac{d\mathbf{s}}{dt}\right)_{e^-} &= \boldsymbol{\mu}\times(\mathbf{B}-\mathbf{v}\times\mathbf{E})-\boldsymbol{\omega}_T\times\mathbf{s}\\ &= \boldsymbol{\mu}\times\left(\mathbf{B}-\mathbf{v}\times\mathbf{E}-\frac{2M}{ge}\boldsymbol{\omega}_T\right),\end{aligned}$$

and the interaction energy is

$$U=-\boldsymbol{\mu}\cdot\left(\mathbf{B}-\mathbf{v}\times\mathbf{E}+\frac{2M}{ge}\boldsymbol{\omega}_T\right).$$

Writing the precession frequency (44) in a non relativistic limit (as electrons bound to atomic orbitals do no possess relativistic energies), and plugging the expression for the acceleration gives

$$\boldsymbol{\omega}_T=\frac{\gamma}{\gamma+1}\mathbf{v}\times\left(-\frac{e}{M}\frac{dV}{dr}\frac{\mathbf{r}}{r}\right)\approx\frac{e}{2M^2r}\frac{dV}{dr}\mathbf{L},$$

which results in the correct interaction energy accounting for both the splitting of the multiplets and the Zeeman effect:

$$U=-\frac{ge}{2M}\mathbf{s}\cdot\mathbf{B}+\frac{(g-1)}{2M^2r}\frac{dV}{dr}\mathbf{s}\cdot\mathbf{L}. \tag{46}$$

4.2 Bargmann-Michel-Telegdi Equation

The Bargmann-Michel-Telegdi (BMT) equation is a more general formulation of the dynamics of spin in a covariant form [13]. It basically states the same as what was derived in the previous subsection, but can be approached more generally. In particular, the relativistic correction derived by Thomas is a natural consequence of this more general equation. It can also be used in an experimental setting to predict what the rate of spin precession is when a charged particle is placed in an electromagnetic field as observed from any frame, including laboratory frame.
At this point, its derivation needs a couple of assumptions. A spin four vector s^μ is assumed to exist and has its spatial component coincide with the spatial spin $\mathbf{s}$ in the particle's rest frame, $s^\mu_{e^-}=(0,\mathbf{s})$. Spin, like orbital angular momentum has the characteristic of being orthogonal to the velocity. This is also applicable for four-spin and four-velocity,

$$s_\mu u^\mu=0\ \rightarrow\ s^0=\mathbf{s}\cdot\mathbf{v} \tag{47}$$

which, in the rest frame of the electron, reduces to $s^0=0$. The purpose of the derivation is to find a covariant form of the equation of motion

$$\frac{d\mathbf{s}}{dt}=\frac{ge}{2M}\mathbf{s}\times\mathbf{B} \tag{48}$$

The relevant quantities would then naturally be covariant ones, i.e $\frac{ds^\mu}{d\tau}$, u^μ, $F^{\mu\nu}$ and a^μ, where the last two are the electromagnetic field strength tensor and four-acceleration, which could be caused by electromagnetic forces or other, non-electromagnetic ones. In order to construct the equation, one should notice that only linear terms in the fields and spin are present in equation (48). Hence it is expected that the covariant expression will be a superposition of all possible four vectors that are linear in those terms.

$$\frac{ds^\mu}{d\tau}=AF^{\mu\nu}s_\nu+B(s_\alpha F^{\alpha\beta}u_\beta)u^\mu+C\left(s_\alpha\frac{du^\alpha}{d\tau}\right)u^\mu. \tag{49}$$

By taking the rate of change of the contraction $s_\alpha u^\alpha=0$, the following relation is obtained

$$u^\alpha\frac{ds_\alpha}{d\tau}=-s_\alpha\frac{du^\alpha}{d\tau}=-\frac{1}{M}\left(s_\alpha F^\alpha+es_\alpha F^{\alpha\beta}u_\beta\right),$$

where the last equality follows from writing the acceleration in terms of the forces (F^α is the non-electromagnetic force). Contracting equation (49) with u_μ, and plugging the previous expression results in

$$\begin{aligned}
-\frac{1}{M}s_\alpha F^\alpha &= -\frac{1}{M}Cs_\beta F^\beta \\
-\frac{e}{M}u_\alpha F^{\alpha\beta}s_\beta &= Au_\beta F^{\beta\alpha}s_\alpha - B\left(s_\mu F^{\mu\nu}u_\nu\right) - \frac{eC}{M^2}\left[s_\beta\left(\frac{M}{e}F^\beta + u_\alpha F^{\alpha\beta}\right)\right],
\end{aligned}$$

after separating the independent, non-electromagnetic force term. This gives $C = 1$, and $(A + B)u_\alpha F^{\alpha\beta}s_\beta = 0$. By considering a situation in which the 3-velocity is $\mathbf{0}$ (a rest frame) and the electric field is absent in $F^{\alpha\beta}$, the spatial part of equation (49) becomes

$$\frac{ds^i}{dt} = AF^{ij}s_j \rightarrow \frac{d\mathbf{s}}{dt} = A\left(\mathbf{s}\times\mathbf{B}\right),$$

which, when compared to equation (48), suggests $A = \frac{ge}{2M}$. A more detailed calculation of the coefficients A,B and C can be found in [11], and the final result is the BMT equation:

$$\frac{ds^\mu}{d\tau} = \frac{ge}{2M}F^{\mu\nu}s_\nu + \frac{(g-2)e}{2M}u^\mu(F^{\alpha\beta}s_\alpha u_\beta). \tag{50}$$

Alternatively, one can derive equation (50) without the need of imposing linearity conditions on the covariant vectors aforementioned. The interested reader is referred to a paper by Krzysztof Rebilas [14].

4.3 Thomas Precession in VSR and the BMT equation

The idea behind VSR is that it could be the actual symmetry of nature, rather than SR, where the latter would then just be an approximate description in most observable circumstances. As such, its aim is to reproduce the symmetries of SR, and where SR fails, VSR should still hold ground. The fact that Lorentz violating phenomena are very weak in nature [5] would be reflected in any departure from SR, which would also be weak. This means that, predictions made by VSR should only slightly differ from those of SR, and this is expected to show up in some small parameter that will mark a departure from Lorentz invariance.
This section will discuss derivations of the VSR version of Thomas precession, as done by S. Das and S.Mohanty [15], and a previous Bachelor thesis [16]. It will then be followed by a presentation of the results from J.Alfaro and V.O. Rivelles [17], which performed a derivation of a VSR-extended BMT equation based on quantum field theoretical tools. An analysis of their derivations will indicate which group has more sensible results.

4.3.1 A Classical Derivation

By choosing the parameters of $HOM(2)$ such that a SR boost that takes an object from a rest frame to a frame moving at velocity $\mathbf{v} = (v_x, v_y, v_z)$ is reproduced, the following transformation is obtained [15]

$$L(\mathbf{v}) = \begin{pmatrix} \gamma & \frac{v_x}{1-v_z} & \frac{v_y}{1-v_z} & \gamma\frac{v_z-v^2}{1-v_z} \\ v_x\gamma & 1 & 0 & -\gamma v_x \\ v_y\gamma & 0 & 1 & -\gamma v_y \\ v_z\gamma & \frac{v_x}{1-v_z} & \frac{v_y}{1-v_z} & \gamma\frac{1-v^2}{1-v_z} \end{pmatrix}, \tag{51}$$

where the parameters of the transformation $L(\mathbf{v}) = \exp(-i\alpha T_1)\exp(-i\beta T_2)\exp(-i\chi_3 K_3)$ are

$$\begin{aligned}
\alpha &= \frac{v_x}{1-v_z}, \\
\beta &= \frac{v_y}{1-v_z}, \\
\chi_3 &= -\ln(\gamma - \gamma v_z).
\end{aligned}$$

The choice of parameters ensures that the velocity addition rule of SR is obeyed [15]. It should be noted that by specifying these parameters, the trajectory the particle travels on is already restricted. As such, it is possible that such a transformation does not properly describe the rotational motion of the electron. When applying the same reasoning as the classical derivation from Thomas — where the transformation that relates the frame S', at time t with velocity $\mathbf{v}$, to a frame S'' at a time $t+\delta t$, with velocity $\mathbf{v}+\delta\mathbf{v}$, is given by $L(S' \to S'') = L(\mathbf{v}+\delta\mathbf{v})$, and the transformation relating the lab frame to the rest frame at time $t+\delta t$ by $L(S \to S'') = L(\mathbf{v}+\delta\mathbf{v})[L(\mathbf{v})]^{-1}$ — the following is obtained

$$L(S \to S'') = \begin{pmatrix} 1 & \gamma^2\delta v_x & \delta v_y & -\gamma^2 v_x\delta v_x \\ \gamma^2\delta v_x & 1 & 0 & -\gamma^2\delta v_x \\ \delta v_y & 0 & 1 & -\delta v_y \\ -\gamma^2 v_x\delta v_x & \gamma^2\delta v_x & \delta v_y & 1 \end{pmatrix}. \tag{52}$$

If looking at it as a product of an infinitesimal rotation and an infinitesimal boost, like in equation (43), the angle is identified as $\delta\boldsymbol{\theta} = (-\delta v_y, \gamma^2\delta v_x, 0)$, and the VSR precession frequency as

$$\boldsymbol{\omega}_{VSR} = -\frac{\delta\boldsymbol{\theta}}{\delta t} = \frac{y}{Mr}\frac{dV}{dr}\hat{\boldsymbol{x}}, \tag{53}$$

where it is assumed that the instantaneous acceleration is only in the y direction. This procedure does not really warrant a correct identification of the angular change, as the inifinitesimal generators of the VSR groups are different from the pure rotation and pure boost generators of the proper Lorentz group.
This ends up with an interaction energy given by

$$U = -\frac{ge}{2M}\mathbf{s}\cdot\mathbf{B} + \frac{g}{2M^2 r}\frac{dV}{dr}\mathbf{s}\cdot\mathbf{L} + s_x\frac{y}{Mr}\frac{dV}{dr}, \tag{54}$$

which is wrong by a factor of two when the spin direction is aligned with the z-axis.

Proceeding with a specific experimental setting, where an external field $\mathbf{B} = B_z\hat{\mathbf{z}}$ is applied and the charged particle is orbiting in the xy plane, the lab frame equation of motion for the normal SR Thomas precession reads

$$\left(\frac{d\mathbf{s}}{dt}\right)_{Lab} = \left(\frac{d\mathbf{s}}{dt}\right)_{e^-} + \ \boldsymbol{\omega}_T \times \mathbf{s} = (\boldsymbol{\omega}_L + \boldsymbol{\omega}_T)\times\mathbf{s},$$

where $\boldsymbol{\omega}_L$ is known as the Larmor frequency. Summing up the two frequencies, one obtains for an ultra-relativistic ($\gamma \to \infty$) frequency magnitude:

$$\omega_{\text{total}} = \frac{eB_z}{2M}\left|g-2\right|. \tag{55}$$

In the VSR case, there is no Thomas-like precession in the ultra relativistic setting, and one ends up with

$$\omega_{\text{total}} = \frac{geB_z}{2M}, \tag{56}$$

which is a factor of order 10^3 larger than the frequency in equation (55), when plugging in the value of g (≈ 2.00232). This means there is a relatively large disagreement between the predictions of SR and those of VSR for highly energetic particles placed in, for example, a particle accelerator.
A similar situation occurs with the predictions made by K. Hakvoort [16], where he also tries to reproduce a SR transformation using VSR symmetry tools. He derives a Thomas precession frequency magnitude given by $\omega_T = \frac{av}{2c^2}$, and a VSR frequency given by $\omega_{VSR} = \frac{a}{v} + \frac{5av^3}{8c^4}$, where a is the particle's acceleration and v its speed as measured in the lab frame. For the sake of argumentation, the speed of light c is included and not set to $c=1$. The two differ by a factor of order 10^4 when the speed v is approximated with an order of magnitude given by $\alpha c \approx c/137$,

which is again too large for a theory that is expected to act as a correction of SR. Moreover, when looked at from the non-relativistic limit, where $c \to \infty$, Thomas precession effects are expected to vanish, as $\omega_T = \frac{av}{2c^2}$. But the VSR derived expression for the frequency does not show this behavior, pointing towards wrong non-relativistic results.

4.3.2 A Quantum Derivation

Taking a look at the last derivation of the dynamics of spins under the VSR groups, a different picture emerges, which agrees with how the deviations should behave in principle. This one relies on Quantum Field Theory to obtain the Lorentz violating results. It builds on a technique in which Lorentz violating terms are constructed by using the VSR invariant ratios of dot products of kinematic variables with a fixed null vector, as discussed in section 2. The inclusion of such a null vector violates space isotropy, as there is now a preferred direction in space. Lorentz symmetry becomes restricted to $E(2)$, but is broadened to $SIM(2)$ when the generator K_z is included. In the paper [17], a reference is made to how standard model fields are coupled to Lorentz invariance violating background fields from a more fundamental theory. Those background fields would then be a manifestation of the preferred direction in VSR.

The Dirac equation for a massive fermion of mass M in VSR is described by the following equation [18]

$$\left(i\partial\!\!\!/ - \frac{1}{2}m^2\frac{n\!\!\!/}{n^\mu P_\mu} - M\right)\psi(\underline{\boldsymbol{x}}) = 0, \tag{57}$$

where the modification can be found in the middle term, characterized by the VSR invariant ratio, and the VSR mass scale m. The slashed notation means a contraction with a four-vector's worth of gamma matrices, $O\!\!\!/ \equiv \gamma_\mu O^\mu$, and $P^\mu \equiv i\partial^\mu$. The gamma matrices are defined as any matrices satisfying the Clifford algebra relation

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{1}.$$

The Hamiltonian is then also modified to accomodate the VSR mass scale and becomes

$$H = \frac{1}{2}\left(P^2 - m^2 - M^2\right) \tag{58}$$

which is obtained through a supersymmetry constraint $\mathcal{S}$ that closes on the Hamiltonian through Poisson brackets [17]. This shifts the original, special relativity based descriptions by a squared VSR mass term.

Maxwell's equations are elegantly packed into one covariant equation by use of the electromagnetic field strength tensor $F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu$, where $\underline{\boldsymbol{A}} = (\phi, \mathbf{A})$ is the electromagnetic four-potential, composed of the electric scalar potential ϕ and the magnetic vector potential $\mathbf{A}$. In vacuum, and in the absence of sources, the equations take the form

$$\partial^\alpha\partial_\alpha F^{\mu\nu} \equiv \Box F^{\mu\nu} = 0 \tag{59}$$

Maxwell's equations also need to be modified to accommodate the VSR mass. The authors of the paper [17] have derived such an expression and it is given by the field equation

$$\left(\partial^\mu + \frac{i}{2}m^2\frac{n^\mu}{n^\lambda P_\lambda}\right)F_{\mu\nu} = 0 \tag{60}$$

which ends up as an extended Maxwell equation:

$$\Box F_{\mu\nu} + m^2 F_{\mu\nu} = 0. \tag{61}$$

The modified version of Maxwell's equations indicates that the fields $F_{\mu\nu}$ have a mass m (massive photons), and are not invariant under the usual gauge transformations of the form $A'_\mu = A_\mu + \partial_\mu \Lambda$. However, by introducing the differential operator

$$D_\mu = \partial_\mu + \frac{i}{2} m^2 \frac{n_\mu}{n^\lambda P_\lambda} \tag{62}$$

the fields are then given by $F_{\mu\nu} = D_\mu A_\nu - D_\nu A_\mu$, and they obey an "updated" version of the gauge transformations:

$$A'_\mu = A_\mu + D_\mu \Lambda. \tag{63}$$

Thus, after modifying electromagnetism to fit VSR, a massive photon field is obtained, described by a field equation $D^\mu F_{\mu\nu}$ and a modified gauge invariance given by equation (63). At this point, it should be noted that SR results are recovered when $m \to 0$, as is expected. The result also gives an indication of the scale of VSR contributions, as upper bounds on the photon mass have been computed from many groups recently [19] [20].
In order to derive the BMT equation, a choice is imposed on the constraints. The spinning particle is coupled to the modified field by the substitution $P^\mu \to P^\mu - qA^\mu$, and the standard BMT equation is obtained in terms of an axial spin vector defined as

$$s^\mu = \epsilon^{\mu\nu\alpha\beta} u_\nu \Psi_\alpha \Psi_\beta, \tag{64}$$

where Ψ^μ and Ψ_5 are fermionic fields, satisfying the anti-commutation relations

$$\{\Psi^\mu, \Psi^\nu\} = \frac{i}{2}\eta^{\mu\nu} \qquad\qquad \{\Psi_5, \Psi_5\} = -\frac{i}{2}.$$

In the VSR case, the supersymmetry constraint is also modified such that the Hamiltonian becomes a relatively complicated equation [17],

$$\begin{aligned}
\mathcal{H} =& \frac{1}{2}\left[(P-qA)^2 - m^2 - M^2\right] - i\left(q + 2\mu M\right) F_{\mu\nu}\Psi^\mu\Psi^\nu \\
& - 4i\mu F_{\mu\nu}\left(P^\mu - qA^\mu\right)\Psi^\nu\Psi_5 + 2\mu^2 (F_{\mu\nu}\Psi^\mu\Psi^\nu)^2 \\
& + iqm^2\Psi^\lambda n_\lambda \frac{F_{\mu\nu}\Psi^\mu n^\nu}{\left[\left(P^\alpha + eA^\alpha\right)n_\alpha\right]^2} - 2i\mu m^2 \frac{F_{\mu\nu}\Psi^\mu n^\nu}{\left(P^\alpha - qA^\alpha\right)n_\alpha}\Psi_5 \\
& + 2\mu m^2 \Psi^\lambda n_\lambda \frac{n^\rho \partial_\rho F_{\mu\nu}\Psi^\mu\Psi^\nu}{\left[\left(P^\alpha - qA^\alpha\right)n_\alpha\right]^2}\Psi_5,
\end{aligned} \tag{65}$$

where q is the particle's charge and $\mu = \frac{q}{2M}\left(\frac{g}{2} - 1\right)$ its magnetic moment. From that, a specific gauge is chosen and expressed in terms of the particle's mass $\sqrt{m^2 + M^2}$. When computing the rate of change of equation (64), the expression no longer gives a short and concise form to the BMT equation, and as such, a modification of s^μ is required. One such choice, satisfying equation (47) involves the fixed null vector n^μ:

$$\tilde{S}^\mu = \frac{1}{u^\alpha n_\alpha}\epsilon^{\mu\nu\rho\sigma} u_\nu n_\rho \Psi_\sigma \Psi_5 \tag{66}$$

The authors of the paper [17] have found that the only combination of equations (64) and (66) that result in a simplification of the modified BMT equation is

$$S^\mu_T = s^\mu - \frac{m^2}{M\sqrt{m^2 + M^2}}\tilde{S}^\mu \tag{67}$$

Considering only terms up to order m^2/M^2 (as $m^2 \ll M^2$) and after a tedious calculation, the VSR-modified BMT equation takes the form:

$$\begin{aligned}\frac{dS_T^\mu}{d\tau} =& \frac{1}{M}\left(1-\frac{1}{2}\frac{m^2}{M^2}\right)(q+2\mu M)\,F^{\mu\nu}(S_T)_\nu \\ &+2\mu\left(u^\mu-\frac{1}{2}\frac{m^2}{M^2}\frac{n^\mu}{u^\alpha n_\alpha}\right)F_{\sigma\tau}S_T^\sigma u^\tau+\mu\frac{m^2}{M^2}F^{\mu\nu}u_\nu\frac{S_T^\alpha n_\alpha}{u^\beta n_\beta} \\ &+\frac{qm^2}{2M^3}F^{\mu\nu}n_\nu\frac{S_T^\alpha n_\alpha}{(u^\beta n_\beta)^2}+\frac{qm^2}{2M^3}\left(u^\mu-\frac{n^\mu}{u^\alpha n_\alpha}\right)\frac{F_{\sigma\tau}S_T^\sigma n^\tau}{u^\beta n_\beta} \\ &-\frac{qm^2}{2M^3}u^\mu(F_{\lambda\omega}u^\lambda n^\omega)\frac{S_T^\alpha n_\alpha}{(u^\beta n_\beta)^2},\end{aligned} \tag{68}$$

which, as can easily be seen when expressed in this form, is equivalent to equation (50) in the limit where $m \to 0$.
To check that their derivations are consistent and in the right direction, they thought of an other way to derive the extended BMT equation. This time, they made use of a distribution function for the spinning particle and calculated the expectation value of the four-spin. It is found that $\langle S_T^\mu \rangle$ reduces to equation (68) when $m^2 \ll M^2$, providing the team with a consistency check.

Comparing the derivations of sections 4.3.1 and 4.3.2, it is found that the results of J. Alfaro and V.O. Rivelles [17] are more sensible than the ones from S. Das and S.Mohanty [15], or those from K. Hakvoort [16]. This may be due to the procedure adapted by the latter, in which the derivation mimics the classical derivation done by Thomas. Reducing the description of the circular motion of the electron to the VSR symmetry group might be the cause of the significant disagreement between the results of the two groups. In the case of Thomas' derivation, the transformations are indeed part of the Lorentz symmetry group. However if the fundamental symmetries of nature are more restricted than Lorentz, it does not mean that only those transformations are accessible to describe the motion of objects. Moreover, another indication that the classical results might not be correct is the fact that the coupling to a VSR preferred direction is not exhibited throughout the derivation, and neither is a tunable parameter that could set the scale of VSR corrections. On top of that, K.Hakvoort's derivations [16] exhibit some strange behavior in that the VSR precession frequency does not vanish in the non-relativistic limit ($c \to \infty$).

5 Conclusion

Starting with an overview of SR, this bachelor thesis' aim was to investigate the idea that a better description of the symmetries of nature might be given by subgroups of Lorentz groups, which together with spacetime translations, make up what is known as VSR. To this end, characteristics of the subgroups proposed by Glashow and Cohen have been uncovered. This made it possible to differentiate them from the full Lorentz group and between themselves.
First, the identification of isomorphisms that relate the groups to actions on a plane were determined. It was straightforward for the first two subgroups $T(2)$ and $E(2)$, but more subtle for $HOM(2)$ and $SIM(2)$, as they needed to be expressed in terms of the $SL(2,\mathbb{C})$ group. As these subgroups had a more restricted number of parameters determining the transformations acting on submodules of the Minkowski space, it was expected that the metric would have more freedom than the Minkowski metric. However this was only true for $T(2)$, $E(2)$, as the two other had the same signature as the Minkowski metric. A peculiar property of the VSR subgroups, when compared to other subgroups of the Lorentz group (e.g. $SO(3)$), is that the complete inhomogeneous Lorentz group is obtained when the discrete symmetries P and/or T are adjoined to any of the VSR subgroups. After calculating the invariant vectors and tensors, it was also found that the groups admit a preferred direction direction in space, which, except for a scaling along the preferred direction in the case of $HOM(2)$ and $SIM(2)$, is kept invariant through all possible symmetry transformations of VSR groups. This difference is important as it naturally connects to the possible existence of weak background fields obeying VSR symmetry, to which fields obeying Lorentz symmetry can be coupled, and where a departure from Lorentz symmetry could be attributed to their effects.
The second part of the thesis studied results obtained in VSR from the description of the dynamics of a charged particle in an electromagnetic field. It was found by the first team [15] that VSR is incompatible with Thomas precession, while the results provided by the second team [17] support the opposite. The derivations obtained by the second team were found to be more sensible as they exhibit a small mass parameter m, originating from the VSR invariant background field, which tunes VSR contributions as corrections to SR predictions. From the BMT-extended equation (68), the VSR corrections to Thomas precession can be calculated and these are expected to be small, as they are of order m^2/M^2. Searches for Lorentz invariance violating phenomena, such as photon polarization anisotropies over astronomical scales [21], or predictions on photon mass [19] [20] could be a measure of the scale at which VSR effects, if they exist at all, kick in, and as such these effects could be tested in principle.

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